Abstract Models of Memory Management*

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Abstract

Most specifications of garbage collectors concentrate on the low-level algorithmic details of how to find and preserve accessible objects. Often, they focus on bit-level manipulations such as “scanning stack frames,” “marking objects,” “tagging data,” etc. While these details are important in some contexts, they often obscure the more fundamental aspects of memory management: what objects are garbage and why?

We develop a series of calculi that are just low-level enough that we can express allocation and garbage collection, yet are sufficiently abstract that we may formally prove the correctness of various memory management strategies. By making the heap of a program syntactically apparent, we can specify memory actions as rewriting rules that allocate values on the heap and automatically dereference pointers to such objects when needed. This formulation permits the specification of garbage collection as a relation that removes portions of the heap without affecting the outcome of the evaluation.

Our high-level approach allows us to specify in a compact manner a wide variety of memory management techniques, including standard trace-based garbage collection (i.e., the family of copying and mark/sweep collection algorithms), generational collection, and type-based, tag-free collection. Furthermore, since the definition of garbage is based on the semantics of the underlying language instead of the conservative approximation of inaccessibility, we are able to specify and prove the idea that type inference can be used to collect some objects that are accessible but never used.

1 Memory Safety

Advanced programming languages manage memory allocation and deallocation automatically. Automatic memory managers, or garbage collectors, significantly facilitate the programming process because programmers can rely on the language implementation for the delicate tasks of finding and freeing unneeded objects. Indeed, the presence of a garbage collector ensures memory safety in the same way that a type system guarantees type safety: no program written in an advanced programming language will crash due to dangling pointer problems while allocation, access, and deallocation are transparent. However, in contrast to type systems, memory management strategies and particularly garbage collectors rarely come with a compact formulation and a formal proof of soundness. Since garbage collectors work on the machine representations of abstract values, the very idea of providing a proof of memory safety sounds unrealistic given the lack of simple models of memory operations.

The recently developed syntactic approaches to the specification of language semantics by Felleisen and Hieb [11] and Mason and Talcott [18, 19] are the first execution models that are intensional enough to permit the specification of memory management actions and yet are sufficiently abstract to permit compact proofs of important properties. Starting from the $\Lambda$-S calculus of Felleisen and Hieb, we design compact specifications of a number of memory management ideas and prove several correctness theorems.

The basic idea underlying the development of our garbage collection calculi is the representation of a program’s run-time memory as a global series of syntactic declarations. The program evaluation rules allocate large objects in the global declaration, which represents the heap, and automatically dereference pointers to such objects when needed. As a result, garbage collection can be specified as any relation that removes portions of the current heap without affecting the result of a program’s execution.

In Section 2, we present a small functional programming language, $\text{Age}$, with a rewriting semantics that makes allocation explicit. We define a semantic notion of garbage collection for $\text{Age}$ and prove that there is no optimal collection strategy that is compatible. In Section 3, we specify the “free-variable” garbage collection rule which models trace-based collectors including mark/sweep and copying collectors. We prove that the free-variable rule is correct and provide two “implementations” at the syntactic level: the first corresponds to a copying collector, the second to a generational one.

In Section 4, we formalize so-called “tag-free” collection algorithms for explicitly-typed, monomorphic languages such as Pascal and Algol [7, 29, 8]. We show how to recover

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necessary shape information about values from types during garbage collection. We are able to prove the correctness of the garbage collection algorithm by using a well known type preservation argument.

In Section 5, we justify our semantic definition of garbage by showing that Milner-style type inference can be used to prove that an object is semantically garbage even though the object is still reachable. While previous authors have sketched this idea [3, 5, 14, 12], we are the first to present a formal proof of this result. The proof is obtained by casting the well known interpretation of types as logical relations into our framework.

Section 6 discusses related work and Section 7 closes with a summary.

Due to a lack of space, most of the proofs for lemmas are omitted in this paper. However, full details may be recovered from our companion technical report [20].

2 Modeling Allocation: \( \lambda gc \)

Syntax: The syntax of \( \lambda gc \) (see Figure 1) is that of a conventional, higher-order, applicative programming language based on the \( \lambda \)-calculus. Following the tradition of functional programming, a \( \lambda gc \) program \( P \) consists of some mutually recursive definitions \( \{ H \} \) and an expression \( e \). The global definitions are useful for defining mutually recursive procedures, but their primary purpose here is to represent the run-time heap of a program. In general, there can be cycles in a heap, so we use letrec instead of let as the binding form. Expressions are either variables \( x \), integers \( i \), pairs \( (e_1,e_2) \), projections \( e_i \), abstractions \( \lambda x.e \), or applications \( e_1 e_2 \) (denoted \( e \)). Formally, the heap is a series of pairs, called bindings, consisting of variables and heap values. Heap values are a semantically significant subset of expressions. The order of the bindings is irrelevant and each variable must be bound to at most one heap value in a heap. Hence, we treat heaps as sets and, when convenient, as finite functions. We write \( \text{Dom}(H) \) to denote the bound variables of \( H \), and \( \text{Rng}(H) \) to denote the heap values bound in \( H \). We sometimes refer to variables bound in the heap as locations or pointers.

The language contains two binding constructs: \( \lambda x.e \) binds \( x \) in \( e \) and letrec \( H \) in \( e \) binds the variables in \( \text{Dom}(H) \) to the expressions \( \text{Rng}(H) \) in both the values in \( \text{Rng}(H) \) and in \( e \). Following convention, we consider programs to be equivalent up to a consistent \( \alpha \)-conversion of bound variables.

Considering programs equivalent modulo \( \alpha \)-conversion and the treatment of heaps as sets instead of sequences hides many of the complexities of memory management. In particular, programs are automatically considered equivalent if the heap is re-arranged and locations are re-named as long as the “graph” of the program is preserved. This abstraction allows us to focus on the issues of determining what bindings in the heap are garbage without specifying how such bindings are represented in a real machine.

Mathematical Notation: We use \( X \uplus X' \) to denote the union of two disjoint sets, \( X \) and \( X' \). We use \( H \uplus H' \) to denote the union of two heaps whose domains are disjoint. We use \( \{ e_1 / x \} e_2 \) to denote capture-avoiding substitution of the expression \( e_1 \) for the free variable \( x \) in the expression \( e_2 \). We use \( X \setminus X' \) to denote \( \{ x \in X \mid x \notin X' \} \).

Semantics: The rewriting semantics for \( \lambda gc \) is an adaptation of the standard reduction function of the \( \lambda \)-S calculus [11]. Roughly speaking, this kind of semantics describes an abstract machine whose states are programs and whose instructions are relations between programs. The desired final state of this abstract machine is an answer program \( (A) \) whose body is a pointer to some value, such as an integer, in the heap.

Each rewriting step of a program letrec \( H \) in \( e \) proceeds according to a simple algorithm. If the body of the program, \( e \), is not a variable, it is partitioned into an evaluation context \( E \) (an expression with a hole \( \{ E \} \) in the place of a sub-expression), which represents the control state, and an instruction expression \( f \), which roughly corresponds to a program counter, \( c = E[U] \). The instruction expression determines the next expression \( c' \) and any changes to the heap resulting in a new heap \( H' \). Putting the pieces together yields the next program in the evaluation sequence: letrec \( H' \) in \( E[c'] \). Each instruction determines one transition rule of the abstract machine. Formally, a rule denotes a relation between programs. A set of rules denotes the union of the respective relations.

