Essays in Mechanism and Market Design

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Abstract

This thesis consists of three essays on mechanism and market design.

The first chapter studies the question of when we can eliminate investment inefficiency in a general mechanism design model with transferable utility. We show that when agents make investments only before participating in the mechanism, inefficient investment equilibria cannot be ruled out whenever an allocatively efficient social choice function is implemented. We then allow agents to make investments before *and after* participating in the mechanism. When *ex post* investments are possible and an allocatively constrained-efficient social choice function is implemented, efficient investments can be fully implemented in perfect Bayesian Nash equilibria if and only if the social choice function is *commitment-proof* (a weaker requirement than strategy-proofness). Our result implies that in the provision of public goods, implementation of efficient investments and efficient allocations is possible even given a budget-balance requirement.

The second chapter analyzes the implementability of efficient investments for two commonly used mechanisms in single-item auctions: the first-price auction and the English auction. We allow uncertain *ex ante* investment and further *ex post* investment. Under private values, we show that both the first-price auction and the English auction implement efficient investments in equilibrium.

In the third chapter, we study the controlled school choice problem employing the lower and upper bounds of type-specific constraints. In such a problem, it is known that the set of feasible, fair and non-wasteful assignments may be empty (Ehlers et al., 2014).

We find that a *common priority condition* of a schools' priority profile plays a key role for the non-emptiness. A schools' priority profile is said to have a common priority order if a student has higher priority than any given student of the same type whenever she does so in some other school. We show that this condition is sufficient for the existence of a feasible assignment that is fair and non-wasteful, and also that it is necessary in a weak sense. To show the sufficiency part, we introduce an algorithm called the type-proposing deferred acceptance (TDA) algorithm.

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Chapter 1

Implementation of Efficient Investments in Mechanism Design

1.1 Introduction

Can an auction, like the spectrum auction, be designed to induce efficient investments as well as efficient allocations? A standard assumption in the mechanism design literature is that the values that the participants get out of the possible outcomes are exogenously given. In many real-life applications however, there are opportunities to invest in the values of the outcomes outside of the mechanism. In the spectrum auction, telecom companies make investments in new technologies or build base stations in anticipation of winning the spectrum licenses. In a procurement auction, participating firms make efforts to reduce the cost of production in preparation for bidding (Tan, 1992; Bag, 1997; Arozamena and Cantillon, 2004). Moreover, the firms in these auctions not only make *ex ante* investments but also make further investments if they win the auction (Piccione and Tan, 1996). These investments endogenously form the valuations of the allocations that are determined by the auction. At the same time, the incentives of both *ex ante* and *ex post* investments are affected by the structure of the allocation mechanism. Therefore, to seek an efficient mechanism, we should take account of the efficiency of the outside investments it induces, in addition to its standard efficiency within the mechanism.

The goal of this chapter is to analyze when we can fully implement efficient investments, i.e., under what mechanisms *every* equilibrium of the investment game will be efficient.[1] To do this, we consider a general mechanism design model with transferable utility. This includes several important applications such as auctions, matching with transfers and the provision of public goods. The valuation functions of agents at the market clearing stage are endogenously determined. We examine the following two environments: (i) agents make investments only before the mechanism, and (ii) they make investments before and after the mechanism. In either environment, we analyze the implementability of full efficiency, which requires that given that an allocatively efficient social choice function is implemented, every equilibrium of the investment game should maximize the total expected utility of agents inclusive of the cost of investments. In particular, we characterize the social choice functions for which efficient investments are implementable in every equilibrium. The main results are summarized as follows: first, with only *ex ante* investments, we show that efficient investments are not implementable for any allocatively efficient social choice function (Theorem 1). Next, allowing for *ex post* investments, we show that a new concept of *commitment-proofness* is sufficient and necessary for implementing efficient investments when an allocatively efficient social choice function is implemented (Theorem 2).

Furthermore, as a variant of the main model, we consider the provision of public goods with budget balance. In this environment, we show that there exists a commitment-proof, allocatively efficient and budget-balanced social choice function (Proposition 1). This implies that even with budget-balance requirement, it is always possible to implement efficient investments and efficient allocations at the same time.

The investigation of full implementation advances the traditional question asked in the literature: under what mechanisms does there exist an efficient pre-mechanism investment

[1]When we simply say "implementation" in this chapter, this refers to full implementation. See Definition 3 and 6 for the mathematical expressions.

equilibrium? Rogerson (1992) initiated this field by showing that when agents make investments prior to the mechanism, there is a socially efficient Nash equilibrium investment profile for any strategy-proof and allocatively efficient mechanisms. Hatfield et al. (2015) complemented Rogerson (1992)'s findings to show that strategy-proofness is also necessary for the existence of an efficient investment equilibrium when the mechanism is allocatively efficient. In the context of information acquisition (Milgrom, 1981; Obara, 2008), Bergemann and Välimäki (2002) indicate the link between *ex ante* efficiency and strategy-proofness; the VCG mechanism ensures *ex ante* efficiency under private values. Overall, in order to induce efficient *ex ante* investment incentives, strategy-proofness is essential because the privately optimal investment choice always becomes socially optimal given other agents' investment choices.

With only *ex ante* investments, however, there may exist another inefficient equilibrium even under strategy-proof mechanisms. Many authors in the literature pointed out this problem in a particular example, but they have not developed a general result.[2] The multiplicity of equilibria is not a trivial problem because an inefficient equilibrium may not be eliminated by employing stronger equilibrium concepts such as trembling-hand perfection. Consider an example where telecom firms are competing for a spectrum license, and suppose they know the competitors' cost functions for investments. When investments are observable, the *ex ante* investment may work as a commitment device even for a firm whose investment is more costly than other firms. If it is the only firm that makes an investment, at the market clearing stage, the value of the license can be higher than the values for any other firms because the cost of investment has been sunk. Therefore, there is an equilibrium at which the firm makes a lot of costly *ex ante* investments and deters its competitors from investing. This role of *ex ante* investment has also been studied as an entry-deterring behavior for an incumbent firm in an oligopolistic market (Spence, 1977, 1979; Salop, 1979; Dixit, 1980). This intuition is generalized by our first result; when agents

[2] For example, see Example 4 of Hatfield et al. (2015). This motivated the spectrum auction example which will be introduced in the next section.

invest only before the mechanism, inefficient investment equilibria cannot be ruled out whenever an allocatively efficient social choice function is implemented (Theorem 1).

In order to eliminate such investment inefficiency while securing allocative efficiency, we consider a setting where agents can invest before *and after* participating in the mechanism. In many applications, agents make further investments after the market clearing stage to maximize the value of the outcome realized in the mechanism. In the context of bidding for government contracts, firms invest in cost reduction once they are selected by the government to perform the task (McAfee and McMillan, 1986; Laffont and Tirole, 1986, 1987). For simplicity, we model investments as an explicit choice of valuation functions. *Ex ante* and *ex post* investments are modeled in the following way. First, agents choose their own valuation functions over the outcomes prior to the mechanism. The cost of each valuation function is determined by each agent's cost type. Each agent knows her own cost type, but does not know the realization of the cost types of other agents. These *ex ante* investments are irreversible, but after participating in the mechanism, agents may make further investments by revising their valuations to more costly ones. Note that equilibrium *ex post* investments are always socially optimal given the outcome of the mechanism as we assume no externality of investments. Therefore, if agents could not make any *ex ante* investments, the problem of implementing efficient investments falls within the scope of the classical mechanism design theory. However, this is not the case when *ex ante* investments are possible. Our main theorem characterizes allocatively efficient social choice functions for which investment efficiency is guaranteed in every equilibrium; given that an allocatively constrained-efficient social choice function is implemented, commitment-proofness of the social choice function is sufficient and necessary for implementing efficient investments in any perfect Bayesian Nash equilibrium (Theorem 2).

We introduce a novel concept called *commitment-proofness* which is illustrated in the following (hypothetical) scenario. Suppose that a participant in a mechanism makes a contract with a third party, in which the agent pays some amount to the third party before the mechanism, and then the third party returns some or all of the payment to the agent contingent on the outcome of the mechanism. Since this contract manipulates the value of each outcome (based on the amount of money returned to the agent), it allows the agent to commit to behaving as a different type in the mechanism. Commitment-proofness of a social choice function requires that no agent be able to benefit from making such a commitment.[3] This is a natural requirement since the third party would always be (weakly) better off from entering this contract. The concept thus precludes an important class of *ex ante* commitments which can potentially be made in a wide range of environments.

Then, how does the possibility of *ex post* investment help us obtain a positive result together with commitment-proofness? First, as we discussed above, investment efficiency is achieved by any allocatively efficient mechanism if no agent makes *ex ante* investments. Therefore, we need to find out under what conditions no agent will have the incentive to make positive *ex ante* investments for any cost type.[4] Consider a firm whose investment is more costly than other firms in the spectrum auction explained above. Suppose that no other firms make any *ex ante* investments. The values of the spectrum license for these firms would be low if there were no *ex post* investment opportunities. But now the value for each firm should be equal to the maximum net profit from the license inclusive of the cost of investment because any firm would make the optimal investment *ex post* if it wins the auction. Thus, in order for the firm with costly investment to win, it needs to beat its competitors who value the license more than the costly firm's potential profit from the license. To completely suppress the incentive of this firm to win out by investing *ex ante*, there must be a sufficient amount of payment for the license. Commitment-proofness of social choice functions characterizes such transfer payments that are sufficient and necessary for suppressing the incentives to invest *ex ante* in a general environment. In this way, the information of firms' cost types are revealed by the presence of *ex post* investment, and commitment-proofness eliminates the incentives for making *ex ante* investment which

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[3] As we will show in Section 1.4, this property is weaker than the well-known strategy-proofness condition.

[4] In the main model, we introduce a (slight) time discounting between two investment stages so that given that the allocation rule is efficient, investment efficiency is achieved only when no agents make costly *ex ante* investments.

works as a commitment device.

In our model, the difficulty of implementing efficient investments stems from the combination of the following assumptions: (i) investments are not verifiable, (ii) investments are irreversible, and (iii) the agents' cost types are not known to the mechanism designer. First, if investments were verifiable to a third party, they could just be part of the outcome of mechanisms and the standard implementation theory applies. However, investment behaviors are usually difficult to describe; they are multi dimensional and they involve the expenditure of time and effort as well as the expenditure of money (Hart, 1995). These non-contractible investments have also been a central concern in the hold-up problems (Klein et al., 1978; Williamson, 1979, 1983; Hart and Moore, 1988). Second, if investments were reversible, the efficiency of allocations would not be affected by the choice of *ex ante* investments. Therefore, we could apply mechanisms proposed by the standard implementation theory and (virtually) implement efficient allocations. Finally, it is obvious that investments would be efficient if the mechanism designer knew the agents' cost types and specified the first-best allocation because investments do not have any externalities.

Unlike related papers that analyze specific mechanisms such as the first-price auction and the second-price auction (Tan, 1992; Piccione and Tan, 1996; Stegeman, 1996; Bag, 1997; Arozamena and Cantillon, 2004), we consider the entire space of social choice functions. Also, we focus on the equilibrium analysis of the investment game outside of the mechanism. That is, the analysis of the game within the mechanism to implement a social choice function is set apart from the discussion. This is because we know that a large class of social choice functions are implementable both under complete and incomplete information. For example, any social choice function can be implemented by an extensive form mechanism in subgame-perfect equilibria under quasi-linear utility and complete information environments (Moore and Repullo, 1988; Maskin and Tirole, 1999). For incomplete information cases, it is known that a large class of social choice functions are virtually implementable by a static mechanism (Abreu and Matsushima, 1992). Therefore, most of the social choice rules considered here can be (virtually) implemented by some mechanism. Hence, our theorem gives a general guideline to distinguish whether an allocatively efficient mechanism, which may have not been analyzed well, implements efficient investments. In order to detect whether a specific mechanism (which has a non-truth-telling equilibrium) implements efficient investments from our results, we need one more step to check if it implements a commitment-proof and allocatively efficient social choice function.

There is also large literature on investment incentives before competition or two-sided matching (Gul, 2001; Cole et al., 2001a,b; Felli and Roberts, 2002; de Meza and Lockwood, 2010; Mailath et al., 2013; Nöldeke and Samuelson, 2015). Although these papers have a common interest with ours, there are two major differences in the modeling choices. First, they often assume that the investments of the two sides of agents have externalilties. Therefore, it is difficult to eliminate inefficient investment equilibria in their framework due to coordination failure. Moreover, they often consider situations where trade takes place in the market clearing stage. In such contexts, it is not plausible to consider the possibility of *ex post* investments. In short, our positive result may not be directly applied to their models because of these differences in the assumptions.

The rest of Chapter 1 is organized as follows. In Section 1.2, we explain a numerical example of the spectrum auction to provide intuition for the results. Section 1.3 introduces the formal model and defines implementability of efficient investments. In Section 1.4, commitment-proofness is introduced, and the impossibility results without *ex post* investments and the possibility results with *ex post* investments are presented. Provision of public goods is discussed as an application of our model in Section 1.5. Section 1.6 concludes. All proofs are in Appendix A.

