Derived categories and birational geometry of Gushel-Mukai varieties

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Abstract

We study the derived categories of coherent sheaves on Gushel–Mukai varieties. In the derived category of such a variety, we isolate a special semiorthogonal component, which is a K3 or Enriques category according to whether the dimension of the variety is even or odd. We analyze the basic properties of this category using Hochschild homology, Hochschild cohomology, and the Grothendieck group.

We study the K3 category associated to a Gushel–Mukai fourfold in more detail. Namely, we show that this category is equivalent to the derived category of a K3 surface for a certain codimension 1 family of rational fourfolds, and to the K3 category of a birational cubic fourfold for a certain codimension 3 family. The first of these results verifies a special case of a duality conjecture which we formulate. We discuss our results in the context of the rationality problem for Gushel–Mukai varieties, which was one of the main motivations for this work.

Contents

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Chapter 1

Introduction

This thesis studies the derived categories of coherent sheaves on Gushel–Mukai varieties, with a special focus on the relation to birational geometry and the case of fourfolds.

1.1 Background

We will study the following class of varieties.

Definition 1.1. A *Gushel–Mukai (GM) variety* is a smooth n-dimensional intersection

$$X = \mathrm{Cone}(\mathrm{Gr}(2,5)) \cap \mathbf{P}^{n+4} \cap Q, \qquad 2 \leq n \leq 6,$$

where $\mathrm{Cone}(\mathrm{Gr}(2,5)) \subset \mathbf{P}^{10}$ is the cone over the Grassmannian $\mathrm{Gr}(2,5) \subset \mathbf{P}^9$ in its Plücker embedding, $\mathbf{P}^{n+4} \subset \mathbf{P}^{10}$ is a linear subspace, and $Q \subset \mathbf{P}^{n+4}$ is a quadric hypersurface.

We note that a more general definition of GM varieties, which includes singular varieties and curves, is given in [7, Definition 2.1]. However, the definition there agrees with ours after imposing the condition that a GM variety is smooth of dimension at least 2, see [7, Proposition 2.26].

The classification results of Gushel [14] and Mukai [38], generalized and simplified in [7], show that the class of GM varieties coincides with the class of all smooth Fano varieties of

Picard number 1, coindex 3, and degree 10, together with Brill–Noether general polarized K3 surfaces of degree 10.

The interest in GM varieties is based on several observations. In the Fano–Iskovskikh–Mori–Mukai classification of Fano threefolds, GM threefolds have an intermediate position between complete intersections in weighted projective spaces and linear sections of homogeneous varieties. They have many special features, and a particularly rich birational geometry.

The case of GM fourfolds is even more interesting, and was our original source of motivation. These fourfolds are similar to cubic fourfolds from several points of view — birational geometry, Hodge theory, and as we will see, derived categories.

In terms of birational geometry, both types of fourfolds are unirational and rational examples are known. On the other hand, a very general fourfold of either type is expected to be irrational, but to date irrationality has not been shown for a single example.

At the level of Hodge theory, a fourfold of either type has middle cohomology of K3 type. Moreover, there is a classification of Noether–Lefschetz loci where the middle cohomology contains (a Tate twist of) the primitive Hodge structure of a polarized K3 surface. This is due to Hassett for cubics [15], and Debarre–Iliev–Manivel [6] for GM fourfolds.

Finally, Kuznetsov studied the derived categories of cubic fourfolds in [25]. There, for any cubic fourfold X', a "K3 category" $\mathcal{A}_{X'}$ is constructed as a semiorthgonal component of the derived category $\mathrm{D^b}(X')$. Further, Kuznetsov conjectured that $\mathcal{A}_{X'}$ is equivalent to the derived category of an actual K3 surface whenever X' is rational, and proved that this holds for many rational X'. This conjecture has recently received a great deal of attention.

1.2 Results

We show in this thesis that the parallel between GM and cubic fourfolds persists at the level of derived categories. In fact, for any GM variety X — not necessarily of dimension 4 — we define a semiorthogonal component $\mathcal{A}_X$ of its derived category as the orthogonal to an exceptional sequence of vector bundles. Namely, projection from the vertex of $\mathrm{Cone}(\mathrm{Gr}(2,5))$

gives a morphism $f\colon X \to \mathrm{Gr}(2,5)$, which corresponds to a rank 2 bundle $\mathcal{U}_X$ on X. If $n = \dim X$, we show there is a semiorthogonal decomposition

$$\mathrm{D^b}(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \mathcal{U}_X^\vee, \mathcal{O}_X(H), \mathcal{U}_X^\vee(H), \ldots, \mathcal{O}_X((n-3)H), \mathcal{U}_X^\vee((n-3)H) \rangle.$$

The *GM category* $\mathcal{A}_X$ is the main object of study of this thesis. Its properties depend on the parity of the dimension n. For instance, we show that in terms of Serre functors, $\mathcal{A}_X$ is a "K3 category" or "Enriques category" according to whether n is even or odd. We also compute the Hochschild homology, Hochschild cohomology, and (in the very general case) the Grothendieck group of GM categories. Our computation of Hochschild cohomology relies on some new general results on equivariant Hochschild cohomology, which are proved in Appendix A. We deduce from our computations structural properties of GM categories. Notably, we show that for any GM variety of odd dimension or for a very general GM variety of even dimension greater than 2, the category $\mathcal{A}_X$ is not equivalent to the derived category of any variety (Proposition 5.17).

In the rest of the thesis, we investigate in detail the structure of $\mathcal{A}_X$ for two interesting families of GM fourfolds. Our first main result gives a family of GM fourfolds whose K3 category is equivalent to the derived category of a K3 surface.

Theorem 1.2. *Let X be an ordinary GM fourfold containing a quintic del Pezzo surface. Then there is a K3 surface Y such that $\mathcal{A}_X \simeq \mathrm{D^b}(Y)$.*

For a more precise statement, see Theorem 6.1. The definition of an "ordinary" GM variety is given in Section 3.1.2. Here we simply mention that a generic GM fourfold is ordinary, and that fourfolds as in the theorem form a 23-dimensional (codimension 1 in moduli) family. We also note that Y is constructed explicitly from the geometry of X, and is actually a GM surface. In fact, Theorem 1.2 is a special case of a duality conjecture (Conjecture 4.8) that we formulate, which relates the categories $\mathcal{A}_X$ for GM varieties of possibly different dimension (but of the same dimension parity). We will address the general case of this conjecture in forthcoming work.

GM fourfolds as in the theorem are rational (see Lemma 6.7), so our result is very much in the spirit of [25]. Indeed, Theorem 1.2 can be considered as one more piece of evidence in favor of the relation between the birational geometry of a variety and the structure of its derived category (see [28] for a review of this circle of ideas). This viewpoint predicts that the derived category of a rational fourfold admits a semiorthogonal decomposition whose components are admissible subcategories of derived categories of points, curves, or surfaces. In the case of a GM fourfold, we expect this to be the same as the condition that $\mathcal{A}_X$ is equivalent to the derived category of a K3 surface (see Conjecture 4.14). As mentioned above, $\mathcal{A}_X$ does not satisfy this condition for a very general GM fourfold, so the prediction is that such fourfolds are irrational.

Our second main result shows that the K3 categories attached to GM and cubic fourfolds are not only analogous, but in some cases even coincide.

Theorem 1.3. *Let X be a generic GM fourfold containing a plane of type* $\mathrm{Gr}(2, 3)$*. Then there is a cubic fourfold X' such that $\mathcal{A}_X \simeq \mathcal{A}_{X'}$.*

For a more precise statement, see Theorem 7.9. We note that GM fourfolds as in the theorem form a 21-dimensional (codimension 3 in moduli) family. The cubic fourfold X' associated to such an X is given explicitly by a construction of Debarre–Iliev–Manivel [6]. In fact, X' is birational to X and we use the structure of this birational isomorphism to establish the result. Theorem 1.3 can be considered as a step toward a 4-dimensional analogue of [23], which exhibits mysterious coincidences among the derived categories of Fano threefolds.

1.3 Further directions

The above results relate the K3 categories attached to three different types of varieties: GM fourfolds, cubic fourfolds, and K3 surfaces (in the last case the K3 category is the whole derived category). We call two such varieties X_1 and X_2 *derived partners* if their K3 categories are equivalent. There is also a notion of X_1 and X_2 being *Hodge-theoretic partners.*

Roughly, this means that there is an "extra" integral middle-degree Hodge class α_i on X_i, such that if $K_i \subset \mathrm{H}^{\dim(X_i)}(X_i, \mathbf{Z})$ denotes the lattice generated by α_i and certain tautological algebraic cycles on X_i, then the orthogonals $K_1^{\perp}$ and $K_2^{\perp}$ are isomorphic as polarized Hodge structures (up to a Tate twist). This notion was studied in [15], [6], under the terminology that "X_2 is associated to X_1". While somewhat less natural than its derived counterpart, the Hodge-theoretic notion of partners has the advantage of being amenable to lattice theoretic techniques. Using this, countably many families of GM fourfolds with Hodge-theoretic K3 and cubic fourfold partners are produced in [6].

We expect that a GM fourfold has a derived partner of a given type if and only if it has a Hodge-theoretic partner of the same type. Theorems 1.2 and 1.3 can be thought of as evidence for this expectation, since by [6, Sections 7.5 and 7.2] a GM fourfold as in Theorem 1.2 or Theorem 1.3 has a Hodge-theoretic K3 or cubic fourfold partner, respectively. Addington and Thomas [1] proved (generically) the analogous expectation for K3 partners of cubic fourfolds. Their method is deformation theoretic, and requires as a starting point an analogue of Theorem 1.2 for cubic fourfolds. In a follow-up work, we plan to extend their method to GM fourfolds.

Finally, let us note that there are other Fano fourfolds which fit into the above story, i.e. whose derived category contains a K3 category. For instance, in [19] Küchle classified Fano fourfolds of index 1 which can be represented as zero loci of equivariant vector bundles on Grassmannians. Among these, three types — labeled (c5), (c7), and (d3) in [19] — have middle Hodge structure of K3 type. In [29] it was shown that fourfolds of type (d3) are isomorphic to the blowup of $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ in a K3 surface, and those of type (c7) are isomorphic to the blowup of a cubic fourfold in a Veronese surface. In particular, these fourfolds do indeed have a K3 category in their derived category, but they reduce to known examples. Fourfolds of type (c5), however, conjecturally give rise to genuinely new K3 categories (see [30] for a discussion of the geometry of these fourfolds).

1.4 Organization of the thesis

In Chapter 2, we review the theory of semiorthogonal decompositions. In Chapter 3, we review some aspects of the geometry of GM varieties. In Chapter 4, we introduce GM categories, formulate the duality conjecture mentioned above, and discuss relations to the rationality problem. In Chapter 5, we compute the Hochschild homology, Hochschild cohomology, and Grothendieck group of GM categories, and use our computations to deduce some structural properties of these categories. In Chapter 6, we prove Theorem 1.2. In Chapter 7, we prove Theorem 1.3. Finally, in Appendix A, we prove some results on Hochschild cohomology that are used in Chapter 5.

1.5 Notation and conventions

We work over an algebraically closed field $\mathbf{k}$ of characteristic 0. A variety is an integral, separated scheme of finite type over $\mathbf{k}$. A vector bundle on a variety X is a finite locally free $\mathcal{O}_X$-module. The projective bundle of a vector bundle $\mathcal{E}$ on a variety X is

$$\mathbf{P}(\mathcal{E}) = \mathrm{Proj}(\mathrm{Sym}^\bullet(\mathcal{E}^\vee)) \xrightarrow{\pi} X,$$

with $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ normalized so that $\pi_*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1) = \mathcal{E}^\vee$. We often commit the following convenient abuse of notation: given a divisor D on a variety X, we denote still by D its pullback to any variety mapping to X. Throughout, we use V_n to denote an n-dimensional vector space. Further, we denote by $\mathbf{G} = \mathrm{Gr}(2, V_5)$ the Grassmannian of 2-dimensional subspaces of V_5. For our conventions on derived categories, see Chapter 2.

Chapter 2

Preliminaries on semiorthogonal decompositions

In this thesis, triangulated categories are **k**-linear and functors between them are **k**-linear and exact. For a variety X, by the *derived category* $\mathrm{D}^{\mathrm{b}}(X)$ we mean the bounded derived category of coherent sheaves on X, regarded as a triangulated category. For a morphism of varieties $f\colon X \to Y$, we write

$$f_*\colon \mathrm{D}^{\mathrm{b}}(X) \to \mathrm{D}^{\mathrm{b}}(Y)$$

for the derived pushforward (provided f is proper), and

$$f^*\colon \mathrm{D}^{\mathrm{b}}(Y) \to \mathrm{D}^{\mathrm{b}}(X)$$

for the derived pullback (provided f has finite Tor-dimension). Similarly, for $\mathcal{F}, \mathcal{G} \in \mathrm{D}^{\mathrm{b}}(X)$, we write $\mathcal{F} \otimes \mathcal{G}$ for the derived tensor product.

2.1 Semiorthogonal decompositions

We recall some basic definitions and results about semiorthogonal decompositions, which will be used freely throughout this work. For more details, see for instance [3] and [4].

Definition 2.1. Let $\mathcal{T}$ be a triangulated category. A *semiorthogonal decomposition*

$$\mathcal{T} = \langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle$$

is a sequence of full triangulated subcategories $\mathcal{A}_1, \dots, \mathcal{A}_n$ of $\mathcal{T}$ — called the *components* of the decomposition — such that:

(1) $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) = 0$ for all $\mathcal{F} \in \mathcal{A}_i, \mathcal{G} \in \mathcal{A}_j$ and $i > j$.

(2) For any $\mathcal{F} \in \mathcal{T}$, there is a sequence of morphisms

$$0 = \mathcal{F}_n \to \mathcal{F}_{n-1} \to \cdots \to \mathcal{F}_1 \to \mathcal{F}_0 = \mathcal{F},$$

such that $\mathrm{Cone}(\mathcal{F}_i \to \mathcal{F}_{i-1}) \in \mathcal{A}_i$ for $1 \leq i \leq n$.

If only condition (1) is satisfied, we say the sequence $\mathcal{A}_1, \dots, \mathcal{A}_n$ is *semiorthogonal*.

Remark 2.2. Condition (1) of the definition implies the "filtration" in (2) and its "factors" are unique and functorial.

A full triangulated subcategory $\mathcal{A} \subset \mathcal{T}$ is called *admissible* if the embedding functor $\alpha \colon \mathcal{A} \to \mathcal{T}$ admits right and left adjoints $\alpha^! \colon \mathcal{T} \to \mathcal{A}$ and $\alpha^* \colon \mathcal{T} \to \mathcal{A}$. This condition guarantees $\mathcal{A}$ is a component of a semiorthogonal decomposition of $\mathcal{T}$. Indeed, let $\mathcal{A}^\perp$ and ${}^\perp\mathcal{A}$ be the *right* and *left orthogonal* categories to $\mathcal{A}$, i.e. the full subcategories of $\mathcal{T}$ defined by

$$\mathcal{A}^\perp = \{\mathcal{F} \in \mathcal{T} \mid \mathrm{Hom}(\mathcal{G}, \mathcal{F}) = 0 \text{ for all } \mathcal{G} \in \mathcal{A}\},$$
$${}^\perp\mathcal{A} = \{\mathcal{F} \in \mathcal{T} \mid \mathrm{Hom}(\mathcal{F}, \mathcal{G}) = 0 \text{ for all } \mathcal{G} \in \mathcal{A}\}.$$

Then if $\mathcal{A}$ is admissible, we have $\mathcal{T} = \langle \mathcal{A}^\perp, \mathcal{A} \rangle$ and $\mathcal{T} = \langle \mathcal{A}, {}^\perp\mathcal{A} \rangle$. More generally, we have:

Lemma 2.3. *Let $\mathcal{A}_1, \dots, \mathcal{A}_n$ be a semiorthogonal sequence of admissible subcategories of $\mathcal{T}$. Then for each $0 \leq k \leq n$, there is a semiorthogonal decomposition*

$$\mathcal{T} = \langle \mathcal{A}_1, \dots, \mathcal{A}_k, {}^\perp\langle \mathcal{A}_1, \dots, \mathcal{A}_k \rangle \cap \langle \mathcal{A}_{k+1}, \dots, \mathcal{A}_n \rangle^\perp, \mathcal{A}_{k+1}, \dots, \mathcal{A}_n \rangle.$$

We note that for a smooth proper variety X, the components of any semiorthogonal decomposition of $\mathrm{D^b}(X)$ are admissible. This follows from the results of [4].

The simplest admissible subcategories are those generated by an exceptional object. Recall that an object $\mathcal{E} \in \mathcal{T}$ of a triangulated category is *exceptional* if

$$\mathrm{Hom}(\mathcal{E}, \mathcal{E}[p]) = \begin{cases} \mathbf{k} & \text{if } p = 0, \\ 0 & \text{if } p \neq 0. \end{cases}$$

If X is a smooth proper variety and $\mathcal{E} \in \mathrm{D^b}(X)$ is an exceptional object, then the triangulated subcategory $\langle \mathcal{E} \rangle \subset \mathrm{D^b}(X)$ generated by $\mathcal{E}$ is an admissible subcategory, which is equivalent to the derived category $\mathrm{D^b}(\mathbf{k})$ of vector spaces. Indeed, the functor $\mathrm{D^b}(\mathbf{k}) \to \mathrm{D^b}(X)$ given by $V \mapsto V \otimes \mathcal{E}$ is an equivalence onto $\langle \mathcal{E} \rangle$, with right and left adjoints given by $\mathcal{F} \mapsto \mathrm{RHom}(\mathcal{E}, \mathcal{F})$ and $\mathcal{F} \mapsto \mathrm{RHom}(\mathcal{F}, \mathcal{E})^\vee$. To simplify notation, we write $\mathcal{E}$ in place of $\langle \mathcal{E} \rangle$ when $\langle \mathcal{E} \rangle$ appears as a component in a semiorthognal decomposition, i.e. instead of $\mathcal{T} = \langle \dots, \langle \mathcal{E} \rangle, \dots \rangle$ we write $\mathcal{T} = \langle \dots, \mathcal{E}, \dots \rangle$. A sequence $\mathcal{E}_1, \dots, \mathcal{E}_n$ of exceptional objects in a triangulated category $\mathcal{T}$ is called an *exceptional sequence* if $\mathrm{Hom}(\mathcal{E}_i, \mathcal{E}_j[p]) = 0$ for $i > j$ and p arbitrary. Note that if $\mathcal{T} = \mathrm{D^b}(X)$ for a smooth proper variety X, then an exceptional sequence gives rise to a semiorthogonal sequence $\langle \mathcal{E}_1 \rangle, \dots, \langle \mathcal{E}_n \rangle$ of admissible subcategories of $\mathrm{D^b}(X)$.

2.2 Lefschetz decompositions

Lefschetz decompositions are semiorthogonal decompositions of a special form, which were introduced in [22]. We will only need so-called rectangular Lefschetz decompositions, so we restrict our discussion to this case.

Given a line bundle $\mathcal{O}_X(1)$ on a variety X and a subcategory $\mathcal{B} \subset \mathrm{D^b}(X)$, for any integer t we denote by $\mathcal{B}(t)$ the image of $\mathcal{B}$ under the autoequivalence $\mathcal{F} \mapsto \mathcal{F}(t) \coloneqq \mathcal{F} \otimes \mathcal{O}_X(t)$ of $\mathrm{D^b}(X)$.

Definition 2.4. A *rectangular Lefschetz decomposition* of $\mathrm{D^b}(X)$ with respect to a line bundle $\mathcal{O}_X(1)$ is a semiorthogonal decomposition of the form $\mathrm{D^b}(X) = \langle \mathcal{B}, \mathcal{B}(1), \dots, \mathcal{B}(m-1) \rangle$.

A key feature of such a decomposition is that it induces semiorthogonal decompositions of varieties constructed from X. For instance, there are induced semiorthogonal decompositions of divisors in linear systems given by a multiple of $\mathcal{O}_X(1)$, and of cyclic covers of X ramified over such divisors. In the induced decompositions, there are a number of components equivalent to $\mathcal{B}$, and one "new" component, see [31, Lemmas 5.1 and 5.5]. In Section 4.1, we will use this to study the derived categories of GM varieties.

It should also be mentioned that Lefschetz decompositions play an important role in the theory of homological projective duality, see [22], [26]. This theory will be a key ingredient in our proof of Theorem 1.2.

2.3 Mutations

Let $\alpha\colon \mathcal{A} \hookrightarrow \mathcal{T}$ be the embedding functor of an admissible subcategory. Then for any $\mathcal{F} \in \mathcal{T}$, the counit $\alpha\alpha^!(\mathcal{F}) \to \mathcal{F}$ and unit $\mathcal{F} \to \alpha\alpha^*(\mathcal{F})$ morphisms can be completed to distinguished triangles

$$\alpha\alpha^!(\mathcal{F}) \to \mathcal{F} \to \mathrm{L}_{\mathcal{A}}(\mathcal{F}) \qquad \text{and} \qquad \mathrm{R}_{\mathcal{A}}(\mathcal{F}) \to \mathcal{F} \to \alpha\alpha^*(\mathcal{F}).$$

These triangles are functorial, and define the *left* and *right mutation functors* $\mathrm{L}_{\mathcal{A}}\colon \mathcal{T} \to \mathcal{T}$ and $\mathrm{R}_{\mathcal{A}}\colon \mathcal{T} \to \mathcal{T}$ of $\mathcal{A} \subset \mathcal{T}$.

Remark 2.5. Let $\mathcal{E} \in \mathrm{D^b}(X)$ be an exceptional object in the derived category of a smooth proper variety X. Then the mutation functors of $\langle \mathcal{E} \rangle$ are given by the distinguished triangles

$$\mathrm{RHom}(\mathcal{E},\mathcal{F}) \otimes \mathcal{E} \to \mathcal{F} \to \mathrm{L}_{\mathcal{E}}(\mathcal{F}) \qquad \text{and} \qquad \mathrm{R}_{\mathcal{E}}(\mathcal{F}) \to \mathcal{F} \to \mathrm{RHom}(\mathcal{F},\mathcal{E})^{\vee} \otimes \mathcal{E}.$$

Both mutation functors $\mathrm{L}_{\mathcal{A}}$ and $\mathrm{R}_{\mathcal{A}}$ annihilate $\mathcal{A}$, and their restrictions $\mathrm{L}_{\mathcal{A}}|_{{}^{\perp}\mathcal{A}}\colon {}^{\perp}\mathcal{A} \to \mathcal{A}^{\perp}$ and $\mathrm{R}_{\mathcal{A}}|_{\mathcal{A}^{\perp}}\colon \mathcal{A}^{\perp} \to {}^{\perp}\mathcal{A}$ are mutually inverse equivalences ([4, Lemma 1.9]). This gives the $n = 2$ case of the next result (which in fact easily reduces to this case) describing the action of mutation functors on semiorthogonal decompositions.

Lemma 2.6. *Let $\mathcal{T} = \langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle$ be a semiorthogonal decomposition with admissible com-*

ponents. Then for $1 \leq i \leq n-1$ there is a semiorthogonal decomposition

$$\mathcal{T} = \langle \mathcal{A}_1, \dots, \mathcal{A}_{i-1}, \mathrm{L}_{\mathcal{A}_i}(\mathcal{A}_{i+1}), \mathcal{A}_i, \mathcal{A}_{i+2}, \dots, \mathcal{A}_n \rangle,$$

and for $2 \leq i \leq n$ there is a semiorthogonal decomposition

$$\mathcal{T} = \langle \mathcal{A}_1, \dots, \mathcal{A}_{i-2}, \mathcal{A}_i, \mathrm{R}_{\mathcal{A}_i}(\mathcal{A}_{i-1}), \mathcal{A}_{i+1}, \dots, \mathcal{A}_n \rangle.$$

The following observation is useful for computing mutations. Let $\mathcal{A} \subset \mathcal{T}$ be a full triangulated subcategory. If for every $\mathcal{F} \in \mathcal{T}$ there exists a distinguished triangle

$$\mathcal{G} \to \mathcal{F} \to \mathcal{G}' \tag{2.1}$$

with $\mathcal{G} \in \mathcal{A}$ and $\mathcal{G}' \in \mathcal{A}^{\perp}$, then the embedding functor $\alpha \colon \mathcal{A} \to \mathcal{T}$ admits a right adjoint $\alpha^!$ given on objects by $\alpha^!(\mathcal{F}) = \mathcal{G}$. Similarly, if for every $\mathcal{F} \in \mathcal{T}$ there exists a distinguished triangle

$$\mathcal{H}' \to \mathcal{F} \to \mathcal{H} \tag{2.2}$$

with $\mathcal{H} \in \mathcal{A}$ and $\mathcal{H}' \in {}^{\perp}\mathcal{A}$, then α admits a left adjoint α^* given on objects by $\alpha^*(\mathcal{F}) = \mathcal{H}$. In particular, $\mathcal{A} \subset \mathcal{T}$ is admissible if and only if for all $\mathcal{F} \in \mathcal{T}$ distinguished triangles as in (2.1) and (2.2) exist, and in this case the mutation functors are given by $\mathrm{L}_{\mathcal{A}}(\mathcal{F}) \cong \mathcal{G}'$ and $\mathrm{R}_{\mathcal{A}}(\mathcal{F}) \cong \mathcal{H}'$. The following three lemmas are easy consequences of these remarks.

Lemma 2.7. *Let $\mathcal{A}_1, \dots, \mathcal{A}_n$ be a semiorthogonal sequence of admissible subcategories of $\mathcal{T}$. Then $\langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle \subset \mathcal{T}$ is admissible, and there are isomorphisms of functors*

$$\mathrm{L}_{\langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle} \cong \mathrm{L}_{\mathcal{A}_1} \circ \mathrm{L}_{\mathcal{A}_2} \circ \cdots \circ \mathrm{L}_{\mathcal{A}_n},$$
$$\mathrm{R}_{\langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle} \cong \mathrm{R}_{\mathcal{A}_n} \circ \mathrm{R}_{\mathcal{A}_{n-1}} \circ \cdots \circ \mathrm{R}_{\mathcal{A}_1}.$$

Proof. Consider the chain of morphisms $\mathcal{F} \to \mathrm{L}_{\mathcal{A}_n}(\mathcal{F}) \to \cdots \to (\mathrm{L}_{\mathcal{A}_1} \circ \cdots \circ \mathrm{L}_{\mathcal{A}_n})(\mathcal{F})$. By definition, their cones are contained in the subcategories $\mathcal{A}_n, \dots, \mathcal{A}_1$, hence by the octahedral axiom the cone of their composition is contained in $\langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle$. Thus there is a distinguished triangle

$$\mathcal{G} \to \mathcal{F} \to (\mathrm{L}_{\mathcal{A}_1} \circ \dots \circ \mathrm{L}_{\mathcal{A}_n})(\mathcal{F})$$

with $\mathcal{G} \in \langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle$. Similarly, there is a distinguished triangle

$$(\mathrm{R}_{\mathcal{A}_n} \circ \dots \circ \mathrm{R}_{\mathcal{A}_1})(\mathcal{F}) \to \mathcal{F} \to \mathcal{H}$$

with $\mathcal{H} \in \langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle$. So the above discussion applies. □

Lemma 2.8. *Let $\mathcal{T} = \langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle$ be a semiorthogonal decomposition with admissible components. Assume for some i the components $\mathcal{A}_i$ and $\mathcal{A}_{i+1}$ are completely orthogonal, i.e. $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) = \mathrm{Hom}(\mathcal{G}, \mathcal{F}) = 0$ for all $\mathcal{F} \in \mathcal{A}_i$, $\mathcal{G} \in \mathcal{A}_{i+1}$. Then $\mathrm{L}_{\mathcal{A}_i}(\mathcal{G}) = \mathcal{G}$ for any $\mathcal{G} \in \mathcal{A}_{i+1}$, and $\mathrm{R}_{\mathcal{A}_{i+1}}(\mathcal{F}) = \mathcal{F}$ for any $\mathcal{F} \in \mathcal{A}_i$. In particular, there is a semiorthogonal decomposition*

$$\mathcal{T} = \langle \mathcal{A}_1, \dots, \mathcal{A}_{i-1}, \mathcal{A}_{i+1}, \mathcal{A}_i, \mathcal{A}_{i+2}, \dots, \mathcal{A}_n \rangle.$$

Lemma 2.9. *Let $F \colon \mathcal{T}_1 \to \mathcal{T}_2$ be an equivalence of triangulated categories. Let $\mathcal{A} \subset \mathcal{T}_1$ be an admissible subcategory. Then $F \circ \mathrm{L}_{\mathcal{A}} \cong \mathrm{L}_{F(\mathcal{A})} \circ F$ and $F \circ \mathrm{R}_{\mathcal{A}} \cong \mathrm{R}_{F(\mathcal{A})} \circ F$.*

In the presence of a Serre functor, there is another mutation functor whose action is easy to describe. Recall from [4] that a *Serre functor* for $\mathcal{T}$ is an autoequivalence $\mathrm{S}_{\mathcal{T}} \colon \mathcal{T} \to \mathcal{T}$ with a bifunctorial isomorphism

$$\mathrm{Hom}(\mathcal{F}, \mathrm{S}_{\mathcal{T}}(\mathcal{G})) \cong \mathrm{Hom}(\mathcal{G}, \mathcal{F})^{\vee}$$

for all $\mathcal{F}, \mathcal{G} \in \mathcal{T}$. If a Serre functor exists, it is unique. If X is a smooth proper variety, $\mathrm{D}^{\mathrm{b}}(X)$ has a Serre functor given by the formula

$$\mathrm{S}_{\mathrm{D}^{\mathrm{b}}(X)}(\mathcal{F}) = \mathcal{F} \otimes \omega_X[\dim X].$$

Lemma 2.10 ([4, Proposition 3.6]). *Let $\mathcal{T} = \langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle$ be a semiorthogonal decomposition with admissible components. If the category $\mathcal{T}$ admits a Serre functor $\mathrm{S}_{\mathcal{T}}$, then we have $\mathrm{L}_{\langle \mathcal{A}_1, \dots, \mathcal{A}_{n-1} \rangle}(\mathcal{A}_n) = \mathrm{S}_{\mathcal{T}}(\mathcal{A}_n)$ and $\mathrm{R}_{\langle \mathcal{A}_2, \dots, \mathcal{A}_n \rangle}(\mathcal{A}_1) = \mathrm{S}_{\mathcal{T}}^{-1}(\mathcal{A}_1)$.*

Remark 2.11. The equalities of Lemma 2.10 do not hold objectwise. In other words, they describe the action of mutation functors on subcategories, not on objects. For a formula that holds objectwise, the Serre functors of $\mathcal{A}_n$ and $\mathcal{A}_1$ need to be taken into account. Namely, there are isomorphisms of functors

$$\mathrm{L}_{\langle \mathcal{A}_1, \dots, \mathcal{A}_{n-1} \rangle} \cong \mathrm{S}_{\mathcal{T}} \circ \mathrm{S}_{\mathcal{A}_n}^{-1} \colon \mathcal{A}_n \to \mathcal{T} \qquad \text{and} \qquad \mathrm{R}_{\langle \mathcal{A}_2, \dots, \mathcal{A}_n \rangle} \cong \mathrm{S}_{\mathcal{T}}^{-1} \circ \mathrm{S}_{\mathcal{A}_1} \colon \mathcal{A}_1 \to \mathcal{T}.$$

Chapter 3

Geometry of GM varieties

In this chapter, we discuss several aspects of the geometry of GM varieties: their classification, moduli, and associated EPW varieties.

