Degenerations, Log K3 Pairs and Low Genus Curves on Algebraic Varieties

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Degenerations, Log K3 Pairs and Low Genus Curves on Algebraic Varieties

A dissertation presented

by

Adrian Ioan Zahariuc

to

The Department of Mathematics

in partial fulfillment of the requirements
for the degree of Doctor of Philosophy
in the subject of Mathematics

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Degenerations, Log K3 Pairs and Low Genus Curves on Algebraic Varieties

Abstract

We investigate several questions pertaining to the enumerative and deformation-theoretic behavior of low-genus curves on algebraic varieties, using specialization techniques.
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Finally, none of the above would have been possible without the infinite love with which my parents brought me up.
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Overview and Summary

Generally speaking, the goal of this thesis is to study the properties of low genus curves on algebraic varieties. The questions concerning these objects that motivate the present work are quite general and have a rather elementary flavor. For instance, we are interested in their deformation theoretic behavior, i.e. we want to understand how flexible or rigid curves on algebraic varieties are, and in their enumerative behavior, that is, if the number of curves satisfying certain properties (e.g. passing through a specified number of points) is finite, we want to know what that number is.

As general questions usually don’t have general answers, sometimes it is necessary to approach them in a case-by-case manner. This thesis consists of three mostly logically independent chapters on the subject outlined in the previous paragraph, with the following common theme: the use of specialization techniques. The format is also partly motivated by the choice of the method. In an imperfect analogy, we may compare specialization techniques with deformation theory in that they are great tools in specific situations, but a completely general theory is almost impossible to write down for obvious reasons.

Very roughly speaking, the blueprint for the sort of analysis we are interested in carrying out has been laid out in the work of Caporaso and Harris [CH98, Ha86] in the case of plane curves of given degree and geometric genus, that is, the study
of the so-called Severi varieties. To enumerate only a few other important works which follow similar principles: [Ch02, Va00, Ga02]. Some of these can be viewed from the point of view of the general theory developed in [Li01, Li02], which we are also going to rely on heavily in chapters 2 and 3 and could have been invoked in Chapter 1 as well, although we chose not to.

The flavor of this thesis is highly geometric and we will try to illustrate the idea that carefully chosen specializations or degenerations can lead to combinatorially manageable degenerate behavior. Beyond the common theme of using specializations, the three chapters share a common motif: the appearance of what we call log K3 pairs, i.e. pairs $(S, E)$ consisting of a surface $S$ and a smooth divisor $E$ such that $K_S + E \sim 0$. Although there are a few other types of log K3 pairs, we will only encounter pairs in which the surface is rational. The interplay between rational curves on $S$ and the group structure of the boundary $E$ (note that $p_a(E) = 1$) is a recurrent phenomenon in all chapters, most strongly emphasized in the final one.

The first chapter is devoted to the study of rational curves on Fano $n$-folds of index $n - 1$. Theorem 1.1.1 is morally concerned with the elementary deformation theory of rational curves, since the deformation of embedded curves is governed by the normal bundles. We prove that certain types of rational curves have balanced normal bundles, which of course implies that the property holds generically, since the property of being balanced is open in families. Theorem 1.1.2 or equivalently the formula in subsection 1.1.2 give a very simple formula for relating the number of rational curves on an index 2 Fano threefold passing through the suitable number of points and the analogous invariants for del Pezzo surfaces. The latter are very easy to compute using quantum cohomology, similar to the famous case of rational plane curves [KM94]. This chapter is based on the preprint [Za15a].

The goal of the second chapter is to analyze the limits of curves on K3 surfaces
when the K3 surfaces undergo a degeneration of a well-known type to a union of two rational surfaces. The nature of the chapter is mostly descriptive and we will think of it only as a resource for potential applications, which will not be pursued.

The third chapter is concerned with the study of algebraic curves on quintic threefolds. The idea that one could gain insight into questions about rational (or higher genus) curves on Calabi-Yau threefolds by degenerating the underlying threefold has been around for a long time, but not much tangible progress has been made, despite considerable effort. The natural first step is the description of the space of genus zero stable morphisms to the degenerate threefold, understood in a suitable sense. The main goal is to exhibit a degeneration of quintic threefolds, in which this natural first step can be carried out. As an elementary application of the analysis, we prove the existence of rigid stable maps with smooth sources of arbitrary genus and sufficiently high degree to very general quintics, generalizing the known case of curves of genus $g \leq 22$. This chapter is based on the preprint [Za15b].
Chapter 1

Rational Curves on Del Pezzo Manifolds

1.1 Introduction

1.1.1 Motivation and Main Results

The index of a smooth Fano variety $X$ is the largest integer by which $-K_X$ is divisible in Pic $X$. It can be proved that the index is at most $\dim X + 1$, and moreover, it is equal to $\dim X + 1$ only for projective spaces, respectively to $\dim X$ only for smooth quadric hypersurfaces. We will be concerned with the study of the rational curves on $X$. Since $\mathbb{P}^n$ and $\mathbb{Q}^n$ are convex, implying that $\overline{\mathcal{M}}_{0,0}(X, \beta)$ is a smooth stack, and, moreover, easily understood from the point of view of quantum cohomology, it is natural to go one step further and study rational curves on varieties of index $\dim X - 1$. In this paper, we will use a simple specialization technique specific to the index $\dim X - 1$ case, to address to some questions concerning the enumerative and deformation theoretic behavior of such curves.

A smooth Fano algebraic variety $X$ such that $-K_X$ is divisible by $n - 1$ in
Pic $X$, where $n$ is the complex dimension of $X$, is sometimes called a del Pezzo manifold. The polarization $\mathcal{O}(1)$ is defined by $-K_X = \mathcal{O}(n - 1)$. We call $\mathcal{O}(1)$ the hyperplane class. The degree $d$ of $X$ is the $n$-fold self-intersection of $\mathcal{O}(1)$. Del Pezzo manifolds have been completely classified by Fujita and Iskovskikh, cf. [Fu80, Fu90, Isk78, Isk80]. For the reader’s convenience we reproduce here their classification in dimension three and larger.

1. $d = 1$, then $X$ is a sextic hypersurface in $\mathbb{P}^3$; 
2. $d = 2$, then $X$ is a double cover of $\mathbb{P}^n$ ramified along a quartic; 
3. $d = 3$, then $X$ is a cubic hypersurface in $\mathbb{P}^{n+1}$; 
4. $d = 4$, then $X$ is a complete intersection of two quadrics in $\mathbb{P}^{n+2}$; 
5. $d = 5$, then $X$ is a linear section of the Grassmannian $G(1, 4) \subset \mathbb{P}^9$; 
6. $d = 6$, then $X$ is either (6a) $\mathbb{P}T_{\mathbb{P}^2}$, (6b) $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, or (6c) $\mathbb{P}^2 \times \mathbb{P}^2$; 
7. $d = 7$, then $X$ is the blowup of $\mathbb{P}^3$ at one point; 
8. $d = 8$, then $X$ is $\mathbb{P}^3$. 

In the last case, the polarizing line bundle $\mathcal{O}(1)$ is what we would normally call $\mathcal{O}_{\mathbb{P}^3}(2)$ and the index is 4 rather than 2. In this paper, we will mainly be concerned with the classes of Picard rank one and base point free polarization, which amounts to $d \in \{2, 3, 4, 5, 8\}$, to avoid excessive bookkeeping when $d \in \{6, 7\}$ respectively some technical issues when $d = 1$. However, most of the arguments still go through.

We denote a del Pezzo manifold of dimension $n$ and type $\theta$ by $d\mathbb{P}[n, \theta]$, where $\theta$ is one of the 10 types listed above. Furthermore, if $d\mathbb{P}[m, \theta]$ is a smooth plane section of $d\mathbb{P}[n, \theta]$, which is still del Pezzo by adjunction, we denote the inclusion map by $j^m_n[\theta] : d\mathbb{P}[m, \theta] \to d\mathbb{P}[n, \theta]$. The purpose of the paper is to study the rational curves on $d\mathbb{P}[n, \theta]$ using a certain elementary specialization technique.

Let us roughly describe the specialization technique in the case of a cubic hypersurface $X = d\mathbb{P}[n, (3)] \subset \mathbb{P}^{n+1}$. The generalization to other cases is completely
straightforward. Consider the incidence correspondence $\Sigma$ between degree $e$ rational curves on $X$ and $e$-tuples of points on $X$ such that all $e$ points lie on the curve. We are being imprecise for now regarding the compactification of $R_e(X)$.

$$\Sigma = \{(C, \xi_1, \ldots, \xi_e) | \xi_i \in C\}$$

The idea is to specialize the $e$ points to $e$ distinct points on $E = dP[1, (3)]$, a smooth plane cubic curve on $X$ obtained by cutting $X$ with a 2-plane. Then the curves containing the $e$ special points are forced to lie on cubic surfaces obtained by cutting $X$ with a 3-plane. Moreover, the divisor class of the rational curve on the corresponding cubic surface restricts on $E$ to $\mathcal{O}_E(\xi_1 + \ldots + \xi_e)$. These allow us to describe the special fibers $\alpha^{-1}(\xi_1, \ldots, \xi_e)$, $(\xi_1, \ldots, \xi_e) \in E^e$, which can be used to prove the results below.

**Theorem 1.1.1.** Assume that $j_{2n}[\theta] : dP[2, \theta] \to dP[n, \theta]$ is a sufficiently general plane section of a del Pezzo manifold of degree $d \in \{2, 3, 4, 5, 8\}$. Let $f : \mathbb{P}^1 \to dP[2, \theta]$ be an unramified degree $e$ morphism such that $f_*[\mathbb{P}^1] \notin c_1(dP[2, \theta])\mathbb{Q}$. Then the normal bundle $\mathcal{N}_{dP[2] / dP[1]}$ of $f$ relative to $dP[n, \theta]$ is isomorphic to $\mathcal{O}(e - 1)^{\oplus 2} \oplus \mathcal{O}(e)^{\oplus (n-3)}$.

This statement generalizes an old result of Harris, Hulek and Eisenbud and Van de Ven [EV81, Hu81] which says that a type $(1, k)$ rational curve on a smooth quadric surface has balanced normal bundle relative to the ambient $\mathbb{P}^3$ if $k > 1$. An immediate corollary is that, with few obvious exceptions which are left to the reader, for a homology class $\beta \in H_2(dP[n, \theta], \mathbb{Z})$, there exists an irreducible component\(^1\) of $\overline{\mathcal{M}}_{0,0}(dP[n, \theta], \beta)$ such that a general stable map in that component

\(^1\)Irreducibility is actually known to hold for the cubic hypersurface case $n \geq 4$ [CS09] and it
The reader may wish to compare this with \cite{EVV81, Ran07, Sh12} due to some overlap.

**Theorem 1.1.2.** Let $dP[3, \theta]$ be a Fano threefold of index 2 and degree $d \in \{2, 3, 4, 5\}$, $\beta \in H_2(dP[3, \theta], \mathbb{Z})$ and $e = \deg \beta$. Then we have

$$
\langle [\text{pt}]^e \rangle_{0, \beta}^{dP[3, \theta]} = \frac{1}{d(9 - d)} \sum_{\beta_2[\theta], \gamma = \beta} \left( (K \cdot \gamma)^2 - (K^2) (\gamma^2) \right) \langle [\text{pt}]^{e-1} \rangle_{0, \gamma}^{dP[2, \theta]}
$$

where $K$ is the canonical divisor of the del Pezzo surface $dP[2, \theta]$.

Some comments are in order. The invariants above can be computed using various enumerative methods; in particular, the ones on the right hand side can very easily be computed using standard quantum cohomology, similar to the famous example of $\mathbb{P}^2$. However, the identity itself is new to the best of the author’s knowledge. The most mysterious feature of this identity is the constant $d(9 - d)$, whose existence is natural from the point of view of the proof of Theorem 1.1.2, but whose nice exact value seems to come down to some numerological coincidences. It would be interesting to find a clear explanation, if one exists.

There is an analogous identity in the case of $\mathbb{P}^3$, which was found (actually conjectured, as the author acknowledges) \cite{Co83} long before the development of modern enumerative geometry. The argument in loc. cit. is different. Very recently, the identity for $\mathbb{P}^3$ was proved by Brugallé and Georgieva \cite{BG15}, following an approach suggested by Kollár \cite{Ko14}, which is similar to the one used here.

**Remark 1.1.3.** An immediate calculation gives the following numbers

is totally conceivable that similar arguments work for other classes.
for the number of degree $e$ curves through $e$ points. It can be checked that for $d \in \{3, 4\}$ and $e \leq 3$, our numbers agree with those computed in [Be95].

1.1.2 The Generating Functions

We conclude the introduction with the simple observation that the formula in theorem 1.1.2 can be naturally expressed using generating functions. Consider the rings of formal power series on $H^2(dP[n, \theta], \mathbb{R})$ with an additional formal variable $R_{n,\theta}[[t]] = \mathbb{R}[[t]] \otimes_{\mathbb{R}} \widehat{\text{Sym}} H^2(dP[n, \theta], \mathbb{R})$ for $n = 2, 3$. The pairing on $H^2(dP[2, \theta], \mathbb{R})$ induced by the cup product allows the introduction of several familiar differential operators for $n = 2$. The ones we will be concerned with are the directional derivative $\nabla_v$, defined for all $v \in H^2(dP[2, \theta], \mathbb{R})$, and the d’Alembert operator $\Box$ associated with the Minkowski metric. We leave the precise definitions in the present circumstances to the reader.

Returning to the generating function for the number of rational curves through suitable number of points, let $u$ be a variable in $H^2(dP[n, \theta], \mathbb{R})$ and define

$$
\Theta_{dP[n,\theta]}(u, t) = \sum_{\gamma \in H_2(dP[n, \theta], \mathbb{Z})} \langle [\mathcal{P}]_{\text{deg} \gamma - \epsilon_n} \rangle_{0, \gamma} e^{\langle \gamma, u \rangle} t^{\text{deg} \gamma} \in R_{n,\theta}[[t]],
$$

where $\epsilon_n = 1$ if $n = 2$, respectively 0 if $n = 3$. Then the formula exhibited in theorem 1.1.2 can be expressed equivalently as $\Theta_{dP[3,\theta]}(u, t) = D\Theta_{dP[2,\theta]}(j^3_2[\theta]^* u, t)$.
where

\[ \mathcal{D} = \frac{\nabla_\omega \nabla_\omega - d\Box}{d(9 - d)}. \]

Here, \( \omega = -c_1(d\text{P}[2, \theta]) \), regarded as an element of \( H^2(d\text{P}[2, \theta], \mathbb{R}) \) and \( \nabla_\omega \nabla_\omega \) of course stands for the second directional derivative along \( \omega \).

## 1.2 Preliminaries

### 1.2.1 Genus Zero Maps to log K3 Pairs

In this section and the next, we prove some easy preliminary results which are more convenient to state independently of the rest of the argument. Let \( \Sigma = d\text{P}[2, \theta] \) be a complex del Pezzo surface and \( E \subset \Sigma \) a smooth anticanonical divisor. By adjunction, \( E \) has genus one. We will show, roughly, that there exist only finitely many degree \( e \) rational curves on \( \Sigma \) which cut \( E \) in any predetermined collection of \( e \) distinct points. For a homology class \( \beta \in H_2(\Sigma, \mathbb{Z}) \), let \( \overline{\mathcal{M}}_{0,e}(\Sigma, \beta) \) be the space of genus zero stable maps to \( \Sigma \) with \( e \) marked points, where we will choose \( e = (E \cdot \beta) \). The evaluation map is \( \text{ev} : \overline{\mathcal{M}}_{0,e}(\Sigma, \beta) \to \Sigma^e \). Let \( \xi_1, ..., \xi_e \in E \) be distinct closed points on \( E \).

**Lemma 1.2.1.** The family of curves over \( \overline{\mathcal{M}}_{0,e}(\Sigma, \beta) \times_{\Sigma^e} \{ (\xi_1, ..., \xi_e) \} \) sweeps out a locus of dimension (at most) one on \( \Sigma \).

If, moreover, for every proper subset \( I \subset [e] = \{1, 2, ..., e\} \), we have

\[ \mathcal{O}_E \left( \sum_{i \in I} \xi_i \right) \notin \text{Im}(\text{Pic}(\Sigma) \to \text{Pic}(E)) \]

then \( \overline{\mathcal{M}}_{0,e}(\Sigma, \beta) \times_{\Sigma^e} \{ (\xi_1, ..., \xi_e) \} \) has dimension zero if nonempty. Moreover, the sources of all the stable maps it parametrized are smooth.
Proof. If this was not the case, by properness, this family of curves would have to sweep out the whole surface $\Sigma$. Let $\xi'$ be a point on $E$, different from all the $\xi_i$. Let $(C, f, p_1, \ldots, p_e) \in \overline{\mathcal{M}}_{0,e}(\Sigma, \beta) \times_{\Sigma^e} \{(\xi_1, \ldots, \xi_e)\}(\mathbb{C})$ whose image hits the point $\xi'$. The section $1 \in H^0(C, f^*\mathcal{O}_E) \otimes_{\mathcal{O}_C} \mathcal{O}_C(\xi_1, \ldots, \xi_e)$ vanishes on at least $e + 1$ points in different fibers of $f$, namely $p_1, \ldots, p_e$ and another point $p'$ mapping to $\xi'$, so, by degree considerations, it must vanish on some irreducible component of $C$ which is not contracted by $f$. Hence $C$ has an irreducible component $C_0$ which maps nonconstantly to $E$, which is impossible since $C_0$ has genus zero.

For the second part, note that the condition implies that for any $(C, f, p_1, \ldots, p_e) \in \overline{\mathcal{M}}_{0,e}(\Sigma, \beta) \times_{\Sigma^e} \{(\xi_1, \ldots, \xi_e)\}(\mathbb{C})$, there is at most one irreducible component of $C$ which is not contracted by $f$. (Here we are implicitly using that $-K_\Sigma = \mathcal{O}_\Sigma(E)$ is ample, so $E$ intersects any divisor on $\Sigma$.) Since $C$ has arithmetic genus zero, it is not hard to deduce that $C$ has no contracted components, so it must in fact be irreducible. Indeed, any contracted component had at most one marked point, so the corresponding vertex in the dual graph has degree at least 2, but the one-vertex edgeless graph is the unique tree with only one vertex of degree less than 2 thus proving the claim. Moreover, since $\xi_1, \ldots, \xi_e$ are distinct, $f$ cannot factor as a multiple cover. Then finiteness follows easily from the previous result. □

Remark 1.2.2. Without further hypotheses, $\overline{\mathcal{M}}_{0,e}(\Sigma, \beta) \times_{\Sigma^e} \{(\xi_1, \ldots, \xi_e)\}$ may fail to be reduced, e.g. when it contains ramified stable maps, such as the normalization of a cuspidal curve.

Note that in order for $\overline{\mathcal{M}}_{0,e}(\Sigma, \beta) \times_{\Sigma^e} \{(\xi_1, \ldots, \xi_e)\}$ to be nonempty, it is necessary that the unique line bundle on $\Sigma$ whose first Chern class is Poincaré dual to
restricts to $\mathcal{O}_E(\xi_1 + \ldots + \xi_e)$ on $E$. If that is the case, then the forgetful map

$$\overline{M}_{0,e}(\Sigma, \beta) \times_{\Sigma^e} \{(\xi_1, \ldots, \xi_e)\} \to \overline{M}_{0,e-1}(\Sigma, \beta) \times_{\Sigma^{e-1}} \{(\xi_1, \ldots, \xi_{e-1})\}$$

is an isomorphism, since the image of any map belonging to the second space necessarily intersects $E$ transversally at $\xi_e$. If, additionally the hypothesis in the second part of lemma 2.1 is satisfied, then

$$\deg [\overline{M}_{0,e}(\Sigma, \beta) \times_{\Sigma^e} \{(\xi_1, \ldots, \xi_e)\}] = \langle [\text{pt}]^{e-1}\rangle_{0,\beta}^{\Sigma}.$$ 

This simple fact will be used later in the proof of Theorem 1.1.2.

### 1.2.2 Lines on Fano Threefolds of Index 2

Let $X = d\mathbb{P}[3, \theta]$ be a smooth Fano threefold of index 2, Picard rank one and base point free polarization, i.e. $d \in \{2, 3, 4, 5\}$. We will need to know the number $\lambda_{3,1}^d$ of lines (i.e. curves of degree 1 relative to the polarization by one-half the canonical class) through a general point $x \in X$.

There are several ways to carry out this count, here we sketch a combinatorial approach. Consider the morphism $X \to \mathbb{P}^{d+1}$. This is an embedding if $d \geq 3$, respectively a ramified covering if $d = 2$. A general hyperplane section of $X$ is a del Pezzo surface of degree $d$ and it can be described as the blowup of $\mathbb{P}^2$ at $9 - d \leq 7$ points, no three of which are collinear and no five of which lie on the same conic. If, $d \geq 3$ let $H \subset \mathbb{P}^{d+1}$ be a general hyperplane containing the projective tangent space to $X$. If $d = 2$, we assume that $x$ lies in the ramification locus of $X \to \mathbb{P}^3$ and choose $H$ to be the plane tangent to the branch locus at the image of $x$. The case $x \in X$ general will still follow by a semicontinuity argument, which
we skip. The pullback $X \times_{\mathbb{P}^{d+1}} H$ has a simple double point at $x$ and is smooth elsewhere. Moreover, it will contain any line on $X$ passing through $x$. Resolving this singularity, we obtain a surface $S$ with a $(-2)$-curve. The lines on $X$ through $x$ correspond to $(-1)$-curves on $S$ intersecting the $(-2)$-curve.

The surface $S$ also can be described as the blowup of $\mathbb{P}^2$ at $9 - d$ points, but with the feature (for instance) that 3 of the points have become collinear. Of course, the $(-2)$-curve is simply the proper transform of the line through these 3 points. Let $A, B, C \in \mathbb{P}^2$ be the three collinear points and $P_1, ..., P_{6-d} \in \mathbb{P}^2$ the remaining $6 - d$ points. The only $(-1)$-curves intersecting the $(-2)$-curve are the three exceptional divisors of the blow up at $A$, $B$ and $C$, the proper transforms of the lines $P_i P_j$, $i < j$, and the proper transforms of conics passing through one of the points $A, B, C$ and 4 of the $P_i$, so the answer to our question is

$$\chi^d_{3,1} = 3 \left( \begin{array}{c} 6 - d \\ 4 \end{array} \right) + \left( \begin{array}{c} 6 - d \\ 2 \end{array} \right) + 3$$

which is 12 for $d = 2$, 6 for $d = 3$, 4 for $d = 4$ respectively 3 for $d = 5$. Note, in particular, that these counts are compatible with the formula in theorem 1.1.2.

1.3 Homology Classes on Surfaces in a Pencil

1.3.1 The Lefschetz Pencil

As in section 2.2, we consider a smooth Fano threefold $X$ of index 2 and degree $d$, with polarization $\mathcal{O}_X(1)$. Assume that Pic($X$) = $\mathbb{Z}$ and the polarization is base-point free, which amounts to $d \in \{2, 3, 4, 5\}$. Consider a general pencil of sections of the polarizing line bundle with base locus $E$ and total space $\rho : W \to \mathbb{P}^1$, where $W$ is the blowup of $X$ along $E$. By adjunction, $p_a(E) = 1$. The members of the
pencil are generically smooth del Pezzo surfaces of the same degree \( d \) as \( X \). Let \( \mathbb{P}^\circ \subset \mathbb{P}^1 \) parametrize smooth del Pezzo fibers and \( W^\circ \subset W \) its preimage. Choose a closed point \( b \in \mathbb{P}^\circ \) and \( \beta_b \in H_2(W_b, \mathbb{Z}) \). The Poincaré dual of \( \beta_b \) is the first Chern class of a (uniquely determined) line bundle \( \mathcal{L}_b \) on the surface \( W_b \).