1.2 Example: Spectrum Auction

Before introducing the general model, we provide intuition for our main theorems (Theorem 1 and 2) using a simple example of an auction. Consider a situation where two firms, A and B, are competing for a single spectrum license. The spectrum license is sold in the English auction, in which the price rises continuously from zero and each firm can drop out of the bidding. (We also consider another mechanism in the last part of the section.) The potential value of the spectrum license is in $[0, 10]$. Each firm $i = \mathrm{A}, \mathrm{B}$ makes investments to increase its own value a^i of the license outside the auction mechanism. Here, we model the investment behavior as the explicit choice of a value from the interval $[0, 10]$.[5] In order to realize $a^{\mathrm{A}}, a^{\mathrm{B}} \in [0, 10]$, each firm incurs the cost of investment which is represented by the following cost functions:

$$c^{\mathrm{A}}(a^{\mathrm{A}}) = \frac{1}{6}(a^{\mathrm{A}})^2,$$
$$c^{\mathrm{B}}(a^{\mathrm{B}}) = \frac{1}{4}(a^{\mathrm{B}})^2.$$

For simplicity, we assume that there is only one cost type for each agent. We alo assume that cost functions are common knowledge between firms and investments are observable (but not verifiable). Therefore, the information is complete between firms in the games which will be defined below.[6] The mechanism designer does not observe either their investments or cost types.

First, consider efficient investments and allocation which maximize the sum of each firm's profit from the license inclusive of the cost of investments (i.e., the social welfare). If firm A obtains the license, the optimal investment would be

$$\arg\max_{a^{\mathrm{A}} \in [0,10]} \left\{ -\frac{1}{6}(a^{\mathrm{A}})^2 + a^{\mathrm{A}} \right\} = 3.$$

The maximum net profit for firm A in this case is

$$\max_{a^{\mathrm{A}} \in [0,10]} \left\{ -\frac{1}{6}(a^{\mathrm{A}})^2 + a^{\mathrm{A}} \right\} = \frac{3}{2}.$$

Similarly, for firm B, the optimal investment would be

$$\arg\max_{a^{\mathrm{B}} \in [0,10]} \left\{ -\frac{1}{4}(a^{\mathrm{B}})^2 + a^{\mathrm{B}} \right\} = 2.$$

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[5] This means that we are assuming no externality for investments.

[6] In the general model, the complete information assumption of the cost types will be relaxed. But we still assume that investments are observable among agents.

The maximum net profit for firm A in this case is

$$\max_{a^{\mathrm{B}} \in [0,10]} \left\{ -\frac{1}{4}(a^{\mathrm{B}})^2 + a^{\mathrm{B}} \right\} = 1.$$

Since there is a single license, it is clear that only one of the firms should make a positive investment to achieve investment efficiency. Therefore, the unique profile of efficient investments is $(a^{*\mathrm{A}}, a^{*\mathrm{B}}) = (3, 0)$ and we should allocate the license to firm A. The maximum social welfare is $\frac{3}{2}$.

Now we define the investment stage as a game between these two firms, and examine whether every equilibrium of the investment game achieves efficiency. The following two settings are considered: [1] firms make investments only before the mechanism, and [2] they make investments before and after the mechanism. We analyze the English auction in both cases, and also analyze another mechanism in the second setting. We consider trembling-hand perfect equilibrium (in the agent-normal form) in this section to exclude unintuitive equilibria of the English auction.[7]

[1] Investments only before the English auction.

In this case, we model the *ex ante* investment stage as a simultaneous move game where each firm chooses its own valuation.[8] The timeline of the investment and the auction is as follows:

1. Each firm $i =$ A, B chooses its own valuation a^i from $[0, 10]$ simultaneously. The cost of investment $c^i(a^i)$ is paid.

2. They participate in the English auction given the valuations $(a^{\mathrm{A}}, a^{\mathrm{B}})$.

First, consider the English auction stage. The unique trembling-hand perfect equilibrium is that each firm drops out when its value is reached.[9] Since the valuations of the license

[7] In the next section, we employ perfect Bayesian Nash equilibria for the analysis of the investment game.

[8] My main results do not heavily rely on the simultaneity of investments. For example, the inefficient equilibrium in the first setting is also achieved when firm *B* moves first. In addition, the efficiency result in the second setting under the English auction is robust to the sequential moves of firms because firm *B* would not want to invest whatever the sequence of the move is.

[9] Under complete information, there are other subgame-perfect equilibria. For example, a firm whose

for firms are $(a^{\mathrm{A}}, a^{\mathrm{B}})$, firm $i \in \{A, B\}$ whose valuation is higher than the other, i.e., $a^i \geq a^j$ where $j \neq i$, wins the license and pays a^j in the unique equilibrium. Therefore, given the equilibrium of the English auction, for any choice of investments $(a^{\mathrm{A}}, a^{\mathrm{B}}) \in [0, 10]^2$, the net profit of firm $i =$ A, B is written as

$$-c^i(a^i) + (a^i - a^j)\mathbb{1}_{\{a^i \geq a^j\}}$$

where j is the other firm.[10]

Next, analyze the equilibrium of the investment stage. First, it is easy to see that the socially efficient investments $(a^{*\mathrm{A}}, a^{*\mathrm{B}}) = (3, 0)$ are achieved in equilibrium. Consider another investment profile $(a^{\mathrm{A}}, a^{\mathrm{B}}) = (0, 2)$ where firm A makes no investment and firm B chooses 2 *ex ante*. Consider firm A's incentive given $a^{\mathrm{B}} = 2$. If firm A wins the auction, the payment in the English auction would be 2, which exceeds the maximum net profit of $\frac{3}{2}$ for firm A;

$$-\frac{1}{6}(a^{\mathrm{A}})^2 + (a^{\mathrm{A}} - 2)\mathbb{1}_{\{a^{\mathrm{A}} \geq 2\}} \leq \frac{3}{2} - 2 < 0$$

for any $a^A \in [0, 10]$. Thus, firm A does not have the incentive to win the auction by making a positive investment. For firm B, it is clear that choosing 2 is optimal given that firm A does not make any investments because B will obtain the license in the auction. Therefore, this profile $(a^{\mathrm{A}}, a^{\mathrm{B}}) = (0, 2)$ is an equilibrium of the *ex ante* investment game. However, this is not an efficient investment profile because it gives less social welfare than $(a^{*\mathrm{A}}, a^{*\mathrm{B}}) = (3, 0)$. Thus, we can conclude that there is a socially inefficient trembling-hand perfect equilibrium. $\square$

This is an example where the English auction failed to fully implement efficient invest-

valuation is lower than the other drops out at price zero in a subgame-perfect equilibrium because dropping out at any low price is indifferent for the losing firm with complete information. However, it is not a trembling-hand perfect equilibrium because dropping out at its own valuation is strictly better when every action of the other firm is taken with a positive probability.

[10] $\mathbb{1}$ is an indicator function. For any proposition p, $\mathbb{1}_{\{p\}}$ is defined by

$$\mathbb{1} = \begin{cases} 1 & \text{if } p \text{ is true,} \\ 0 & \text{otherwise.} \end{cases}$$

ments. Unfortunately, we show that not only the English auction but any other mechanism fails to implement efficient investments in the general model when there are no *ex post* investment opportunities and the allocation is selected efficiently (Theorem 1). Next, let's consider what will happen with *ex post* investments when the same English auction is used.

[2-1] Investments before and after the English auction.

When *ex post* investments are possible, another investment stage for revising their own valuations is added after the mechanism. The timeline of the investment and the auction in this case is:

1. Each firm $i = \mathrm{A}, \mathrm{B}$ chooses its own valuation a^i from $[0, 10]$ simultaneously. The cost of investment $c^i(a^i)$ is paid.

2. They participate in the English auction.

3. Each firm $i = \mathrm{A}, \mathrm{B}$ again chooses its own valuation $\bar{a}^i$ from $[a^i, 10]$. The cost of additional investment $c^i(\bar{a}^i) - c^i(a^i)$ is paid.

As we discuss in the next section, we assume the irreversibility of investments; $\bar{a}^i$ can be only chosen from $[a^i, 10]$. Also, the cost function is assumed to be unchanged over time so that for a fixed total amount $\bar{a}^i$, the total cost of investment is $c^i(\bar{a}^i)$ and choosing any *ex ante* investments $a^i \in [0, \bar{a}^i]$ is indifferent if the allocation is fixed. However, since we consider an auction mechanism to determine the allocation, *ex ante* choices matter as they change the outcome of the auction. The net profit of firm $i = \mathrm{A}, \mathrm{B}$ is written as

$$-c^i(a^i) + (\bar{a}^i - p)\mathbb{1}_{\{i \text{ wins the auction}\}} - (c^i(\bar{a}^i) - c^i(a^i))$$

where p is the payment in the auction, whose equilibrium value will be computed below.

Although the investment game is different from the first setting, efficient investments and allocation are unchanged; firm A should obtain the license and it makes investments $(a^{*\mathrm{A}}, \bar{a}^{*\mathrm{A}}) \in [0, 10]^2$ such that $a^{*\mathrm{A}} \leq \bar{a}^{*\mathrm{A}} = 3$. Firm B should not make any investment, i.e., $(a^{*\mathrm{B}}, \bar{a}^{*\mathrm{B}}) = (0, 0)$.

The equilibrium is solved by backward induction. Consider firm A's optimal strategy in the *ex post* investment stage. Given any *ex ante* valuation choice $a^{\mathrm{A}} \in [0, 10]$, the profit from the license in the last stage is

$$\bar{a}^{\mathrm{A}} - \left(c^{\mathrm{A}}(\bar{a}^{\mathrm{A}}) - c^{\mathrm{A}}(a^{\mathrm{A}})\right).$$

Thus, it makes further investment only when it obtains the license and a^{A} is less than 3. The optimal *ex post* investment strategy given a^{A} is

$$\bar{a}^{\mathrm{A}} = \begin{cases} \max\{3, a^{\mathrm{A}}\} & \text{if firm A obtains the license,} \\ a^{\mathrm{A}} & \text{otherwise.} \end{cases}$$

Similarly, firm B's optimal *ex post* investment strategy given a^{B} is

$$\bar{a}^{\mathrm{B}} = \begin{cases} \max\{2, a^{\mathrm{B}}\} & \text{if firm B obtains the license,} \\ a^{\mathrm{B}} & \text{otherwise.} \end{cases}$$

Next, analyze the English auction. Again, in the unique trembling-hand perfect equilibrium, the firm with the higher willingness to pay should win and it pays the other firm's valuation. Let $b^i(a^i)$ be the value of the license in the auction stage when firm i chooses a^i *ex ante*. The following two things should be noted in calculating it; (i) $b^i(a^i)$ takes account of the optimal strategy in the *ex post* stage, and (ii) the cost of *ex ante* investment is sunk. For each $a^{\mathrm{A}} \in [0, 10]$, it is

$$b^{\mathrm{A}}(a^{\mathrm{A}}) = \max_{\bar{a}^{\mathrm{A}} \in [a^{\mathrm{A}}, 10]} \left\{ \bar{a}^{\mathrm{A}} - \left(c^{\mathrm{A}}(\bar{a}^{\mathrm{A}}) - c^{\mathrm{A}}(a^{\mathrm{A}})\right) \right\} = \begin{cases} \frac{3}{2} + \frac{1}{6}(a^{\mathrm{A}})^2 & \text{if } a^{\mathrm{A}} \in [0, 3) \text{ and} \\ a^{\mathrm{A}} & \text{if } a^{\mathrm{A}} \in [3, 10], \end{cases}$$

and for each $a^{\mathrm{B}} \in [0, 10]$,

$$b^{\mathrm{B}}(a^{\mathrm{B}}) = \max_{\bar{a}^{\mathrm{B}} \in [a^{\mathrm{B}}, 10]} \left\{ \bar{a}^{\mathrm{B}} - \left(c^{\mathrm{B}}(\bar{a}^{\mathrm{B}}) - c^{\mathrm{B}}(a^{\mathrm{B}})\right) \right\} = \begin{cases} 1 + \frac{1}{4}(a^{\mathrm{B}})^2 & \text{if } a^{\mathrm{B}} \in [0, 2) \text{ and} \\ a^{\mathrm{B}} & \text{if } a^{\mathrm{B}} \in [2, 10]. \end{cases}$$

Intuitively, when firm i's initial investment a^i is more than the optimal value 3, $b^i(a^i)$ is equal to a^i as there is no further investment. If a^i is less than the optimal value 3, $b^i(a^i)$ is increasing in a^i exactly by the amount of $c^i(a^i)$ because more *ex ante* investment means less cost of additional investment when the license is awarded to the firm. Under the unique equilibrium of the English auction, if firm A wins the license, the payment will be $b^{\mathrm{B}}(a^{\mathrm{B}})$ and vice versa.