We use the following conventions: Let \( G \) be a set of program relations, and let \( P \) and \( P' \) be programs. Then,

\[
P \xrightarrow{G} P' \text{ means } P \text{ rewrites to } P' \text{ according to one of the rules in } G \text{ and } \xrightarrow{G}^* \text{ is the reflexive, transitive closure of } \xrightarrow{G}.
\]

\( G + r \) is the union of \( G \) with the rule \( r \). A program \( P \) is irreducible with respect to \( G \) if there is no rule in \( G \) and no \( P' \) such that \( P \xrightarrow{G} P' \). \( P \xrightarrow{G}^* P' \) means that \( P \xrightarrow{G}^* P' \) and \( P' \) is irreducible with respect to \( G \).

\[
P \xrightarrow{G}^* \text{ means that there exists an infinite sequence of programs } P_0, \text{ such that } P \xrightarrow{G} P_1 \xrightarrow{G} P_2 \xrightarrow{G} \cdots.
\]

Figure 1 defines the set of evaluation contexts and instruction expressions for \( \lambda gc \). The definition of evaluation contexts \( (E) \) reflects the left-to-right, call-by-value evaluation order of the language. All terms to the left of the path from the root to the hole are variables; the terms on the right are arbitrary. Instruction expressions \( (I) \) consist of heap values \( (h) \), applications of \( \{ \text{pointers to heap-allocated} \} \) procedures to \( \{ \text{pointers to heap-allocated} \} \), and projections of \( \{ \text{pointers to heap-allocated} \} \) tuples.

The evaluation rules for \( \lambda gc \) reflect the intentions behind our choice of instruction expressions. The transition rule alloc models the allocation of values in the heap by binding the value to a new variable and using this variable in its place in the program. Note that the “\( \text{in} \)” notation carries the implicit requirement that the newly allocated variable \( x \) cannot be in the domain of the heap \( H \). The transition rule \( \text{proj} \) specifies how a projection instruction extracts the appropriate component from a pointer to a heap-allocated pair. Similarly, \( \text{app} \) is a transliteration of the conventional \( \beta \)-rule into our modified setting. It binds the formal parameter of a heap-allocated procedure to the value of the pointer given as the actual argument, and places the expression part of the procedure into the evaluation context. Multiple applications of the same procedure require \( \alpha \)-conversion to ensure that the formal parameter does not conflict with bindings already in the heap.\(^1\) We use \( R \) to abbreviate the union of the rules alloc, proj, and app.

\(^1\)An alternative rule for application substitutes the actual argument \( y \) for the formal \( x \) within \( e \) and performs no allocation. This rule is essentially equivalent to \( \text{app} \), but the definition above simplifies the proofs of Section 5.
Programs:

(variables) \(x, y, z \in \text{Var}\)
(integers) \(i \in \text{Int} \quad ::= \quad 0 \mid 1 \mid 2 \mid \cdots\)
(expressions) \(e \in \text{Exp} \quad ::= \quad x \mid i \mid \langle e_1, e_2 \rangle \mid \pi_1 e \mid \pi_2 e \mid \lambda x.e \mid e_1 e_2\)
(heap values) \(h \in \text{Hval} \quad ::= \quad i \mid \langle x_1, x_2 \rangle \mid \lambda x.e\)
(heaps) \(H \in \text{Heap} \quad ::= \quad \{x_1 = h_1, \ldots, x_n = h_n\}\)
(programs) \(P \in \text{Prog} \quad ::= \quad \text{letrec}\ H \ \text{in}\ e\)
(answers) \(A \in \text{Ans} \quad ::= \quad \text{letrec}\ H \ \text{in}\ x\)

Evaluation Contexts and Instruction Expressions:

\[
\begin{align*}
\text{(contexts)} & \quad E \in \text{Ctx} \quad ::= \quad \[\] \mid \langle E, e \rangle \mid \langle x, E \rangle \mid \pi_i E \mid E \mid E \mid x \mid y\ \\
\text{(instructions)} & \quad I \in \text{Instr} \quad ::= \quad h \mid \pi_i x \mid x \mid y
\end{align*}
\]

Rewriting Rules

(aloc) \(\text{letrec}\ H \ \text{in}\ E[D] \xrightarrow{\text{alloc}} \text{letrec}\ H \ \bowtie \ \{x = h\} \ \text{in}\ E[x]\)
(proj) \(\text{letrec}\ H \ \text{in}\ E[\pi_i x] \xrightarrow{\pi_i} \text{letrec}\ H \ \text{in}\ E[x_i]\)
(app) \(\text{letrec}\ H \ \text{in}\ E[x\ y] \xrightarrow{x\ y} \text{letrec}\ H \ \bowtie \ \{z = H(y)\} \ \text{in}\ E[e]\)

Figure 1: The Syntax and Operational Semantics of \(\lambda gc\)

The irreducible programs of \(\lambda gc\) are either answers or stuck programs. The latter correspond to machine states that result from the misapplication of primitive program operations or unbound variables.

**Definition 2.1 (Stuck Programs)** A program is stuck if it is one of the following forms:

- \(\text{letrec}\ H \ \text{in}\ E[\pi_i x] \quad (x \not\in \text{Dom}(H) \ \text{or} \ H(x) \neq \langle x_1, x_2 \rangle)\)
- \(\text{letrec}\ H \ \text{in}\ E[x\ y] \quad (x \not\in \text{Dom}(H) \ \text{or} \ H(x) \neq \lambda z.e \ \text{or} \ y \not\in \text{Dom}(H))\)

All programs either diverge or evaluate to an answer or a stuck program. Put differently, the evaluation process defines a partial function from \(\lambda gc\) programs to irreducible programs [11, 30].

**A Semantic Definition of Garbage** Since the semantics of \(\lambda gc\) makes the allocation of values explicit, including the implicit pointer dereferencing in the language, we can also define what it means to garbage collect a value in the heap and then analyze some basic properties. A binding \(x = h\) in the heap of a program is garbage if removing the binding has no “observable” effect on running the program. In our case, we consider only integer results and non-termination to be observable.

**Definition 2.2 (Kleene Equivalence)** \(P_1, G_1 \simeq (P_2, G_2)\) means \(P_1 \upharpoonright G_1 \text{letrec}\ H_1 \ \text{in}\ x \ \text{where} \ H_1(x) = i \ \text{if and only if} \ P_2 \upharpoonright G_2 \text{letrec}\ H_2 \ \text{in}\ y \ \text{and} \ H_2(y) = i.\)

If \(G_1 = G_2 = R\), then we simply write \(P_1 \simeq P_2\).

A binding is garbage if removing it results in a program that is Kleene equivalent to the original program.

**Definition 2.3 (Garbage)** If \(P = \text{letrec}\ H \ \bowtie \ \{x = h\} \ \text{in}\ e, \) then the binding \(x = h\) is garbage with respect to \(P\) if \(P \bowtie \text{letrec}\ H \ \text{in}\ e\). A collection of a program is the same program with some garbage bindings removed. An optimal collection of a program is a program with as many garbage bindings removed as possible.

Unfortunately, there can be no optimal garbage collector because determining whether a binding is garbage or not is undecidable.

**Proposition 2.4 (Garbage Undecidable)** Determining if a binding is garbage in an arbitrary closed \(\lambda gc\) program is undecidable.

**Proof (sketch):** We can reduce the halting problem to an optimal collector by taking an arbitrary program, adding a binding to the heap and modifying the program so that if it terminates, it accesses the extra binding. An optimal collector will collect the binding if and only if the original program does not terminate. \(\square\)

### 3 Reachability-Based Garbage Collection

Since computing an optimal collection is undecidable, a garbage collection algorithm must conservatively approximate the set of garbage bindings. Most garbage collectors compute the reachable set of bindings in a program given the variables in use in the current instruction expression and control state. All reachable bindings are preserved; the others are eliminated.