First, we introduce some notation. We will consider objects \((N, \langle \cdot, \cdot \rangle, \nu)\) consisting of the following data: (1) a finitely generated free abelian group \( N \); (2) a bilinear map \( \langle \cdot, \cdot \rangle : N \times N \to \mathbb{Z} \); and (3) a nonzero distinguished element \( \nu \in N \). A morphism between two such objects \((N, \langle \cdot, \cdot \rangle_1, \nu)\) and \((M, \langle \cdot, \cdot \rangle_2, \mu)\) is a map \( \varphi : N \to M \) such that \( \varphi(\nu) = \mu \) and \( \langle \varphi(v), \varphi(w) \rangle_2 = \langle v, w \rangle_1 \). We denote the resulting category by \( \mathcal{D} \). The purpose of this category is merely to simplify language.

Set \( r = 9 - d \). Let \((H_r, \langle \cdot, \cdot \rangle, \omega)\) be an object of \( \mathcal{D} \) defined by \( H_r = \mathbb{Z}\ell_0 \oplus \mathbb{Z}\ell_1 \oplus ... \oplus \mathbb{Z}\ell_r \),

\[
\langle \ell_i, \ell_j \rangle = \begin{cases} 
0 & \text{if } i \neq j, \\
1 & \text{if } i = j = 0, \\
-1 & \text{if } i = j \geq 1
\end{cases} \tag{3.1}
\]

and \( \omega = -3\ell_0 + \ell_1 + \ell_2 + ... + \ell_r \). Note that this is isomorphic to \( H^2(\Sigma, \mathbb{Z}) \) of a degree \( d \) del Pezzo surface with the cup product and canonical class. The abelian subgroup \( \omega^\perp = \{ v : \langle v, \omega \rangle = 0 \} \) with the negative pairing is the lattice \( E_r \), if \( d \in \{2, 3, 4, 5\} \). We are abusing notation by writing \( E_r \) for \( r = 4, 5 \) instead of \( A_4 \) and \( D_5 \) respectively. The complexification \( E_r \otimes \mathbb{C} = \omega^\perp \) is the (dual) Cartan subalgebra \( \mathfrak{h}^\vee \) of the corresponding Lie algebra and the restriction of the pairing is the Killing form. Of course, the Killing form identifies \( \mathfrak{h}^\vee \) and \( \mathfrak{h} \), so we will simply write \( \mathfrak{h} \) despite the fact that we sometimes mean the dual. Recall that the automorphism group \( \text{Aut}_\mathcal{D}(H_r, \langle \cdot, \cdot \rangle, \omega) \) is the Weyl group \( G = W(E_r) \). Another folklore fact which will be used is the following: up to scalars, the Killing form is
the only $G$-invariant bilinear map on $\mathfrak{h}$.

Consider the monodromy action $\pi_1(\mathbb{P}^o, b) \to \text{Aut}_\mathfrak{D}(H^2(W_b, \mathbb{Z}), \cup, c_1(\mathcal{T}_{W_b}^\vee))$. As $W \to \mathbb{P}^1$ is a Lefschetz pencil, it is well-known that the image of the monodromy map is the full Weyl group $G$. Let $B^o$ parametrize pairs $(t, \varphi_t)$ consisting of a closed point $t \in \mathbb{P}^o$ and a $\mathfrak{D}$-isomorphism of $(H^2(W_t, \mathbb{Z}), \cup, c_1(\mathcal{T}_{W_t}^\vee))$ with $(H_r, \langle \cdot, \cdot \rangle, \omega)$. Then $B^o \to \mathbb{P}^o$ is finite and étale. A sketch of an algebraic construction of $B^o$ goes as follows. Let $F \to \mathbb{P}^o$ be the relative Fano scheme of lines in the smooth fibers of the pencil. The locus

$$\{(\ell_1, \ldots, \ell_r) : \ell_i \cap \ell_j = \emptyset \} \subset F \times_{\mathbb{P}^o} F \times_{\mathbb{P}^o} \ldots \times_{\mathbb{P}^o} F$$

parametrizing $r$-tuples of mutually disjoint lines in the same fiber is both open and closed. We define $B^o$ to be this locus and $B$ to be its nonsingular completion. By properness, the étale map $B^o \to \mathbb{P}^o$ extends to a (ramified) finite map $B \to \mathbb{P}^1$.

Let $W_B^o$ be the pullback of $W$ to $B^o$ and $W_B$ the pullback of $W$ to $B$. Although we won’t use it, we should point out that all singular points of $W_B$ are ordinary double points so, after small resolutions, the family $W_B \to B$ becomes a smooth family $Y \to B$.\footnote{This family is clearly not a topological fibration, since the new fibers have other intersection forms.} By construction, the relative Picard scheme $\text{Pic}(W_B^o/B^o)$ is simply $H_r \times B^o$, so every $\beta \in H_r$ naturally induces a section $\sigma(\beta) : B^o \to \text{Pic}(W_B^o/B^o)$. Before stating the main enumerative problem of this section, we prove a lemma which will turn out very useful later in ruling out potential multiplicities.

Lemma 1.3.1. The composition $B^o \to \text{Pic}(W_B^o/B^o) \to \text{Pic}(E)$ of $\sigma(\beta)$ with the restriction map is constant if and only if $\beta$ is a rational multiple of $\omega$.

Proof. The if direction is trivial since $-K_{W_t} = \mathcal{O}_{W_t}(1)$ restricts to $\mathcal{O}_E(1)$ on
$E \subset W_t$. To prove the converse, we use monodromy. Let $\Lambda \subset H_r$ be the set of all $\lambda \in H_r$ for which the composition in the statement of the lemma is actually constant. Of course, $\Lambda \neq \emptyset$ since it contains $\omega$. It is not hard to check that $\Lambda$ has the following properties: (1) $\Lambda$ is a subgroup of $H_r$, (2) $\Lambda$ is $G$-invariant and (3) if $m \neq 0$ is an integer and $m\lambda \in \Lambda$, then $\lambda \in \Lambda$.

An easy exercise in lattice theory proves that the properties above imply that either $\Lambda$ consists precisely of the multiples of $\omega$, or $\Lambda = H_r$. We claim that the latter is impossible. Indeed, we will show that $\ell = \ell_1 \notin \Lambda$. Geometrically, $\ell$ is the class of a line on a del Pezzo surface of degree $d$. In section 1.2.2, we’ve shown that there exist lines passing through each point of the original threefold $X$. Applying this to points of $E$ and noting that any line on $X$ intersecting $E$ has to lie inside some $W_t$, we conclude that $\ell \notin \Lambda$. \qed

We will also use the infinitesimal version of the lemma: the differential of the composition above is generically nonzero, unless $\beta \in \mathbb{Q}\omega$.

### 1.3.2 Counting Homology Classes

Let $\beta \in H_r$ be an element corresponding to the chosen $\beta_b \in \text{H}_2(W_b, \mathbb{Z})$ under a suitable isomorphism. Fix $\mathcal{L}_E \in \text{Pic}^e(E)$. In this section, we want to address the following enumerative problem.

**Problem 1.3.2.** Assuming sufficiently general choices, how many pairs $(t, \mathcal{L}_t)$ consisting of a closed point $t \in \mathbb{P}^o$ and $\mathcal{L}_t \in \text{Pic}(W_t) \cong \text{H}^2(W_t, \mathbb{Z})$ such that

- $\mathcal{L}_t$ restricts on $E$ to the line bundle $\mathcal{L}_E$; and

- there is a $\mathcal{D}$-isomorphism $(\text{H}^2(W_t, \mathbb{Z}), \cup, c_1(\mathcal{R}_{W_t}^\omega)) \cong (H_r, \langle \cdot, \cdot \rangle, \omega)$ mapping $\mathcal{L}_t \mapsto \beta$
are there?

The way we will answer 1.3.2 is similar to the way most enumerative questions are answered: by doing intersection theory on a moduli space. Consider the functor $\text{Sch}_{\mathbb{C}}^{\text{op}} \rightarrow \text{Set}$ mapping a scheme $S$ over $\mathbb{C}$ to the set of homogeneous $S$-group scheme homomorphisms $H_r \times S \rightarrow \text{Pic}(E) \times S$ which send $\omega$ to $\mathcal{O}_E(-1)$ fiberwise. We require this to be homogeneous with respect to the grading on $H_r$ given by pairing with $-\omega$ and the natural grading by degree on $\text{Pic}(E)$. The map on arrows is defined in the obvious way by pullback.

It is not hard to prove that the functor defined above is represented by an abelian variety $A$ of complex dimension $r$. The tangent bundle of $A$ is naturally isomorphic to $\mathfrak{h} \otimes_{\mathbb{C}} \mathcal{O}_A = E_r \otimes_{\mathbb{Z}} \mathcal{O}_A$. If we defined a similar functor without the requirement that $\omega \mapsto \mathcal{O}_E(-1)$ but not dropping the homogeneity condition, we’d have obtained an abelian variety of dimension $r + 1$ which we’ll denote by $V$. It is clear that $A$ sits naturally inside $V$.

**Fact 1.3.3.** Let $T$ be a real (topological) torus, i.e a power of the circle. It is easy to describe the cohomology ring of the torus $T$: for any commutative ring $R$, there is an isomorphism

$$
(H^*(T, R), \cup) \cong \left( \bigwedge H^1(T, R), \wedge \right)
$$

of graded-commutative $R$-algebras. The isomorphism is most explicit if we take $R$ to be the field of real or complex numbers and interpret cohomology as deRham cohomology. Assume now that $T$ is a complex torus. For $R = \mathbb{C}$, there are natural Hodge structures on both sides of (3.2): on the right hand side, the Hodge structure is obtained by algebraically taking exterior powers of the Hodge structure on $H^1(T, \mathbb{C})$, while on the left hand side, it is simply the Hodge structure on the cohomology ring of $T$. The two structures coincide under the isomorphism above.
In particular, there is a canonical isomorphism $H^{1,1}(T) = H^{1,0}(T) \otimes H^{0,1}(T)$, which we will use later.

The necessary moduli space for solving problem 1.3.2 directly is $[A/G]$, but to get around some totally avoidable technical issues, we will work simply with $A$. The family $E \times B^\circ \subset W^\circ_B \to B^\circ$ induces a morphism $B^\circ \to A$. By properness, this map is canonically extended to a $G$-equivariant morphism $\tilde{\alpha} : B \to A$, which is a lift of the analogous map $\alpha : \mathbb{P}^1 \to [A/G]$. The evaluation of a homogeneous homomorphism $H_\tau \to \text{Pic}(E)$ at $\beta$ is a morphism of abelian varieties $\hat{\beta} : A \to \text{Pic}^e(E)$. Now we can answer 1.3.2 theoretically. An equivalent formulation is to ask for the number of triples $(t; \mathcal{L}_t, \varphi_t)$ where $\varphi_t$ is an isomorphism as in the second bullet; the answer to this problem is simply $\deg \hat{\beta} \circ \tilde{\alpha}$. To obtain the answer in the original formulation, simply divide by $|\text{Stab}_G(\beta)|$.

Consider the pullback on cohomology $\tilde{\alpha}^* : H^{1,1}(A) \to H^{1,1}(B)$. The domain of definition is canonically isomorphic to $\mathfrak{h} \otimes \overline{\mathfrak{h}}$, so we obtain a $G$-invariant map

$$\tilde{\alpha}^* : \mathfrak{h} \otimes \overline{\mathfrak{h}} \to H^{1,1}(B).$$

The crucial observation is that by the uniqueness up to scalars of $G$-invariant bilinear maps, $\tilde{\alpha}^*$ factors through the map $v \otimes \overline{w} \mapsto \langle v, w \rangle$, where the inner product is the Killing form. Note that we are actually using the general fact stated above for sesquilinear maps, which is equivalent since the action of $G$ on $\mathfrak{h}$ is real, i.e. it preserves the $\mathbb{R}$-span of $E_\tau$. It follows that

$$\tilde{\alpha}^*(v \otimes \overline{w}) = -\langle v, w \rangle \tilde{\alpha}^*(\ell_1 \otimes \ell_1),$$

so we’ve boiled down the calculation to computing $\tilde{\alpha}^*(\ell_1 \otimes \ell_1) \in H^{1,1}(B)$ and
Lemma 1.3.4. Let $\beta^\perp$ be the projection of $\beta$ to the orthogonal complement of $\omega$. The pullback morphism $H^{1,1}(\text{Pic}^e(E)) \to H^{1,1}(A)$ maps the Poincaré dual of the class of a point to the pure tensor $\beta^\perp \otimes \beta^\perp \in \mathfrak{h} \otimes \overline{\mathfrak{h}} \cong H^{1,1}(A)$.

Proof. First, we should point out that all $H^{i,j}(\text{Pic}^e(E))$ can be naturally identified with $\mathbb{C}$, since $H^0(E, \mathcal{O}_E) \cong \mathbb{C}$. Consider the pullback maps

$$\hat{\beta}^*: H^{i,j}(\text{Pic}^e(E)) \to H^{i,j}(A)$$

for all $i, j \in \{0, 1\}$. Note that $\hat{\beta}_{1,1}^* = \hat{\beta}_{1,0}^* \otimes \hat{\beta}_{0,1}^*$ and the two factors are naturally complex conjugate. Consider the dual differential

$$d\hat{\beta}^\vee: \hat{\beta}^* \Omega^1 \text{Pic}^e(E) \to \Omega^1(A).$$

This map is in fact simply the morphism $\mathcal{O}_A \to \mathcal{O}_A \otimes \mathfrak{h}$ given by $s \mapsto s \otimes \beta^\perp$. This will be proved shortly, let’s just assume it for now. Taking global sections of $d\hat{\beta}^\vee$ we recover the map $\hat{\beta}_{1,0}^*$, which was therefore the unique linear map $\mathbb{C} \to \mathfrak{h}$ such that $1 \mapsto \beta^\perp$. Having understood $\hat{\beta}_{1,0}^*$, we extrapolate to $\hat{\beta}_{0,1}^*$ by conjugation and then to $\hat{\beta}_{1,1}^*$ by tensoring, obtaining the desired conclusion. $\square$

Claim 1.3.5. The map $d\hat{\beta}^\vee$ is just $\mathcal{O}_A \to \mathcal{O}_A \otimes \mathfrak{h}$, $s \mapsto s \otimes \beta^\perp$.

Proof. Note that there is a related map $d\hat{\beta}^\vee: \hat{\beta}^* \Omega^1 \text{Pic}^e(E) \to \Omega^1(V)$. This is easily seen to be simply tensoring with $\beta$. To obtain the original $d\hat{\beta}^\vee$, we need to restrict to $A$ and compose this map with the restriction $\Omega^1(V) \otimes \mathcal{O}_A \to \Omega^1(A)$, which is simply the sheafified version of the projection map $H_r \otimes \mathbb{C} \to \mathfrak{h}$ to the orthogonal complement of $\omega$. $\square$
Proof of 1.3.2. Denote \( \Delta(\omega, \beta) = (\omega \cdot \beta)^2 - (\omega^2)(\beta^2) \). By (3.3) and lemma 3.4, we get

\[
\deg \hat{\beta} \hat{\alpha} = \hat{\alpha}^* \left( \hat{\beta}^* ([\text{pt}]) \right) = \hat{\alpha}^* (\beta^\perp \otimes \beta^\perp) = -\frac{\Delta(\omega, \beta)}{(\omega^2)} \hat{\alpha}^* (\ell_1 \otimes \ell_1),
\]

for all \( \beta \in H_r \). Plugging \( \beta = \ell_1 \) in (3.4), we obtain \( \deg(\hat{\ell}_1 \hat{\alpha}) = -(1 + 1/d) \hat{\alpha}^* (\ell_1 \otimes \ell_1) \). However, recall that all lines on \( X \) intersecting \( E \) actually lie in one of the fibers \( W_t \) of the pencil. Therefore, the argument so far for \( e = 1 \) shows that

\[
\lambda_{2,0}^d \deg \hat{\ell}_1 \hat{\alpha} = |G| \lambda_{3,1}^d, \text{ hence } \hat{\alpha}^* (\ell_1 \otimes \ell_1) = -|G| \frac{d \lambda_{3,1}^d}{(d+1) \lambda_{2,0}^d},
\]

where \( \lambda_{2,0}^d = |G \cdot \ell_1| \) is the number of lines on a degree \( d \) del Pezzo surface and \( \lambda_{3,1}^d \) is, as in section 1.2.2, the number of lines through a point on a degree \( d \) del Pezzo threefold.

| Type                             | \( d \) | \( \lambda_{2,0}^d \) | \( \lambda_{3,1}^d \) | \( |G| \)  | \( \frac{d \lambda_{3,1}^d}{(d+1) \lambda_{2,0}^d} \) |
|----------------------------------|---------|------------------------|------------------------|------|----------------------------------------|
| double cover ramified along a quartic | 2       | 56                     | 12                     | 2903040 | 1/7                                   |
| cubic threefold                  | 3       | 27                     | 6                      | 51840 | 1/6                                    |
| \( 2,2 \)-complete intersection  | 4       | 16                     | 4                      | 1920  | 1/5                                    |
| plane section of Grassmannian \( \mathbb{G}(1,4) \) | 5       | 10                     | 3                      | 120   | 1/4                                    |

The final entry is \( 1/r \) in all four cases. Therefore, by (3.4) and (3.5), we get

\[
\deg \hat{\beta} \hat{\alpha} = |G| \frac{\Delta(\omega, \beta)}{d(9 - d)},
\]

from which we obtain the answer by correcting with the stabilizer factor. \( \square \)

Remark 1.3.6. The Hodge index theorem implies that \( \Delta(\omega, \beta) \geq 0 \), with equality if and only if \( \beta \) is a multiple of \( \omega \).
Finally, we will rearrange the enumerative problem we’ve just solved in a more convenient form. Let $S(L, \beta)$ be the set of solutions to question 1.3.2. As a scheme, this lives on $\text{Pic}(W^\circ/\mathbb{P}^\circ)$ for general $L$, since there are only countably many divisor classes on $E$ obtained by restricting divisor classes on the singular fibers of $W \to \mathbb{P}^1$. Moreover, it is reduced by Lemma 1.3.1 and its infinitesimal version, since no $L_t$ can be a multiple of $K_{W_t}$ if $L_E(-m)$ is not torsion for all integers $m$. In the curve counting problem, we will encounter a $G$-constant counting function $N : H_r \to \mathbb{N}$. (In that case $N$ counts the number of class $\beta$ genus 0 stable maps through $e - 1 = \deg \beta - 1$ points on a degree $d$ del Pezzo surface $\Sigma$.) Then, as purely combinatorial statement, we have

$$\sum_{\beta \in H_r/G} |S(L, \beta)|N(\beta) = \sum_{\beta \in H_r} \frac{|G|\Delta(\omega, \beta)}{d(9 - d)} \frac{1}{|\text{Stab}_G(\beta)|} \frac{N(\beta)}{|G \cdot \beta|} =$$

$$= \frac{1}{d(9 - d)} \sum_{\beta \in H_r} \Delta(\omega, \beta)N(\beta), \quad (3.6)$$

where $H_r^e$ is the degree $e$ piece of $H_r$.

### 1.4 Interpolating Points on a Genus One Curve

Let $X$ be a del Pezzo variety of dimension $n$ and degree $d \in \{2, 3, 4, 5, 8\}$, a homology class $\beta \in H_2(X, \mathbb{Z})$ and $\overline{\mathcal{M}}_{0,0}(X, \beta)$ the space of class $\beta$ genus 0 stable maps to $X$. It is easy to check that that $H_2(X, \mathbb{Z})$ is torsion free. The hyperplane class $\mathcal{O}(1) \in \text{Pic } X$ is uniquely determined by the property $-K_X = \mathcal{O}(n - 1)$. Let $e = (\beta \cdot \mathcal{O}(1))$ be the degree of $\beta$.

Recall from the introduction that we want to analyze the incidence correspondence between $m$-tuples of points on $X$ and rational curves containing them. In
the stable map compactification, the correspondence translates naturally as the evaluation map \( \text{ev} : \overline{\mathcal{M}}_{0,m}(X, \beta) \to X^m \). Roughly, the evaluation map is dominant if and only if it is possible to interpolate \( m \) general points on \( X \) with a rational curve of class \( \beta \). This is expected to happen when \( \text{vdim} \overline{\mathcal{M}}_{0,m}(X, \beta) \geq \dim X^m \), or equivalently,

\[
m \leq \frac{(-K_X \cdot \beta) - 2}{\dim X - 1} + 1 = \frac{(n - 1)e - 2}{n - 1} + 1,
\]

(4.1)

since \( \text{vdim} \overline{\mathcal{M}}_{0,m}(X, \beta) = \dim X - (K_X \cdot \beta) + m - 3 \), so the largest integer \( m = m_{\text{max}} \) for which the inequality (4.1) holds is

\[
m_{\text{max}} = \begin{cases} 
e & \text{if } n \geq 3, \\ e - 1 & \text{if } n = 2. \end{cases}
\]

(4.2)

Before outlining the approach, we introduce some notation.

Let \( X \) be any smooth variety, \( \beta \in H_2(X, \mathbb{Z})/\text{torsion} \) a homology class, \( Y \) a closed subvariety of \( X \) and \( m \leq n \). We write \( \overline{\mathcal{M}}_{g,n}^{[m]}(X, Y; \beta) = \overline{\mathcal{M}}_{g,n}(X, \beta) \times_{X^m} Y^m \), where the map from \( \overline{\mathcal{M}}_{g,n}(X, \beta) \) to \( X^m \) is \( (\text{ev}_1, \ldots, \text{ev}_m) \). If \( U \subset Y^m \) is open, let

\[
\overline{\mathcal{M}}_{g,n}^{[m]}(X, Y; \beta)|_U = \overline{\mathcal{M}}_{g,n}(X, Y; \beta) \times_{Y^m} U.
\]

If \( m = n \), we drop the superscript. If \( n = m + 1 \), we can think of \( \overline{\mathcal{M}}_{g,m+1}^{[m]}(X, Y; \beta) \) as the universal curve over \( \overline{\mathcal{M}}_{g,m}(X, Y; \beta) \) and we denote it by \( \overline{\mathcal{C}}_{g,m}(X, Y; \beta) \). If \( g = 0 \), which is the only case treated in this paper, we drop the subscript indicating the genus.

Let us return to the problem. Let \( E \subset X \) be a section of \( X \) by \( n - 1 \) general hyperplanes. By adjunction and Bertini, the property of being del Pezzo is
preserved at each step, so $E$ is a smooth genus one curve. Set

$$V = H^0(X, \mathcal{I}_{E/X} \otimes \mathcal{O}(1)) \subset H^0(X, \mathcal{O}(1)).$$

Roughly, the main observation is the following: a curve of class $\beta$ which meets $E$ at $e$ distinct points is forced to lie in a surface on $X$ containing $E$, obtained by cutting $X$ with $n-2$ hyperplanes. To use this observation, we have to formalize it in families. Let $S$ be any finite type scheme over $\mathbb{C}$ and

$$(C, \pi, f, p_1, p_2, ..., p_e) \in \overline{\mathcal{M}}_e(X, E; \beta)|_{\Delta^e(S)},$$

where $\Delta^e = E^e \setminus \Delta$ and $\Delta$ is the big diagonal of $E^e$. Denote by $D_i \subset C$ the image of the closed embedding $p_i : S \to C$ and set $D = \sum D_i$. We start by proving the observation for of stable maps, then proceed with the formalism in families. Much of the formalism below is forced by the possibility that $S$ is not reduced. Since we can't say a priori that $\overline{\mathcal{M}}_e(X, E; \beta)$ is generically reduced (i.e. that it is not contained in the ramification locus of the evaluation map), this is a difficulty we are forced to face. The generic smoothness of $\overline{\mathcal{M}}_e(X, E; \beta)$ will be a corollary of the subsequent analysis.