Given these equilibrium strategies, we can analyze the first investment stage. Consider firm B's incentive. If it wins the license in the English auction, the payment is at least $\frac{3}{2}$ because $b^{\mathrm{A}}(a^{\mathrm{A}}) \geq \frac{3}{2}$ holds for any $a^{\mathrm{A}} \in [0, 10]$. However, since the maximum net profit from the spectrum license is 1 for firm B, it does not have the incentive to win by choosing $a^{\mathrm{B}} > \frac{3}{2}$;

$$\max\{2, a^{\mathrm{B}}\} - b^{\mathrm{A}}(a^{\mathrm{A}}) - \frac{1}{4}\big(\max\{2, a^{\mathrm{B}}\}\big)^2 \leq 1 - \frac{3}{2} < 0.$$

Therefore, firm B refrains from making investments in equilibrium, and chooses $a^{*\mathrm{B}} = 0$. Since firm A always wins the auction with the payment $b^{\mathrm{B}}(0) = 1$, it is indifferent to choose any investments $(a^{*\mathrm{A}}, \bar{a}^{*\mathrm{A}})$ such that $a^{*\mathrm{A}} \leq \bar{a}^{*\mathrm{A}} = 3$.[11] Therefore, investment efficiency is achieved in any trembling-hand perfect equilibrium. □

Now allowing for *ex post* investments, any trembling-hand perfect equilibrium achieves investment efficiency in the English auction. Why did this become possible? Intuitively, with only *ex ante* investments, if firm A has not made any investment, it will drop out at price zero in the English auction and firm B will choose an investment $a^{\mathrm{B}} = 2$ to maximize its profit. Furthermore, firm A will optimally choose not to make any investment given $a^{\mathrm{B}} = 2$ because firm B will stay too long in the English auction for firm A to make a profit from any positive investment. On the other hand, with *ex post* investments, firm A will stay in the English auction until the price reaches $\frac{3}{2}$ because firm A can make a profit when firm B drops out before $\frac{3}{2}$. Now, since firm B's payment exceeds $\frac{3}{2}$ if it wins the auction, it cannot make a profit from any positive investment.

However, when we consider other mechanisms, allowing *ex post* investment does not

[11] When there is a strict time discounting as we consider in the general model, the unique optimal investment is $(a^{*\mathrm{A}}, \bar{a}^{*\mathrm{A}}) = (0, 3)$.

always solve the problem. More importantly, this is not because the mechanism fails to allocate the license efficiently, but because an inefficient investment equilibrium exists even though the mechanism always selects an efficient allocation (according to the valuations in the auction stage).

To introduce such an example of a mechanism, we review the literature of (subgame-perfect) implementation. A seminal paper by Moore and Repullo (1988) showed that under complete information and quasi-linear utility functions, any social choice function is subgame-perfect implementable. This implies that by their mechanism, we can implement an efficient allocation rule with any transfer rule. Consider here one such mechanism: a Moore-Repullo mechanism which always chooses an efficient allocation according to $(b^{\mathrm{A}}(a^{\mathrm{A}}), b^{\mathrm{B}}(a^{\mathrm{B}}))$ and does not impose any transfers.[12]

[2-2] Investments before and after the efficient Moore-Repullo mechanism with no transfers.

The timeline of the investment game is the same as in the previous case [2-1]. The English auction is replaced by the following mechanism.

Stage 1:

1-1. Firm A announces its own valuation $\bar{b}^{\mathrm{A}}$.

1-2. Firm B decides whether to challenge firm A's announcement $\bar{b}^{\mathrm{A}}$.

If firm B does not challenge it, go to stage 2.

If firm B challenges, firm A pays 20 to the mechanism designer. Firm B receives 20 if the challenge is successful, but pays 20 to the mechanism designer if it is a failure. Whether it is a success or a failure is determined by the following game: The license is sold in the second-price auction. Firm B chooses some $\bar{b}^{\mathrm{B}}$ to submit to the auction and a positive value $\eta > 0$, and asks firm A to choose one of them:

(i) submitting any value,

(ii) submitting $\bar{b}^{\mathrm{A}}$ and receiving an additional transfer η.

[12] In some countries such as Japan, spectrum licenses are still allocated to firms for free once they are screened by the government. Although this process is not a mechanism, if the government correctly observes the valuations $(b^{\mathrm{A}}(a^{\mathrm{A}}), b^{\mathrm{B}}(a^{\mathrm{B}}))$, it is exactly the social choice function implemented by this Moore-Repullo mechanism.

The challenge is successful only if firm A picks (i). Stop.

Stage 2: Same as stage 1, but the roles of A and B are switched.

Stage 3: If there are no challenges in stage 1 and 2, the license is given for free to firm i such that $\bar{b}^i \geq \bar{b}^j$ where j is the other firm.

Given the optimal strategies in the *ex post* investment stage, for any profile of *ex ante* investments $(a^{\mathrm{A}}, a^{\mathrm{B}})$, it is shown that the unique subgame-perfect (and also trembling-hand perfect) equilibrium of this mechanism is such that each firm $i = \mathrm{A}, \mathrm{B}$ announces its true valuation $b^i(a^i)$, and no firm challenges the other firm's claim (Moore and Repullo, 1988). The intuitive reason is that in the challenge phase, the other firm j can choose some $\bar{b}^j$ and $\eta > 0$ so that the challenge is successful (firm i optimally chooses (i)) whenever the announcement $\bar{b}^i$ of firm i is different from $b^i(a^i)$. Also, the other firm's challenge would never be successful when the announcement is truthful since (ii) is always chosen by a truthful firm. Therefore, the allocation is always determined efficiently and no transfer is imposed in equilibrium.

Consider firm B's incentive in the first investment stage. Now firm B has the incentive to invest more than firm A as long as A's investment is socially efficient, i.e., $a^{\mathrm{A}} \leq 3$. This is because the price of the license is zero in the mechanism and firm B would still earn a positive profit by winning the auction: for some $a^{\mathrm{B}} \in (3, 4)$,

$$\max\{2, a^{\mathrm{B}}\} - 0 - \frac{1}{4}\big(\max\{2, a^{\mathrm{B}}\}\big)^2 > 0.$$

Actually, there is a mixed strategy equilibrium in which $a^{\mathrm{B}} > 0$ occurs with a positive probability. Thus, efficient investments are not implemented by this allocatively efficient Moore-Repullo mechanism with no transfers. $\square$

In the English auction with *ex post* investments, firm B could not make a profit by investing $a^{\mathrm{B}} = 2$ because the price of the license was greater than $\frac{3}{2}$. However, in this zero-payment mechanism, $a^{\mathrm{B}} = 2$ remains profitable because firm B does not pay anything in the auction. This shows that the range of the price of the license is critical for inducing the right incentive for firm B. Suppose that the allocation is always efficiently determined, and that firm A does not make any *ex ante* investment, i.e., $a^{\mathrm{A}} = 0$. Then, firm B would lose the auction when choosing $a^{\mathrm{B}} = 0$, but would win the auction if it chooses $a^{\mathrm{B}} = 2$. In order to prevent firm B from choosing 2, the price p of the license when firms choose $(a^{\mathrm{A}}, a^{\mathrm{B}}) = (0, 2)$ *ex ante* should satisfy

$$0 \geq b^{\mathrm{B}}(2) - c^{\mathrm{B}}(2) - p \;\Leftrightarrow\; p \geq 1.$$

Obviously, the English auction satisfied this condition, but the Moore-Repullo mechanism with no transfers violated it. This idea of disincentivizing *ex ante* investment with a right transfer rule can be applied to more general environments. Our main contribution is to discover a property of a social choice function, which we call *commitment-proofness*, in the general model and to show that it is sufficient and necessary for implementing efficient investments.

1.3 General Model

There is a finite set I of agents and a finite set Ω of alternatives. A valuation function of agent $i \in I$ is $v^i : \Omega \to \mathbb{R}$. The valuation function is endogenously determined by each agent's investment decision as described below. The set of possible valuation functions is $V^i \subseteq \mathbb{R}^{\Omega}$. Assume that V^i is a compact set. Denote the profile of the sets of valuations by $V \equiv \times_{i \in I} V^i$. We assume that investments are not verifiable to a third party. Therefore, a mechanism chooses an alternative and transfers, but does not choose agents' investment behaviors. We discuss the relationship between social choice rules and mechanisms later in this section.

Each agent makes an investment decision to determine her own valuation over alternatives. The investment is modeled as an explicit choice of a valuation function with the cost of investment determined by a cost function $c^i : V^i \times \Theta^i \to C^i \subseteq \mathbb{R}_+$ where Θ^i is a finite set of cost types of agent i. Each agent i knows her own cost type $\theta^i \in \Theta^i$, but may be unsure about $\theta^{-i} \equiv (\theta^j)_{j \in I \setminus \{i\}}$. There is a common prior distribution on $\Theta \equiv \times_{i \in I} \Theta^i$, denoted p. Conditional on knowing her own cost type θ^i, agent i's posterior distribution over $\Theta^{-i} \equiv \times_{j \in I \setminus \{i\}} \Theta^j$ is denoted $p(\cdot|\theta^i)$. $p(\cdot|\theta^i)$ is computed by Bayes rule whenever θ^i occurs with a positive probability, i.e., $\sum_{\theta^{-i} \in \Theta^{-i}} p(\theta^i, \theta^{-i}) > 0$.[13] Assume that C^i is a compact set and $0 \in C^i$. Denote the profile of the sets of possible costs by $C \equiv \times_{i \in I} C^i$. Without loss of generality, the cost of investment is assumed to be non-negative, and we also assume that for each $\theta^i \in \Theta^i$, there is $v^i \in V^i$ such that $c^i(v^i, \theta^i) = 0$. There are two investment stages; before and after participating in the mechanism. We model each of the investment stages as a simultaneous move game by all agents. Assume that the investment is irreversible; if agent i with cost type θ^i chooses $v^i \in V^i$ before the mechanism, she can only choose a valuation function from the set $\{\bar{v}^i \in V^i | c^i(\bar{v}^i, \theta^i) \geq c^i(v^i, \theta^i)\}$ in the second investment stage.[14] To clarify, the timeline of the investment game induced by a mechanism is:

0. Each agent i observes her own cost type $\theta^i \in \Theta^i$.
1. Each agent makes a prior investment by choosing a valuation function $v^i \in V^i$ simultaneously.
2. Agents participate in a mechanism.
3. After the mechanism is run, each agent can make an additional investment, i.e., each agent chooses a valuation function from $\{\bar{v}^i \in V^i | c^i(\bar{v}^i, \theta^i) \geq c^i(v^i, \theta^i)\}$.

We assume that chosen valuation functions are observable among agents (but not verifiable). Also, assume that cost functions are common knowledge among agents and the mechanism designer. However, each agent only knows her own cost type and the distribution of other agents' cost types. The mechanism designer does not know the realized cost type vector θ or the common prior distribution p. The investment game is an incomplete information game if p is a non-degenerate distribution. We allow for the complete information case where $p(\theta) = 1$ for some $\theta \in \Theta$. Throughout the analyses in this chapter, we

[13] If $\sum_{\theta^{-i} \in \Theta^{-i}} p(\theta^i, \theta^{-i}) = 0$, we assign any arbitrary posterior distribution $p(\cdot|\theta^i)$.

[14] The essential assumption is actually that the cost of *ex ante* investment is sunk, rather than the (physical) irreversibility of an investment itself. However, we maintain the assumption of irreversibility since it keeps the analysis simple and easy to understand.

fix the set I of agents, the set Ω of alternatives, the set Θ of cost types and the common prior distribution p.

The *ex ante* utility function of an agent has the following three components: the valuation functions she chooses in the first and the second investment stages, the cost function and a discount factor. Let $\beta \in (0,1]$ be a discount factor which discounts the utility realized in the second stage and later.[15] For an alternative $\omega \in \Omega$, a transfer vector $t \equiv (t^i)_{i\in I} \in \mathbb{R}^I$ and an investment schedule $(v^i, \bar{v}^i) \in (V^i)^2$ where v^i is the valuation function chosen before the mechanism and $\bar{v}^i$ is the final valuation function, the *ex ante* utility of agent i with cost type θ^i is defined by:

$$-c^i(v^i,\theta^i) + \beta\Big[\bar{v}^i(\omega) - t^i - \big(c^i(\bar{v}^i,\theta^i) - c^i(v^i,\theta^i)\big)\Big]. \tag{1.1}$$

In the first stage, only the cost $c^i(v^i,\theta^i)$ of *ex ante* investment is paid. In the second stage, the outcome (ω, t) of the mechanism is evaluated by the final valuation function $\bar{v}^i$. And in the last stage, the additional cost $c^i(\bar{v}^i,\theta^i) - c^i(v^i,\theta^i) \geq 0$ of revising the valuation function is paid.[16] Throughout this chapter, we consider this quasi-linear utility function, i.e., utility to be perfectly transferable.

When agents face the mechanism in the second stage, the cost of investment made in the first stage is already sunk. Moreover, in any equilibrium, an alternative $\omega \in \Omega$ is evaluated by a valuation function which is the optimal choice of the *ex post* investment. Therefore, we can define the valuations of agents at the time of the mechanism as follows using the notation b^{c^i,θ^i,v^i} for any cost function $c^i : V^i \times \Theta^i \to C^i$, cost type $\theta^i \in \Theta^i$ and the prior investment $v^i \in V^i$.

Definition 1. *The valuation function $b^{c^i,\theta^i,v^i} : \Omega \to \mathbb{R}$ at the time of the mechanism* given a cost function $c^i : V^i \times \Theta^i \to C^i$, a cost type $\theta^i \in \Theta^i$ and a valuation function $v^i \in V^i$ is defined

[15]There is no time discounting between the mechanism stage and the *ex post* investment stage, but this is without loss of generality.