3.1 GM varieties

Let V_5 be a 5-dimensional vector space and $\mathbf{G} = \mathrm{Gr}(2, V_5)$ the Grassmannian of 2-dimensional subspaces. Consider the Plücker embedding $\mathbf{G} \hookrightarrow \mathbf{P}(\wedge^2 V_5)$ and let $\mathrm{Cone}(\mathbf{G}) \subset \mathbf{P}(\mathbf{k} \oplus \wedge^2 V_5)$ be the cone over $\mathbf{G}$. Further, let

$$W \subset \mathbf{k} \oplus \wedge^2 V_5$$

be a linear subspace of dimension $n+5$ with $2 \leq n \leq 6$, and $Q \subset \mathbf{P}(W)$ a quadric hypersurface. By Definition 1.1, if the intersection

$$X = \mathrm{Cone}(\mathbf{G}) \cap Q \tag{3.1}$$

is smooth and transverse, then X is a GM variety of dimension n, and every GM variety can be written in this form.

There is a natural polarization H on a GM variety X, given by the restriction of the hyperplane class on $\mathbf{P}(\mathbf{k} \oplus \wedge^2 V_5)$. It is straightforward to check that

$$H^n = 10 \quad \text{and} \quad -K_X = (n-2)H. \tag{3.2}$$

Together with a restriction on the Picard group, these numerical conditions give an intrinsic characterization of GM varieties:

Theorem 3.1 ([14, 38, 7])**.** *Let X be a smooth n-dimensional projective variety with an ample divisor H, such that one of the following holds:*

(1) X is Fano with $\mathrm{Pic}(X) = \mathbf{Z}H$, $-K_X = (n-2)H$, *and* $H^n = 10$.

(2) (X, H) is a Brill–Noether general polarized K3 surface of degree 10.

Then X is canonically isomorphic to a GM variety, and H is given by the restriction of the hyperplane class on $\mathbf{P}(\mathbf{k} \oplus \wedge^2 V_5)$*. Conversely, every GM variety with its natural polarization satisfies either (1) or (2).*

3.1.1 Canonical data

We highlight some important data associated to a GM variety X. The intersection $\mathrm{Cone}(\mathbf{G}) \cap Q$ does not contain the vertex of the cone, since X is smooth. Hence projection from the vertex defines a regular map

$$f \colon X \to \mathbf{G},$$

called the *Gushel map*. Let $\mathcal{U}$ be the rank 2 tautological subbundle on $\mathbf{G}$. Then $\mathcal{U}_X = f^*\mathcal{U}$ is a rank 2 vector bundle on X, called the *Gushel bundle*. By [7], the Gushel map and the Gushel bundle are canonically associated to X, i.e. only depend on the abstract polarized variety (X, H) and not on the particular realization (3.1). In particular, so is the space V_5 (being the dual of the space of sections of $\mathcal{U}_X^\vee$), and we will sometimes write it as $V_5(X)$ to emphasize this.

The line bundle $\mathcal{O}_X(1) \coloneqq \mathcal{O}_X(H)$ is the ample generator of $\mathrm{Pic}(X)$ if $\dim(X) \geq 3$, or the

Brill–Noether general polarization of X if $\dim(X) = 2$. It is in fact very ample as it gives the embedding $X \hookrightarrow \mathbf{P}(W)$.

The intersection

$$M_X = \mathrm{Cone}(\mathbf{G}) \cap \mathbf{P}(W)$$

is called the *Grassmannian hull* of X. Note that $X = M_X \cap Q$ is a quadric section of M_X. Let W' be the projection of W to $\wedge^2 V_5$. Below it will be convenient to also consider the intersection

$$M'_X = \mathbf{G} \cap \mathbf{P}(W'), \tag{3.3}$$

called the *projected Grassmannian hull* of X. Again by [7], both M_X and M'_X are canonically associated to X.

On the contrary, the quadric hypersurface $Q \subset \mathbf{P}(W)$ is not canonically associated to X (see the discussion in Section 3.1.3), nor is the embedding $W \hookrightarrow \mathbf{k} \oplus \wedge^2 V_5$ (although the composition $W \hookrightarrow \mathbf{k} \oplus \wedge^2 V_5 \twoheadrightarrow \wedge^2 V_5$ is canonical).

3.1.2 The two types of GM varieties

The Gushel map is either an embedding or two-to-one onto its image. These two different possibilities correspond to the position of the linear subspace $W \subset \mathbf{k} \oplus \wedge^2 V_5$: either W does not meet the vertex $\mathbf{k}$, or contains it. In the first case, $W \cong W'$ and $M_X \cong M'_X$. Then considering Q as a subvariety of $\mathbf{P}(W')$, we have

$$X \cong M'_X \cap Q. \tag{3.4}$$

That is, X is a quadric section of a linear section of the Grassmannian $\mathbf{G}$. A GM variety of this type is called *ordinary*.

If W contains $\mathbf{k}$, then $\mathbf{P}(W) = \mathrm{Cone}(\mathbf{P}(W'))$ and $M_X = \mathrm{Cone}(M'_X)$. As Q does not contain the vertex of the cone (by smoothness of X), projection from the vertex gives a double cover

$$X \xrightarrow{2:1} M'_X. \tag{3.5}$$

That is, X is a double cover of a linear section of the Grassmannian $\mathbf{G}$. A GM variety of this type is called *special*.

By Lemma 3.2 below, if X is special then M'_X is smooth. Further, the branch divisor of the double cover (3.5) is the smooth intersection $X' = \mathbf{G} \cap Q'$, where $Q' = Q \cap \mathbf{P}(W')$ is a quadric hypersurface in $\mathbf{P}(W')$. Hence, as long as $n \geq 3$, the branch divisor X' of (3.5) is an ordinary GM variety of dimension $n-1$. This gives rise to an operation taking a GM variety of one type to the opposite type, by defining in this situation

$$X^{\mathrm{op}} = X' \qquad \text{and} \qquad (X')^{\mathrm{op}} = X. \tag{3.6}$$

Lemma 3.2 ([7, Proposition 2.20])**.** *Let X be a GM variety of dimension n. Then the intersection* (3.3) *defining M'_X is dimensionally transverse. Moreover:*

(1) If $n \geq 3$, or if $n = 2$ and X is special, then M'_X is smooth.

(2) If $n = 2$ and X is ordinary, then M'_X has at worst rational double point singularities.

3.1.3 The choice of quadric

As opposed to the other data discussed above, the quadric Q defining X in (3.1) is not unique. The space of quadrics in $\mathbf{P}(W)$ containing X is a 6-dimensional vector space, denoted $V_6(X)$. Indeed, the Grassmannian hull M_X is the intersection of (the restrictions to W of) the Plücker quadrics, which span a 5-dimensional space naturally identified with $V_5(X)$. As X is a quadric section of M_X, the space $V_6(X)$ is thus 6-dimensional, and contains $V_5(X)$. We call the subspace $V_5(X) \subset V_6(X)$ the *Plücker hyperplane* of X, and the corresponding point

$$\mathbf{p}(X) \in \mathbf{P}(V_6(X)^\vee)$$

the *Plücker point* of X.

The quadrics Q cutting out X as in (3.1) are thus parameterized by the affine space $\mathbf{P}(V_6(X)) \setminus \mathbf{P}(V_5(X))$. That is, such a Q corresponds to the choice of a *quadric point*

$$\mathbf{q} \in \mathbf{P}(V_6(X))$$

such that $(\mathbf{q}, \mathbf{p}(X))$ does not lie on the incidence divisor in $\mathbf{P}(V_6(X)) \times \mathbf{P}(V_6(X)^\vee)$. The symmetry between the Plücker point $\mathbf{p}(X)$ and the quadric point $\mathbf{q}$ is the basis for the duality of GM varieties, which we discuss in Section 4.2.

3.2 Moduli of GM varieties

Let $(\mathrm{Sch}/\mathbf{k})$ denote the category of $\mathbf{k}$-schemes.

Definition 3.3. For $2 \leq n \leq 6$, the moduli stack $\mathcal{M}_n$ of n-dimensional GM varieties is the fibered category over $(\mathrm{Sch}/\mathbf{k})$ whose fiber over $S \in (\mathrm{Sch}/\mathbf{k})$ is the groupoid of pairs $(\pi\colon X \to S, \mathcal{L})$, where $\pi\colon X \to S$ is a smooth proper morphism of schemes and $\mathcal{L} \in \mathrm{Pic}_{X/S}(S)$, such that for every geometric point $\bar{s} \in S$ the pair $(X_{\bar{s}}, \mathcal{L}_{\bar{s}})$ is isomorphic to an n-dimensional GM variety with its natural polarization (equivalently, $(X_{\bar{s}}, \mathcal{L}_{\bar{s}})$ satisfies condition (1) or (2) of Theorem 3.1 with H the divisor corresponding to $\mathcal{L}_{\bar{s}}$). A morphism from $(\pi'\colon X' \to S', \mathcal{L}')$ to $(\pi\colon X \to S, \mathcal{L})$ is a fiber product diagram

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "\pi'"'] & X \arrow[d, "\pi"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

such that $(g')^*(\mathcal{L}) = \mathcal{L}' \in \mathrm{Pic}_{X'/S'}(S')$.

The following result gives the basic properties of the moduli stack $\mathcal{M}_n$. An explicit description of $\mathcal{M}_n$ will be given in [9]. We follow [52] for our conventions on algebraic stacks.

Proposition 3.4. *The moduli stack* $\mathcal{M}_n$ *is a smooth and irreducible Deligne–Mumford stack of finite type over* $\mathbf{k}$*. Its dimension is given by* $\dim \mathcal{M}_n = 25 - (6-n)(5-n)/2$*, i.e.*

$$\dim \mathcal{M}_2 = 19, \ \ \dim \mathcal{M}_3 = 22, \ \ \dim \mathcal{M}_4 = 24, \ \ \dim \mathcal{M}_5 = 25, \ \ \dim \mathcal{M}_6 = 25.$$

We will use the following lemma.

Lemma 3.5. *Let* X *be a GM variety of dimension* $n \geq 3$*. Then:*

(1) The automorphism group scheme $\mathrm{Aut}_{\mathbf{k}}(X)$ *is finite and reduced.*

(2) $\mathrm{H}^i(X, \mathrm{T}_X) = 0$ *for* $i \neq 1$*.*

(3) $\dim \mathrm{H}^1(X, \mathrm{T}_X) = 25 - (6-n)(5-n)/2$*.*

Proof. As our base field $\mathbf{k}$ has characteristic 0, $\mathrm{Aut}_{\mathbf{k}}(X)$ is automatically reduced by a theorem of Cartier [39, Lecture 25], and it is finite by [7, Proposition 3.19(c)]. Hence $\mathrm{H}^0(X, \mathrm{T}_X)$, being the tangent space to $\mathrm{Aut}_{\mathbf{k}}(X)$ at the identity, vanishes. Further, $\mathrm{T}_X \cong \Omega_X^{n-1}(n-2)$ by (3.2) and hence $\mathrm{H}^i(X, \mathrm{T}_X) = 0$ for $i \geq 2$ by Kodaira–Akizuki–Nakano vanishing. Finally, the dimension of $\mathrm{H}^1(X, \mathrm{T}_X)$ is straightforward to compute using Riemann–Roch. $\square$

Proof of Proposition 3.4. First consider the case $n = 2$. Then by Theorem 3.1, $\mathcal{M}_2$ is the Brill–Noether general locus (and hence Zariski open) in the moduli stack of polarized K3 surfaces of degree 10. It is well-known that all the properties in the proposition hold for the moduli stack of primitively polarized K3 surfaces of a fixed degree (see [16, Chapter 5]), so they also hold for $\mathcal{M}_2$.

From now on assume $n \geq 3$. A standard Hilbert scheme argument shows that $\mathcal{M}_n$ is an algebraic stack of finite type over $\mathbf{k}$, whose diagonal is affine and of finite type. To prove $\mathcal{M}_n$ is Deligne–Mumford, by [52, Tag 06N3] it suffices to show its diagonal is unramified. As a finite type morphism is unramified if and only if all of its geometric fibers are finite and reduced, we are done by Lemma 3.5(1) (note that for a GM variety of dimension $n \geq 3$, all automorphisms preserve the natural polarization).

Next we check smoothness of $\mathcal{M}_n$. Let $(X, \mathcal{L})$ be a point of $\mathcal{M}_n$, i.e. X is a GM n-fold and $\mathcal{L} \in \mathrm{Pic}(X)$ is the ample generator. Let $\mathfrak{A}_{\mathcal{L}}$ be the Atiyah extension of $\mathcal{L}$, i.e. the extension

$$0 \to \mathcal{O}_X \to \mathfrak{A}_{\mathcal{L}} \to \mathrm{T}_X \to 0$$

given by the Atiyah class of $\mathcal{L}$. Further, recall that $\mathrm{H}^1(X, \mathfrak{A}_{\mathcal{L}})$ classifies first order deformations of the pair $(X, \mathcal{L})$, and $\mathrm{H}^2(X, \mathfrak{A}_{\mathcal{L}})$ is the obstruction space for such deformations (see [50, Section 3.3.3]). Taking cohomology in the above sequence shows that $\mathrm{H}^i(X, \mathfrak{A}_{\mathcal{L}}) = \mathrm{H}^i(X, \mathrm{T}_X)$ for $i \geq 1$. In particular, $\mathrm{H}^2(X, \mathfrak{A}_{\mathcal{L}}) = 0$ by Lemma 3.5(2), so the formal deformation space of $\mathcal{M}_n$ at $(X, \mathcal{L})$ is smooth of dimension $\dim \mathrm{H}^1(X, \mathfrak{A}_{\mathcal{L}}) = \dim \mathrm{H}^1(X, \mathrm{T}_X)$. This implies the smoothness of $\mathcal{M}_n$ and, using Lemma 3.5(3), the formula for its dimension.

It remains to show that $\mathcal{M}_n$ is irreducible. This follows from the defining expression (3.1) of any GM variety. Indeed, let P_n be the space of pairs (W, Q) where $W \subset \mathbf{k} \oplus \wedge^2 V_5$ is an $(n+5)$-dimensional linear subspace and $Q \subset \mathbf{P}(W)$ is a quadric hypersurface, and let $U_n \subset P_n$ be the open subset where $\mathrm{Cone}(\mathbf{G}) \cap Q$ is smooth of dimension n. The projection $P_n \to \mathrm{Gr}(n+5, \mathbf{k} \oplus \wedge^2 V_5)$ is a projective bundle, hence P_n and U_n are irreducible. On the other hand, by (3.1), U_n maps surjectively onto $\mathcal{M}_n$. Hence $\mathcal{M}_n$ is irreducible as well. $\square$

Several times throughout this thesis we will consider a *very general* GM variety X of dimension n. This means that the moduli point $[X] \in \mathcal{M}_n(\mathbf{k})$ lies in the complement of countably many proper closed substacks of $\mathcal{M}_n$.

3.3 GM varieties and EPW varieties

Let V_6 be a 6-dimensional vector space. Its exterior power $\wedge^3 V_6$ has a natural $\det(V_6)$-valued symplectic form, given by $\omega(\xi_1, \xi_2) = \xi_1 \wedge \xi_2$. For any Lagrangian subspace $\mathsf{A} \subset \wedge^3 V_6$, we consider the following stratification of $\mathbf{P}(V_6)$:

$$\mathsf{Y}^k_{\mathsf{A}} = \{v \in \mathbf{P}(V_6) \mid \dim(\mathsf{A} \cap (v \wedge (\wedge^2 V_6))) = k\} \subset \mathbf{P}(V_6).$$

We write $\mathsf{Y}_{\mathsf{A}}^{\geq k}$ for the union of all $\mathsf{Y}_{\mathsf{A}}^{i}$ for $i \geq k$, and Y_{A} for $\mathsf{Y}_{\mathsf{A}}^{\geq 1}$. The variety Y_{A} is called an *EPW sextic* (for Eisenbud, Popescu, and Walter, who first defined it), and the sequence $\mathsf{Y}_{\mathsf{A}}^{k}$ is called the *EPW stratification.*

We say A has *no decomposable vectors* if $\mathbf{P}(\mathsf{A})$ does not intersect $\mathrm{Gr}(3, V_6) \subset \mathbf{P}(\wedge^3 V_6)$. O'Grady [41, 42, 43, 44, 45, 46] extensively investigated the geometry of EPW sextics, and proved in particular that (see also [7, Theorem B.2]) if A has no decomposable vectors, then:

- $\mathsf{Y}_{\mathsf{A}} = \mathsf{Y}_{\mathsf{A}}^{\geq 1}$ is a normal irreducible sextic hypersurface, smooth along $\mathsf{Y}_{\mathsf{A}}^{1}$;
- $\mathsf{Y}_{\mathsf{A}}^{\geq 2} = \mathrm{Sing}(\mathsf{Y}_{\mathsf{A}})$ is a normal irreducible surface of degree 40, smooth along $\mathsf{Y}_{\mathsf{A}}^{2}$;
- $\mathsf{Y}_{\mathsf{A}}^{3} = \mathrm{Sing}(\mathsf{Y}_{\mathsf{A}}^{\geq 2})$ is at most finite, and for general A is empty;
- $\mathsf{Y}_{\mathsf{A}}^{\geq 4} = \varnothing$.

For any Lagrangian subspace $\mathsf{A} \subset \wedge^3 V_6$, its orthogonal $\mathsf{A}^{\perp} = \ker(\wedge^3 V_6^{\vee} \to \mathsf{A}^{\vee}) \subset \wedge^3 V_6^{\vee}$ is also Lagrangian, and A has no decomposable vectors if and only if the same is true of $\mathsf{A}^{\perp}$. In particular, $\mathsf{A}^{\perp}$ gives rise to an EPW sequence of subvarieties of $\mathbf{P}(V_6^{\vee})$, which can be written in terms of A as follows:

$$\mathsf{Y}_{\mathsf{A}^{\perp}}^{k} = \{V_5 \in \mathbf{P}(V_6^{\vee}) \mid \dim(\mathsf{A} \cap \wedge^3 V_5) = k\} \subset \mathbf{P}(V_6^{\vee}).$$

By O'Grady's work $\mathsf{Y}_{\mathsf{A}^{\perp}}$ is projectively dual to Y_{A}, and for this reason is called the *dual EPW sextic* to Y_{A}. We note that $\mathsf{Y}_{\mathsf{A}^{\perp}}$ is not isomorphic to Y_{A} for general A (see [42, Theorem 1.1]).

One of the main results of [7] is the following description of the set of all isomorphism classes of smooth ordinary GM varieties.

Theorem 3.6 ([7])**.** *There is a natural bijection between*

(1) the set of ordinary GM varieties X of dimension $n \geq 2$ whose Grassmannian hull M_X is smooth, up to isomorphism, and

(2) the set of pairs $(\mathsf{A}, \mathbf{p})$, where $\mathsf{A} \subset \wedge^3 V_6$ is a Lagrangian subspace with no decomposable vectors and $\mathbf{p} \in \mathsf{Y}_{\mathsf{A}^{\perp}}^{5-n}$, up to the action of $\mathrm{PGL}(V_6)$.

Note that by Lemma 3.2, M_X is automatically smooth if $n \geq 3$.

To an ordinary GM variety X as in (1), the correspondence of the theorem associates a pair $(\mathsf{A}(X), \mathbf{p}(X))$ where $\mathsf{A}(X) \subset \wedge^3 V_6(X)$ is Lagrangian. As indicated by the notation, $V_6(X)$ is indeed the space of quadrics containing X and $\mathbf{p}(X) \in \mathbf{P}(V_6(X)^\vee)$ is the Plücker point of X discussed in Section 3.1.

The association $X \mapsto (\mathsf{A}(X), \mathbf{p}(X))$ extends to all GM varieties, see [7]. If X is special of dimension at least 3, then $\mathsf{A}(X) = \mathsf{A}(X^{\mathrm{op}})$ is given by the Lagrangian of the ordinary GM variety X^{op} defined by (3.6). We thus have the following version of Theorem 3.6 for special GM varieties.

Corollary 3.7. *There is a natural bijection between*

(1) the set of special GM varieties of dimension $n \geq 3$, up to isomorphism, and

(2) the set of pairs $(\mathsf{A}, \mathbf{p})$, where $\mathsf{A} \subset \wedge^3 V_6$ is a Lagrangian subspace with no decomposable vectors and $\mathbf{p} \in \mathsf{Y}^{6-n}_{\mathsf{A}^\perp}$, up to the action of $\mathrm{PGL}(V_6)$.

Remark 3.8. To include GM surfaces into the above bijection, we must allow a more general class of Lagrangian subspaces in (2), namely those that contain finitely many decomposable vectors, see [7, Theorem 3.14 and Remark 3.15].

Remark 3.9. Theorem 3.6 and Corollary 3.7 suggest there is a morphism from the moduli stack $\mathcal{M}_n$ of n-dimensional GM varieties to the quotient stack $\mathrm{LG}(\wedge^3 V_6)/\mathrm{PGL}(V_6)$ — given by $X \mapsto \mathsf{A}(X)$ at the level of points — whose fiber over a point A is the union of two EPW strata $\mathsf{Y}^{5-n}_{\mathsf{A}^\perp} \sqcup \mathsf{Y}^{6-n}_{\mathsf{A}^\perp}$, modulo the action of the stabilizer of A in $\mathrm{PGL}(V_6)$. This map will be discussed in [9]. Let us simply note that it gives a way to compute $\dim \mathcal{M}_n$ geometrically (cf. Proposition 3.4). Namely, the quotient stack $\mathrm{LG}(\wedge^3 V_6)/\mathrm{PGL}(V_6)$ has dimension 20, and the fibers of the supposed morphism have dimension 5, 5, 4, or 2 for $n = 6, 5, 4,$ or 3, respectively. Finally, for $n = 2$ the morphism is no longer dominant, as its image is the divisor of those A such that $\mathsf{Y}^3_{\mathsf{A}^\perp} \neq \varnothing$, and its fibers are finite.

The EPW stratification arising from a GM variety has the following interpretation.

Proposition 3.10 ([7, Proposition 3.9(b)]). *Let X be a GM variety. Under the identification of the affine space $\mathbf{P}(V_6(X)) \setminus \mathbf{P}(V_5(X))$ with the space of non-Plücker quadrics containing X, the stratum*

$$\mathsf{Y}^k_{\mathsf{A}(X)} \cap (\mathbf{P}(V_6(X)) \setminus \mathbf{P}(V_5(X)))$$

corresponds to the quadrics Q such that $\dim(\ker(Q)) = k$.

For any Lagrangian $\mathsf{A} \subset \wedge^3 V_6$, Iliev-Kapustka-Kapustka-Ranestad [17] associated another important sequence of varieties, which this time stratify the Grassmannian $\mathrm{Gr}(3, V_6)$:

$$\mathsf{Z}^k_{\mathsf{A}} = \{V_3 \subset V_6 \mid \dim(\mathsf{A} \cap ((\wedge^2 V_3) \wedge V_6)) = k\} \subset \mathrm{Gr}(3, V_6).$$

As above, we write $\mathsf{Z}^{\geq k}_{\mathsf{A}}$ for the union of all $\mathsf{Z}^i_{\mathsf{A}}$ for $i \geq k$, and Z_{A} for $\mathsf{Z}^{\geq 1}_{\mathsf{A}}$. If A has no decomposable vectors, then:

- $\mathsf{Z}_{\mathsf{A}} = \mathsf{Z}^{\geq 1}_{\mathsf{A}}$ is a normal irreducible quartic hypersurface, smooth along $\mathsf{Z}^1_{\mathsf{A}}$;
- $\mathsf{Z}^{\geq 2}_{\mathsf{A}} = \mathrm{Sing}(\mathsf{Z}_{\mathsf{A}})$ is a normal irreducible sixfold, smooth along $\mathsf{Z}^2_{\mathsf{A}}$;
- $\mathsf{Z}^{\geq 3}_{\mathsf{A}} = \mathrm{Sing}(\mathsf{Z}^{\geq 2}_{\mathsf{A}})$ is a normal irreducible threefold, smooth along $\mathsf{Z}^3_{\mathsf{A}}$;
- $\mathsf{Z}^4_{\mathsf{A}} = \mathrm{Sing}(\mathsf{Z}^{\geq 3}_{\mathsf{A}})$ is at most finite, and for general A is empty;
- $\mathsf{Z}^{\geq 5}_{\mathsf{A}} = \varnothing$.

We call Z_{A} the *EPW quartic* of A. In contrast to EPW sextics, $\mathsf{Z}_{\mathsf{A}^\perp} = \mathsf{Z}_{\mathsf{A}}$ holds for all A.

Much of the geometry of a GM variety X is encoded by the stratified EPW varieties $(\mathsf{Y}_{\mathsf{A}(X)}, \mathsf{Y}_{\mathsf{A}(X)^\perp}, \mathsf{Z}_{\mathsf{A}(X)})$. We will discuss throughout the thesis how to express some geometric conditions on X in terms of these varieties.

Chapter 4

Derived categories of GM varieties

Our first goal in this chapter is to define the GM category of a GM variety, and to describe its Serre functor. Next we formulate a conjecture that identifies the GM categories of "generalized dual" varieties. Finally, we discuss the rationality problem for GM varieties from the perspective of derived categories.

4.1 The GM category and its Serre functor

By the discussion in Section 3.1.2, any GM variety X is obtained from a linear section of $\mathbf{G}$ by taking a quadric section or a branched double cover. To describe a natural semiorthogonal decomposition of $\mathrm{D}^{\mathrm{b}}(X)$, we first recall that $\mathbf{G}$ and its smooth linear sections of codimension at most 3 admit rectangular Lefschetz decompositions of their derived categories. Note that $\mathcal{O}_{\mathbf{G}}, \mathcal{U}^{\vee}$ form an exceptional pair in $\mathrm{D}^{\mathrm{b}}(\mathbf{G})$ (recall $\mathcal{U}$ denotes the tautological rank 2 bundle). Let

$$\mathcal{B} = \langle \mathcal{O}_{\mathbf{G}}, \mathcal{U}^{\vee} \rangle \subset \mathrm{D}^{\mathrm{b}}(\mathbf{G})$$

be the triangulated subcategory they generate. The following result holds by [21, Section 6.1].

Lemma 4.1. *Let M be a smooth linear section of $\mathbf{G} \subset \mathbf{P}(V)$ of dimension $n \geq 3$. Let $i \colon M \hookrightarrow \mathbf{G}$ be the inclusion.*

(1) *The functor* $i^*\colon \mathrm{D^b}(\mathbf{G}) \to \mathrm{D^b}(M)$ *is fully faithful on* $\mathcal{B} \subset \mathrm{D^b}(\mathbf{G})$*.*

(2) *Denoting the essential image of* $\mathcal{B}$ *by* $\mathcal{B}_M$*, there is a rectangular Lefschetz decomposition*

$$\mathrm{D^b}(M) = \langle \mathcal{B}_M, \mathcal{B}_M(1), \dots, \mathcal{B}_M(n-2) \rangle. \tag{4.1}$$

The next result gives a semiorthogonal decomposition of the derived category of a GM variety.

Proposition 4.2. *Let* X *be a GM variety of dimension* $n \geq 3$*. Let* $f\colon X \to \mathbf{G}$ *be the Gushel map.*

(1) *The functor* $f^*\colon \mathrm{D^b}(\mathbf{G}) \to \mathrm{D^b}(X)$ *is fully faithful on* $\mathcal{B} \subset \mathrm{D^b}(\mathbf{G})$*.*

(2) *Denoting the essential image of* $\mathcal{B}$ *by* $\mathcal{B}_X$*, there is a semiorthogonal decomposition*

$$\mathrm{D^b}(X) = \langle \mathcal{A}_X, \mathcal{B}_X, \mathcal{B}_X(1), \dots, \mathcal{B}_X(n-3) \rangle, \tag{4.2}$$

where $\mathcal{A}_X$ *is the right orthogonal category to* $\langle \mathcal{B}_X, \dots, \mathcal{B}_X(n-3) \rangle \subset \mathrm{D^b}(X)$*.*

Thus $\mathrm{D^b}(X)$ *has a semiorthogonal decomposition with* $\mathcal{A}_X$ *and* $2(n-2)$ *exceptional objects as components.*

Remark 4.3. If $n = 2$ we set $\mathcal{A}_X = \mathrm{D^b}(X)$, so that (4.2) still holds.

Proof. The Gushel map factors through the map $X \to M'_X$ to the projected Grassmannian hull M'_X defined by (3.3). By Lemma 3.2, M'_X is smooth and has dimension $n+1$ if X is ordinary, or dimension n if X is special. In particular, $\mathrm{D^b}(M'_X)$ has a rectangular Lefschetz decomposition given by Lemma 4.1. Further, $X \to M'_X$ realizes X as a quadric section (3.4) if X is ordinary, or as a double cover (3.5) if X is special. Now applying [31, Lemmas 5.1 and 5.5] gives the result. $\square$

Definition 4.4. Let X be a GM variety. The *GM category* of X is the category $\mathcal{A}_X$ defined by the semiorthogonal decomposition (4.2).

The GM category $\mathcal{A}_X$ is the main object of study of this thesis. As we will see below, its properties depend strongly on the parity of $\dim(X)$. For this reason, we sometimes emphasize the parity of $\dim(X)$ by calling $\mathcal{A}_X$ an *even* or *odd GM category* according to whether $\dim(X)$ is even or odd.

Using [27], we can describe the Serre functor of GM categories:

Proposition 4.5. *Let X be a GM variety of dimension n.*

(1) If n is even, the Serre functor of the GM category $\mathcal{A}_X$ satisfies $\mathrm{S}_{\mathcal{A}_X} \cong [2]$.

(2) If n is odd, the Serre functor of the GM category $\mathcal{A}_X$ satisfies $\mathrm{S}_{\mathcal{A}_X} \cong \sigma \circ [2]$ for a nontrivial involutive autoequivalence σ of $\mathcal{A}_X$. If in addition X is special, then σ is induced by the involution of the double cover (3.5).