**Lemma 1.4.1.** If $S = \text{Spec} \, \mathbb{C}$, then $f^* \mathcal{O}(1) \cong \mathcal{O}_C(D)$. Moreover, there is no irreducible component of $C$ mapped constantly to a point on $E$.

**Proof.** Let $M$ be a finite set indexing all maximal connected curves of arithmetic genus zero $C'_\mu \subset C$ which are contracted by $f$ to a point on $E$ and let $C = \bigsqcup_{\mu \in M} C'_\mu \cup C_0$ with $C_0$ possibly disconnected. The dual graph $\Gamma_\mu$ of each $C'_\mu$ is a tree. We further decorate each dual graph with "legs" for each intersection point with $C_0$. For all $\mu \in M$, let $\nu_\mu$ be the number of $i$ such that $p_i \in C'_\mu$ and $\lambda_\mu$ the
number of legs in the dual graph $\Gamma_\mu$. Since $f(p_i) \neq f(p_j)$ for $i \neq j$, $\nu_\mu \leq 1$ for all $\mu \in M$. By stability, this implies that no vertex of any $\Gamma_\mu$ can be a leaf of the dual graph of $C$. Therefore, there is at least one leg attached to each leaf of $\Gamma_\mu$, so $\lambda_\mu \geq 2$. In particular, $\lambda_\mu > \nu_\mu$. Similarly, we let $\nu_0$ be the number of $i$ such that $p_i \in C_0$ and $\lambda_0 = \sum \lambda_\mu$. Let $D_0$ be the restriction of $D$ to $C_0$ and $D_\lambda$ the divisor on $C_0$ consisting of the $\lambda_0$ “bridge points” to the union of the components on which $f$ maps constantly to $E$. Clearly, $D_0$ and $D_\lambda$ are reduced and have disjoint supports. Let $f_0$ be the restriction of $f$ to $C_0$. We claim that the line bundle

$$L_0 := f_0^* \mathcal{O}(1) \otimes \mathcal{O}(D_0 - D_\lambda)$$

admits sections with finitely many zeroes. We may argue using the pullback map on global sections $H^0(X, \mathcal{I}_{E/X}(1)) \to H^0(C_0, L_0)$. Indeed, we may choose a section of $\mathcal{I}_{E/X}(1)$ which is not identically zero on the image of any component of $C_0$ and correspondingly map it to a section of $L_0$ with finitely many zeroes. From here, we get the inequality $e - \nu_0 - \lambda_0 \geq 0$, or $\nu_0 + \mu_0 \leq e$. Since $\nu_0 + \sum \nu_\mu = e$, it follows that

$$\sum \lambda_\mu = \lambda_0 \leq \sum \nu_\mu,$$

which contradicts $\lambda_\mu > \nu_\mu$ for all $\mu \in M$, unless $M = \emptyset$. Regardless, the line bundle $L_0$, now $f^* \mathcal{O}(1) \otimes \mathcal{O}(D)$, still has sections with finitely many zeroes and visibly has degree 0, so it must be trivial. $\square$

**Lemma 1.4.2.** As above, define $L = f^* \mathcal{O}(1) \otimes \mathcal{O}(D)$. Then $\pi_*L$ is invertible. Moreover, if

$$\varphi_S : V \otimes \mathcal{O}_S \to \pi_*L$$

is the natural $\mathcal{O}_S$-modules map, there exists a unique morphism $\psi_S : S \to \mathbb{P}V$ such
that \( \text{Ker } \varphi_S = \psi^*_S \mathcal{U}_{P} \), where \( \mathcal{U}_{P} \) is the tautological subbundle of \( V \otimes \mathcal{O}_{P} \).

**Proof.** First, let us spell out the construction of the map \( \varphi_S \) in the statement. The \( \mathcal{O}_X \)-modules homomorphism \( H^0(\mathcal{O}(1)) \otimes \mathcal{O}_X \rightarrow \mathcal{O}(1) \) pulls back via \( f \) to a map \( H^0(\mathcal{O}(1)) \otimes \mathcal{O}_C \rightarrow f^* \mathcal{O}(1) \). By the adjoint property, we get an \( \mathcal{O}_S \)-modules homomorphism

\[
H^0(\mathcal{O}(1)) \otimes \mathcal{O}_S \rightarrow \pi_* f^* \mathcal{O}(1).
\]

This map composed further with \( \pi_* f^* \mathcal{O}(1) \rightarrow \pi_* (f^* \mathcal{O}(1)|_D) \) vanishes on \( V \otimes \mathcal{O}_S \), so it induces an \( \mathcal{O}_S \)-modules homomorphism \( \varphi_S : V \otimes \mathcal{O}_S \rightarrow \pi_* (f^* \mathcal{O}(1) \otimes \mathcal{O}_C(-D)) = \pi_* \mathcal{L} \). For any closed point \( s \in S \), the map

\[
V \otimes_C \kappa(s) \rightarrow \pi_* \mathcal{L} \otimes \mathcal{O}_S \kappa(s) \rightarrow H^0(C_s, \mathcal{L}_s)
\]

is nonzero; otherwise, the image of \( f_s \) would be contained inside \( E \), which is impossible. However, \( \mathcal{L}_s \) is trivial by Lemma 1.4.1 meaning that the rightmost term above is 1-dimensional, so the composed map above is surjective. Furthermore, the second map in the composition has to be surjective as well, so by the cohomology and base change theorem, it is actually an isomorphism. By the same theorem, this property extends automatically to the non-closed points. The \( \pi \)-pushorward of any torsion-free sheaf on \( C \) is torsion free as well, so \( \pi_* L \) is torsion free. Together with

\[
\dim_{\kappa(s)} \pi_* \mathcal{L} \otimes \mathcal{O}_S \kappa(s) = 1
\]

for closed \( s \in S \), this proves that \( \pi_* \mathcal{L} \) is invertible. Indeed, the corresponding stalk of \( \pi_* \mathcal{L} \) at \( s \) is generated by a single element as an \( \mathcal{O}_{s,S} \)-module by Nakayama’s lemma and, since it is torsion free, it has to be free of rank one. Finally, since

\[\text{We are using the Grothendieck convention for projective spaces, which says that the closed points of } \mathbb{P} V \text{ correspond to codimension one, rather than dimension one, subspaces of } V.\]
\((\varphi_s)_s : V \otimes_C \kappa(s) \to \pi_*\mathcal{L} \otimes_{\mathcal{O}_S} \kappa(s)\) is surjective for closed points \(s \in S\), we can define a morphism of schemes \(\psi_S : S \to \mathbb{P}V\) such that \(\text{Ker} \ \varphi_S = \psi_S^*\mathcal{U}_{\mathbb{P}V}\), where \(\mathcal{U}_{\mathbb{P}V}\) is the tautological subbundle of \(V \otimes \mathcal{O}_{\mathbb{P}V}\).

**Corollary 1.4.3.** The sheaf \((\psi_S \circ \pi)^*\mathcal{U}_{\mathbb{P}V}\), regarded as an \(\mathcal{O}_C\)-submodule of \(H^0(\mathcal{O}(1)) \otimes \mathcal{O}_C\), is contained in the kernel of \(H^0(X, \mathcal{O}(1)) \otimes \mathcal{O}_C \to f^*(\mathcal{O}(1))\).

**Proof.** By the definition of \(\varphi_S\), \(\text{Ker} \ \varphi_S\) lies inside the kernel of the map \(H^0(\mathcal{O}(1)) \otimes \mathcal{O}_S \to \pi_* f^* \mathcal{O}(1)\). First, \(\pi^*\) is left exact because \(\pi\) is flat, so \(\pi^* \text{Ker} \ \varphi_S = (\psi_S \circ \pi)^*\mathcal{U}_{\mathbb{P}V}\) can be regarded as an \(\mathcal{O}_C\)-submodule of \(H^0(\mathcal{O}(1)) \otimes \mathcal{O}_C\). Moreover, it is contained in the kernel of the map

\[
H^0(\mathcal{O}(1)) \otimes \mathcal{O}_C \longrightarrow \pi^* \pi_* f^*(\mathcal{O}(1)).
\]

Composing the last map with \(\pi^* \pi_* f^* \mathcal{O}(1) \to f^* \mathcal{O}(1)\), we recover the obvious pullback homomorphism \(H^0(\mathcal{O}(1)) \otimes \mathcal{O}_C \to f^* \mathcal{O}(1)\). □

Next, we lift the map \(\psi_S\) constructed above to the total space of the family of curves, \(C\). The idea is obviously that \(C_s\) is mapped by \(f_s\) into \(X_{\psi_S(s)}\), the vanishing locus of the codimension one subspace of \(V \subset H^0(\mathcal{O}(1))\) corresponding to \(\psi_S(s) \in \mathbb{P}V\), but the formal statement will be given later in proposition 4.5. We introduce \(W\), the blowup of \(X\) along \(E\). As it is always the case with blowups, the graded \(\mathcal{O}_X\)-algebras homomorphism

\[
\text{Sym} \ V \otimes \mathcal{O}_X \longrightarrow \bigoplus_{k=0}^{\infty} \mathcal{I}^k_{E/X}
\]

induces the natural morphism of schemes \(\tau = \tau_X \times \tau_{\mathbb{P}V} : W \to X \times \mathbb{P}V\). Note that there is a natural map \(w : \text{proj}_{\mathbb{P}V}^* \mathcal{U}_{\mathbb{P}V} \to \text{proj}_X^* \mathcal{O}_X(1)\), where \(\text{proj}_X\) and \(\text{proj}_{\mathbb{P}V}\) denote the projections to the respective factors of \(X \times \mathbb{P}V\).
Claim 1.4.4. The scheme-theoretic vanishing locus of $w$, regarded as a section of the vector bundle $\mathcal{H}om(\text{proj}^* \mathcal{U}_{\mathbb{P}V}, \text{proj}^* \mathcal{O}_X(1)) = \mathcal{O}_X(1) \otimes \mathcal{U}_V^\vee$, is precisely $W$.

Proof. Let $\chi : X \to \mathbb{P}H^0(X, \mathcal{O}_X)$ be the embedding associated with $\mathcal{O}_X(1)$ and $P$ the blowup of $\mathbb{P}H^0(X, \mathcal{O}_X(1))$ along the projectivization of the cokernel of

$$V = H^0(X, \mathcal{I}_{E/X}(1)) \to H^0(X, \mathcal{O}_X(1)).$$

Again, $P$ sits naturally inside $\mathbb{P}H^0(X, \mathcal{O}_X(1)) \times \mathbb{P}V$. Replacing $X$ with $\mathbb{P}H^0(X, \mathcal{O}_X)$, there is an analogous way to define a natural section $p$ of $\mathcal{O}(1) \otimes \mathcal{U}_{\mathbb{P}V}$. The analogous statement, that the vanishing locus of $p$ is $P$, is clear. Since $w$ is the pullback of $p$ to $X \times \mathbb{P}V$, the claim follows simply by pulling back via $\chi \times \text{Id}_{\mathbb{P}V}$. □

Proposition 1.4.5. There exists a lift $\bar{f} : C \to W$ of $f$ along $\tau_X$ such that $\psi_S \circ \pi = \tau_{\mathbb{P}V} \circ \bar{f}$, i.e. we require

$$
\begin{array}{cccc}
\text{C} & \stackrel{\bar{f}}{\longrightarrow} & W & \stackrel{\tau}{\longrightarrow} & X \times \mathbb{P}V \\
\downarrow & & \downarrow & & \downarrow \\
S & \stackrel{\psi_S}{\longrightarrow} & \mathbb{P}V & & \\
\end{array}
$$

to be a commutative diagram.

Proof. If we show that the morphism $f \times (\psi_S \circ \pi) : C \to X \times \mathbb{P}V$ factors through $\tau$, we may take $\bar{f}$ to be the quotient morphism. The important point is that pullback by $f \times (\psi_S \circ \pi)$ kills $w$. Indeed, the pullback map $(f \times (\psi_S \circ \pi))^*w : (\psi_S \circ \pi)^*\mathcal{U}_{\mathbb{P}V} \to f^*\mathcal{O}_X(1)$ is the restriction to $(\psi_S \circ \pi)^*\mathcal{U}_{\mathbb{P}V}$ of the homomorphism $H^0(X, \mathcal{O}_X(1)) \otimes \mathcal{O}_C \to f^*\mathcal{O}_X(1)$, which is zero, by Corollary 1.4.3. Then 1.4.4
shows that $f \times (\psi_S \circ \pi) : C \to X \times \mathbb{P}V$ indeed factors through $\tau$, completing the proof. □

The crucial step is to understand the image of $\Phi_S$. The idea is very simple: if some $W_t$ contains some rational curve of some class $\tilde{\beta}$ (lift of $\hat{\beta}$) through all $\xi_i$, then the rational curve will cut the copy of $E$ inside $W_t$ precisely in the divisor $\xi_1 + \ldots + \xi_e$ by degree considerations. The existence of divisor classes on $W_t$ restricting to a predetermined divisor class on $E$ imposes one condition on $t$.

Let $(C, \pi, f, p_1, p_2, \ldots, p_e) \in \overline{M}_e(X, E; \beta)|_{\Delta e}(S)$ such that the map $\pi$ is smooth, and $W_S := W \times_S \mathbb{P}V \to S$ is smooth. The square in the diagram of proposition 1.4.5 induces an $S$-morphism $\hat{f} : C \to W_S$. Note that, since we’re over the complement of $\Delta$, any component of the source of any individual map is either contracted, or mapped birationally onto its image. Let $\mathcal{K}$ be the kernel of $\mathcal{O}_{W_S} \to \hat{f}_* \mathcal{O}_C$. The fact that $C$ and $W_S$ are flat over $S$ easily implies that $\mathcal{K}$ is also flat over $S$. However, since the restriction of $\mathcal{K}$ to $W_s$ for closed points $s \in S$ is invertible by obvious geometric considerations, it follows by A.1.2 (the reader is referred to the appendix) that $\mathcal{K}$ is itself invertible. The section of $\mathcal{H}om(\mathcal{K}, \mathcal{O}_{W_S})$ corresponding to the inclusion $\mathcal{K} \to \mathcal{O}_{W_S}$ restricts on $E \times S$ to the section of $\mathcal{H}om(\mathcal{I}_{p(S)}, \mathcal{O}_{E \times S}) \cong \mathcal{O}_{E \times S}(p(S))$ corresponding to the inclusion $\mathcal{I}_{p(S)} \to \mathcal{O}_{E \times S}$, so we have a commutative square with Pic($W_S/S$), $S$, $E^e$ and Pic($E$) and the natural maps between them.

By well-known properties, $H_2(W, \mathbb{Z}) \cong H_2(X, \mathbb{Z}) \oplus \mathbb{Z}$. Moreover, there exists a unique $(\tau_X)_*$-lift, $\tilde{\beta} \in H_2(W, \mathbb{Z})$ of $\beta$, such that $(\tau_{\mathbb{P}V})_* \tilde{\beta} = 0$. It is not hard to see that the mapping defined in proposition 1.4.5 $(C, \pi, f, p_1, p_2, \ldots, p_e) \mapsto (C, \pi, \tilde{f}, p_1, p_2, \ldots, p_e)$ induces a morphism of Deligne-Mumford stacks

$$\Phi : \overline{M}_e(X, E; \beta)|_{\Delta e} \to \overline{M}_e(W, E \times \mathbb{P}V; \tilde{\beta}). \quad (4.3)$$
Since the obvious map \( j : \overline{M}_e(W, E \times \mathbb{P}V; \hat{\beta}) \to \overline{M}_e(X, E; \beta) \) is inverse to \( \Phi \), the morphism \( \Phi \) induces an isomorphism \( \overline{M}_e(X, E; \beta)|_{\Delta^c} \to \overline{M}_e(W, E \times \mathbb{P}V; \hat{\beta})|_{\Delta^c \times (\mathbb{P}V)^e} \). To conclude, there is a commutative diagram

\[
\begin{array}{ccc}
\overline{C}_e(X, E; \beta)|_{\Delta^c} & \xrightarrow{\Phi_C} & \overline{C}_e(W, E \times \mathbb{P}V; \hat{\beta}) \\
\downarrow & & \downarrow \\
\overline{M}_e(X, E; \beta)|_{\Delta^c} & \xrightarrow{\Phi} & \overline{M}_e(W, E \times \mathbb{P}V; \hat{\beta}) \\
\downarrow & & \downarrow \\
E^c & \xrightarrow{ev_{X,E}} & (E \times \mathbb{P}V)^e \\
\end{array}
\]

and \( \Phi \) is an open embedding. Let \( \xi \in E^c \) be a general \( c \)-tuple of closed points on \( E \), so \( \xi \in \Delta^c \). The diagram above shows that

\[ ev_X^{-1}(\xi) = ev_{X,E}^{-1}(\xi) \cong ev_{W,E \times \mathbb{P}V}^{-1}(\xi \times \mathbb{P}V). \]

Let \( \mathcal{M}_\xi \) denote the last space and \( \mathcal{C}_\xi \) its universal family, constructed in the obvious way by restricting the universal family above.

Let \( U \subset \mathbb{P}V \) be an open subset over which \( W \to \mathbb{P}V \) is smooth. By a slight abuse of notation we write \( \overline{M}_e(W, E \times \mathbb{P}V; \hat{\beta})|_U \) where we actually mean restriction to \( \Psi^{-1}(U) \cap (\Delta^c \times (\mathbb{P}V)^e) \). Denote by \( \overline{M}_e(W, E \times \mathbb{P}V; \hat{\beta})|_U \) the open locus where the source curve is smooth. By the discussion above, there is a map \( \omega : \overline{M}_e(W, E \times \mathbb{P}V; \hat{\beta})|_U \to \text{Pic}(W_U/U) \), where \( W_U = \tau_{\mathbb{P}V}^{-1}(U) \). Let \( \rho \) be the restriction map \( \rho : \text{Pic}(W_U/U) \to U \times \text{Pic}(E) \). Similar to the construction in section 1.3, let \( U' \) parametrize pairs \( (t, \varphi_t) \), where \( t \in U \) is a closed point and \( \varphi_t \) is a \( \mathcal{O}_t \)-isomorphism of \( (H^2(W_t, \mathbb{Z}), \cup, c_1(\mathcal{R}_W^{\vee})) \) with \( (H_r, \langle \cdot, \cdot \rangle, \omega) \). Then \( U' \to U \) is an étale morphism. The data is summarized in the following diagram.
Finally, we remark that the analogue of lemma 1.3.1, namely the statement that the section $U' \to \text{Pic}(W_{U'/U'})$ corresponding to an element $\beta \in H_r$ which is not a multiple of $\omega \in H_r$ composed with the map $\text{Pic}(W_{U'/U'}) \to \text{Pic}(E)$ is nonconstant, follows from 1.3.1 itself simply by restricting to a line $\mathbb{P}^1 \subset \mathbb{P}V$.

Proof of Theorem 1.1.1. We use the current notation. Let $(C, \tilde{f}, p_1, ..., p_e) \in \mathcal{M}_\xi(\mathbb{C})$, which is the $\Phi$-image of some stable map $(C, f, p_1, ..., p_e) \in \text{ev}_{X,E}^{-1}(\xi) \subset \overline{\mathcal{M}}_e(X, E; \beta) \subset \overline{\mathcal{M}}_e(X; \beta)$. Let

$$\mathcal{N}_{f,X} = \mathcal{O}(a_1) \oplus \mathcal{O}(a_2) \oplus ... \oplus \mathcal{O}(a_{n-1})$$

with $a_1 \geq a_2 \geq ... \geq a_{n-1}$. Then, as we’ve seen, $f$ maps to the surface $W_t \hookrightarrow X$.

We denote the tangent space to $\mathcal{M}_\xi$ at $(C, \tilde{f}, p_1, ..., p_e)$ by $\text{Def}^l_{W_\xi}(\tilde{f})$. Similarly, $\text{Def}^l_{W_\xi}(\tilde{f})$ denotes the space of first order deformations of $\tilde{f}$ which remain inside $W_t$ and $\text{Def}^l_{X,\xi}(f)$ denotes the space of first order deformations of $f$, subject to the condition that the marked points map to $\xi$. By the observation preceding the proof and the assumption that $W_t$ is general, we have an exact sequence

$$0 \to \text{Def}^l_{W_t, \xi}(\tilde{f}) \to \text{Def}^l_{W, \xi}(\tilde{f}) \to \mathcal{T}_t \mathbb{P}V \to \mathbb{C} \to 0.$$ 

It follows that $\dim \text{Def}^l_{W_\xi}(\tilde{f}) = \dim \text{Def}^l_{W_t, \xi}(\tilde{f}) + \dim \mathbb{P}V - 1$. We claim that $\text{Def}^l_{W_t, \xi}(\tilde{f}) = 0$. Recall that $\tilde{f}$ is unramified, so the normal bundle $\mathcal{N}_{f,W_t}$ of $\tilde{f}$ relative to $W_t$ is
locally free of rank one. From the standard sequence

\[ 0 \to \mathcal{T}_C \to \tilde{f}^* \mathcal{T}_{W_t} \to \mathcal{N}_{f,W_t} \to 0, \]

we infer that \( c_1(\mathcal{N}_{f,W_t}) = \tilde{f}^*c_1(W_t) - c_1(\mathcal{T}_C) \), so \( \deg \mathcal{N}_{f,W_t} = e - 2 \). Therefore,

\[ \text{Def}^1_{W_t,\xi}(\tilde{f}) = \text{H}^0(\mathcal{N}_{f,W_t} \otimes \mathcal{O}_C(-p_1 - ... - p_e)) = 0, \]

as desired. Therefore, \( \dim \text{Def}^1_{W_t,\xi}(\tilde{f}) = n - 3 \), so \( \text{Def}^1_{X,\xi}(f) = n - 3 \) as well, since \( \Phi \) is an open immersion. Using once more the interplay between first order deformations and normal bundles, we conclude that \( h^0(\mathcal{N}_{f,X}(-p_1 - ... - p_e)) = n - 3 \). Therefore,

\[ \dim \bigoplus_{i=1}^{n-1} \text{H}^0(\mathcal{O}(a_i - e)) = n - 3, \]

and combining with the constrain \( a_1 + a_2 + ... + a_{n-1} = \deg c_1(\mathcal{N}_{f,X}) = (n-1)e - 2 \), we can infer that \( a_{n-1} \geq e - 1 \).