Following Felleisen and Hieb [11], reachability in \(\lambda gc\) is formalized by considering free variables. The following “free-variable” GC rule describes bindings as garbage if there are no references to these bindings in the other bindings, nor in the currently evaluating expression:

\[
\text{(fv)} \quad \text{letrec}\ H_1 \ \bowtie \ H_2 \ \text{in}\ e \xrightarrow{\text{fv}} \text{letrec}\ H_1 \ \text{in}\ e \quad \text{if} \ Dom(H_2) \cap \text{FV}(|\text{letrec}\ H_1 \ \text{in}\ e|) = \emptyset
\]

The fv rule is correct in that it only removes garbage, and thus computes valid collections. The keys to the proof of correctness of fv are a postponement lemma and a diamond lemma.

**Lemma 3.1 (Postponement)** If \(P_1 \xrightarrow{\text{fv}} P_2 \xrightarrow{\text{fv}} P_3\), then there exists a \(P'_2\) such that \(P_1 \xrightarrow{\text{fv}} P'_2 \xrightarrow{\text{fv}} P_3\).

**Proof (sketch):** By cases on the elements of \(R\). \(\square\)
Lemma 3.2 (Diamond) If $P_1 \xrightarrow{n} P_2$ and $P_2 \xrightarrow{m} P_2'$, then there exists a $P_3$ such that $P_2 \xrightarrow{2} P_3$ and $P_2' \xrightarrow{2} P_3$.

Proof (sketch): Assume $P_1 = \text{letrec } H_1 \uplus H_2 \in E[\Pi]$, $P_2 = \text{letrec } H_1 \in E[\Pi]$, and $P_1 \xrightarrow{t} P_2$. We can easily show by case analysis on the elements of $R$ that if $P_1 \xrightarrow{n} P_2'$ where $P_2' = \text{letrec } H_1 \uplus H_2 \uplus H_3 \in E[\Pi]$, for some $H_3$ and $e$, then $P_2' \xrightarrow{2} P_3$ and $P_2 \xrightarrow{2} P_3$ where $P_3 = \text{letrec } H_1 \uplus H_3 \in E[\Pi]$.

With the Postponement and Diamond Lemmas in hand, it is straightforward to show that $\text{fv}$ is a correct GC rule.

Theorem 3.3 (Correctness of $\text{fv}$) If $P \xrightarrow{\text{fv}} P'$, then $P'$ is a collection of $P$.

Proof: Let $P = \text{letrec } H_1 \uplus H_2 \in e$ and let $P' = \text{letrec } H_1 \in e$ such that $P \xrightarrow{\text{fv}} P'$. We must show $P'$ evaluates to an integer value if $P$ evaluates to the same integer. Suppose $P' \xrightarrow{R} \text{letrec } H' \in x$ and $H'(x) = i$. By induction on the number of rewriting steps using the Postponement Lemma, we can show that $P \xrightarrow{R} \text{letrec } H \uplus H_2 \in x$ and clearly $(H \uplus H_2)(x) = H'(x) = i$. Now suppose $P \xrightarrow{R} \text{letrec } H \in x$ and $H(x) = i$. By induction on the number of rewriting steps using the Diamond Lemma, we know that there exists an $H'$ such that $P \xrightarrow{R} \text{letrec } H' \in x$ and $H'(x) = i$. Thus, $x$ must be bound in $H'$ and since $\text{fv}$ only drops bindings, $H'(x) = i$.

This theorem shows that a single application of $\text{fv}$ results in a Kleene equivalent program. A real implementation interleave garbage collection with evaluation. The following theorem shows that adding $\text{fv}$ to $P$ preserves evaluation.

Theorem 3.4 For all programs $P$, $(P,R) \simeq (P,R + \text{fv})$.

Proof: Clearly any evaluation under $R$ can be simulated by $R + \text{fv}$ simply by not performing any $\text{fv}$ steps. Thus, if $P \xrightarrow{R} A$ then $P \xrightarrow{R + \text{fv}} A$. Now suppose $P \xrightarrow{R + \text{fv}} \text{letrec } H_1 \in x_1$ and $H_1(x_1) = i$. Then there exists a finite rewriting sequence using $R + \text{fv}$ as follows:

$$P \xrightarrow{\text{fv}} P_1 \xrightarrow{\text{fv}} P_2 \xrightarrow{\text{fv}} \cdots \xrightarrow{\text{fv}} \text{letrec } H_1 \in x_1$$

We can show by induction on the number of rewriting steps in this sequence, using the Postponement Lemma, that all $\text{fv}$ steps can be performed at the end of the evaluation sequence. This provides us with an alternative evaluation sequence where all the $R$ steps are performed at the beginning:

$$P \xrightarrow{R \uplus \text{fv}} P_1 \xrightarrow{\text{fv}} P_2 \xrightarrow{\text{fv}} \cdots \xrightarrow{\text{fv}} \text{letrec } H_1 \in x_1$$

Since $\text{fv}$ does not affect the expression part of a program and only removes bindings from the heap, $P'_n = \text{letrec } H_1 \uplus H_2 \in x$ for some $H_2$. Thus, $P \xrightarrow{R} \text{letrec } H_1 \uplus H_2 \in x$. Since $H_1(x) = i$, $(H_1 \uplus H_2)(x) = i$.

3.1 The Free-Variable Tracing Algorithm

The free-variable GC rule is a specification of a garbage collection algorithm. It assumes some mechanism for partitioning the set of bindings into two disjoint pieces such that one set of bindings is unreachable from the second set of bindings and the body of the program. Real garbage collection algorithms need a deterministic mechanism for generating this partitioning. It is possible to formulate an abstract version of such a mechanism, the free-variable tracing algorithm, by lifting the ideas of mark-sweep and copying collectors to the level of program syntax.

We adopt the terminology of copying collection in the description of the free-variable tracing algorithm. We use two heaps and a set: a “from-heap” ($H_f$), a “scan-set” ($S$), and a “to-heap” ($H_t$). The from-heap is the set of bindings in the current program and the to-heap will become the set of bindings preserved by the algorithm. The scan-set records the set of variables reachable from the to-heap that have not yet been moved from the from-heap to the to-heap.

The body of the algorithm proceeds as follows: A variable $x$ is removed from $S$ such that $H_t$ has a binding for $x$. If no such locations are in $S$, the algorithm terminates. Otherwise, it scans the heap value $h$ to which $x$ is bound in the from-set $H_f$, looking for free variables. For each $y \in FV(h)$, it checks to see if $y$ has already been forwarded to the to-set $H_t$. Only if $y$ is not bound in $H_t$ does it add the variable to the scan-set $S$. This ensures that a variable moves at most once from the from-heap to the scan-set.

