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Lower bound on the blow-up rate of the axisymmetric Navier-Stokes equations

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Abstract

Consider axisymmetric strong solutions of the incompressible Navier-Stokes equations in $\mathbb{R}^3$ with non-trivial swirl. Such solutions are not known to be globally defined, but it is shown in [11, 1] that they could only blow up on the axis of symmetry. Let z denote the axis of symmetry and r measure the distance to the z-axis. Suppose the solution satisfies the pointwise scale invariant bound $|v(x,t)| \le C_*(r^2 - t)^{-1/2}$ for $-T_0 \le t < 0$ and $0 < C_* < \infty$ allowed to be large, we then prove that v is regular at time zero.

1 Introduction

The incompressible Navier-Stokes equations in *cartesian coordinates* are given by

$$\partial_t v + (v \cdot \nabla)v + \nabla p = \Delta v, \quad \operatorname{div} v = 0. \tag{N-S}$$

The velocity field is $v(x,t) = (v_1, v_2, v_3) : \mathbb{R}^3 \times [-T_0, 0) \to \mathbb{R}^3$ and $p(x,t) : \mathbb{R}^3 \times [-T_0, 0) \to \mathbb{R}$ is the pressure. It is a long standing open question to determine if solutions with large smooth initial data of finite energy remain regular for all time.

In this paper we consider the special class of solutions which are *axisymmetric*. This means, in *cylindrical coordinates* r, θ, z with $(x_1, x_2, x_3) = (r\cos\theta, r\sin\theta, z)$, that the solution is of the form

$$v(x,t) = v_r(r,z,t)e_r + v_\theta(r,z,t)e_\theta + v_z(r,z,t)e_z. \tag{1.1}$$

The components v_r, v_θ, v_z do not depend upon θ and the basis vectors e_r, e_θ, e_z are

$$e_r = \left(\frac{x_1}{r}, \frac{x_2}{r}, 0\right), \quad e_\theta = \left(-\frac{x_2}{r}, \frac{x_1}{r}, 0\right), \quad e_z = (0,0,1).$$

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The main result of our paper shows that axisymmetric solutions must blow up faster than the scale invariant rate which appears in (1.2) below.

For $R > 0$ define $B(x_0, R) \subset \mathbb{R}^3$ as the ball of radius R centered at x_0. The parabolic cylinder is $Q(X_0, R) = B(x_0, R) \times (t_0 - R^2, t_0) \subset \mathbb{R}^{3+1}$ centered at $X_0 = (x_0, t_0)$. If the center is the origin we use the abbreviations $B_R = B(0, R)$ and $Q_R = Q(0, R)$.

Theorem 1.1 *Let (v, p) be an axisymmetric solution of the Navier-Stokes equations* (N-S) *in $D = \mathbb{R}^3 \times (-T_0, 0)$ for which $v(x, t)$ is smooth in x and Hölder continuous in t. Suppose the pressure satisfies $p \in L^{5/3}(D)$ and v is pointwise bounded as*

$$|v(x,t)| \le C_*(r^2 - t)^{-1/2}, \quad (x,t) \in D. \tag{1.2}$$

The constant $C_* < \infty$ is allowed to be large. Then $v \in L^\infty(B_R \times [-T_0, 0])$ for any $R > 0$.

We remark that the exponent $5/3$ for the norm of p can be replaced. However, it is the natural exponent occurring in the existence theory for weak solutions, see e.g. [2626, 26], [11, 1].

Recall the natural scaling of Navier-Stokes equations: If (v, p) is a solution to (N-S), then for any $\lambda > 0$ the following rescaled pair is also a solution:

$$v^\lambda(x,t) = \lambda v(\lambda x, \lambda^2 t), \quad p^\lambda(x,t) = \lambda^2 p(\lambda x, \lambda^2 t). \tag{1.3}$$

Suppose a solution $v(x, t)$ of the Navier-Stokes equations blows up at $X_0 = (x_0, t_0)$. Leray [1616, 16] proved that the blow up rate in time is at least

$$\|v(\cdot, t)\|_{L^\infty_x} \ge C(t_0 - t)^{-1/2}.$$

Caffarelli, Kohn, and Nirenberg [11, 1] showed that for such a blow-up solution the average of $|v|$ over $Q(X_0, R)$ satisfies

$$\left(\frac{1}{|Q_R|} \int_{Q(X_0,R)} |v|^3 + |p|^{3/2} dxdt \right)^{1/3} \ge \frac{C}{R}.$$

See also [1919, 192222, 223737, 37]. Thus, the natural rate for blow-up is at least

$$|v(x,t)| \sim \frac{O(1)}{[(x_0 - x)^2 + t_0 - t]^{1/2}}. \tag{1.4}$$

Both this and the rate (1.2) are invariant under the natural scaling (1.3).

The Serrin type criteria [2828, 281313, 1355, 577, 72929, 293131, 3188, 8] states that v is regular if it satisfies

$$\|v\|_{L^s_t L^q_x(Q_1)} < \infty, \quad \frac{3}{q} + \frac{2}{s} \le 1, \; s, q \in (2, \infty), \quad \text{or} \quad (s, q) = (2, \infty). \tag{1.5}$$

Above, for a domain $\Omega \subset \mathbb{R}^3$, we use the definition

$$\|v\|_{L^s_t L^q_x(\Omega \times (t_1, t_2))} := \| \|v(x,t)\|_{L^q_x(\Omega)} \|_{L^s_t(t_1, t_2)}.$$

For any $X_0 = (x_0, t_0) \in Q_1$, (1.5) implies the following local smallness of v:

$$\lim_{R \downarrow 0} \|v\|_{L_t^s L_x^q (Q(X_0,R))} = 0. \tag{1.6}$$

Therefore (1.5) is a so-called ϵ-regularity criterion since it implies that the norm is locally small. For $(q, s) = (3, \infty)$, (1.6) does not follow from (1.5). Hence the $(q, s) = (3, \infty)$ end point regularity criterion (1.5) proved in [2727, 2744, 4] is not an ϵ-regularity type theory.

However these criteria do not rule out blow-up with the natural scaling rate (1.4). It is a fundamental problem in the study of the incompressible Navier-Stokes equations to determine if solutions to (N-S) with the following scale invariant bound are regular

$$|v(x,t)| \leq \frac{C}{[(x_0 - x)^2 + t_0 - t]^{1/2}}. \tag{1.7}$$

If a self-similar solution satisfies this bound then it is known to be zero [3535, 35] (the self-similar solution from [2222, 22] belongs to $L_t^\infty L_x^3$).

Theorem 1.1 rules out singular axisymmetric solutions satisfying the bound (1.7). In fact (1.2) is considerably weaker than (1.7) and is also not a borderline case of the Serrin type criterion. For example (1.2) implies that $v \in L^q(Q_1)$ for $q < 4$, but not for $q \in [4, 5)$. The borderline of the Serrin type criterion, on the other hand, is $v \in L^5(Q_1)$.

We now recall the previous results on the regularity of axisymmetric solutions to the Navier-Stokes equations. Global in time regularity was first proved under the *no swirl* assumption, $v_\theta = 0$, independently by Ukhovskii-Yudovich [3636, 36] and Ladyzhenskaya [1414, 14]. See [1515, 15] for a refined proof and [1111, 11] for similar results in the half space setting.

When the swirl component v_θ is not assumed to be trivial, global regularity is unknown. But it follows from the partial regularity theory of [11, 1] that singular points can only lie on the axis of symmetry. Any off axis symmetry would imply a whole circle of singular points, which contradicts [11, 1]. Neustupa-Pokorný [2323, 232424, 24] proved regularity assuming the zero dimensional condition $v_r \in L_t^s L_x^q$ with $3/q + 2/s = 1$, $3 < q \leq \infty$. Regularity criteria can also be put on the vorticity field $\omega = \operatorname{curl} v$:

$$\omega(x,t) = \omega_r e_r + \omega_\theta e_\theta + \omega_z e_z, \tag{1.8}$$

where

$$\omega_r = -\partial_z v_\theta, \quad \omega_\theta = \partial_z v_r - \partial_r v_z, \quad \omega_z = (\partial_r + r^{-1}) v_\theta.$$

Chae-Lee [22, 2] proved regularity assuming finiteness of another zero-dimensional integral: $\omega_\theta \in L_t^s L_x^q$ with $3/q + 2/s = 2$. Jiu-Xin [1010, 10] proved regularity if the sum of the zero-dimensional *scaled* norms $\int_{Q_R} (R^{-1}|\omega_\theta|^2 + R^{-3}|v_\theta|^2) dz$ is sufficiently small for $R > 0$ small enough. Recently, Hou-Li [99, 9] constructed a family of global solutions with large initial data.

The main idea of our proof is as follows. The bound (1.2) ensures that the first blow up time is no earlier than $t = 0$. For $t \in (-T_0, 0)$ we show that the swirl component v_θ gains a modicum of regularity: For some small $\alpha = \alpha(C_*) > 0$, (1.2) enables us to conclude that

$$|v_\theta(t, r, z)| \leq C r^{\alpha - 1}. \tag{1.9}$$

We prove (1.9) in Section 3. This estimate breaks the scaling, thereby transforming the problem from order one to ϵ-regularity, which is shown to be sufficient in Section 2.

2 Proof of main theorem

In this section we prove Theorem 1.1. First we show that our solutions are in fact suitable weak solutions. Then we make use of (1.9), to establish our main theorem.

2.1 Suitable weak solution

We recall from [2626, 2611, 11919, 19] that a *suitable weak solution* of the Navier-Stokes equations in a domain $Q \subset \mathbb{R}^3 \times \mathbb{R}$ is defined to be a pair (v, p) satisfying

$$v \in L_t^\infty L_x^2(Q), \quad \nabla v \in L^2(Q), \quad p \in L^{3/2}(Q). \tag{2.1}$$

Further (v, p) solve (N-S) in the sense of distributions and satisfy the *local energy inequality*:

$$2 \int_Q |\nabla v|^2 \varphi \leq \int_Q \left\{ |v|^2 (\partial_t \varphi + \Delta \varphi) + (|v|^2 + 2p) v \cdot \nabla \varphi \right\}, \quad \forall \varphi \in C_c^\infty(Q), \ \varphi \geq 0. \tag{2.2}$$

To prove interior regularity, we do not need to specify the initial or boundary data.