Let \( \xi_{e+1} \in X \setminus E = W \setminus E \times \mathbb{P}V \), say \( \xi_{e+1} \in W_t \). We consider stable maps with one additional marked point. Let \( \xi_1, ..., \xi_e \) as before, \( \xi' = (\xi_1, ..., \xi_e, \xi_{e+1}) \) and \( \text{ev}^{-1}_X(\xi') \) the space of stable maps \( f \) to \( X \) such that \( f(p_i) = \xi_i \). We will only sketch this part of the argument, being similar to the analysis above. The condition \( f(p_{e+1}) = \xi_{e+1} \) forces \( f \) to map to \( W_t \), so \( \text{Def}^1_{X,\xi'}(f) \) is now isomorphic to \( \text{Def}^1_{W_t,\xi'}(\tilde{f}) \), but, as above, \( \text{Def}^1_{W_t,\xi'}(\tilde{f}) = 0 \), so \( \text{Def}^1_{X,\xi'}(f) = 0 \). In terms of normal bundles, \( h^0(\mathcal{N}_{f,X}(-p_1 - ... - p_{e+1})) = 0 \), implying that \( a_1 \leq e \). Combining with the previous inequality, we conclude that \( a_1 = a_2 = e \) and \( a_3 = ... = a_{n-1} = e - 1 \), as desired. \( \square \)

**Proof of Theorem 1.1.2.** We are in the situation analyzed in section 1.3, \( X = \text{dP}[3,\theta] \) a Fano threefold of index 2 and degree \( d \in \{2,3,4,5\} \) and \( W \to \mathbb{P}^1 \) a
Lefschetz pencil. Let $\mathcal{L}_E := \mathcal{O}_E(\xi_1 + \ldots + \xi_e) \in \operatorname{Pic}^e(E)$. As in section 3, let $S(\xi) = S(\mathcal{L}_E)$ be the set of pairs $(t, \mathcal{L}_t)$ such that $t \in \mathbb{P}^e = U$ and $\mathcal{L}_t$ restricts to $L_E$ on $E$. To each such $\mathcal{L}_t$, we can associate $\tilde{\beta}_t \in H_2(W_t, \mathbb{Z})$, the Poincaré dual to $c_1(\mathcal{L}_t)$. The condition in the second part of lemma 1.2.1 is satisfied for a general choice of $\xi$. Indeed, for $e \geq 2$, we may move the $e$ points around preserving $\mathcal{L}_E$ (and therefore $S(\xi)$) to avoid the finitely many prohibited situations. For $e = 1$, the condition is satisfied vacuously.

Of course, $S(\xi)$ splits as a disjoint union of $S(\mathcal{L}_E, \gamma)$ consisting of those pairs $(t, \mathcal{L}_t)$ such that $\tilde{\beta}_t$ corresponds to $\gamma$ under a suitable $\mathcal{O}$-isomorphism, where $\gamma$ varies over a set of representatives of $H^e_2/G$. From the discussion at the end of section 1.3, $S(\mathcal{L}_E, \gamma)$ is reduced and we have

$$
\mathcal{M}_\xi \cong \bigsqcup_{(t, \mathcal{L}_t) \in S(\xi)} \mathcal{M}_e(W_t, \tilde{\beta}_t) \times W_t^\xi \{\xi\} \cong \bigsqcup_{(t, \mathcal{L}_t) \in S(\xi)} \mathcal{M}_{e-1}(W_t, \tilde{\beta}_t) \times W_t^{e-1}\{((\xi_1, \ldots, \xi_{e-1})\}. 
$$

Taking the degrees of these 0-cycles, we obtain

$$
\deg [\mathcal{M}_\xi] = \sum_{(t, \mathcal{L}_t) \in S(\xi)} \deg \left(\mathcal{M}_{e-1}(W_t, \tilde{\beta}_t) \times W_t^{e-1}\{((\xi_1, \ldots, \xi_{e-1})\}\right),
$$

and hence

$$
\langle [pt]^e\rangle_{0, \beta} = \sum_{(t, \mathcal{L}_t) \in S(\xi)} \langle [pt]^{e-1}\rangle_{0, \beta}\rangle_{W_t} = \sum_{\gamma \in H^e_2/G} \sum_{(t, \mathcal{L}_t) \in S(\mathcal{L}_E, \gamma)} \langle [pt]^{e-1}\rangle_{0, \beta}\rangle_{W_t}.
$$

However, all $W_t$ are deformation equivalent to any degree $d$ del Pezzo surface $\Sigma$, so $\langle [pt]^{e-1}\rangle_{0, \beta} = \langle [pt]^{e-1}\rangle_{0, \gamma}^{\Sigma}$ for $(t, \mathcal{L}_t) \in S(\mathcal{L}_E, \gamma)$. By (3.5), we conclude that

$$
\langle [pt]^e\rangle_{0, \beta} = \sum_{\gamma \in H^e_2/G} |S(\mathcal{L}_E, \gamma)| \langle [pt]^{e-1}\rangle_{0, \gamma}^{\Sigma} = \frac{1}{d(9-d)} \sum_{\gamma \in H^e_2} \Delta(\omega, \gamma) \langle [pt]^{e-1}\rangle_{0, \gamma}^{\Sigma},
$$

completing the proof of the formula. □
Chapter 2

Severi Varieties of K3 Surfaces

2.1 Introduction

In this very short (and purely descriptive) chapter, we will analyze the degeneration of curves on K3 surfaces, when the K3 surfaces themselves undergo a certain type of degeneration. Morally, this chapter serves as a warm-up for the next and final chapter, in which a similar task will be undertaken for curves on a well-known class of higher-dimensional Calabi-Yau varieties, namely on quintic threefolds. Like in both Chapter 1 and Chapter 3, rational curves on "log K3" pairs will play a central role.

Degenerations of K3 surfaces with normal crossing singularities are in a sense completely understood due classical to work of Kulikov [Ku77] and later Scattone [Sc87], and also somewhat more recent work of Olsson [Ols04]. They fall into 3 categories:

- Type I. These are the trivial degenerations: the special fiber is still smooth.
- Type II. The special fiber consists of a chain of surfaces with normal crossings: the surfaces at either end are rational and the ones in between are ruled elliptic.
Moreover, the double curves are smooth of genus one.

- Type III. The special fiber consists of a configuration of rational surfaces, whose dual graph (or rather dual 2-simplex) is a triangulation of the sphere $S^2$. The double curves are rational in this case.

Consider $S$, a genus $g$ principally polarized K3 surface. For $h \leq g$, we can consider the space of sections of the polarizing line bundle which are curves of geometric genus $h$. The associated classical Severi variety is defined as the closure of this locus. Now we can state the main goal of this chapter more precisely. Given a degeneration of genus $g$ polarized K3 surfaces, which we can choose according to our needs, we can consider the family of Severi varieties in the smooth (general) fibers and take its limit as $t \to 0$. The specific degeneration we will use is a certain type II degeneration. The question we would like to address is the following: what is this limit of the Severi varieties? We will give a reasonably clean answer to this question, up to the problem of understanding the multiplicities of the components of the degeneration.

The arguments we will use are somewhat inspired by the seminal work [CH98] of Caporaso and Harris, in the case of Severi varieties of $\mathbb{P}^2$. In the cited work, the analysis takes the form of understanding special plane sections of the Severi variety, rather than degenerations of it. The two approaches are actually very much related, but we will not get into the details of explaining this relation.

Even more related is the work of X. Chen [Ch99, Ch02], in which he uses a similar approach to analyze the singularities of the rational curves in the polarizing class. Chen’s approach ultimately yields the very difficult theorem that all rational curves in the polarizing class of a very general genus $g$ polarized K3 surface have only nodal singularities. In principle, this chapter is an alternative to the relevant arguments in [Ch99, Ch02], with some caveats: (1) we will use principally polarized
K3’s rather than trigonal ones and a different degenerate polarization; (2) we will also consider the case $h > 0$; (3) the appearance of the final answer here is quite simple, due to the choices made.

2.2 A Polarized Type II Degeneration (Sketch)

In this section, we sketch the construction of the degeneration at the level of K3 surfaces. The existence of such a degeneration follows from Olsson’s work [Ols04], on which we will ultimately rely, but given that most of the required ingredients for our particular degeneration can be seen very concretely, a brief down-to-earth account seems in order.

We begin with some straightforward calculations involving a certain class of log K3 pairs. Let $E \subset \mathbb{P}^2$ be a smooth plane cubic and $\Sigma$ the blowup of $\mathbb{P}^2$ at 9 general points on $E$ - equivalently, start with nine general points in the plane and find the smooth cubic interpolating them. By a slight abuse of notation, we continue to call the birational transform simply $E$. Note that $-K_\Sigma \sim E$, so in particular it is effective.

**Step 1.** $\Sigma$ admits no nontrivial global regular vector field, that is $h^0(\mathcal{T}_\Sigma) = 0$.

*Proof.* We can interpret global vector fields as first-order infinitesimal automorphisms of $\Sigma$. Since (-1)-curves are rigid, any such infinitesimal automorphism must keep the exceptional curves fixed. We can blow down 9 of these exceptional curves to obtain an infinitesimal automorphism of $\mathbb{P}^2$ with at least 9 fixed points in sufficiently general position. Since $\chi_{\text{top}}(\mathbb{P}^2) = 3 < 9$, this is impossible. □

**Step 2.** The sheaf $\mathcal{R}_\Sigma(-E) = \mathcal{R}_\Sigma \otimes \mathcal{I}_{E/\Sigma}$ has $h^0 = h^2 = 0$ and $h^1 = 10$. 35
Proof. The previous step automatically implies $h^0 = 0$. For $h^2$, by Serre duality we have $h^2(\mathcal{F}_\Sigma(-E)) = h^0(\Omega^2_\Sigma(E) \otimes \Omega^2_\Sigma) = h^0(\Omega^2_\Sigma) = h^{1,0}(\Sigma) = b_1(\Sigma)/2 = 0$. To compute $h^1$, it will suffice to compute the sheaf Euler characteristic and to that end we can apply Hirzebruch-Riemann-Roch. With the standard notations, we can compute

$$\chi(\mathcal{F}_\Sigma(-E)) = \int_{\Sigma} \text{ch}(\mathcal{F}_\Sigma(-E)) \text{td}(\Sigma) = \int_{\Sigma} \text{ch}(\Sigma) \text{ch}(\mathcal{I}_E) \text{td}(\Sigma) =$$

$$= \int_{\Sigma} \text{ch}(\Sigma) \text{td}(\Sigma) \left( 1 - c_1(\Sigma) + \frac{c_2^2(\Sigma)}{2} \right) = -\frac{5}{6} \int_{\Sigma} c_2(\Sigma) = -10$$

since $\int_{\Sigma} c_1^2(\Sigma) = (K^2_\Sigma) = 0$ and $\int_{\Sigma} c_2(\Sigma) = \chi_{\text{top}}(\Sigma) = 12$. Therefore, $h^1(\mathcal{F}_\Sigma(-E)) = 10$. □

Step 3. The same holds for the sheaf $\mathcal{F}_\Sigma(-\log E)$. Moreover, the homomorphism $H^1\psi : H^1(\Sigma, \mathcal{F}_\Sigma(-\log E)) \to H^1(E, \mathcal{F}_E)$ is nonzero.

Proof. To clarify, it is the morphism coming from the short exact sequence

$$0 \to \mathcal{F}_\Sigma(-E) \to \mathcal{F}_\Sigma(-\log E) \to \iota_* \mathcal{F}_E \to 0$$

Taking the associated long exact sequence, we have

$$0 \to H^0(E, \mathcal{F}_E) \to H^1(\Sigma, \mathcal{F}_\Sigma(-E)) \to H^1(\Sigma, \mathcal{F}_\Sigma(-\log E)) \to H^1(E, \mathcal{F}_E) \to 0$$

since $H^2(\Sigma, \mathcal{F}_\Sigma(-E)) = 0$ by Step 2. This implies the second statement and the first statement for $H^1$. The first statement for $H^2$ also follows since the respective cohomology group sits between two 0’s in the long exact sequence and the first statement for $H^0$ actually follows automatically from Step 1. □
The purpose of the calculations above is to construct an explicit type II degenerate K3 surface, which we can smooth. Let \((Y_i, E)\) be two pairs consisting of a rational surface with \((K_{Y_i}^2) = 0, i = 1, 2\) and \(E\) a smooth anticanonical divisor inside each surface. We assume the two anticanonical divisors are abstractly isomorphic. Let \(W_0 = Y_1 \cup_E Y_2\) glued transversally. In order to say that \(W_0\) smoothes to a K3 surface, we need to impose the logarithmic \(d\)-semistability condition, which amounts to

\[ N_{E/Y_1} \otimes N_{E/Y_2} \cong \mathcal{O}_E. \tag{2.1} \]

Then the condition \(H^2(\mathcal{R}_{W_0}(- \log E)) = 0\) of having a trivial obstruction space will suffice to smooth \(W_0\), cf [KN94]. That is, there exists a family of smooth K3 surfaces over the punctured disk \(\Delta^*\) which specializes to \(W_0\) at \(0 \in \Delta\).

**Step 4.** The sheaf \(\mathcal{R}_{W_0}(- \log E)\) has \(h^0 = 0, h^1 = 20\) and \(h^2 = 0\).

*Proof.* Consider the short exact sequence

\[ 0 \to \mathcal{R}_{W_0}(- \log E) \to \iota_* \mathcal{R}_{Y_1}(- \log E) \oplus \iota_* \mathcal{R}_{Y_2}(- \log E) \to \iota_* \mathcal{R}_E \to 0. \]

We are abusing notation by writing \(\iota\) for all closed immersions in sight. Step 3 above implies that \(H^1\) on the last nonzero map is nonzero and also that \(H^2\) of the sheaf in the middle vanishes. Therefore, \(H^2(\mathcal{R}_{W_0}(- \log E)) = 0\). Analyzing the long exact sequence we get the other claims as well. \(\Box\)

However, this is not quite the end of the story. The careful reader will have noticed that we have only constructed a degeneration in the complex analytic category. In fact, for general \(W_0\) constructed as above, there can be no algebraic degenerations, since the general \(W_0\) does not admit any nontrivial algebraic line bundles. However, the condition of \(W_0\) admitting such a line bundle boils down
to imposing one linear condition in $\text{Pic}(E)$ (of countably many possible ones) on the 18 points on $E$ which are being blown up in either component. Simply put, the fiber product $\text{Pic}(Y_1) \times_{\text{Pic}(E)} \text{Pic}(Y_2)$ has to be nonempty. We will make a very concrete and particular choice for the compatible $\mathcal{L}_i \in \text{Pic}(Y_i)$ and $\mathcal{L}_E \in \text{Pic}(E)$, which will suit our future needs.

Recall that we have natural distinguished blowdown maps $\rho_i : Y_i \to \mathbb{P}^2$ contracting the 9 exceptional divisors which intersect $E \subset Y_i$. Let $E_{i,1}, E_{i,2}, \ldots, E_{i,k_i}$ be the exceptional divisors obtained by blowing up $\mathbb{P}^2$ at the points $p_{i,j}$, such that $E_{i,j}$ intersects $E$ at the point $p_{i,j} \in E$, now viewed on $E \subset Y_i$, for $i = 1, 2$ and $j = 1, 2, \ldots, 9$. We can define the complete linear system $|\mathcal{L}_i|$ as the $\rho_i$-pullback of the (incomplete) linear system $|\mathcal{P}_i^k|$ given by

$$|\mathcal{P}_i^k, \text{even}| = \text{Pic}^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(k) \otimes \mathcal{I}_{p_{i,1}}^{k-1}), \text{ or}$$

$$|\mathcal{P}_i^k, \text{odd}| = \text{Pic}^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(k) \otimes \mathcal{I}_{p_{i,1}}^{k-1} \otimes \mathcal{I}_{p_{i,2}}),$$

depending on whether $g = 2k$ or $2k - 1$. Here, we are using the classical convention for $\mathbb{P}$. These are simply degree $k$ plane curves with multiplicity at least $k - 1$ at $p_{i,1}$, which are also required to pass through $p_{i,2}$ if $g$ is odd. Generically, they are rational and irreducible, but they can split 0, 1, 2, 3, ..., $k - 2$ or $k$ lines through $p_{i,1}$. Alternatively, we can simply write $\mathcal{L}_i = \mathcal{O}_{Y_i}(kH_i - (k - 1)E_{i,1} - E_{i,2})$ if $g = 2k - 1$ and similarly for $g = 2k$, where $H_i \in \text{Pic}(Y_i)$ is the pullback of the hyperplane class of the $i$th copy of $\mathbb{P}^2$ to $Y_i$.

With the notation above, logarithmic $d$-semistability amounts to

$$\sum_{(i,j) \in [2] \times [9]} p_{i,j} \sim 3H_1|_E + 3H_2|_E$$

(2.2)
in Pic(E). Finally, we need to impose the condition \( L_1|_E \cong L_2|_E \), which defines the line bundle \( L_E \). This condition simply reads

\[
kH_1|_E - (k - 1)p_{1,1} - \epsilon p_{1,2} \sim kH_2|_E - (k - 1)p_{2,1} - \epsilon p_{2,2}
\]

(2.3)

for \( g = 2k - \epsilon, \epsilon = 0, 1 \). To review, all we need is to choose 18 general points \( p_{i,j} \) and two classes \( H_1, H_2 \in \text{Pic}^3(E) \) satisfying conditions (2.2) and (2.3) and then we can run the construction described so far. Note that the line bundles have been constructed such that \( (\mathcal{L}_i^2) = g - 1, (\mathcal{L}_i \cdot -K_{Y_i}) = \deg \mathcal{L}_E = g + 1, p_a(\mathcal{L}_i) = 0 \) and the restriction map \( |\mathcal{L}_i| \to |\mathcal{L}_E| \), which is clearly well defined since \( p_a(\mathcal{L}_i) = 0 \) and \( p_a(E) = 1 \), is an isomorphism.

The final and most difficult step (which will only be sketched, relying on Olsson’s general moduli space construction) is to construct a degeneration as above within the realm of genus \( g \) polarized K3 surfaces, rather than general complex-analytic K3 surfaces. Explicitly, we want to prove the existence of a degeneration \( \pi : W \to \Delta, \mathcal{L} \in \text{Pic}(W) \) over a smooth affine curve \( \Delta \) with a distinguished point \( 0 \in \Delta \) of a family of genus \( g \) polarized K3 surfaces, i.e. \( (W_t = \pi^{-1}(t), \mathcal{L}_t) \) is a genus \( g \) smooth (primitively) polarized K3 and such that the pair \( (W_0, \mathcal{L}_0) \) where \( \mathcal{L}_0 = (\mathcal{L}_1, \mathcal{L}_2) \in \text{Pic}(Y_1) \times_{\text{Pic}(E)} \text{Pic}(Y_2) = \text{Pic}(W_0) \) has all the properties described above. This simply follows from Theorem 6.2 in [Ols04]. Indeed, if we use the notation \( \overline{\mathcal{K}}_g \) for Spec(\( \mathbb{C} \))-pullback of the moduli stack \( \mathcal{M}_{2g-2} \) constructed by Olsson, then simply \( (W_0, \mathcal{L}_0) \in \overline{\mathcal{K}}_g(\mathbb{C}) \), so we can draw an arc from this point into the open subset parametrizing smooth polarized surfaces by the smoothness of \( \overline{\mathcal{K}}_g \), so we’re done.

Finally, we claim that we can construct a degeneration as in the previous paragraph, with the additional property that the total space \( W \) is smooth. This is
always true at points of $W\setminus E$, but not necessarily along $E$. Let $U$ be a (smooth, since there are no obstructions) versal deformation space of $W_0$ in the complex analytic setting and $V$ a small complex analytic neighborhood of $(W_0, \mathcal{L}_0)$ in $\overline{\mathcal{M}}_g(C)^{an}$.

Then there exists a map $\varphi : V \to U$ obtained by simply forgetting the polarization. Both $U$ and $V$ contain smooth (for $V$ smoothness also follows from Theorem 6.2, [Ols04]) codimension one loci $R_U$ and $R_V$ corresponding to reducible deformations of $W_0$. Of course, $\varphi(R_V) \subseteq R_U$. Note that $R_V$ is generically smooth (hence generically reduced) since it’s locally described simply by equations (2.2) and (2.3) over the smooth surface parametrizing triples $(E, H_1|_E, H_2|_E)$ with $H_i|_E \in \text{Pic}^3(E)$, so $R_U \times_U V$ is generically a transversal intersection. Since the total space of the versal family over $U$ is smooth, a simple local calculation shows that the last statement is exactly what we need to infer smoothness in the ”polarized” loci.

## 2.3 Degenerations of Severi Varieties

### 2.3.1 Setup

We consider the degeneration of polarized K3 surfaces $(W, \mathcal{L}) \to \Delta$ over the smooth curve $\Delta$, as described above. Note that $H^0(W_t, \mathcal{L}_t) = g + 1$ for all $t$ in a Zariski neighborhood of 0, so, by shrinking $\Delta$ we may assume it holds everywhere, hence $\pi_*\mathcal{L}$ is locally free of rank $g + 1$. Now let’s introduce the main objects of study - the Severi varieties of the K3 surfaces. If $(S, \mathcal{L}_S)$ is a smooth genus $g$ primitively polarized K3 surface, we let $V_h(S, \mathcal{L}_S)$ be the closure of the locus in $|\mathcal{L}_S|$ of all curves of geometric genus equal to $h$, for all $0 \leq h \leq g$. Note that all curves $|\mathcal{L}_S|$ are reduced and irreducible because the polarization is primitive. There is a well-defined cycle-map $\overline{\mathcal{M}}_{h,0}(S, \beta_S) \to V_h(S, \mathcal{L}_S)$ from the moduli space
of stable maps to the Severi variety, where $\beta_S$ is the Poincaré dual to $c_1(\mathcal{L}_S)$. Note that these maps are surjective by the definition of the Severi variety and the properness of the Kontsevich space.

The main goal of this chapter is to exhibit a degeneration of such Severi varieties. After shrinking further if necessary, the Severi varieties of the smooth fibers fit into a proper flat family $V_{\text{rel}}^h(W_0^\circ, \mathcal{L}_0^\circ)$. To construct the degeneration, simply take the closure of the total space $V_{\text{rel}}^h(W_0^\circ, \mathcal{L}_0^\circ)$ inside the projective bundle $P = \text{Proj}_{\Delta} \text{Sym}(\pi_* \mathcal{L})^\vee$ and denote the resulting locus by $V_{\text{rel}}^h(W, \mathcal{L})$ and its fiber over $0 \in \Delta$ by $V_h(W_0, \mathcal{L}_0)$. The purpose of what follows is to understand $V_h(W_0, \mathcal{L}_0) \subseteq |\mathcal{L}_0|$, up to issues relating to the multiplicities of its components.

### 2.3.2 The Main Criterion

We will use the theory of relative stable maps and stable maps to degenerations introduced by Li [Li01, Li02]. Although this theory is of great importance in Gromov-Witten theory, largely thanks to the fact that the moduli spaces admit a perfect obstruction theory, in this chapter it merely plays the role of a fancy, highly efficient and lazy language for semistable reduction. In the next chapter, it will play a more substantial role, including the existence of perfect obstruction theories. Let $\pi_\mathcal{M} : \mathcal{M}(\mathfrak{W}, \Gamma) \to \Delta$ be the degeneration of the (usual) moduli spaces of stable maps $\overline{\mathcal{M}}_{h,0}(W_t, \beta_t), t \neq 0, \beta_t$ the Poincaré dual of $c_1(\mathcal{L}_t)$, constructed in loc. cit.

Choose an element $D_0 = D_1 + D_2 \in V_h(W_0, \mathcal{L}_0)(\mathbb{C})$, with $D_i \subset Y_i$. We claim that there exists a degenerate genus $h$ stable map $(C, f) \in \mathcal{M}_{h,0}(\mathfrak{W}_0, \Gamma)(\mathbb{C})$ whose image-cycle in $W_0$ is precisely $D_1 + D_2$. This is automatic from the definitions: with possible (temporary) base changes at each step, first choose $D_t \in V_h(W_t, \mathcal{L}_t)(\mathbb{C}) \to D_0$ as $t \to 0$, then lift $D_t$ to $(C_t, f_t) \in \overline{\mathcal{M}}_{h,0}(W_t, \beta_t)$ and, by properness [Li1] of the
family $\pi_M$, we deduce the claim.

Note that any non-contracted component of $C$ mapping to $Y_1$ or $Y_2$ has to be of genus zero, since $p_a(|\mathcal{L}|) = 0$. For components mapping to the intermediary elliptic ruled surfaces, they have to be either contracted, or mapped to the fibers of the ruling. Indeed, the fact that $\mathcal{L}_i \otimes \mathcal{O}_{Y_i}(-E)$ is not effective, implies that the composition of $f$ restricted to such a component with the map to $E$ given by the ruling cannot be dominant.