Proof. If $n = 2$, then $\mathcal{A}_X = \mathrm{D}^{\mathrm{b}}(X)$ and X is a K3 surface, so the result is clear. If $n \geq 3$, then as in the proof of Proposition 4.2 we may express X as a quadric section or double cover of the smooth variety M'_X. Moreover, it is easy to see the length m of the Lefschetz decomposition of $\mathrm{D}^{\mathrm{b}}(M'_X)$ given by Lemma 4.1 satisfies $K_{M'_X} = -mH$, where H is the restriction of the ample generator of $\mathrm{Pic}(\mathbf{G})$. Hence we may apply [27, Corollaries 3.7 and 3.8] to see that the Serre functors have the desired form. If σ were trivial, then the Hochschild homology $\mathrm{HH}_{-2}(\mathcal{A}_X)$ would be nontrivial (see Proposition 5.5), which contradicts the computation of Lemma 5.3 below. $\square$

The proposition shows that even GM categories can be regarded as "noncommutative K3 surfaces", and odd GM categories can be regarded as "noncommutative Enriques surfaces". This analogy goes further than the relation between Serre functors. For instance, any Enriques surface (in characteristic 0) is the quotient of a K3 surface by an involution. Similarly, the results of [31] show that odd GM categories can be described as "quotients" of even GM categories by involutions. To state this precisely, recall from Section 3.1.2 that if X is a GM variety of dimension at least 3, then there is an associated GM variety X^{op} of the opposite type and parity of dimension.

Proposition 4.6. *Let X be a GM variety with $\dim(X) \geq 3$. Then there is a $\mathbf{Z}/2$-action on $\mathcal{A}_X$ such that if $\mathcal{A}_X^{\mathbf{Z}/2}$ denotes the equivariant category, then there is an equivalence*

$$\mathcal{A}_X^{\mathbf{Z}/2} \simeq \mathcal{A}_{X^{\mathrm{op}}}.$$

If $\dim(X)$ is odd, then the $\mathbf{Z}/2$-action is given by the autoequivalence $\sigma = \mathrm{S}_{\mathcal{A}_X} \circ [-2]$ of Proposition 4.5.

Proof. The proof is contained in [31, Section 8.2]. If X is special, the $\mathbf{Z}/2$-action is induced by the involution of the canonical double cover $X \to M'_X$. If X is ordinary, the action is given by a so-called *rotation functor*, which if $\dim(X)$ is odd coincides with σ. $\square$

4.2 The duality conjecture

Recall from Section 3.3 that if X is a GM variety, then there is an associated Lagrangian subspace $\mathsf{A}(X) \subset \wedge^3 V_6(X)$, where $V_6(X)$ is the space of quadrics cutting out X. The following definition extends [7, Definition 3.24].

Definition 4.7. Let X_1 and X_2 be GM varieties such that there exists an isomorphism $V_6(X_1) \cong V_6(X_2)^\vee$ identifying $\mathsf{A}(X_1) \subset \wedge^3 V_6(X_1)$ with $\mathsf{A}(X_2)^\perp \subset \wedge^3 V_6(X_2)^\vee$. Then we say:

- X_1 and X_2 are *dual* if $\dim(X_1) = \dim(X_2)$, and
- X_1 and X_2 are *generalized dual* if $\dim(X_1) \equiv \dim(X_2) \pmod 2$.

One of the main results of [7, Section 4] is that dual GM varieties of dimension at least 3 are birational. Our motivation for defining generalized duality is the following conjecture.

Conjecture 4.8. *If X_1 and X_2 are generalized dual GM varieties, then there is an equivalence $\mathcal{A}_{X_1} \simeq \mathcal{A}_{X_2}$ of GM categories.*

By Proposition 4.5 the equality of the parities of the dimensions of GM varieties is necessary for an equivalence of their GM categories. As evidence for Conjecture 4.8, we prove in

Chapter 6 the special case where X_1 is an ordinary GM fourfold and X_2 is a (suitably generic) generalized dual surface. In fact, the approach of Chapter 6 can be used to attack the full conjecture, but is quite unwieldy to carry out in the general case.

Remark 4.9. If X is a GM variety, then either $\mathsf{A}(X)$ does or does not contain decomposable vectors, and these two cases are preserved by generalized duality. The first case happens only when X is an ordinary surface with singular Grassmannian hull or X is a special surface, see [7, Theorem 3.14 and Remark 3.15].

We discuss here some consequences of the duality conjecture.

Let X be an n-dimensional GM variety, and assume $\mathsf{A}(X)$ has no decomposable vectors (which is almost always the case by Remark 4.9). Then any quadric point $\mathbf{q} \in \mathbf{P}(V_6(X))$ corresponds to a generalized dual $X_{\mathbf{q}}^{\vee}$ of X. Indeed, $\mathbf{q}$ lies in the stratum $\mathsf{Y}^k_{\mathsf{A}(X)}$ for some k. If $5-k \equiv n \pmod 2$, then the ordinary GM variety $X_{\mathbf{q}}^{\vee}$ of dimension $5-k$ associated by Theorem 3.6 to the pair $(\mathsf{A}(X)^{\perp}, \mathbf{q})$ (with $V_6 = V_6(X)^{\vee}$) is generalized dual to X. Otherwise $6-k \equiv n \pmod 2$, and the special GM variety $X_{\mathbf{q}}^{\vee}$ of dimension $6-k$ associated by Corollary 3.7 to the pair $(\mathsf{A}(X)^{\perp}, \mathbf{q})$ (with $V_6 = V_6(X)^{\vee}$) is generalized dual to X. In fact, reversing the above argument shows that any generalized dual to X arises as $X_{\mathbf{q}}^{\vee}$ for some $\mathbf{q} \in \mathbf{P}(V_6(X))$. More precisely, the set of isomorphism classes of generalized duals to X can be identified with $\mathbf{P}(V_6(X))$, modulo the action of the stabilizer of $\mathsf{A}(X)$ in $\mathrm{PGL}(V_6(X))$.

Below we list more explicitly the type of $X_{\mathbf{q}}^{\vee}$ according to k and the parity of n.

Table 1: Types of generalized dual varieties

k	$X_{\mathbf{q}}^{\vee}$ for n even	$X_{\mathbf{q}}^{\vee}$ for n odd
0	special sixfold	ordinary fivefold
1	ordinary fourfold	special fivefold
2	special fourfold	ordinary threefold
3	ordinary surface	special threefold

Recall that the stratum $\mathsf{Y}^k_{\mathsf{A}(X)}$ is always nonempty for $k = 0, 1, 2$, generically empty for $k = 3$, and always empty for $k \geq 4$. In fact, the condition that $\mathsf{Y}^3_{\mathsf{A}(X)}$ is nonempty is divisorial in $\mathcal{M}_n$ (see Remark 6.3).

The duality conjecture says there is an equivalence

$$\mathcal{A}_X \simeq \mathcal{A}_{X^\vee_{\mathbf{q}}}$$

for every $\mathbf{q} \in \mathbf{P}(V_6(X))$. In particular, it predicts that often GM categories are equivalent to those of lower-dimensional GM varieties, namely that:

(1) If X is a sixfold, then its GM category is equivalent to a fourfold's GM category.

(2) If X is a fivefold, then its GM category is equivalent to a threefold's GM category.

(3) If X is a fourfold such that $\mathsf{Y}^3_{\mathsf{A}(X)} \neq \varnothing$, then its GM category is equivalent to the derived category of a GM surface.

As mentioned above, in Chapter 6 we prove case (3) of the duality conjecture, under the slightly stronger assumption $\mathsf{Y}^3_{\mathsf{A}(X)} \cap (\mathbf{P}(V_6(X)) \setminus \mathbf{P}(V_5(X))) \neq \varnothing$.

Remark 4.10. Using Theorem 3.6 and Corollary 3.7, it is easy to see that to prove the equivalence $\mathcal{A}_X \simeq \mathcal{A}_{X^\vee_{\mathbf{q}}}$ for all X and $\mathbf{q} \in \mathbf{P}(V_6(X))$, it is enough to prove it for all X and $\mathbf{q} \in \mathbf{P}(V_6(X)) \setminus \mathbf{P}(V_5(X))$.

Remark 4.11. A GM variety X as in (1)–(3) above is rational (see the discussion below and Lemma 6.7). It seems likely that for such an X there is a rationality construction that involves a blowup of a generalized dual variety of dimension 2 less, and gives rise to an equivalence of GM categories. Our approach to (3) in Chapter 6 takes a completely different route.

Finally, we define another relation between GM varieties, which is similar to generalized duality.

Definition 4.12. Let X_1 and X_2 be GM varieties such that there exists an isomorphism $V_6(X_1) \cong V_6(X_2)$ identifying $\mathsf{A}(X_1) \subset \wedge^3 V_6(X_1)$ with $\mathsf{A}(X_2) \subset \wedge^3 V_6(X_2)$. Then we say:

- X_1 and X_2 are *period partners* if $\dim(X_1) = \dim(X_2)$, and
- X_1 and X_2 are *generalized period partners* if $\dim(X_1) \equiv \dim(X_2) \pmod 2$.

The (non-generalized) notion of period partners was first introduced in [7], where it is shown that period partners of dimension at least 3 are birational. By analogy with the duality conjecture, it is natural to expect that generalized period partners have equivalent GM categories. In fact, it is easy to see that this follows from the duality conjecture.

4.3 Relation to rationality

Let us recall what is known about rationality of GM varieties. A general GM threefold is irrational by [2, Theorem 5.6], while every GM fivefold or sixfold is rational by [7, Proposition 4.2] (for a *general* GM fivefold or sixfold this was already known to Roth). The intermediate case of GM fourfolds is more mysterious, and closely parallels the situation for cubic fourfolds: some rational examples are known [6], but while a very general GM fourfold is expected to be irrational, it has not been proven that a single GM fourfold is irrational.

In the spirit of [28], we discuss the above state of affairs from the point of view of derived categories. As in [28], for any triangulated category $\mathcal{A}$, the *geometric dimension* $\mathrm{gdim}(\mathcal{A})$ is defined as the minimal integer m such that there exists an m-dimensional connected smooth projective variety M and an admissible embedding $\mathcal{A} \hookrightarrow \mathrm{D^b}(M)$. If X is a rational smooth projective variety, then there is a sequence of blowups and blowdowns along smooth centers connecting X to a projective space. Analyzing the behavior of semiorthogonal decompositions under blowups then suggests the following conjecture.

Conjecture 4.13 ([28])**.** *Let X be a rational smooth projective variety with $\dim(X) \geq 2$. Then $\mathrm{D^b}(X)$ admits a semiorthogonal decomposition with all components of geometric dimension at most $\dim(X) - 2$.*

Let X be a GM variety. Then there is a semiorthogonal decomposition of $\mathrm{D^b}(X)$ with $\mathcal{A}_X$ and a sequence of exceptional objects as components. Since the category generated by an exceptional object has geometric dimension 0, the maximal geometric dimension of the components is achieved by the GM category $\mathcal{A}_X$.

If X is a threefold, we prove in Lemma 5.18 that $\mathcal{A}_X$ does not admit a semiorthogonal decomposition with all components of geometric dimension at most 1. We view this as evidence that $\mathrm{D}^{\mathrm{b}}(X)$ does not admit such a decomposition, so that Conjecture 4.13 would predict the irrationality of X. If X is a fivefold or sixfold, then by the discussion at the end of Section 4.2, the duality conjecture implies $\mathrm{gdim}(\mathcal{A}_X) \leq \dim(X) - 2$. Hence, assuming the duality conjecture, Conjecture 4.13 is consistent with the rationality of X.

Finally, for fourfolds we make the following refinement of Conjecture 4.13.

Conjecture 4.14. *If X is a rational GM fourfold, then the GM category $\mathcal{A}_X$ is equivalent to the derived category of a K3 surface.*

Remark 4.15. Conjecture 4.14 is a GM fourfold analogue of [25, Conjecture 1.1]. It should be thought of as claiming the conclusion of Conjecture 4.13 holds for the *particular* decomposition (4.2). Indeed:

(1) The condition $\mathrm{gdim}(\mathcal{A}_X) \leq 2$ is equivalent to $\mathrm{gdim}(\mathcal{A}_X) = 2$, since $\mathcal{A}_X$ being 2-Calabi–Yau implies the reverse inequality always holds ([27, Theorem 5.7]).

(2) If $\mathcal{A}_X$ is equivalent to the derived category of a K3 surface then $\mathrm{gdim}(\mathcal{A}_X) = 2$, and a general conjecture on Calabi–Yau categories [27, Conjecture 5.8] combined with Proposition 5.17 implies the converse is true.

One of our main results, Theorem 1.2, verifies Conjecture 4.14 for a certain family of rational GM fourfolds. We also prove that the GM category of a very general GM fourfold is not equivalent to the derived category of a K3 surface (Proposition 5.19). Hence Conjecture 4.14 is consistent with the expectation that a very general GM fourfold is irrational.

Chapter 5

Algebraic invariants of GM categories

In this chapter, we compute the Hochschild homology, Hochschild cohomology, and (in the very general case) the Grothendieck group of GM categories. Then we use these computations to prove some structural results about these categories.

5.1 The Hochschild homology of GM categories

Given a suitably enhanced triangulated category $\mathcal{A}$, there is an invariant $\mathrm{HH}_\bullet(\mathcal{A})$ called its *Hochschild homology*, which is a certain graded $\mathbf{k}$-vector space. For the definition in the context of $\mathbf{k}$-linear stable ∞-categories, see Definition A.25. We will exclusively be interested in admissible subcategories of the derived category of a smooth projective variety. Such a category has a natural stable ∞-category enhancement which admits a Serre functor, so Definition A.25 applies. However, in this situation it is also possible to give a more down-to-earth definition, see [24]. In fact, we recall all of the properties of Hochschild homology that we need below, so the reader may take the precise definition as a black box.

If $\mathcal{A} = \mathrm{D}^{\mathrm{b}}(X)$, we write $\mathrm{HH}_\bullet(X)$ for $\mathrm{HH}_\bullet(\mathcal{A})$. The Hochschild–Kostant–Rosenberg (HKR) isomorphism gives the following explicit description of Hochschild homology in this case.

Theorem 5.1 ([37]). *Let X be a smooth projective variety. Then for all i there is an isomorphism of vector spaces*

$$\mathrm{HH}_i(X) \cong \bigoplus_{q-p=i} \mathrm{H}^q(X, \Omega_X^p).$$

An important property of Hochschild homology is that it is additive under semiorthogonal decompositions.

Theorem 5.2 ([24, Theorem 7.3]). *Let X be a smooth projective variety. Given a semiorthogonal decomposition $\mathrm{D}^{\mathrm{b}}(X) = \langle \mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m \rangle$, there is an isomorphism*

$$\mathrm{HH}_\bullet(X) \cong \bigoplus_{i=1}^{m} \mathrm{HH}_\bullet(\mathcal{A}_i).$$

By combining this additivity property with the HKR isomorphism, we can compute the Hochschild homology of GM categories.

Lemma 5.3. *Let X be a GM variety of dimension n. Then*

$$\mathrm{HH}_\bullet(\mathcal{A}_X) \cong \begin{cases} \mathbf{k}[2] \oplus \mathbf{k}^{22}[0] \oplus \mathbf{k}[-2] & \text{if } n \text{ is even,} \\ \mathbf{k}^{10}[1] \oplus \mathbf{k}^{2}[0] \oplus \mathbf{k}^{10}[-1] & \text{if } n \text{ is odd.} \end{cases}$$

Here and below we use $[-]$ to denote a shift of grading. Thus, the first line means that $\mathrm{HH}_i(\mathcal{A}_X)$ has dimension 1 for $i = \pm 2$, and dimension 22 for $i = 0$.

Proof. By Proposition 4.2, there is a semiorthogonal decomposition of $\mathrm{D}^{\mathrm{b}}(X)$ with $\mathcal{A}_X$ and $2(n-2)$ exceptional objects as components. Since the category generated by an exceptional object is equivalent to the derived category of a point, its Hochschild homology is $\mathbf{k}[0]$. Hence by additivity,

$$\mathrm{HH}_\bullet(X) \cong \mathrm{HH}_\bullet(\mathcal{A}_X) \oplus \mathbf{k}^{2(n-2)}[0].$$

By Theorem 5.1, the graded dimension of $\mathrm{HH}_\bullet(X)$ can be computed by summing the columns of the Hodge diamond of X, which looks as follows (see [33], [18], [40], and [8]).

Table 2: Hodge diamonds of GM varieties

$\dim(X) = 2$	$\dim(X) = 3$	$\dim(X) = 4$	$\dim(X) = 5$	$\dim(X) = 6$
$\begin{array}{c} 1 \\ 0 \quad 0 \\ 1 \quad 20 \quad 1 \\ 0 \quad 0 \\ 1 \end{array}$	$\begin{array}{c} 1 \\ 0 \quad 0 \\ 0 \quad 1 \quad 0 \\ 0 \quad 10 \quad 10 \quad 0 \\ 0 \quad 1 \quad 0 \\ 0 \quad 0 \\ 1 \end{array}$	$\begin{array}{c} 1 \\ 0 \quad 0 \\ 0 \quad 1 \quad 0 \\ 0 \quad 0 \quad 0 \quad 0 \\ 0 \quad 1 \quad 22 \quad 1 \quad 0 \\ 0 \quad 0 \quad 0 \quad 0 \\ 0 \quad 1 \quad 0 \\ 0 \quad 0 \\ 1 \end{array}$	$\begin{array}{c} 1 \\ 0 \quad 0 \\ 0 \quad 1 \quad 0 \\ 0 \quad 0 \quad 0 \quad 0 \\ 0 \quad 0 \quad 2 \quad 0 \quad 0 \\ 0 \quad 0 \quad 10 \quad 10 \quad 0 \quad 0 \\ 0 \quad 0 \quad 2 \quad 0 \quad 0 \\ 0 \quad 0 \quad 0 \quad 0 \\ 0 \quad 1 \quad 0 \\ 0 \quad 0 \\ 1 \end{array}$	$\begin{array}{c} 1 \\ 0 \quad 0 \\ 0 \quad 1 \quad 0 \\ 0 \quad 0 \quad 0 \quad 0 \\ 0 \quad 0 \quad 2 \quad 0 \quad 0 \\ 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\ 0 \quad 0 \quad 1 \quad 22 \quad 1 \quad 0 \quad 0 \\ 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\ 0 \quad 0 \quad 2 \quad 0 \quad 0 \\ 0 \quad 0 \quad 0 \quad 0 \\ 0 \quad 1 \quad 0 \\ 0 \quad 0 \\ 1 \end{array}$

Now the lemma follows by inspection. □

5.2 The Hochschild cohomology of GM categories

Given a suitably enhanced triangulated category $\mathcal{A}$, there is also an invariant $\mathrm{HH}^\bullet(\mathcal{A})$ called its *Hochschild cohomology*, which has the structure of a graded $\mathbf{k}$-algebra (see Definition A.7). Again, for $\mathcal{A}$ an admissible subcategory of the derived category of a smooth projective variety, a more down-to-earth definition can be found in [24].

If $\mathcal{A} = \mathrm{D}^{\mathrm{b}}(X)$, we write $\mathrm{HH}^\bullet(X)$ for $\mathrm{HH}^\bullet(\mathcal{A})$. There is the following version of the HKR isomorphism for Hochschild cohomology.

Theorem 5.4 ([37])**.** *Let X be a smooth projective variety. Then for all i there is an isomorphism of vector spaces*

$$\mathrm{HH}^i(X) \cong \bigoplus_{p+q=i} \mathrm{H}^q(X, \wedge^p \mathrm{T}_X).$$

Hochschild cohomology is not additive under semiorthogonal decompositions, and so it is generally much harder to compute than Hochschild homology. There is, however, a case when the computation simplifies considerably. Recall that a triangulated category $\mathcal{A}$ is called *n-Calabi–Yau* if the shift functor $[n]$ is a Serre functor for $\mathcal{A}$.

Proposition 5.5 ([27, Proposition 5.3])**.** *Let $\mathcal{A}$ be an admissible subcategory of $\mathrm{D}^{\mathrm{b}}(X)$ for a smooth projective variety X. If $\mathcal{A}$ is an n-Calabi–Yau category, then for each i there is an* *isomorphism of vector spaces*

$$\mathrm{HH}^i(\mathcal{A}) \cong \mathrm{HH}_{i-n}(\mathcal{A}).$$

This immediately applies to even GM categories, as by Proposition 4.5 they are 2-Calabi–Yau.

Corollary 5.6. *Let X be a GM variety of even dimension. Then*

$$\mathrm{HH}^\bullet(\mathcal{A}_X) \cong \mathbf{k}[0] \oplus \mathbf{k}^{22}[-2] \oplus \mathbf{k}[-4].$$

The Hochschild cohomology of odd GM categories is significantly harder to compute. Our strategy is to exploit the fact that there is a $\mathbf{Z}/2$-action on such a category, with invariants an even GM category. By the results of Appendix A, this reduces us to a problem involving the Hochschild cohomology of an even GM category and the Hochschild *homology* of an odd GM category. We note that this same technique can be applied more generally to relate the Hochschild cohomology of a fractional Calabi–Yau category to that of its "canonical Calabi–Yau cover" (in the sense of Remark A.21).

Proposition 5.7. *Let X be a GM variety of odd dimension. Then*

$$\mathrm{HH}^\bullet(\mathcal{A}_X) \cong \mathbf{k}[0] \oplus \mathbf{k}^{20}[-2] \oplus \mathbf{k}[-4].$$

Proof. Recall that there is a $\mathbf{Z}/2$-action on $\mathcal{A}_X$ such that if $\sigma\colon \mathcal{A}_X \to \mathcal{A}_X$ denotes the corresponding involutive autoequivalence, then:

(1) $\mathrm{S}_{\mathcal{A}_X} = \sigma \circ [2]$ is a Serre functor for $\mathcal{A}_X$ (Proposition 4.5).

(2) $\mathcal{A}_X^{\mathbf{Z}/2} \simeq \mathcal{A}_{X^{\mathrm{op}}}$, where X^{op} is the opposite variety to X (Proposition 4.6).

These are results at the level of triangulated categories, but in fact their proofs go through at the enhanced level, to show the following. There is a $\mathbf{Z}/2$-action on the $\mathbf{k}$-linear stable enhancement $\mathcal{A}_X^{\mathrm{enh}}$ of $\mathcal{A}_X$ (see Remark A.1), such that if $\sigma^{\mathrm{enh}}\colon \mathcal{A}_X^{\mathrm{enh}} \to \mathcal{A}_X^{\mathrm{enh}}$ denotes the corresponding involutive autoequivalence, then (1) and (2) above hold with $\mathcal{A}_X$, $\mathrm{S}_{\mathcal{A}_X}$, σ, and $\mathcal{A}_{X^{\mathrm{op}}}$ replaced by their enhanced versions, and (1) and (2) are recovered by passing to homotopy categories. Hence we may apply Corollary A.26 to find

$$\mathrm{HH}^\bullet(\mathcal{A}_{X^{\mathrm{op}}}) = \mathrm{HH}^\bullet(\mathcal{A}_X) \oplus (\mathrm{HH}_\bullet(\mathcal{A}_X)^{\mathbf{Z}/2}[-2]) \tag{5.1}$$

where the $\mathbf{Z}/2$-action on $\mathrm{HH}_\bullet(\mathcal{A}_X)$ is induced by σ.

Since X has odd dimension (and hence X^{op} has even dimension), by Corollary 5.6 we have

$$\mathrm{HH}^\bullet(\mathcal{A}_{X^{\mathrm{op}}}) \cong \mathbf{k}[0] \oplus \mathbf{k}^{22}[-2] \oplus \mathbf{k}[-4],$$

and by Lemma 5.3 we have

$$\mathrm{HH}_\bullet(\mathcal{A}_X) \cong \mathbf{k}^{10}[1] \oplus \mathbf{k}^2[0] \oplus \mathbf{k}^{10}[-1].$$

Combined with (5.1), this immediately shows $\mathrm{HH}_\bullet(\mathcal{A}_X)^{\mathbf{Z}/2}$ is concentrated in degree 0, i.e. $\mathrm{HH}_\bullet(\mathcal{A}_X)^{\mathbf{Z}/2} \cong \mathbf{k}^d[0]$ for some $0 \leq d \leq 2$, and

$$\mathrm{HH}^\bullet(\mathcal{A}_X) \cong \mathbf{k}[0] \oplus \mathbf{k}^{22-d}[-2] \oplus \mathbf{k}[-4].$$

To prove $d = 2$, we apply [49, Corollary 3.11], which gives an equality

$$\sum_i (-1)^i \dim \mathrm{HH}^i(\mathcal{A}_X) = \sum_i (-1)^i \operatorname{Tr}((\mathrm{S}_{\mathcal{A}_X}^{-1})_* \colon \mathrm{HH}_i(\mathcal{A}_X) \to \mathrm{HH}_i(\mathcal{A}_X)). \tag{5.2}$$

Note that since $\mathrm{S}_{\mathcal{A}_X} = \sigma \circ [2]$, the map $(\mathrm{S}_{\mathcal{A}_X}^{-1})_* \colon \mathrm{HH}_i(\mathcal{A}_X) \to \mathrm{HH}_i(\mathcal{A}_X)$ induced by $\mathrm{S}_{\mathcal{A}_X}^{-1}$ on Hochschild homology coincides with the map induced by σ, and in particular squares to the identity. It follows that the right side of (5.2) is bounded above by $\sum_i \dim \mathrm{HH}_i(\mathcal{A}_X) = 22$. But the left side of (5.2) equals $24 - d$ where $0 \leq d \leq 2$, so $d = 2$. $\square$

Remark 5.8. It is possible to show $d = 2$ in the above proof without appealing to the equality (5.2), as follows. Note that the statement is deformation invariant, since it is equivalent to the Euler characteristic $\sum_i (-1)^i \dim \mathrm{HH}^i(\mathcal{A}_X)$ being 22. So we may assume X is special. Then the $\mathbf{Z}/2$-action on $\mathcal{A}_X$ is induced by the involution i of the double cover $X \to M'_X$. We want to show that $\mathbf{Z}/2$ acts trivially on $\mathrm{HH}_0(\mathcal{A}_X)$. But $\mathrm{HH}_\bullet(\mathcal{A}_X)$ is canonically a summand of $\mathrm{HH}_\bullet(X)$, and we claim that $i^* \colon \mathrm{D^b}(X) \to \mathrm{D^b}(X)$ acts trivially on $\mathrm{HH}_0(X)$. Indeed, since X is odd-dimensional, pullback under $X \to M'_X$ induces a surjection on even-degree cohomology and hence on HH_0. The claim follows.

5.3 The Grothendieck group of GM categories

For a triangulated category $\mathcal{T}$, the *Grothendieck group* $\mathrm{K}_0(\mathcal{T})$ is the free group on isomorphism classes $[\mathcal{F}]$ of objects $\mathcal{F} \in \mathcal{T}$, modulo the relations $[\mathcal{G}] = [\mathcal{F}] + [\mathcal{H}]$ for every distinguished triangle $\mathcal{F} \to \mathcal{G} \to \mathcal{H}$.

Assume $\mathcal{T}$ is *proper*, i.e. that $\bigoplus_i \mathrm{Hom}(\mathcal{F}, \mathcal{G}[i])$ is finite dimensional for all $\mathcal{F}, \mathcal{G} \in \mathcal{T}$. For instance, this holds if $\mathcal{T}$ is admissible in the derived category of a smooth projective variety. Then for $\mathcal{F}, \mathcal{G} \in \mathcal{T}$, we set

$$\chi(\mathcal{F}, \mathcal{G}) = \sum_i (-1)^i \dim \mathrm{Hom}(\mathcal{F}, \mathcal{G}[i]).$$

This descends to a bilinear form $\chi \colon \mathrm{K}_0(\mathcal{T}) \times \mathrm{K}_0(\mathcal{T}) \to \mathbf{Z}$, called the *Euler form*. In general this form is neither symmetric nor antisymmetric. However, if $\mathcal{T}$ admits a Serre functor (e.g. if $\mathcal{T}$ is admissible in the derived category of a smooth projective variety), then the left and right kernels of the form χ agree, and we denote this common subgroup of $\mathrm{K}_0(\mathcal{T})$ by $\ker(\chi)$. In this situation, the *numerical Grothendieck group* is the quotient

$$\mathrm{K}_0(\mathcal{T})_{\mathrm{num}} = \mathrm{K}_0(\mathcal{T}) / \ker(\chi).$$

Note that $\mathrm{K}_0(\mathcal{T})_{\mathrm{num}}$ is torsion free, since any torsion element is contained in the kernel of χ.

If X is a smooth projective variety, we write

$$\mathrm{K}_0(X) = \mathrm{K}_0(\mathrm{D^b}(X)) \quad \text{and} \quad \mathrm{K}_0(X)_{\mathrm{num}} = \mathrm{K}_0(\mathrm{D^b}(X))_{\mathrm{num}}.$$

Further, let $\mathrm{CH}(X)$ and $\mathrm{CH}(X)_{\mathrm{num}}$ denote the Chow rings of cycles modulo rational and numerical equivalence. The following well-known consequence of Hirzebruch–Riemann–Roch computes the (numerical) Grothendieck group of X in terms of its (numerical) Chow group.

Lemma 5.9. *Let X be a smooth projective variety. Then there are isomorphisms*

$$\mathrm{K}_0(X) \otimes \mathbf{Q} \cong \mathrm{CH}(X) \otimes \mathbf{Q} \qquad \textit{and} \qquad \mathrm{K}_0(X)_{\mathrm{num}} \otimes \mathbf{Q} \cong \mathrm{CH}(X)_{\mathrm{num}} \otimes \mathbf{Q}.$$

Proof. The isomorphisms are induced by the Chern character $\mathrm{ch}\colon \mathrm{K}_0(X) \to \mathrm{CH}(X) \otimes \mathbf{Q}$. For the first, see [11, Example 15.2.16(b)]. The second then follows from the observation that, by Riemann–Roch, the kernel of the Euler form is precisely the preimage under the Chern character of the subring of numerically trivial cycles. □

The following well-known lemma says that Grothendieck groups are additive.

Lemma 5.10. *Let X be a smooth projective variety. Given a semiorthogonal decomposition $\mathrm{D^b}(X) = \langle \mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m \rangle$, there are isomorphisms*

$$\mathrm{K}_0(X) \cong \bigoplus_{i=1}^{m} \mathrm{K}_0(\mathcal{A}_i) \qquad \textit{and} \qquad \mathrm{K}_0(X)_{\mathrm{num}} \cong \bigoplus_{i=1}^{m} \mathrm{K}_0(\mathcal{A}_i)_{\mathrm{num}}.$$

Proof. The embedding functors $\mathcal{A}_i \hookrightarrow \mathrm{D^b}(X)$ induce a map $\bigoplus_i \mathrm{K}_0(\mathcal{A}_i) \to \mathrm{K}_0(X)$, whose inverse is the map induced by the projection functors $\mathrm{D^b}(X) \to \mathcal{A}_i$. This isomorphism also descends to numerical Grothendieck groups. □

Now let X be a GM variety. If X is a surface then $\mathcal{A}_X = \mathrm{D^b}(X)$, so the Grothendieck group of $\mathcal{A}_X$ coincides with that of X. Below we describe $\mathrm{K}_0(\mathcal{A}_X)_{\mathrm{num}}$ if X is odd dimensional, or if X is a fourfold or sixfold which is not "Hodge-theoretically special" in the following sense.