We define a solution $v(x, t)$ to be *regular* at a point X_0 if $v \in L^\infty(Q(X_0, R))$ for some $R > 0$. Otherwise $v(x, t)$ is *singular* at X_0. We will use the following regularity criterion.

Lemma 2.1 *Suppose that (v, p) is a suitable weak solution of* (N-S) *in $Q(X_0, 1)$. Then there exists an $\epsilon_1 > 0$ so that X_0 is a regular point if*

$$\limsup_{R \downarrow 0} \frac{1}{R^2} \int_{Q(X_0, R)} |v|^3 \leq \epsilon_1. \tag{2.3}$$

This regularity criterion, which is a variant of the criterion in [11, 1], was proven in [3434, 34]; see [88, 8] for more general results. The condition (2.3) does not explicitly involve the pressure, but one does require $p \in L^{3/2}(Q(X_0, 1))$ because the pair (v, p) is assumed to be a suitable weak solution.

2.2 Preliminary estimates

In this subsection we show that the solution (v, p) in Theorem 1.1 is sufficiently integrable to be a suitable weak solution, and we derive estimates depending only upon C_* of (1.2).

We estimate the pressure with weighted singular integral estimates. We therefore first estimate v in weighted spaces. Fix $\beta \in (1, 5/3)$. For $t \in (-T_0, 0)$ by (1.2) we have

$$\int_{\mathbb{R}^3} \frac{|v(x,t)|^4}{|x|^\beta}\, dx \leq \int_{\mathbb{R}^3} \frac{1}{|x|^\beta} \frac{C_* r dr dz}{(r^2 - t)^2} = \int_{|z|\geq 1} + \int_{|z|<1, r>1} + \int_{|z|<1, r<1} = I_1 + I_2 + I_3.$$

Each of these integrals can be estimated as follows

$$\begin{aligned} |I_1| &\leq \int_{|z|>1} \frac{dz}{|z|^\beta} \int_0^\infty \frac{C_* r dr}{(r^2 - t)^2} \leq c|t|^{-1}, \\ |I_2| &\leq \int_1^\infty r^{-\beta} \frac{C_*}{(r^2 - t)^2} r dr \leq c, \\ |I_3| &\leq \int_0^1 (1 + r^{1-\beta}) \frac{c}{(r^2 - t)^2} r dr \leq c|t|^{-(1+\beta)/2}. \end{aligned}$$

Summing the estimates and using $\beta > 1$ we get

$$\int_{\mathbb{R}^3} \frac{1}{|x|^\beta} |v(x,t)|^4\, dx \leq c + c|t|^{-(1+\beta)/2}.$$

Define R_i's to be the Riesz transforms: $R_i = \frac{\partial_i}{\sqrt{-\Delta}}$. We consider the singular integral

$$\tilde{p}(x,t) = \int \sum_{i,j} \partial_i \partial_j (v_i v_j)(y) \frac{1}{4\pi |x-y|}\, dy = \sum_{i,j} R_i R_j (v_i v_j).$$

To show that this singular integral is well defined for every t, we use the $L^q(\mathbb{R}^3)$-estimates for singular integrals with A_q weight [3030, 30]. Specifically, we use $q = 2$ and the A_2 weight function $|x|^{-\beta}$. We have the estimate

$$\int \frac{1}{|x|^\beta} |\tilde{p}(x,t)|^2\, dx \leq c \int \frac{1}{|x|^\beta} |v(x,t)|^4\, dx \leq c + c|t|^{-(1+\beta)/2}. \tag{2.4}$$

Choose $\gamma \in (1/2 + 5\beta/6, 3)$. Hölder's inequality gives us the bound

$$\int_{|x|>1} \frac{|\tilde{p}(x,t)|^{5/3}}{|x|^\gamma} dx \leq \left(\int_{|x|>1} \frac{|\tilde{p}(x,t)|^2}{|x|^\beta} dx \right)^{5/6} \left(\int_{|x|>1} |x|^{-(\gamma - \frac{5}{6}\beta)6} dx \right)^{1/6} < \infty.$$

We will use these bounds to show that the pressure p can be identified with $\tilde{p}$.

Let $h(x,t) = p(x,t) - \tilde{p}(x,t)$. Then h is harmonic in x, $\Delta_x h(x,t) = 0$, and by assumption $p(\cdot, t) \in L^{5/3}(\mathbb{R}^3)$ for almost every t. For each such t we have

$$\int_{|x|>1} \frac{|h(x,t)|^{5/3}}{|x|^\gamma} dx \leq c \int_{|x|>1} |p(x,t)|^{5/3} dx + c \int_{|x|>1} \frac{|\tilde{p}(x,t)|^{5/3}}{|x|^\gamma} dx < \infty.$$

We may thus conclude from using a Liouville theorem that $h(x,t) = 0$ for all x if $\gamma < 3$.

To see the last assertion, fix a radial smooth function $\phi(x) \geq 0$ supported in $2 < |x| < 4$ satisfying $\int \phi = 1$. For any $x \in \mathbb{R}^3$ with $R > |x|$ we have

$$h(x,t) = \int h(y,t) R^{-3} \phi(x + y/R)\, dy.$$

This is the mean value theorem for harmonic functions. Define $A = B_{5R} - B_R$, then

$$|h(x,t)| \leq cR^{-3} \int_A |h(y,t)| dy \leq cR^{-3+(6+3\gamma)/5} \left(\int_A |y|^{-\gamma} |h(y,t)|^{5/3} dy \right)^{3/5}.$$

This clearly vanishes as $R \to \infty$. Thus $p(x,t) = \tilde{p}(x,t)$ for all x and for almost every t.

Next we show that (v,p) form a suitable weak solution. From Hölder's inequality, (2.4) and $\beta < 5/3$ we conclude that

$$\int_{Q_1} |p(x,t)|^{3/2} \, dxdt \leq c \int_{-1}^{0} \left(\int_{B_1} \frac{1}{|x|^\beta} |p(x,t)|^2 \, dx \right)^{3/4} dt \leq c. \tag{2.5}$$

The pointwise estimate (1.2) on v implies

$$v \in L^s_t L^q_x(Q_1), \quad \frac{1}{q} + \frac{1}{s} > \frac{1}{2}. \tag{2.6}$$

We will use $(s,q) = (3,3)$. We also see from (1.2) that $v \in L^4(B_1 \times (-T_0, -\epsilon))$ for any small $\epsilon > 0$. Thus the vector product of (N-S) with $u\varphi$ for any $\varphi \in C^\infty_c(Q_1)$ is integrable in Q_1 and we can integrate by parts to get the local energy inequality (2.2) with $Q = Q_1$. In fact we have equality.

Now, for any $R \in (0,1)$ and $t_0 \in (-R^2, 0)$, we can choose a sequence of φ which converges a.e. in Q_R to $H(t_0 - t)$, the Heviside function that equals 1 for $t < t_0$ and 0 for $t > t_0$. Since the limit of $\partial_t \varphi$ is the negative delta function in t, this gives us the estimate

$$\operatorname*{ess\,sup}_{-R^2 < t < 0} \int_{B_R} |v(x,t)|^2 dx + \int_{Q_R} |\nabla v|^2 \leq C_R \int_{Q_1} (|v|^3 + |p|^{3/2}). \tag{2.7}$$

These estimates show that (v,p) is a suitable weak solution of (N-S) in Q_R. Note that these bounds depend on C_* of (1.2) only, not on $\|p\|_{L^{5/3}(\mathbb{R}^3 \times (-T_0, 0))}$.

2.3 Scaling limit

To show Theorem 1.1, it suffices to show that every point on the z-axis is regular. Suppose now a point $x_* = (0,0,x_3)$ on the z-axis is a singular point of v. We will derive a contradiction. Define $X_* = (x_*, 0)$. Let (v^λ, p^λ) be rescaled solutions of (N-S) defined by

$$v^\lambda(x,t) = \lambda v(\lambda(x - x_*), \lambda^2 t), \quad p^\lambda(x,t) = \lambda^2 p(\lambda(x - x_*), \lambda^2 t). \tag{2.8}$$

By Lemma 2.1, there is a sequence λ_k, $k \in \mathbb{N}$, so that $\lambda_k \to 0$ as $k \to \infty$ and

$$\int_{Q_1} |v^{\lambda_k}|^3 = \frac{1}{\lambda_k^2} \int_{Q(X_*,\lambda_k)} |v|^3 > \epsilon_1. \tag{2.9}$$

We will derive a contradiction to this statement.

For (v^λ, p^λ) with $0 < \lambda < 1$, the pointwise estimate (1.2) is preserved:

$$|v^\lambda(x,t)| \le C_*(r^2 - t)^{-1/2}, \quad (x,t) \in \mathbb{R}^3 \times (-T_0, 0).$$

We also have by rescaling

$$p^\lambda(x,t) = \int \sum_{i,j} \partial_i \partial_j (v_i^\lambda v_j^\lambda)(y) \frac{1}{4\pi|x-y|}\, dy,$$

The argument in the previous subsection provides the uniform bounds for $q \in (1,4)$:

$$\int_{Q_1} |v^\lambda|^q + |p^\lambda|^{3/2} \le C, \quad \operatorname*{ess\,sup}_{-R^2<t<0} \int_{B_R} |v^\lambda(x,t)|^2 dx + \int_{Q_R} |\nabla v^\lambda|^2 \le C. \tag{2.10}$$

Above the bound for p_λ follows from (2.5), the bound for $|v^\lambda|^q$ follows from (1.2), and the energy bound then follows from (2.7).

Thus from the sequence λ_k we can extract a subsequence, still denoted by λ_k, so that $(v^{\lambda_k}, p^{\lambda_k})$ weakly converges to some limit function $(\bar v, \bar p)$

$$v^{\lambda_k} \rightharpoonup \bar v \quad \text{in } L^q(Q_R), \quad \nabla v^{\lambda_k} \rightharpoonup \nabla \bar v \quad \text{in } L^2(Q_R), \quad p^{\lambda_k} \rightharpoonup \bar p \quad \text{in } L^{3/2}(Q_R).$$

Moreover since (v^λ, p^λ) solves (N-S) with bound (2.10), we also have the uniform bound

$$\|\partial_t v^\lambda\|_{L^{3/2}((-R^2,0);H^{-2}(B_R))} < C.$$

We can then apply Theorem 2.1 of [3333; 33, chap. III] to conclude that the v^{λ_k} remain in a compact set of $L^{3/2}(Q_R)$. Therefore (a further subsequence of) $v^{\lambda_k} \to \bar v$ strongly in $L^{3/2}(Q_R)$. Since the v^{λ_k} remain bounded in $L^q(Q_R)$ for all $q < 4$, we deduce that $v^{\lambda_k} \to \bar v$ strongly in $L^q(Q_R)$ for all $1 \le q < 4$.