Formulating the free-variable tracing algorithm as a rewriting system is easy. It requires only one rule that relates triples of from-sets, scan-sets, and to-sets:

$$\{(H_f \uplus \{x = h\}, S \uplus \{x = h\}, H_t) \Rightarrow (H_f \uplus S \uplus FV(h) \setminus (\text{Dom}(H_t) \cup \{x\}), H_t \uplus \{x = h\})\}$$

Initially the free variables of the evaluation context and instruction expression, which correspond to the “roots” of a computation, are placed in $S$. Computing the free variables of the context represents the scanning of the “stack” of a conventional implementation while computing the free variables of the instruction expression corresponds to scanning the “registers.” The initial triple is re-written until we reach a state where no variable in the scan-set is bound in the from-heap. At this point, we have forwarded enough bindings to the to-heap. This leads to the following free-variable tracing algorithm rule:

$$\text{letrec } H \in e \xrightarrow{\text{fv}} \text{letrec } H' \in e$$

(fva) if $\langle H, FV(e) \uplus \emptyset \rangle \Rightarrow^* \langle H'', S, H' \rangle$

and $\text{Dom}(H'') \cap S = \emptyset$

Clearly, the algorithm always terminates since the size of the from-heap strictly decreases with each step. Furthermore, this new rewriting rule is a subrule of the rule $\text{fv}$, which implies the correctness of the algorithm.

Theorem 3.5 If $P \xrightarrow{\text{fv}} P'$, then $P \xrightarrow{\text{fv}} P'$.

Proof: Let $P = \text{letrec } H \in e$ be a λ-expr program. The first step is to prove the basic invariants of the garbage collection rewriting system: If $(H, FV(e), \emptyset) \Rightarrow^* (H_f, S, H_t)$, then $H_f \uplus H_t \uplus H_f \uplus FV(e) \in e$. Now let $P' = \text{letrec } H_1 \in x$ and suppose $P \xrightarrow{\text{fv}} P'$. Then,

$$\langle H_1 \uplus H_2, FV(e), \emptyset \rangle \Rightarrow^* (H_2, S, H_1)$$

and $\text{Dom}(H_2) \cap S = \emptyset$. By the invariants, $FV(\text{letrec } H_1 \in e) = S$, so $\text{Dom}(H_1) \cap FV(P') = \emptyset$. Consequently, $P \xrightarrow{\text{fv}} P'$.

If we require that a collection algorithm produce a closed program, then $\text{fv}$ is "optimal" in the following weak sense:
if \( P \) is a closed program and \( P \xrightarrow{\text{letrec}} P' \), then \( P' \) has the fewest bindings needed to keep the program closed without affecting evaluation. Assuming each step in the free-variable tracing algorithm takes time proportional to the size (in symbols) of the heap object forwarded to the to-heap, the time cost of the algorithm is proportional to the amount of data preserved, not the total amount of data in the original heap.

### 3.2 Generational Garbage Collection

The free-variable tracing algorithm examines all of the reachable bindings in the heap to determine that a set of bindings may be removed. By carefully partitioning the heap into smaller heaps, a garbage collector can scan less than the whole heap and still free significant amounts of memory. A generational partition of a program’s heap is a sequence of sub-heaps ordered in such a way that “older” generations never have pointers to “younger” generations.

**Definition 3.6 (Generational Partition)**

A generational partition of a heap \( H \) is a sequence of heaps \( H_1, H_2, \ldots, H_n \) such that \( H = H_1 \uplus H_2 \uplus \cdots \uplus H_n \) and for all \( i \) such that \( 1 \leq i \leq n \), \( \text{Dom}(H_{i+1}) \cap \text{Dom}(H_{i+2}) \uplus \cdots \uplus \text{Dom}(H_n) = \emptyset \). The \( H_i \) are referred to as generations and \( H_i \) is said to be an older generation than \( H_j \) if \( i < j \).

Given a generational partition of a program’s heap, a free-variable based garbage collector can eliminate a set of bindings in younger generations without looking at any older generations.

**Theorem 3.7 (Generational Collection)**

Let \( H_1, \ldots, H_n \) be a generational partition of the heap of \( P \) = letrec \( H \) in \( e \). Suppose \( H_i = (H_i^1 \uplus H_i^2) \), and \( \text{Dom}(H_i^2) \cap \text{fv}(\text{letrec} \ H_i^1 \uplus H_{i+1} \uplus \cdots \uplus H_n) \in e) = \emptyset \). Then \( P \xrightarrow{\text{letrec}} \text{letrec} \ (H \setminus H_i^2) \) in \( e \).

**Proof:** We must show that \( \text{Dom}(H_i^2) \cap \text{fv}(\text{letrec} \ (H \setminus H_i^2)) \in e) = \emptyset \). Since \( H_1, \ldots, H_n \) is a generational partition of \( H \), for all \( j, 1 \leq j < i \), \( \text{fv}(H_j) \cap \text{Dom}(H_{j+1}^2 \uplus \cdots \uplus H_n) = \emptyset \). Hence, \( \text{fv}(H_j^1 \uplus \cdots \uplus H_{j-1}^1) \cap \text{Dom}(H_j^2) = \emptyset \). Now,

\[
\text{fv}(\text{letrec} \ H_j^1 \uplus H_j^2) \in e) \cap \text{Dom}(H_j^2)
\]

This shows the soundness of the type system for letrec gc-monos, an explicitly typed, monomorphic variant of letrec gc. We show how to adapt the free-variable tracing algorithm to recover types of objects in the heap and to use these types in the traversal of heap objects instead of abstract syntax. The proof of correctness for this garbage collection algorithm is given by extending the proof of soundness of the type system for letrec gc-monos.

### 4 Garbage Collection via Type Recovery

The delimiters and other tokens of the abstract syntax mark or “tag” heap values with enough information that we can distinguish pairs from functions, pointers from integers, etc. This allows us to navigate through the memory unambiguously, but placing tags on heap values and stripping them off to perform a computation can impose a heavy overhead on the running time and space requirements of programs [26].

An alternative to tagging is the use of types to determine the shape of an object. If types are determined at compile time and evaluation maintains enough information that the types of reachable objects can always be recovered, then there is no need to tag values.

A number of researchers have made attempts to explore this alternative [7, 8, 3, 13, 27, 2], but none of them presented concise characterizations of the underlying techniques with correctness proofs. In this section, we present the basic idea behind type-recovery based garbage collection. We then introduce letrec gc-monos, an explicitly typed, monomorphic variant of letrec gc. We show how to adapt the free-variable tracing algorithm to recover types of objects in the heap and to use these types in the traversal of heap objects instead of abstract syntax. The proof of correctness for this garbage collection algorithm is given by extending the proof of soundness of the type system for letrec gc-monos.

#### 4.1 letrec gc-monos

Letrec gc-monos is an explicitly typed, monomorphic variant of letrec gc. The set of types \( \tau \) of letrec gc-monos contains the conventional basic types and constructed types for typing a functional programming language like letrec gc. The expressions of letrec gc-monos are the same as for letrec gc, except that each raw expression \( (u) \) is paired with some type information (see Figure 2). Heap values are not paired with their type, reflecting the fact that the memory is “almost tag free.” Of course, some type information is recorded within heap values that are functions. In a low-level model that uses closures (an explicit value environment paired with code), this corresponds to maintaining a type environment, recording the types of the values in the closure’s environment. The type environment for a closure can be computed at compile time and shared among multiple instances, just like the code.

The evaluation contexts \( (E) \) consist of a raw context \( (U) \) and a type \( \tau \). Raw contexts are filled with instructions \( (I) \)
which are a subset of the raw expressions \((u)\). The evaluation rules for \(\lambda g\)-mono, named \(RM\), are variants of the rules for \(\lambda g\) that largely ignore the type information on the sub-expressions of the program's body. Allocation of integers, tuples and functions strips the type tag off the heap value before placing it in the heap. Allocation of tuples also removes the tags from the components of the tuple. The removal of type information corresponds to the passage of a value from code to data and does not necessarily reflect a runtime cost. Projection and application are essentially the same as for \(\lambda g\). Note that substitution of a result expression for an instruction occurs "in place," and hence the type of the instruction is ascribed to the expression.