Let $r$ be the number of distinguished marked points of $(C, f)$. Then $r = \ell(\Gamma^{\text{red}})$, where $\Gamma = D_1 \cap E = D_1 \cap D_2$ and $\ell$ will denote in this chapter the length of a zero dimensional scheme. Let

$$\Gamma = \Gamma^{\text{red}} + \Gamma^{\text{ex}},$$

where $\Gamma^{\text{red}}$ is the sum of all points of $\Gamma$ taken without any multiplicities and $\Gamma^{\text{ex}}$ is the leftover, i.e. the "excess part" of $\Gamma$.

The key to the combinatorial analysis below are two rather subtle numerical measures of the degeneracy of $D_1 \cup D_2$. First, let $\nu : |\mathcal{L}_i| \to \mathbb{N} = \{0, 1, 2, 3, \ldots\}$ be one less the number of components of the image in $\mathbb{P}^2$ under the natural blowdown of $R \in |\mathcal{L}_i|$, taken with multiplicities, plus the number of components which are contracted under this blowdown, taken without multiplicities. Explicitly, if

$$R = m_1 R_1 + \ldots + m_\alpha R_\alpha + m'_1 E_{i,1} + \ldots + m'_9 E_{i,9}$$

with all $R_l$ irreducible and different from the $E_{i,j}$, then

$$\nu(R) = \sum_{l=1}^{\alpha} m_l + \sum_{j=1}^{9} \text{sgn}(m'_j) - 1,$$

where $\text{sgn}$ is the sign function, i.e. 1 if $m'_j > 0$ and 0 if $m'_j = 0$. Note that $\nu$ is
upper semi-continuous in the Zariski topology.

Second, we let $\mu_0(D_1+D_2)$ be the number of arcs of $D_1+D_2$ in a formal/complex analytic neighborhood of $E = Y_1 \cap Y_2 \subset W_0$, taken without multiplicities. Let

$$
\mu(D_1 + D_2) = \mu_0(D_1 + D_2) - \ell(\Gamma_{\text{red}}).
$$

Since there are at least two distinct arcs near any point of $\Gamma = D_i \cap E$, we have $\mu(D_1 + D_2) \geq \ell(\Gamma_{\text{red}})$.

We will construct a certain curve $\tilde{C} \subset C$ as follows. For $i = 1, 2$, let $C_i$ be the union (inside $C$) of all components of $C$ which map nonconstantly to $Y_i$ but not to some $E_{i,j}$ and of exactly one curve mapping nonconstantly to each $E_{i,j}$, if one exists. For each of the $\ell(\Gamma_{\text{red}})$ points of $\Gamma$, there exists a "bridge" (a tree in the dual graph - not necessarily a chain) joining $C_1$ with $C_2$, whose number of leaves is equal to the number of arcs of $D_1 + D_2$ at that point and all leaves are connected to either $C_1$ or $C_2$. We define $\tilde{C}$ to be the union inside $C$ of $C_1$, $C_2$ and these $\ell(\Gamma_{\text{red}})$ disjoint bridges.

Then, we have $p_a(\tilde{C}) \leq p_a(C)$ since erasing components from the connected semistable (nodal) curve $C$ can only decrease the arithmetic genus. Moreover, we have $p_a(C_1) + p_a(C_2) + \mu(D_1 + D_2) - 1 \leq p_a(\tilde{C})$ from the usual formula for the arithmetic genus of nodal curves (equality does occur if all components of the chain are rational, but it’s not entirely clear if this is always the case). Hence, we
obtain the key inequality
\[ p_a(C_1) + p_a(C_2) + \mu(D_1 + D_2) - 1 \leq p_a(C) = h. \] (2.4)

Moreover, we have \( p_a(C_i) \geq -\nu(D_i) \) for \( i = 1, 2 \), which also follows from the formula for the arithmetic genus of a reducible curve so we obtain the inequality
\[ \nu(D_1) + \nu(D_2) + (g + 1 - \mu(D_1 + D_2)) \geq g - h. \] (2.5)

Note that this inequality involves only terms related to \( D_1 + D_2 \) rather than terms involving \( C \) in any unavoidable way, so relative/degenerate stable maps are already out of the picture. Since \( V_h(W_0, \mathcal{L}_0) \subseteq V'_h(W_0, \mathcal{L}_0) \), this inequality applies to any \( D_1 + D_2 \in V_h(W_0, \mathcal{L}_0)(\mathbb{C}) \).

Given that \( \mu(D_1 + D_2) \geq \ell(\Gamma^\text{red}) \) and \( \ell(\Gamma^\text{red}) + \ell(\Gamma^\text{ex}) = g + 1 \), we also obtain the weaker, but simpler, inequality
\[ \nu(D_1) + \nu(D_2) + \ell(\Gamma^\text{ex}) \geq g - h. \] (2.5b)

The rest of the chapter will be dedicated to interpreting inequalities (2.5) and (2.5b) in terms of the explicit description of \( Y_i, \mathcal{L}_i \) and \( E \). Denote the \( \nu \)-stratification on \( |\mathcal{L}_i| \) by \( \Omega_i \) and the open stratum of dimension \( d \) by \( \Omega_i[d] \). Recalling the description of the linear systems \( |\mathcal{L}_i| \), we can understand explicitly the two stratifications. Similarly, on \( |\mathcal{L}_E|(|\mathbb{C}) \) we can define the loci \( \Omega_E[e] := \{ \Gamma \in |\mathcal{L}_E|(|\mathbb{C}) : \ell(\Gamma^\text{ex}) = e \} \).

Finally, the restriction maps \( H^0(\mathcal{L}_i) \to H^0(\mathcal{L}_E) \) are isomorphisms by construction, so we can canonically identify their classical projectivizations \( |\mathcal{R}_i^k| \) and \( |\mathcal{L}_E| \).
We will call this projective space simply $\mathbb{P}^g$. Moreover, since

$$H^0(W_0, \mathcal{L}_0) = H^0(Y_1, \mathcal{L}_1) \times_{H^0(E, \mathcal{L}_E)} H^0(Y_2, \mathcal{L}_2),$$

the space of sections $|\mathcal{L}_0|$ is also naturally identified with $\mathbb{P}^g$. The purpose of all this is to view $\Omega_1$, $\Omega_2$ and $\Omega_E$ simultaneously on the same space. We can restate the weaker criterion (2.5b) in the following simple form.

**Theorem 2.2.1.** With the identification $|\mathcal{L}_0| \cong \mathbb{P}^g$ as above, the locus $V_h(W_0, \mathcal{L}_0)^{\text{red}}$ is contained inside the union

$$\bigcup_{d_1 + d_2 + e = g - h} \Omega_1[d_1] \cap \Omega_2[d_2] \cap \Omega_E[e]$$

of all triple intersections of closed strata of total codimension $g - h$. Moreover, the points of $V_h(W_0, \mathcal{L}_0)^{\text{red}}$ have to also satisfy the stronger inequality (2.5).

### 2.4 Example: $(g, h) = (2, 0)$

In this final section, we concretely illustrate theorem 2.2.1 in the case $g = 2$, $h = 0$, i.e. rational curves on a genus 2 K3 surface. It is a very well-known fact that a general genus 2 K3 surface contains exactly 324 rational curves. To check this number, we also need to take into account the multiplicities of the components of $V_h(W_0, \mathcal{L}_0)^{\text{red}}$. We will not go rigorously into this question, but the general philosophy, as reflected in many sources including [CH98] and [Li02], is that the multiplicity of a component should equal the product of the multiplicities of the points of $\Gamma$ at a general curve in the respective component.

In our concrete case, the linear system $|\mathcal{L}|$ is simply the pullback of the linear
system of lines on the $i$th copy $\mathbb{P}^2[i]$ of the projective plane. We will rather think of the sections of this series as living in the respective projective plane. Applying the description in theorem 2.2.1, we get the following possibilities for the limits in $W_0$ of the rational curves.

<table>
<thead>
<tr>
<th>$(d_1, d_2, e)$</th>
<th>number</th>
<th>brief description</th>
<th>expected multiplicity</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(2,0,0)$</td>
<td>$\left(\frac{9}{2}\right) = 36$</td>
<td>line through $p_{1,i}$ and $p_{1,j}$ in $\mathbb{P}^2[1]$ (and hence same in $\mathbb{P}^2[2]$)</td>
<td>1</td>
</tr>
<tr>
<td>$(0,2,0)$</td>
<td>$\left(\frac{9}{2}\right) = 36$</td>
<td>line through $p_{2,i}$ and $p_{2,j}$ in $\mathbb{P}^2[2]$ (and hence same in $\mathbb{P}^2[1]$)</td>
<td>1</td>
</tr>
<tr>
<td>$(0,0,2)$</td>
<td>9</td>
<td>tangent lines at flex points in both $\mathbb{P}^2[1]$ and $\mathbb{P}^2[2]$</td>
<td>3</td>
</tr>
<tr>
<td>$(1,1,0)$</td>
<td>$9 \times 9 = 81$</td>
<td>line through $p_{1,i}$ in $\mathbb{P}^2[1]$ and line through $p_{2,j}$ in $\mathbb{P}^2[2]$</td>
<td>1</td>
</tr>
<tr>
<td>$(1,0,1)$</td>
<td>$9 \times 4 = 36$</td>
<td>line through $p_{1,i}$ in $\mathbb{P}^2[1]$, tangent to $E \subset \mathbb{P}^2[1]$ at other point</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>tangent line to $E \subset \mathbb{P}^2[1]$ at $p_{1,i}$</td>
<td>0</td>
</tr>
<tr>
<td>$(0,1,1)$</td>
<td>$9 \times 4 = 36$</td>
<td>line through $p_{2,i}$ in $\mathbb{P}^2[2]$, tangent to $E \subset \mathbb{P}^2[2]$ at other point</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>tangent line to $E \subset \mathbb{P}^2[2]$ at $p_{2,i}$</td>
<td>0</td>
</tr>
</tbody>
</table>

The reason why two of the classes above have multiplicity 0, that is, they do not correspond to real limits of rational curves, is simply that they violate the stronger inequality (2.5), whereas the organizing principle of the table is the weaker (2.5b).

In conclusion, we obtain the expected grand total of

$$\left(\frac{9}{2}\right) \times 1 + \left(\frac{9}{2}\right) \times 1 + 9 \times 3 + 9 \cdot 9 \times 1 + 9 \cdot 4 \times 4 + 9 \times 0 + 9 \cdot 4 \times 4 + 9 \times 0 = 324.$$
Chapter 3

Quintic Threefolds and the Chordal Variety

3.1 Introduction

3.1.1 Algebraic Curves on Calabi-Yau Threefolds

In the study of algebraic curves on Calabi-Yau manifolds, one of the first practical obstacles to understanding their elementary geometric properties is the fact that such curves can rarely be deformed holomorphically. However, this expected rigidity is at the same time one of the premises to such a rich theory.

Let $X$ be a smooth complex projective variety. A stable map $f : C \to X$ of class $\beta$ and genus $g$ will be called rigid, if the following condition is satisfied: for any flat family of stable maps $\mathcal{C} \to S$, $\mathcal{C} \to X$ parametrized by an integral $\mathbb{C}$-scheme $S$, if there exists a closed point $\text{Spec}(\mathbb{C}) \to X$ such that $\text{Spec}(\mathbb{C}) \times_X \mathcal{C} \to X$ is isomorphic to $f$, then the condition holds for all closed points of $S$. Equivalently, $(C, f)$ is the only point in a connected component of the moduli stack $\overline{\mathcal{M}}_g(X, \beta)$ of stable maps. The slightly stronger notion of infinitesimal rigidity amounts to $f$
having only trivial infinitesimal deformations.

The naive expectation is that such rigid stable maps occur when

$$v \dim \overline{M}_g(X, \beta) = \int_\beta c_1(X) + (g - 1)(3 - \dim X) = 0,$$

so we expect an abundance of such rigid stable maps if and only if $X$ is a Calabi-Yau threefold. Of course, if $v \dim \overline{M}_g(X, \beta) > 0$, then rigid stable maps cannot exist, whereas if $v \dim \overline{M}_g(X, \beta) < 0$, then they must be obstructed.

Although the whole point of the Gromov-Witten theory of Calabi-Yau varieties is to avoid questions concerning rigidity such as the Clemens conjecture [Cl86] (which states that a very general quintic threefold contains countably many rational curves), such problems remain interesting and mostly wide open, especially in large degree. For quintics, Clemens proved the existence of smooth infinitesimally rigid rational curves of arbitrarily large degree as an ingredient in his theorem on algebraic versus homological equivalence of 1-cycles [Cl83] and later Katz strengthened the argument [Ka86a] to show that such rational curves occur in all degrees. This approach was taken further by Knutsen [Kn12] to prove the existence of smooth infinitesimally rigid curves of genera $g \leq 22$, if $d$ is sufficiently large.

The main application of the construction presented in the rest of the chapter concerns the existence of rigid stable maps when $g > 0$, which requires methods different from those used in the case $g \leq 22$. Nevertheless, much of the chapter is written with the rational case as the guiding target, which is the best suited to our construction.

**Theorem 3.1.1.** Let $g \geq 0$, $d \gg_g 0$ and $Q$ a general quintic threefold. Then $Q$ admits a rigid, degree $d$ stable map $f : C \to Q$ whose source $C$ is a smooth curve of genus $g$. 
The possibility of $f$ factoring through an unramified cover of smooth curves with source $C$ is not ruled out in general, but this situation can only occur for special pairs $(d, g)$. Pushing further the arguments below could fix this shortcoming, as well as prove infinitesimal rigidity, but such polishings will not be pursued here.

### 3.1.2 The Approach by Specialization

The idea that one could gain insight into questions about rational curves (or more generally, arbitrary algebraic curves) on Calabi-Yau threefolds by degenerating the underlying threefold has been around for a long time. Folklore states that the idea originated as a potential approach to the Clemens conjecture. Very naively, if one could exhibit a sufficiently general family of quintics specializing to a reducible variety and locate the flat limits of the curves on the degenerate variety, finiteness would obviously follow. In enumerative geometry, the idea to use toric degenerations and tropical geometry to understand mirror symmetry is known as the Gross-Siebert program [GS06, GS10, GS09, Bu10].

A simple but already very interesting example is the degeneration of a family quintics to the reducible hypersurface $X_0X_1X_2X_3X_4 = 0$, the union of all coordinate hyperplanes. The project of describing explicitly the limits of the rational curves from the general fibers has been carried out in degree $d = 1$, cf. [Ka86b, Ni15]. However, extending this analysis to arbitrary degree is a difficult task. The main purpose of this chapter is to exhibit an alternative degeneration of quintic threefolds which is not toric, but provides a different type of leverage in the hunt for the limits of the rational curves and to show some definite progress towards this goal.

The degeneration we will consider is obtained by running semistable reduction on a very general pencil of quintic threefolds specializing to the chordal variety of
a normal elliptic curve in the ambient projective space. Let $E \subset \mathbb{P}^4$ be a smooth nondegenerate curve of genus 1 and degree 5 and let $\Theta$ be the chordal variety of $E$, that is, the singular threefold swept out by all lines connecting pairs of point on $E$. Consider the diagram

$$
\begin{array}{ccc}
\tilde{\Theta} & \longrightarrow & \mathcal{U} \\
\downarrow & & \downarrow \\
E^{(2)} & \longrightarrow & G(1, 4)
\end{array}
$$

where $E^{(k)}$ is the $k$th symmetric power of $E$ or equivalently the Hilbert scheme of length $k$ schemes on $E$ and $\mathcal{U} \to G(1, 4)$ is the universal family of lines in $\mathbb{P}^4$. The map $E^{(2)} \to G(1, 4)$ is defined by sending a length two subscheme of $E$ to the unique line in $\mathbb{P}^4$ which contains it. Define $\tilde{\Theta} = E^{(2)} \times_{G(1, 4)} \mathcal{U}$ and $\Theta$ its image in $\mathbb{P}^4$. It is not hard to check that $\Theta$ has degree 5. We’ll briefly review this calculation and other geometric properties early in section 3.2.1.

Consider a very general pencil of quintic threefolds whose fiber over $0 \in \mathbb{A}^1 \subset \mathbb{P}^1$ is the secant variety $\Theta$. The total space of the pencil is the variety $\{F_\Theta + tF = 0 \} \subset \mathbb{A}_t^1 \times \mathbb{P}^4$ where $F_\Theta$ is the homogeneous polynomial of $\Theta$ and $F \in H^0(\mathcal{O}(5))$. Seen over $\mathbb{A}_t^1$, this is a family of quintics degenerating to $\Theta$. The main observation is that running semistable reduction on this family yields a central fiber with two (smooth and transverse) components both admitting nonconstant maps to a smooth algebraic curve of genus one.

Section 3.2.2 is devoted to the semistable reduction calculation. There are two steps: (1) make a base change of order 3 totally ramified at $0 \in \mathbb{A}^1$; and (2) blow up the curve $E \times \{0\}$ in the total space of the family. Said differently, $W$ is the proper transform of $Z = \{F_\Theta + t^3F = 0 \}$ in the blowup of $\mathbb{A}^1 \times \mathbb{P}^4$ along $\{0\} \times E$. We will check:
(a) The total space $W$ is smooth.

(b) Its central fiber $W_0$ has two normal crossing smooth components:

- $Y_1 = \tilde{\Theta}$, which is fibered over $\text{Pic}^2(E)$ with fibers $F^1$ and
- the exceptional divisor $Y_2$, which admits a map to $E$, whose fibers are
  - smooth (with 25 exceptions) "cyclic" (see 3.2.3) cubic surfaces and
  - 25 projective cones over a smooth place cubic, above certain $p_1, \ldots, p_{25}$.

(c) The intersection $Y_1 \cap Y_2$ is isomorphic to $E \times E$.

(d) The two maps from $Y_i$ to $\text{Pic}^2(E)$ or $E$ restrict on the intersection to the tensor product map $E \times E \to \text{Pic}^2(E)$ and projection to one of the factors respectively.

It may be useful to abstract away the essential features of the configuration above. What we have is a family $\pi : W \to \Delta$ over a smooth curve $\Delta$, with smooth total space, such that $W_0 = Y_1 \cup Y_2$ with $Y_1 \cap Y_2 = E_1 \times E_2$ and $E_1 \cong E_2$ isomorphic smooth genus 1 curves, $Y_i$ smooth and admitting a regular map $\varphi_i : Y_i \to E_i$ such that the diagram

$$
\begin{array}{ccc}
E_1 \times E_2 & \longrightarrow & E_i \\
\downarrow & & \downarrow \\
Y_i & \longrightarrow &
\end{array}
$$

in which the vertical arrow is the closed embedding, the diagonal arrow is $\varphi_i$ and the horizontal arrow is the projection to the respective factor, is commutative for $i = 1, 2$. Moreover, the fibers of the pair $(Y_i, E_1 \times E_2)$ over $E_i$ are generically log K3 pairs, i.e. pairs consisting of a smooth surface and a smooth effective anticanonical divisor. In this thesis we will only be concerned with such a family of quintic threefolds, but it seems very likely cf. [GP00] that degenerations fitting into the pattern above also exist for other deformation classes of Calabi-Yau threefolds.
The (purely aesthetic) change of perspective from $E \times E$ to $E_{1} \times E_{2}$ is obtained by considering the isomorphism $(\otimes, \text{proj}_{1}) : E \times E \rightarrow E_{1} \times E_{2}$, where $E_{1} = \text{Pic}^{2}(E)$ and $E_{2} = E$. A choice of one of the 25 hyperexes of $E \subset \mathbb{P}^{4}$ gives a convenient identification of $E_{1} = \text{Pic}^{2}(E)$ with $E$. If $h$ is the hyperflex, the isomorphism is the map $p \mapsto \mathcal{O}_{E}(p + h)$.

3.1.3 Limits of Rational Curves

As we have explained in the previous section, the purpose of the construction above is to analyze the reducible curves in the central fiber $W_{0}$ which are limits of rational curves from nearby fibers $W_{t}, t \neq 0$. The fact that both components admit maps to genus one curves confines all the irreducible components of any limit to the fibers of these maps. There is some ambiguity about what we mean by limit: for now we will think of the limits as being 1-cycles in the obvious way, while in the rest of the paper we will use the theory of relative stable maps. Consider the map

$$
\left\{ \begin{array}{l}
\text{1-cycles on } W_{0} \text{ which are limits} \\
\text{of rational curves from nearby fibers}
\end{array} \right\} \xrightarrow{\Phi} \left\{ \begin{array}{l}
\text{finite collections of} \\
\text{closed points on } E
\end{array} \right\}
$$

which associates to each 1-cycle $L = a_{1}D_{1} + ... + a_{n}D_{n} \in Z^{1}(W_{0})$ which is a limit of rational curves the unordered collection of all points $p \in E_{1}$ and $p \in E_{2}$ such that $\varphi_{i}^{-1}(p)$ contains some irreducible component $D_{j}$. Let $\Phi_{d}$ be the restriction of $\Phi$ to the subset of limits of degree $d$ curves.

It is easy to prove that $\Phi_{d}$ is finite-to-one. Therefore, the set of limits is finite (which is obviously equivalent to the Clemens conjecture being true) if and only if $\text{Im}(\Phi_{d})$ is finite. It appears that $\text{Im}(\Phi_{d})$ is governed by complicated, but purely combinatorial, patterns related to the group structure of $E$. This speculation will
be illustrated in the following (perhaps somewhat pretentious) unsolved question, which can be proved for $d \leq 5$ using the methods presented here.

**Question 3.1.2.** *Is it necessarily true that all entries of any element of $\text{Im}(\Phi)$ are rational linear combinations\(^1\) of the 25 intersection points of the original normal elliptic curve $E$ with the base locus of the pencil?*

The main goal of this chapter is to describe the spaces to genus zero stable maps to $W_0$ (or more precisely $\mathcal{M}_0$), as defined by Li [Li01, Li02]. We will obtain a manageable description of these spaces for any degree $d$, involving certain spaces of rational curves on del Pezzo surfaces and some combinatorics involving discrete differential equations for torus-valued functions on a graph. A remarkable and totally unexpected, but possibly misleading observation, cf. remark 3.3.3, is that families of maps sweeping out a locus of dimension two or more on $W_0$ have, in a certain sense, contribution $0$ to the virtual count. Currently, there are some serious technical obstacles to even stating this remark in a meaningful way, but further investigation seems worthwhile. This remark, together with the fact that "finite" classes do satisfy the property above, could also count as evidence in favor of 3.1.2.

Finally, we would like to briefly discuss the $g > 0$ case. The analysis above can be easily generalized only for degenerate stable maps whose dual graph has the property $\text{rank } H^1(G, \mathbb{Z}) = g$, since this purely topological condition forces all components to be rational. This can be used to prove Theorem 3.1.1. After some rather messy reductions, we boil down the proof to the existence of graphs with rank $\text{H}^1(G, \mathbb{Z}) = g$ and $d$ edges, having the (Laplace) eigenvalue $\frac{5}{3}$ in some $\mathbb{F}_p$ but not in $\mathbb{Q}$ and satisfying some additional properties. This final combinatorial step was computer-assisted in the following sense: the examples exhibited in the

\(^1\)By this we mean $l_0 \sim k_1 p_1 + \ldots + k_{25} p_{25}$ for $l, k_1, \ldots, k_{25} \in \mathbb{Z}$, $l \neq 0$. 53
appendix were found using a computer, but their validity can be ascertained by hand, given sufficient patience.