First, we note that if n denotes the dimension of X, the Gushel morphism $f\colon X \to \mathbf{G}$ induces an injection

$$\mathrm{H}^n(\mathbf{G}, \mathbf{Z}) \hookrightarrow \mathrm{H}^n(X, \mathbf{Z}).$$

If n is odd, then $\mathrm{H}^n(\mathbf{G}, \mathbf{Z})$ simply vanishes. But if $n = 4$ or 6, then $\mathrm{H}^n(\mathbf{G}, \mathbf{Z}) = \mathbf{Z}^2$ is generated by Schubert cycles, and the *vanishing cohomology* $\mathrm{H}^n_{\mathrm{van}}(X, \mathbf{Z})$ is defined as the orthogonal to $\mathrm{H}^n(\mathbf{G}, \mathbf{Z}) \subset \mathrm{H}^n(X, \mathbf{Z})$ with respect to the intersection form.

Definition 5.11 ([6])**.** Let X be a GM variety of dimension $n = 4$ or 6. Then X is *Hodge-special* if

$$\mathrm{H}^{\frac{n}{2},\frac{n}{2}}(X) \cap \mathrm{H}^n_{\mathrm{van}}(X, \mathbf{Q}) \neq 0.$$

Lemma 5.12 ([6])**.** *If X is a very general GM fourfold or sixfold, then X is not Hodge-special.*

Proof. In the fourfold case, this is [6, Corollary 4.6]. The main point of the proof is the computation that the local period map for GM fourfolds is a submersion. The sixfold case can be proved by the same argument. □

Lemma 5.13. *Let X be a GM variety of dimension $n \geq 3$. If n is even assume also that X is not Hodge-special. Then $\mathrm{K}_0(\mathcal{A}_X)_{\mathrm{num}} \simeq \mathbf{Z}^2$.*

Proof. The proof is similar to that of Lemma 5.3. First, note that by Proposition 4.2 there is a semiorthogonal decomposition of $\mathrm{D}^{\mathrm{b}}(X)$ with $\mathcal{A}_X$ and $2(n-2)$ exceptional objects as components. Since the category generated by an exceptional object is equivalent to the derived category of a point, both its usual and numerical Grothendieck group is $\mathbf{Z}$. Hence by additivity,

$$\mathrm{K}_0(X)_{\mathrm{num}} \cong \mathrm{K}_0(\mathcal{A}_X)_{\mathrm{num}} \oplus \mathbf{Z}^{2(n-2)}.$$

On the other hand, $\mathrm{K}_0(X)_{\mathrm{num}} \otimes \mathbf{Q} \cong \mathrm{CH}(X)_{\mathrm{num}} \otimes \mathbf{Q}$. But under our assumptions on X, the rational Hodge classes on X are spanned by the restrictions of Schubert cycles on $\mathbf{G}$. In particular, the Hodge conjecture holds for X. So numerical equivalence coincides with homological equivalence, and

$$\mathrm{CH}(X)_{\mathrm{num}} \otimes \mathbf{Q} \cong \bigoplus_k \mathrm{H}^{k,k}(X, \mathbf{Q})$$

where $\mathrm{H}^{k,k}(X, \mathbf{Q}) = \mathrm{H}^{k,k}(X) \cap \mathrm{H}^{2k}(X, \mathbf{Q})$. Thus using the Hodge diamond of X (recorded in Lemma 5.3) and the assumption that X is not Hodge-special if n is even, we find

$$\dim(\mathrm{K}_0(X)_{\mathrm{num}} \otimes \mathbf{Q}) = 2n - 2.$$

Combined with the above, this shows the rank of $\mathrm{K}_0(\mathcal{A}_X)_{\mathrm{num}}$ is 2. Since $\mathrm{K}_0(\mathcal{A}_X)_{\mathrm{num}}$ is torsion free, we conclude $\mathrm{K}_0(\mathcal{A}_X)_{\mathrm{num}} \cong \mathbf{Z}^2$. □

Remark 5.14. Let X be a GM variety of dimension $n = 4$ or 6. The proof of the proposition shows that

$$\mathrm{rank}(\mathrm{K}_0(\mathcal{A}_X)_{\mathrm{num}}) = \dim_{\mathbf{Q}} \mathrm{H}^{\frac{n}{2},\frac{n}{2}}(X, \mathbf{Q})$$

if the Hodge conjecture holds for X. Since the Hodge conjecture holds for any uniruled smooth projective fourfold [5], for $n = 4$ the above equality is unconditional.

Lemma 5.15. *Let X be a GM variety as in Lemma 5.13. Then in a suitable basis, the Euler form on $\mathrm{K}_0(\mathcal{A}_X)_{\mathrm{num}} = \mathbf{Z}^2$ is given by*

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \text{ if } n = 3, \qquad \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \text{ if } n = 4.$$

Proof. For $n = 3$, this is shown in the proof of [23, Proposition 3.9].

For $n = 4$, we sketch the proof. First, note that any GM variety contains a line, since by taking hyperplane sections we reduce to the case of dimension 3, where the result is well-known. Let $P \in X$ be a point, $L \subset X$ a line, Σ be the zero locus of a regular section of $\mathcal{U}_X^{\vee}$, S a complete intersection of two hyperplanes in X, and H a hyperplane section of X. The key claim is that

$$\mathrm{K}_0(X)_{\mathrm{num}} = \mathbf{Z}\langle[\mathcal{O}_P], [\mathcal{O}_L], [\mathcal{O}_\Sigma], [\mathcal{O}_S], [\mathcal{O}_H], [\mathcal{O}_X]\rangle, \tag{5.3}$$

i.e. the structure sheaves of these subvarieties give an integral basis of $\mathrm{K}_0(X)_{\mathrm{num}}$. Once this is known, as in the proof of [23, Proposition 3.9], the lemma reduces to a (tedious) computation, which we omit.

Using [23, Remark 5.9] it is easy to see X is AK-compatible in the sense of [23, Definition 5.1], hence to prove the claim it is enough to show that

$$\mathrm{CH}(X)_{\mathrm{num}} = \mathbf{Z}\langle[P], [L], [\Sigma], [S], [H], 1\rangle.$$

Clearly, this is equivalent to $\mathrm{CH}^2(X)_{\mathrm{num}} = \mathbf{Z}\langle[\Sigma], [S]\rangle$. But $\mathrm{CH}^2(X)_{\mathrm{num}}$ coincides with the group $\mathrm{CH}^2(X)_{\mathrm{hom}} \subset \mathrm{H}^4(X, \mathbf{Z})$ of 2-cycles modulo homological equivalence (see the proof of Lemma 5.13), and $\mathbf{Z}\langle[\Sigma], [S]\rangle$ is the image of the inclusion $\mathrm{H}^4(\mathbf{G}, \mathbf{Z}) \hookrightarrow \mathrm{CH}^2(X)_{\mathrm{hom}}$. Hence it suffices to show the cokernel of this inclusion is torsion free. Even better, the cokernel of

$$\mathrm{H}^4(\mathbf{G}, \mathbf{Z}) \hookrightarrow \mathrm{H}^4(X, \mathbf{Z})$$

is torsion free. Indeed, we may assume X is ordinary, and then the statement holds by the proof of the Lefschetz hyperplane theorem, see [32, Example 3.1.18]. $\square$

Remark 5.16. The duality conjecture (Conjecture 4.8) implies that if X is as in Lemma 5.13, then for $n = 5$ or 6 the lattice $\mathrm{K}_0(\mathcal{A}_X)_{\mathrm{num}} = \mathbf{Z}^2$ is isomorphic to the lattice described in Lemma 5.15 for $n = 3$ or 4, respectively.

5.4 Applications

Now we gather some consequences of the above computations. First, we consider the question of when $\mathcal{A}_X$ can be equivalent to the derived category of a variety.

Proposition 5.17. *Let X be a GM variety of dimension n.*

(1) If n is even and S is a variety such that $\mathcal{A}_X \simeq \mathrm{D^b}(S)$, then S is a K3 surface.

(2) If n is odd, then $\mathcal{A}_X$ is not equivalent to the derived category of any variety.

Proof. Suppose S is a variety such that $\mathcal{A}_X \simeq \mathrm{D^b}(S)$. Then S is smooth by [21, Lemma D.22], and proper by [48, Proposition 3.30]. In particular, $\mathrm{D^b}(S)$ has a Serre functor given by

$$\mathrm{S}_{\mathrm{D^b}(S)}(\mathcal{F}) = \mathcal{F} \otimes \omega_S[\dim(S)],$$

which is unique up to isomorphism. Thus by Proposition 4.5, S is a surface with trivial (if n is even) or 2-torsion (if n is odd) canonical class. Hence S is a K3, Enriques, abelian, or bielliptic surface. Using the HKR isomorphism and the Hodge diamonds of such surfaces, we find

$$\mathrm{HH}_\bullet(S) = \begin{cases} \mathbf{k}[2] \oplus \mathbf{k}^{22} \oplus \mathbf{k}[-2] & \text{if } S \text{ is K3,} \\ \mathbf{k}^{12}[0] & \text{if } S \text{ is Enriques,} \\ \mathbf{k}[2] \oplus \mathbf{k}^4[1] \oplus \mathbf{k}^6 \oplus \mathbf{k}^4[-1] \oplus \mathbf{k}[-2] & \text{if } S \text{ is abelian,} \\ \mathbf{k}^2[1] \oplus \mathbf{k}^4 \oplus \mathbf{k}^2[-1] & \text{if } S \text{ is bielliptic.} \end{cases}$$

Now the lemma follows by comparing with $\mathrm{HH}_\bullet(\mathcal{A}_X)$ as given by Lemma 5.3. □

In view of Conjecture 4.13, it is interesting to determine in which cases a GM category $\mathcal{A}_X$ (and hence also $\mathrm{D^b}(X)$) admits a semiorthogonal decomposition with components of geometric dimension at most $\dim(X) - 2$. For fivefolds and sixfolds, the duality conjecture (Conjecture 4.8) implies this always holds. For threefolds, however, we show this never holds:

Lemma 5.18. *Let X be a GM threefold. Then $\mathcal{A}_X$ does not admit a semiorthogonal decomposition with all components of geometric dimension at most* 1.

Proof. It is easy to see that any category of geometric dimension 0 is equivalent to the derived category of a point. Further, by [47] any category of geometric dimension 1 is equivalent to the derived category of a curve. Note that $\mathrm{HH}_\bullet(\mathrm{Spec}(\mathbf{k})) = \mathbf{k}[0]$, and if C is a curve of genus

g then

$$\mathrm{HH}_\bullet(C) = \mathbf{k}^g[1] \oplus \mathbf{k}^2[0] \oplus \mathbf{k}^g[-1].$$

Thus if $\mathcal{A}_X$ has a semiorthogonal decomposition with all components of geometric dimension at most 1, Lemma 5.3 and Theorem 5.2 imply $\mathcal{A}_X \simeq \mathrm{D^b}(C)$ for a genus 10 curve C. This cannot happen by Proposition 5.17. □

The following result fits well with Conjecture 4.14 and the expectation that a very general GM fourfold is irrational.

Proposition 5.19. *If X is a GM fourfold or sixfold and there is a K3 surface S such that $\mathcal{A}_X \simeq \mathrm{D^b}(S)$, then X is Hodge-special. In particular, there is no such S for a very general X.*

Proof. If $\mathcal{A}_X \simeq \mathrm{D^b}(S)$, then $\mathrm{K}_0(\mathcal{A}_X)_{\mathrm{num}} \cong \mathrm{K}_0(S)_{\mathrm{num}}$. But on a projective surface powers of the hyperplane class give 3 independent elements in $\mathrm{CH}(S)_{\mathrm{num}} \otimes \mathbf{Q} \cong \mathrm{K}_0(S)_{\mathrm{num}} \otimes \mathbf{Q}$. Hence by Lemma 5.13, X is Hodge-special. The last claim of the proposition follows from Lemma 5.12. □

Next we discuss the indecomposability of GM categories. Recall that a triangulated category $\mathcal{T}$ is called *indecomposable* if it admits no nontrivial semiorthogonal decompositions, i.e. if $\mathcal{T} = \langle \mathcal{A}_1, \mathcal{A}_2 \rangle$ implies either $\mathcal{A}_1 \simeq 0$ or $\mathcal{A}_2 \simeq 0$. In general, there are very few techniques for proving indecomposability of a triangulated category. However, for Calabi–Yau categories, we recall a simple criterion below. First, if $\mathcal{A}$ is an admissible subcategory of the derived category of a smooth projective variety, we say $\mathcal{A}$ is *connected* if $\mathrm{HH}^0(\mathcal{A}) = \mathbf{k}$. This definition is motivated by the fact that a smooth projective variety X is connected if and only if

$$\mathrm{HH}^0(X) = \mathrm{H}^0(X, \mathcal{O}_X) = \mathbf{k}.$$

By Corollary 5.6 and Proposition 5.7, all GM categories are connected.

Lemma 5.20 ([27, Lemma 2.9, Proposition 5.1])**.** *Let $\mathcal{A}$ be a connected admissible subcategory of the derived category of a smooth projective variety. Then $\mathcal{A}$ admits no nontrivial completely* *orthogonal decompositions. If furthermore $\mathcal{A}$ is Calabi–Yau, then $\mathcal{A}$ is indecomposable.*

Proposition 5.21. *Let X be a GM variety of dimension n.*

(1) If n is even, then $\mathcal{A}_X$ is indecomposable.

(2) If n is odd, then $\mathcal{A}_X$ admits no nontrivial completely orthogonal decompositions.

Proof. This follows from Lemma 5.20, the connectivity of $\mathcal{A}_X$, and the fact that $\mathcal{A}_X$ is Calabi–Yau if n is even. □

Remark 5.22. It is plausible that $\mathcal{A}_X$ is indecomposable if X is an odd-dimensional GM variety, but we do not know how to prove this.

Chapter 6

Fourfold-to-surface duality

In this chapter, we prove Conjecture 4.8 for ordinary fourfolds with a generalized dual surface corresponding to a non-Plücker quadric point. More precisely, recall that for any GM fourfold X and quadric point $\mathbf{q} \in \mathbf{P}(V_6(X))$, we associated in Section 4.2 a generalized dual variety $X^\vee_{\mathbf{q}}$, which is an ordinary GM surface if $\mathbf{q} \in \mathsf{Y}^3_{\mathsf{A}(X)}$.

Theorem 6.1. *Let X be an ordinary GM fourfold such that*

$$\mathsf{Y}^3_{\mathsf{A}(X)} \cap (\mathbf{P}(V_6(X)) \setminus \mathbf{P}(V_5(X))) \neq \varnothing.$$

Then for any point $\mathbf{q} \in \mathsf{Y}^3_{\mathsf{A}(X)} \cap (\mathbf{P}(V_6(X)) \setminus \mathbf{P}(V_5(X)))$, there is an equivalence

$$\mathcal{A}_X \simeq \mathrm{D}^{\mathrm{b}}(X^\vee_{\mathbf{q}}).$$

See Lemma 6.4 below for other characterizations of such fourfolds.

Remark 6.2. With suitable modifications, the arguments in this chapter can be used to prove Theorem 6.1 also holds for special GM fourfolds.

Remark 6.3. GM fourfolds X as in the theorem form a 23-dimensional (codimension 1 in moduli) family. This can be seen using Theorem 3.6. Indeed, by [45, Proposition 2.2] Lagrangian subspaces $\mathsf{A} \subset \wedge^3 V_6$ with no decomposable vectors such that $\mathsf{Y}^3_{\mathsf{A}} \neq \varnothing$ form a divisor in the moduli space of all A, and hence form a 19-dimensional family. Having fixed such an A there are finitely many $\mathbf{q} \in \mathsf{Y}^3_{\mathsf{A}}$, and in order for $\mathbf{q} \in \mathbf{P}(V_6(X)) \setminus \mathbf{P}(V_5(X))$ the Plücker point $\mathbf{p}$ of X can be any point of $\mathsf{Y}^1_{\mathsf{A}^\perp}$ such that $(\mathbf{q}, \mathbf{p})$ is not on the incidence divisor. The first condition on $\mathbf{p}$ is closed and gives a 4-dimensional space of choices, while the second condition is open and does not change the dimension.

Recall from Section 3.1 that if X is an ordinary GM fourfold, there is a (canonical) hyperplane $W \subset \wedge^2 V_5(X)$ and a (non-canonical) quadric $Q \subset \mathbf{P}(W)$ such that $X = \mathbf{G} \cap Q$. The fourfolds satisfying the assumption of Theorem 6.1 admit several different characterizations.

Lemma 6.4. *Let X be an ordinary GM fourfold. The following are equivalent:*

(1) $\mathsf{Y}^3_{\mathsf{A}(X)} \cap (\mathbf{P}(V_6(X)) \setminus \mathbf{P}(V_5(X))) \neq \varnothing$.

(2) There is a rank 6 quadric $Q \subset \mathbf{P}(W)$ such that $X = \mathbf{G} \cap Q$.

(3) X contains a quintic del Pezzo surface, i.e. a smooth codimension 4 linear section of the Grassmannian $\mathbf{G} \subset \mathbf{P}(\wedge^2 V_5(X))$.

Proof. The equivalence of (1) and (2) follows from Proposition 3.10 since $\dim W = 9$. Note that since $\mathsf{Y}^4_{\mathsf{A}(X)} = \varnothing$, the same proposition also shows that if $X = \mathbf{G} \cap Q$ then $\operatorname{rank}(Q) \geq 6$.

We show (2) is equivalent to (3). First assume (2) holds. Then a maximal isotropic space $I \subset W$ for Q has dimension 6, so $\mathbf{G} \cap \mathbf{P}(I)$ is a quintic del Pezzo contained in X, provided this intersection is transverse. By the argument of [7, Lemma 4.1] (or by Lemma 6.6 below), this is true for a general I.

Conversely, assume (1) holds, i.e. assume there is a 6-dimensional subspace $I \subset W$ such that $S = \mathbf{G} \cap \mathbf{P}(I) \subset X$ is a quintic del Pezzo. The restriction map $V_6(X) \to \mathrm{H}^0(\mathcal{I}_{S/\mathbf{P}(I)}(2))$ from quadrics containing X to those containing S is surjective with one-dimensional kernel. If $Q \subset \mathbf{P}(W)$ is the quadric corresponding to this kernel, then $X = \mathbf{G} \cap Q$ and $\mathbf{P}(I) \subset Q$. It follows that $\operatorname{rank}(Q) \leq 6$. But as we noted above, the reverse inequality also holds. $\square$

For the rest of the chapter, we fix an ordinary GM fourfold X satisfying the equivalent conditions of Lemma 6.4 and a point $\mathbf{q} \in \mathsf{Y}^3_{\mathsf{A}(X)} \cap (\mathbf{P}(V_6(X)) \setminus \mathbf{P}(V_5(X)))$. Further, to ease notation, we set

$$Y = X_{\mathbf{q}}^{\vee}.$$

6.1 Setup and outline of the proof

The proof of Theorem 6.1 takes the rest of this chapter. We outline our approach here.

The starting point for our proof is the following explicit geometric relation between X and Y. By Proposition 3.10, the point $\mathbf{q}$ corresponds to a rank 6 quadric Q cutting out X, and the Plücker point $\mathbf{p}(X) \in \mathbf{P}(V_6(X)^{\vee}) \cong \mathbf{P}(V_6(Y))$ of X corresponds to a quadric Q' cutting out Y. Because X and Y are ordinary, we may regard Q as a subvariety of $\mathbf{P}(\wedge^2 V_5(X))$ and Q' as a subvariety of $\mathbf{P}(\wedge^2 V_5(Y))$. Then [7, Proposition 3.26] (which is stated for dual varieties but works just as well for generalized duals) says that there is an isomorphism $V_5(X) \cong V_5(Y)^{\vee}$ identifying $Q' \subset \mathbf{P}(\wedge^2 V_5(Y))$ with the projective dual to $Q \subset \mathbf{P}(\wedge^2 V_5(X))$. Hence, fixing $V_5 = V_5(X)$, our setup is as follows: there is a hyperplane $W \subset \wedge^2 V_5$ and a rank 6 quadric $Q \subset \mathbf{P}(W)$ such that

$$X = \mathbf{G} \cap Q \qquad \text{and} \qquad Y = \mathbf{G}^{\vee} \cap Q^{\vee},$$

where $Q^{\vee} \subset \mathbf{P}(\wedge^2 V_5^{\vee})$ is the projectively dual quadric to $Q \subset \mathbf{P}(\wedge^2 V_5)$, and

$$\mathbf{G}^{\vee} = \mathrm{Gr}(2, V_5^{\vee}) \subset \mathbf{P}(\wedge^2 V_5^{\vee})$$

is the dual Grassmannian.

From this starting point, the main steps of the proof are as follows. First, by considering families of maximal linear subspaces of Q and $Q^{\vee}$, we find $\mathbf{P}^1$-bundles $\widehat{X} \to X$ and $\widehat{Y} \to Y$, together with morphisms $\widehat{X} \to \mathbf{P}^3$ and $\widehat{Y} \to \mathbf{P}^3$ realizing $\widehat{X}$ and $\widehat{Y}$ as families of mutually orthogonal linear sections of $\mathbf{G}$ and $\mathbf{G}^{\vee}$. This allows us to apply homological projective duality to obtain a semiorthogonal decomposition of $\mathrm{D^b}(\widehat{X})$ with $\mathrm{D^b}(\widehat{Y})$ as a component. By comparing this (via mutation functors) with the decomposition of $\mathrm{D^b}(\widehat{X})$ coming from its

$\mathbf{P}^1$-bundle structure over X, we show $\mathrm{D^b}(\widehat{Y})$ has a decomposition into two copies of $\mathcal{A}_X$. On the other hand, as $\widehat{Y} \to Y$ is a $\mathbf{P}^1$-bundle, $\mathrm{D^b}(\widehat{Y})$ also decomposes into two copies of $\mathrm{D^b}(Y)$. We show these two decompositions of $\mathrm{D^b}(\widehat{Y})$ coincide, and hence $\mathcal{A}_X \simeq \mathrm{D^b}(Y)$. Our proof gives an explicit functor inducing this equivalence, see (6.13).

6.2 Maximal linear subspaces of the quadrics

We start by discussing a geometric relation between Q and $Q^\vee$. Let $K \subset W$ be the kernel of Q, regarded as a symmetric linear map $W \to W^\vee$. Since $\dim W = 9$ and $\mathrm{rank}(Q) = 6$, we have $\dim K = 3$. The filtration

$$0 \subset K \subset W \subset \wedge^2 V_5$$

induces a filtration

$$0 \subset W^\perp \subset K^\perp \subset \wedge^2 V_5^\vee$$

where $K^\perp$ and $W^\perp$ are the annihilators of K and W, so that $\dim K^\perp = 7$ and $\dim W^\perp = 1$. The pairing between the dual spaces induces a nondegenerate pairing between W/K and $K^\perp/W^\perp$, and hence an isomorphism

$$K^\perp/W^\perp \cong (W/K)^\vee.$$

The quadric Q induces a smooth quadric $\bar{Q}$ in the 5-dimensional projective space $\mathbf{P}(W/K)$. The quadric $\bar{Q}$ can be identified with the Grassmannian $\mathrm{Gr}(2,4)$; more precisely, fixing an isomorphism

$$W/K \cong \wedge^2 S$$

for a 4-dimensional vector space S, we have an identification

$$\bar{Q} = \mathrm{Gr}(2, S) \subset \mathbf{P}(\wedge^2 S).$$

The projective dual of $\bar{Q}$ is then the dual Grassmannian

$$\bar{Q}^\vee = \mathrm{Gr}(2, S^\vee) \subset \mathbf{P}(\wedge^2 S^\vee) = \mathbf{P}((W/K)^\vee) = \mathbf{P}(K^\perp/W^\perp).$$

It follows that the projective dual of the quadric

$$Q = \mathrm{Cone}_{\mathbf{P}(K)} \bar{Q} \subset \mathbf{P}(\wedge^2 V_5) \tag{6.1}$$

is the quadric

$$Q^\vee = \mathrm{Cone}_{\mathbf{P}(W^\perp)} \bar{Q}^\vee \subset \mathbf{P}(\wedge^2 V_5^\vee). \tag{6.2}$$

Projective 3-space $\mathbf{P}(S)$ is (a connected component of) the space of maximal linear subspaces of the quadric $\bar{Q} = \mathrm{Gr}(2, S)$. The universal family is the flag variety $\mathrm{Fl}(1, 2; S) \to \mathbf{P}(S)$, with fiber over a point $s \in \mathbf{P}(S)$ the plane $\mathbf{P}(s \wedge S) \subset \mathbf{P}(\wedge^2 S)$. Analogously, the same flag variety $\mathrm{Fl}(2, 3; S^\vee) \cong \mathrm{Fl}(1, 2; S)$ is (a connected component of) the space of maximal linear subspaces of $\bar{Q}^\vee = \mathbf{G}(2, S^\vee)$, this time with fiber over a point $s \in \mathbf{P}(S)$ being the plane $\mathbf{P}(\wedge^2 s^\perp) \subset \mathbf{P}(\wedge^2 S^\vee)$. Note that the fibers of these two correspondences over a point $s \in \mathbf{P}(S)$ are mutually orthogonal with respect to the pairing between $\wedge^2 S$ and $\wedge^2 S^\vee$. We summarize this discussion by the diagram

$$\begin{tikzcd}
& \mathrm{Fl}(1,2;S) \arrow[dl, "p_{\bar{Q}}"'] \arrow[dr, "\pi_{\bar{Q}}"] & & \mathrm{Fl}(2,3;S^\vee) \arrow[dl, "\pi_{\bar{Q}^\vee}"'] \arrow[dr, "p_{\bar{Q}^\vee}"] & \\
\bar{Q} & & \mathbf{P}(S) & & \bar{Q}^\vee
\end{tikzcd} \tag{6.3}$$

with the inner arrows being $\mathbf{P}^2$-bundles with mutually orthogonal fibers, and the outer arrows being $\mathbf{P}^1$-bundles.

By (6.1) every maximal isotropic subspace of $\bar{Q}$ gives a maximal isotropic subspace of Q by taking its preimage under the projection $W \to W/K = \wedge^2 S$. Analogously, by (6.2) every maximal isotropic subspace of $\bar{Q}^\vee$ gives a maximal isotropic subspace of $Q^\vee$ by taking its preimage under the projection $K^\perp \to K^\perp/W^\perp = \wedge^2 S^\vee$. Note that for the pairing between

W and $K^\perp$ induced by the pairing between $\wedge^2 V_5$ and $\wedge^2 V_5^\vee$, the subspace $K \subset W$ is the left kernel, and the subspace $W^\perp \subset K^\perp$ is the right kernel. Hence any $s \in \mathbf{P}(S)$ gives mutually orthogonal maximal isotropic spaces $\mathcal{I}_s$ and $\mathcal{I}_s^\perp$ of Q and $Q^\vee$ respectively. These spaces form the fibers of vector bundles $\mathcal{I}$ and $\mathcal{I}^\perp$ over $\mathbf{P}(S)$ of ranks 6 and 4, which are orthogonal subbundles of $\wedge^2 V_5 \otimes \mathcal{O}_{\mathbf{P}(S)}$ and $\wedge^2 V_5^\vee \otimes \mathcal{O}_{\mathbf{P}(S)}$. We can summarize this discussion by the following diagram

$$\begin{tikzcd} & \mathbf{P}_{\mathbf{P}(S)}(\mathcal{I}) \arrow[dl, "p_Q"'] \arrow[dr, "\pi_Q"] & & \mathbf{P}_{\mathbf{P}(S)}(\mathcal{I}^\perp) \arrow[dl, "\pi_{Q^\vee}"'] \arrow[dr, "p_{Q^\vee}"] & \\ Q & & \mathbf{P}(S) & & Q^\vee \end{tikzcd} \tag{6.4}$$

Here the inner arrows are $\mathbf{P}^5$- and $\mathbf{P}^3$-bundles with mutually orthogonal fibers, and the outer arrows are $\mathbf{P}^1$-bundles (induced by the $\mathbf{P}^1$-bundles of diagram (6.3)) away from the vertices $\mathbf{P}(K)$ and $\mathbf{P}(W^\perp)$ of the quadrics (over which the fibers are isomorphic to $\mathbf{P}(S)$).

6.3 Families of linear sections of the Grassmannians

Now define

$$\widehat{X} := \mathbf{G} \times_{\mathbf{P}(\wedge^2 V_5)} \mathbf{P}_{\mathbf{P}(S)}(\mathcal{I}) \qquad \text{and} \qquad \widehat{Y} := \mathbf{G}^\vee \times_{\mathbf{P}(\wedge^2 V_5^\vee)} \mathbf{P}_{\mathbf{P}(S)}(\mathcal{I}^\perp) \tag{6.5}$$

to be the induced families of linear sections of $\mathbf{G}$ and $\mathbf{G}^\vee$. They fit into a diagram

$$\begin{tikzcd} & \widehat{X} \arrow[dl, "p_X"'] \arrow[dr, "\pi_X"] & & \widehat{Y} \arrow[dl, "\pi_Y"'] \arrow[dr, "p_Y"] & \\ X & & \mathbf{P}(S) & & Y \end{tikzcd} \tag{6.6}$$

with the maps induced by those in (6.4) (remember that $X = \mathbf{G} \cap \mathbf{Q}$ and $Y = \mathbf{G}^\vee \cap Q^\vee$).

We will denote by H, H', and h the ample generators of $\mathrm{Pic}(\mathbf{G})$, $\mathrm{Pic}(\mathbf{G}^\vee)$, and $\mathbf{P}(S)$.

Lemma 6.5. *There are rank 2 vector bundles $\mathcal{S}_X$ and $\mathcal{S}_Y$ on X and Y and isomorphisms*

$$\widehat{X} \cong \mathbf{P}_X(\mathcal{S}_X) \qquad \textit{and} \qquad \widehat{Y} \cong \mathbf{P}_Y(\mathcal{S}_Y),$$

such that $\mathcal{O}_{\mathbf{P}_X(\mathcal{S}_X)}(1) = \pi_X^*\mathcal{O}_{\mathbf{P}(S)}(h)$ and $\mathcal{O}_{\mathbf{P}_Y(\mathcal{S}_Y)}(1) = \pi_Y^*\mathcal{O}_{\mathbf{P}(S)}(h)$. In particular, $\widehat{X}$ is a smooth fivefold, $\widehat{Y}$ is a smooth threefold, and

$$K_{\widehat{X}} = -H - 2h \qquad \textit{and} \qquad K_{\widehat{Y}} = H' - 2h. \tag{6.7}$$

Proof. Since X and Y are smooth, they do not intersect the vertices $\mathbf{P}(K)$ and $\mathbf{P}(W^\perp)$ of the quadrics Q and $Q^\vee$, hence the maps p_X and p_Y are $\mathbf{P}^1$-fibrations induced by those in diagram (6.3). In other words, we have pullback squares

$$\begin{tikzcd} \widehat{X} \arrow[r] \arrow[d, "p_X"'] & \mathrm{Fl}(1,2;S) \arrow[d, "p_{\bar{Q}}"] \\ X \arrow[r] & \bar{Q} \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} \widehat{Y} \arrow[r] \arrow[d, "p_Y"'] & \mathrm{Fl}(2,3;S^\vee) \arrow[d, "p_{\bar{Q}^\vee}"] \\ Y \arrow[r] & \bar{Q}^\vee \end{tikzcd} .$$

The map $p_{\bar{Q}}$ is the projectivization of the tautological subbundle of $S \otimes \mathcal{O}$ on $\bar{Q} = \mathrm{Gr}(2, S)$, and $p_{\bar{Q}^\vee}$ is the projectivization of the annihilator of the tautological subbundle of $S^\vee \otimes \mathcal{O}$ on $\bar{Q}^\vee = \mathrm{Gr}(2, S^\vee)$. So we can take $\mathcal{S}_X$ and $\mathcal{S}_Y$ to be the pullbacks to X and Y of these bundles.