The notion of a stuck state is adapted in accordance with the type structure of the language.

**Definition 4.1 (\(\lambda g\)-mono Stuck Programs)** A program is stuck if it is of one of the following forms:

\[
\begin{align*}
\text{letrec } & H \text{ in } E \{ x : \tau \} \quad (x \not\in \text{Dom}(H) \text{ or } H(x) \neq (x_1, x_2)) \\
\text{letrec } & H \text{ in } E \{ x:\tau \} \quad (x \not\in \text{Dom}(H) \text{ or } H(x) \neq \lambda y : \tau . e)
\end{align*}
\]

The static semantics of \(\lambda g\)-mono consists of four judgements about program components. The first judgement, \(\Gamma \vdash e\), is a binary relation between a type assignment \(\Gamma\), mapping a finite set of variables to types, and a typed expression \(e\). Figure 3 contains the fairly conventional inference rules for expressions that generate the static semantics. Typing heaps and complete programs requires three additional judgements. The first is \(\Gamma \vdash h : \tau\) which asserts that the heap value \(h\) has type \(\tau\) under the assumptions in \(\Gamma\). The judgement is defined using inference rules (not shown here) similar to the expression-level rules. The second is \(\Gamma \vdash H : \Gamma'\), which asserts that the variables given type \(\tau\) in \(\Gamma'\) are bound to heap values in \(H\) of the appropriate type, under the assumptions in \(\Gamma\). The judgement's definition via the \(\text{heap}\) rule requires "guessing" the types of the values in the heap and verifying these guesses simultaneously. The third judgement, \(\triangleright P\), asserts the well-typing of a complete program.

The calculus \(\lambda g\)-mono is \(\text{type sound}\) in that evaluation of well formed programs cannot get stuck [30]. A key to the proof of soundness is a \(\text{type preservation}\) lemma:

**Lemma 4.2 (Type Preservation)** If \(\triangleright P\) and \(P \overset{RM}{\rightarrow} P'\), then \(\triangleright P'\).

**Theorem 4.3 (Type Soundness)** If \(\triangleright P\) then either \(P\) is an answer or else there exists some \(P'\) such that \(P \overset{RM}{\rightarrow} P'\) and \(\triangleright P'\).

### 4.2 Using Types in Garbage Collection

Specifying a valid garbage collection rule that exploits types is straightforward. The key property that the rule must preserve is \(\text{type preservation}\). That is, if \(P \overset{RM}{\rightarrow} P'\) and \(P\) is well typed, then \(P'\) must be well typed. One way to achieve this goal is to make it into a side-condition of the GC rule:

\[
(\text{mono}) \quad \text{letrec } H_1 \equiv H_2 \text{ in } e \overset{\text{nm}}{\rightarrow} \text{letrec } H_1 \text{ in } e \quad \text{if } \triangleleft \text{letrec } H_1 \text{ in } e
\]

**Theorem 4.4** For all well formed \(\lambda g\)-mono programs \(P\), \((P, RM) \simeq (P, RM + \text{mono})\).

**Proof:** Since \(\text{mono}\) preserves types, the type soundness proof trivially extends to the dynamic semantics with \(\text{mono}\). This implies that, since \(P\) is well formed, \(P\) cannot get stuck under either system. Since bindings are only dropped and not updated by \(\text{mono}\), the results of evaluating under either system must be the same. \(\square\)

Like the free-variable rule of \(\lambda g\), \(\text{mono}\) needs to be refined before it can serve as the basis for an implementation.
The refinement is similar to the one from \textsc{fv} to \textsc{fva} but uses the types embedded in programs to traverse heap values. The basis for the collection algorithm is a function that determines a minimal, with respect to set-inclusion, type assignment \( \Gamma \) for any expression \( e \) such that \( \Gamma \vdash e \):

\[
\begin{align*}
\text{(var)} & \quad \Gamma \vdash (u : \tau_1) & \text{(int)} & \quad \Gamma \vdash (i : \text{int}) \\
\text{(pair)} & \quad \Gamma \vdash (u_1 : \tau_1, u_2 : \tau_2) & \text{(proj)} & \quad \Gamma \vdash (u : \tau_1 \times \tau_2) \\
\text{(fn)} & \quad \Gamma \vdash \lambda x : \tau_1. (u : \tau_2) & \text{(app)} & \quad \Gamma \vdash (u_1 : \tau_1 \rightarrow \tau_2, u_2 : \tau_1) \\
\text{(heap)} & \quad \forall x \in \text{Dom}(\Gamma^t). \Gamma \vdash \Gamma^t \triangleright H(x) : \Gamma^t \\
\end{align*}
\]

Figure 3: The Static Semantics of \(\alpha\)-mono

where \( \Gamma^t = (\Gamma^t \cup \text{MTA}(\text{Tag}^{\Gamma^t}(h))) \setminus \Gamma_t \)

The algorithm is initialized by taking the program heap \( H \) as the initial from-heap and the minimal type assignment of the program expression as \( \Gamma_0 \). At each step in the algorithm, \( \Gamma_s \) describes the types of all locations that are immediately reachable from \( e \) or \( H \), but have not yet been forwarded to \( H_s \). \( \Gamma_t \) describes the types of all locations that have been forwarded to \( H_t \).

When a variable \( x \) is found in \( \Gamma_s \) with type \( \tau \) and \( x \) is bound in \( H_f \) to the heap value \( h \), the collector forwards the binding \( x = h \) to \( H_t \) and adds \( x : \tau \) to \( \Gamma_t \). It then uses \( \text{Tag}^{\Gamma^t} \) to traverse \( h \), placing the necessary type information on the components so that \( \text{MTA} \) can determine the heap value’s minimal type assignment. This step provides the locations (and their types) that are immediately reachable from \( h \). Finally, the collector adds each of these locations to \( \Gamma_s \) unless they have already been forwarded to \( H_t \).

Using this algorithm instead of an \textit{a priori} partitioning of the heap, \textit{mono-\alpha} becomes a high-level specification of a collector whose traversal of heap values uses types instead of tag information:

\[
\begin{align*}
\text{letrec} & \quad H' \in e \\
\text{if} & \quad (H, \text{MTA}(e), \emptyset, \emptyset) \xrightarrow{\text{mono-\alpha}} (H'', \emptyset, H', \Gamma)
\end{align*}
\]

The following definition gives the primary invariants of the algorithm:

\textbf{Definition 4.7 \textit{(mono-\alpha) Invariants}} \hspace{0.5cm} (\textbf{H}_f, \Gamma_s, H_t, \Gamma_t) satisfies the \textit{mono-\alpha} invariants with respect to a program \textbf{letrec} \( H \) in \( e \) and type assignment \( \Gamma_0 \) iff:

1. \( H = H_f \triangleright H_t \)
2. \( \Gamma_s \triangleright \Gamma_t \triangleright e \)
3. \( \text{Dom}(\Gamma_s) \subseteq \text{Dom}(H_f) \)
4. \( \Gamma_s \triangleright H_t : \Gamma_t \)
5. \( \Gamma_s \triangleright \Gamma_t \subseteq \Gamma_0 \)

Intuitively, the invariants guarantee (1) that each binding is accounted for, (2) every binding needed for \( e \) is in the from-set or to-heap, (3) the from-set contains all bindings in the from-heap, (4) the from-set contains all free variables in the to-heap, and (5) the from-set and to-heap agree with \( \Gamma_0 \).