3.2 Explicit Description of the Degeneration

3.2.1 Chordal Variety of a Normal Elliptic Curve

We begin with an elementary review of the chordal variety of a normal elliptic curve in $\mathbb{P}^4$. We use the same notation as above: $E$ is the normal elliptic curve and $\Theta$ is the chordal variety, which by definition is the singular threefold swept out by all the lines intersecting $E$ in a scheme of length 2. Let $\tilde{P}$ be the blow up of $\mathbb{P}^4$ along the curve $E$. We have an obvious tower of $\mathbb{P}^1$-bundles $\tilde{\Theta} \to \text{Sym}^2 E \to \text{Pic}^2 E$ and thus $\tilde{\Theta}$ is smooth. It follows that for any $D \in \text{Pic}^2 E$, the fiber $\tilde{\Theta}_D$ is some rational ruled surface $\mathbb{F}_n$.

**Lemma 3.2.1.** The degree of $\Theta$ in $\mathbb{P}^4$ is equal to 5.

*Sketch of proof.* We may project away from a general line $\ell \subset \mathbb{P}^4$ onto $\mathbb{P}^2$ and note that intersection points of $\ell$ with $\Theta$ correspond to the nodes of the projection of $E$. Since the projection of $E$ to $\mathbb{P}^2$ has degree 5 and geometric genus 1, the genus-degree formula implies that the number of nodes is equal to 5. □

**Lemma 3.2.2.** The map $j : \tilde{\Theta} \to \tilde{P}$ is a closed embedding.

*Sketch of Proof.* It suffices to show that $j$ is injective and $dj : T\tilde{\Theta} \to j^*T\tilde{P}$ is fiberwise injective. For the first claim, assume that there exist two distinct points $p, q \in \tilde{\Theta}$ such that $j(p) = j(q)$. Let $\ell_p, \ell_q \subset \tilde{\Theta}$ be the two secant lines containing $p$ and $q$ and let $\ell_p, \ell_q$ be their images in $\mathbb{P}^4$. Clearly, $\ell_p \neq \ell_q$. Assume first that $\ell_p \cap \ell_q \notin E$. Then the length 4 subscheme $D_{p,q} = E \cap (\ell_p \cup \ell_q)$ of $E$ is contained
in the projective subspace of \( \mathbb{P}^4 \) spanned by \( \ell_p \) and \( \ell_q \), which has dimension 2. However, by Riemann-Roch,

\[
h^0(\mathcal{O}_E(1) \otimes \mathcal{O}_E(-D_{p,q})) = 1,
\]

leading to contradiction. If \( \ell_p \cap \ell_q \in E \), then \( \tilde{\ell}_p \cap \tilde{\ell}_q = \emptyset \), which is also impossible.

The second claim is an infinitesimal version of the first. Assume that there is a tangent vector of to \( \tilde{\Theta} \) which is killed by \( d_j \). Then there exists a line and a first order thickening \( \tilde{\ell} \subset \hat{\ell} \subset \tilde{\Theta} \) such that the corresponding section of \( \mathcal{N}_{\tilde{\ell}/\hat{\ell}} \) vanishes at some point, or equivalently, \( \ell' \) is contained in a plane, where \( \ell \subset \ell' \) is the image in \( \mathbb{P}^4 \) of \( \tilde{\ell} \subset \hat{\ell} \). Then an argument similar to the one above leads to contradiction. \( \square \)

**Lemma 3.2.3.** All the fibers of \( \tilde{\Theta} \to \text{Pic}^2 E \) are isomorphic to \( \mathbb{F}_1 \).

**Proof.** One way to prove this is via an indirect argument starting with the Hirzebruch surface. Let \( f, e_\infty, e_0 \in \text{Pic} \mathbb{F}_1 \) be the classes of the fibration, of the directrix and of the preimage of a general line from the projective plane obtained by blowing down the directrix respectively. These classes generate \( \text{Pic} \mathbb{F}_1 \), subject to the relation \( e_0 = e_\infty + f \). The line bundle \( \mathcal{O}_{\mathbb{F}_1}(e_\infty + 2f) \) defines the standard embedding of \( \mathbb{F}_1 \) into \( \mathbb{P}^4 \).

Consider the anticanonical class \(-K_{\mathbb{F}_1} = \mathcal{O}_{\mathbb{F}_1}(2e_\infty + 3f)\). The divisors in the associated linear system are curves of arithmetic genus one, intersecting the directrix once and the fibers of the ruling twice. It is not hard to see that we can find such a divisor, which is abstractly isomorphic to our curve \( E \) and such that the divisor class on \( E \) cut by the ruling is precisely \( D \). Then embedding the pair \( E \subset \mathbb{F}_1 \) in \( \mathbb{P}^4 \), it is not hard to see that \( E \) is an elliptic normal curve and that the \( \mathbb{F}_1 \) surface is precisely the surface swept out by the lines connecting pairs of points...
on $E$ whose sum as divisor classes is precisely $D$. Since the action of $\text{PGL}(5, \mathbb{C})$ on the set of normal elliptic curves of a fixed $j$-invariant is transitive, this argument proves the claim that $\tilde{\Theta}_D$ is an $\mathbb{P}_1$ surface. □

Alternative Proof. The claim also follows from the observation that $\tilde{\Theta}_D$ is a minimal degree variety, due to the classification proved in [EH87]. □

Note that the intersection $S$ of $\tilde{\Theta}$ with the exceptional divisor $T$ of $\tilde{P}$ is canonically isomorphic to $E \times E$. Roughly, any point in the intersection is specified by choosing a point on $E$ and an infinitesimal direction inside $\Theta$ from the chosen point, the latter also being specified by a point on $E$. The restriction of $\tilde{\Theta} \to \text{Pic}^2 E$ to this locus is the tensor product map $\mu$ by construction. In this section, we order the factors of $S = E \times E$ such that the restriction $S \to E$ of the blowdown map is projection to the first factor. We will usually identify $\mu^{-1}(D)$ with $E$ in this way.

Lemma 3.2.4. (a) The point where the directrix of $\tilde{\Theta}_D$ intersects $\mu^{-1}(D) \cong E$ is the one whose divisor class is $\mathcal{O}_E(1) \otimes \mathcal{O}_E(-2D)$.

(b) The restriction of the hyperplane class of $T_p$ to $S_p \cong E$ is $\mathcal{O}_E(1) \otimes \mathcal{O}_E(-2p)$, for all closed $p \in E$.

Proof. Part (a) follows from the proof of 3.2.1. For part (b), simply pick a hyperplane containing the line tangent to $E \subset \mathbb{P}^4$ at $p$ and project away from the tangent line. Note that $S_p$ sits inside $T_p \cong \mathbb{P}^2$ as a smooth plane cubic. □

Denote the blowdown map $\tilde{P} \to \mathbb{P}^4$ by $\rho$ and its restriction to $T$ by $\tau : T \to E$. Of course, $T$ is the projectivization of the normal bundle of $E$, i.e. $T = \text{Proj Sym } \mathcal{N}^\vee_{E/\mathbb{P}^4}$ and as such comes equipped with a tautological line bundle $\mathcal{O}_T(-1)$ which is isomorphic to $\mathcal{O}_T(3) \otimes \tau^* \mathcal{O}_E(5)$.

Claim 3.2.5. We have $\rho^* \mathcal{O}_{\mathbb{P}^4}(\Theta) \cong \mathcal{O}_{\tilde{P}}(\tilde{\Theta} + 3T)$ and $\mathcal{O}_T(S) \cong \mathcal{O}_T(3) \otimes \tau^* \mathcal{O}_E(5)$.
Proof. We must have $\rho^*\Theta = \hat{\Theta} + kT$ for some $k$ as Cartier divisors. For a line $L$ contained inside the fibers of $\tau$ we have $(\rho^*\Theta \cdot L) = 0$, $(\hat{\Theta} \cdot L) = 3$ since $S_p \subset T_p$ has degree 3 and $(T \cdot L) = -1$, hence $k = 3$. For the second part, tensor both sides with $\mathcal{O}_T$. The left hand side and the right hand side become $\rho^*\mathcal{O}_{P^4}(5) \otimes \mathcal{O}_T \cong \tau^*\mathcal{O}_{E}(5)$ respectively $\mathcal{O}_{\hat{T}}(\hat{\Theta} + 3T) \otimes \mathcal{O}_T \cong \mathcal{O}_T(-3) \otimes \mathcal{O}_T(S)$, proving the claim. □

Lemma 3.2.6. The ideal sheaf $\mathcal{I}_{\Theta/P^4}$, viewed as a subsheaf of the structure sheaf of $P^4$, is contained inside $\mathcal{I}_{E/P^4}^3$.

Proof. Is it clear from (2.1.a) that there exists a natural homomorphism of $\mathcal{O}_{\hat{T}}$-modules $\mathcal{O}_{\hat{T}}(3T) \rightarrow \rho^*\mathcal{O}_{P^4}(\Theta)$, dual to some homomorphism $\rho^*\mathcal{I}_{\Theta/P^4} \rightarrow \mathcal{I}_{T/\hat{P}}^3$. By the adjoint property, we get a map $\mathcal{I}_{\Theta/P^4} \rightarrow \rho_*\mathcal{I}_{T/\hat{P}}^3$ and it is not hard to check that the latter sheaf is simply $\mathcal{I}_{E/P^4}^3$. The verification of the fact that the triangle formed by the $\mathcal{O}_{P^4}$-module homomorphism defined above and the inclusion maps into the structure sheaf commutes is skipped. □

Side Remark 3.2.7. Since degenerations of quintics to unions of five hyperplanes were mentioned in 3.1.2, it’s hard to resist the temptation of saying that \{X_0X_1X_2X_3X_4 = 0\} is in fact a flat limit of a family of such chordal varieties.

Indeed, let $\ell_{i,j}$ be the line \{[\alpha e_i + \beta e_j] : \alpha, \beta \in \mathbb{C}\}. Then $E^t = \ell_{0,1} \cup \ell_{1,2} \cup \ell_{2,3} \cup \ell_{3,4} \cup \ell_{4,0}$ is a reducible curve of degree 5 and arithmetic genus 1 whose chordal variety

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is simply \( \{X_0 X_1 X_2 X_3 X_4 = 0\} \). It is not hard to check that \( E^\dagger \) is smoothable, as required.

### 3.2.2 Base Change and Blowup

The previous lemma suggests that in order to obtain a good central fiber, it is necessary to perform a base change of order 3 before blowing up the singular locus. Set \( \mathcal{N} = \mathcal{N}_{E \times \{0\}/\mathbb{P}^4} \). The normal bundle of \( E \times \{0\} \) in \( \mathbb{P}^4 \times \mathbb{A}^1 \) is isomorphic to \( \mathcal{N} \oplus \mathcal{T}_E \). To avoid being confusing later, we should clarify that the summand \( \mathcal{T}_E \) should actually be written as \( \mathcal{T}_E = \mathcal{O}_E \otimes \mathcal{T} = \mathcal{O}_E \otimes \mathbb{C}\langle t \rangle^\vee \), where \( t \) is the affine coordinate of \( \mathbb{A}^1 \) and \( \mathcal{T} = T_0 \mathbb{A}^1 \).

Let \( X \) be the blowup of \( \mathbb{P}^4 \times \mathbb{A}^1 \) along \( E \times \{0\} \). From now on we will usually write just \( E \) instead of \( E \times \{0\} \). The central fiber \( W_0 \) has two components: \( X_{0,1} \cong \bar{P} \) and \( X_{0,2} = \text{Proj} \text{ Sym} (\mathcal{N} \oplus \mathcal{O}_E)^\vee \). Note that the inclusion \( \mathcal{O}_E \subset \mathcal{O}_E \oplus \mathcal{N} \) induces a section \( \sigma : E \to X_{0,2} \) whose image is geometrically the intersection of the proper transform of \( E \times \mathbb{A}^1 \) with the central fiber. Dually, it defines a distinguished section \( t \in H^0(\mathcal{O}_{X_{0,2}}(T)) \) which we may identify with the affine coordinate above.

After a base change of order 3, a pencil of quintics specializing to \( \Theta \) has total space \( Z = V\{F_\Theta + t^3 F = 0\} \) where we choose \( F \) to be a very general homogeneous quintic polynomial and \( F_\Theta \) is the homogeneous equation of \( \Theta \). The central fiber \( W_0 \) of the proper transform \( W \) of \( Z \) correspondingly has two components: \( Y_1 = \bar{\Theta} \subset X_{0,1} \) and \( Y_2 \subset X_{0,2} \). There seems to be little choice in describing \( Y_2 \) besides writing explicit equations. We start from the definition of \( Y_2 \)

\[
Y_2 = \text{Proj}_E \bigoplus_{k \geq 0} \mathcal{I}^k_{E/Z} / \mathcal{I}^{k+1}_{E/Z} \tag{2.2}
\]

We will show that \( Y_2 \) is the total space of a family of cubic surfaces in the fibers.
of the projective bundle $\varphi_2 : X_{0,2} \to E$.

Note that Lemma 3.2.6 further implies that $\mathcal{I}_{Z/P^4 \times A^1} \subset \mathcal{I}_{E/P^4 \times A^1}$. Indeed, given the shape of the equation defining $Z$, we have

$$\mathcal{I}_{Z/P^4 \times A^1} \subset \mathcal{I}_{\Theta/P^4} \otimes \mathcal{O}_{A^1} + \mathcal{I}_{E/P^4 \times A^1} \subset \mathcal{I}_{E/P^4 \times A^1}.$$

Consider the following commutative diagram of coherent sheaves on $E$

$$\begin{array}{ccc}
\mathcal{I}_{Z/P^4 \times A^1} \otimes \mathcal{O}_E & \longrightarrow & \mathcal{I}_{E/P^4 \times A^1} \otimes \mathcal{O}_E \\
\downarrow & & \downarrow \\
\mathcal{I}_{\Theta/P^4} \otimes \mathcal{O}_E & \longrightarrow & \mathcal{I}_{E/P^4} \otimes \mathcal{O}_E \\
\downarrow & & \downarrow \\
\mathcal{I}_{E/P^4} \otimes \mathcal{O}_E & \longrightarrow & \mathcal{I}_{E/P^4} \otimes \mathcal{O}_E \end{array}$$

Sym$^3(\mathcal{N} \oplus \mathcal{O}_E)^\vee$

The left vertical arrow is an isomorphism as it is simply the restriction to $E$ of the isomorphism $\mathcal{I}_{Z/P^4 \times A^1} \{t=0\} \cong \mathcal{I}_{\Theta/P^4}$. The horizontal arrows on the right side are also isomorphisms by well known properties. If we, so to speak, contract the isomorphisms in the diagram above, we can write the diagram more succinctly as a commutative triangle with entries $\mathcal{O}_E(-5)$, $\varphi_2^* \mathcal{O}_{X_{0,2}}(3)$ and $\tau_* \mathcal{O}_T(3)$. By push-pull we get a global section $y_2 \in \text{H}^0(\mathcal{O}_{X_{0,2}}(3) \otimes \varphi_2^* \mathcal{O}_E(5))$ which restricts on $T$ to the global section $s \in \text{H}^0(\mathcal{O}_T(3) \otimes \tau^* \mathcal{O}_E(5))$ defining $S$ inside $T$, due to the second part of 3.2.5.

We claim that the scheme-theoretic vanishing locus of $y_2$ is precisely $Y_2$. First, the fact that $y_2$ vanishes on $Y_2$ can be seen from construction as follows. We have started with the fact that $\mathcal{I}_{Z/P^4 \times A^1} \subset \mathcal{I}_{E/P^4 \times A^1}$ but more can be said: $\mathcal{I}_{Z/P^4 \times A^1} \otimes \mathcal{O}_E$ actually lies in the kernel of the map $\mathcal{I}_{E/P^4 \times A^1} \otimes \mathcal{O}_E \to \mathcal{I}_{E/Z} \otimes \mathcal{O}_E$, which is a piece of the map of graded $\mathcal{O}_E$-algebras

$$\bigoplus_{k \geq 0} \mathcal{I}_{E/P^4 \times A^1}^k \otimes \mathcal{O}_E \longrightarrow \bigoplus_{k \geq 0} \mathcal{I}_{E/Z}^k \otimes \mathcal{O}_E$$
corresponding to the closed immersion $Y_2 \to X_{0,2}$, thus proving that $y_2$ vanishes on $Y_2$. The fact that $y_2$ is precisely the defining equation of $Y_2$ follows from the observation that $y_2$ restricts on $T$ to the section $s$ defining $S$.

For the purpose of giving explicit equations, it is most convenient to use the description of our sheaves as symmetric powers of normal bundles. Standard multilinear algebra gives a canonical decomposition

$$\text{Sym}^3(\mathcal{N} \oplus \mathcal{I} \otimes \mathcal{O}_E)^\vee \cong \bigoplus_{k=0}^{3} \text{Sym}^k \mathcal{N}^\vee \otimes (t^{3-k}) \mathcal{O}_E$$

(we have reverted to writing $\mathcal{I} \otimes \mathcal{O}_E$ instead of just $\mathcal{O}_E$), or equivalently

$$\varphi_2^* \mathcal{O}_{X_{0,2}}(3) \cong \bigoplus_{k=0}^{3} \tau_* \mathcal{O}_T(k) \otimes (t^{3-k}) \mathcal{O}_E,$$

which once more can be conveniently rearranged using push-pull as

$$H^0(\mathcal{O}_{X_{0,2}}(3) \otimes \varphi_2^* \mathcal{O}_E(5)) \cong \bigoplus_{k=0}^{3} H^0(\mathcal{O}_T(k) \otimes \tau^* \mathcal{O}_E(5)) \otimes \mathbb{C}(t^{3-k}). \quad (2.3)$$

The main point is that, by construction,

$$y_2 = \tau^* F|_E \otimes t^3 + s \otimes 1 \quad (2.4)$$

with the identifications above, where $F|_E$ is the restriction of $F$ to $E$ and $\tau^* F|_E$ is its pullback to $T$. In conclusion, we have obtained an explicit equation for $Y_2$ inside the projective bundle $X_{0,2}$.

**Proposition 3.2.8.** The scheme $Y_2$ is smooth and irreducible. For closed $p \in E$, the fiber of $\varphi_2 : Y_2 \to E$ over $p$ is a cubic surface inside the corresponding fiber of $W_{0,2}$ which is either a cone over $E$ with vertex at $\sigma(p)$ if $p \in \{p_1, p_2, \ldots, p_{25}\}$ or a
special smooth cubic surface which can be expressed as a triple cover of $T_p$ totally ramified at $S_p \cong E \subset T_p$ via projection from $\sigma(p)$ otherwise.

**Proof.** Most statements follow immediately from (2.4) since the formula shows that the equation of each fiber of $Y_2$ inside the corresponding fiber of $X_{0,2}$ is of the form $c_p T^3 + F_3(X,Y,Z) = 0$ for some constant $c_p$ which is zero precisely when $p \in \{p_1, p_2, ..., p_{25}\}$. The only somewhat nontrivial thing that’s left to check is the smoothness of $Y_2$ at the 25 points $\sigma(p_i)$, but this follows easily from the fact that $\partial c/\partial p(p_i) \neq 0$, which in turn follows from the fact that $p_1, p_2, ..., p_{25}$ are distinct. The derivative above is actually well-defined since sections of line bundles admit derivatives at vanishing points. □

**Proposition 3.2.9.** For general choices, $Y_1$ and $Y_2$ are smooth irreducible divisors on $W$ meeting transversally and $W$ is smooth in a neighborhood of the central fiber.

**Proof.** It’s worth noting that, since $W$ is a Cartier divisor on $X$, there will be no need to worry about embedded components in this proof. Smoothness and irreducibility of $Y_1$ and $Y_2$ have already been checked.

First, let’s verify smoothness of the total space. It suffices to check smoothness at closed $p \in W_0$, say $p \in Y_i$. The idea now is to use once more the embedding in $X$. If $W$ were singular at $p$, then the intersection $Y_i = W \cap X_{0,i}$ would also be singular at $p$, which is a contradiction. Slightly more precisely, since ”fiber products commute with tangent spaces” by functor of points nonsense, $T_p Y_i = T_p W \times_{T_p X} T_p X_{0,i}$ and since $T_p Y_i \neq T_p X_{0,i}$ then also $T_p W \neq T_p X$ proving smoothness. Transversality of $Y_1$ and $Y_2$ is automatic, since we already know that their intersection is $S$, which is smooth. □
3.2.3 Triple Cyclic Covers of $\mathbb{P}^2$

We conclude this section some easy elementary remarks on the special class of cubic surfaces encountered earlier. If $S$ is a cubic surface whose equation in $\mathbb{P}^3$ is given in coordinates by

$$F_3(X, Y, Z) + T^3 = 0$$

for some homogeneous degree 3 polynomial $F_3 \in \mathbb{C}[X, Y, Z]$ which defines a smooth plane cubic $E$, then projection from the point $[0 : 0 : 0 : 1]$ to the plane $\{T = 0\}$ exhibits $S$ as a triple cover totally branched along $E$ and étale elsewhere.

As it is the case with all smooth cubic surfaces, $\text{Pic} \, S$ is (over-)generated by the divisor classes of the 27 lines. However, the configuration of lines is special: the 27 lines come in 9 triples of coplanar lines passing through each of the 9 flex points of $E$. Indeed, the plane determined by the center of projection and any of the triple tangents to $E$ cuts $S$ along a plane cubic with a triple point at the corresponding flex of $E$ and it therefore has to be a union of 3 lines. Consequently, the image of the restriction map $\text{Pic} \, S \to \text{Pic} \, E$ consists precisely of the line bundles on $E$ whose cube is some $\mathcal{O}_E(k)$. Finally, we prove some basic existence results which will be used in the proof of the existence of rigid stable maps of arbitrary genus.

**Lemma 3.2.10.** Let $f_1$ and $f_2$ be two distinct flex points of $E$ and $\ell_1$ an arbitrary line in $S$ passing through $f_1$. Then there exists a unique line $\ell_2 \subset S$ passing through $f_2$ which intersects $\ell_1$.

**Proof.** We will give an indirect argument proving uniqueness first. If there were two such lines $\ell_2$ and $\ell_1'$, then $\ell_1$ would have to lie in the plane spanned by $\ell_2$ and $\ell_1'$. However, the only other line on $S$ contained in the plane spanned by $\ell_2$ and $\ell_1'$ is just the third line on $S$ passing through $f_2$, which is different from $\ell_1$ thus
proving uniqueness. Existence can be inferred from the fact that \( \ell_1 \) has to intersect 10 other lines on \( S \). Two of them are the other lines passing through \( f_1 \), so there are 8 more lines to account for. However, there are precisely 8 flex points of \( E \) besides \( f_1 \), so there can only be precisely one line intersecting \( \ell_1 \) through each such point. \( \square \)

**Lemma 3.2.11.** Let \( k \leq 3 \) and \( D_E \) be an effective divisor on \( E \) such that \( \mathcal{O}_E(3D_E) \cong \mathcal{O}_E(k) \). Then there exists an effective divisor \( D_S \) on \( S \) such that \( D_S \cap E = D_E \) and the complete linear system \( |D_S| \) is either: (1) a singleton consisting of a line if \( k = 1 \); (2) a pencil of conics if \( k = 2 \); (3) either (3a) a net of twisted cubics or (3b) the anticanonical linear system, if \( k = 3 \). Moreover, in (3b), \( D_S \) can be chosen to be singular.

**Proof.** If \( k = 1 \), we may just pick any of the 3 lines through \( D_E = p \). If \( k = 2 \), there exist two distinct flex points of \( E \) \( p \) and \( q \) such that \( D_E \sim p + q \). Let \( \ell_p \) be any line through \( p \). By the previous lemma, there exists a line \( \ell_q \) through \( q \) which intersects \( \ell_p \). Then \( |\ell_p + \ell_q| \) is a pencil of conics and it is easy to see that the restriction \( |\ell_p + \ell_q| \rightarrow |D_E| \) is bijective, so there exists \( D_S \in |\ell_p + \ell_q| \) such that \( D_S \cap E = D_E \).