[16]Here, we assume that the same cost function is used for both investment stages. Some of the main results, however, still hold when the cost functions differ across time. For example, the sufficiency part of our possibility theorem (Theorem 2) holds as long as the *ex post* cost function is weakly lower than the *ex ante* cost function.

by

$$b^{c^i,\theta^i,v^i}(\omega) = \max_{\bar{v}^i \in \{\tilde{v}^i \in V^i | c^i(\tilde{v}^i,\theta^i) \geq c^i(v^i,\theta^i)\}} \left\{ \bar{v}^i(\omega) - c^i(\bar{v}^i,\theta^i) \right\} + c^i(v^i,\theta^i)$$

for each $\omega \in \Omega$. Let $b^{c,\theta,v} \equiv (b^{c^i,\theta^i,v^i})_{i \in I}$.

The equation is taken from the second term of equation (1.1), and takes account of each agent's optimal *ex post* investment choice given the cost type. Given a prior investment $v^i \in V^i$ and an alternative $\omega \in \Omega$, the optimal choice of the *ex post* investment should be $\bar{v}^i \in V^i$ which maximizes the net value $\bar{v}^i(\omega) - c^i(\bar{v}^i,\theta^i)$ among the set of feasible valuation functions, which is $\{\tilde{v}^i \in V^i | c^i(\tilde{v}^i,\theta^i) \geq c^i(v^i,\theta^i)\}$.[17]

A social choice function $h : \mathbb{R}^{\Omega \times I} \to \Omega \times \mathbb{R}^I$ is defined as a mapping from the potential set $\mathbb{R}^{\Omega \times I}$ of valuation functions at the time of the mechanism to the set Ω of alternatives and the set $\mathbb{R}^I$ of transfer vectors. A social choice function $h \equiv (h_\omega, h_t)$ has the following two components; $h_\omega : \mathbb{R}^{\Omega \times I} \to \Omega$ is called an allocation rule and $h_t : \mathbb{R}^{\Omega \times I} \to \mathbb{R}^I$ is called a transfer rule. The transfer rule for each agent is denoted by $h_t^i : \mathbb{R}^{\Omega \times I} \to \mathbb{R}$ and $h_t(b) = (h_t^i(b))_{i \in I}$ holds for any $b \in \mathbb{R}^{\Omega \times I}$. Note that the domain $\mathbb{R}^{\Omega \times I}$ of social choice functions is not restricted by V, but defined to include any potential valuation functions at the time of the mechanism. Therefore, a social choice function is defined only for a tuple (I, Ω). As we see below, when we define the implementability of efficient investments given a social choice function, we consider any possible set $V \subseteq \mathbb{R}^{\Omega \times I}$ of valuation functions and a profile of cost functions $c : V \times \Theta \to C$.

We are interested in whether efficient investments are fully implementable in perfect Bayesian Nash equilibria when an allocatively efficient social choice function is implemented. In this chapter, we focus on the analysis of an investment game induced by a social choice function, and do not explicitly consider mechanisms to implement the social choice function. Although we do not discuss whether a specific social choice function is implementable, the literature has shown several positive results under both complete and

[17] If the cost of *ex ante* investments is refundable, the valuation function at the time of the mechanism only shifts by a constant for any choice of *ex ante* investment (since the first term of $b^{c^i,\theta^i,v^i}(\omega)$ would then be fixed). This means that concepts such as allocative efficiency (defined shortly) are not essentially affected by the *ex ante* investment behaviors. Therefore, we focus on the non-trivial cases where *ex ante* investment is irreversible.

incomplete information. For example, Moore and Repullo (1988) showed that any social choice function is subgame-perfect implementable by their extensive form mechanism under transferable utility and complete information environments.[18] Their extensive form mechanism only works under complete information, but even under incomplete information, Abreu and Matsushima (1992) showed that a large class of social choice functions are virtually implementable. Therefore, we take these positive theorems as given, and simply consider the entire space of social choice functions in this chapter. We leave the equilibrium analysis within a mechanism outside the scope of this chapter, and concentrate on finding out the properties of social choice functions which enable us to implement efficient investments.

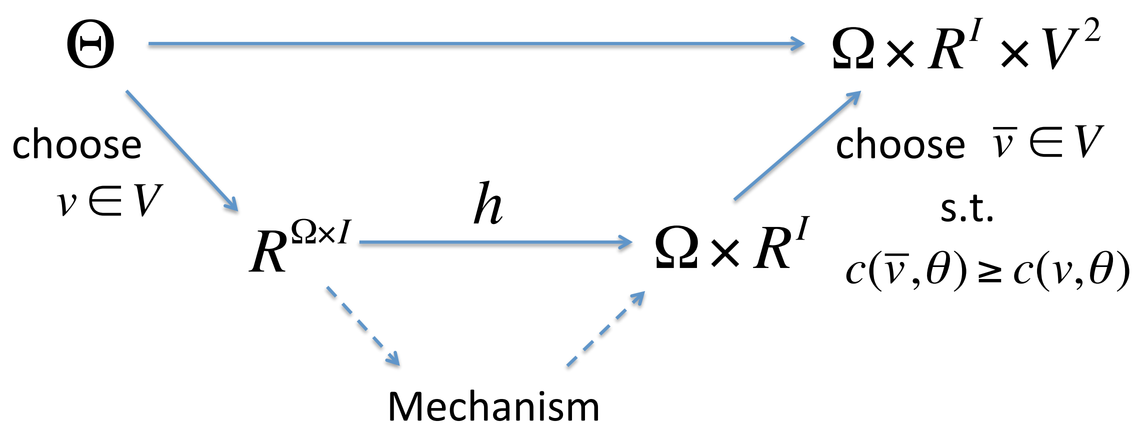

Figure 1.1: *The structure of a social choice function and the investment game.*

To introduce the implementability of efficient investments, we first define a perfect Bayesian Nash equilibrium of an investment game induced by a given social choice function. For the set of strategies in the first investment stage, we denote the set of all mappings from Θ^i to V^i by Σ^i. For the set of strategies in the last investment stage, we denote the set of all mappings from $V^i \times \Omega \times \Theta^i$ to V^i as $\mathcal{M}^i$. Let $\Sigma \equiv \times_{i \in I} \Sigma^i$ and $\mathcal{M} \equiv \times_{i \in I} \mathcal{M}^i$.

Definition 2. For any $V \subseteq \mathbb{R}^{\Omega \times I}$ and any profile of cost functions $c : V \times \Theta \to C$, a profile

[18]To make use of the Moore-Repullo mechanism, the utility of agents must be uniformly bounded. Thus, the amount of penalty used in this mechanism needs to depend on (V, C), but it can be appropriately chosen in this setting because V and C are both bounded.

of investment strategies $(\sigma^*, \mu^*) \in \Sigma \times \mathcal{M}$ is a *perfect Bayesian Nash equilibrium (PBNE) of the investment game* given a social choice function $h : \mathbb{R}^{\Omega \times I} \to \Omega \times \mathbb{R}^I$ and a discount factor $\beta \in (0, 1]$ if for each $i \in I$ and $\theta^i \in \Theta^i$,

1. $\mu^{*i}(v^i, \omega, \theta^i) \in \underset{\bar{v}^i \in \{\bar{v}^i \in V^i | c^i(\bar{v}^i, \theta^i) \geq c^i(v^i, \theta^i)\}}{\arg\max} \left\{ \bar{v}^i(\omega) - c^i(\bar{v}^i, \theta^i) \right\}$
   for any $v^i \in V^i$ and $\omega \in \Omega$, and

2. $$\sigma^{*i}(\theta^i) \in \underset{v^i \in V^i}{\arg\max} \Big\{ - c^i(v^i, \theta^i) + \beta \textstyle\sum_{\theta^{-i} \in \Theta^{-i}} p(\theta^{-i} | \theta^i)$$
   $$\Big[\mu^{*i}(v^i, h_\omega(b^{c^i,\theta^i,v^i}, b^{-i}), \theta^i)(h_\omega(b^{c^i,\theta^i,v^i}, b^{-i})) - h_t^i(b^{c^i,\theta^i,v^i}, b^{-i})$$
   $$-c^i(\mu^{*i}(v^i, h_\omega(b^{c^i,\theta^i,v^i}, b^{-i}), \theta^i), \theta^i) + c^i(v^i, \theta^i)\Big]\Big\}$$
   where $b^{-i} \equiv b^{c^{-i},\theta^{-i},\sigma^{*-i}(\theta^{-i})}$

hold.

The first condition of a PBNE is the optimality in the *ex post* investment stage. Since the investment does not have an externality, this is simply an individual maximization problem. Therefore, it is defined for each information set in the last stage, which is characterized by the choice of the first stage investment, the realized alternative and the cost type of the agent. The second condition requires that σ^* forms a Bayesian Nash equilibrium of the first stage investment game, given the optimal *ex post* investment strategy μ^* and the social choice function h.

Full implementation of efficient investments requires that any PBNE of the investment game should be socially efficient. More precisely, given that a social choice function h is implemented, efficient investments are said to be implementable in PBNE if for any profile of the sets of valuations and cost functions, any PBNE of the investment game given h and a discount factor β maximizes the sum of expected utility of agents inclusive of the cost of investments given h and β.

Definition 3. Given a social choice function $h : \mathbb{R}^{\Omega \times I} \to \Omega \times \mathbb{R}^I$ and a discount factor $\beta \in (0, 1]$, *efficient investments are implementable in perfect Bayesian Nash equilibria* if for any

$V \subseteq \mathbb{R}^{\Omega \times I}$ and any profile of cost functions $c : V \times \Theta \to C$, any perfect Bayesian Nash equilibrium $(\sigma^*, \mu^*) \in \Sigma \times \mathcal{M}$ satisfies the following equation:

$$(\sigma^*, \mu^*) \in \underset{(\sigma,\mu)\in\Sigma\times\mathcal{M}}{\arg\max} \sum_{\theta\in\Theta} p(\theta) \sum_{i\in I} \Big\{ - c^i(\sigma^i(\theta^i), \theta^i)$$
$$+\beta\Big[\mu^i(\sigma^i(\theta^i), h_\omega(b^{c,\theta,\sigma(\theta)}), \theta^i)(h_\omega(b^{c,\theta,\sigma(\theta)})) - c^i(\mu^i(\sigma^i(\theta^i), h_\omega(b^{c,\theta,\sigma(\theta)}), \theta^i), \theta^i) + c^i(\sigma^i(\theta^i), \theta^i)\Big]\Big\}.$$

Next, we define the properties of social choice functions. There are two versions of allocative efficiency. The first definition of allocative efficiency is standard; the allocation rule chooses an alternative to maximize the sum of the valuation of agents. A social choice function $h : \mathbb{R}^{\Omega \times I} \to \Omega \times \mathbb{R}^I$ is allocatively efficient if for any $b \in \mathbb{R}^{\Omega \times I}$,

$$h_\omega(b) \in \underset{\omega\in\Omega}{\arg\max} \sum_{i\in I} b^i(\omega).$$

Our main theorem (Theorem 2) holds for a weaker notion of allocative efficiency, which is called allocative constrained-efficiency. This guarantees allocative efficiency within a certain subset of alternatives.

Definition 4. A social choice function $h : \mathbb{R}^{\Omega \times I} \to \Omega \times \mathbb{R}^I$ is *allocatively constrained-efficient for $\Omega' \subseteq \Omega$ with $\Omega' \neq \emptyset$* if for any $b \in \mathbb{R}^{\Omega \times I}$, the allocation rule satisfies

$$h_\omega(b) \in \underset{\omega\in\Omega'}{\arg\max} \sum_{i\in I} b^i(\omega).$$

Note that Ω' in the definition above can be a singleton set. Thus a constant social choice function $\bar{h} : \mathbb{R}^{\Omega \times I} \to \Omega \times \mathbb{R}^I$ such that $\bar{h}_\omega(b) = \bar{\omega} \in \Omega$ for any $b \in \mathbb{R}^{\Omega \times I}$ also satisfies allocative constrained-efficiency for $\Omega' \equiv \{\bar{\omega}\}$. We also say that an allocation rule $h_\omega : \mathbb{R}^{\Omega \times I} \to \Omega$ is allocatively (constrained-) efficient if a social choice function $h \equiv (h_\omega, h_t)$ is allocatively (constrained-) efficient.

As mentioned in the introduction, a new concept called commitment-proofness plays a crucial role in our possibility theorem (Theorem 2). Since it will need a careful explanation, we will defer the definition of commitment-proofness to subsection 1.4.2 where we begin to discuss the possibility of implementing efficient investments.

1.4 Implementation of Efficient Investments

1.4.1 Impossibility without Ex Post Investments

In the literature, it is often assumed that investments are made only before the mechanism. In such a situation, Rogerson (1992) and Hatfield et al. (2015) showed that we can find an efficient equilibrium of the investment game given allocatively efficient and strategy-proof social choice functions. But at the same time, another inefficient equilibrium exists in many examples. This is due to the fact that the *ex ante* investment stage incentivizes some agents to make more investments than at the efficient level and generates a multiplicity of equilibria. To see if this observation can be generalized, we consider the implementability of efficient investments without the post-mechanism investments in our model. For this purpose, we need to redefine the implementability of efficient investments for this environment accordingly.

When *ex post* investments are not allowed, the investment game induced by a social choice function is a one-shot game which takes place before the mechanism. Thus, the equilibrium concept we employ in the investment game reduces to a Bayesian Nash equilibrium in this case.