2.4 The limit solution

The convergence established at the end of Section 2.3 is sufficient to conclude that the limit function $(\bar v, \bar p)$ is a suitable weak solution of the Navier-Stokes equations in Q_R, as in [11, 11919, 19]. Since v satisfies (1.2) so does $\bar v$. Hence $\bar v$ is regular at any interior point of Q_R, and $t = 0$ is the first time when $\bar v(x,t)$ could develop a singularity.

To gather more information we use axisymmetry. We will argue in this subsection and the next that the estimate (1.9) (proven in the Section 3) is enough to conclude that our solution is regular. In particular (1.9) tells us that

$$\int_{Q_R} \left| v_\theta^\lambda \right| \le C\lambda^\alpha \to 0 \quad \text{as} \quad \lambda \downarrow 0.$$

Thus the limit $\bar{v}$ has no-swirl, $\bar{v}_\theta = 0$.

Let $\bar{\omega} = \nabla \times \bar{v}$ be the vorticity of $\bar{v}$. The θ component of $\bar{\omega}$, $\bar{\omega}_\theta = \partial_z \bar{v}_r - \partial_r \bar{v}_z$, solves

$$\left(\partial_t + \bar{b} \cdot \nabla - \Delta + \frac{1}{r^2}\right) \bar{\omega}_\theta - \frac{\bar{v}_r}{r} \bar{\omega}_\theta = 0.$$

We have used $\bar{v}_\theta = 0$. Above

$$\bar{b} = \bar{v} = \bar{v}_r e_r + \bar{v}_z e_z, \quad \bar{b} \cdot \nabla = \bar{v}_r \partial_r + \bar{v}_z \partial_z, \quad \operatorname{div} \bar{b} = 0.$$

We record the Laplacian for axisymmetric functions

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}.$$

Next define $\Omega = \bar{\omega}_\theta / r$. Then Ω solves

$$\left(\partial_t + \bar{b} \cdot \nabla - \Delta - \frac{2}{r}\partial_r\right) \Omega = 0. \tag{2.11}$$

We now derive L^q estimates on Ω using estimates for the Stokes system.

Since $\bar{v}$ satisfies (1.2), it also satisfies (2.6). We will use both $(s,q) = (5/2, 5)$ and $(s,q) = (5/4, 5/4)$ below. We rewrite (N-S) as a Stokes system with force

$$(\partial_t - \Delta)\bar{v}_i + \nabla_i \bar{p} = \partial_j f_{ij}, \quad \operatorname{div} \bar{v} = 0, \quad f_{ij} = -\bar{v}_i \bar{v}_j.$$

By the interior estimates of Stokes system (shown in the Appendix) we have

$$\|\nabla \bar{v}\|_{L_t^{5/4} L_x^{5/2}(Q_{5/8})} \le C \|\bar{v}\|^2_{L_t^{5/2} L_x^5(Q_{3/4})} + C\|\bar{v}\|_{L^{5/4}(Q_{3/4})} \le C.$$

Hence Ω has the bound

$$\|\Omega\|_{L^{20/19}(Q_{5/8})} \le \|\nabla \bar{v}\|_{L_t^{5/4} L_x^{5/2}(Q_{5/8})} \|1/r\|_{L_t^\infty L_x^{20/11}(Q_{5/8})} \le C. \tag{2.12}$$

In Section 2.6, we obtain $\Omega \in L^\infty$ from (2.11), (2.12) and a local maximum estimate. Then in Section 2.7 we show that this is sufficient to conclude Theorem 1.1.

2.5 Energy Estimates

We derive parabolic De Giorgi type energy estimates for (2.11). To do this we assume that

$$|\bar{b}(r,z,t)| \le C_*/r.$$

This assumption on $\bar{b}$ is substantially weaker than the one from Theorem 1.1.

Consider a test function $0 \le \zeta(x,t) \le 1$ defined on Q_R for which $\zeta = 0$ on $\partial B_R \times [-R^2, 0]$ and $\zeta = 1$ on $Q_{\sigma R}$ for $0 < \sigma < 1$. Define $(u)_\pm = \max\{\pm u, 0\}$ for a scalar function u. Multiply (2.11) by $p(\Omega - k)_\pm^{p-1}\zeta^2$ for $1 < p \le 2$ and $k \ge 0$ to obtain

$$\begin{aligned} &\int_{B_R} \zeta^2 (\Omega - k)_\pm^p \Big|_{-R^2}^{0} + \frac{4(p-1)}{p} \int_{Q_R} |\nabla((\Omega - k)_\pm^{p/2} \zeta)|^2 \\ &= 2 \int_{Q_R} (\Omega - k)_\pm^p \left(\zeta \frac{\partial \zeta}{\partial t} + |\nabla \zeta|^2 + \frac{2-p}{p} \zeta \Delta \zeta - 2\zeta \frac{\partial_r \zeta}{r} + \bar{b} \cdot \zeta \nabla \zeta \right) \\ &\qquad -2 \int dt \int dz \ \zeta^2 (\Omega - k)_\pm^p \Big|_{r=0} . \end{aligned}$$

Notice that the last term has a good sign.

To estimate the term involving b we use Young's inequality

$$\int_{\mathbb{R}^3} v_\pm^2 b \zeta \cdot \nabla \zeta \le \delta \frac{R^{-1+\epsilon}}{1+\epsilon} \int_{\mathbb{R}^3} v_\pm^2 \zeta^2 |b|^{1+\epsilon} + C_\delta \frac{\epsilon R^{-2+(1+\epsilon)/\epsilon}}{1+\epsilon} \int_{\mathbb{R}^3} v_\pm^2 \zeta^2 \left[\frac{|\nabla \zeta|}{\zeta} \right]^{(1+\epsilon)/\epsilon} .$$

This holds for small $\delta > 0$ and $\epsilon > 0$ to be chosen. Further choose ζ to decay like $(1 - |x|/R)^n$ near the boundary of B_R. If n is large enough (depending on ϵ) we have

$$C_\delta \frac{\epsilon R^{-2+(1+\epsilon)/\epsilon}}{1+\epsilon} \int_{\mathbb{R}^3} v_\pm^2 \zeta^2 \left[\frac{|\nabla \zeta|}{\zeta} \right]^{(1+\epsilon)/\epsilon} \le C R^{-2} \int_{B_R} v_\pm^2.$$

We also use the Hölder and Sobolev inequalities to obtain

$$\begin{aligned} \delta \frac{R^{-1+\epsilon}}{1+\epsilon} \int_{\mathbb{R}^3} v_\pm^2 \zeta^2 |b|^{1+\epsilon} &\le \delta \left(R^{(-1+\epsilon)3/2} \int_{B_R} |b|^{(1+\epsilon)3/2} \right)^{2/3} \int_{\mathbb{R}^3} |\nabla (v_\pm \zeta)|^2 \\ &\le \delta C \int_{\mathbb{R}^3} |\nabla (v_\pm \zeta)|^2. \end{aligned}$$

The last inequality is satisfied for example if $|b| \le C_*/r$ and $\epsilon < 1/3$. We conclude

$$\int_{\mathbb{R}^3} v_\pm^2 b \zeta \cdot \nabla \zeta \le \delta C \int_{\mathbb{R}^3} |\nabla (v_\pm \zeta)|^2 + C R^{-2} \int_{B_R} v_\pm^2. \tag{2.13}$$

The key point which we used here to control the more singular drift term was to split b from the main part of the term $v_\pm \zeta$, using the Young and Sobolev inequalities instead of standard techniques which utilize the Hardy inequality type spectral gap estimate to control $|b| v_\pm^2 \zeta^2$ in one step. We choose δ sufficiently small in order to absorb this term into the dissipation.

We have $\partial_r \zeta / r = \partial_\rho \zeta / \rho$ since ζ is radial; so that the singularity $1/\rho$ is effectively $1/R$. We thus have

$$\begin{aligned} \sup_{-\sigma^2 R^2 < t < 0} \int_{B_{\sigma R} \times \{t\}} |(\Omega - k)_\pm|^p + \int_{Q_{\sigma R}} |\nabla (\Omega - k)_\pm^{p/2}|^2 \\ \le \frac{C}{(1-\sigma)^2 R^2} \int_{Q_R} |(\Omega - k)_\pm|^p. \end{aligned} \tag{2.14}$$

Our goal will be to establish L^p to L^∞ bounds for functions in this energy class.

2.6 Local maximum estimate

The estimates in this section will be proven for a general function $u = \Omega$ satisfying (2.14):

Lemma 2.2 *Suppose $u = \Omega$ satisfies* (2.14) *for $1 < p \leq 2$. Then*

$$\sup_{Q_{R/2}} u_\pm \leq C(p, C_*) \left(R^{-3-2} \int_{Q_R} |u_\pm|^p \right)^{1/p}.$$

This estimate can be found in [1818, 18] for $p = 2$. The proof below is similar and we include it so that the proof of Theorem 3.1, which uses Lemma 2.2, is self-contained. Our choice of p is made merely because those are the ones we need although others are possible.

Proof. For $K > 0$ to be determined and N a positive integer we define

$$k_N = k_N^\pm = (1 \mp 2^{-N})K,\ R_N = (1 + 2^{-N})R/2,\ \rho_N = \frac{R}{2^{N+3}},$$
$$R_{N+1} < \bar{R}_N = (R_N + R_{N+1})/2 < R_N.$$

Notice that

$$R_N - \bar{R}_N = (R_N - R_{N+1})/2 = (2^{-N} - 2^{-N-1})R/4 = \rho_N.$$

Define $Q_N = Q(R_N)$ and $\bar{Q}_N = Q(\bar{R}_N) \subset Q_N$. Choose a smooth test function ζ_N satisfying $\zeta_N \equiv 1$ on $\bar{Q}_N$, $\zeta \equiv 0$ outside Q_N and vanishing on it's spatial boundary, $0 \leq \zeta_N \leq 1$ and $|\nabla \zeta_N| \leq \rho_N^{-1}$ in Q_N. Further let

$$A^\pm(N) = \{X \in Q_N : \pm(u - k_{N+1})(X) > 0\}.$$

And $A_{N,\pm} = |A^\pm(N)|$. Let $v_\pm = \zeta_N (u - k_{N+1})_\pm^{p/2}$.