\textbf{Lemma 4.8} If \( \emptyset \triangleright H : \Gamma_0 \), \( T \) has the \textit{mono-\alpha} invariant properties with respect to \( P \) and \( \Gamma_0 \), and \( T \xrightarrow{\text{mono-\alpha}} T' \), then \( T' \)

\footnote{The garbage collection rewriting system only maintains \( \Gamma_t \) to simplify the presentation and proof; an implementation will not have to construct \( \Gamma_t \).}
has the mono-a invariant properties with respect to $P$ and $\Gamma_0$.

The correctness of the algorithmic type-based garbage collection rule can now easily be verified.

**Theorem 4.9** If $P$ is a well-formed program and $P \overset{\text{mono}}{\rightarrow} P'$, then $P \overset{\text{gc-mono}}{\rightarrow} P'$.

**Proof:** We must verify that the mono-a garbage collection rule preserves typability in order for mono to apply. That is, we must show that if $P$ is a well-formed program, and $P \overset{\text{mono}}{\rightarrow} P'$, then $P \overset{\text{gc-mono}}{\rightarrow} P'$.

Let $P = \text{letrec } H \in e$ and suppose $P \overset{\text{gc-mono}}{\rightarrow} P'$. Then, for some $\Gamma_0$, $\emptyset \triangleright H : \Gamma_0$ and $\Gamma_0 \triangleright e$. If

$$\langle H, \text{MTA}(e), \emptyset, \emptyset \rangle \overset{\text{mono}}{\rightarrow}^* \langle H', \emptyset, H', \Gamma_1 \rangle,$$

then by Lemma 4.8 taking $\Gamma_s = \emptyset$, we know that $\emptyset \triangleright H' : \Gamma_s$ and $\Gamma_s \triangleright e$. Thus, by the $\text{prog}$ rule of the static semantics, $P \overset{\text{gc-mono}}{\rightarrow} P'$.

Furthermore, the algorithm never gets stuck, so it always applies:

**Theorem 4.10** If $P$ is a well-formed program, then there exists a program $P'$ such that $P \overset{\text{gc-mono}}{\rightarrow} P'$.

**Proof:** If the algorithm takes a step, the size of the from-heap strictly decreases, so we know the collection either terminates or gets stuck. Here we show that the collection cannot get stuck, so it must terminate. Suppose $P = \text{letrec } H \in e$ and $\langle H, \text{MTA}(e), \emptyset, \emptyset \rangle \overset{\text{mono}}{\rightarrow}^* \langle H', \emptyset, H', \Gamma_s \rangle$. By Lemma 4.8, we know that $x \in \text{Dom}(H_s)$, since the domain of the scan type-assignment is a subset of the from-heap. Consequently, there exists an $h$ such that $H_f = H'_f \triangleright \{ x = h \}$ and $\langle H_f, \Gamma_s \triangleleft \{ x : \tau \}, H_f, \Gamma_s \rangle \overset{\text{mono}}{\rightarrow}^* \langle H'_f, \Gamma_s, H_f \triangleright \{ x = h \}, \Gamma_f \rangle$ with appropriate $\Gamma_f$ and $\Gamma'_s$.

Extending the mono-a collection algorithm to work for a language with explicit polymorphism, where types are passed to polymorphic routines at run time as suggested by various language implementors [21, 1, 27, 15], is straightforward because enough information is preserved by evaluation to always reconstruct the type of a polymorphic object. In our technical report [20], we show how this may be accomplished.

## 5 Collecting Reachable Garbage Using Type Inference

Thus far, we have only considered specifications and algorithms for collecting unreachable bindings. In this section, we show that by using type inference during the garbage collection process, some bindings that are reachable can still be safely collected. That is, type inference can be used to prove that an object is garbage even though it is reachable.

For a simple example, consider the following λgc program:

```plaintext
letrec \{ x_1 = 1, x_2 = 2, x_3 = \{ x_2, x_2 \}, x_4 = \{ x_1, x_3 \} \} in \pi_1 x_4
```

Every binding in the heap is accessible from the program's expression $(\pi_1 x_4)$, so the free-variable based collection rules can collect nothing. But clearly the program will never dereference $x_3$ or $x_2$. The inference-based collection scheme described in this section will allow us to conclude that replacing the binding $x_3 = \{ x_2, x_2 \}$ with $x_3 = 0$ (or any other binding) will have no observable effect on evaluation. That is, the inference collection scheme shows that