Definition 5. For any $V \subseteq \mathbb{R}^{\Omega \times I}$ and any profile of cost functions $c : V \times \Theta \to C$, a profile of investment strategies $\sigma^* \in \Sigma$ is a *Bayesian Nash equilibrium of the ex ante investment game* given a social choice function $h : \mathbb{R}^{\Omega \times I} \to \Omega \times \mathbb{R}^I$ and a discount factor $\beta \in (0, 1]$ if for each $i \in I$ and $\theta^i \in \Theta^i$,

$$\sigma^{*i}(\theta^i) \in \arg\max_{v^i \in V^i} \left\{ -c^i(v^i, \theta^i) + \beta \sum_{\theta^{-i} \in \Theta^{-i}} p(\theta^{-i}|\theta^i) \left[v^i(h_\omega(v^i, \sigma^{*-i}(\theta^{-i}))) - h^i_t(v^i, \sigma^{*-i}(\theta^{-i})) \right] \right\}$$

holds.

Implementability of efficient investments is redefined in the following way. In this environment, investment efficiency requires that the total expected utility of agents be maximized given that agents cannot revise their original choices of valuation functions after the mechanism.

Definition 6. Given a social choice function $h : \mathbb{R}^{\Omega \times I} \to \Omega \times \mathbb{R}^I$ and a discount factor $\beta \in (0, 1]$, *efficient ex ante investments are Bayesian Nash implementable* if for any $V \subseteq \mathbb{R}^{\Omega \times I}$ and any profile of cost functions $c : V \times \Theta \to C$, any Bayesian Nash equilibrium $\sigma^* \in \Sigma$ satisfies the following equation:

$$\sigma^* \in \arg\max_{\sigma \in \Sigma} \sum_{\theta \in \Theta} p(\theta) \sum_{i \in I} \Big\{ -c^i(\sigma^i(\theta^i), \theta^i) + \beta\sigma^i(\theta^i)(h_\omega(\sigma(\theta))) \Big\}.$$

The question is whether efficient *ex ante* investments are Bayesian Nash implementable given certain social choice functions. Unfortunately, the result is negative when we require allocative efficiency; for any allocatively efficient social choice function, there is a profile of the sets of valuations and cost functions under which there exists an inefficient equilibrium of the *ex ante* investment game.

Theorem 1. *Suppose $|I| \geq 2$ and $|\Omega| \geq 2$. Given any allocatively efficient social choice function $h : \mathbb{R}^{\Omega \times I} \to \Omega \times \mathbb{R}^I$ and any discount factor $\beta \in (0, 1]$, there exists $V \subseteq \mathbb{R}^{\Omega \times I}$ and a profile of cost functions $c : V \times \Theta \to C$ such that an inefficient Bayesian Nash equilibrium of the ex ante investment game exists, which means that efficient ex ante investments are not Bayesian Nash implementable.*

We show Theorem 1 by considering the following two cases: when the social choice function h is strategy-proof and when it is not. Here strategy-proofness plays a key role because *ex post* investments are not allowed and hence the model has the same structure as those considered by Rogerson (1992) and Hatfield et al. (2015). Therefore, there exists an efficient Bayesian Nash equilibrium of the *ex ante* investment game if h is strategy-proof, and there may not if it is not strategy-proof. In both cases, we construct an example where the cost functions are constant across any type profile $\theta \in \Theta$ of agents, so that the investment game is under complete information.

When h is not strategy-proof, the logic follows Theorem 1 and 2 of Hatfield et al. (2015) who show that for an allocatively efficient social choice function h, if agent i's *ex ante* choice of a valuation that maximizes her own utility always maximizes the social welfare given other agents' valuations, then h must be strategy-proof for i. Therefore, when it is not strategy-proof, we can construct a profile of cost functions under which, given other agents' valuations, the privately optimal *ex ante* investment choice for agent i does not achieve investment efficiency.[19]

On the other hand, for any strategy-proof social choice function, the logic of the English auction example in the previous section applies. Thus, we can always construct a case where an inefficient investment equilibrium exists in addition to the efficient one. This is because the *ex ante* investment stage gives commitment power to more than one agents although their cost functions are different. Once some agent makes a large investment, then other more efficient agents may refrain from making investments as it is costly to compete with them in the mechanism. Hence, the mechanism allows them to achieve a socially inefficient outcome in equilibrium. In the next subsection, we introduce *commitment-proofness* to eliminate such incentives when further investments are possible after the mechanism.

1.4.2 Commitment-proofness

The previous subsection demonstrated that inefficient equilibria cannot be ruled out if there are no *ex post* investment opportunities. In this chapter, we seek the possibility of implementation by allowing the *ex post* investment opportunities. When investments are possible both *ex ante* and *ex post*, there are two opposing forces which influence the implementability of efficient investments. The *ex post* investment stage helps to achieve it by allowing agents to reflect the information of their cost types onto the valuations at the time of the mechanism. As we saw in Theorem 1 however, the *ex ante* investment stage does the opposite by preventing us from extracting the information of their cost types. Which of these two forces dominates the other depends on the characteristics of the social choice function to be implemented. To answer this question, we introduce a new concept of a social choice function called *commitment-proofness*.

[19] Note that the construction of cost functions is slightly different from Hatfield et al. (2015) because the cost of investment in our model is non-negative whereas it is not assumed as such in their paper.

Definition 7. A social choice function $h : \mathbb{R}^{\Omega \times I} \to \Omega \times \mathbb{R}^I$ is *commitment-proof* if for any $i \in I$, $b \in \mathbb{R}^{\Omega \times I}$, $\tilde{b}^i \in \mathbb{R}^\Omega$ and $x \geq 0$ such that $\tilde{b}^i(\omega) \leq b^i(\omega) + x$ for all $\omega \in \Omega$,

$$\tilde{b}^i(h_\omega(\tilde{b}^i, b^{-i})) - h_t^i(\tilde{b}^i, b^{-i}) - x \leq b^i(h_\omega(b)) - h_t^i(b). \tag{1.2}$$

The concept of commitment-proofness involves a manipulation of an agent's true valuation through a certain commitment behavior given that the social choice function is implemented. Conceptually, this is distinct from a misreport of valuations when the social choice function is regarded as a direct mechanism, but there is indeed a close relationship with the strategy-proofness condition. This point will be demonstrated shortly. In equation (1.2), the non-negative value x can be interpreted as the cost of commitment. Consider a situation where each agent can make a contract with a third party that the agent will pay x to the third party in advance, and x or less will be paid back to the agent depending on the alternative chosen by the social choice function (the payback can be negative). We argue that the supposition of this situation (or other situations which bring about the same effect) is not demanding because the third party wouldn't lose anything from this contract. By making this agreement, the agent can commit to having a different valuation in the mechanism because the value from each realization of $\omega \in \Omega$ has been manipulated even though the genuine value of ω is unchanged. When agent i's original valuation function is b^i, her new valuation function given this contract will be $\tilde{b}^i$ which satisfies $\tilde{b}^i(\omega) \leq b^i(\omega) + x$ for all $\omega \in \Omega$. Equation (1.2) requires that no agent be able to benefit from such a commitment under h.

The following example gives a numerical illustration of a commitment $(\tilde{b}^i, x)$ for a given b^i, and shows how we detect commitment-proofness of social choice functions.

Example 1. Consider an auction with a single item and two bidders. Let $I = \{i, j\}$ and $\Omega = \{\omega^i, \omega^j\}$ where ω^i and ω^j each represent the alternatives where i and j obtain the item respectively. Suppose that the original valuation function (at the time of the mechanism) of agent i is $b^i : \Omega \to \mathbb{R}$ such that

$$b^i(\omega^i) = 10,$$
$$b^i(\omega^j) = 0.$$

Consider $x = 5$ and another valuation function (at the time of the mechanism) $\tilde{b}^i : \Omega \to \mathbb{R}$ such that

$$\tilde{b}^i(\omega^i) = 15,$$
$$\tilde{b}^i(\omega^j) = 0.$$

These x and $\tilde{b}^i$ satisfy the condition that $\tilde{b}^i(\omega) \leq b^i(\omega) + x$ for all $\omega \in \Omega$. Thus, $(\tilde{b}^i, x)$ is one of the commitments given b^i.

Suppose agent j's valuation function is fixed to $b^j : \Omega \to \mathbb{R}$ such that

$$b^j(\omega^i) = 0,$$
$$b^j(\omega^j) = 11.$$

Consider the following two social choice functions:[20]

1. The second-price auction h^{SPA} which gives the item to who values it most and has the winner pay the other agent's value, and
2. The half-price auction h^{half} which gives the item to who values it most and has the winner pay the half of her own value.

We examine whether the equation (1.2) holds for the example of valuation functions $b^i, \tilde{b}^i$ and b^j above.

[1] Under h^{SPA}, the RHS of equation (1.2) is 0 because agent i loses the auction. On the LHS, i wins the auction when her true valuation is $\tilde{b}^i$, and the utility from the auction is $\tilde{b}^i(h^{SPA}_{\omega}(\tilde{b}^i, b^j)) - h^{SPA,i}_{t}(\tilde{b}^i, b^j) = 15 - 11 = 4$. However, including the cost of commitment

[20]The social choice function should be defined for $\mathbb{R}^{\Omega \times I}$ in general in this chapter, but for this example we only consider the following domains, $B^i = \{a\mathbb{1}_{\{\omega=\omega^i\}} | a \in \mathbb{R}_+\}$ and $B^j = \{a\mathbb{1}_{\{\omega=\omega^j\}} | a \in \mathbb{R}_+\}$, to simplify the exposition of auction rules.

$x = 5$, we have

$$\tilde{b}^i(h_\omega^{SPA}(\tilde{b}^i, b^j)) - h_t^{SPA,i}(\tilde{b}^i, b^j) - x = -1 < 0 = b^i(h_\omega^{SPA}(b)) - h_t^{SPA,i}(b).$$

Thus, equation (1.2) holds for this example of valuation functions.[21]

[2] Under h^{half}, the RHS of equation (1.2) is again 0 for the same reason. On the LHS, i wins the auction when her true valuation is $\tilde{b}^i$, and the utility from the auction is $\tilde{b}^i(h_\omega^{half}(\tilde{b}^i, b^j)) - h_t^{half,i}(\tilde{b}^i, b^j) = 15 - 7.5 = 7.5$. Then, even with the cost of commitment $x = 5$, we have

$$\tilde{b}^i(h_\omega^{half}(\tilde{b}^i, b^j)) - h_t^{half,i}(\tilde{b}^i, b^j) - x = 2.5 > 0 = b^i(h_\omega^{half}(b)) - h_t^{half,i}(b).$$

Therefore, we know that the half-price auction h^{half} is not commitment-proof. □

Commitment-proofness is defined as a property of a social choice function and is not directly related to the structure of the investment game. Our main theorem establishes a strong connection between this concept and the implementability of efficient investments; commitment-proofness is sufficient and necessary for implementing efficient investments in PBNE. Intuitively, for any sets of valuation functions and cost functions, it will be shown that the cost of any costly *ex ante* investment corresponds to the cost of commitment (x) in the definition of commitment-proofness. Thus, no agent has the incentive to make a costly investment before the mechanism is run, and investment efficiency is achieved. As we will see in more detail in the next subsection, commitment-proofness works as a dividing ridge for understanding the interaction of two investment stages: (only) when commitment-proof social choice functions are implemented, the role of the *ex post* investment stage outweighs that of the *ex ante* investment stage.

As mentioned above, commitment-proofness has an interesting relationship with the more well-known strategy-proofness; any strategy-proof social choice function is commitment-proof. To see this, we first define strategy-proofness. A social choice function $h : \mathbb{R}^{\Omega \times I} \to$

[21] Indeed, it is shown that this holds for any other valuation functions concerned in the definition of commitment-proofness, and that the second-price auction is commitment-proof.

$\Omega \times \mathbb{R}^I$ is strategy-proof if for any $i \in I$, $b \in \mathbb{R}^{\Omega \times I}$ and $\tilde{b}^i \in \mathbb{R}^\Omega$,

$$b^i(h_\omega(\tilde{b}^i, b^{-i})) - h_t^i(\tilde{b}^i, b^{-i}) \leq b^i(h_\omega(b)) - h_t^i(b)$$

Showing that commitment-proofness is implied by strategy-proofness is straightforward: for any $i \in I$, $b \in \mathbb{R}^{\Omega \times I}$, $\tilde{b}^i \in \mathbb{R}^\Omega$ and $x \geq 0$ such that $\tilde{b}^i(\omega) \leq b^i(\omega) + x$ for all $\omega \in \Omega$,

$$\tilde{b}^i(h_\omega(\tilde{b}^i, b^{-i})) - h_t^i(\tilde{b}^i, b^{-i}) - x \leq b^i(h_\omega(\tilde{b}^i, b^{-i})) - h_t^i(\tilde{b}^i, b^{-i}) \leq b^i(h_\omega(b)) - h_t^i(b),$$

where the first inequality follows from the definition of $\tilde{b}^i$, and the second inequality holds from the strategy-proofness of h. Commitment-proofness concerns behaviors to manipulate the agents' true types outside the mechanism, rather than their misreports in the mechanism. Nonetheless, the fact that commitment-proofness is weaker than strategy-proofness implies that the consequence of commitments considered in this definition is translated into a type of misreports when the social choice function is regarded as a direct mechanism.