Hölder's inequality gives us

$$\begin{aligned} \int_{Q_{N+1}} |(u - k_{N+1})_\pm|^p &\leq \int_{\bar{Q}_N} |v_\pm|^2 \\ &\leq \left(\int_{\bar{Q}_N} |v_\pm|^{2(n+2)/n} \right)^{n/(n+2)} A_{N,\pm}^{2/(n+2)}. \end{aligned}$$

We will use the following parabolic Sobolev inequality:

$$\int_{Q_R} |u|^{2(n+2)/n} \leq C(n) \left(\sup_{-R^2 < t < 0} \int_{B_R \times \{t\}} |u|^2 \right)^{2/n} \int_{Q_R} |\nabla u|^2.$$

See [1818; 18, Theorem 6.11, p.112]. We are interested in the form

$$\int_{Q_R} |u^{p/2}|^{2(n+2)/n} \leq C(n) \left(\sup_{-R^2 < t < 0} \int_{B_R \times \{t\}} |u|^p \right)^{2/n} \int_{Q_R} |\nabla u^{p/2}|^2.$$

As in the above followed by Young's inequality then followed by (2.14) we obtain

$$
\begin{aligned}
&\left(\int_{\bar{Q}_N} |v_\pm|^{2(n+2)/n}\right)^{n/(n+2)} \\
&\le C\left(\sup_{-R_N^2<t<0}\int_{B(R_N)\times\{t\}} |v_\pm|^2\right)^{2/(n+2)} \left(\int_{Q_N} |\nabla v_\pm|^2\right)^{n/(n+2)} \\
&\le C\left(\sup_{-R_N^2<t<0}\int_{B(R_N)\times\{t\}} |v_\pm|^2 + \int_{Q_N} |\nabla v_\pm|^2\right) \\
&\le C\left(\sup_{-R_N^2<t<0}\int_{B(R_N)\times\{t\}} |(u-k_{N+1})_\pm|^p + \int_{Q_N} |\nabla (u-k_{N+1})_\pm^{p/2}|^2\right) \\
&\quad + \frac{C_*}{\rho_N^2}\int_{Q_N} |(u-k_{N+1})_\pm|^p \\
&\le \frac{C_*}{\rho_N^2}\int_{Q_N} |(u-k_{N+1})_\pm|^p \le \frac{C_*}{\rho_N^2}\int_{Q_N} |(u-k_N)_\pm|^p.
\end{aligned}
$$

Further assume $K^p \ge R^{-n-2}\int_{Q(R)} |u_\pm|^p$. And define

$$
Y_N \equiv K^{-p}R^{-n-2}\int_{Q_N} |(u-k_N)_\pm|^p.
$$

Since $k_N^\pm$ are increasing for $+$ or decreasing for $-$ and Q_N are decreasing, Y_N is decreasing.

Chebyshev's inequality tells us that

$$
\begin{aligned}
A_{N,\pm} &= \left|\{Q_N : \pm(u-k_{N+1}^\pm) > 0\}\right| = \left|\{Q_N : \pm(u-k_N^\pm) > \pm(k_{N+1}^\pm - k_N^\pm)\}\right| \\
&= \left|\{Q_N : \pm(u-k_N) > K/2^{N+1}\}\right| \le 2^{p(N+1)}R^{n+2}Y_N.
\end{aligned}
$$

Putting all of this together yields

$$
\begin{aligned}
\int_{Q_{N+1}} |(u-k_{N+1})_\pm|^p &\le \left(\int_{\bar{Q}_N} |v_\pm|^{2(n+2)/n}\right)^{n/(n+2)} A_{N,\pm}^{2/(n+2)}, \\
&\le \left(\frac{C_*}{\rho_N^2}\int_{Q_N} |(u-k_N)_\pm|^p\right)\left(2^{p(N+1)}R^{n+2}Y_N\right)^{2/(n+2)} \\
&\le \left(\frac{C_*}{\rho_N^2}K^pR^{n+2}Y_N\right)\left(2^{p(N+1)}R^{n+2}Y_N\right)^{2/(n+2)} \\
&= C_*K^p2^{2(N+3)}2^{2p(N+1)/(n+2)}R^{n+2}Y_N^{1+\frac{2}{n+2}}.
\end{aligned}
$$

We have thus shown that

$$
Y_{N+1} \le C(N)Y_N^{1+\frac{2}{n+2}}
$$

Here $C(N) = C_* 2^{2(N+3)} 2^{2p(N+1)/(n+2)}$. We now choose K as

$$K^p = \left(1 + \frac{1}{C_0}\right) R^{-n-2} \int_{Q_0} |u_\pm|^p.$$

Above the constant C_0 is chosen to ensure that $Y_N \to 0$ as $N \to \infty$. □

2.7 Regularity of the original solution

The limiting solution Ω satisfies (2.11), (2.12) and (2.14). We conclude from Lemma 2.2 that

$$\Omega \in L^\infty(Q_{5/16}).$$

We further know that $\operatorname{curl} \bar{v} = \bar{\omega} e_\theta \in L^\infty(Q_{5/16})$ from the above estimate on Ω since $\bar{v}_\theta = 0$. Also $\operatorname{div} \bar{v} = 0$ from the equation. Next $\bar{v} \in L_t^\infty L_x^1(Q_{5/16})$ by (1.2). We thus conclude $\nabla \bar{v} \in L_t^\infty L_x^4(Q_{1/4})$ by Lemma A.1. Thus $\bar{v} \in L^\infty(Q_{1/4})$ by embedding.

Now we can deduce regularity of the original solution from the regularity of the limit solution. Since $\bar{v} \in L^\infty(Q_{1/4})$ for R sufficiently small we have

$$\frac{1}{R^2} \int_{Q_R} |\bar{v}|^3 \le \epsilon_1/2,$$

where ϵ_1 is the small constant in Lemma 2.1. Fix one such $R > 0$. Since $v^{\lambda_k} \to \bar{v}$ strongly in L^3 for k sufficiently large we have

$$\frac{1}{R^2} \int_{Q_R} |v^{\lambda_k}|^3 \le \epsilon_1.$$

But this is a contradiction to (2.9). Thus every point x_* on the z-axis is regular; that is, there is a radius $R_{x_*} > 0$ so that $v \in L^\infty(Q(x_*, R_{x_*}))$. Since any finite portion of the z-axis can be covered by a finite subcover of $\{Q(x_*, R_{x_*})\}$, we have proved Theorem 1.1.

The rest of the paper is devoted to proving the key Theorem 3.1.

3 Hölder estimate for axisymmetric solutions

We now move from cartesian to cylindrical coordinates via the standard change of variables $x = (x_1, x_2, x_3) = (r\cos\theta, r\sin\theta, z)$. For axisymmetric solutions (v, p) of the form (1.1), the Navier-Stokes equations (N-S) take the form

$$\begin{aligned}
\frac{\partial v_r}{\partial t} + b \cdot \nabla v_r - \frac{v_\theta^2}{r} + \frac{\partial p}{\partial r} &= \left(\Delta - \frac{1}{r^2}\right) v_r, \\
\frac{\partial v_\theta}{\partial t} + b \cdot \nabla v_\theta + \frac{v_\theta v_r}{r} &= \left(\Delta - \frac{1}{r^2}\right) v_\theta, \\
\frac{\partial v_z}{\partial t} + b \cdot \nabla v_z + \frac{\partial p}{\partial z} &= \Delta v_z, \\
\frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \frac{\partial v_z}{\partial z} &= 0.
\end{aligned}$$

The vector b is given by

$$b = v_r e_r + v_z e_z, \quad \operatorname{div} b = 0.$$

The equations of the vorticity $\omega = \operatorname{curl} v$, decomposed in the form (1.8), are

$$\frac{\partial \omega_r}{\partial t} + b \cdot \nabla \omega_r - \omega_r \partial_r u_r - \omega_z \partial_z u_r = \left(\Delta - \frac{1}{r^2}\right) \omega_r,$$
$$\frac{\partial \omega_\theta}{\partial t} + b \cdot \nabla \omega_\theta - 2\frac{u_\theta}{r} \partial_z u_\theta - \frac{u_r}{r} \omega_\theta = \left(\Delta - \frac{1}{r^2}\right) \omega_\theta,$$
$$\frac{\partial \omega_z}{\partial t} + b \cdot \nabla \omega_z - \omega_z \partial_z u_z - \omega_r \partial_r u_z = \Delta \omega_z.$$

Although we do not use them. We are interested in the equation for v_θ, which is independent of the pressure.

Consider the change of variable $\Gamma = r v_\theta$, which is well known (see the references in the introduction). The function Γ is smooth and satisfies

$$\frac{\partial \Gamma}{\partial t} + b \cdot \nabla \Gamma - \Delta \Gamma + \frac{2}{r} \frac{\partial \Gamma}{\partial r} = 0. \tag{3.1}$$

Note that the sign of the term $\frac{2}{r} \frac{\partial \Gamma}{\partial r}$ is opposite to that of (2.11). It follows directly from (1.2) that $\|\Gamma\|_{L^\infty_{t,x}} \le C_*$; see [22,2] for related estimates. Since v is smooth, we have $\Gamma(t, 0, z) = 0$ for $t < 0$. The smoothness and axisymmetry assumptions also imply $v_\theta(t, 0, z) = 0$, but we will not use this fact. The main result of this section is the following.

Theorem 3.1 *Suppose that $\Gamma(x, t)$ is a smooth bounded solution of (3.1) in Q_2 with smooth $b(x, t)$, both may depend on θ, and*

$$\Gamma|_{r=0} = 0, \quad \operatorname{div} b = 0, \quad |b| \le C_*/r \quad \text{in } Q_2.$$

Then there exist constants C and $\alpha > 0$ which depend only upon C_* such that

$$|\Gamma(x, t)| \le C \|\Gamma\|_{L^\infty_{t,x}(Q_2)} r^\alpha \quad \text{in } Q_1.$$

We remark that the condition above is substantially weaker than (1.2), and we do not need Γ to be axisymmetric. In the rest of this section, we will prove the theorem. Here we are facing two difficulties: First, the condition $\Gamma|_{r=0} = 0$ precludes a direct lower bound on the fundamental solution and a Harnack inequality on Γ (since, when $b = 0$, $\Gamma = r^2$ is a nonnegative solution which does not satisfy the usual Harnack inequality.) Second, the condition $b \le C/r$ is weaker than the standard assumption $b \le C/|x|$ (see the discussion below). It turns out that one can develop new techniques incorporating the methods introduced by De Giorgi [33,3] and Moser [2020,20] to over come these two points. However, we do not know if one can follow the approach of Nash [2121,2166,6] which relies critically on a Gaussian lower bound of the fundamental solution. The proof of Theorem 3.1 is independent of the rest of the paper.