To compute the canonical classes, note that the determinant of the tautological bundle (and of its annihilator) on $\mathrm{Gr}(2, S)$ is $\mathcal{O}_{\mathrm{Gr}(2,S)}(-1)$, hence $\mathrm{c}_1(\mathcal{S}_X) = -H$ and $\mathrm{c}_1(\mathcal{S}_Y) = -H'$. Now apply the standard formula for the canonical bundle of the projectivization of a vector bundle, taking into account that $K_X = -2H$ and $K_Y = 0$ by (3.2). $\square$

Lemma 6.6. *The map $\pi_X \colon \widehat{X} \to \mathbf{P}(S)$ is flat with general fiber a smooth quintic del Pezzo surface. The map $\pi_Y \colon \widehat{Y} \to \mathbf{P}(S)$ is generically finite of degree* 5.

Proof. The fiber of π_X over a point $s \in \mathbf{P}(S)$ is the intersection $\mathbf{G} \cap \mathbf{P}(\mathcal{I}_s)$, where the subspace $\mathbf{P}(\mathcal{I}_s) \subset \mathbf{P}(\wedge^2 V_5)$ has codimension 4. Thus the dimension of $\pi_X^{-1}(s)$ is at least 2. If the dimension were greater than 2, this fiber would give a divisor in X of degree at most 5, but by

Theorem 3.1 every divisor in X has degree divisible by 10. Thus every fiber is a dimensionally transverse intersection, and flatness follows.

Furthermore, since $\widehat{X}$ is smooth, the general fiber of π_X is a smooth quintic del Pezzo surface. Then by [7, Proposition 2.22] the general fiber of π_Y is a dimensionally transverse and smooth linear section of $\mathbf{G}^\vee$ of codimension 6, hence is just 5 reduced points. $\square$

As a byproduct of the above, we obtain:

Lemma 6.7. *The variety X is rational.*

Proof. The same argument as in [7, Proposition 4.2] works. Let $\widetilde{X} \subset \widehat{X}$ be the preimage under the map π_X of a general hyperplane $\mathbf{P}^2 \subset \mathbf{P}(S)$. By Lemma 6.6, the general fiber of $\widetilde{X} \to \mathbf{P}^2$ is a smooth quintic del Pezzo surface. Hence by a theorem of Enriques [51], $\widetilde{X}$ is rational over $\mathbf{P}^2$, and so over $\mathbf{k}$ as well. On the other hand, the map $\widetilde{X} \to X$ is birational (in fact, it is a blowup of a quintic del Pezzo surface), so X is rational. $\square$

6.4 Homological projective duality

Homological projective duality (HPD) is a key tool in the proof of Theorem 6.1. Very roughly, HPD relates the derived categories of linear sections of a given variety to those of orthogonal linear sections of an "HPD variety". We refer to [22] for the details of this theory, and to [26] or [53] for an overview. For us, the relevant point is that the dual Grassmannian $\mathbf{G}^\vee$ is HPD to $\mathbf{G}$. We spell out the precise consequence of this that we need below.

Recall that by Lemma 4.1 there is a rectangular Lefschetz decomposition

$$\mathrm{D^b}(\mathbf{G}) = \langle \mathcal{B}, \mathcal{B}(H), \mathcal{B}(2H), \mathcal{B}(3H), \mathcal{B}(4H) \rangle.$$

Let

$$i \colon \mathrm{I}(\mathbf{G}, \mathbf{G}^\vee) \hookrightarrow \mathbf{G} \times \mathbf{G}^\vee$$

be the incidence divisor defined by the canonical section of $\mathcal{O}(H + H')$. Recall that $\mathcal{U}$ denotes the tautological rank 2 bundle on $\mathbf{G}$, and let $\mathcal{V}$ denote the tautological rank 2 bundle on $\mathbf{G}^\vee$. The following was proved in [21, Section 6.1]. See [22, Definition 6.1] for the definition of HPD.

Theorem 6.8. *The Grassmannian $\mathbf{G}^\vee \to \mathbf{P}(\wedge^2 V_5^\vee)$ is HPD to $\mathbf{G} \to \mathbf{P}(\wedge^2 V_5)$ with respect to the above Lefschetz decomposition. Moreover, the duality is implemented by a sheaf $\mathcal{E}$ on $\mathrm{I}(\mathbf{G}, \mathbf{G}^\vee)$ whose pushforward to $\mathbf{G} \times \mathbf{G}^\vee$ fits into an exact sequence*

$$0 \to \mathcal{O}_{\mathbf{G}} \boxtimes \mathcal{V} \to \mathcal{U}^\vee \boxtimes \mathcal{O}_{\mathbf{G}^\vee} \to i_*\mathcal{E} \to 0.$$

Note that the natural map

$$\widehat{X} \times_{\mathbf{P}(S)} \widehat{Y} \to X \times Y \to \mathbf{G} \times \mathbf{G}^\vee$$

factors through $\mathrm{I}(\mathbf{G}, \mathbf{G}^\vee)$. Indeed, the fiber of $\widehat{X} \times_{\mathbf{P}(S)} \widehat{Y}$ over any point $s \in \mathbf{P}(S)$ is

$$(\mathbf{P}(\mathcal{I}_s) \times \mathbf{P}(\mathcal{I}_s^\perp)) \cap (\mathbf{G} \times \mathbf{G}^\vee) \subset \mathrm{I}(\mathbf{G}, \mathbf{G}^\vee).$$

Note also that

$$\dim(\widehat{X} \times_{\mathbf{P}(S)} \widehat{Y}) = 5, \tag{6.8}$$

since by Lemma 6.6 the map $\widehat{X} \times_{\mathbf{P}(S)} \widehat{Y} \to \widehat{Y}$ is flat of relative dimension 2, and $\dim(\widehat{Y}) = 3$ by Lemma 6.5.

Denote by $\widehat{\mathcal{E}}$ the pullback of the HPD object $\mathcal{E}$ to $\widehat{X} \times_{\mathbf{P}(S)} \widehat{Y}$ and by $\widehat{\Phi}\colon \mathrm{D^b}(\widehat{Y}) \to \mathrm{D^b}(\widehat{X})$ the corresponding Fourier–Mukai functor. Note that $\widehat{\Phi}$ is $\mathbf{P}(S)$-linear, i.e.

$$\widehat{\Phi}(\mathcal{F} \otimes \pi_Y^* \mathcal{G}) \cong \widehat{\Phi}(\mathcal{F}) \otimes \pi_X^* \mathcal{G}$$

for all $\mathcal{F} \in \mathrm{D^b}(\widehat{Y})$ and $\mathcal{G} \in \mathrm{D^b}(\mathbf{P}(S))$. By Lemma 6.5 and (6.8), the families $\widehat{X}$ and $\widehat{Y}$ of linear sections of $\mathbf{G}$ and $\mathbf{G}^\vee$ satisfy the dimension assumptions of [22, Theorem 6.27]. Hence we obtain:

Proposition 6.9. *The functor* $\widehat{\Phi}\colon \mathrm{D^b}(\widehat{Y}) \to \mathrm{D^b}(\widehat{X})$ *is fully faithful, and there is a semiorthogonal decomposition*

$$\mathrm{D^b}(\widehat{X}) = \langle \widehat{\Phi}(\mathrm{D^b}(\widehat{Y})), \mathcal{B}_X(H) \boxtimes \mathrm{D^b}(\mathbf{P}(S)) \rangle, \tag{6.9}$$

where $\mathcal{B}_X(H) \boxtimes \mathrm{D^b}(\mathbf{P}(S))$ *denotes the triangulated subcategory generated by objects of the form* $p_X^*(\mathcal{F}) \otimes \pi_X(\mathcal{G})$ *for* $\mathcal{F} \in \mathcal{B}_X(H)$ *and* $\mathcal{G} \in \mathrm{D^b}(\mathbf{P}(S))$.

6.5 Mutations

Since $p_X\colon \widehat{X} \to X$ is a $\mathbf{P}^1$-bundle (Lemma 6.5), we also have a semiorthogonal decomposition

$$\begin{aligned} \mathrm{D^b}(\widehat{X}) &= \langle p_X^* \mathrm{D^b}(X), p_X^* \mathrm{D^b}(X)(h) \rangle \\ &= \langle \mathcal{A}_{\widehat{X}}, \mathcal{B}, \mathcal{B}(H), \mathcal{A}_{\widehat{X}}(h), \mathcal{B}(h), \mathcal{B}(H+h) \rangle. \end{aligned}$$

Here and below, to ease notation we write $\mathcal{A}_{\widehat{X}}$ for $p_X^* \mathcal{A}_X$ and simply $\mathcal{B}$ for $p_X^* \mathcal{B}_X$. We find a sequence of mutations bringing this decomposition into the form of (6.9). In the course of the proof we will use several times $K_X = -2H$, which holds by (3.2), and $K_{\widehat{X}} = -H - 2h$, which holds by (6.7).

Step 1. Mutate $\mathcal{B}(H)$ to the left of $\mathcal{A}_{\widehat{X}}$. Since this is a mutation in $p_X^* \mathrm{D^b}(X)$ and $K_X = -2H$, by Lemma 2.10 we get

$$\mathrm{D^b}(\widehat{X}) = \langle \mathcal{B}(-H), \mathcal{A}_{\widehat{X}}, \mathcal{B}, \mathcal{A}_{\widehat{X}}(h), \mathcal{B}(h), \mathcal{B}(H+h) \rangle.$$

Step 2. Mutate $\mathcal{B}(H+h)$ to the far left. Since $K_{\widehat{X}} = -H - 2h$, by Lemma 2.10 we get

$$\mathrm{D^b}(\widehat{X}) = \langle \mathcal{B}(-h), \mathcal{B}(-H), \mathcal{A}_{\widehat{X}}, \mathcal{B}, \mathcal{A}_{\widehat{X}}(h), \mathcal{B}(h) \rangle.$$

Step 3. Mutate $\mathcal{B}(-H)$ to the left of $\mathcal{B}(-h)$. Since these two subcategories are completely orthogonal (see the lemma below), by Lemma 2.8 we get

$$\mathrm{D^b}(\widehat{X}) = \langle \mathcal{B}(-H), \mathcal{B}(-h), \mathcal{A}_{\widehat{X}}, \mathcal{B}, \mathcal{A}_{\widehat{X}}(h), \mathcal{B}(h) \rangle.$$

Lemma 6.10. *The categories $\mathcal{B}(-H)$ and $\mathcal{B}(-h)$ in $\mathrm{D^b}(\widehat{X})$ are completely orthogonal.*

Proof. By Step 2, the pair $(\mathcal{B}(-h), \mathcal{B}(-H))$ is semiorthogonal. On the other hand, by Serre duality and (6.7), semiorthogonality of $(\mathcal{B}(-H), \mathcal{B}(-h))$ is equivalent to that of $(\mathcal{B}(-h), \mathcal{B}(2h))$, which follows from (6.9) as $(\mathcal{O}(-h), \mathcal{O}(2h))$ is semiorthogonal in $\mathrm{D^b}(\mathbf{P}(S))$. $\square$

Step 4. Mutate $\mathcal{B}(-H)$ to the far right. Again by Lemma 2.10, we get

$$\mathrm{D^b}(\widehat{X}) = \langle \mathcal{B}(-h), \mathcal{A}_{\widehat{X}}, \mathcal{B}, \mathcal{A}_{\widehat{X}}(h), \mathcal{B}(h), \mathcal{B}(2h) \rangle.$$

Step 5. Mutate $\mathcal{A}_{\widehat{X}}$ and $\mathcal{A}_{\widehat{X}}(h)$ to the far left. We get

$$\begin{aligned} \mathrm{D^b}(\widehat{X}) &= \langle \mathrm{L}_{\mathcal{B}(-h)}(\mathcal{A}_{\widehat{X}}), \mathrm{L}_{\langle \mathcal{B}(-h), \mathcal{B} \rangle}(\mathcal{A}_{\widehat{X}}(h)), \mathcal{B}(-h), \mathcal{B}, \mathcal{B}(h), \mathcal{B}(2h) \rangle \\ &= \langle \mathrm{L}_{\mathcal{B}(-h)}(\mathcal{A}_{\widehat{X}}), \mathrm{L}_{\langle \mathcal{B}(-h), \mathcal{B} \rangle}(\mathcal{A}_{\widehat{X}}(h)), \mathcal{B}_X \boxtimes \mathrm{D^b}(\mathbf{P}(S)) \rangle, \end{aligned}$$

where we used the standard decomposition $\mathrm{D^b}(\mathbf{P}(S)) = \langle \mathcal{O}(-h), \mathcal{O}, \mathcal{O}(h), \mathcal{O}(2h) \rangle$.

Step 6. Twist the decomposition by $\mathcal{O}(H)$. We get

$$\mathrm{D^b}(\widehat{X}) = \langle \mathrm{L}_{\mathcal{B}(H-h)}(\mathcal{A}_{\widehat{X}}(H)), \mathrm{L}_{\langle \mathcal{B}(H-h), \mathcal{B}(H) \rangle}(\mathcal{A}_{\widehat{X}}(H+h)), \mathcal{B}_X(H) \boxtimes \mathrm{D^b}(\mathbf{P}(S)) \rangle. \tag{6.10}$$

We used here Lemma 2.9 with $F = (- \otimes \mathcal{O}(H))$ to rewrite the first two components.

Finally, we obtain:

Proposition 6.11. *The functor $\widehat{\Phi}^* \circ (- \otimes \mathcal{O}(H)) \colon \mathrm{D^b}(\widehat{X}) \to \mathrm{D^b}(\widehat{Y})$ induces an equivalence*

$$\langle \mathcal{A}_{\widehat{X}}, \mathcal{A}_{\widehat{X}}(h) \rangle \simeq \mathrm{D^b}(\widehat{Y}),$$

where $\widehat{\Phi}^*\colon \mathrm{D^b}(\widehat{X}) \to \mathrm{D^b}(\widehat{Y})$ *denotes the left adjoint of* $\widehat{\Phi}$.

Proof. Comparing the decompositions (6.10) and (6.9), we see that $\widehat{\Phi}$ induces an equivalence

$$\widehat{\Phi}\colon \mathrm{D^b}(\widehat{Y}) \xrightarrow{\sim} \langle \mathrm{L}_{\mathcal{B}(H-h)}(\mathcal{A}_{\widehat{X}}(H)), \mathrm{L}_{\langle \mathcal{B}(H-h), \mathcal{B}(H)\rangle}(\mathcal{A}_{\widehat{X}}(H+h))\rangle.$$

Therefore its left adjoint $\widehat{\Phi}^*$ gives an inverse equivalence. On the other hand, by semiorthogonality of (6.9) the functor $\widehat{\Phi}^*$ vanishes on $\mathcal{B}(H-h)$ and $\mathcal{B}(H)$, hence its composition with mutation functors through these categories is isomorphic to $\widehat{\Phi}^*$. Thus $\widehat{\Phi}^*$ induces an equivalence between $\langle \mathcal{A}_{\widehat{X}}(H), \mathcal{A}_{\widehat{X}}(H+h)\rangle$ and $\mathrm{D^b}(\widehat{Y})$. This is equivalent to the claim. □

6.6 Proof of the theorem

Since $p_Y\colon \widehat{Y} \to Y$ is a $\mathbf{P}^1$-bundle (Lemma 6.5), we have

$$\mathrm{D^b}(\widehat{Y}) = \langle p_Y^*\mathrm{D^b}(Y), p_Y^*\mathrm{D^b}(Y)(h)\rangle. \tag{6.11}$$

We aim to prove that this semiorthgonal decomposition coincides with the one obtained by applying the fully faithful functor $(-\otimes \mathcal{O}(-h)) \circ \widehat{\Phi}^* \circ (-\otimes \mathcal{O}(H))$ to $\langle \mathcal{A}_{\widehat{X}}, \mathcal{A}_{\widehat{X}}(h)\rangle$. For this, we consider the composition of functors

$$\mathrm{F} \coloneqq p_{Y*} \circ (-\otimes \mathcal{O}(-2h)) \circ \widehat{\Phi}^* \circ (-\otimes \mathcal{O}(H)) \circ p_X^*\colon \mathrm{D^b}(X) \to \mathrm{D^b}(Y). \tag{6.12}$$

Proposition 6.12. *The functor* F *vanishes on the subcategory* $\mathcal{A}_X \subset \mathrm{D^b}(X)$.

Before proving the proposition, let us show how it implies the equivalence $\mathcal{A}_X \simeq \mathrm{D^b}(Y)$.

Proof of Theorem 6.1. We claim that

$$p_{Y*} \circ (-\otimes \mathcal{O}(-h)) \circ \widehat{\Phi}^* \circ (-\otimes \mathcal{O}(H)) \circ p_X^*\colon \mathrm{D^b}(X) \to \mathrm{D^b}(Y) \tag{6.13}$$

induces an equivalence $\mathcal{A}_X \simeq \mathrm{D^b}(Y)$. Note that the functor p_X^* is fully faithful on $\mathcal{A}_X$. So by Proposition 6.11 the functor $(- \otimes \mathcal{O}(-h)) \circ \widehat{\Phi}^* \circ (- \otimes \mathcal{O}(H)) \circ p_X^*$ gives a fully faithful embedding $\mathcal{A}_X \hookrightarrow \mathrm{D^b}(\widehat{Y})$, whose image $\mathcal{A}$ satisfies

$$\mathrm{D^b}(Y) = \langle \mathcal{A}, \mathcal{A}(h) \rangle. \tag{6.14}$$

On the other hand, by Proposition 6.12 the functor p_{Y*} annihilates $\mathcal{A}(-h)$. But the kernel of the functor p_{Y*} is $p_Y^* \mathrm{D^b}(Y)(-h)$, so $\mathcal{A}(-h) \subset p_Y^* \mathrm{D^b}(Y)(-h)$, and thus

$$\mathcal{A} \subset p_Y^* \mathrm{D^b}(Y) \qquad \text{and} \qquad \mathcal{A}(h) \subset p_Y^* \mathrm{D^b}(Y)(h).$$

In view of the decompositions (6.14) and (6.11), we see that equality holds in the above inclusions. Since p_{Y*} induces an equivalence $p_Y^* \mathrm{D^b}(Y) \simeq \mathrm{D^b}(Y)$, this finishes the proof. $\square$

Now we turn to the proof of Proposition 6.12. Let $f_X \colon X \to \mathbf{G}$ and $f_Y \colon Y \to \mathbf{G}^\vee$ be the Gushel maps, and let $p_{XY} \colon \widehat{X} \times_{\mathbf{P}(S)} \widehat{Y} \to X \times Y$ be the natural morphism. Recall from Section 6.4 that the composition

$$\widehat{X} \times_{\mathbf{P}(S)} \widehat{Y} \xrightarrow{p_{XY}} X \times Y \xrightarrow{f_X \times f_Y} \mathbf{G} \times \mathbf{G}^\vee$$

factors through the incidence divisor $\mathrm{I}(\mathbf{G}, \mathbf{G}^\vee)$. Hence there is a commutative diagram

$$\begin{tikzcd}
\widehat{X} \times_{\mathbf{P}(S)} \widehat{Y} \arrow[r, "p"] \arrow[dr, "\widehat{g}"'] & \mathrm{I}(X,Y) \arrow[r, "j"] \arrow[d, "g"] & X \times Y \arrow[d, "f_X \times f_Y"] \\
& \mathrm{I}(\mathbf{G}, \mathbf{G}^\vee) \arrow[r, "i"] & \mathbf{G} \times \mathbf{G}^\vee
\end{tikzcd} \tag{6.15}$$

where $\mathrm{I}(X,Y)$ is by definition the pullback of $\mathrm{I}(\mathbf{G}, \mathbf{G}^\vee)$ along $f_X \times f_Y$, and $p_{XY} = j \circ p$. We will need the following two lemmas.

Lemma 6.13. *There is an isomorphism* $p_* \mathcal{O}_{\widehat{X} \times_{\mathbf{P}(S)} \widehat{Y}} \cong \mathcal{O}_{\mathrm{I}(X,Y)}$.

Proof. We have a diagram

$$\begin{tikzcd}
\widehat{X} \times_{\mathbf{P}(S)} \widehat{Y} \arrow[r, "\widehat{\Delta}"] \arrow[d] & \widehat{X} \times \widehat{Y} \arrow[r, "p_X \times p_Y"] \arrow[d, "\pi_X \times \pi_Y"] & X \times Y \\
\mathbf{P}(S) \arrow[r, "\Delta"] & \mathbf{P}(S) \times \mathbf{P}(S) &
\end{tikzcd}$$

where the square is cartesian, and also Tor-independent as the fiber product has expected dimension by (6.8). To prove the lemma, we must show $(p_X \times p_Y)_*(\widehat{\Delta}_* \mathcal{O}_{\widehat{X} \times_{\mathbf{P}(S)} \widehat{Y}}) \cong \mathcal{O}_{\mathrm{I}(X,Y)}$. By Tor-independence, we have an isomorphism

$$\widehat{\Delta}_* \mathcal{O}_{\widehat{X} \times_{\mathbf{P}(S)} \widehat{Y}} \cong (\pi_X \times \pi_Y)^* \Delta_* \mathcal{O}_{\mathbf{P}(S)}.$$

Pulling back the standard resolution of the diagonal on $\mathbf{P}(S) \times \mathbf{P}(S)$, we obtain an exact sequence

$$\begin{aligned} 0 \to \pi_X^* \mathcal{O}_{\mathbf{P}(S)}(-3h) \boxtimes \pi_Y^* \Omega^3_{\mathbf{P}(S)}(3h) \to \pi_X^* \mathcal{O}_{\mathbf{P}(S)}(-2h) \boxtimes \pi_Y^* \Omega^2_{\mathbf{P}(S)}(2h) \to \\ \to \pi_X^* \mathcal{O}_{\mathbf{P}(S)}(-h) \boxtimes \pi_Y^* \Omega^1_{\mathbf{P}(S)}(h) \to \mathcal{O}_{\widehat{X} \times \widehat{Y}} \to \widehat{\Delta}_* \mathcal{O}_{\widehat{X} \times_{\mathbf{P}(S)} \widehat{Y}} \to 0 \end{aligned}$$

on $\widehat{X} \times \widehat{Y}$. Using the identifications $p_X \colon \widehat{X} = \mathbf{P}_X(\mathcal{S}_X) \to X$ and $p_Y \colon \widehat{Y} = \mathbf{P}_Y(\mathcal{S}_Y) \to Y$ of Lemma 6.5, it is easy to compute:

$$\begin{aligned} p_{Y*} \pi_Y^* \Omega^3_{\mathbf{P}(S)}(3h) &= p_{Y*} \pi_Y^* \mathcal{O}(-h) = 0, \\ p_{X*} \pi_X^* \mathcal{O}_{\mathbf{P}(S)}(-2h) &= \det(\mathcal{S}_X)[-1] = \mathcal{O}_X(-H)[-1], \\ p_{Y*} \pi_Y^* \Omega^2_{\mathbf{P}(S)}(2h) &= p_{Y*} \pi_Y^* \mathrm{T}_{\mathbf{P}(S)}(-2h) = \det(\mathcal{S}_Y) = \mathcal{O}_Y(-H'), \\ p_{X*} \pi_X^* \mathcal{O}_{\mathbf{P}(S)}(-h) &= 0, \\ (p_X \times p_Y)_*(\mathcal{O}_{\widehat{X} \times \widehat{Y}}) &= \mathcal{O}_{X \times Y}. \end{aligned}$$

It follows that in the spectral sequence computing $(p_X \times p_Y)_*(\widehat{\Delta}_* \mathcal{O}_{\widehat{X} \times_{\mathbf{P}(S)} \widehat{Y}})$ from the above resolution, the only nontrivial terms are

$$\mathrm{R}^1(p_X \times p_Y)_*(\pi_X^*\mathcal{O}(-2h) \boxtimes \pi_Y^*\Omega^2_{\mathbf{P}(S)}(2h)) = \mathcal{O}_{X\times Y}(-H-H'),$$
$$\mathrm{R}^0(p_X \times p_Y)_*(\mathcal{O}_{\widehat{X}\times\widehat{Y}}) = \mathcal{O}_{X\times Y},$$

and we get an exact sequence

$$0 \to \mathcal{O}_{X\times Y}(-H-H') \to \mathcal{O}_{X\times Y} \to (p_X\times p_Y)_*(\widehat{\Delta}_*\mathcal{O}_{\widehat{X}\times_{\mathbf{P}(S)}\widehat{Y}}) \to 0,$$

which gives the required isomorphism $(p_X\times p_Y)_*(\widehat{\Delta}_*\mathcal{O}_{\widehat{X}\times_{\mathbf{P}(S)}\widehat{Y}}) \cong \mathcal{O}_{\mathrm{I}(X,Y)}$. □

Lemma 6.14. *The functor* $\mathrm{F}[-2]$ *is given by a Fourier–Mukai kernel* $\mathcal{K} \in \mathrm{D^b}(X\times Y)$*, which fits into a distinguished triangle*

$$\mathcal{U}_X(-H)\boxtimes\mathcal{O}_Y(-H') \to \mathcal{O}_X(-H)\boxtimes\mathcal{V}_Y^\vee(-H') \to \mathcal{K}.$$

Proof. The main term in the definition (6.12) of F is the left adjoint $\widehat{\Phi}^*$ of $\widehat{\Phi}$. By definition $\widehat{\Phi}$ is given by the Fourier–Mukai kernel $\widehat{\mathcal{E}} \in \mathrm{D^b}(\widehat{X}\times_{\mathbf{P}(S)}\widehat{Y})$, so by Grothendieck duality we find that $\widehat{\Phi}^*$ is given by the kernel

$$\widehat{\mathcal{E}}^\vee \otimes \omega_{\widehat{X}\times_{\mathbf{P}(S)}\widehat{Y}/\widehat{Y}}[2] = \widehat{\mathcal{E}}^\vee(2h-H)[2] \in \mathrm{D^b}(\widehat{X}\times_{\mathbf{P}(S)}\widehat{Y}),$$

where $\widehat{\mathcal{E}}^\vee = \mathrm{R}\mathcal{H}om(\widehat{\mathcal{E}},\mathcal{O})$ is the derived dual. Using this, it follows easily from the definition of F that $\mathrm{F}[-2]$ is given by the kernel

$$\mathcal{K} := p_{XY*}(\widehat{\mathcal{E}}^\vee) \in \mathrm{D^b}(X\times Y).$$

Using the diagram (6.15) and the definition of $\widehat{\mathcal{E}}$, we can rewrite this as

$$\begin{aligned}
\mathcal{K} &= j_* p_* \, \mathrm{R}\mathcal{H}om(p^* g^* \mathcal{E}, \mathcal{O}_{\widehat{X} \times_{\mathbf{P}(S)} \widehat{Y}}) \\
&= j_* \, \mathrm{R}\mathcal{H}om(g^* \mathcal{E}, p_* \mathcal{O}_{\widehat{X} \times_{\mathbf{P}(S)} \widehat{Y}}) \\
&= j_* \, \mathrm{R}\mathcal{H}om(g^* \mathcal{E}, \mathcal{O}_{\mathrm{I}(X,Y)}),
\end{aligned}$$

where the second line holds by the local adjunction between p^* and p_*, and the third by Lemma 6.13. Now Grothendieck duality for the inclusion $j \colon \mathrm{I}(X,Y) \to X \times Y$ of the incidence divisor (which has class $H + H'$) gives

$$\mathcal{K} \cong j_* \, \mathrm{R}\mathcal{H}om(g^* \mathcal{E}, j^! \mathcal{O}_{\mathrm{I}(X,Y)}(-H - H')[1]) \cong \mathrm{R}\mathcal{H}om(j_* g^* \mathcal{E}, \mathcal{O}_{\mathrm{I}(X,Y)}(-H - H')[1]).$$

On the other hand, the fiber square in diagram (6.15) is Tor-independent because $\mathrm{I}(X,Y)$ has expected dimension. Hence we have an isomorphism

$$j_* g^* \mathcal{E} \cong (f_X \times f_Y)^* i_* \mathcal{E},$$

and so, by the explicit resolution of $i_* \mathcal{E}$ from Theorem 6.8, a distinguished triangle

$$\mathcal{O}_X \boxtimes \mathcal{V}_Y \to \mathcal{U}_X^\vee \boxtimes \mathcal{O}_Y \to j_* g^* \mathcal{E}.$$

Dualizing, twisting by $\mathcal{O}(-H - H')$, and rotating this triangle, we obtain a distinguished triangle

$$\mathcal{U}_X(-H) \boxtimes \mathcal{O}_Y(-H') \to \mathcal{O}_X(-H) \boxtimes \mathcal{V}_Y^\vee(-H') \to \mathrm{R}\mathcal{H}om(j_* g^* \mathcal{E}, \mathcal{O}_{\mathrm{I}(X,Y)}(-H - H')[1]),$$

which combined with the above expression for $\mathcal{K}$ finishes the proof. $\square$

Finally, we prove Proposition 6.12.

Proof of Proposition 6.12. By Lemma 6.14, it suffices to show the Fourier–Mukai functors with kernels

$$\mathcal{U}_X(-H) \boxtimes \mathcal{O}_Y(-H') \qquad \text{and} \qquad \mathcal{O}_X(-H) \boxtimes \mathcal{V}_Y^\vee(-H')$$

vanish on $\mathcal{A}_X$. This is equivalent to the vanishing

$$\mathrm{H}^\bullet(X, \mathcal{U}_X(-H) \otimes \mathcal{F}) = 0 \quad \text{and} \quad \mathrm{H}^\bullet(X, \mathcal{O}_X(-H) \otimes \mathcal{F}) = 0$$

for all $\mathcal{F} \in \mathcal{A}_X$, which holds since $\mathcal{A}_X$ is right orthogonal to $\mathcal{B}_X(H) = \langle \mathcal{O}_X(H), \mathcal{U}_X^\vee(H) \rangle$ by definition. □

Chapter 7

Derived cubic fourfold partners

Recall that for any smooth cubic fourfold $X' \subset \mathbf{P}^5$, there is a semiorthogonal decomposition

$$\mathrm{D^b}(X') = \langle \mathcal{A}_{X'}, \mathcal{O}_{X'}, \mathcal{O}_{X'}(1), \mathcal{O}_{X'}(2) \rangle, \tag{7.1}$$

where $\mathcal{A}_{X'}$ is a K3 category (see [20, Corollary 4.3] or [27, Corollary 4.1]). Moreover, in [25] it is shown that for all known examples of rational X', the category $\mathcal{A}_{X'}$ is equivalent to the derived category of a K3 surface.