From this relationship, we know that the VCG auction, which is known to be strategy-proof, satisfies commitment-proofness. The VCG social choice function h^{VCG} is defined as follows: for any $b \in \mathbb{R}^{\Omega \times I}$,

$$\begin{aligned} h_\omega^{VCG}(b) &\in \arg\max_{\omega \in \Omega} \sum_{i \in I} b^i(\omega), \\ h_t^{VCG,i}(b) &= \max_{\omega \in \Omega} \sum_{j \in I \setminus \{i\}} b^j(\omega) - \sum_{j \in I \setminus \{i\}} b^j(h_\omega^{VCG}(b)) \text{ for any } i \in I. \end{aligned}$$

The second-price auction is a special case of the VCG auction, so it is also commitment-proof.

Since commitment-proofness is weaker than strategy-proofness, there exists a non-strategy-proof social choice function which is commitment-proof. Consider a class of social choice functions $h^\alpha : \mathbb{R}^{\Omega \times I} \to \Omega \times \mathbb{R}^I$ parameterized by $\alpha \in [0, 1)$ such that the alternative is efficiently chosen and the payment is a convex combination of the VCG payment and each agent's own valuation from the alternative itself: for any $b \in \mathbb{R}^{\Omega\times I}$,

$$h^{\alpha}_{\omega}(b) \in \arg\max_{\omega\in\Omega} \sum_{i\in I} b^i(\omega),$$
$$h^{\alpha,i}_{t}(b) = \alpha\Big\{ \max_{\omega\in\Omega} \sum_{j\in I\setminus\{i\}} b^j(\omega) - \sum_{j\in I\setminus\{i\}} b^j(h^{\alpha}_{\omega}(b)) \Big\} + (1-\alpha) b^i(h^{\alpha}_{\omega}(b)) \text{ for any } i \in I$$

for some $\alpha \in [0,1)$. This h^{α} is not strategy-proof because for some valuations of other agents, an agent will be strictly better off by decreasing her report of valuation without changing the alternative chosen by h^{α}. However, this is shown to be commitment-proof. The first part of the payment is exactly the VCG payment, and we know that the VCG social choice function satisfies equation (1.2). Regarding the second part of the payment, it is easy to see that for any $i \in I$, $b \in \mathbb{R}^{\Omega\times I}$, $\tilde{b}^i \in \mathbb{R}^{\Omega}$ and $x \geq 0$ such that $\tilde{b}^i(\omega) \leq b^i(\omega) + x$ for all $\omega \in \Omega$,

$$\tilde{b}^i(h_{\omega}(\tilde{b}^i, b^{-i})) - h^i_t(\tilde{b}^i, b^{-i}) - x = -x \leq 0 = b^i(h_{\omega}(b)) - h^i_t(b)$$

holds. Therefore, equation (1.2) is satisfied when the transfer rule is a convex combination of these two, and hence h^{α} is commitment-proof.

1.4.3 Possibility with Ex Ante and Ex Post Investments

Now we formally present the possibility theorem in our original model. In what follows, we demonstrate how commitment-proofness makes it possible to implement efficient investments when *ex post* investments are allowed.

First, for the purpose of the main theorem, we prove the following lemma.

Lemma 1. *For any agent $i \in I$, $V^i \subseteq \mathbb{R}^{\Omega\times I}$ and a cost function $c^i : V^i \times \Theta^i \to C^i$,*

$$c^i(v^i, \theta^i) \geq \max_{\omega\in\Omega} \left\{ b^{c^i,\theta^i,v^i}(\omega) - b^{c^i,\theta^i,v^{0i}}(\omega) \right\}$$

holds for any $\theta^i \in \Theta^i$, $v^i \in V^i$, and $v^{0i} \in V^i$ such that $c^i(v^{0i}) = 0$.

Proof: From the definition of the valuation at the time of the mechanism,

$$\begin{aligned} b^{c^i,\theta^i,v^{0i}}(\omega) &= \max_{\bar{v}^i \in V^i} \left\{ \bar{v}^i(\omega) - c^i(\bar{v}^i,\theta^i) \right\} \\ &\geq \max_{\bar{v}^i \in \{\bar{v}^i \in V^i | c^i(\bar{v}^i,\theta^i) \geq c^i(v^i,\theta^i)\}} \left\{ \bar{v}^i(\omega) - c^i(\bar{v}^i,\theta^i) \right\} \\ &= b^{c^i,\theta^i,v^i}(\omega) - c^i(v^i,\theta^i) \end{aligned}$$

holds for any $\omega \in \Omega$. Thus, we have $c^i(v^i,\theta^i) \geq \max_{\omega\in\Omega} \left\{ b^{c^i,\theta^i,v^i}(\omega) - b^{c^i,\theta^i,v^{0i}}(\omega) \right\}$. □

This lemma shows that the cost of changing the original valuation $b^{c^i,\theta^i,v^{0i}}$ with least costly *ex ante* investment v^{0i} to another valuation b^{c^i,θ^i,v^i} with some *ex ante* investment v^i is at least as large as the maximum element of the difference between $b^{c^i,\theta^i,v^{0i}}$ and b^{c^i,θ^i,v^i}. This is useful when we connect the definition of commitment-proofness to the structure of the investment game in the following theorem.

The next result is the main theorem of this chapter which identifies when efficient investments are implementable; for allocatively constrained-efficient social choice functions, commitment-proofness is sufficient and necessary for implementing efficient investments in PBNE for any discount factor $\beta \in (0,1)$.

Theorem 2. *Consider any I, Ω and any social choice function $h : \mathbb{R}^{\Omega\times I} \to \Omega \times \mathbb{R}^I$ which is allocatively constrained-efficient for some $\Omega' \subseteq \Omega$ with $\Omega' \neq \emptyset$. Given the social choice function h, efficient investments are implementable in PBNE for any discount factor $\beta \in (0,1)$ if and only if h is commitment-proof.*

The proof consists of the following two parts; (i) commitment-proofness of h as sufficient for implementing efficient investments, and (ii) it also being necessary. First, we characterize the set of PBNE when h is commitment-proof. We show that under commitment-proof social choice functions, no agent has the incentive to make a costly investment *ex ante* for any cost type. This is because the cost of any (costly) investment corresponds to x in the definition of commitment-proofness as shown in Lemma 1, and every agent i prefers to have the valuation $b^{c^i,\theta^i,v^{0i}}$ with least costly *ex ante* investment at the mechanism stage. And we show that any such PBNE maximizes the expected social welfare when the social choice function h is allocatively constrained-efficient. For the necessity part, we show that if h is not commitment-proof for agent i, there is a set of valuations and a profile of cost functions under which agent i has the incentive to make a costly investment *ex ante*, which is socially inefficient. Therefore, we conclude that only under commitment-proof social choice functions, the incentive for making a commitment through *ex ante* investment is completely suppressed by the presence of the *ex post* investment stage, and efficient investments are implemented.

Regarding the two distinct features of our main result that (i) inefficient investment equilibria are eliminated when (ii) post-mechanism investments are allowed, Piccione and Tan (1996) provided a closely related result in the literature. They analyze a procurement auction in which firms make R&D investments prior to the auction and the firm that wins the procurement contract exerts an additional effort to reduce costs. One of the main results of their paper is that the full-information solution (in which investments and alternative are efficient) can be uniquely implemented by the first-price and second-price auctions when the R&D technology exhibits decreasing returns to scale. Although the model is similar to ours, the focus of their theorem is different. Their result determines the structure of cost functions which enable unique implementation under those two common auction rules. On the other hand, we characterize the set of social choice functions for which efficient investments are implementable. Also, our cost functions allow any arbitrary heterogeneity among agents, which is not allowed in Piccione and Tan (1996), but we assume a certain relationship between *ex ante* and *ex post* cost functions. (See footnote 16.) Since we do not analyze the equilibrium of specific mechanisms such as the first-price auction, it would be an interesting direction to analyze such mechanisms and see how the result relates to Piccione and Tan (1996).

In the rest of the section, we provide two examples to show the importance of (i) β being strictly less than one and (ii) the allocative constrained-efficiency of h in Theorem 2.

First, a strict time discounting plays an important role. Although commitment-proofness implies implementability of efficient investments for any β which is arbitrarily close to one, it does not when β is exactly one. Intuitively, this is because when β is one, there are cases where the choice between investing *ex ante* and *ex post* is indifferent and there exists an equilibrium in which more than one agents chooses costly *ex ante* investments, which is socially inefficient. We provide an example where given $\beta = 1$ and a VCG social choice function (see subsection 1.4.2 for the definition), which is allocatively efficient and strategy-proof, efficient investments are not implementable in PBNE.

Observation 1. *Suppose $|I| \geq 2$, $|\Omega| \geq 2$. Given a VCG social choice function $h^{VCG} : \mathbb{R}^{\Omega \times I} \to \Omega \times \mathbb{R}^I$ and $\beta = 1$, efficient investments are not implementable in perfect Bayesian Nash equilibria.*

Example 2. Let $\{i, j\} \subseteq I$ and $\{\omega_1, \omega_2\} \subseteq \Omega$. Consider the following sets of valuations:

$$V^i = \{b^i, \tilde{b}^i\},$$

$$V^j = \{b^j, \tilde{b}^j\},$$

$$V^k = \{0\} \text{ for any } k \in I \setminus \{i, j\}$$

where

$$b^i(\omega_1) = b^j(\omega_1) = 5,\ b^i(\omega_2) = b^j(\omega_2) = 4,\ b^i(\omega) = b^j(\omega) = 0 \text{ for any } \omega \in \Omega \setminus \{\omega_1, \omega_2\}$$

$$\tilde{b}^i(\omega_1) = \tilde{b}^j(\omega_1) = 0,\ \tilde{b}^i(\omega_2) = \tilde{b}^j(\omega_2) = 6,\ \tilde{b}^i(\omega) = \tilde{b}^j(\omega) = 0 \text{ for any } \omega \in \Omega \setminus \{\omega_1, \omega_2\}.$$

Consider the following cost functions: for any $\theta \in \Theta$ with $p(\theta) > 0$,

$$c^i(b^i, \theta^i) = c^j(b^j, \theta^j) = 0,$$

$$c^i(\tilde{b}^i, \theta^i) = c^j(\tilde{b}^j, \theta^j) = 2,$$

$$c^k(0, \theta^k) = 0 \text{ for any } k \in I \setminus \{i, j\}.$$

Since the only choice of valuation is 0 for any $k \in I \setminus \{i, j\}$, we can ignore these agents. Given a VCG social choice function h^{VCG}, the most efficient investment schedules of agents i and j is $((b^i, b^j), (b^i, b^j))$. This is because it achieves the maximum social welfare $\beta(5+5) = \beta 10 = 10$ as h^{VCG} chooses ω_1 for (b^i, b^j), and the cost of (b^i, b^j) is zero for any $\theta \in \Theta$ which occurs with a positive probability.

Next, consider an investment strategy $(\tilde{\sigma}^l, \mu^l) \in \Sigma^l \times \mathcal{M}^l$ for each agent $l = i, j$ where

$\tilde{\sigma}^l(\theta^l) = \tilde{b}^l$ and μ^l is the optimal *ex post* investment strategy. First, because $c^l(\tilde{b}^l, \theta^l) > c^l(b^l, \theta^l)$ for each agent $l = i, j$,

$$\mu^l(\tilde{b}^l, \omega, \theta^l) = \tilde{b}^l$$

holds for any $\omega \in \Omega$ and $\theta^l \in \Theta^l$. Thus, for the *ex ante* investment strategy $\tilde{\sigma}^l(\theta^l) = \tilde{b}^l$, the valuation at the time of the mechanism is $\tilde{b}^l$.

Suppose that agent j takes this investment strategy $(\tilde{\sigma}^j, \mu^j) \in \Sigma^j \times \mathcal{M}^j$, and consider agent i's incentive. When she chooses b^i in the first stage, since $b^i(\omega) \geq \tilde{b}^i(\omega) - c^i(\tilde{b}^i)$ holds for any $\omega \in \Omega$, the valuation at the time of the mechanism is

$$b^{c^i, b^i}(\omega) = \max_{\bar{v}^i \in \{b^i, \tilde{b}^i\}} \left\{ \bar{v}^i(\omega) - c^i(\bar{v}^i) \right\} = b^i(\omega)$$

for each $\omega \in \Omega$. In this case, the outcome of the social choice function should be

$$h^{VCG}_{\omega}(b^i, \tilde{b}^j, 0) = \omega_2, \text{ and}$$
$$h^{VCG,i}_{t}(b^i, \tilde{b}^j, 0) = 0.$$

The total utility of agent i would be $4\beta = 4$. On the other hand, when she chooses $\tilde{b}^j$ in the first stage, the outcome of the social choice function will be

$$h^{VCG}_{\omega}(\tilde{b}^i, \tilde{b}^j, 0) = \omega_2, \text{ and}$$
$$h^{VCG,i}_{t}(\tilde{b}^i, \tilde{b}^j, 0) = 0.$$

The total utility of agent i would be $6\beta - 2 = 4$. Since these choices are indifferent, choosing $\tilde{b}^i$ in the first stage can be a best response for agent i. Therefore, the same logic applies to agent j, and $\{(\tilde{\sigma}^l, \mu^l) \in \Sigma^l \times \mathcal{M}^l\}_{l=i,j}$ constitutes a PBNE of the investment game. But $\{(\tilde{\sigma}^l, \mu^l) \in \Sigma^l \times \mathcal{M}^l\}_{l=i,j}$ gives the social welfare of 8, which is less than that of $\{(\sigma^l, \mu^l) \in \Sigma^l \times \mathcal{M}^l\}_{l=i,j}$ such that $\sigma^l(\theta^l) = b^l$ for any $\theta^l \in \Theta^l$, which gives the social welfare of 10. Thus, efficient investments are not implementable in PBNE given h^{VCG} and $\beta = 1$. □

As a second observation, the sufficiency of commitment-proofness in Theorem 2 no longer holds if the social choice function is not allocatively constrained-efficient for any $\Omega' \subseteq \Omega$. The next example demonstrates that efficient investments are not implementable given a strategy-proof (and hence, commitment-proof) social choice function which is not allocatively constrained-efficient.