The following related equation has been previously studied by Zhang [3939, 39]:

$$\frac{\partial u}{\partial t} + b \cdot \nabla u - \Delta u = 0.$$

He has shown among other things Hölder continuity of solutions to this equation if $b = b(x)$ is independent of time and b satisfies an integral condition which is fulfilled if say b is controlled by $1/|x|$. His proof makes use of Moser iteration and Gaussian bounds.

3.1 Notation, Reformulation, and Energy inequalities

Let $X = (x, t)$. Define the modified parabolic cylinder at the origin

$$Q(R, \tau) = \{X : |x| < R, -\tau R^2 < t < 0\}.$$

Here $R > 0$ and $\tau \in (0, 1]$. We sometimes for brevity write $Q_R = Q(R) = Q(R, 1)$. Let

$$m_2 \equiv \inf_{Q(2R)} \Gamma, \quad M_2 \equiv \sup_{Q(2R)} \Gamma, \quad M \equiv M_2 - m_2 > 0.$$

Notice that $m_2 \le 0 \le M_2$ since $\Gamma|_{r=0} = 0$.

Now we reformulate the problem in $Q(2R)$ into a new function, u, which will be zero when $|\Gamma|$ is at its maximum value. Specifically, we define

$$u \equiv \begin{cases} 2(\Gamma - m_2)/M & \text{if} \quad -m_2 > M_2, \\ 2(M_2 - \Gamma)/M & \text{else.} \end{cases} \tag{3.2}$$

In either case u solves (3.1) and $0 \le u \le 2$ in $Q(2R)$. We will further use

$$a \equiv u|_{r=0} = \frac{2}{M}\left(\sup_{Q(2R)} |\Gamma|\right) = \frac{2}{M}\max\{M_2, -m_2\} \ge 1,$$

which follows from our conditions.

We now derive energy estimates for (3.1). Define $v_\pm = (u - k)_\pm$ with $k \ge 0$. We have $v_+ \le (2 - k)_+$ and $v_- \le k$. Consider a radial test function $0 \le \zeta(x, t) \le 1$ for which $\zeta = 0$ on $\partial B_R \times [-\tau R^2, 0]$ and $\frac{\partial \zeta}{\partial r} \le 0$. We multiply (3.1) for $u - k$ with $\zeta^2 v_\pm$ and integrate over $\mathbb{R}^3 \times [t_0, t]$ to obtain

$$\begin{aligned} \frac{1}{2}\left[\int_{\mathbb{R}^3} |\zeta v_\pm|^2\right]_{t_0}^{t} &+ \int_{t_0}^{t}\int_{\mathbb{R}^3} |\nabla(\zeta v_\pm)|^2 \\ &= \int_{t_0}^{t}\int_{\mathbb{R}^3} v_\pm^2 \left(b\zeta \cdot \nabla\zeta + \zeta\frac{\partial \zeta}{\partial t} + |\nabla \zeta|^2 + \frac{2\zeta}{r}\frac{\partial \zeta}{\partial r}\right) \\ &\quad + 2\pi[(a-k)_\pm]^2 \int_{t_0}^{t}\int_{\mathbb{R}} dz\ \zeta^2|_{r=0}. \end{aligned} \tag{3.3}$$

We need to estimate all the terms in parenthesis.

Choose $\sigma \in (1/4, 1)$, we require that the test function satisfies $\zeta \equiv 1$ on $Q(\sigma R, \tau)$. If we further choose $\zeta(x, t_0) = 0$ then, using (2.13), we estimate (3.3) as follows

$$\begin{aligned} \sup_{-\tau\sigma^2 R^2 < t < 0} \int_{B(\sigma R)\times\{t\}} v_\pm^2 + \int_{Q(\sigma R,\tau)} |\nabla v_\pm|^2 \\ \leq \frac{C_{**}}{\tau(1-\sigma)^2 R^2} \int_{Q(R,\tau)} v_\pm^2 + C\tau R^3 [(a-k)_\pm]^2. \end{aligned} \tag{3.4}$$

If we alternatively choose $\zeta = \zeta(x)$ then (3.3) takes the form

$$\begin{aligned} \sup_{t_0 < s < t} \int_{B(\sigma R)\times\{s\}} v_\pm^2 + \int_{t_0}^{t} \int_{B(\sigma R)} |\nabla v_\pm|^2 - \int_{B_R\times\{t_0\}} v_\pm^2 \\ \leq \frac{C_{**}}{(1-\sigma)^2 R^2} \int_{t_0}^{t} \int_{B_R} v_\pm^2 + C\tau R^3 [(a-k)_\pm]^2. \end{aligned} \tag{3.5}$$

Notice that there is no τ^{-1} appearing in this energy inequality (3.5) compared to (3.4).

The energy estimates (3.4) and (3.5) are the standard parabolic De Giorgi classes except for the last term. Our goal will be to use them to show that the set where Γ is very close to its largest absolute value or, equivalently, the set where u is almost zero is as small as you wish. We establish this fact in the following series of Lemma's.

3.2 Initial Estimates

Later on we will use the two standard Lemma's below in a non-standard iteration scheme of sorts to show that the set where u is almost zero has very small Lebesgue measure.

Lemma 3.2 *Suppose there exists a* $t_0 \in [-\tau R^2, 0]$*,* $K > 0$ *and* $\gamma \in (0, 1)$ *so that*

$$|\{x \in B_R : u(x, t_0) \leq K\}| \leq \gamma |B_R|.$$

Further suppose that u *satisfies* (3.5) *for* v_-*. Then for all* $\eta \in (0, 1 - \sqrt{\gamma})$ *and* $\mu \in (\gamma/(1-\eta)^2, 1)$ *there exists* $\theta \in (0, 1)$ *such that*

$$|\{x \in B_R : u(x, t) \leq \eta K\}| \leq \mu |B_R|, \quad \forall t \in [t_0, t_0 + (\tau \wedge \theta) R^2].$$

Here θ *depends only on the constants in* (3.5) *and* γ*.*

We note that the proof shows that $\theta(\gamma) \to 0$ as $\gamma \uparrow 1$, but if τ is sufficiently small then we may take $\theta = \tau$ when γ is close enough to zero. And if γ is small, then μ can be taken almost as small.

Proof. We consider $v_- = (u-K)_-$ and assume without loss of generality that $K < 1$. The energy inequality (3.5) for this function is

$$\int_{B(\sigma R)\times\{t\}} v_-^2 \le \int_{B_R\times\{t_0\}} v_-^2 + \frac{C_{**}}{(1-\sigma)^2R^2}\int_{t_0}^t\int_{B_R} v_-^2 + C\tau R^3[(a-K)_-]^2$$
$$\le K^2|B_R|\left(\gamma + \frac{C_{**}(\tau\wedge\theta)}{(1-\sigma)^2}\right).$$

We have used $(a-K)_- = 0$. The Chebyshev inequality tells us that

$$A \equiv |\{x \in B(\sigma R) : u(x,t) \le \eta K\}| \cdot (K-\eta K)^2 \le \int_{B(\sigma R)\times\{t\}} v_-^2.$$

The quantity $1-\sigma^3$ is an upper bound for the measure of A^c, which grants the following general inequality

$$\frac{|\{x \in B_R, u(x,t) \le \eta K\}|}{|B_R|} \le \frac{|\{x \in B(\sigma R), u(x,t) \le \eta K\}|}{|B_R|} + (1-\sigma^3),$$
$$\le (1-\eta)^{-2}\left(\gamma + \frac{C_{**}(\tau\wedge\theta)}{(1-\sigma)^2}\right) + (1-\sigma^3).$$

Now let σ be so close to one that $\frac{\gamma}{(1-\eta)^2} + (1-\sigma^3) < \mu$. Then, with τ fixed, choose θ small enough that the whole thing is $\le \mu$. □

The Lemma above shows continuity in time of the Lebesgue measure of the set where u is small and the lemma below shows that if the set where u is small is less than the whole set, then the set where u is even smaller can be made tiny. This is an extremely weak way to measure diffusion.

Lemma 3.3 *Suppose that $u(x,t)$ satisfies (3.4) for v_-. In addition*

$$|\{x \in B_R : u(x,t) \le K\}| \le \gamma|B_R|, \quad \forall t \in [t_0, t_0+\theta R^2] = I,$$

where $K, \theta > 0$, $\gamma \in (0,1)$ and $B_R \times I \subset Q(R,\tau)$. Then for all $\epsilon \in (0,1)$ there exists a $\delta \in (0,1)$ such that

$$|\{X \in B_R \times I : u(X) \le \delta\}| \le \epsilon|B_R \times I|.$$

Proof. We denote, for $n = 0,1,2,3,\ldots$,

$$A_n(t) = \left\{x \in B_R : u(x,t) \le 2^{-n}K\right\}, \quad A_n = \cup_{t\in I} A_n(t).$$

Clearly $|A_{n+1}| \le |A_n| \le |A_0| \le \gamma|B_R \times I|$. And

$$|A_n^c(t)| = \left|\left\{x \in B_R : u(x,t) > 2^{-n}K\right\}\right| = |B_R| - |A_n(t)| \ge (1-\gamma)|B_R|.$$

Since $\gamma < 1$, we know that $A_n^c(t)$ does not have measure zero.