Finally, if \( k = 3 \), there exist 3 distinct flex points of \( E \) \( p, q, r \) such that \( D_E \sim p + q + r \). Let \( \ell_p \) be a line through \( p \) and \( \ell_q, \ell_r \) the lines passing through \( q \) and \( r \) which intersect \( \ell_p \). Then there are two cases:

- **(3a) \( \ell_q \cap \ell_r = \emptyset \).** In this case, \( |\ell_p + \ell_q + \ell_r| \) is a net of twisted cubics and the restriction \( |\ell_p + \ell_q + \ell_r| \rightarrow |D_E| \) is again bijective, so there exists \( D_S \in |\ell_p + \ell_q + \ell_r| \) such that \( D_S \cap E = D_E \).

- **(3b) \( \ell_q \cap \ell_r \neq \emptyset \).** Now \( |\ell_p + \ell_q + \ell_r| \) is the anticanonical linear system, i.e. the linear system of hyperplane sections. In this case, the (now rational) restriction
map $|\ell_p + \ell_q + \ell_r| \to |D_E|$ is surjective with fibers of dimension one. The fiber of $D_E$ is a pencil of hyperplane sections, so $D_S$ can be any singular member of this pencil. □

3.3 Degenerate Stable Maps to the Central Fiber

3.3.1 Preliminary Remarks

As emphasized in the introduction, the motivation for the choice of the specific degeneration worked out in the previous section is the presence of the two nonconstant maps to $E$, which greatly restricts the position of the limits of the nearby rational curves and, to a lower extent, of nearby curves of arbitrary genus. The theoretical framework for working with stable maps to degenerations has been laid out in [Li01, Li02]. The degeneration formula for Gromov-Witten invariants proved in loc. cit. will be less important to us than the construction of the moduli spaces, namely the extension of the family of moduli spaces of stable maps over the singular fiber. Very roughly, this degenerate moduli space parametrizes locally deformable (to nearby fibers) maps to $W_0$ with semistable source - with a twist, namely the idea to allow an ”expansion” of $W_0$ by inserting a chain of ruled varieties between $Y_1$ and $Y_2$ with the purpose of avoiding contracted components (of the source curve) to the singular locus of $W_0$.

Recall [Li01] that to the total space $W$ and the pair $Y_i^{rel} = (Y_i, S)$ we may associate the (Artin) stack of expanded degenerations $\mathcal{W}$, respectively the stack of expanded pairs $\mathcal{Y}_i^{rel}$. The stacks of expanded degenerations and expanded pairs allow one to define the spaces of stable maps to expanded degenerations and relative stable maps, which will be denoted by $\mathcal{M}(\mathcal{W}, \Gamma)$ and $\mathcal{M}(\mathcal{Y}_i^{rel}, \Gamma_i)$ respectively. The topological data $\Gamma$ consists simply of the triple $(g, b, k = 0)$, where $g$ is the
arithmetic genus of the source, $d$ is the degree of the stable maps relative to some predetermined relatively ample line bundle $H$ on $W$ and $k$ is the number of ordinary marked points. The discrete data specified in the $\Gamma_i$ is the following:

- a decorated graph with vertices $V(\Gamma_i)$ corresponding to the connected components of the stable maps, roots $R(\Gamma_i)$ corresponding to the $r$ distinguished marked points attached to the suitable vertices and no edges;
- a function $\mu : R(\Gamma_i) \to \mathbb{Z}^+$ assigning the suitable order of intersection with $S$ at each distinguished marked point;
- functions $\beta_i : V(\Gamma_i) \to H_2(Y_i, \mathbb{Z})$ and $g_i : V(\Gamma_i) \to \mathbb{N}$ prescribing the homology class in $Y_i$, respectively the arithmetic genus of each connected component.

The total space forms a family $\pi^\mathcal{M} : \mathcal{M}(\mathfrak{M}, \Gamma) \to \Delta = \mathbb{A}^1$ which is [Li01] separated and proper over $\Delta$.

There exist distinguished evaluation morphisms $q_i : \mathcal{M}(\mathfrak{M}_i, \Gamma_i) \to S^r$ (the distinguished marked points get artificially ordered as part of the topological data $\Gamma_i$) for $i = 1, 2$. Whenever $\Gamma_1$ and $\Gamma_2$ are compatible in the sense of having the same number $r$ of roots and the corresponding roots are weighted identically, their fiber product admits a morphism

$$
\Phi_\Gamma : \mathcal{M}(\mathfrak{M}_1, \Gamma_1) \times_{S^r} \mathcal{M}(\mathfrak{M}_2, \Gamma_2) \longrightarrow \mathcal{M}(\mathfrak{M}_1 \sqcup \mathfrak{M}_2, \Gamma_1 \sqcup \Gamma_2) \quad (3.1)
$$

to a closed substack of $\mathcal{M}(\mathfrak{M}_0, \Gamma)$, which glues two relative stable maps along their distinguished marked points. For each compatible pair $\eta = (\Gamma_1, \Gamma_2)$, there exist line bundles $\mathbb{L}_\eta$ on the total space $\mathcal{M}(\mathfrak{M}, \Gamma)$ with global sections $s_\eta$ such that the vanishing locus of $s_\eta$ is a closed substack $\mathcal{M}(\mathfrak{M}_0, \eta)$ of $\mathcal{M}(\mathfrak{M}_0, \Gamma)$ which is topologically identical to $\mathcal{M}(\mathfrak{M}_1 \sqcup \mathfrak{M}_2, \Gamma_1 \sqcup \Gamma_2)$.

Having briefly recalled the main objects, we can return to the specifics of the
degeneration described in the previous section. For notational purposes, it is useful to introduce the usual space of stable maps \( \overline{\mathcal{M}}_{0,0}(W,d) \) and the Chow variety \( \text{Chow}_d(W) \). First, we have a composition \( \mathcal{M}(\mathfrak{Y}, \Gamma) \to \overline{\mathcal{M}}_{0}(W,d) \to \text{Chow}_d(W) \), which we denote by \( \gamma_{\mathfrak{Y}} \). There is a similar sequence of morphisms for each \( Y_i \)

\[
\mathcal{M}(\mathfrak{Y}_i^{\text{rel}}, \Gamma_i) \to \prod_{v \in V(\Gamma_i)} \overline{\mathcal{M}}_{0}(Y_i, \beta(v)) \to \prod_{v \in V(\Gamma_i)} \text{Chow}_{\beta(v)}(Y_i), \tag{3.2}
\]

where \( v \) ranges over all vertices of the graph specified by \( \Gamma_i \) and \( \beta : V(\Gamma_i) \to H_2(Y_i, Z) \) gives the class in \( Y_i \) of the image of each component.

The expectation that \( \mathcal{M}(\mathfrak{Y}_0, \Gamma) \) has dimension zero is unrealistic because of the presence of multiple covers. However, we may ask whether its image under \( \gamma_{\mathfrak{Y}} \) has dimension zero (this is false in general as well). Then three questions arise:

(Q1) Are the images of the (closed) fibers of \( q_i \) under \( \gamma_{\mathfrak{Y}} \) finite?

(Q2) Do the images of \( q_i \) have the expected dimensions?

(Q3) Do these images intersect dimensionally transversally for \( i = 1, 2 \)?

The answer to (Q1) is affirmative. The answer to (Q2) is also morally affirmative. It seems possible that under exceptional circumstances the dimensions of the images of \( q_i \) drop below the expected ones, but this is of no concern. We will prove that they never exceed the expected dimension. Unfortunately, the answer to the third question is in general negative. Nevertheless, the third point turns out to be of purely combinatorial nature and as such, it is possible to determine algorithmically when it does holds.

### 3.3.2 Relative Stable Maps to \( Y_1 \) and \( Y_2 \)

In this section we will describe the moduli spaces \( \mathcal{M}(\mathfrak{Y}_i^{\text{rel}}, \Gamma_i) \) of relative stable maps to the \( \mathfrak{Y}_i \), when \( g_i \equiv 0 \), i.e. all connected components have arithmetic
genus 0. Since there are no nonconstant maps from a rational curve to a curve of geometric genus 1, all irreducible components of any relative stable map are sent either inside the fibers of the maps \( \varphi_i : Y_i \to E_i \), or to the lines in the rulings of the intermediary ruled varieties reviewed earlier, possibly as multiple covers. Formally, we have a morphism

\[
\varphi_i^M : \mathcal{M}(\mathcal{Y}_i^{\text{rel}}, \Gamma_i) \to E_i^{V(\Gamma_i)}
\]

specifying the \( \varphi_i \)-fiber containing each connected component. We will show that the conditions the \( r \)-tuples of images in \( S \) of the distinguishes marked points need to satisfy boil down to a collection of linear equations in \( \text{Pic}(E) \). These conditions are obtained by restricting information about rational equivalence on the surfaces \( \varphi_i^{-1}(p) \) to information about rational equivalence on the corresponding copy of \( E_j \subset \varphi_i^{-1}(p) \).

First, we will address (Q1) above. To avoid mentioning the awkward maps to Chow varieties at each step, we introduce some ad hoc terminology. Let \( \pi : C \to S \) be a flat family of semistable curves over a scheme (or perhaps DM-stack) and \( f : C \to X \) a morphism to some projective variety \( X \), inducing a morphism \( S \to \text{Chow}(X) \). Let \( g : S \to Y \) be a proper morphism to a (complex) scheme \( Y \).

We say that \( g \) is cycle-finite relative to the family of maps \( C \to X \) if the image in \( \text{Chow}(X) \) of any closed fiber \( g^{-1}(y) \) is zero-dimensional. In this language question (Q1) becomes: are the distinguished evaluation morphisms \( q_i \) cycle-finite? We prove that this is the case.

**Proposition 3.3.1.** The distinguished evaluation morphisms \( q_i \) are cycle-finite relative to the universal family over \( \mathcal{M}(\mathcal{Y}_i^{\text{rel}}, \Gamma_i) \) mapping into \( Y_i \).

**Proof.** Roughly, positive-dimensional families would be forced to split off a component in \( S \), which is impossible since \( S \) doesn’t contain rational curves.
Denote the universal family by \( \pi_1 : \mathcal{M}(\mathcal{Y}_i^{\text{rel}}, \Gamma'_i) \rightarrow \mathcal{M}(\mathcal{Y}_i^{\text{rel}}, \Gamma_i) \). The notation \( \Gamma'_i \) denotes a topological type identical to \( \Gamma_i \), with the exception of the existence of a single ordinary marked point. Let \( \text{ev}_1 : \mathcal{M}(\mathcal{Y}_i^{\text{rel}}, \Gamma'_i) \rightarrow Y_i \) be the evaluation morphism associated to this point. Let \( s = (s_1, ..., s_r) \in S^r \) be a closed point. Our claim is essentially that the image \( \Sigma \) of \( \pi_1^{-1}(q_i^{-1}(s)) \) under \( \text{ev}_1 \) in \( Y_i \) has dimension one. The crucial observation is this: since \( S = E \times E \) contains no rational curves, all components of the relative stable maps which map to the intermediary ruled threefolds over \( S \) will be mapped as covers of some fibers of these rulings; this implies that \( \Sigma \cap S \) is supported on \( \{s_1, ..., s_r\} \), so \( \Sigma \) can’t have dimension 2 or more. □

Choose once and for all an arbitrary flex point of \( E \subset \mathbb{P}^4 \) to be the zero element of \( (E, +) \). This gives natural identifications of all connected components of \( \text{Pic}(E) \) with \( E \) and in particular, a preferred isomorphism \( E_1 \cong E \). Consider the pushforward maps \( \varphi_{i*} : H_2(Y_i, \mathbb{Z}) \rightarrow H_2(E_i, \mathbb{Z}) \cong \mathbb{Z} \). Let \( K_1 \cong \mathbb{Z}[\text{fiber}] \oplus \mathbb{Z}[\text{directrix}] \subset H_2(Y_1, \mathbb{Z}) \) generated by the classes of the directrix respectively any line in the ruling of any fiber of \( \varphi_1 \) and \( K_2 \) generated by the classes of all lines on any smooth fiber of \( \varphi_2 \). Then \( \text{Im}(\beta_i) \subset K_i \), for \( i = 1, 2 \).

For any \( v \in V(\Gamma_i) \), we denote by \( \Gamma_i|_v \) the restriction of the combinatorial data to the vertex \( v \), i.e. the collection of data consisting of the graph with the unique vertex \( v \), the roots which were previously attached to \( v \), the class \( \beta_i(v) \), the genus function \( g_i(v) = 0 \) and no legs, corresponding to no ordinary marked points. Consider the moduli stack \( \mathcal{M}(\mathcal{Y}_i^{\text{rel}}, \Gamma_i|_v) \) of relative stable maps of this topological type. Again, there is a morphism

\[
\varphi_{i|_v}^\mathcal{M} : \mathcal{M}(\mathcal{Y}_i^{\text{rel}}, \Gamma_i|_v) \rightarrow E_i.
\]
Let $\Gamma_i^v$ denote the topological type which is identical to $\Gamma_i^v$, with the exception of the existence of one ordinary marked point. We obtain the universal family $\mathcal{M}(\mathcal{Y}_{i,\text{rel}}^{\text{rel}}, \Gamma_i^v)$ over the previous moduli space. We have a map $\mathcal{M}(\mathcal{Y}_{i,\text{rel}}^{\text{rel}}, \Gamma_i^v) \to \mathcal{M}(\mathcal{Y}_{i,\text{rel}}^{\text{rel}}, \Gamma_i^v) \times_{E_i, Y_i}$. Define the maps

$$\omega_i : \mathcal{M}(\mathcal{Y}_{i,\text{rel}}^{\text{rel}}, \Gamma_i^v) \to \text{Pic}^{\sum_{\alpha \in R(v)} \mu(\alpha)}(E_j)$$

which associate to each relative stable map $(C, f, (q_\alpha)_{\alpha \in R(v)})$ the divisor class cut out by $f(C)$ on $\{\varphi_i^M(f)\} \times E_j$, i.e. $\mathcal{O}_{E_j}(\sum_{\alpha \in R(v)} \mu(\alpha)f(q_\alpha))$. The degree $\sum_{\alpha \in R(v)} \mu(\alpha)$ will be denoted by $\sigma(v)$. The first step is to understand the image of the map $\omega_i \times \varphi_i^M$ in $\text{Pic}^{\sigma(v)}(E_j) \times E_i$. Identify the target with $E_j \times E_i = E_1 \times E_2$ and let $(x, y)$ be coordinates on $E_1 \times E_2$.

**Lemma 3.3.2.** (a) Let $i = 1$ and $\beta_1(v) = k_f[\text{line}] + k_\infty[\text{directrix}] \in K_1$. Then the image is contained inside the locus in $E_1 \times E_2$ given by the equation $y = (k_f - 2k_\infty)x = 0$.

(b) Let $i = 2$ and $\beta_2(v) \in K_2$ of degree $k_1$ relative to $\mathcal{O}_{X_{n,2}}(1)$. Then the image is contained in the union of the curve of equation $3x = k_1y$ with $E_1 \times \{p_1, ..., p_{25}\}$.

**Proof.** (a) This follows from part (a) of lemma 3.2.4. Indeed, for $\mathcal{D} \in \text{Pic}^2(E) = E_1$, 2.4.(a) says that the divisor class $\mathcal{O}_E(f)^{\otimes k_f} \otimes \mathcal{O}_E(e_\infty)^{\otimes k_\infty}$ on the fiber $\Sigma = \varphi_i^{-1}(\mathcal{D})$ restricts on the corresponding copy of $E_2 = E$ sitting inside $\Sigma$ to $\mathcal{O}_{p_4}(k_\infty)|_{E} \otimes \mathcal{D}^{\otimes (k_f - 2k_\infty)}$. Given that the implicit isomorphism of the corresponding connected component of $\text{Pic}(E)$ with $E$ is induced by a hyperflex point, the first terms disappears, so the image lies inside the locus $y = (k_f - 2k_\infty)x$.

(b) Similarly, this follows from part (b) of 3.2.4 and the discussion in section 3.2.3. Let $p \in E_2 = E$. Assume that $p \notin \{p_1, p_2, ..., p_{25}\}$. Then the fiber $\Sigma = \varphi_2^{-1}(p)$ of $\varphi_2$ is a smooth cubic surface, as in section 3.2.3. By the discussion in
that section, any divisor class on $\Sigma$ which pushes forward to $k_i$ line in $H_2(Y_2, \mathbb{Z})$ has the property that its cube is the class $\mathcal{O}_{\mathbb{P}^1}(k_i)|_E \otimes \mathcal{O}_E(-2k_iP)$, so, with the implicit identifications, we get the desired constrain $3(x - k_i y) + 2k_i y = 0$. The reason for the term $x - k_i y$ is the following: the coordinate $x$ on $E_1 \times E_2$ is actually the coordinate $\mu$ on the canonical $E \times E$ and the corrections will stack in $\text{Pic}^{k_i}(E_1)$.

Let $I(\Gamma_{i|v})$ be the loci in $\text{Pic}^{\sigma(v)}(E_j) \times E_i$ described above. Let $q_{i,v}$ is the evaluation morphism restricted to $v$ and $h$ be the map defined by

$$(y_\alpha)_{\alpha \in R(v)} \mapsto \sum_{\alpha \in R(v)} \mu(\alpha)y_\alpha \in \text{Pic}^{\sigma(v)}(E_j).$$

With these notations, consider the diagram

\[
\begin{array}{ccc}
(E_1 \times E_2)^{R(v)} & \overset{\text{Id}_{E_j^{R(v)} \times \text{Diag}_{E_i}}}{\leftarrow} & E_j^{R(v)} \times E_i \\
\downarrow \text{Id}_{E_j^{R(v)}} & & \downarrow h \times \text{Id}_{E_i} \\
M(\mathfrak{M}_{i|v}^{\text{rel}}, \Gamma_{i|v}) & \overset{\rho_i^\Gamma}{\leftarrow} & \omega_i \times \varphi_i^{\mathfrak{M}}
\end{array}
\]

Define $A(\Gamma_{i|v})$ to be the image of $(h \times \text{Id}_{E_i})^{-1}(I(\Gamma_{i|v}))$ in $(E_1 \times E_2)^{R(v)}$. Then $A(\Gamma_{i|v})$ is an equidimensional union of abelian subvarieties of $(E_1 \times E_2)^{R(v)}$ of dimension

$$\dim A(\Gamma_{i|v}) = \dim E_j^{R(v)} \times E_i - 1 = |R(v)|$$

and the image of $q_{i|v}$ is contained in $A(\Gamma_{i|v})$. Let $A(\Gamma_i) \subset (E_1 \times E_2)^r$ be the direct product of all $A(\Gamma_{i|v})$ over all $v \in V(\Gamma_i)$. Then the image of $q_i$ is contained in $A(\Gamma_i)$, which is a union of abelian subvarieties of $(E_1 \times E_2)^r$ of pure dimension

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\[ \sum_{v \in V(\Gamma_i)} |R(v)| = r. \]  
Note that \( A(\Gamma_1) \) and \( A(\Gamma_2) \) have complementary dimensions in \( S^r \), as we should expect.

**Remark 3.3.3.** Given any abelian variety and and any two abelian subvarieties of complementary dimensions, either the two abelian subvarieties intersect transversally, or the intersection of the associated homology classes is zero. This can be seen, for instance, from the excess intersection formula.

From the point of view of the degeneration formula proved by Li, this seems to have intriguing consequences: any potentially infinite\(^2\) family of limits is necessarily associated with the vanishing of the corresponding part of the contribution to the virtual count. However, a huge technical difficulty is the following: for \( i = 2 \) the image consists not of an abelian subvariety, but a singular union of subvarieties and it is entirely unclear whether it is possible to separate the contributions from each individual component in a geometrically meaningful way.

### 3.3.3 Combinatorics of the Degenerate Maps

In this section, we write explicitly the combinatorial laws governing the stable maps to \( \mathcal{M}_0 \). Let \( \eta = (\Gamma_1, \Gamma_2) \) be a pair of compatible topological types, i.e. \( \Gamma_1 \) and \( \Gamma_2 \) have the same number \( r \) of distinguished marked points and the identically indexed distinguished marked points have the same \( \mu \)-weight. As we said, we will only be concerned with the case \( g_1 \equiv g_2 \equiv 0 \), which covers all possibilities in the case \( g = 0 \) respectively some of the possibilities when \( g > 0 \).

To avoid notation becoming unnecessarily cryptic, we will work with only one map instead of families. Fix a single stable map \( (C, f, q_1, ..., q_r) \in \mathcal{M}(\mathcal{M}_0, \eta)(\mathbb{C}) \). Let \( G \) be the graph with vertices \( V(G) = V(\Gamma_1) \cup V(\Gamma_2) \) and \( r \) edges obtained by

\(^2\)By infinite we mean not cycle-finite in our terminology, so for instance multiple covers of a fixed curve don’t count.
glueing the roots with identical indexing. Then \( \text{rank } H^1(G) = g \) since

\[
g = \text{rank } H^1(G) + \sum_{v \in V(\Gamma_1)} g_1(v) + \sum_{v \in V(\Gamma_2)} g_2(v)
\]

and we are assuming \( g_1 \equiv g_2 \equiv 0 \). From now on, we will call the elements of \( V(\Gamma_1) \) red vertices and the elements of \( V(\Gamma_2) \) blue vertices. As in lemma 3.2.3, let \( k_f, k_\infty : V(\Gamma_1) \to \mathbb{N} \) such that \( \beta_1 \equiv k_f[\text{line}] + k_\infty[\text{directrix}] \) and \( k_l : V(\Gamma_2) \to \mathbb{N} \) such that \( \beta_2 \) has degree \( k_l \) relative to \( \mathcal{O}_{X_{0,2}}(1) \).

Let \( \lambda : V(G) \to (E, +) \cong (\mathbb{R}^2/\mathbb{Z}^2, +) \) be the map giving the components of \( \varphi^M_1([f]) \) and \( \varphi^M_2([f]) \). It is possible to merge \( i = 1 \) and \( i = 2 \) as we are simultaneously identifying \( E_1 \) and \( E_2 \) with \( E \). The point is that lemma 3.3.2 and the discussion thereafter imposes \( |V(G)| \) linear conditions on \( \lambda \), which are usually linearly independent. First, let \( v \) be a red vertex. Then the condition imposed on \( \lambda \) is

\[
\sum_{w \in V(v)} \mu([vw])\lambda(w) = (k_f(v) - 2k_\infty(v))\lambda(v)
\]

with the numerical constrain on the multiplicities

\[
\sum_{w \in V(v)} \mu[vw] = 2k_f(v) + k_\infty(v),
\]

since \( (k_f[\text{line}] + k_\infty[\text{directrix}] \cdot -K_{\mathbb{F}_1})_{\mathbb{F}_1} = 2k_f^f(v) + k_\infty^f(v) \) by a straightforward calculation. We are denoting the set of neighbors of \( v \) in \( G \) by \( V(v) \).

Now let \( v \) be a blue vertex. We have two cases: either \( \lambda(v) \in \{p_1, p_2, ..., p_{25}\} \) in which nothing more can be said, or otherwise, again from the computation in
3.3.2 and the following discussion, we get

\[ 3 \sum_{w \in V(v)} \mu([vw]) \lambda(w) = k_I(v) \lambda(v) \]

and the numerical constrain on the weights

\[ \sum_{w \in V(v)} \mu([vw]) = k_I(v) \]

Note that the latter constrain still holds in the case \( \lambda(v) \in \{p_1, p_2, ..., p_{25}\} \). We introduce some (fairly standard) notation to state the equations above in a slightly more pleasant form.