letrec \{ $x_1 = 1, x_2 = 2, x_3 = 0, x_4 = \{ x_1, x_3 \} \} in \pi_1 x_4
```

is Kleene equivalent to the original program. Now by applying the free-variable rule, we can conclude that the binding $x_2 = 2$ can be safely collected.

We start by considering the original λgc language as an implicitly typed, monomorphic language, where the types of the language are the same as for λgc-mono except for the addition of type variables:

$$\text{(types) } \tau \in \text{Type } ::= \text{t | int | } t_1 \times t_2 \mid \text{t} \rightarrow t_2$$

By implicitly typed, we mean that the terms of the language are not decorated with types as in λgc-mono. We add variables to the set of types so that each well-formed expression has a principal or most general type (explained below).

Type inference is the process of decorating λgc programs with types so that the resulting program type checks under the λgc-mono rules. (Refer to Figure 1 for the syntax and dynamic semantics of λgc and Figure 3 for the typing rules for λgc-mono.) Alternatively, we may directly specify a set of typing rules for λgc programs by taking the typing rules for λgc-mono and enforcing the type information from the terms, resulting in the inference system of Figure 4. These rules define judgements of the form $\Gamma \vdash e : \tau$, where $\Gamma$ is a type assignment, $e$ is a λgc expression, and $\tau$ is a type.

A given λgc expression can have multiple typing derivations according to these rules and consequently multiple typings, but an expression's typings may be ordered so that there is a most general, or principal typing and every other typing is an instance of this principal typing and is thus derivable.

We will show that if we can find a typing for a program that assigns a heap location a type variable, then the contents of that location has no effect on the rest of evaluation. Consequently, any pointers contained in the location's binding do not need to be scanned and traced during garbage collection. The intuition behind the theorem is that a location's type is unconstrained only if the location is not used in some manner that would constrain the type. Consequently, we can replace the binding in the heap with any binding we choose. In particular, if the location is bound to a large heap value, we can bind the location to an integer or dummy piece of code without affecting evaluation. This replacement allows us to collect any bindings that used to be reachable through this binding without knowing anything about the shape of the original heap value.

The proof of the theorem relies upon a semantic interpretation of types as logical relations. In our case, the relations are a type-indexed family of binary relations relating programs to programs, answers to answers, and heaps to heaps. The relations are contrived so that, if two programs are related, then they are Kleene equivalent, so one program converges to an answer iff the other converges to a related answer and related answers at base type (int) yield equal values. Roughly speaking, the relations are logically extended to relate answers of functional type ($\rightarrow$) if, whenever such answers are applied to appropriately related answers, the resulting computations yield related results. Our proof only covers programs without cycles in their heaps, but it should be possible to extend our arguments to all programs (see below for more details).

The relations are defined in Figure 5 by induction on types. The definitions are parameterized by an arbitrary
relational interpretation of type variables, \( \Theta \). If \( t \) is a type variable, then \( \Theta(t) \) determines some fixed, but arbitrary relation between answer programs. This is consistent with the idea that well-typed programs have an implicit "\( \forall \)" quantifier for the type variables in a program. The parameterization of the interpretation of type variables makes it straightforward to extend the definition of the relations to account for predicative polymorphism.

Two well-typed programs \( P_1 \) and \( P_2 \) are related at a type \( \tau \), written \( \Theta \vdash P_1 \sim P_2 : \tau \) iff, whenever one of the programs terminates with an answer, then the other program terminates with a related answer at type \( \tau \).

Two answers \( A_1 \) and \( A_2 \) are related at type \( \tau \), written \( \Theta \vdash A_1 \sim A_2 : \tau \) as follows: If \( \tau \) is a type variable \( t \), then the answers are related iff they are in the relation \( \Theta(t) \). The relation is extended to the other types in a natural fashion. For example, if \( \tau \) is an arrow type \( \tau_1 \rightarrow \tau_2 \), the answers are related iff, whenever we apply the answer variables to related arguments at type \( \tau_1 \), we get related programs at type \( \tau_2 \). Even though the relations between programs and answers are defined in terms of one another, the relations are well-founded because the size of the type index always decreases when one relation refers to another.

The definition of the relations ensures that related programs remain related even if more bindings are added to the programs’ heaps. Since evaluation only adds new bindings and leaves existing bindings intact, it is clear that evaluation preserves the relations. If the language permitted assignment, then this property would not necessarily hold.

For the statement of the following lemma, we need to extend the logical relation on answers to heaps. Two heaps, \( H_1 \) and \( H_2 \), are related at a context \( \Gamma \), written \( \Theta \vdash H_1 \approx H_2 : \Gamma \) if for all variables \( x \) in \( \Gamma \), the answers \( \text{letrec } H_1 \) in \( x \) and \( \text{letrec } H_2 \) in \( x \) are related at \( \Gamma(x) \).

The following lemma is the key to establishing our result: an expression is related to itself in the context of any two related heaps.

**Lemma 5.1** For all \( \Theta : T \rightarrow P(\text{Ans} \times \text{Ans}) \), if \( \Gamma \vdash e : \tau \) and \( \Theta \vdash H_1 \approx H_2 : \Gamma \), then \( \Theta \vdash \text{letrec } H_1 \in e \sim \text{letrec } H_2 \in e : \tau \).

**Proof (sketch):** By induction on the derivation of \( \Gamma \vdash e : \tau \). Here we give the interesting case (fin). Suppose \( \Gamma \vdash \lambda x.e : \tau_1 \rightarrow \tau_2 \) and \( \Theta \vdash H_1 \approx H_2 : \Gamma \). By alloc,

\[
\begin{align*}
\text{letrec } H_1 \text{ in } \lambda x.e & \rightarrow \text{letrec } H_1 \uplus (y_1 = \lambda x.e) \text{ in } y_1 \\
\text{letrec } H_2 \text{ in } \lambda x.e & \rightarrow \text{letrec } H_2 \uplus (y_2 = \lambda x.e) \text{ in } y_2
\end{align*}
\]

for some fresh \( y_1 \) and \( y_2 \). We must show that the two resulting answers are appropriately related.

Let \( H'_1 \) and \( H'_2 \) be heaps such that \( H_1 \uplus \{ y_1 = \lambda x.e \} \subseteq H'_1 \) for \( i = 1, 2 \), and

\[
\Theta \vdash \text{letrec } H'_1 \text{ in } z_1 \approx \text{letrec } H'_2 \text{ in } z_2 : \tau_0
\]

for some \( z_1 \) and \( z_2 \). We must show that

\[
\Theta \vdash \text{letrec } H'_1 \text{ in } y_1 z_1 \sim \text{letrec } H'_2 \text{ in } y_2 z_2 : \tau_2.
\]

Taking \( H''_1 = H'_1 \uplus \{ x = H'_1(z_1) \} \), this follows if

\[
\Theta \vdash \text{letrec } H''_1 \text{ in } e \sim \text{letrec } H''_2 \text{ in } e : \tau_2.
\]

By the induction hypothesis, it suffices to show that

\[
\Theta \vdash H''_1 \approx H''_2 : \Gamma \uplus \{ x : \tau_1 \}.
\]

since \( e \) has a smaller typing derivation than \( \lambda x.e \). By the lemma’s hypothesis, we know that \( \Theta \vdash H_1 \approx H_2 : \Gamma \), and since heaps remain related under extensions, \( \Theta \vdash H''_1 \approx H''_2 : \Gamma \). By assumption,

\[
\Theta \vdash \text{letrec } H''_1 \text{ in } z_1 \approx \text{letrec } H''_2 \text{ in } z_2 : \tau_1.
\]

Since \( H''_1 = H'_1 \uplus \{ x = H'_1(z_1) \} \), it is easy to show that

\[
\Theta \vdash \text{letrec } H''_1 \text{ in } x \approx \text{letrec } H''_2 \text{ in } x : \tau_1.
\]

Consequently, \( H''_1 \) and \( H''_2 \) have related heap-values for each variable in \( \Gamma \uplus \{ x : \tau_1 \} \) and thus \( \Theta \vdash H''_1 \approx H''_2 : \Gamma \uplus \{ x : \tau_1 \} \).

The other cases follow in a similar manner. \( \square \)

Our goal is to show that if \( \Gamma \) is a type assignment, \( e \) an expression, and \( H \) a heap such that \( \Gamma \vdash e : \tau \) and \( \emptyset \vdash H : \Gamma \), then \( \text{letrec } H' \text{ in } e \) is Kleene equivalent to \( \text{letrec } H \text{ in } e \) where \( H' \) is defined as follows:

\[
H' = \{ x = H(x) \mid \Gamma(x) \not\in T \text{Var} \} \uplus \{ x = 0 \mid \Gamma(x) \in T \text{Var} \}
\]

This follows from Lemma 5.1 if we can show that, taking \( \Theta(t) \) to be the everywhere-defined relation on answer programs, \( \Theta(t) \vdash H \approx H' : \Gamma \). This in turn follows if we can show that \( \Theta(t) \vdash H \approx H : \Gamma \) (if \( H \) is related to itself), since \( \Theta(t) \) relates every program and \( H'(x) \) differs from \( H(x) \) only when \( \Gamma(x) = t \).