We invoke the following well known version of the Poincaré inequality. For any $v \in W^{1,1}(B_R)$ and for any $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$ we have

$$|\{x \in B_R : v(x) \leq \alpha\}| \leq \frac{CR^{3+1}/(\beta - \alpha)}{|\{x \in B_R : v(x) > \beta\}|} \int_{B_R \cap \{\alpha < u \leq \beta\}} |\nabla v|,$$

where $C > 0$ only depends on the dimension. Let $\beta = 2^{-n}K$ and $\alpha = 2^{-n-1}K$. We have

$$|A_{n+1}(t)| \leq \frac{C2^{n+1}R}{K(1-\gamma)} \int_{A_n(t) - A_{n+1}(t)} |\nabla u| = \frac{C2^{n+1}R}{K(1-\gamma)} \int_{A_n(t) - A_{n+1}(t)} |\nabla (u-a)^-|.$$

We use the Cauchy-Schwartz inequality to bound this integral as

$$\begin{aligned} |A_{n+1}| = \int_I |A_{n+1}(t)| &\leq \frac{C2^{n+1}R}{K(1-\gamma)} \int_{A_n - A_{n+1}} |\nabla (u-a)^-| \\ &\leq \frac{C2^{n+1}R}{K(1-\gamma)} |A_n - A_{n+1}|^{1/2} \left(\int_{A_n - A_{n+1}} |\nabla (u-a)^-|^2 \right)^{1/2}. \end{aligned}$$

The energy inequality (3.4), with σR and R replaced by R and $2R$ results in

$$\begin{aligned} |A_{n+1}| &\leq \frac{C2^{n+1}R}{K(1-\gamma)} |A_n - A_{n+1}|^{1/2} \left(\frac{C}{\tau R^2} \int_{Q(2R,\tau)} |(u-a)^-|^2 \right)^{1/2} \\ &\leq \frac{C2^{n+1}R}{K(1-\gamma)} |A_n - A_{n+1}|^{1/2} |B(2R)|^{1/2} a = \frac{CR^{5/2}}{K(1-\gamma)} |A_n - A_{n+1}|^{1/2}. \end{aligned}$$

Square both sides of this inequality and dividing by $|B_R \times I|^2$ to obtain

$$\frac{|A_{n+1}|^2}{|B_R \times I|^2} \leq \frac{C}{\theta K^2 (1-\gamma)^2} \left(\frac{|A_n|}{|B_R \times I|} - \frac{|A_{n+1}|}{|B_R \times I|} \right).$$

Summing in n, we get

$$\begin{aligned} n \frac{|A_n|^2}{|B_R \times I|^2} \leq \sum_{j=1}^{n} \frac{|A_j|^2}{|B_R \times I|^2} &\leq \frac{C}{\theta K^2 (1-\gamma)^2} \sum_{j=1}^{n} \left(\frac{|A_{j-1}|}{|B_R \times I|} - \frac{|A_j|}{|B_R \times I|} \right) \\ &= \frac{C}{\theta K^2 (1-\gamma)^2} \left(\frac{|A_0|}{|B_R \times I|} - \frac{|A_n|}{|B_R \times I|} \right) \\ &\leq \frac{C}{\theta K^2 (1-\gamma)^2} \frac{|A_0|}{|B_R \times I|} \leq \frac{C\gamma}{\theta K^2 (1-\gamma)^2}. \end{aligned}$$

We complete the proof by choosing n sufficiently large. □

3.3 Estimate on the measure of the set where u is small

The next lemma allows us to apply all the machinery above.

Lemma 3.4 *There exists a* $\kappa \in (0,1)$ *such that* $0 < \lambda < \min\{\kappa\tau, 1/8\}$ *implies*

$$\left|\left\{X \in Q(R,\tau) : u(X) \leq \lambda^2\right\}\right| \leq (1-4\lambda)|Q(R,\tau)|.$$

Proof. We establish a contradiction using energy estimates. Suppose the opposite

$$\left|\left\{X \in Q(R,\tau) : u(X) \leq \lambda^2\right\}\right| > (1-4\lambda)|Q(R,\tau)|.$$

Or equivalently

$$\left|\left\{X \in Q(R,\tau) : u(X) > \lambda^2\right\}\right| < 4\lambda|Q(R,\tau)|. \tag{3.6}$$

This condition will imply a contradiction to the size condition on $a \geq 1$.

We will test the equation (3.1) with $pu^{p-1}\zeta^2$ for $0 < p < 1$ and $\zeta \geq 0$. Since $u = 0$ sometimes, in general we should test (3.1) for $u + \epsilon$ with $p(u+\epsilon)^{p-1}\zeta^2$ and then send $\epsilon \downarrow 0$ to obtain our estimates. However, since the result is the same, to simplify the presentation we will omit these details. We have

$$\int_{Q(R,\tau)} pu^{p-1}\zeta^2 \frac{\partial u}{\partial t} = \left[\int_{B_R} \zeta^2 u^p\right]_{t_1}^{0} - \int_{Q(R,\tau)} u^p 2\zeta \frac{\partial \zeta}{\partial t} \equiv I_1 + I_2,$$

$$\begin{aligned}\int_{Q(R,\tau)} pu^{p-1}\zeta^2(-\Delta u) =& \frac{4(p-1)}{p}\int_{Q(R,\tau)} |\nabla(u^{p/2}\zeta)|^2 \\ &+ \int_{Q(R,\tau)} 2u^p\left[-|\nabla\zeta|^2 + \frac{p-2}{p}\zeta\Delta\zeta\right] \equiv I_3 + I_4,\end{aligned}$$

$$\int_{Q(R,\tau)} pu^{p-1}\zeta^2 b\cdot\nabla u = -\int_{Q(R,\tau)} 2u^p b\cdot\zeta\nabla\zeta \equiv I_5,$$

$$\begin{aligned}\int_{Q(R,\tau)} pu^{p-1}\zeta^2\frac{2}{r}\partial_r u =& -\int_{Q(R,\tau)} 4u^p\zeta\zeta_\rho/\rho - \int_{-\tau R^2}^{0} dt\int_{\mathbb{R}} dz\ 2(\zeta^2u^p)|_{r=0} \\ \equiv& I_6 + I_7.\end{aligned}$$

In the computation of I_6 we have used $\zeta_r/r = \zeta_\rho/\rho$, which follows if $\zeta = \zeta(\rho,t)$ where $\rho = |x| = \sqrt{r^2+z^2}$. Notice that $\sum_{j=1}^{7} I_j = 0$. For arbitrary $p \in (0,1)$, we see that I_3 and I_7 are both non-positive.

We choose $\zeta = \zeta_1(\rho)\zeta_2(t)$ where $\zeta_1(\rho) = 1$ in $B(R/2)$ and $\zeta_1(\rho)$ has compact support in B_R; also $\zeta_2(t) = 1$ if $t \in [-\frac{7}{8}\tau R^2, -\frac{1}{8}\tau R^2]$ and $\zeta_2(t)$ has compact support in $(-\tau R^2, 0)$. Thus $I_1 = 0$ and we have

$$\frac{6}{4}\tau R^3 a^p \leq -I_7 = \sum_{j=2}^{6} I_j.$$

We estimate each of the terms I_2 through I_6 to obtain a contradiction.

By the argument in (2.13), we have

$$|I_5| \le \frac{2(1-p)}{p} \int_{Q(R,\tau)} |\nabla(u^{p/2}\zeta)|^2 + \frac{C}{R^2}\int_{Q(R,\tau)} u^p.$$

Also note $\nabla\zeta = 0$ in $B(R/2)$ and so the singularity $1/\rho$ is effectively $1/R$. Thus,

$$I_2 \le \frac{C}{\tau R^2}\int_{Q(R,\tau)} u^p, \quad \sum_{j=3}^{6} I_j \le \frac{C}{R^2}\int_{Q(R,\tau)} u^p.$$

Assuming (3.6) and using $0 \le u \le 2$, we have

$$a^p \le \frac{C}{\tau^2 R^5}\int_{Q(R,\tau)} u^p \le \frac{C}{\tau^2 R^5}\left\{\lambda^{2p}|Q(R,\tau)| + 2^p(4\lambda|Q(R,\tau)|)\right\} \le \frac{C_2}{\tau}(\lambda^{2p}+\lambda).$$

Here $C_2 = C_2(C_*)$. Take $p = 1/2$ and $\kappa = \frac{1}{4C_2}$ to get $a^p < 1$, a contradiction. □

Lemma 3.4 is the starting point of our iteration scheme. From this Lemma we know that there is a $t_1 \in [-\tau R^2, -2\lambda\tau R^2]$ so that

$$|\left\{x \in B_R : u(x,t_1) \le \lambda^2\right\}| \le (1-2\lambda)|B_R|. \tag{3.7}$$

Then apply Lemma 3.2 with $K = \lambda^2$ to (3.7) to see, for say $\eta = \lambda$ and $\mu = 1-\lambda$, that

$$|\left\{x \in B_R : u(x,t) \le \lambda^3\right\}| \le (1-\lambda)|B_R|, \quad \forall t \in [t_1, t_1+\theta_* R^2] \equiv I_*.$$

Here $\theta_* = \theta \wedge \tau$ and θ is the constant chosen in Lemma 3.2. From here Lemma 3.3 allows us to conclude

$$|\{X \in B_R \times I_* : u(X) \le \delta_*\}| \le \frac{\epsilon_*}{2}|B_R \times I_*|,$$

where $\epsilon_* > 0$ is as small as you want and $\delta_* = \delta_*(\epsilon_*)$.

Then, as in (3.7), there exists a $t_2 \in I_*$ (so that $t_2 \le -\lambda\tau R^2$) such that

$$|\{x \in B_R : u(x,t_2) \le \delta_*\}| \le \epsilon_*|B_R|. \tag{3.8}$$

Up till now all the small parameters that we have chosen depend upon τ. But above ϵ_* can be taken arbitrarily small independent of the size of τ. This is the key point that enables us to proceed. It is the reason why we are required to do this procedure twice.

Now suppose $1-\sigma^3 = 1/4$ and choose first $\tau < 1/8$ so that $C_{**}\tau/(1-\sigma)^2 \le 1/4$. Then take δ_* from (3.8) with $\epsilon_* < 1/16$ above playing the role of γ in Lemma 3.2. Also $\eta < 1/2$. With all this, from Lemma 3.2, we can choose $\mu < 1$ so that

$$|\{x \in B_R : u(x,t) \le \eta\delta_*\}| \le \mu|B_R|, \quad \forall t \in [t_2, t_2+\tau R^2] \equiv I.$$

Further, it is safe to assume that $\theta_* \le \lambda$; we see that $t_2 \le -\lambda\tau R^2$ and so $[-\lambda\tau R^2, 0] \subset I$. Finally apply Lemma 3.3 again to obtain

$$|\{X \in Q(R,\lambda\tau) : u(X) \le \delta\}| \le \epsilon|Q(R,\lambda\tau)|, \tag{3.9}$$

with $\epsilon > 0$ arbitrarily small. This is a key step in what follows.