**Definition 3.3.4.** Let \( G \) be any simple connected graph without isolated vertices and let \( \mu : E(G) \to \mathbb{Z}^+ \) be a weight function on the set of edges. First, we naturally define the weighted degrees of the vertices as

\[ \text{deg}_\mu(v) = \sum_{w \in V(v)} \mu([vw]). \]

For any function \( f \) defined on the set of vertices \( V(G) \) with values in some abelian group, we define the \( \mu \)-weighted unnormalized \( G \)-Laplacian of \( f \) by

\[ \Delta_\mu^u f(v) := \sum_{w \in V(v)} \mu([vw])(f(v) - f(w)) \]

The normalized \( \mu \)-weighted \( G \)-Laplacian \( \Delta_\mu \) is defined by the same formula, but dividing by \( \text{deg}_\mu(v) \), assuming that \( \text{deg}_\mu(v) \) is invertible in the target group. If \( \mu \equiv 1 \), we suppress this subscript.
A basic computation allows us to rewrite the conditions above as follows:

\[
\begin{align*}
\Delta^u_{\mu} \lambda(v) &= \left(3 \deg_{\mu}(v) - 5k_f(v)\right) \lambda(v) \quad \text{if } v \text{ is red}, \\
3\Delta^u_{\mu} \lambda(v) &= 2\deg_{\mu}(v)\lambda(v) \text{ or } \lambda(v) \in \{p_1, p_2, \ldots, p_{25}\} \quad \text{if } v \text{ is blue}.
\end{align*}
\] (3.4)

In conclusion, (cycle-)finiteness of \( \mathcal{M}(\mathcal{W}_0, \eta) \) boils down to the system of linear equations in \((E, +)\) above having only finitely many solution. For \( g = 0 \), the smallest counterexample to finiteness occurs in degree \( d = 6 \).

**Example 3.3.5.** As soon as \( r = 2 \) we may find examples in which the system (3.4) is special. Let \( G \) have two red vertices \( R_1, R_2 \) and a blue vertex \( B \) of ”smooth” type. Let \( k_f(R_i) = 2, k_\infty(R_i) = 1, k_l(B) = 10 \) and let both edges have weight 5.

An immediate calculation shows that (3.4) is special in this case. Geometrically, the red vertices correspond to twisted cubics which are sections of the corresponding \( F_1 \) surfaces by the same osculating hyperplane in \( \mathbb{P}^4 \) to a hyperflex of \( E \).

### 3.3.4 Existence of Rigid Stable Maps

**3.3.3. Existence of Rigid Stable Maps.** In this section, we prove the main theorem 3.1.1 by explicitly exhibiting rigid degenerate stable maps to \( \mathcal{W}_0 \), of arithmetic genus \( g \) and degree \( d \gg g \) with certain additional properties and invoking the existence of a perfect obstruction theory on \( \mathcal{M}(\mathcal{W}, \Gamma) \) to infer that they are indeed limits of rigid stable maps to nearby fibers of \( W \to \mathbb{A}^1 \). Smoothness of

---

\(^3\text{This example was found by Gabriel Bujokas.}\)
the source follows easily from the smoothness of the connected components of the maps to \( \mathcal{V}_1^{rel} \) and \( \mathcal{V}_2^{rel} \).

The chaotic nature of the combinatorial problem encountered earlier works to our advantage, in that we have an enormous amount of freedom in choosing the topological types. The first main assumption is \( \beta_1 \equiv \text{[line]} \), that is, we are assuming that all red vertices correspond to fibers of the rulings in the corresponding Hirzebruch surfaces. Furthermore, we assume \( \mu \equiv 1 \), hence all red vertices have degree 2. Then the first branch of equation (3.4) simply reads \( \lambda(v) = \lambda(w_1) + \lambda(w_2) \), where \( v \) is any red vertex and \( w_1, w_2 \) are its two neighbors.

Let \( G' \) be the graph obtained from \( G \) by suppressing all red vertices and replacing any length 2 chain in \( G \) between two blue vertices with an edge. Then the second branch of equation (3.4) becomes simply \( 3 \Delta^u \lambda(v) = 5 \deg(v) \lambda(v) \), where the discrete Laplacian is now taken relative to \( G' \). We further assume that \( \deg(v) \leq 3 \) for all vertices of \( G' \) and proceed to construct the degenerate curve.

**Claim 3.3.6.** For \( d, g \geq 1, d \gg g \), there exists a set of data as follows:

- A graph simple connected graph \( G = (V, E) \) with \( |E| = d \) and \( h^1(G) = g \), hence \( |V| = d - g + 1 \). We require that \( \deg(v) \leq 3 \) for all \( v \in V \).
- If \( \mathbb{K} \) is some field of characteristic \( \neq 2, 3 \), consider the discrete equation

\[
\Delta \lambda = \frac{5}{3} \lambda
\]  

in \( \text{Fun}(V, \mathbb{K}) \). We require that (3.5) has only the trivial solution \( \lambda = 0 \) when \( \mathbb{K} = \mathbb{R} \), i.e. \( 5/3 \) is not an eigenvalue of the Laplacian.

- A prime number \( p \geq 7 \) and a solution \( \hat{\lambda} \) of (3.5) for \( \mathbb{K} = \mathbb{F}_p \) such that:
  1. \( \hat{\lambda}(v) \neq \hat{\lambda}(w) \) if \( 1 \leq \text{dist}(v, w) \leq 2 \); and
  2. \( \hat{\lambda} \) is a strongly irreducible solution of (3.5), in the following sense. For any
induced subgraph \( H \subset G \), consider the similar discrete differential equation

\[
F_H \lambda := \Delta_H \lambda - \frac{5}{3} \lambda = 0.
\]

Then we require that \( \hat{\lambda} \) has the following property: if \( F_H \hat{\lambda}(v) = 0 \), then \( \deg_H v \) is equal to either \( \deg_G v \) or 0.

**Lemma 3.3.7.** If no two degree 3 vertices of \( G \) are adjacent, then (3.5) only has the trivial solution 0 for \( \mathbb{K} = \mathbb{R} \) or equivalently \( \mathbb{Q} \).

**Proof.** Let \( \mathbb{K} = \mathbb{Q} \) and let \( v_3 : \mathbb{Q} \to \mathbb{Z} \cup \{+\infty\} \) be the valuation at the prime 3.

Pick any solution \( \lambda \) of (3.5). Then we have

\[
v_3(\lambda(v)) = v_3 \left( \sum_{w \in V(v)} \lambda(w) \right) - v_3(\deg v) + 1 \geq \min_{w \in V(v)} v_3(\lambda(w)) + 1 - v_3(\deg v).
\]

Hence any vertex \( v \) has a neighbor with strictly smaller \( v_3 \circ \lambda \), if \( \deg v \leq 2 \) and \( \lambda(v) \neq 0 \), respectively no greater \( v_3 \circ \lambda \), if \( \deg v = 3 \). This easily implies \( v_3 \circ \lambda \equiv +\infty \), so \( \lambda \equiv 0 \). \( \square \)

**Claim 3.3.8.** There exists a prime \( p \geq 7 \) and three graphs \( G_1, G_2, G_3 \) with eigenvectors \( \hat{\lambda}_i \) in \( \mathbb{F}_p \) satisfying the conditions in 3.6 and 3.7 (\( d \) and \( g \) are not fixed) and two elements \( a, b \in \mathbb{F}_p \) such that:

- \( |E(G_1)| \) and \( |E(G_2)| \) are coprime;
- \( h^1(G_1) = h^1(G_2) = 1 \) and \( h^1(G_3) = 2 \);
- all graphs \( G_i \) contain an edge \([vw]\) such that \( v \) and \( w \) both have degree 2 in \( G_i \) and \( \hat{\lambda}_i(v) = a \), \( \hat{\lambda}_i(w) = b \).

**Proof.** The proof is by explicit example. See the appendix. \( \square \)
**Proof of 3.3.6.** We will use the graphs $G_1, G_2, G_3$ constructed in 3.3.8 as the building blocks for constructing $G$. For each $G_i$, we may define a graph $H_i$ with $|E(H_i)| = |E(G_i)| + 1$ and $h^1(H_i) = h^1(G_i) - 1$ obtained by duplicating the edge $[vw]$ and then separating the two ends, i.e. $H_i$ will begin and end with a copy $[vw]$, like in our pictorial representation of the $G_i$.

Starting with some $G_i$, we may insert copies of various $H_j$’s at the edges which are copies of $[vw]$. Note that we may glue the functions from vertices to $\mathbb{F}_p$ and crucially, the property of being eigenvectors with eigenvalue $5/3$ is preserved since any collections of vertices consisting of some arbitrary vertex and all its neighbors is contained isomorphically in some $G_i$. It is also not hard to see that the conditions $\hat{\lambda}(v) \neq \hat{\lambda}(w)$ if $1 \leq \text{dist}(v, w) \leq 2$ respectively $\deg v \neq 3$ if $v$ has some neighbor of degree 3 are preserved as well through this process. To ensure $h^1(G) = g$, it is enough to insert $H_3$ precisely $g - 1$ times throughout the process. Finally, since $|E(G_1)|$ and $|E(G_2)|$ are coprime, the elementary ”coin problem” ensures that $G$ may have any sufficiently high number of edges. □

**Lemma 3.3.9.** For $d \gg g$, $g \geq 1$ there exists a stable map $f_0 : C_0 \to W_0$, $[f_0] \in \mathcal{M}(\mathfrak{M}_0, \Gamma)$ obtained by glueing two relative stable maps $f_1 \in \mathcal{M}(\mathfrak{Y}_1^{\text{rel}}, \Gamma_1)$ and $f_2 \in \mathcal{M}(\mathfrak{Y}_2^{\text{rel}}, \Gamma_2)$ mapping to $Y_1[0]$ respectively $Y_2[0]$ such that all connected components of the source of $f_i$ are smooth and are mapped generically one-to-one onto their images and $[f_0]$ is an isolated point of $\mathcal{M}(\mathfrak{M}_0, \Gamma)$.

**Proof.** We will avoid going into the details of this proof, since the desired degenerate map is obtained simply by reversing the discussion so far, with few significant new ingredients. First, we use the graphs constructed in 3.3.6. as the $G'$ in the discussion at the beginning of this section. Using these, we may reconstruct $G$, the dual graph of $C$. Second, we use the solution $\hat{\lambda}$ of (3.5) in $\mathbb{F}_p$ to reconstruct the
desired solution $\lambda$ of (3.4) as follows: first, we extend $\hat{\lambda}$ to $(E, +)$ by first embedding $\mathbb{F}_p \cong \mathbb{Z}/p\mathbb{Z}$ diagonally into $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, which naturally sits inside $(\mathbb{R}^2/\mathbb{Z}^2, +) \cong (E, +)$, then define $\lambda$ on the red vertices simply by $\lambda(v) = \lambda(w_1) + \lambda(w_2)$, where $w_1, w_2$ are the two green neighbors of some red $v$.

Next, we construct the components $f_v : C_v \to Y_i$ of the stable map. We start with $v$ blue. Consider the divisor $D = \sum_{w \in V(v)} \lambda(w)$ on the copy of $E$ sitting inside the special cubic surface $\Sigma = \varphi_2^{-1}(\lambda(v))$. The fact that $\lambda$ is a solution of (3.4) implies that we are in the situation of lemma 3.2.11. Let $D_\Sigma$ be the divisor on $\Sigma$ such that $D_\Sigma \cap E = D$, as constructed in the said lemma. The crucial point (which will only be sketched) is that $D_\Sigma$ is irreducible. This is where the (combinatorial) strong irreducibility condition comes into play. Indeed, if the divisor were reducible, then either component would intersect the boundary $E$ at points which satisfy the analogous linear conditions in $(E, +)$ - precisely what is ruled out by the strong irreducibility assumption. We are implicitly relying on the equivalence of the linear conditions for $\hat{\lambda}$ and for $\lambda$. In conclusion, we can define $f_v : C_v \cong \mathbb{P}^1 \to Y_2$ to be the normalization of the divisor above. Finally, the distinguished marked points on $C_v$ are the preimages of $E$ under $f_v$. The fact that they are distinct is ensured by the condition $\hat{\lambda}(v) \neq \hat{\lambda}(w)$ if $\text{dist}(v, w) = 2$ in $G'$.

The case $v$ red is similar but less laborious. Again, the fact that $\lambda$ is a solution of (3.4) ensures that we may find such a line. The property that the distinguished marked points are distinct is ensured by $\hat{\lambda}(v) \neq \hat{\lambda}(w)$ if $v$ and $w$ are neighbors in $G'$.

All in all, we may glue $f_1$ and $f_2$ to obtain a morphism $f_0 : C_0 \to W_0$, where $C_0$ is obtained by glueing nodally all components of $C_v$ along the pair of identically indexed distinguished marked points. Note that the fact that the distinguished marked points were obtained distinct directly from the construction above means
that the target of $f_0$ is indeed $X_0$ rather than some higher $W_0[n]$. It is clear that $\text{Aut}(f_0)$ is trivial, so $f_0$ is stable. The crucial condition that $[f_0]$ is isolated in $\mathcal{M}(\mathfrak{W}_0, \Gamma)$ follows from the second bullet in 3.3.6. □

Proof of Theorem 3.1.1. If $g = 0$, the statement is known by work of Katz, as explained in the introduction. If $g \geq 1$, the previous lemma all but completes the proof of the theorem. Since $\mathcal{M}(\mathfrak{W}, \Gamma)$ admits a perfect obstruction theory of the expected dimension [Li02], the irreducible component of $\mathcal{M}(\mathfrak{W}, \Gamma)$ containing the point $[f_0]$ must have dimension at least one. In fact, since the intersection of this locus with the central fiber has an isolated point at $[f_0]$, it must be of dimension precisely 1 and, moreover, it must not be contained in the central fiber. It follows that we can find an $\mathbb{A}^1$-morphism $h : B \to \mathcal{M}(\mathfrak{W}, \Gamma)$ from a smooth affine curve $B$ with a distinguished point $b_0 \in B$ admitting a map $\psi : B \to \mathbb{A}^1$ such that $h(b_0) = [f_0]$ lies in the central fiber, but no other $h(b), b \in B$ does. We may choose $B$ such that all $h(b_t)$ are isolated points of $\overline{\mathcal{M}}_{0,0}(W_{(t), d})$.

Finally, we have to argue that the source $C_t$ of $h(b_t) = [(C_t, f_t)]$ is smooth for all $t$ in a punctured neighborhood of $b_0 \in B$, where $C \to B$ is the pullback to $B$ of the universal family of stable maps. Consider the commutative diagram

\[
\begin{array}{ccc}
C & \xrightarrow{f} & W \\
\downarrow & & \downarrow \\
B & \xrightarrow{\psi} & \mathbb{A}^1
\end{array}
\]

inducing a map $C \to W \times_{\mathbb{A}^1} B$. Any singular point of $C_0$ is a distinguished marked point $q$, so, in particular it is a node. The versal deformation space of a node is the germ at the origin of the family $\text{Spec } \mathbb{C}[x, y, s]/(s - xy) \to \text{Spec } \mathbb{C}[s]$, so it suffices to prove that $C$ is locally irreducible near $q$ in the complex-analytic topology. If
it wasn’t, then it would have two components \( Z_1 \) and \( Z_2 \), indexed such that \( Z_i \) contains the branch of \( f_0 \) near \( q \) mapping to \( Y_i \). Then the pullback of the line bundle \( \mathcal{O}_W(Y_1) \) with the section \( 1 \in H^0(\mathcal{O}_W(Y_1)) \) restricts on \( Z_2 \) to a line bundle with a section vanishing only at \( q \), which is impossible. \( \square \)
Appendix

4.1 A Lemma in Commutative Algebra

Here we state and prove a lemma in commutative algebra which was invoked in the previous section.

Fact A.1.1. Let $f : (A, \mathfrak{m}) \to (B, \mathfrak{n})$ be a flat local homomorphism of local Noetherian rings. If $I \subseteq \mathfrak{m}^e$ is an ideal of $B$ such that $B/I$ is flat as an $A$-module, then $I = (0)$.

Proof. First, tensoring the short exact sequence $0 \to \mathfrak{m} \to A \to A/\mathfrak{m} \to 0$ with $B$ we obtain an exact sequence of $A$-modules

$0 \to \mathfrak{m} \otimes_A B \to A \otimes_A B \to A/\mathfrak{m} \otimes_A B \to 0$.

The middle term is $B$ and the next term is $B/\mathfrak{m}B = B/\mathfrak{m}^e$, so flatness of $B$ over $A$ implies that the $A$-module homomorphism $B \otimes_A \mathfrak{m} \to \mathfrak{m}^e$ is actually an isomorphism. Second, tensoring the short exact sequence $0 \to \mathfrak{m} \to A \to A/\mathfrak{m} \to 0$
with $B/I$ we obtain an exact sequence of $A$-modules

$$0 \rightarrow \mathfrak{m} \otimes_A B/I \rightarrow A \otimes_A B/I \rightarrow A/\mathfrak{m} \otimes_A B/I \rightarrow 0.$$

First, $A/\mathfrak{m} \otimes_A B/I$ is $(B/I)/\mathfrak{m}(B/I) = (B/I)/\mathfrak{m}^e(B/I)$. The kernel of the surjective composition $B \rightarrow B/I \rightarrow (B/I)/\mathfrak{m}^e(B/I)$ is just $\mathfrak{m}^e$ hence $A/\mathfrak{m} \otimes_A B/I \cong B/\mathfrak{m}^e$. We therefore have an isomorphism $\mathfrak{m} \otimes_A B/I \cong \mathfrak{m}^e/I$.

However, we have a well defined surjective $A$-module homomorphism $\mathfrak{m} \otimes_A B/I \rightarrow \mathfrak{m}^e/\mathfrak{m}^e I$, which fits in the following commutative diagram of $A$-modules.

The vertical lateral maps are simply quotients. The lower left horizontal map is surjective and the all other horizontal maps are isomorphisms. It follows that $I = \mathfrak{m}^e I$. Inductively, $I = (\mathfrak{m}^e)^k I$ for all positive integers $k$, hence

$$I \subseteq \bigcap_{k \geq 0} (\mathfrak{m}^e)^k = (0),$$

by Krull’s intersection theorem, so $I = (0)$, as desired. □

**Corollary A.1.2.** As above, let $f : (A, \mathfrak{m}) \rightarrow (B, \mathfrak{n})$ be a flat local homomorphism of local Noetherian rings. Let $M$ be a $B$-module such that:

- $M$ is flat as an $A$-module; and

- $B/\mathfrak{m}^e \otimes_B M$ is a free rank one module over $B/\mathfrak{m}^e$.

Then $M$ is a free rank one module over $B$.

**Proof.** This follows immediately from A.1.1 and Nakayama’s lemma. □
4.2 Graphs with Eigenvalue $5/3$ in $\mathbb{F}_p$

We have deferred the proof of 3.8 to this appendix. The examples below were found using a computer program in C++. In principle, they can be checked by hand, but I have written a second (much simpler) computer program available on my personal website, only for the purpose of verifying these examples. The prime number is $p = 23$. Below, we draw $G_1, G_2, G_3$ and write the corresponding value of $\hat{\lambda}_i$ inside each vertex. For formatting reasons, $G_1$ and $G_2$ are actually drawn as trees; the actual $G_1$ and $G_2$ are obtained by gluing the distinguished $[1 - 2]$ edges. Note that $G_1, G_2, G_3$ have 11, 18 respectively 43 edges.

obtained by completing with the branches $A, B, C$ given below.
Insert the 3 branches $A, B, C$ to the previous diagram as indicated to obtain $G_3$.

### 4.2.1 Some Easy Code

Here, we write the code used to verify the examples written above. Unlike the code used to find them, this is logically necessary. The reader will have to remove each pair of comments `/* */` around the data corresponding to each of the three cases and run the program a total of 3 times.

```c
#include <stdio.h>
int a[100][100], b[100][100], deg[100];
int main() {
    int noedge, novert;
    int p,v[100],LapV[100];
    bool solution = true, different = true;
    bool distant = true, minimal = true, no30 = true;

    // introduce data here
    p=23;
    /*
     noedge = 11;
     novert = 11;
    
    a[0][1] = a[1][2] = a[2][3] = a[3][4] = 1;
    */
```

a[9][10] = a[0][10] = 1; a[6][7] = 1;


/*
noedge = 18;
novert = 18;

a[0][1] = a[1][2] = a[2][3] = a[3][4] = 1;


a[12][13] = a[13][14] = a[0][14] = 1;

a[6][7] = 1;


v[14]=12;


*/

/*

noedge = 25;
novert = 24;


a[9][10] = a[5][10] = a[1][5] = 1;


a[21][22] = a[7][21] = a[1][7] = 1;


*/
noedge = 43;
novert = 42;

a[34][35] = a[35][36] = a[36][37] = a[37][38] = a[38][39] = a[39][40] =
a[40][5] = a[5][40] = 1;
a[0][3] = a[3][12] = a[12][17] = a[12][13] = a[13][14];
a[1][7] = 1;

v[16] = 10; v[17] = 11; v[18] = 8; v[41] = 5;
v[28] = 6; v[29] = 12; v[30] = 1; v[31] = 2;
v[32] = 4; v[33] = 17; v[34] = 19; v[35] = 9;
v[36] = 15; v[37] = 17; v[38] = 16; v[39] = 22;
v[40] = 21;

// complete the incidence matrix

for (int i = 0; i < novert - 1; i++)
    for (int j = i + 1; j < novert; j++) a[j][i] = a[i][j];

// compute degrees

for (int i = 0; i < novert; i++)
    for (int j = 0; j < novert; j++)
        deg[i] += a[i][j];

// find which vertices have common neighbors

for (int i = 0; i < novert; i++)
    for (int j = 1; j < novert; j++)
for (int k = 1; k < novert; k++)
if(a[i][k] == 1 && a[j][k] == 1 && i != j) b[i][j] = 1;

// unnormalized Laplacian as in notes
for (int i = 0; i < novert; i++) {
    LapV[i] = 0;
    for (int j = 0; j < novert; j++)
        if(a[i][j] == 1) LapV[i] += v[j];
    LapV[i] -= deg[i] * v[i];
}

// is it a solution?
for (int i = 0; i < novert; i++)
    if ( (3*LapV[i] + 5*deg[i]*v[i]) % p != 0 )
solution = false;

if (solution) printf("OK. Solution!\n");
else printf("It’s NOT a solution!!\n");

// are they sufficiently different?
for (int i = 0; i < novert; i++)
    for (int j = 0; j < novert; j++) if(a[i][j] == 1 || b[i][j] == 1)
        if(v[i] == v[j]) different = false;

if (different) printf("OK. Different!\n");
else printf("NOT Different!\n");

// are the 3’s separated?
for (int i = 0; i < novert; i++)
    for (int j = 0; j < novert; j++) if(a[i][j] == 1)
        if(deg[i] == 3 && deg[j] == 3) distant = false;

if(distant) printf("OK. The 3’s are far away!\n");
else printf("The 3’s are NOT alright!\n");

// is it a minimal solution?
for (int i = 0; i < novert; i++)
    for (int j = 0; j < novert; j++) if(a[i][j] == 1)
if((2 * v[i] + 3 * v[j]) % p == 0 && deg[i]>1)  
    minimal = false;

if(minimal) printf("OK. Minimal!\n");
else printf("NOT Minimal!\n");

// is every degree 3 different from 0?
for (int i = 0; i < novert; i++)
    if (deg[i] == 3 && v[i] == 0) no30 = false;

if(no30)
    printf("OK. No 0 in degree 3 position!\n");
else
    printf("NOT good - 0 in degree 3 position!\n");
}

4.2.2 Some Harder Code

The code used to find the examples above is more complicated and logically unnecessary, and as such will not be printed here.
Bibliography