Observation 2. *Suppose* $|I| \geq 2$ *and* $|\Omega| \geq 2$*. There is a strategy-proof social choice function* $h : \mathbb{R}^{\Omega \times I} \to \Omega \times \mathbb{R}^I$ *which is not allocatively constrained-efficient for any* $\Omega' \subseteq \Omega$ *such that efficient investments are not implementable in perfect Bayesian Nash equilibria given* h *and some* $\beta \in (0, 1)$*.*

Example 3. Let $\{i, j\} \subseteq I$ and $\{\omega_1, \omega_2\} \subseteq \Omega$. Consider a social choice function $h : \mathbb{R}^{\Omega \times I} \to \Omega \times \mathbb{R}^I$ such that for any $b \in \mathbb{R}^{\Omega \times I}$,

$$h_\omega(b) \in \arg\max_{\omega \in \Omega} \left\{ b^i(\omega) \right\},$$
$$h_t^k(b) = 0 \text{ for any } k \in I.$$

This means that the best alternative for agent i is always chosen and no transfer is made under h. This h is strategy-proof because i does not have the incentive to manipulate her type and j's report does not affect the outcome. It is clear that h is not allocatively constrained-efficient because other agents' valuations are not taken into account. Consider the following sets of valuations:

$$V^i = \{b^i, \tilde{b}^i\},$$
$$V^j = \{b^j\},$$
$$V^k = \{0\} \text{ for any } k \in I \setminus \{i, j\}$$

where

$$b^i(\omega_1) = 5,\ b^i(\omega_2) = 4,\ b^i(\omega) = 0 \text{ for any } \omega \in \Omega \setminus \{\omega_1, \omega_2\}$$
$$\tilde{b}^i(\omega_1) = 5,\ \tilde{b}^i(\omega_2) = 6,\ \tilde{b}^i(\omega) = 0 \text{ for any } \omega \in \Omega \setminus \{\omega_1, \omega_2\}$$
$$b^j(\omega_1) = 0,\ b^j(\omega_2) = 3,\ b^j(\omega) = 0 \text{ for any } \omega \in \Omega \setminus \{\omega_1, \omega_2\}.$$

Also consider the following cost functions: for any $\theta \in \Theta$ with $p(\theta) > 0$,

$$c^i(b^i, \theta^i) = 0,\ c^i(\tilde{b}^i, \theta^i) = 3,$$
$$c^j(b^j, \theta^j) = 0,$$
$$c^k(0, \theta^k) = 0 \text{ for any } k \in I \setminus \{i, j\}.$$

Since the only choice of valuation is 0 for any $k \in I \setminus \{i, j\}$, we can ignore these agents. For j, the only choice of valuation is b^j.

Consider the optimal choice for agent i in the second investment stage for any $\theta^i \in \Theta^i$ which occurs with a positive probability. If i chooses b^i before the mechanism, since $b^i(\omega) > \tilde{b}^i(\omega) - c^i(\tilde{b}^i, \theta^i)$ holds for any $\omega \in \Omega$, her optimal valuation after the mechanism is b^i. If i chooses $\tilde{b}^i$ before the mechanism, then the only valuation she can choose after the mechanism is $\tilde{b}^i$ because $c^i(\tilde{b}^i, \theta^i) > c^i(b^i, \theta^i)$. In either case, when the same valuation is taken *ex ante* and *ex post*, the valuation at the time of the mechanism is also that valuation. To summarize, agent i's optimal *ex post* investment strategy and the valuation at the time of the mechanism is as follows:

Ex Ante Valuation	Valuation at the Mechanism	Optimal *Ex Post* Valuation
b^i	b^i	ω_1: b^i ω_2: b^i
$\tilde{b}^i$	$\tilde{b}^i$	ω_1: $\tilde{b}^i$ ω_2: $\tilde{b}^i$

Thus, we can compare two investment choices b^i and $\tilde{b}^i$ of agent i in the first stage to analyze the investment efficiency and the equilibrium.

First, we show that $\tilde{b}^i$ gives higher social welfare than b^i for sufficiently large $\beta \in (0, 1)$. Given j's valuation b^j, the social welfare when i chooses $\tilde{b}^i$ is

$$-3 + \beta(6 + 3) = 9\beta - 3.$$

The social welfare when i chooses b^i is

$$0 + \beta(5 + 0) = 5\beta.$$

Since the former is larger for $\beta > \frac{3}{4}$, choosing $\tilde{b}^i$ is socially efficient, and choosing b^i is not for such β.

Next, consider the incentive of agent i. Given j's valuation b^j, compare the utility of i when she chooses $\tilde{b}^i$ and b^i in the first stage. When i chooses $\tilde{b}^i$, her utility is $6\beta - 3$ whereas it is 5β when i chooses b^i. Since

$$6\beta - 3 < 5\beta \text{ for any } \beta \in (0, 1),$$

agent i chooses b^i in a PBNE. Thus, agent i chooses b^i in a PBNE of the investment game, but it does not maximize the social welfare for $\beta > \frac{3}{4}$. Therefore, efficient investments are not implementable in PBNE given h and such β. □

1.5 Provision of Public Goods

In this section, we consider a variant of the original problem; providing public goods through the finances of agents. The provision of public goods is represented by a choice of an alternative $\omega \in \Omega$ in our model. We still assume perfectly transferable utility and allow for transfers $(t^i)_{i \in I}$ in the mechanism. The only difference from the original model is that we require a budget balance for social choice functions, i.e., the sum of the transfers must be equal to zero.

Definition 8. A social choice function h is *budget-balanced* if

$$\sum_{i \in I} h_t^i(b) = 0$$

for any $b \in \mathbb{R}^{\Omega \times I}$.

Budget balance is considered to be part of allocative efficiency because the transfer collected by the mechanism designer is regarded as the loss of welfare in this problem. In this environment, it is known that there is no social choice function that is strategy-proof, allocatively efficient and budget-balanced (Green and Laffont, 1977; Hölmstrom, 1979; Walker, 1980). Therefore, when there is only an *ex ante* investment stage, it is impossible to even ensure the existence of efficient investment equilibria if we require budget balance and allocatively efficiency of the social choice function (Hatfield et al., 2015). However, we can show that commitment-proofness is compatible with these two properties; there is a social choice function which is commitment-proof, allocatively efficient and budget-balanced.

Proposition 1. *For any I, Ω and an efficient allocation rule $h_\omega : \mathbb{R}^{\Omega\times I} \to \Omega$, there exists a transfer rule $h_t : \mathbb{R}^{\Omega\times I} \to \mathbb{R}^I$ with which $h = (h_\omega, h_t)$ is commitment-proof and budget-balanced.*

Proposition 1 is shown by proposing a specific transfer rule h_t: for any agent $i \in I$, h_t^i is defined by

$$h_t^i(b) = b^i(h_\omega(b)) - \frac{1}{n}\sum_{i\in I} b^i(h_\omega(b)).$$

By this transfer rule, the maximized social welfare is equally divided to all agents. Consider the definition of commitment-proofness. Under this transfer rule, the value $\tilde{b}^i(h_\omega(\tilde{b}^i, b^{-i})) - h_t^i(\tilde{b}^i, b^{-i})$ from the social choice function h under type $\tilde{b}^i$ increases from the original value $b^i(h_\omega(b)) - h_t^i(b)$ under type b^i by only $\frac{1}{n}$ of the increment of the social welfare. On the other hand, since x satisfies $x \geq \max_{\omega\in\Omega}\{\tilde{b}^i(\omega) - b^i(\omega)\}$, x should be larger than the increment of social welfare. Therefore, the equation of commitment-proofness is satisfied under this transfer rule. It is easy to see that this h is not strategy-proof because agents have the incentive to underreport their valuations to reduce the payment.

By the result of Theorem 2, we obtain the following corollary; with the *ex post* investments, budget balance does not preclude the implementation of efficient investments.

Corollary 1. *For any I and Ω, there exists an allocatively efficient and budget-balanced social choice function $h : \mathbb{R}^{\Omega\times I} \to \Omega \times \mathbb{R}^I$ such that efficient investments are implementable in perfect Bayesian Nash equilibria given h and any discount factor $\beta \in (0, 1)$.*

1.6 Concluding Remarks

Our main result shows that allowing for *ex post* investments, commitment-proofness is equivalent to the implementability of efficient investments for allocatively efficient social choice functions. This has the following two implications. First, whenever it is possible, the mechanism should be run sufficiently before the actual production or consumption is carried out. This allows agents to reflect the information of their cost types onto their valuations at the mechanism stage through the optimal behavior in the *ex post* investment stage. Otherwise, according to Theorem 1, we cannot eliminate the possibility of inefficient equilibria. Second, commitment-proofness of the mechanism is essential. This ensures that no agent has the incentive to commit to having a different valuation in the mechanism by making prior investments. Moreover, this is not a restrictive concept since it is much weaker than the strategy-proofness condition.

In this chapter, we allow for incomplete information about the cost types of other agents, but we assume that every agent knows her own cost function and it is unchanged over time. This assumption allows us to characterize the set of PBNE, in which no agent makes a costly *ex ante* investment. Although out result still holds for some systematic changes of cost functions after the mechanism (see footnote 16), we do not know what will happen if the *ex post* cost function is uncertain *ex ante*. When agents are unsure about thier own *ex post* cost functions, they may need to make some investments *ex ante* to improve their own *ex post* cost functions. This uncertain investment model will be related to Piccione and Tan (1996) and other papers on information acquisition (Bergemann and Välimäki, 2002; Obara, 2008). Under this uncertain investment setting, we hope to obtain conditions on social choice functions or cost structures which make the implementation of investment efficiency possible.

Chapter 2

Auctions That Implement Efficient Investments

2.1 Introduction

In the literature of auction theory, a number of papers have examined how the auction mechanism affects investment incentives (Tan, 1992; Piccione and Tan, 1996; Arozamena and Cantillon, 2004). However, there has not been enough research about when we can fully implement efficient investments, i.e., when investments are socially efficient in *every* equilibrium. Chapter 1 provided a general result to this question: a new concept *commitment-proofness* is sufficient and necessary for fully implementing efficient investments when *ex post* investment is possible and the allocation is efficiently determined given *ex ante* investments. The first limitation of the analysis in Chapter 1 is that only allocatively efficient social choice functions are considered. Therefore, we do not know the consequences of mechanisms that may be allocatively inefficient such as an asymmetric first-price auction. Second, we did not allow for uncertain investments in Chapter 1, which may add more complication to the analysis.

In this article, we analyze the implementability of efficient investments for two commonly used auction mechanisms: the first-price auction and the English auction, allowing for uncertain investments. We consider single-item and private-value auctions throughout this chapter. Investment is modeled as a choice of (a distribution of) private valuations of the item. The timeline of the game is as follows: each agent simultaneously chooses a distribution over the potential valuations of the item before the auction. For simplicity, we assume that the cost types are complete information among agents. They participate in the auction with the knowledge of their own realized valuations and the distribution of other agents' valuations. After the auction, agents may make further deterministic investments. In this setup, we show two positive theorems: the first-price auction implements efficient investments in any perfect Bayesian Nash equilibrium, and the English auction implements efficient investments in any trembling-hand perfect equilibrium. Our model allows any *ex ante* heterogeneity in the cost structure of investments.

To obtain the result on the English auction, we can simply extend Theorem 2 in Chapter 1 to uncertain investments since the English auction is allocatively efficient and commitment-proof. For the first-price auction, however, we also need to make use of the equilibrium characterization results of Maskin and Riley (2000) and Maskin and Riley (2003) to analyze the equilibrium of investments.

There are three important assumptions for these results. First, once the mechanism designer commits to the auction mechanism, the timing of investment is flexible; investment can be made before or after the mechanism is run. Indeed, the first-price auction and the English auction implement efficient investments because all the losers refrain from making *ex ante* investments in equilibrium. Contrary to this result, Arozamena and Cantillon (2004) show that the first-price auction induces less investment than the second-price auction. This discrepancy comes from the fact that they only consider one firm's incentive and moreover, they do not allow *ex post* investments. Secondly, we assume that the cost of deterministic investment does not change before and after the auction. Under uncertainty, the cost of a distribution of investments is modeled as the expected cost of realized investments. Lastly, our model assumes that investments are observable as opposed to the models of Tan (1992), Piccione and Tan (1996) and Bag (1997). Overall, the combination of these assumptions gives a natural environment in which both the first-price auction and the English auction fully implement efficient investments.