Let $U = \delta - u$, where δ is the constant from (3.9). U is clearly a solution of (3.1) and $U|_{r=0} = \delta - a < 0$. We apply (3.4) to U on $Q(2d)$ (with $\tau = 1$) to get

$$\sup_{-\sigma^2 d^2 < t < 0} \int_{B(\sigma d)\times\{t\}} |(U-k)^+|^2 + \int_{Q(\sigma d)} |\nabla (U-k)^+|^2 \\ \leq \frac{C_{**}}{(1-\sigma)^2 d^2} \int_{Q(d)} |(U-k)^+|^2.$$

This holds for all $k > 0$ and $\sigma \in (0,1)$. So we can apply Lemma 2.2 to conclude

$$\sup_{Q(d/2)} (\delta - u) \leq \left(\frac{C}{|Q(d)|} \int_{Q(d)} |(\delta - u)^+|^2 \right)^{1/2}. \tag{3.10}$$

This inequality combined with (3.9) will produce a lower bound.

3.4 Regularity from a lower bound

Let $d = \sqrt{\lambda\tau} R$ so that $Q(d) \subset Q(R, \lambda\tau)$. By (3.10) and (3.9),

$$\delta - \inf_{Q(d/2)} u \leq \left(\frac{C}{|Q(d)|} \int_{Q(d)} |(\delta - u)^+|^2 \right)^{1/2} \\ \leq \left(\frac{C\delta^2 \epsilon |Q(R,\lambda\tau)|}{|Q(d)|} \right)^{1/2} = C\delta\epsilon^{1/2} (\lambda\tau)^{-3/4},$$

which is less than $\frac{\delta}{2}$ if ϵ is chosen sufficiently small. We conclude

$$\inf_{Q(d/2)} u \geq \frac{\delta}{2}.$$

This is the lower bound we seek. From it we will deduce an oscillation estimate.

This entails a bit of algebra. We define

$$m_d \equiv \inf_{Q(d/2)} \Gamma, \quad M_d \equiv \sup_{Q(d/2)} \Gamma.$$

Then from (3.2) we have

$$\inf_{Q(d/2)} u = \begin{cases} 2(m_d - m_2)/M & \text{if} \quad -m_2 > M_2, \\ 2(M_2 - M_d)/M & \text{else}, \end{cases}$$

Notice that both expressions above are non-negative in any case; thus we can add them together to observe that

$$\frac{\delta}{2} \leq \frac{2}{M} \{M - \operatorname{osc}(\Gamma, d/2)\}$$

Here $\mathrm{osc}(\Gamma, d/2) = M_d - m_d$ and $\mathrm{osc}(\Gamma, 2R) = M_2 - m_2 = M$. We rearrange the above

$$\mathrm{osc}(\Gamma, d/2) \leq \left(1 - \frac{\delta}{4}\right) \mathrm{osc}(\Gamma, 2R).$$

This is enough to produce the desired Hölder continuity via the following.

3.5 Iteration Argument

Suppose we have a non-decreasing function ω on an interval $(0, R_0]$ which satisfies

$$\omega(\tau R) \leq \gamma \omega(R),$$

with $0 < \gamma, \tau < 1$. Then for $R \leq R_0$ we have

$$\omega(R) \leq \frac{1}{\gamma} \left(\frac{R}{R_0}\right)^{\alpha} \omega(R_0), \tag{3.11}$$

where $\alpha = \log \gamma / \log \tau > 0$.

Iterating, as in (3.11), we get, for $C_\Gamma = \left(1 - \frac{\delta}{4}\right)^{-1} \sup_{Q(1)} \Gamma$, that

$$\mathrm{osc}(\Gamma, R) \leq C_\Gamma R^{\alpha}, \quad \forall R \in (0, 1), \tag{3.12}$$

for $\alpha = 2\log(1 - \frac{\delta}{4})/\log(\lambda\tau/16) > 0$. Thus Γ is Hölder continuous near the origin. We have proved Theorem 3.1.

Appendix

Here we collect some estimates needed for Section 2.

Lemma A.1 *Let $B_{R_2} \subset B_{R_1} \subset \mathbb{R}^3$ be concentric with $0 < R_2 < R_1$. Let v be a vector field defined in B_{R_1}. Let $1 < q < \infty$ and $0 < \alpha < 1$. Then for $k = 0, 1, \ldots$ there is a constant c depending on R_2, R_1, q, α, k so that*

$$\|\nabla^{k+1} u\|_{L^q(B_{R_2})} \leq c\|\nabla^k \mathrm{div}\, u\|_{L^q(B_{R_1})} + c\|\nabla^k \mathrm{curl}\, u\|_{L^q(B_{R_1})} + c\|u\|_{L^1(B_{R_1})}.$$

and

$$\|\nabla^{k+1} u\|_{C^\alpha(B_{R_2})} \leq c\|\nabla^k \mathrm{div}\, u\|_{C^\alpha(B_{R_1})} + c\|\nabla^k \mathrm{curl}\, u\|_{C^\alpha(B_{R_1})} + c\|u\|_{L^1(B_{R_1})}.$$

This is well-known, see [2222, 22].

Lemma A.2 (Interior estimates for Stokes system) *Fix $R \in (0, 1)$. Let $1 < s, q < \infty$ and $f = (f_{ij}) \in L^s_t L^q_x(Q_1)$. Assume that $v \in L^s_t L^1_x(Q_1)$ is a weak solution of the Stokes system*

$$\partial_t v_i - \Delta v_i + \partial_i p = \partial_j f_{ij}, \quad \mathrm{div}\, v = 0 \quad \textit{in } Q_1.$$

Then v satisfies, for some constant $c = c(q, s, R)$,

$$\|\nabla v\|_{L_t^s L_x^q(Q_R)} \le c\|f\|_{L_t^s L_x^q(Q_1)} + c\|v\|_{L_t^s L_x^1(Q_1)}. \tag{A.1}$$

If instead v is a weak solution of

$$\partial_t v_i - \Delta v_i + \partial_i p = g_i, \quad \operatorname{div} v = 0 \quad \textit{in } Q_1,$$

then

$$\|\nabla^2 v\|_{L_t^s L_x^q(Q_R)} \le c\|g\|_{L_t^s L_x^q(Q_1)} + c\|v\|_{L^s L^1(Q_1)}. \tag{A.2}$$

An important feature of these estimates is that a bound of the pressure p is not needed in the right side. A similar estimate for the time-independent Stokes system appeared in [3232, 32]. Note that these estimates improve the spatial regularity only. One cannot improve the temporal regularity, in view of Serrin's example of a solution $v(x,t) = f(t)\nabla h(x)$ where $h(x)$ is harmonic.

Proof. Denote by P the Helmholtz projection in $\mathbb{R}^3$, $(Pg)_i = g_i - R_i R_k g_k$, where R_i is the i-th Riesz transform. Let $\tau = R^{1/4} \in (R, 1)$ and choose $\zeta(x,t) \in C^\infty(\mathbb{R}^4)$, $\zeta \ge 0$, $\zeta = 1$ on Q_τ and $\zeta = 0$ on $\mathbb{R}^3 \times (-\infty, 0] - Q_1$. For a fixed i, define

$$\tilde v_i(x,t) = \int_{-1}^t \Gamma(x-y, t-s)\, \partial_j(F_{ij})(y,s)\, dy\, ds,$$

where Γ is the heat kernel and $F_{ij} = f_{ij}\zeta - R_i R_k(f_{kj}\zeta)$. The function $\tilde v_i$ satisfies

$$(\partial_t - \Delta)\tilde v_i = \partial_j F_{ij} = [P\partial_j \zeta (f_{kj})_{k=1}^3]_i, \quad \operatorname{div} \tilde v = 0.$$

The $L_t^s L_x^q$-estimates for the parabolic version of singular integrals and potentials (see [1717, 172525, 253030, 30], also see [3838, 38], [1212, 12] and their references), and the usual version of L^q-estimates for singular integrals, give

$$\|\nabla \tilde v\|_{L_t^s L_x^q(Q_1)} + \|\tilde v\|_{L_t^s L_x^q(Q_1)} \le c\|F\|_{L_t^s L_x^q} \le c\|f\|_{L_t^s L_x^q(Q_1)}. \tag{A.3}$$

Furthermore, for some function $\tilde p(x,t)$,

$$(\partial_t - \Delta)\tilde v + \nabla \tilde p = \partial_j(\zeta f_{ij}), \quad \operatorname{div} \tilde v = 0.$$

The differences $u = v - \tilde v$ and $\pi = p - \tilde p$ satisfy the homogeneous Stokes system

$$\partial_t u - \Delta u + \nabla \pi = 0, \quad \operatorname{div} v = 0 \quad \text{in } Q_\tau.$$

Its vorticity $\omega = \operatorname{curl} u$ satisfies the heat equation $(\partial_t - \Delta)\omega = 0$. Let $W = \zeta_\tau \omega$, where $\zeta_\tau(x,t) = \zeta(x/\tau, t/\tau^2)$. It satisfies

$$(\partial_t - \Delta)W = G := w(\partial_t - \Delta)\zeta_\tau - 2(\partial_m \zeta_\tau)\partial_m \omega.$$

And thus, for $(x,t) \in Q_{\tau^2}$,

$$\omega_i(x,t) = W_i(x,t) = \int_{-1}^{t} \int \Gamma(x-y,t-s)\, G_i(y,s)\, dyds = \int_{-1}^{t} \int H^{i,j}_{x,y}(y,s)\, u_j(y,s)\, dyds$$

where, using $\omega_i = -\delta_{ijk}\partial_k u_j$,

$$H^{i,j}_{x,y}(y,s) = \partial_{y_k}\delta_{ijk}\left\{\Gamma(x-y,t-s)(\partial_t - \Delta)\zeta_\tau + 2\,\mathrm{div}[\Gamma(x-y,t-s)\nabla\zeta_\tau]\right\}.$$

The functions $H^{i,j}_{x,y}$ are smooth with uniform L^∞-bound for $(x,t) \in Q_{\tau^3}$. Thus

$$\|\,\mathrm{curl}\, u\|_{L^\infty(Q_{\tau^3})} \le C\|u\|_{L^1(Q_1)}.$$

Since $\mathrm{div}\, u = 0$, we have for any $q < \infty$, using Lemma A.1,

$$\|\nabla u\|_{L^s_t L^q_x(Q_R)} \le c\|u\|_{L^s_t L^1_x(Q_1)} \le c\|v\|_{L^s_t L^1_x(Q)} + c\|\tilde{v}\|_{L^s_t L^1_x(Q)}. \tag{A.4}$$