In this chapter, we show that the K3 categories attached to GM and cubic fourfolds not only behave similarly, but sometimes even coincide. For this, we will consider GM fourfolds satisfying the following equivalent conditions.

Lemma 7.1. *Let X be an ordinary GM fourfold. The following are equivalent:*

(1) There is a 3*-dimensional subspace* $V_3 \subset V_5(X)$ *such that* $\mathrm{Gr}(2, V_3) \subset X$*.*

(2) The EPW quartic $\mathsf{Z}_{\mathsf{A}(X)} \subset \mathrm{Gr}(3, V_6(X))$ *has a point of multiplicity* 4 *on* $\mathrm{Gr}(3, V_5(X))$*, i.e.* $\mathsf{Z}^4_{\mathsf{A}(X)} \cap \mathrm{Gr}(3, V_5(X)) \neq \varnothing$*.*

Proof. Follows from [7, Proposition 4.10]. □

Remark 7.2. GM fourfolds as in the lemma form a 21-dimensional (codimension 3 in moduli) family. This can be seen using Theorem 3.6. Indeed, by [17, Lemma 3.6] Lagrangian subspaces

$\mathsf{A} \subset \wedge^3 V_6$ with no decomposable vectors such that $\mathsf{Z}^4_{\mathsf{A}} \neq \varnothing$ form a divisor in the moduli space of all A, and hence form a 19-dimensional family. Having fixed such an A there are finitely many points $V_3 \in \mathsf{Z}^4_{\mathsf{A}} \subset \mathrm{Gr}(3, V_6)$, and in order for $V_3 \subset V_5(X)$ for an ordinary GM fourfold X, the Plücker point $\mathbf{p}$ of X can be any point of $\mathsf{Y}^1_{\mathsf{A}^\perp} \cap \mathbf{P}(V_3^\perp)$, where $V_3^\perp \subset V_6^\vee$ is the orthogonal to $V_3 \subset V_6$. By the following lemma, the intersection $\mathsf{Y}^1_{\mathsf{A}^\perp} \cap \mathbf{P}(V_3^\perp)$ is an open subset of the 2-dimensional space $\mathbf{P}(V_3^\perp)$.

Lemma 7.3. *Let $\mathsf{A} \subset \wedge^3 V_6$ be a Lagrangian subspace, and suppose $V_3 \in \mathsf{Z}^{\geq 4}_{\mathsf{A}} \subset \mathrm{Gr}(3, V_6)$. Then there is an inclusion $\mathbf{P}(V_3^\perp) \subset \mathsf{Y}_{\mathsf{A}^\perp}$.*

Proof. A point of $\mathbf{P}(V_3^\perp) \subset \mathbf{P}(V_6^\vee)$ corresponds to a 5-dimensional subspace $V_5 \subset V_6$ such that $V_3 \subset V_5$. The condition $V_3 \in \mathsf{Z}^{\geq 4}_{\mathsf{A}}$ means $\dim(\mathsf{A} \cap ((\wedge^2 V_3) \wedge V_6)) \geq 4$. Since the last term in the exact sequence

$$0 \to (\wedge^2 V_3) \wedge V_5 \to (\wedge^2 V_3) \wedge V_6 \to (\wedge^2 V_3) \otimes (V_6/V_5) \to 0$$

has dimension 3, it follows that $\dim(\mathsf{A} \cap ((\wedge^2 V_3) \wedge V_5)) \geq 1$. But $(\wedge^2 V_3) \wedge V_5 \subset \wedge^3 V_5$, hence this gives $\mathsf{A} \cap (\wedge^3 V_5) \neq 0$, i.e. $V_5 \in \mathsf{Y}_{\mathsf{A}^\perp}$. $\square$

For the rest of this chapter, we fix an ordinary GM fourfold X satisfying the equivalent conditions of Lemma 7.1. From now on we write $V_5 = V_5(X)$, and we fix a 3-dimensional subspace $V_3 \subset V_5$ such that $\mathrm{Gr}(2, V_3) \subset X$. We associate to X a birational cubic fourfold X' following [6, Section 7.2], and provided X' is smooth (which is generically the case), we prove there is an equivalence $\mathcal{A}_X \simeq \mathcal{A}_{X'}$ (Theorem 7.9). The cubic X' is simply the image of the linear projection from the plane $\mathrm{Gr}(2, V_3)$ in X. It is convenient to first study this projection as a map from the entire Grassmannian $\mathbf{G}$.

7.1 A linear projection of the Grassmannian

Set

$$P = \mathbf{P}(\wedge^2 V_3) = \mathrm{Gr}(2, V_3) \subset \mathbf{G}.$$

Choose a complement V_2 to V_3, and set

$$B = \wedge^2 V_5 / \wedge^2 V_3 = \wedge^2 V_2 \oplus (V_2 \otimes V_3).$$

Then the linear projection from P gives a birational isomorphism from $\mathbf{G}$ to $\mathbf{P}(B)$. Its structure can be described as follows.

Lemma 7.4. *Let $p \colon \widetilde{\mathbf{G}} \to \mathbf{G}$ be the blowup with center in P. Then the linear projection from P induces a regular map $q \colon \widetilde{\mathbf{G}} \to \mathbf{P}(B)$ which identifies $\mathbf{G}$ with the blowup of $\mathbf{P}(B)$ in $\mathbf{P}(V_2) \times \mathbf{P}(V_3) \subset \mathbf{P}(V_2 \otimes V_3) \subset \mathbf{P}(B)$. In other words, we have a diagram*

$$\begin{tikzcd}
 & E \arrow[r] \arrow[dl] & \widetilde{\mathbf{G}} \arrow[dl, "p"'] \arrow[dr, "q"] & E' \arrow[l] \arrow[drr] & & \\
P \arrow[r] & \mathbf{G} & & \mathbf{P}(B) & & \mathbf{P}(V_2) \times \mathbf{P}(V_3) \arrow[ll]
\end{tikzcd} \tag{7.2}$$

where

- *E is the exceptional divisor of the blowup p, and is mapped birationally by q onto the hyperplane $\mathbf{P}(V_2 \otimes V_3) \subset \mathbf{P}(B)$.*
- *E' is the exceptional divisor of the blowup q, and is mapped birationally by p onto the Schubert variety*

$$\Sigma = \{U \in \mathbf{G} \mid U \cap V_3 \neq 0\} \subset \mathbf{G}$$

Proof. Straightforward. □

We denote by H and H' the ample generators of $\mathrm{Pic}(\mathbf{G})$ and $\mathrm{Pic}(\mathbf{P}(B))$.

Lemma 7.5. *On $\widetilde{\mathbf{G}}$ we have the relations*

$$\begin{cases} H' = H - E, \\ E' = H - 2E, \end{cases} \qquad \text{or equivalently} \qquad \begin{cases} H = 2H' - E', \\ E = H' - E', \end{cases} \tag{7.3}$$

as divisors modulo linear equivalence. Moreover, we have

$$K_{\widetilde{\mathbf{G}}} = -5H + 3E = -7H' + 2E'. \tag{7.4}$$

Proof. The equalities (7.4) follow from the standard formula for the canonical class of a blowup, and the equality $H' = H - E$ holds by definition of p. Using these, the other equalities in (7.3) follow directly (note that $\mathrm{Pic}(\widetilde{\mathbf{G}}) \cong \mathbf{Z}^2$ is torsion free). $\square$

Later in this chapter we will need an expression for the vector bundle $p^*\mathcal{U}^\vee$ on $\widetilde{\mathbf{G}}$. For this, we consider the composition

$$\begin{aligned}\phi\colon (V_2^\vee \oplus V_3^\vee) \otimes \mathcal{O}_{\mathbf{P}(B)} \hookrightarrow V_2 \otimes V_2^\vee \otimes (V_2^\vee \oplus V_3^\vee) \otimes \mathcal{O}_{\mathbf{P}(B)} \to \\ \to V_2 \otimes (\wedge^2 V_2^\vee \oplus (V_2^\vee \otimes V_3^\vee)) \otimes \mathcal{O}_{\mathbf{P}(B)} \to V_2 \otimes \mathcal{O}_{\mathbf{P}(B)}(H'),\end{aligned}$$

where the first morphism is induced by the map $\mathbf{k} \to V_2 \otimes V_2^\vee$ corresponding to the identity of V_2, the second is induced by the map $V_2^\vee \otimes V_2^\vee \to \wedge^2 V_2^\vee$, and the third is induced by the composition

$$(\wedge^2 V_2^\vee \oplus (V_2^\vee \otimes V_3^\vee)) \otimes \mathcal{O}_{\mathbf{P}(B)} \hookrightarrow \wedge^2 V_5^\vee \otimes \mathcal{O}_{\mathbf{P}(B)} \to \mathcal{O}_{\mathbf{P}(B)}(H').$$

Lemma 7.6. *The cokernel of ϕ is the sheaf $\mathcal{O}_{\mathbf{P}(V_2)\times\mathbf{P}(V_3)}(2,1)$.*

Proof. Write

$$\begin{aligned}\phi'\colon V_2^\vee \otimes \mathcal{O}_{\mathbf{P}(B)} \to V_2 \otimes \mathcal{O}_{\mathbf{P}(B)}(H'),\\ \phi''\colon V_3^\vee \otimes \mathcal{O}_{\mathbf{P}(B)} \to V_2 \otimes \mathcal{O}_{\mathbf{P}(B)}(H'),\end{aligned}$$

for the components of ϕ. The morphism ϕ' is an isomorphism away from the hyperplane $\mathbf{P}(V_2 \otimes V_3) \subset \mathbf{P}(B)$, and zero on it. Hence $\mathrm{coker}(\phi') = V_2 \otimes \mathcal{O}_{\mathbf{P}(V_2 \otimes V_3)}(H')$. It follows that the cokernel of ϕ coincides with the cokernel of the morphism

$$\phi''_{|\mathbf{P}(V_2\otimes V_3)} \colon V_3^\vee \otimes \mathcal{O}_{\mathbf{P}(V_2\otimes V_3)} \to V_2 \otimes \mathcal{O}_{\mathbf{P}(V_2\otimes V_3)}(H').$$

But the morphism $\phi''_{|\mathbf{P}(V_2\otimes V_3)}$ is generically surjective with degeneracy locus the Segre subvariety $\mathbf{P}(V_2)\times\mathbf{P}(V_3)\subset\mathbf{P}(V_2\otimes V_3)$, and its restriction to this locus factors as the composition

$$V_3^\vee \otimes \mathcal{O}_{\mathbf{P}(V_2)\times\mathbf{P}(V_3)} \twoheadrightarrow \mathcal{O}_{\mathbf{P}(V_2)\times\mathbf{P}(V_3)}(0,1) \hookrightarrow V_2 \otimes \mathcal{O}_{\mathbf{P}(V_2)\times\mathbf{P}(V_3)}(1,1) = V_2 \otimes \mathcal{O}_{\mathbf{P}(V_2)\times\mathbf{P}(V_3)}(H').$$

It follows that the cokernel of $\phi''_{|\mathbf{P}(V_2\otimes V_3)}$ is isomorphic to $\mathcal{O}_{\mathbf{P}(V_2)\times\mathbf{P}(V_3)}(2,1)$. □

Let F denote the class of a fiber of the natural projection $E' \to \mathbf{P}(V_2)\times\mathbf{P}(V_3) \to \mathbf{P}(V_2)$.

Proposition 7.7. *On $\widetilde{\mathbf{G}}$ there is an exact sequence*

$$0 \to p^*\mathcal{U}^\vee \to V_2 \otimes \mathcal{O}_{\widetilde{\mathbf{G}}}(H') \to \mathcal{O}_{E'}(H'+F) \to 0. \tag{7.5}$$

Proof. By Lemma 7.6, we have an exact sequence

$$V_5^\vee \otimes \mathcal{O}_{\mathbf{P}(B)} \xrightarrow{\phi} V_2 \otimes \mathcal{O}_{\mathbf{P}(B)}(H') \to \mathcal{O}_{\mathbf{P}(V_2)\times\mathbf{P}(V_3)}(2,1) \to 0.$$

Pulling back to $\widetilde{\mathbf{G}}$, we obtain an exact sequence

$$V_5^\vee \otimes \mathcal{O}_{\widetilde{\mathbf{G}}} \to V_2 \otimes \mathcal{O}_{\widetilde{\mathbf{G}}}(H') \to \mathcal{O}_{E'}(H'+F) \to 0,$$

Since E' is a divisor on $\widetilde{\mathbf{G}}$, the kernel $\mathcal{K}$ of the epimorphism $V_2\otimes\mathcal{O}_{\widetilde{\mathbf{G}}}(H') \to \mathcal{O}_{E'}(H'+F)$ is a rank 2 vector bundle on $\widetilde{\mathbf{G}}$, which by the above exact sequence is a quotient of the trivial bundle $V_5^\vee\otimes\mathcal{O}_{\widetilde{\mathbf{G}}}$. Hence $\mathcal{K}$ induces a morphism $\widetilde{\mathbf{G}} \to \mathbf{G}$. This morphism can be checked to agree with the blowdown morphism p, so $\mathcal{K} \cong p^*\mathcal{U}^\vee$. □

7.2 Setup and statement of the result

Recall that X is an ordinary GM fourfold containing the plane $P = \mathrm{Gr}(2, V_3)$. The following proposition describes the structure of the rational map from X to $\mathbf{P}^5$ given by projection from P. We slightly abuse notation by using the same symbols for the exceptional divisors and blowup morphisms as in the above discussion of $\mathbf{G}$.

Proposition 7.8. *Let $p\colon \widetilde{X} \to X$ be the blowup with center in P. Then the linear projection from P induces a regular map $q\colon \widetilde{X} \to X'$ to a cubic fourfold X' containing a smooth cubic surface scroll T, and identifies $\widetilde{X}$ as the blowup of X' in T. In other words, we have a diagram*

$$\begin{tikzcd}
 & E \arrow[r, "i"] \arrow[dl, "p_E"'] & \widetilde{X} \arrow[dl, "p"'] \arrow[dr, "q"] & E' \arrow[l, "j"'] \arrow[dr, "q_{E'}"] & \\
P \arrow[r] & X & & X' & T \arrow[l]
\end{tikzcd}$$

where p and q are blowups with exceptional divisors E and E'. Moreover, the relations (7.3) *continue to hold on $\widetilde{X}$, and*

$$K_{\widetilde{X}} = -2H + E = -3H' + E'. \tag{7.6}$$

Finally, if X does not contain planes of the form $\mathbf{P}(V_1 \wedge V_4)$ where $V_1 \subset V_3 \subset V_4 \subset V_5$, then X' is smooth.

Proof. By Section 3.1 there is a hyperplane $\mathbf{P}(W) \subset \mathbf{P}(\wedge^2 V_5)$ and a quadric hypersurface $Q \subset \mathbf{P}(W)$ such that $X = \mathbf{G} \cap Q$ and $P \subset Q$. Let

$$C = W/\wedge^2 V_3$$

so that $\mathbf{P}(C) \subset \mathbf{P}(B)$ is a hyperplane. We claim that the corresponding hyperplane section

$$T = (\mathbf{P}(V_2) \times \mathbf{P}(V_3)) \cap \mathbf{P}(C)$$

of $\mathbf{P}(V_2) \times \mathbf{P}(V_3) \subset \mathbf{P}(B)$ is a smooth cubic surface scroll. For this it is enough to show that $\mathbf{P}(C) \cap \mathbf{P}(V_2 \otimes V_3)$ is a hyperplane in $\mathbf{P}(V_2 \otimes V_3)$ whose equation, considered as an element in $V_2^\vee \otimes V_3^\vee \cong \mathrm{Hom}(V_3, V_2^\vee)$, has rank 2. Assume on the contrary that the rank of this equation is at most 1. Then its kernel is a subspace of V_3 of dimension at least 2, which is contained in the kernel of the skew form ω on V_5 defining W. So the rank of ω is 2. But then the Grassmannian hull $M = \mathbf{G} \cap \mathbf{P}(W)$ of X is singular along $\mathbf{P}^2 = \mathrm{Gr}(2, \ker(\omega))$, and X is singular along $\mathbf{P}^2 \cap Q$. This contradiction proves the claim.

The proper transform of the Grassmannian hull $M = \mathbf{G} \cap \mathbf{P}(W)$ under $p\colon \widetilde{\mathbf{G}} \to \mathbf{G}$ coincides with the proper transform of $\mathbf{P}(C)$ under $q\colon \widetilde{\mathbf{G}} \to \mathbf{P}(B)$. Thus if $\widetilde{M} = \mathrm{Bl}_P(M) \to M$ is the blowup in P, then projection from P gives an identification $\widetilde{M} \cong \mathrm{Bl}_T(\mathbf{P}(C)) \to \mathbf{P}(C)$. Further, the proper transform of $X = M \cap Q$ under $\widetilde{M} \to M$ is cut out by a section of the line bundle

$$\mathcal{O}_{\widetilde{M}}(2H - E) = \mathcal{O}_{\widetilde{M}}(3H' - E'),$$

and therefore coincides with the proper transform under $\widetilde{M} \to \mathbf{P}(C)$ of a cubic fourfold $X' \subset \mathbf{P}(C)$ containing T. This proves the first part of the lemma.

The relations (7.3) clearly restrict to $\widetilde{X}$, and the equalities (7.6) follow from the standard formula for the canonical class of a blowup.

It remains to show that X' is smooth if X does not contain planes of the form $\mathbf{P}(V_1 \wedge V_4)$ where $V_1 \subset V_3 \subset V_4 \subset V_5$. For this, first note that the blowup of X' in T is smooth, since it coincides with the blowup of X in P. Therefore, X' is smooth away from T. On the other hand, T is also smooth, so it is enough to check that $T \subset X'$ is a locally complete intersection, i.e. that its conormal sheaf is locally free. Since $E' \to T$ is the exceptional divisor of the blowup of X' in T, it is enough to check that the map $E' \to T$ is a $\mathbf{P}^1$-bundle. Since E' is cut out in the exceptional divisor of (7.2) by fiberwise linear conditions, it is enough to show that there are no points in $T \subset \mathbf{P}(V_2) \times \mathbf{P}(V_3)$ over which the fiber of E' is isomorphic to $\mathbf{P}^2$. But such a point would correspond to a choice of a $V_1 \subset V_3$ (giving a point in $\mathbf{P}(V_3)$) and $V_3 \subset V_4$ (giving a point of $\mathbf{P}(V_5/V_3) = \mathbf{P}(V_2)$), such that the the plane $\mathbf{P}(V_1 \wedge V_4)$ is in X. Since we assumed there are no such planes in X, we conclude that X' is smooth. $\square$

Our goal is to prove the following result.

Theorem 7.9. *Assume the cubic fourfold X' associated to X by Proposition 7.8 is smooth. Then there is an equivalence $\mathcal{A}_X \simeq \mathcal{A}_{X'}$.*

Note that, by the final statement of Proposition 7.8, the assumption of the theorem is generically satisfied.

Remark 7.10. Theorem 7.9 is of an essentially different nature than Theorem 6.1, in that it does not "come from" K3 surfaces. More precisely, for a very general GM fourfold X satisfying the equivalent conditions of Lemma 7.1, the category $\mathcal{A}_X$ is not equivalent to the derived category of a K3 surface. This follows from the fact that the construction of Proposition 7.8 dominates the locus of cubic fourfolds containing a smooth cubic surface scroll, combined with [1, Theorem 1.1] and [15, Theorem 5.1.3 and Example 4.1.2].

7.3 Strategy of the proof

From now on, we assume the hypothesis of Theorem 7.9 is satisfied. The proof of this theorem occupies the rest of this chapter. Here is our strategy.

By Orlov's decomposition of the derived category of a blowup, we have

$$\mathrm{D^b}(\widetilde{X}) = \langle p^*\mathrm{D^b}(X), i_* p_E^* \mathrm{D^b}(P)\rangle.$$

The decomposition (4.2) can in our case be written as

$$\mathrm{D^b}(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \mathcal{U}_X^{\vee}, \mathcal{O}_X(H), \mathcal{U}_X^{\vee}(H)\rangle. \tag{7.7}$$

Inserting this and the standard decomposition of $\mathrm{D^b}(P)$ into the above decomposition, we obtain

$$\mathrm{D^b}(\widetilde{X}) = \langle p^*\mathcal{A}_X, \mathcal{O}, \mathcal{U}^{\vee}, \mathcal{O}(H), \mathcal{U}^{\vee}(H), \mathcal{O}_E, \mathcal{O}_E(H), \mathcal{O}_E(2H)\rangle. \tag{7.8}$$

Here and below, to ease notation we write $\mathcal{U}^\vee$ for $p^*\mathcal{U}_X^\vee$. This decomposition of $\mathrm{D^b}(\widetilde{X})$ consists of a copy of $\mathcal{A}_X$ and 7 exceptional objects.

On the other hand, from the expression of $\widetilde{X}$ as a blowup of X', we have

$$\mathrm{D^b}(\widetilde{X}) = \langle q^*\mathrm{D^b}(X'), j_*q_{E'}^*\mathrm{D^b}(T)\rangle.$$

Inserting the decomposition (7.1) for $\mathrm{D^b}(X')$, we obtain

$$\mathrm{D^b}(\widetilde{X}) = \langle q^*\mathcal{A}_{X'}, \mathcal{O}, \mathcal{O}(H'), \mathcal{O}(2H'), j_*q_{E'}^*\mathrm{D^b}(T)\rangle. \tag{7.9}$$

Note that $\mathrm{D^b}(T)$ has a decomposition consisting of 4 exceptional objects, hence the decomposition (7.9) consists of one copy of $\mathcal{A}_{X'}$ and again 7 exceptional objects.

To prove the equivalence $\mathcal{A}_X \simeq \mathcal{A}_{X'}$, we will find a sequence of mutations transforming the exceptional objects of (7.8) into those of (7.9). In doing so, we will explicitly identify a functor giving the desired equivalence, see (7.14).

7.4 Mutations

We perform a sequence of mutations, starting with (7.8).

Step 1. Mutate $\mathcal{U}^\vee(H)$ to the far left in (7.8). Since this is a mutation in $\mathrm{D^b}(X)$ and we have $K_X = -2H$, by Lemma 2.10 the result is

$$\mathrm{D^b}(\widetilde{X}) = \langle \mathcal{U}^\vee(-H), p^*\mathcal{A}_X, \mathcal{O}, \mathcal{U}^\vee, \mathcal{O}(H), \mathcal{O}_E, \mathcal{O}_E(H), \mathcal{O}_E(2H)\rangle.$$

Step 2. Mutate $\mathcal{U}^\vee(-H)$ to the far right. Again by Lemma 2.10 and (7.6), the result is

$$\mathrm{D^b}(\widetilde{X}) = \langle p^*\mathcal{A}_X, \mathcal{O}, \mathcal{U}^\vee, \mathcal{O}(H), \mathcal{O}_E, \mathcal{O}_E(H), \mathcal{O}_E(2H), \mathcal{U}^\vee(H-E)\rangle.$$

Step 3. Left mutate $\mathcal{O}_E$ through $\langle \mathcal{O}, \mathcal{U}^\vee, \mathcal{O}(H)\rangle$. We have

$$\begin{aligned}
\operatorname{Ext}^\bullet(\mathcal{O}(H), \mathcal{O}_E) &= \mathrm{H}^\bullet(P, \mathcal{O}_P(-H)) = 0,\\
\operatorname{Ext}^\bullet(\mathcal{U}^\vee, \mathcal{O}_E) &= \mathrm{H}^\bullet(P, \mathcal{U}_P) = 0,\\
\operatorname{Ext}^\bullet(\mathcal{O}, \mathcal{O}_E) &= \mathrm{H}^\bullet(P, \mathcal{O}_P) = \mathbf{k},
\end{aligned}$$

where in the second line $\mathcal{U}_P$ is the tautological rank 2 bundle on $P = \mathrm{Gr}(2, V_3)$, i.e. the restriction of $\mathcal{U}$ from $\mathbf{G}$ to P. Hence by Lemma 2.7

$$\mathrm{L}_{\langle \mathcal{O}, \mathcal{U}^\vee, \mathcal{O}(H)\rangle}(\mathcal{O}_E) = \operatorname{Cone}(\mathcal{O} \to \mathcal{O}_E) = \mathcal{O}(-E)[1],$$

and the resulting decomposition is

$$\mathrm{D}^{\mathrm{b}}(\widetilde{X}) = \langle p^*\mathcal{A}_X, \mathcal{O}(-E), \mathcal{O}, \mathcal{U}^\vee, \mathcal{O}(H), \mathcal{O}_E(H), \mathcal{O}_E(2H), \mathcal{U}^\vee(H-E)\rangle.$$

Step 4. Left mutate $\mathcal{O}_E(2H)$ through $\langle \mathcal{O}, \mathcal{U}^\vee, \mathcal{O}(H), \mathcal{O}_E(H)\rangle$.

Lemma 7.11. *We have* $\mathrm{L}_{\langle \mathcal{O}, \mathcal{U}^\vee, \mathcal{O}(H), \mathcal{O}_E(H)\rangle}(\mathcal{O}_E(2H)) \cong \mathcal{O}_{E'}(E' - F)[2]$.

Proof. By Lemma 2.7 we may successively left mutate $\mathcal{O}_E(2H)$ through $\mathcal{O}_E(H), \mathcal{O}(H), \mathcal{U}^\vee, \mathcal{O}$.

To compute $\mathrm{L}_{\mathcal{O}_E(H)}(\mathcal{O}_E(2H))$, we may compute $\mathrm{L}_{\mathcal{O}_P(H)}(\mathcal{O}_P(2H))$ and pull back the result. We have $\operatorname{Ext}^\bullet(\mathcal{O}_P(H), \mathcal{O}_P(2H)) = \mathrm{H}^\bullet(P, \mathcal{O}_P(H)) = V_3$, so

$$\mathrm{L}_{\mathcal{O}_P(H)}(\mathcal{O}_P(2H)) = \operatorname{Cone}(\mathcal{O}_P(H) \otimes V_3 \to \mathcal{O}_P(2H)).$$

The morphism $\mathcal{O}_P(H) \otimes V_3 \to \mathcal{O}_P(2H)$ is the twist by H of the tautological morphism, hence it is surjective with kernel $\mathcal{U}_P(H) \cong \mathcal{U}_P^\vee$. Thus the above cone is $\mathcal{U}_P^\vee[1]$, and

$$\mathrm{L}_{\mathcal{O}_E(H)}(\mathcal{O}_E(2H)) = \mathcal{U}_E^\vee[1].$$

Next note $\operatorname{Ext}^\bullet(\mathcal{O}(H), \mathcal{U}_E^\vee) = \mathrm{H}^\bullet(P, \mathcal{U}_P^\vee(-H)) = 0$, hence

$$\mathrm{L}_{\mathcal{O}(H)}(\mathcal{U}_E^\vee) = \mathcal{U}_E^\vee.$$

Further, we have $\operatorname{Ext}^\bullet(\mathcal{U}^\vee, \mathcal{U}_E^\vee) = \mathrm{H}^\bullet(P, \mathcal{U}_P \otimes \mathcal{U}_P^\vee) = \mathbf{k}$, hence

$$\mathrm{L}_{\mathcal{U}^\vee}(\mathcal{U}_E^\vee) = \operatorname{Cone}(\mathcal{U}^\vee \to \mathcal{U}_E^\vee) = \mathcal{U}^\vee(-E)[1].$$

Now we are left with the last and most interesting step — the mutation through $\mathcal{O}$. First, using the exact sequence

$$0 \to \mathcal{O}(-E) \to \mathcal{O} \to \mathcal{O}_E \to 0$$

twisted by $\mathcal{U}^\vee$, we find

$$\operatorname{Ext}^\bullet(\mathcal{O}, \mathcal{U}^\vee(-E)) = \mathrm{H}^\bullet(\widetilde{X}, \mathcal{U}^\vee(-E)) = \ker(V_5^\vee \to V_3^\vee) = V_2^\vee. \tag{7.10}$$

Thus we need to understand the cone of the natural morphism $V_2^\vee \otimes \mathcal{O} \to \mathcal{U}^\vee(-E)$. Restricting (7.5) to $\widetilde{X}$, dualizing, twisting by $H' = H - E$, and using the isomorphism $\mathcal{U}(H) \cong \mathcal{U}^\vee$, we obtain a distinguished triangle

$$V_2^\vee \otimes \mathcal{O} \to \mathcal{U}^\vee(-E) \to \mathcal{O}_{E'}(E' - F).$$

Thus

$$\mathrm{L}_{\mathcal{O}}(\mathcal{U}^\vee(-E)) = \mathcal{O}_{E'}(E' - F), \tag{7.11}$$

which completes the proof of the lemma. □

By the lemma, the result of the above mutation is

$$\mathrm{D}^{\mathrm{b}}(\widetilde{X}) = \langle p^*\mathcal{A}_X, \mathcal{O}(-E), \mathcal{O}_{E'}(E'-F), \mathcal{O}, \mathcal{U}^\vee, \mathcal{O}(H), \mathcal{O}_E(H), \mathcal{U}^\vee(H-E)\rangle.$$

Step 5. Left mutate $\mathcal{O}_E(H)$ through $\mathcal{O}(H)$. We have

$$\mathrm{L}_{\mathcal{O}(H)}(\mathcal{O}_E(H)) = \mathrm{Cone}(\mathcal{O}(H) \to \mathcal{O}_E(H)) = \mathcal{O}(H-E)[1] = \mathcal{O}(H')[1],$$

so the result is

$$\mathrm{D^b}(\widetilde{X}) = \langle p^*\mathcal{A}_X, \mathcal{O}(-E), \mathcal{O}_{E'}(E'-F), \mathcal{O}, \mathcal{U}^\vee, \mathcal{O}(H'), \mathcal{O}(H), \mathcal{U}^\vee(H-E)\rangle.$$

Step 6. Right mutate $\mathcal{U}^\vee$ through $\mathcal{O}(H')$. We have

$$\mathrm{Ext}^\bullet(\mathcal{U}^\vee, \mathcal{O}(H')) = \mathrm{Ext}^\bullet(\mathcal{O}, \mathcal{U}(H-E)) = \mathrm{Ext}^\bullet(\mathcal{O}, \mathcal{U}^\vee(-E)) = V_2^\vee,$$

where the last equality holds by (7.10). Hence

$$\mathrm{R}_{\mathcal{O}(H')}(\mathcal{U}^\vee) = \mathrm{Cone}(\mathcal{U}^\vee \to V_2 \otimes \mathcal{O}(H'))[-1].$$

Now restricting (7.5) to $\widetilde{X}$ shows $\mathrm{R}_{\mathcal{O}(H')}(\mathcal{U}^\vee) = \mathcal{O}_{E'}(H'+F)[-1]$. Thus under the above mutation our decomposition becomes

$$\mathrm{D^b}(\widetilde{X}) = \langle p^*\mathcal{A}_X, \mathcal{O}(-E), \mathcal{O}_{E'}(E'-F), \mathcal{O}, \mathcal{O}(H'), \mathcal{O}_{E'}(H'+F), \mathcal{O}(H), \mathcal{U}^\vee(H-E)\rangle.$$

Step 7. Left mutate $\mathcal{U}^\vee(H-E)$ through $\mathcal{O}(H)$. By (7.11) and Lemma 2.9 we have

$$\mathrm{L}_{\mathcal{O}(H)}(\mathcal{U}^\vee(H-E)) = \mathcal{O}_{E'}(H+E'-F) = \mathcal{O}_{E'}(2H'-F),$$

so the result is

$$\mathrm{D^b}(\widetilde{X}) = \langle p^*\mathcal{A}_X, \mathcal{O}(-E), \mathcal{O}_{E'}(E'-F), \mathcal{O}, \mathcal{O}(H'), \mathcal{O}_{E'}(H'+F), \mathcal{O}_{E'}(2H'-F), \mathcal{O}(H)\rangle.$$

Step 8. Right mutate $p^*\mathcal{A}_X$ through $\langle \mathcal{O}(-E), \mathcal{O}_{E'}(E'-F)\rangle$. The result is

$$\mathrm{D^b}(\widetilde{X}) = \langle \mathcal{O}(-E), \mathcal{O}_{E'}(E'-F), \Psi p^*\mathcal{A}_X, \mathcal{O}, \mathcal{O}(H'), \mathcal{O}_{E'}(H'+F), \mathcal{O}_{E'}(2H'-F), \mathcal{O}(H)\rangle,$$

where $\Psi = \mathrm{R}_{\langle \mathcal{O}(-E), \mathcal{O}_{E'}(E'-F)\rangle}$.