Unfortunately, we cannot directly show that a well-formed heap is related to itself! The problem is that if we attempt to argue by induction on the derivation of \( \emptyset \vdash H : \Gamma \), the uses of the heap rule require that we assume what we
are trying to prove. The same problem is encountered when using logical relations to reason about conventional calculi with recursion or iteration operators.

If we forbid cycles in the heap, then we can transform the derivation of the heap’s well formedness into a derivation that only uses a let-style rule instead of the recursive let-rec-style heap rule:

\[
\begin{align*}
\text{(let-heap)} & \quad \Gamma_1 \parallel \Gamma_2 \vdash h : \tau & \quad \Gamma_1 \vdash H : \Gamma_2 \\
& \quad \Gamma_1 \vdash H \parallel \{x \equiv h\} : \Gamma_2 \parallel \{x : \tau\}
\end{align*}
\]

Consequently, if a heap is cycle free, we may show by induction on the derivation using the let-heap rule that the heap is related to itself.

**Lemma 5.2** If \(\Gamma_1 \vdash H : \Gamma_2\), \(\Theta \models H_1 \approx H_2 : \Gamma_1\) and \(H\) is cycle free, then \(\Theta \models H_1 \parallel H \approx H_2 : \Gamma_1 \parallel \Gamma_2\).

Finally, we can state and prove the following Inference GC specification: Given a cycle-free program, if we can find a typing that assigns a heap location a type variable, then that location can be bound to \(~\_\) without affecting evaluation.

**Theorem 5.3 (Inference GC)** Let
\[
\begin{align*}
\Gamma_1 = \{x_1 : t_1, \ldots, x_n : t_n\} \quad \text{and} \quad H_1 = \{x_1 = h_1, \ldots, x_n = h_n\} \\
\text{and} \quad H_1' = \{x_1 = 0, \ldots, x_n = 0\}. \quad \text{If}
\end{align*}
\]

1. \(\Gamma_1 \parallel \Gamma_2 \vdash e : \tau\) (\(~\_\) \(\notin\) \(\text{Var}\)), and
2. \(\Gamma_1 \vdash H : \Gamma_2\), and
3. \(\exists S \parallel H_1 : S'\), and
4. \(H_2\) is cycle free,

then letrec \(H_1 \parallel H_2\) in \(e\) \(\approx\) letrec \(H_1' \parallel H_2\) in \(e\).

**Proof:** Taking \(\Theta_0(t)\) to be the everywhere defined relation, \(\Theta_0 \models H_1 \approx H_1' : \Gamma_1\) holds trivially. By Lemma 5.2, since \(H_2\) is cycle free and \(\Gamma_1 \vdash H_2 : \Gamma_2\), we know that
\[
\Theta_0 \models H_1 \parallel H_2 \approx H_1' \parallel H_2 : \Gamma_1 \parallel \Gamma_2.
\]

Since \(\Gamma_1 \parallel \Gamma_2 \vdash e : \tau\), we know by Lemma 5.1 that
\[
\Theta_0 \models \text{letrec } H_1 \parallel H_2 \text{ in } e \sim \text{letrec } H_1' \parallel H_2 \text{ in } e : \tau.
\]

Thus, letrec \(H_1 \parallel H_2\) in \(e\) \(\approx\) letrec \(H'_1 \parallel H_2\) in \(e\).

\(\square\)

Using the Inference GC theorem, we can now show that the binding \(x_3 = \{x_2, x_3\}\) in the program:

\[
\text{letrec } \{x_1 = 1, x_2 = 2, x_3 = \{x_2, x_3\}, x_4 = \{x_1, x_3\}\} \text{ in } \pi_1 x_4
\]

can be replaced by \(x_3 = 0\). Taking \(\Gamma_1 = \{x_2 : t_2, x_3 : t_3\}, \Gamma_2 = \{x_1 : \text{int}, x_4 : \text{int} \times t_3\}, H_1 = \{x_2 = 2, x_3 = \{x_2, x_3\}\}, H_2 = \{x_1 = 0, x_4 = \{x_1, x_3\}\}\), it is easy to show that the pre-conditions of the theorem hold. Consequently, replacing \(H_1\) with \(H'_1 = \{x_2 = 0, x_3 = 0\}\) results in a Kleene equivalent program:

\[
\text{letrec } \{x_1 = 1, x_2 = 0, x_3 = 0, x_4 = \{x_1, x_3\}\} \text{ in } \pi_1 x_4
\]

Now by invoking \(\text{fv}\), we can collect the binding \(x_2 = 0\):

\[
\text{letrec } \{x_1 = 1, x_2 = 0, x_4 = \{x_1, x_3\}\} \text{ in } \pi_1 x_4
\]

and we know that the resulting program is Kleene equivalent to the original program.

**6 Related Work**

The literature on garbage collection in sequential programming languages per se contains few papers that attempt to provide a compact characterization of algorithms or correctness proofs. Demers et al. [80] give a model of memory parameterized by an abstract notion of a “points-to” relation. As a result, they can characterize reachability-based algorithms including mark-sweep, copying, generational, “conservative,” and other sophisticated forms of garbage collection. However, their model is intentionally divorced from the programming language and cannot take advantage of
any semantic properties of evaluation, such as type preservation. Consequently, their framework cannot model the type-based collectors of Sections 4 and 5. Nettles [22] provides a concrete specification of a copying garbage collection algorithm using the Larch specification language. Our specification of the free-variable tracing algorithm is essentially a high-level, one-line description of his specification. Hudak gives a denotational model that tracks reference counts for a first-order language [17]. He presents an abstraction of the model and gives an algorithm for computing approximations of reference counts statically. Chirimar, Gunter, and Riecke give a framework for proving invariants regarding memory management for a language with a linear type system [9]. Their low-level semantics specifies explicit memory management based on reference counting. Both Hudak and Chirimar et al. assume a weak approximation of garbage (reference counts). Barendsen and Smetsers give a Curry-like type system for functional languages extended with uniqueness information that guarantees an object is only "locally accessible" [6]. This provides a compiler enough information to determine when certain objects may be garbage collected or overwritten.

Tolmach [27] built a type-recovery collector for a variant of SML that passes type information to polymorphic routines during execution, effectively implementing a polymorphic version of our language and collector described in Section 4. Aditya and Caro gave a type-recovery algorithm for an implementation of Id that uses a technique that appears to be equivalent to type passing [1] and Aditya, Flood, and Hicks extended this work to garbage collection for Id [2].

Over the past few years, a number of papers on inference-based collection in monomorphic [7, 29, 8] and polymorphic [3, 13, 14, 12] languages appeared in the literature. Appel [3] argued informally that "tag-free" collection is possible for polymorphic languages such as SML by a combination of recording information statically and performing what amounts to type inference during the collection process, though the connections between inference and collection were not made clear. Baker [5] recognized that Milner-style type inference can be used to prove that reachable objects can be safely collected, but did not give a formal account of this result. Goldberg and Gloger [14] recognized that it is not possible to reconstruct the concrete types of all reachable values in an implementation of an ML-style language that does not pass types to polymorphic routines. They gave an informal argument based on traversal of stack frames to show that such values are semantically garbage. Fradet [12] gave another argument based on Reynolds' abstraction/parametricity theorem [24]. Fradet's formulation is closer to ours than Goldberg and Gloger's, since he represented the evaluation "stack" as a source-language term. However, none of these papers give a complete formulation of the underlying dynamic and static semantics of the language and thus the proofs of correctness are necessarily ad hoc.

Finally, Purushothaman and Seaman [23, 25] and Launchbury [17] have proposed "natural" semantics for call-by-need (lazy) languages where the semantic objects include an explicit heap. This allows sharing and memoization of computations to be expressed in the semantics. More recently, Ariola et al. [4] have presented a purely syntactic theory of the call-by-need \(\lambda\)-calculus that is largely compatible with our work.

7 Summary and Future Work

Our paper provides a unifying framework for a variety of garbage collection ideas including standard copying and mark-sweep collection, generational collection, tag-free collection, and inference-based collection. By making allocation and the heap explicit, we are able to reason about memory management using traditional \(\lambda\)-calculus techniques. In particular, we are able to make strong connections between garbage collection and type theory. By abstracting away such low-level details as evaluation stacks, registers, and addresses, we are able to formulate complicated collection algorithms in a compact manner and yet give a formal proof of correctness.

The framework is flexible, as demonstrated by the breadth of topics covered in this paper, and we expect that the framework can be extended to deal with other work on garbage collection. In the future, we hope to extend the inference GC theorem of Section 5 to cover programs with cyclic heaps. One approach is to interpret types as COPs and prove that programs have a suitable compactness property: every cyclic heap is appropriately approximated by some finite "unwinding" of the heap. An alternative approach might be to provide a purely syntactic argument that locations assigned a type variable never arise in an "active" position during evaluation. We also hope to provide a proof of correctness for an algorithm that simultaneously infers types and performs garbage collection.

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