The sum of (A.3) and (A.4) gives (A.1). The proof of (A.2) is similar: one defines

$$\tilde{v}_i(x,t) = \int_{-1}^{t} \Gamma(x-y,t-s)\, F_i(y,s)\, dy\, ds, \quad F_i = g_i\zeta - R_i R_k(g_k\zeta)$$

and obtains $\|\nabla^2\tilde{v}\|_{L^s_t L^q_x(Q_1)} + \|\tilde{v}\|_{L^s_t L^q_x(Q_1)} \le c\|F\|_{L^s_t L^q_x} \le c\|g\|_{L^s L^q(Q_1)}$. One then estimates $\|\nabla^2(v-\tilde{v})\|_{L^s_t L^q_x(Q_R)}$ in the same way. □

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References

[1]]cite.MR6738301L. Caffarelli, R. Kohn, and L. Nirenberg, *Partial regularity of suitable weak solutions of the Navier-Stokes equations*, Comm. Pure Appl. Math. **35** (1982), no. 6, 771–831. MR **673830 (84m:**35097)

[2]]cite.MR19020552Dongho Chae and Jihoon Lee, *On the regularity of the axisymmetric solutions of the Navier-Stokes equations*, Math. Z. **239** (2002), no. 4, 645–671. MR **1902055 (2003d:**35205)

[3]]cite.MR00936493Ennio De Giorgi, *Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari*, Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. (3) **3** (1957), 25–43 (Italian). MR 0093649 (20 #172)

[4]]cite.MR20056394L. Escauriaza, G. Seregin, and V. Šverák, *Backward uniqueness for parabolic equations*, Arch. Ration. Mech. Anal. **169** (2003), no. 2, 147–157. MR **2005639 (2005j:**35097)

[5]]cite.MR03169155E. B. Fabes, B. F. Jones, and N. M. Rivière, *The initial value problem for the Navier-Stokes equations with data in L^p*, Arch. Rational Mech. Anal. **45** (1972), 222–240. MR 0316915 (47 #5463)

[6]]cite.MR8557536E. B. Fabes and D. W. Stroock, *A new proof of Moser's parabolic Harnack inequality using the old ideas of Nash*, Arch. Rational Mech. Anal. **96** (1986), no. 4, 327–338. MR **855753 (88b:**35037)

[7]]cite.MR8334167Yoshikazu Giga, *Solutions for semilinear parabolic equations in L^p and regularity of weak solutions of the Navier-Stokes system*, J. Differential Equations **62** (1986), no. 2, 186–212. MR **833416 (87h:**35157)

[8]]cite.GKT8Stephen Gustafson, Kyungkeun Kang, and Tai-Peng Tsai, *Interior regularity criteria for suitable weak solutions of the Navier-Stokes equations*, Comm. Math. Phys. (to appear), available at `arXiv:math.AP/0607114`.

[9]]cite.HouLi9Thomas Y. Hou and Congming Li, *Dynamic Stability of the 3D Axi-symmetric Navier-Stokes Equations with Swirl*, available at `arXiv:math.AP/0608295`.

[10]]cite.MR205584210Quansen Jiu and Zhouping Xin, *Some regularity criteria on suitable weak solutions of the 3-D incompressible axisymmetric Navier-Stokes equations*, Lectures on partial differential equations, New Stud. Adv. Math., vol. 2, Int. Press, Somerville, MA, 2003, pp. 119–139. MR **2055842 (2005i:**35210)

[11]]cite.MR208379311Kyungkeun Kang, *Regularity of axially symmetric flows in a half-space in three dimensions*, SIAM J. Math. Anal. **35** (2004), no. 6, 1636–1643 (electronic). MR **2083793 (2005m:**76046)

[12]]cite.MR186549312N. V. Krylov, *The Calderón-Zygmund theorem and its applications to parabolic equations*, Algebra i Analiz **13** (2001), no. 4, 1–25 (Russian, with Russian summary); English transl., St. Petersburg Math. J. **13** (2002), no. 4, 509–526. MR **1865493 (2002g:**35033)

[13]]cite.MR023654113O. A. Ladyženskaja, *Uniqueness and smoothness of generalized solutions of Navier-Stokes equations*, Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **5** (1967), 169–185 (Russian). MR 0236541 (38 #4836)

[14]]cite.MR024183314______, *Unique global solvability of the three-dimensional Cauchy problem for the Navier-Stokes equations in the presence of axial symmetry*, Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **7** (1968), 155–177 (Russian). MR 0241833 (39 #3170)

[15]]cite.MR171815615S. Leonardi, J. Málek, J. Nečas, and M. Pokorný, *On axially symmetric flows in $\mathbf{R}^3$*, Z. Anal. Anwendungen **18** (1999), no. 3, 639–649. MR **1718156 (2000h:**76038)

[16]]cite.JFM60.0726.0516Jean Leray, *Sur le mouvement d'un liquide visqueux emplissant l'espace.*, Acta Math. **63** (1934), 193–248 (French).

[17]]cite.MR023413017J. E. Lewis, *Mixed estimates for singular integrals and an application to initial value problems in parabolic differential equations*, Singular Integrals (Proc. Sympos. Pure Math., Chicago, Ill., 1966), Amer. Math. Soc., Providence, R.I., 1967, pp. 218–231. MR 0234130 (38 #2449)

[18]]cite.MR146518418Gary M. Lieberman, *Second order parabolic differential equations*, World Scientific Publishing Co. Inc., River Edge, NJ, 1996. MR **1465184 (98k:**35003)

[19]]cite.MR148851419Fanghua Lin, *A new proof of the Caffarelli-Kohn-Nirenberg theorem*, Comm. Pure Appl. Math. **51** (1998), no. 3, 241–257. MR **1488514 (98k:**35151)

[20]]cite.MR017009120Jürgen Moser, *A new proof of De Giorgi's theorem concerning the regularity problem for elliptic differential equations*, Comm. Pure Appl. Math. **13** (1960), 457–468. MR 0170091 (30 #332)

[21]]cite.MR010015821J. Nash, *Continuity of solutions of parabolic and elliptic equations*, Amer. J. Math. **80** (1958), 931–954. MR 0100158 (20 #6592)

[22]]cite.MR139756422J. Nečas, M. Růžička, and V. Šverák, *On Leray's self-similar solutions of the Navier-Stokes equations*, Acta Math. **176** (1996), no. 2, 283–294. MR **1397564 (97f:**35165)

[23]]cite.MR181422423Jiří Neustupa and Milan Pokorný, *An interior regularity criterion for an axially symmetric suitable weak solution to the Navier-Stokes equations*, J. Math. Fluid Mech. **2** (2000), no. 4, 381–399. MR **1814224 (2001m:**76034)

[24]]cite.MR184428424______, *Axisymmetric flow of Navier-Stokes fluid in the whole space with non-zero angular velocity component*, Proceedings of Partial Differential Equations and Applications (Olomouc, 1999), 2001, pp. 469–481. MR **1844284 (2003e:**76030)

[25]]cite.MR044026825N. M. Rivière, *Singular integrals and multiplier operators*, Ark. Mat. **9** (1971), 243–278. MR 0440268 (55 #13146)

[26]]cite.MR045442626Vladimir Scheffer, *Partial regularity of solutions to the Navier-Stokes equations*, Pacific J. Math. **66** (1976), no. 2, 535–552. MR 0454426 (56 #12677)

[27]]cite.MR197200527Gregory Seregin and Vladimir Šverák, *The Navier-Stokes equations and backward uniqueness*, Nonlinear problems in mathematical physics and related topics, II, Int. Math. Ser. (N. Y.), vol. 2, Kluwer/Plenum, New York, 2002, pp. 353–366. MR **1972005 (2005j:**35173)

[28]]cite.MR013688528James Serrin, *On the interior regularity of weak solutions of the Navier-Stokes equations*, Arch. Rational Mech. Anal. **9** (1962), 187–195. MR 0136885 (25 #346)

[29]]cite.MR71628329Hermann Sohr, *Zur Regularitätstheorie der instationären Gleichungen von Navier-Stokes*, Math. Z. **184** (1983), no. 3, 359–375 (German). MR **716283 (85f:**35167)

[30]]cite.MR123219230Elias M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Mathematical Series, vol. 43, Princeton University Press, Princeton, NJ, 1993. With the assistance of Timothy S. Murphy; Monographs in Harmonic Analysis, III. MR **1232192 (95c:**42002)

[31]]cite.MR93323031Michael Struwe, *On partial regularity results for the Navier-Stokes equations*, Comm. Pure Appl. Math. **41** (1988), no. 4, 437–458. MR **933230 (89h:**35270)

[32]]cite.MR178992232Vladimír Šverák and Tai-Peng Tsai, *On the spatial decay of 3-D steady-state Navier-Stokes flows*, Comm. Partial Differential Equations **25** (2000), no. 11-12, 2107–2117. MR **1789922 (2002g:**76040)

[33]]cite.MR060973233Roger Temam, *Navier-Stokes equations. Theory and numerical analysis*, North-Holland Publishing Co., Amsterdam, 1977. Studies in Mathematics and its Applications, Vol. 2. MR 0609732 (58 #29439)

[34]]cite.MR168561034Gang Tian and Zhouping Xin, *Gradient estimation on Navier-Stokes equations*, Comm. Anal. Geom. **7** (1999), no. 2, 221–257. MR **1685610 (2000i:**35166)

[35]]cite.MR164365035Tai-Peng Tsai, *On Leray's self-similar solutions of the Navier-Stokes equations satisfying local energy estimates*, Arch. Rational Mech. Anal. **143** (1998), no. 1, 29–51. MR **1643650 (99j:**35171)

[36]]cite.MR023929336M. R. Ukhovskii and V. I. Iudovich, *Axially symmetric flows of ideal and viscous fluids filling the whole space*, J. Appl. Math. Mech. **32** (1968), 52–61. MR 0239293 (39 #650)

[37]]cite.Vasseur37A. Vasseur, *A new proof of partial regularity of solutions to Navier-Stokes equations*, NoDEA. (to appear).

[38]]cite.MR65750538Wolf von Wahl, *The equation $u' + A(t)u = f$ in a Hilbert space and L^p-estimates for parabolic equations*, J. London Math. Soc. (2) **25** (1982), no. 3, 483–497. MR **657505 (84k:**34065)

[39]]cite.MR203102939Qi S. Zhang, *A strong regularity result for parabolic equations*, Comm. Math. Phys. **244** (2004), no. 2, 245–260. MR **2031029 (2005b:**35116)