Step 9. Mutate $\langle \mathcal{O}(-E), \mathcal{O}_{E'}(E'-F)\rangle$ to the far right. By Lemma 2.10, the result is

$$\mathrm{D^b}(\widetilde{X}) = \langle \Psi p^*\mathcal{A}_X, \mathcal{O}, \mathcal{O}(H'), \mathcal{O}_{E'}(H'+F), \mathcal{O}_{E'}(2H'-F), \mathcal{O}(H), \mathcal{O}(2H'), \mathcal{O}_{E'}(3H'-F)\rangle.$$

Step 10. Right mutate $\mathcal{O}(H)$ through $\mathcal{O}(2H')$. We have

$$\mathrm{Ext}^\bullet(\mathcal{O}(H), \mathcal{O}(2H')) = \mathrm{H}^\bullet(\widetilde{X}, \mathcal{O}(E')) = \mathbf{k}$$

and hence

$$\mathrm{R}_{\mathcal{O}(2H')}(\mathcal{O}(H)) = \mathrm{Cone}(\mathcal{O}(H) \to \mathcal{O}(2H'))[-1].$$

The morphism $\mathcal{O}(H) \to \mathcal{O}(2H')$ is the twist by $2H'$ of $\mathcal{O}(-E') \to \mathcal{O}$, hence

$$\mathrm{R}_{\mathcal{O}(2H')}(\mathcal{O}(H)) = \mathcal{O}_{E'}(2H')[-1].$$

Thus the result of the mutation is a decomposition

$$\mathrm{D^b}(\widetilde{X}) = \langle \Psi p^*\mathcal{A}_X, \mathcal{O}, \mathcal{O}(H'), \mathcal{O}_{E'}(H'+F), \mathcal{O}_{E'}(2H'-F), \mathcal{O}(2H'), \mathcal{O}_{E'}(2H'), \mathcal{O}_{E'}(3H'-F)\rangle.$$

Step 11. Left mutate $\mathcal{O}(2H')$ through $\langle \mathcal{O}_{E'}(H'+F), \mathcal{O}_{E'}(2H'-F)\rangle$. By the semiorthogonality of $q^*\mathrm{D^b}(X')$ and $j_*q_{E'}^*\mathrm{D^b}(T)$ in $\mathrm{D^b}(\widetilde{X})$, this mutation is just a transposition. Thus the result is

$$\mathrm{D^b}(\widetilde{X}) = \langle \Psi p^*\mathcal{A}_X, \mathcal{O}, \mathcal{O}(H'), \mathcal{O}(2H'), \mathcal{O}_{E'}(H'+F), \mathcal{O}_{E'}(2H'-F), \mathcal{O}_{E'}(2H'), \mathcal{O}_{E'}(3H'-F)\rangle.$$

It is straightforward to check that

$$\mathrm{D^b}(T) = \langle \mathcal{O}_T(H'+F), \mathcal{O}_T(2H'-F), \mathcal{O}_T(2H'), \mathcal{O}_T(3H'-F)\rangle,$$

so the above decomposition can be written as

$$\mathrm{D^b}(\widetilde{X}) = \langle \Psi p^*\mathcal{A}_X, \mathcal{O}, \mathcal{O}(H'), \mathcal{O}(2H'), j_*q_{E'}^*\mathrm{D^b}(T)\rangle. \tag{7.12}$$

This completes the proof of Theorem 7.9. Indeed, comparing the decompositions (7.12) and (7.9) shows

$$q_* \circ \mathrm{R}_{\langle \mathcal{O}_{\widetilde{X}}(-E), \mathcal{O}_{E'}(E'-F)\rangle} \circ p^* \colon \mathcal{A}_X \to \mathcal{A}_{X'} \tag{7.13}$$

is an equivalence. $\square$

Remark 7.12. The functor (7.13) is in fact isomorphic to

$$q_* \circ \mathrm{R}_{\mathcal{O}_{\widetilde{X}}(-E)} \circ p^* \colon \mathcal{A}_X \to \mathcal{A}_{X'}. \tag{7.14}$$

To see this, observe that q_* kills $\mathcal{O}_{E'}(E'-F)$: if $j_0 \colon T \hookrightarrow X'$ denotes the inclusion, then

$$q_*(\mathcal{O}_{E'}(E'-F)) = j_{0*}q_{E'*}(\mathcal{O}_{E'}(E'-F)) = j_{0*}(q_{E'*}(\mathcal{O}_{E'}(E')) \otimes \mathcal{O}_T(F)) = 0$$

since $q_{E'*}(\mathcal{O}_{E'}(E')) = 0$. Thus $q_* \circ \mathrm{R}_{\mathcal{O}_{E'}(E'-F)} \cong q_*$, and the claim follows from Lemma 2.7.

Appendix A

Hochschild cohomology

In this appendix, we prove some results on Hochschild cohomology. The first main result (Theorem A.8) states that Serre functors act trivially on Hochschild cohomology. The second main result (Theorem A.22) describes the behavior of Hochschild cohomology under passing to the category of invariants for the action of a finite group. In the case of a $\mathbf{Z}/2$-action given (up to a shift) by a Serre functor, we combine these results to prove a useful formula (Corollary A.26) relating the Hochschild cohomology of the category of invariants to the Hochschild cohomology and homology of the original category. This formula is applied in the body of the text to compute the Hochschild cohomology of odd GM categories (Proposition 5.7).

We remind the reader that $\mathbf{k}$ denotes an algebraically closed field of characteristic 0. We will indicate in the arguments where the assumption on the characteristic is used. In fact, for all of the results, it is enough to assume that the characteristic of $\mathbf{k}$ is coprime to the order of any finite group involved (and sometimes even this is not necessary).

A.1 k-linear stable ∞-categories

We will work in the context of $\mathbf{k}$-linear stable ∞-categories. We assume familiarity with this theory, but below we will summarize some of the key points as we spell out our conventions. An uninitiated reader should be able to follow the main lines of reasoning in this appendix, if they are willing to take for granted that a theory of ∞-categories exists and behaves analogously to ordinary category theory. For background, see [34], [35], [36], or the survey [13, Chapter I.1].

Let $\mathrm{Vect}_{\mathbf{k}}^{\mathrm{fd}}$ denote the ∞-category of finite complexes of finite-dimensional $\mathbf{k}$-vector spaces. The category $\mathrm{Vect}_{\mathbf{k}}^{\mathrm{fd}}$ is stable and has a natural symmetric monoidal structure. In this appendix, we use the term $\mathbf{k}$-*linear stable* ∞-*category* to mean a small idempotent-complete stable ∞-category $\mathcal{C}$ equipped with a module structure over $\mathrm{Vect}_{\mathbf{k}}^{\mathrm{fd}}$, such that the action functor

$$\begin{aligned} \mathrm{Vect}_{\mathbf{k}}^{\mathrm{fd}} \times \mathcal{C} &\to \mathcal{C} \\ (V, X) &\mapsto V \otimes X \end{aligned}$$

is exact in both variables. All of the functors between $\mathbf{k}$-linear stable ∞-categories considered below will be $\mathbf{k}$-linear and exact, so we often omit these adjectives.

A reader unfamiliar with the above language will not lose much by pretending that $\mathcal{C}$ is a small pretriangulated DG category over $\mathbf{k}$. We note that the homotopy category $\mathrm{h}\mathcal{C}$ of $\mathcal{C}$ is indeed a $\mathbf{k}$-linear triangulated category.

The theory of $\mathbf{k}$-linear stable ∞-categories has several technical advantages over the classical theory of DG categories. First, due to the foundations set up in [34], [35], the theory of ∞-categories is a robust generalization of ordinary category theory. In particular, there are ∞-categorical notions of commutativity of a diagram, and of limits and colimits, which are formally very similar to the corresponding notions for ordinary categories. Another important feature is that the collection of all $\mathbf{k}$-linear stable ∞-categories (with morphisms between them the exact $\mathbf{k}$-linear functors) can be organized into an ∞-category $\mathrm{Cat}_{\mathbf{k}}^{\mathrm{St}}$, which admits small limits and colimits.

Remark A.1. Let X be a variety over $\mathbf{k}$. Then there is a natural $\mathbf{k}$-linear stable ∞-category $\mathrm{D^b}(X)^{\mathrm{enh}}$ which is an enhancement of $\mathrm{D^b}(X)$, i.e. the homotopy category of $\mathrm{D^b}(X)^{\mathrm{enh}}$ is equivalent to $\mathrm{D^b}(X)$. For instance, with this notation $\mathrm{Vect}_{\mathbf{k}}^{\mathrm{fd}} = \mathrm{D^b}(\mathrm{Spec}(\mathbf{k}))^{\mathrm{enh}}$. Moreover, the pushforward and pullback functors along a morphism $X \to Y$ lift to morphisms between the enhancements $\mathrm{D^b}(X)^{\mathrm{enh}}$ and $\mathrm{D^b}(Y)^{\mathrm{enh}}$. This is accomplished in two steps. First, an enhancement of the unbounded derived category of quasi-coherent sheaves, compatible with pushforward and pullback functors, is constructed [13, Chapter I.3, Sections 1-2]. Then $\mathrm{D^b}(X)^{\mathrm{enh}}$ is obtained by passing to the full subcategory of bounded complexes with coherent cohomology. We further note that if $\mathcal{A} \subset \mathrm{D^b}(X)$ is an admissible subcategory, then there is a $\mathbf{k}$-linear stable enhancement $\mathcal{A}^{\mathrm{enh}}$ of $\mathcal{A}$, which is an admissible subcategory of $\mathrm{D^b}(X)^{\mathrm{enh}}$, i.e. the inclusion $\mathcal{A}^{\mathrm{enh}} \to \mathrm{D^b}(X)^{\mathrm{enh}}$ admits left and right adjoints.

For objects $X, Y \in \mathcal{C}$ in an ∞-category $\mathcal{C}$, we denote by $\mathrm{Map}_{\mathcal{C}}(X, Y)$ the space of maps from X to Y. Let $\mathrm{Vect}_{\mathbf{k}}$ denote the ∞-category of complexes of $\mathbf{k}$-vector spaces. If $\mathcal{C}$ is a $\mathbf{k}$-linear stable ∞-category, then there is a mapping object $\underline{\mathrm{Map}}_{\mathcal{C}}(X, Y) \in \mathrm{Vect}_{\mathbf{k}}$ characterized by equivalences

$$\mathrm{Map}_{\mathrm{Vect}_{\mathbf{k}}}(V, \underline{\mathrm{Map}}_{\mathcal{C}}(X, Y)) \simeq \mathrm{Map}_{\mathcal{C}}(V \otimes X, Y) \tag{A.1}$$

for $V \in \mathrm{Vect}_{\mathbf{k}}^{\mathrm{fd}}$.

A.2 Ind-completion

We review here the "large" version of $\mathbf{k}$-linear stable ∞-categories. The only time this material will be needed below is in the proof of Proposition A.14 and its attendant lemmas.

Let $\mathrm{Vect}_{\mathbf{k}}$ denote the ∞-category of complexes of $\mathbf{k}$-vector spaces, with its natural symmetric monoidal structure. A *presentable* $\mathbf{k}$*-linear stable* ∞*-category* is a presentable stable ∞-category $\mathcal{D}$ equipped with a module structure over $\mathrm{Vect}_{\mathbf{k}}$. Recall that a presentable ∞-category is one which admits small colimits and satisfies a mild set-theoretic condition — roughly, that $\mathcal{D}$ is generated under sufficiently filtered colimits by a small subcategory (see [34, Chapter 5] for details). There is an ∞-category $\mathrm{PrCat}_{\mathbf{k}}^{\mathrm{St}}$ whose objects are the presentable $\mathbf{k}$-linear stable ∞-categories and whose morphisms are the colimit preserving $\mathbf{k}$-linear functors. Just as $\mathrm{Cat}_{\mathbf{k}}^{\mathrm{St}}$, the category $\mathrm{PrCat}_{\mathbf{k}}^{\mathrm{St}}$ admits small limits and colimits. Moreover, for any $\mathcal{D} \in \mathrm{PrCat}_{\mathbf{k}}^{\mathrm{St}}$ and objects $X, Y \in \mathcal{D}$, there is a mapping object $\underline{\mathrm{Map}}_{\mathcal{C}}(X, Y) \in \mathrm{Vect}_{\mathbf{k}}$ characterized by equivalences as in (A.1), where now V is allowed to be any object in $\mathrm{Vect}_{\mathbf{k}}$.

Remark A.2. In the literature, the term "$\mathbf{k}$-linear stable ∞-category" is often taken to mean a *presentable* $\mathbf{k}$-linear stable ∞-category in the sense described above. We have instead reserved this term for the small version of these categories described in Section A.1, because we will almost exclusively deal with categories of this type.

The small and presentable versions of $\mathbf{k}$-linear stable ∞-categories are related via the operation of Ind-completion. Namely, there is a functor

$$\mathrm{Ind}\colon \mathrm{Cat}_{\mathbf{k}}^{\mathrm{St}} \to \mathrm{PrCat}_{\mathbf{k}}^{\mathrm{St}}$$

which takes $\mathcal{C} \in \mathrm{Cat}_{\mathbf{k}}^{\mathrm{St}}$ to its *Ind-completion* $\mathrm{Ind}(\mathcal{C}) \in \mathrm{PrCat}_{\mathbf{k}}^{\mathrm{St}}$. Roughly, $\mathrm{Ind}(\mathcal{C})$ is obtained from $\mathcal{C}$ by freely adjoining all filtered colimits.

Remark A.3. If X is a smooth variety over $\mathbf{k}$, then $\mathrm{Ind}(\mathrm{D}^{\mathrm{b}}(X)^{\mathrm{enh}})$ is a presentable $\mathbf{k}$-linear stable ∞-category whose homotopy category is equivalent to the unbounded derived category of quasi-coherent sheaves on X.

For the details of Ind-completion, see [34, Chapter 5] or [13, Chapter I.1, Section 7.2]. All that we need for our purposes are the following facts.

Lemma A.4. *Ind-completion satisfies the following properties:*

(1) The functor $\mathrm{Ind}\colon \mathrm{Cat}_{\mathbf{k}}^{\mathrm{St}} \to \mathrm{PrCat}_{\mathbf{k}}^{\mathrm{St}}$ *commutes with colimits.*

(2) There is a natural fully faithful functor $\mathcal{C} \hookrightarrow \mathrm{Ind}(\mathcal{C})$ *of* $\mathrm{Vect}_{\mathbf{k}}^{\mathrm{fd}}$*-module categories whose essential image is the subcategory* $\mathrm{Ind}(\mathcal{C})^c \subset \mathrm{Ind}(\mathcal{C})$ *of compact objects, and which therefore induces an equivalence* $\mathcal{C} \simeq \mathrm{Ind}(\mathcal{C})^c$.

A.3 Serre functors

A $\mathbf{k}$-linear stable ∞-category $\mathcal{C}$ is *proper* if for all $X, Y \in \mathcal{C}$, the mapping object $\underline{\mathrm{Map}}_{\mathcal{C}}(X, Y)$ lies in the essential image of $\mathrm{Vect}_{\mathbf{k}}^{\mathrm{fd}} \to \mathrm{Vect}_{\mathbf{k}}$, i.e. if its total cohomology $\bigoplus_i \mathrm{H}^i(\underline{\mathrm{Map}}_{\mathcal{C}}(X, Y))$

is finite-dimensional. In this situation, we have functors

$$\begin{aligned} \underline{\mathrm{Map}}_{\mathcal{C}} \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} &\to \mathrm{Vect}_{\mathbf{k}}^{\mathrm{fd}} & \qquad \underline{\mathrm{Map}}_{\mathcal{C}}^{\mathrm{op}} \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} &\to (\mathrm{Vect}_{\mathbf{k}}^{\mathrm{fd}})^{\mathrm{op}} \\ (X, Y) &\mapsto \underline{\mathrm{Map}}_{\mathcal{C}}(X, Y) & (X, Y) &\mapsto \underline{\mathrm{Map}}_{\mathcal{C}}(Y, X) \end{aligned}$$

Note that there is an equivalence $(-)^{\vee} \colon (\mathrm{Vect}_{\mathbf{k}}^{\mathrm{fd}})^{\mathrm{op}} \xrightarrow{\sim} \mathrm{Vect}_{\mathbf{k}}^{\mathrm{fd}}$ given by dualization of complexes. Here and below, given an ∞-category $\mathcal{C}$, we denote by $\mathcal{C}^{\mathrm{op}}$ its opposite category.

Definition A.5. Let $\mathcal{C}$ be a proper $\mathbf{k}$-linear stable ∞-category. A *Serre functor* for $\mathcal{C}$ is an autoequivalence $\mathrm{S} \colon \mathcal{C} \xrightarrow{\sim} \mathcal{C}$, such that there is a commutative diagram

$$\begin{tikzcd} \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \arrow[r, "\mathrm{id} \times \mathrm{S}"] \arrow[d, "\underline{\mathrm{Map}}_{\mathcal{C}}^{\mathrm{op}}"'] & \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \arrow[d, "\underline{\mathrm{Map}}_{\mathcal{C}}"] \\ (\mathrm{Vect}_{\mathbf{k}}^{\mathrm{fd}})^{\mathrm{op}} \arrow[r, "(-)^{\vee}"] & \mathrm{Vect}_{\mathbf{k}}^{\mathrm{fd}} \end{tikzcd}$$

In other words, a Serre functor for $\mathcal{C}$ is characterized by the existence of natural equivalences

$$\underline{\mathrm{Map}}_{\mathcal{C}}(X, \mathrm{S}(Y)) \simeq \underline{\mathrm{Map}}_{\mathcal{C}}(Y, X)^{\vee}$$

for all objects $X, Y \in \mathcal{C}$. Note that if S is a Serre functor for $\mathcal{C}$, then it induces an autoequivalence of the homotopy category $\mathrm{h}\mathcal{C}$, which is a Serre functor in the usual sense of triangulated categories (see [4] or Section 2.3).

Remark A.6. If X is a smooth proper variety, then $\mathrm{D}^{\mathrm{b}}(X)^{\mathrm{enh}}$ admits a Serre functor, given by the usual formula (see [12, Section 9], where Serre duality is developed in a much more general setup). Further, if $\mathcal{A} \subset \mathrm{D}^{\mathrm{b}}(X)$ is an admissible subcategory, then $\mathcal{A}^{\mathrm{enh}}$ also admits a Serre functor. Indeed, in general suppose $\mathcal{C}$ is $\mathbf{k}$-linear stable ∞-category with a Serre functor $\mathrm{S}_{\mathcal{C}}$. Then the same argument as in the triangulated case shows that if $i \colon \mathcal{D} \to \mathcal{C}$ is the inclusion of an admissible subcategory with right and left adjoints $i^{!}$ and i^{*}, the category $\mathcal{D}$ has a Serre functor given by $\mathrm{S}_{\mathcal{D}} = i^{!} \circ \mathrm{S}_{\mathcal{C}} \circ i$ with inverse $\mathrm{S}_{\mathcal{D}}^{-1} = i^{*} \circ \mathrm{S}_{\mathcal{C}}^{-1} \circ i$.

A.4 Hochschild cohomology

If $\mathcal{C}$ and $\mathcal{D}$ are $\mathbf{k}$-linear stable ∞-categories, then the $\mathbf{k}$-linear exact functors from $\mathcal{C}$ to $\mathcal{D}$ form the objects of a $\mathbf{k}$-linear stable ∞-category $\mathrm{Fun}_{\mathbf{k}}(\mathcal{C}, \mathcal{D})$.

Definition A.7. Let $\mathcal{C}$ be a $\mathbf{k}$-linear stable ∞-category. The *Hochschild cochain complex* of $\mathcal{C}$ is defined as

$$\mathrm{HC}^\bullet(\mathcal{C}) = \underline{\mathrm{Map}}_{\mathrm{Fun}_{\mathbf{k}}(\mathcal{C},\mathcal{C})}(\mathrm{id}_{\mathcal{C}}, \mathrm{id}_{\mathcal{C}}) \in \mathrm{Vect}_{\mathbf{k}} .$$

The *Hochschild cohomology* $\mathrm{HH}^\bullet(\mathcal{C})$ of $\mathcal{C}$ is the cohomology of this complex.

Hochschild cohomology is not functorial with respect to arbitrary functors of $\mathbf{k}$-linear stable ∞-categories. But, of course, it is functorial with respect to equivalences. Namely, if $\Phi \colon \mathcal{C} \to \mathcal{D}$ is an equivalence, conjugation by Φ induces an equivalence

$$\Phi_* \colon \mathrm{HC}^\bullet(\mathcal{C}) \xrightarrow{\sim} \mathrm{HC}^\bullet(\mathcal{D})$$

and hence an isomorphism $\Phi_* \colon \mathrm{HH}^\bullet(\mathcal{C}) \xrightarrow{\sim} \mathrm{HH}^\bullet(\mathcal{D})$. Explicitly, the image $\Phi_*(a) \in \mathrm{HC}^\bullet(\mathcal{D})$ of an element $a \colon \mathrm{id}_{\mathcal{C}} \to \mathrm{id}_{\mathcal{C}} \in \mathrm{HC}^\bullet(\mathcal{C})$ is determined by the commutative diagram

$$\begin{tikzcd}
\mathrm{id}_{\mathcal{D}} \arrow[r, "\Phi_*(a)"] \arrow[d, "\wr"'] & \mathrm{id}_{\mathcal{D}} \arrow[d, "\wr"] \\
\Phi \circ \mathrm{id}_{\mathcal{C}} \circ \Phi^{-1} \arrow[r, "\Phi a \Phi^{-1}"] & \Phi \circ \mathrm{id}_{\mathcal{C}} \circ \Phi^{-1}
\end{tikzcd}$$

A.5 The Serre functor's action on Hochschild cohomology

We will show that Serre functors act trivially on Hochschild cohomology:

Theorem A.8. *Let $\mathcal{C}$ be a proper $\mathbf{k}$-linear stable ∞-category. Assume $\mathcal{C}$ admits a Serre functor S. Then the induced map $\mathrm{S}_* \colon \mathrm{HH}^\bullet(\mathcal{C}) \to \mathrm{HH}^\bullet(\mathcal{C})$ is the identity.*

We will deduce the theorem from Lemma A.10 below. We need the following lemma, which can be verified by unwinding the definitions.

Lemma A.9. *Let $F, G\colon \mathcal{C} \to \mathcal{D}$ be functors between $\mathbf{k}$-linear stable ∞-categories, and let $a\colon F \to G$ be a point of the mapping space $\mathrm{Map}_{\mathrm{Fun}_{\mathbf{k}}(\mathcal{C},\mathcal{D})}(F, G)$. Assume F and G admit right adjoints $F^!$ and $G^!$. Let $\eta_F\colon \mathrm{id}_{\mathcal{C}} \to F^! \circ F$ be the unit of the adjunction between F and $F^!$, and let $\epsilon_G\colon G \circ G^! \to \mathrm{id}_{\mathcal{D}}$ be the counit of the adjunction between G and $G^!$. Define $a^!\colon G^! \to F^!$ as the composition*

$$a^!\colon G^! = \mathrm{id}_{\mathcal{C}} \circ G^! \xrightarrow{\eta_F G^!} F^! \circ F \circ G^! \xrightarrow{F^! a G^!} F^! \circ G \circ G^! \xrightarrow{F^! \epsilon_G} F^! \circ \mathrm{id}_{\mathcal{D}} = F^!.$$

Then there is functorially in $X, Y \in \mathcal{C}$ a commutative diagram

$$\begin{tikzcd}
\underline{\mathrm{Map}}_{\mathcal{D}}(F(X), Y) \arrow[d, "\wr"'] & \underline{\mathrm{Map}}_{\mathcal{D}}(G(X), Y) \arrow[l] \arrow[d, "\wr"] \\
\underline{\mathrm{Map}}_{\mathcal{C}}(X, F^!(Y)) & \underline{\mathrm{Map}}_{\mathcal{C}}(X, G^!(Y)) \arrow[l]
\end{tikzcd}$$

where the top horizontal arrow is induced by $a\colon F \to G$, the bottom horizontal arrow is induced by $a^!\colon G^! \to F^!$, and the vertical arrows are given by adjunction.

It is well known that a Serre functor commutes with all autoequivalences. The following result gives a sense in which this commutation is functorial with respect to morphisms of autoequivalences.

Lemma A.10. *Let $\mathcal{C}$ be a proper $\mathbf{k}$-linear stable ∞-category, which admits a Serre functor S. Let $F, G\colon \mathcal{C} \to \mathcal{C}$ be autoequivalences of $\mathcal{C}$. Let $a\colon F \to G$ be an object of $\mathrm{Map}_{\mathrm{Fun}_{\mathbf{k}}(\mathcal{C},\mathcal{C})}(F, G)$, and let $(a^!)^!\colon F \to G$ be the morphism obtained by applying the construction of Lemma A.9 twice (note that F and its inverse F^{-1} are mutually left and right adjoint, and similarly for G and G^{-1}). Then there is a commutative diagram of functors*

$$\begin{tikzcd}
\mathrm{S} \circ F \arrow[r, "\mathrm{S}a"] \arrow[d, "\wr"'] & \mathrm{S} \circ G \arrow[d, "\wr"] \\
F \circ \mathrm{S} \arrow[r, "(a^!)^! \mathrm{S}"] & G \circ \mathrm{S}
\end{tikzcd}$$

where the vertical arrows are equivalences.

Proof. We have functorially in $X, Y \in \mathcal{C}$ a diagram

$$\begin{tikzcd}
\underline{\mathrm{Map}}_{\mathcal{C}}(X, \mathrm{S} \circ F(Y)) \arrow[r] \arrow[d, "\wr"'] & \underline{\mathrm{Map}}_{\mathcal{C}}(X, \mathrm{S} \circ G(Y)) \arrow[d, "\wr"] \\
\underline{\mathrm{Map}}_{\mathcal{C}}(F(Y), X)^{\vee} \arrow[r] \arrow[d, "\wr"'] & \underline{\mathrm{Map}}_{\mathcal{C}}(G(Y), X)^{\vee} \arrow[d, "\wr"] \\
\underline{\mathrm{Map}}_{\mathcal{C}}(Y, F^{-1}(X))^{\vee} \arrow[r] \arrow[d, "\wr"'] & \underline{\mathrm{Map}}_{\mathcal{C}}(Y, G^{-1}(X))^{\vee} \arrow[d, "\wr"] \\
\underline{\mathrm{Map}}_{\mathcal{C}}(F^{-1}(X), S(Y)) \arrow[r] \arrow[d, "\wr"'] & \underline{\mathrm{Map}}_{\mathcal{C}}(G^{-1}(X), \mathrm{S}(Y)) \arrow[d, "\wr"] \\
\underline{\mathrm{Map}}_{\mathcal{C}}(X, F \circ S(Y)) \arrow[r] & \underline{\mathrm{Map}}_{\mathcal{C}}(X, G \circ \mathrm{S}(Y)).
\end{tikzcd}$$

The first and third vertical equivalences in the diagram are given by the defining equivalences of a Serre functor, and the second and fourth are given by the adjunctions between F and F^{-1} and G and G^{-1}. The first and second horizontal morphisms are induced by $a\colon F \to G$, the third and fourth by $a^{!}\colon G^{-1} \to F^{-1}$, and the last by $(a^{!})^{!}\colon F \to G$. The first and third squares in the diagram commute since the defining equivalences of a Serre functor are functorial, and the second and fourth squares commute by Lemma A.9. Hence the diagram is commutative. The outer square in the diagram thus induces the desired diagram of functors. $\square$

Proof of Theorem A.8. By the definition of the action of S on $\mathrm{HH}^{\bullet}(\mathcal{C})$, it suffices to show that for any $a\colon \mathrm{id}_{\mathcal{C}} \to \mathrm{id}_{\mathcal{C}} \in \mathrm{HC}^{\bullet}(\mathcal{C})$, there is a commutative diagram

$$\begin{tikzcd}
\mathrm{S} \circ \mathrm{id}_{\mathcal{C}} \arrow[r, "\mathrm{S}a"] \arrow[d, "\mathrm{id}"'] & \mathrm{S} \circ \mathrm{id}_{\mathcal{C}} \arrow[d, "\mathrm{id}"] \\
\mathrm{id}_{\mathcal{C}} \circ \mathrm{S} \arrow[r, "a\mathrm{S}"] & \mathrm{id}_{\mathcal{C}} \circ \mathrm{S}
\end{tikzcd}$$

This follows from Lemma A.10 with $F = G = \mathrm{id}_{\mathcal{C}}$. Indeed, in this case it is easy to see that $(a^{!})^{!} = a$ and that the equivalences $\mathrm{S} \circ F \simeq F \circ \mathrm{S}$ and $\mathrm{S} \circ G \simeq G \circ \mathrm{S}$ constructed in the proof of the lemma are given by the identity. $\square$

A.6 Preliminaries on group actions

Let G be a finite group. We denote by BG the classifying space of G, regarded as an ∞-category (i.e. BG is the nerve of the ordinary category with a single object whose endomorphisms are given by the group G).

Definition A.11. Let $\mathcal{D}$ be an ∞-category, and let $X \in \mathcal{D}$ be an object. An *action* of G on X is a functor $\phi\colon BG \to \mathcal{D}$ which carries the unique object $* \in BG$ to $X \in \mathcal{D}$. Given such an action, the *G-invariants* X^G and *G-coinvariants* X_G are defined by

$$X^G = \lim(\phi) \qquad \text{and} \qquad X_G = \operatorname{colim}(\phi),$$

provided the displayed limit and colimit exist. In this case, we denote by $p\colon X^G \to X$ and $q\colon X \to X_G$ the canonical morphisms.

Remark A.12. What we have called an action of G could more precisely be called a *left action* of G. There is also a notion of a *right action* of G on an object $X \in \mathcal{D}$, namely, a functor $\psi\colon BG^{\mathrm{op}} \to \mathcal{D}$ from the opposite category of BG which carries the basepoint to X. Note that any left action $\phi\colon BG \to \mathcal{D}$ on X gives rise to a right action by composing with the equivalence $BG^{\mathrm{op}} \simeq BG$ induced by inversion in G.

To relate the above definition to the classical notion of a group action, note that ϕ specifies for each $g \in G$ (thought of as an endomorphism of the basepoint $* \in BG$) an equivalence $\phi_g\colon X \to X$. The data of the entire action functor $\phi\colon BG \to \mathcal{D}$ then specifies certain compatibilities among the ϕ_g.

We will be particularly interested in the case of Definition A.11 where $\mathcal{D} = \mathrm{Cat}^{\mathrm{St}}_{\mathbf{k}}$, i.e. where G acts on a $\mathbf{k}$-linear stable ∞-category $\mathcal{C} \in \mathrm{Cat}^{\mathrm{St}}_{\mathbf{k}}$. Since $\mathrm{Cat}^{\mathrm{St}}_{\mathbf{k}}$ admits limits and colimits, the G-invariants $\mathcal{C}^G$ and coinvariants $\mathcal{C}_G$ always exist in this situation.

The objects and morphisms of $\mathcal{C}^G$ can be described in relatively concrete terms, as follows. By the universal property of the projection functor $\mathcal{C}^G \to \mathcal{C}$, an object of $\widetilde{X} \in \mathcal{C}^G$ corresponds to an object $X \in \mathcal{C}$ together with a *linearization*, i.e. a family of equivalences

$\ell_g\colon X \to \phi_g(X)$ satisfying certain compatibilities. By abuse of notation, given an object $X \in \mathcal{C}$ with a linearization, we will often denote by the same symbol X the corresponding object of $\mathcal{C}^G$.

Given another object $\widetilde{Y} \in \mathcal{C}^G$ corresponding to $Y \in \mathcal{C}$ with linearizations $m_g\colon Y \to \phi_g(Y)$, there is a natural action of G on $\underline{\mathrm{Map}}_{\mathcal{C}}(X,Y)$, i.e. a functor $\rho\colon BG \to \mathrm{Vect}_{\mathbf{k}}$ sending the basepoint $* \in BG$ to $\underline{\mathrm{Map}}_{\mathcal{C}}(X,Y) \in \mathrm{Vect}_{\mathbf{k}}$. Concretely, the image of $f\colon X \to Y \in \underline{\mathrm{Map}}_{\mathcal{C}}(X,Y)$ under $\rho_g\colon \underline{\mathrm{Map}}_{\mathcal{C}}(X,Y) \to \underline{\mathrm{Map}}_{\mathcal{C}}(X,Y)$ is determined by the commutative diagram

$$\begin{tikzcd} X \arrow[r, "\rho_g(f)"] \arrow[d, "\ell_g"'] & Y \arrow[d, "m_g"] \\ \phi_g(X) \arrow[r, "\phi_g(f)"] & \phi_g(Y) \end{tikzcd}$$

The morphisms in the category $\mathcal{C}^G$ are then described by the formula

$$\underline{\mathrm{Map}}_{\mathcal{C}^G}(\widetilde{X},\widetilde{Y}) \simeq \underline{\mathrm{Map}}_{\mathcal{C}}(X,Y)^G. \tag{A.2}$$

A.6.1 Induction and restriction

Let G be a finite group and $H \subset G$ a subgroup. Recall that there are natural induction and restriction functors between the classical categories of linear representations of H and G, and that these functors are mutually left and right adjoint. There is an analogous relation between the categories of invariants for group actions on a $\mathbf{k}$-linear stable ∞-category.

Namely, let $\mathcal{C}$ be a $\mathbf{k}$-linear stable ∞-category with a G-action. Then there is an induced H-action on $\mathcal{C}$. The functor $BH \to BG$ induces the *restriction functor*

$$\mathrm{Res}^G_H\colon \mathcal{C}^G \to \mathcal{C}^H.$$

Further, by choosing representatives for the cosets of G/H, we obtain a functor

$$\bigoplus_{[g]\in G/H} \phi_g\colon \mathcal{C} \to \mathcal{C}, \qquad X \mapsto \bigoplus_{[g]\in G/H} \phi_g(X).$$

Precomposing with the projection $p\colon \mathcal{C}^H \to \mathcal{C}$ yields a functor $\mathcal{C}^H \to \mathcal{C}$, which lifts to the *induction functor*

$$\operatorname{Ind}_H^G\colon \mathcal{C}^H \to \mathcal{C}^G.$$

If H is the trivial group, we write $\operatorname{Av}\colon \mathcal{C} \to \mathcal{C}^G$ for the induction functor and call it the *averaging functor*.

In the setting of triangulated categories with a group action, it is well-known that the induction and restriction functors are mutually left and right adjoint (see [10, Lemma 3.8] for the case where H is the trivial group). In our setup, we have the following analogue.

Lemma A.13. *Let $\mathcal{C}$ be a $\mathbf{k}$-linear stable ∞-category with an action by a finite group G, and let $H \subset G$ be a subgroup. Then the functors Ind_H^G and Res_H^G are mutually left and right adjoint.*

A.6.2 The norm functor

Let V be a (ordinary) $\mathbf{k}$-vector space with a G-action. Then the *norm map* $\operatorname{Nm}\colon V_G \to V^G$ is induced by the map $V \to V$ given by $x \mapsto \sum_{g\in G} g(x)$. Using that the order of G is invertible in $\mathbf{k}$ (as $\mathbf{k}$ has characteristic 0), it is easy to see that the norm map is an isomorphism. By the same argument, the analogous statement holds in the ∞-categorical setting where $V \in \operatorname{Vect}_{\mathbf{k}}$ and G acts on V: the norm map $V_G \to V^G$ is an equivalence. Our goal below is to describe what happens when V is replaced by a $\mathbf{k}$-linear stable ∞-category.

Let $\mathcal{C}$ be a $\mathbf{k}$-linear stable ∞-category with a G-action. By the universal properties of $\mathcal{C}_G$ and $\mathcal{C}^G$, the functor $\bigoplus_{g\in G} \phi_g\colon \mathcal{C} \to \mathcal{C}$ induces the *norm functor*

$$\operatorname{Nm}\colon \mathcal{C}_G \to \mathcal{C}^G,$$

which is characterized by the existence of a factorization

$$\bigoplus_{g\in G} \phi_g\colon \mathcal{C} \xrightarrow{q} \mathcal{C}_G \xrightarrow{\operatorname{Nm}} \mathcal{C}^G \xrightarrow{p} \mathcal{C}.$$

Note that the composition $\mathrm{Nm} \circ q \colon \mathcal{C} \to \mathcal{C}^G$ is nothing but the averaging functor Av. We aim to prove:

Proposition A.14. *Let $\mathcal{C}$ be a $\mathbf{k}$-linear stable ∞-category with an action by a finite group G. Then the norm functor $\mathrm{Nm} \colon \mathcal{C}_G \to \mathcal{C}^G$ is an equivalence.*

We will prove the proposition by relating it to the situation for presentable $\mathbf{k}$-linear stable ∞-categories. First note that if $\mathcal{D}$ is such a category with a G-action, then the same construction as above provides a norm functor $\mathrm{Nm} \colon \mathcal{D}_G \to \mathcal{D}^G$.

Lemma A.15. *Let $\mathcal{D}$ be a presentable $\mathbf{k}$-linear stable ∞-category with an action by a finite group G. Then the norm functor $\mathrm{Nm} \colon \mathcal{D}_G \to \mathcal{D}^G$ is an equivalence.*

Proof. We use the following fundamental relation between colimits and limits in the presentable setting. Let $F \colon I \to \mathrm{PrCat}^{\mathrm{St}}_{\mathbf{k}}$ be a functor from a small ∞-category I to $\mathrm{PrCat}^{\mathrm{St}}_{\mathbf{k}}$; we think of F as a diagram of categories $F(i) = \mathcal{C}_i$ for $i \in I$. Assume that for every morphism $a \colon i \to j$, the functor $F(a) \colon \mathcal{C}_i \to \mathcal{C}_j$ admits a right adjoint $F(a)^!$. Then there is a functor $G \colon I^{\mathrm{op}} \to \mathrm{PrCat}^{\mathrm{St}}_{\mathbf{k}}$ obtained by "passing to right adjoints", which satisfies $G(i) = \mathcal{C}_i$ for all $i \in I$, but $G(a) = F(a)^! \colon \mathcal{C}_j \to \mathcal{C}_i$ for every morphism $a \colon i \to j$ in I regarded as a morphism $j \to i$ in I^{op} (see [13, Chapter I.1, Section 2.4]). Then the key fact is that there is an equivalence $\mathrm{colim}(F) \simeq \lim(G)$, induced by the left adjoints $p_i^* \colon \mathcal{C}_i \to \lim(G)$ to the natural functors $p_i \colon \lim(G) \to \mathcal{C}_i$ for $i \in I$ (see [13, Chapter I.1, Section 2.5.8]).

Let us apply this to the functor $\phi \colon BG \to \mathrm{PrCat}^{\mathrm{St}}_{\mathbf{k}}$ encoding the action of G on $\mathcal{D}$. The functor $\psi \colon BG^{\mathrm{op}} \to \mathrm{PrCat}^{\mathrm{St}}_{\mathbf{k}}$ obtained by passing to right adjoints is nothing but the right G-action induced by ϕ, as described in Remark A.12. In particular, since ψ differs from ϕ by composition with the equivalence $BG^{\mathrm{op}} \simeq BG$, we see that $\lim(\psi) \simeq \lim(\phi) = \mathcal{D}^G$. Hence applying the key fact from above, we find that there is an equivalence $\mathcal{D}_G \simeq \mathcal{D}^G$. Moreover, this equivalence is induced by the right adjoint to $p \colon \mathcal{D}^G \to \mathcal{D}$, i.e. by the averaging functor $\mathrm{Av} \colon \mathcal{D} \to \mathcal{D}^G$, and hence is given by the norm functor. $\square$

Remark A.16. The proof of Lemma A.15 shows that it holds without any assumption on the characteristic of $\mathbf{k}$. In contrast, Proposition A.14 does not hold without the assumption that the order of G is coprime to the characteristic of $\mathbf{k}$.

We will also need the following result. We use the notation $\mathcal{D}^c$ for the full subcategory of compact objects of an ∞-category $\mathcal{D}$.

Lemma A.17. *Let $\mathcal{C}$ be a $\mathbf{k}$-linear stable ∞-category with an action by a finite group G. Then the natural fully faithful functor $\mathcal{C} \hookrightarrow \mathrm{Ind}(\mathcal{C})$ induces equivalences $\mathcal{C}_G \simeq (\mathrm{Ind}(\mathcal{C})_G)^c$ and $\mathcal{C}^G \simeq (\mathrm{Ind}(\mathcal{C})^G)^c$, where G acts on $\mathrm{Ind}(\mathcal{C})$ by functoriality.*

Proof. For the coinvariants, note that by Lemma A.4(1) we have $\mathrm{Ind}(\mathcal{C})_G \simeq \mathrm{Ind}(\mathcal{C}_G)$. Hence by Lemma A.4(2) the functor $\mathcal{C}_G \to \mathrm{Ind}(\mathcal{C})_G$ factors through an equivalence $\mathcal{C}_G \simeq (\mathrm{Ind}(\mathcal{C})_G)^c$. For the invariants, observe that the natural functor $\mathcal{C}^G \to \mathrm{Ind}(\mathcal{C})^G$ is fully faithful, by fully faithfulness of $\mathcal{C} \hookrightarrow \mathrm{Ind}(\mathcal{C})$ combined with the description (A.2) (which also holds in the presentable setting) of mapping spaces in a category of invariants. This realizes $\mathcal{C}^G$ as the full subcategory of $\mathrm{Ind}(\mathcal{C})^G$ consisting of objects whose image under $\mathrm{Ind}(\mathcal{C})^G \to \mathrm{Ind}(\mathcal{C})$ is in $\mathcal{C} \simeq \mathrm{Ind}(\mathcal{C})^c \subset \mathrm{Ind}(\mathcal{C})$. Now the equivalence $\mathcal{C}^G \simeq (\mathrm{Ind}(\mathcal{C})^G)^c$ is a consequence of the following lemma. □

Lemma A.18. *Let $\mathcal{D}$ be a presentable $\mathbf{k}$-linear stable ∞-category with an action by a finite group G. Then an object of $\mathcal{D}^G$ is compact if and only if its image under $\mathcal{D}^G \to \mathcal{D}$ is compact.*

Proof. Let $\widetilde{X}$ be an object of $\mathcal{D}^G$ and X its image in $\mathcal{D}$. Assume first that $\widetilde{X}$ is compact. Then by adjointness of induction and restriction, for $Y \in \mathcal{D}$ we have

$$\underline{\mathrm{Map}}_{\mathcal{D}}(X, Y) \simeq \underline{\mathrm{Map}}_{\mathcal{D}^G}(\widetilde{X}, \mathrm{Av}(Y)).$$

The functor Av commutes with colimits because it is a left adjoint, and $\underline{\mathrm{Map}}_{\mathcal{D}^G}(\widetilde{X}, -)$ commutes with filtered colimits by compactness of $\widetilde{X}$. Hence the above equivalence shows that $\underline{\mathrm{Map}}_{\mathcal{D}}(X, -)$ commutes with filtered colimits, i.e. that X is compact.

Now assume that X is compact. Let $p\colon \mathcal{D}^G \to \mathcal{D}$ be the projection. Then for $\widetilde{Y} \in \mathcal{D}^G$, we have

$$\underline{\mathrm{Map}}_{\mathcal{D}^G}(\widetilde{X}, \widetilde{Y}) \simeq \underline{\mathrm{Map}}_{\mathcal{D}}(X, p(\widetilde{Y}))^G \simeq \underline{\mathrm{Map}}_{\mathcal{D}}(X, p(\widetilde{Y}))_G,$$

where the first equivalence is (A.2) and the second is given by the inverse of the norm map (here we use the assumption that $\mathbf{k}$ has characteristic 0). Note that $p\colon \mathcal{D}^G \to \mathcal{D}$ commutes with colimits because it is a left adjoint, $\underline{\mathrm{Map}}_{\mathcal{D}}(X, -)$ commutes with filtered colimits by compactness of X, and taking G-coinvariants commutes with colimits because by definition it is given by a colimit. Hence the above equivalence shows that $\underline{\mathrm{Map}}_{\mathcal{D}^G}(\widetilde{X}, -)$ commutes with filtered colimits, i.e. that $\widetilde{X}$ is compact. □

Proof of Proposition A.14. By Lemma A.15 the norm functor $\mathrm{Nm}\colon \mathrm{Ind}(\mathcal{C})_G \to \mathrm{Ind}(\mathcal{C})^G$ is an equivalence, and hence so is its restriction to the full subcategories of compact objects. But by Lemma A.17, this restriction is identified with $\mathrm{Nm}\colon \mathcal{C}_G \to \mathcal{C}^G$. □

A.6.3 Serre functors of categories of invariants

We include the following result because it fits naturally into our discussion. However, it will not be needed below, and so can safely be skipped.

Lemma A.19. *Let $\mathcal{C}$ be a proper $\mathbf{k}$-linear stable ∞-category with an action by a finite group G. Assume $\mathcal{C}$ admits a Serre functor S. Then the composition*

$$\mathcal{C}^G \xrightarrow{p} \mathcal{C} \xrightarrow{S} \mathcal{C}$$

induces a functor $\mathrm{S}^G\colon \mathcal{C}^G \to \mathcal{C}^G$, which is a Serre functor for $\mathcal{C}^G$.

Proof. First note that $\mathcal{C}^G$ is proper, by the description of its mapping spaces (A.2) and our assumption that $\mathbf{k}$ has characteristic 0. The Serre functor S canonically commutes with autoequivalences, and hence with the G-action, so it induces an autoequivalence $\mathrm{S}^G\colon \mathcal{C}^G \to \mathcal{C}^G$.

We have functorially in $\widetilde{X}, \widetilde{Y} \in \mathcal{C}^G$ equivalences

$$\underline{\mathrm{Map}}_{\mathcal{C}^G}(\widetilde{X}, \mathrm{S}^G(\widetilde{Y})) \simeq \underline{\mathrm{Map}}_{\mathcal{C}}(X, \mathrm{S}(Y))^G \simeq (\underline{\mathrm{Map}}_{\mathcal{C}}(Y, X)^\vee)^G \tag{A.3}$$

By functoriality of the defining equivalences of a Serre functor, the action of G on $\underline{\mathrm{Map}}_{\mathcal{C}}(Y, X)^\vee$ is induced by the action of G on $\underline{\mathrm{Map}}_{\mathcal{C}}(Y, X)$. Moreover, by definition

$$\underline{\mathrm{Map}}_{\mathcal{C}}(Y, X)^\vee = \underline{\mathrm{Map}}_{\mathrm{Vect}_{\mathbf{k}}}(\underline{\mathrm{Map}}_{\mathcal{C}}(Y, X), \mathbf{k}),$$

so since the formation of the mapping space $\underline{\mathrm{Map}}_{\mathrm{Vect}_{\mathbf{k}}}(-,-)$ takes colimits in the first variable to limits, we find

$$(\underline{\mathrm{Map}}_{\mathcal{C}}(Y, X)^\vee)^G \simeq (\underline{\mathrm{Map}}_{\mathcal{C}}(Y, X)_G)^\vee. \tag{A.4}$$

Further, we have

$$\underline{\mathrm{Map}}_{\mathcal{C}}(Y, X)_G \simeq \underline{\mathrm{Map}}_{\mathcal{C}}(Y, X)^G \simeq \underline{\mathrm{Map}}_{\mathcal{C}^G}(\widetilde{Y}, \widetilde{X}), \tag{A.5}$$

where the first equivalence is given by the norm map (using the assumption that $\mathbf{k}$ has characteristic 0) and the second by (A.2). Finally, combining (A.3)-(A.5) gives the required functorial equivalences

$$\underline{\mathrm{Map}}_{\mathcal{C}^G}(\widetilde{X}, \mathrm{S}^G(\widetilde{Y})) \simeq \underline{\mathrm{Map}}_{\mathcal{C}^G}(\widetilde{Y}, \widetilde{X})^\vee. \qquad \square$$

Corollary A.20. *Let $\mathcal{C}$ be a proper $\mathbf{k}$-linear stable ∞-category such that $\mathrm{S} = \sigma \circ [n]$ is a Serre functor, where n is an integer and $\sigma \colon \mathcal{C} \to \mathcal{C}$ is the autoequivalence corresponding to the generator of a $\mathbf{Z}/k$-action on $\mathcal{C}$. Then the shift functor $[n] \colon \mathcal{C}^{\mathbf{Z}/k} \to \mathcal{C}^{\mathbf{Z}/k}$ is a Serre functor for $\mathcal{C}^{\mathbf{Z}/k}$.*

Proof. Lemma A.19 shows $\mathrm{S} = \sigma \circ [n]$ induces a Serre functor for $\mathcal{C}^{\mathbf{Z}/k}$, but σ induces the identity on $\mathcal{C}^{\mathbf{Z}/k}$. $\square$

Remark A.21. In the situation of Corollary A.20, the category $\mathcal{C}^{\mathbf{Z}/k}$ should be regarded as the "canonical Calabi–Yau cover" of $\mathcal{C}$. Indeed, assume X is a smooth projective variety of dimension n, whose canonical bundle satisfies $\omega_X^k \cong \mathcal{O}_X$. Then there is a k-fold étale cover

$\widetilde{X} = \mathrm{Spec}_X(\mathcal{R}) \to X$, where

$$\mathcal{R} = \mathcal{O}_X \oplus \omega_X \oplus \cdots \oplus \omega_X^{k-1}$$

with algebra structure determined by $\omega_X^k \cong \mathcal{O}_X$. The variety $\widetilde{X}$ is Calabi–Yau in the sense that $\omega_{\widetilde{X}} \cong \mathcal{O}_{\widetilde{X}}$. Now note that $\mathcal{C} = \mathrm{D^b}(X)^{\mathrm{enh}}$ satisfies the assumptions of Corollary A.20 with $\mathbf{Z}/k$-action given by tensoring with ω_X, and $\mathcal{C}^{\mathbf{Z}/k} \simeq \mathrm{D^b}(\widetilde{X})$ recovers the derived category of the canonical Calabi–Yau cover of X. In Corollary A.26 below, we will describe an interesting relation between the Hochschild cohomology of $\mathcal{C}$ and $\mathcal{C}^{\mathbf{Z}/k}$ in the case $k = 2$.

A.7 Hochschild cohomology and group invariants

Let $\mathcal{C}$ be a $\mathbf{k}$-linear stable ∞-category with an action by a finite group G. Our goal is to relate the Hochschild cohomology of $\mathcal{C}^G$ to that of $\mathcal{C}$. To state the result, note that there is an induced action of $G \times G$ on $\mathrm{Fun}_{\mathbf{k}}(\mathcal{C}, \mathcal{C})$. Concretely, $(g_1, g_2) \in G \times G$ acts on $\mathrm{Fun}_{\mathbf{k}}(\mathcal{C}, \mathcal{C})$ by sending $F \colon \mathcal{C} \to \mathcal{C} \in \mathrm{Fun}_{\mathbf{k}}(\mathcal{C}, \mathcal{C})$ to $\phi_{g_2} \circ F \circ \phi_{g_1}^{-1}$. Via the diagonal embedding $G \subset G \times G$, this restricts to the conjugation action of G on $\mathrm{Fun}_{\mathbf{k}}(\mathcal{C}, \mathcal{C})$.

Theorem A.22. *Let $\mathcal{C}$ be a $\mathbf{k}$-linear stable ∞-category with an action by a finite group G. Then there is an isomorphism*

$$\mathrm{HH}^\bullet(\mathcal{C}^G) \cong \mathrm{HH}^\bullet(\mathcal{C})^G \oplus \left(\bigoplus_{g \neq 1 \in G} \mathrm{H}^\bullet(\underline{\mathrm{Map}}_{\mathrm{Fun}_{\mathbf{k}}(\mathcal{C},\mathcal{C})}(\mathrm{id}_{\mathcal{C}}, \phi_g)) \right)^G,$$

where $\phi_g \colon \mathcal{C} \to \mathcal{C}$ is the autoequivalence corresponding to $g \in G$, and the G-action on the right side is induced by the conjugation action of G on $\mathrm{Fun}_{\mathbf{k}}(\mathcal{C}, \mathcal{C})$.

Remark A.23. Theorem A.22 holds verbatim for a presentable $\mathbf{k}$-linear stable ∞-category $\mathcal{D}$ in place of $\mathcal{C}$, with the same proof given below.

We will need the following lemma for the proof of the theorem.

Lemma A.24. *Let $\mathcal{C}$ be a $\mathbf{k}$-linear stable ∞-category with an action by a finite group G.*

Then there is an equivalence

$$\mathrm{Fun}_{\mathbf{k}}(\mathcal{C}^G, \mathcal{C}^G) \simeq \mathrm{Fun}_{\mathbf{k}}(\mathcal{C}, \mathcal{C})^{G\times G}$$

under which the identity $\mathrm{id}_{\mathcal{C}^G}$ *corresponds to the functor* $\bigoplus_{g\in G} \phi_g \colon \mathcal{C} \to \mathcal{C}$ *(with the natural* $G \times G$*-linearization).*

Proof. The norm equivalence $\mathrm{Nm}\colon \mathcal{C}_G \to \mathcal{C}^G$ induces an equivalence

$$\mathrm{Fun}_{\mathbf{k}}(\mathcal{C}^G, \mathcal{C}^G) \simeq \mathrm{Fun}_{\mathbf{k}}(\mathcal{C}_G, \mathcal{C}^G).$$

The formation of the functor category $\mathrm{Fun}_{\mathbf{k}}(-,-)$ between $\mathbf{k}$-linear stable ∞-categories takes colimits in the first variable to limits, and limits in the second variable to limits. Applying this to $\mathcal{C}_G = \mathrm{colim}_{BG}\, \mathcal{C}$ and then $\mathcal{C}^G = \lim_{BG} \mathcal{C}$, we find

$$\mathrm{Fun}_{\mathbf{k}}(\mathcal{C}_G, \mathcal{C}^G) \simeq \mathrm{Fun}_{\mathbf{k}}(\mathcal{C}, \mathcal{C}^G)^G \simeq (\mathrm{Fun}_{\mathbf{k}}(\mathcal{C}, \mathcal{C})^G)^G.$$

In the final term, the inner G-action is induced by the action of G on the first copy of $\mathcal{C}$, and the outer G-action by the action of G on the second copy of $\mathcal{C}$. Thus, the inner G-action is identified with the restriction of the $G \times G$-action on $\mathrm{Fun}_{\mathbf{k}}(\mathcal{C}, \mathcal{C})$ to the first factor, and the outer G-action is induced by the restriction of the $G\times G$-action to the second factor. It follows that

$$(\mathrm{Fun}_{\mathbf{k}}(\mathcal{C}, \mathcal{C})^G)^G \simeq \mathrm{Fun}_{\mathbf{k}}(\mathcal{C}, \mathcal{C})^{G\times G}.$$

All together this proves the equivalence stated in the lemma, and tracing through the intermediate equivalences above shows that $\mathrm{id}_{\mathcal{C}^G}$ corresponds to $\bigoplus_{g\in G} \phi_g$. $\square$

Proof of Theorem A.22. By the definition of the Hochschild cochain complex combined with Lemma A.24, we have

$$\mathrm{HC}^\bullet(\mathcal{C}^G) = \underline{\mathrm{Map}}_{\mathrm{Fun}_{\mathbf{k}}(\mathcal{C}^G,\mathcal{C}^G)}(\mathrm{id}_{\mathcal{C}^G}, \mathrm{id}_{\mathcal{C}^G}) \simeq \underline{\mathrm{Map}}_{\mathrm{Fun}_{\mathbf{k}}(\mathcal{C},\mathcal{C})^{G\times G}}(\textstyle\bigoplus_{g\in G} \phi_g, \bigoplus_{g\in G} \phi_g).$$

Recall that G acts on $\mathrm{Fun}_{\mathbf{k}}(\mathcal{C},\mathcal{C})$ via restriction along the diagonal embedding $G \subset G \times G$. Notice that the induction of $\mathrm{id}_{\mathcal{C}} \in \mathrm{Fun}_{\mathbf{k}}(\mathcal{C},\mathcal{C})^G$ along the diagonal is

$$\mathrm{Ind}_G^{G\times G}(\mathrm{id}_{\mathcal{C}}) = \textstyle\bigoplus_{g\in G} \phi_g \in \mathrm{Fun}_{\mathbf{k}}(\mathcal{C},\mathcal{C})^{G\times G}.$$

Hence by adjointness of induction and restriction, we can rewrite the above expression as

$$\begin{aligned}\mathrm{HC}^{\bullet}(\mathcal{C}^G) &\simeq \underline{\mathrm{Map}}_{\mathrm{Fun}_{\mathbf{k}}(\mathcal{C},\mathcal{C})^G}(\mathrm{id}_{\mathcal{C}}, \textstyle\bigoplus_{g\in G}\phi_g) \\ &\simeq \Big(\textstyle\bigoplus_{g\in G} \underline{\mathrm{Map}}_{\mathrm{Fun}_{\mathbf{k}}(\mathcal{C},\mathcal{C})}(\mathrm{id}_{\mathcal{C}}, \phi_g)\Big)^G \\ &\simeq \Big(\mathrm{HC}^{\bullet}(\mathcal{C}) \oplus \textstyle\bigoplus_{g\neq 1\in G} \underline{\mathrm{Map}}_{\mathrm{Fun}_{\mathbf{k}}(\mathcal{C},\mathcal{C})}(\mathrm{id}_{\mathcal{C}}, \phi_g)\Big)^G\end{aligned}$$

Since we are working over a field of characteristic 0, the operation of taking group invariants commutes with taking cohomology. So by taking cohomology, the theorem follows. □

A.8 An application

Recall that in the case of a category admitting a Serre functor, Hochschild homology can be defined as follows.

Definition A.25. Let $\mathcal{C}$ is a proper $\mathbf{k}$-linear stable ∞-category admitting a Serre functor S. The *Hochschild chain complex* of $\mathcal{C}$ is defined as

$$\mathrm{HC}_{\bullet}(\mathcal{C}) = \underline{\mathrm{Map}}_{\mathrm{Fun}_{\mathbf{k}}(\mathcal{C},\mathcal{C})}(\mathrm{id}_{\mathcal{C}}, \mathrm{S}) \in \mathrm{Vect}_{\mathbf{k}}\,.$$

The *Hochschild homology* $\mathrm{HH}_{\bullet}(\mathcal{C})$ of $\mathcal{C}$ is the cohomology of this complex.

Corollary A.26. *Let $\mathcal{C}$ be a proper $\mathbf{k}$-linear stable ∞-category. Assume that $\mathcal{C}$ is equipped with a $\mathbf{Z}/2$-action, such that if $\sigma\colon \mathcal{C} \to \mathcal{C}$ denotes the autoequivalence corresponding to the generator of $\mathbf{Z}/2$, then $\mathrm{S} = \sigma \circ [n]$ is a Serre functor for $\mathcal{C}$ for some integer n. Then there is an isomorphism*

$$\mathrm{HH}^{\bullet}(\mathcal{C}^{\mathbf{Z}/2}) \cong \mathrm{HH}^{\bullet}(\mathcal{C}) \oplus (\mathrm{HH}_{\bullet}(\mathcal{C})^{\mathbf{Z}/2}[-n]),$$

where $\mathbf{Z}/2$ *acts on* $\mathrm{HH}_\bullet(\mathcal{C})$ *via conjugation by* σ *on* $\mathrm{Fun}_{\mathbf{k}}(\mathcal{C},\mathcal{C})$.

Proof. Theorem A.22 gives

$$\mathrm{HH}^\bullet(\mathcal{C}^{\mathbf{Z}/2}) \cong \mathrm{HH}^\bullet(\mathcal{C})^{\mathbf{Z}/2} \oplus \mathrm{H}^\bullet(\underline{\mathrm{Map}}_{\mathrm{Fun}_{\mathbf{k}}(\mathcal{C},\mathcal{C})}(\mathrm{id}_{\mathcal{C}},\sigma))^{\mathbf{Z}/2}.$$

Since $\sigma = \mathrm{S} \circ [-n]$ the second summand is as claimed, and it suffices to see that σ acts trivially on $\mathrm{HH}^\bullet(\mathcal{C})$. But the shift functor $[-n]$ clearly acts trivially on $\mathrm{HH}^\bullet(\mathcal{C})$, and so does S by Theorem A.8. $\square$

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