The rest of Chapter 2 is organized as follows. Section 2.2 introduces the model. Investment incentive under the first-price auction is analyzed in Section 2.3, and we study the case of the English auction in Section 2.4. Section 2.5 concludes.

2.2 Model

We analyze investment incentives in single-item auctions with private values. Consider any finite set $I \equiv \{1, 2, ..., n\}$ of agents. The set Ω of alternatives is defined as $\Omega \equiv \{\omega_i\}_{i \in I}$. For each agent $i \in I$, the realization of ω_i means that i obtains the item. The valuation a_i of the item for each $i \in I$ is in an interval $[0, \alpha]$ where $\alpha \in \mathbb{R}_+$.

Each agent makes an investment decision to determine the distribution of valuations prior to the mechanism. They can also make a further investment to increase the valuation *ex post*. Investments do not have externalities, i.e., they only alter their own private valuations. Let $\mathcal{F}_i$ be the set of cumulative distribution functions of valuations which can be chosen as an *ex ante* investment by agent $i \in I$. We assume that $\mathcal{F}_i$ includes any cumulative distribution function F_i of a_i that satisfies the following conditions: (i) the support of F_i is in $[0, \alpha]$, (ii) F_i is continuously differentiable, and (iii) F_i's derivative is strictly positive on its support.

The costs of uncertain and deterministic investments are defined in the following way. First, the cost of deterministic investment $a_i \in [0, \alpha]$ for each agent $i \in I$ is given by a cost function $c_i : [0, \alpha] \to \mathbb{R}_+$. The cost function $c_i(\cdot)$ is assumed to be strictly increasing, continuously differentiable, and $c_i(0) = 0$. If agent i increases its valuation from a_i to $\tilde{a}_i$ such that $a_i, \tilde{a}_i \in [0, \alpha]$ and $\tilde{a}_i \geq a_i$, then the additional cost is $c_i(\tilde{a}_i) - c_i(a_i)$. The cost of each uncertain investment $F_i \in \mathcal{F}_i$ is defined as the expected cost of valuations that are realized from F_i. In equation, it is denoted by a function $\gamma_i^{c_i} : \mathcal{F}_i \to \mathbb{R}_+$ and defined as

$$\gamma_i^{c_i}(F_i) \equiv \int_0^\alpha c_i(a) dF_i(a).$$

Although investments can be uncertain, we assume that the cost functions $\{c_i(\cdot)\}_{i\in I}$ are complete information among agents.

We assume that the investment is irreversible, i.e., if the valuation realized before the auction is $a_i \in [0, \alpha]$, then she can only choose a new valuation from $[a_i, \alpha]$ after the auction. The timeline of the investment game and the informational assumption are summarized as follows:

1. Each agent $i \in I$ simultaneously chooses a distribution $F_i \in \mathcal{F}_i$. The valuation $a_i \in [0, \alpha]$ of the item for i is drawn from F_i.

2. Agents participate in the auction. Their own valuations and the distributions of other agents' valuations are common knowledge.

3. After the auction, each agent may make an additional investment, i.e., they again choose a valuation $\tilde{a}_i$ from $[a_i, \alpha]$.

For simplicity, we assume that $\max_{a\in[0,\alpha]}\{a - c_i(a)\} \neq \max_{a\in[0,\alpha]}\{a - c_j(a)\}$ holds for any $i, j \in I$.

2.3 First-Price Auction

In the first-price auction, each agent submits a non-negative sealed bid $b_i \in \mathbb{R}_+$. The bidder with the highest bid wins and pays their own bid. If two or more bids tie, we use the *Vickrey tie-breaking rule*, in which each agent submits a non-negative sealed tie-breaker $t_i \in \mathbb{R}_+$. If more than one bidders tie with a bid b, the bidder i with the highest tie-breaker among them wins, and they pay $b + \max_{j\in I\setminus\{i\}}\{t_j | b_j = b\}$. If there is a tie for the highest tie-breaker, we will randomize among those who make this bid with equal probability.

For each agent $i \in I$, a strategy in the first-price auction with an investment game is defined by $(F_i, (\beta_i^F)_{F\in\mathcal{F}}, \tilde{a}_i)$. $F_i \in \mathcal{F}_i$ is the choice of *ex ante* investment. $\beta_i^F : [\underline{a}_i, \bar{a}_i] \to \Delta(\mathbb{R}_+^2)$ is the (mixed) bidding strategy (which also specifies the tie-breaker) for each $F \in \mathcal{F}$ where the support of F_i is $[\underline{a}_i, \bar{a}_i] \subseteq [0, \alpha]$. $\tilde{a}_i : [0, \alpha] \times \Omega \to [0, \alpha]$ is the *ex post* investment strategy which satisfies $\tilde{a}_i(a_i, \omega) \in [a_i, \alpha]$ for any $a_i \in [0, \alpha]$ and $\omega \in \Omega$. Note that the domain of the *ex post* investment strategy is $[0, \alpha] \times \Omega$ since it only depends on the *ex ante* valuation and the allocation of the item (not the information of submitted bids). For any profile of cost functions $c : [0, \alpha]^n \to \mathbb{R}^n_+$ and any profile of strategies $(F, (\beta^F)_{F\in\mathcal{F}}, \tilde{a})$, the interim utility $u_i^{FPA}(F, a_i, (\beta^F)_{F\in\mathcal{F}}, \tilde{a})$ for each realized type a_i is defined as

$$u_i^{FPA}(F, a_i, (\beta^F)_{F\in\mathcal{F}}, \tilde{a}) \equiv \int_0^\alpha ... \int_0^\alpha \int_{\mathbb{R}^2_+} ... \int_{\mathbb{R}^2_+} \Big[\big(\tilde{a}_i(a_i, \omega_i) - b_i\big) \mathbb{1}_{\{i \in \arg\max_{j\in I}\{b_j\}\}} \mathbb{1}_{\{|\arg\max_{j\in I}\{b_j\}|=1\}}$$
$$+ \big(\tilde{a}_i(a_i, \omega_i) - b_i - \max_{j\in I\setminus\{i\}} \{t_j | b_j = b_i\}\big) \sum_{k=1}^{n} \frac{1}{k} \mathbb{1}_{\{i\in\arg\max_{j\in I}\{b_j\}\}} \mathbb{1}_{\{i\in\arg\max_{j\in I}\{t_j|b_j=b_i\}\}} \mathbb{1}_{\{k=|\arg\max_{j\in I}\{t_j|b_j=b_i\}|\}}$$
$$- \big\{c_i(\tilde{a}_i(a_i, \omega_i)) - c_i(a_i)\big\}\Big] \big(\beta_1^F(a_1)\big)(b_1, t_1) d(b_1, d_1) ... \big(\beta_n^F(a_n)\big)(b_n, t_n) d(b_n, t_n)$$
$$dF_1(a_1) ... dF_{i-1}(a_{i-1}) dF_{i+1}(a_{i+1}) ... dF_n(a_n).$$

The interim utility represents each agent's expected payoff for each realized type a_i in the auction stage. The *ex ante* utility $U_i^{FPA}(F, (\beta^F)_{F\in\mathcal{F}}, \tilde{a})$ is defined by taking the expectation of the interim utility with respect to a_i and adding the cost of *ex ante* investment:

$$U_i^{FPA}(F, (\beta^F)_{F\in\mathcal{F}}, \tilde{a}) \equiv -\gamma_i^{c_i}(F_i) + \int_0^\alpha u_i^{FPA}(F, a_i, (\beta^F)_{F\in\mathcal{F}}, \tilde{a}) dF_i(a_i).$$

We define a perfect Bayesian Nash equilibrium of the first-price auction with an investment game.

Definition 9. Given a profile of cost functions $c : [0, \alpha]^n \to \mathbb{R}^n_+$, a *perfect Bayesian Nash equilibrium* of the first-price auction with an investment game is a profile of strategies $(F, (\beta^F)_{F\in\mathcal{F}}, \tilde{a})$ that satisfies the following conditions:

1. For each agent $i \in I$, the *ex post* investment strategy $\tilde{a}_i : [0, \alpha] \times \Omega \to [0, \alpha]$ satisfies

$$\tilde{a}_i(a_i, \omega_i) \in \arg\max_{a\in[a_i,\alpha]} \Big[a - c_i(a)\Big] \text{ and}$$
$$\tilde{a}_i(a_i, \omega) = a_i \text{ for any } \omega \neq \omega_i$$

   for each original valuation $a_i \in [0, \alpha]$.

2. For each agent $i \in I$ and any profile of cumulative distribution functions $F \in \mathcal{F}$, a

bidding strategy $\beta_i^F : [\underline{a}_i, \bar{a}_i] \to \Delta(\mathbb{R}_+^2)$ in the first-price auction satisfies

$$\beta_i^F \in \underset{\tilde{\beta}_i : [\underline{a}_i, \bar{a}_i] \to \Delta(\mathbb{R}_+^2)}{\arg\max} \; u_i^{FPA}(F, a_i, (\tilde{\beta}_i, \beta_{-i}^F), \tilde{a})$$

for any $a_i \in [\underline{a}_i, \bar{a}_i]$ given other agents' bidding strategies β_{-i}^F.

3. For each agent $i \in I$, the choice of cumulative distribution function $F_i \in \mathcal{F}_i$ satisfies

$$F_i \in \underset{\tilde{F}_i \in \mathcal{F}_i}{\arg\max} \; U_i^{FPA}((\tilde{F}_i, F_{-i}), (\beta^F)_{F \in \mathcal{F}}, \tilde{a})$$

given other agents' choices $F_{-i} \in \mathcal{F}_{-i}$.

The first condition says that in the optimal *ex post* investment strategy, each agent maximizes the value of the item only when they obtain it. The second condition ensures that for any choices of *ex ante* investments $F \in \mathcal{F}$, a profile of bidding strategies β^F in the first-price auction constitutes a Bayesian Nash equilibrium given the optimal *ex post* strategy $\tilde{a}$. The third condition requires that the choices of cumulative distribution functions F constitute a Nash equilibrium in the *ex ante* investment stage given the equilibrium bidding strategies $(\beta^F)_{F \in \mathcal{F}}$ and the optimal *ex post* strategy $\tilde{a}$.

Although the belief system does not appear in the definition, the above conditions are consistent with the standard definition of a perfect Bayesian Nash equilibrium. In the *ex post* investment stage, the optimal strategy does not depend on the belief on the other agents' types in any information set. In the auction stage, since we assume that *ex ante* investments are observable, the standard definition of a perfect Bayesian Nash equilibrium requires every agent have a correct belief on other agents' types in any information set. This is represented by the second condition of Definition 9.

We say that the first-price auction with an investment game achieves full efficiency in a perfect Bayesian Nash equilibrium if the equilibrium achieves efficiency in both allocations and investments.

Definition 10. Given a profile of cost functions $c : [0, \alpha]^n \to \mathbb{R}_+^n$, *the first-price auction with an investment game achieves full efficiency in a perfect Bayesian Nash equilibrium* $(F, (\beta^F)_{F \in \mathcal{F}}, \tilde{a})$

if the following conditions are satisfied:

1. Allocative Efficiency: For any a in the support of F, if

$$\int_{\mathbb{R}^2_+} \cdots \int_{\mathbb{R}^2_+} \mathbb{1}_{\{i \in \arg\max_{j \in I}\{b_j\}\}} \mathbb{1}_{\{i \in \arg\max_{j \in I}\{t_j | b_j = b_i\}\}} \big(\beta_1^F(a_1)\big)(b_1, t_1) d(b_1, d_1) \ldots \big(\beta_n^F(a_n)\big)(b_n, t_n) d(b_n, t_n) > 0,$$

then

$$i \in \arg\max_{j \in I} \Big\{ \tilde{a}_j(a_j, \omega_j) - \big[c_j(\tilde{a}_j(a_j, \omega_j)) - c_j(a_j) \big] \Big\}.$$

2. Investment Efficiency:

$$F \in \arg\max_{\tilde{F} \in \mathcal{F}} \Big[-\sum_{i \in I} \gamma_i^{c_i}(\tilde{F}_i) + \int_0^\alpha \ldots \int_0^\alpha \max_{i \in I} \big[\tilde{a}_i(a_i, \omega_i) - \{ c_i(\tilde{a}_i(a_i, \omega_i)) - c_i(a_i) \} \big] d\tilde{F}_1(a_1) \ldots d\tilde{F}_n(a_n) \Big].$$

The first condition ensures the allocative efficiency on the equilibrium path: for any realization of valuations a, the item is allocated to agents who have the highest valuation taking account of the *ex post* investments. The second condition is about the efficiency of *ex ante* investments: the choices of *ex ante* investments maximize the social welfare given that the allocative efficiency is achieved. We do not need to require conditions on the *ex post* investment strategy $\tilde{a}$ because it is always socially optimal in equilibrium.

Our first result shows that the first-price auction with an investment game always achieves full efficiency in any perfect Bayesian Nash equilibrium.

Theorem 3. *For any profile of cost functions $c : [0, \alpha]^n \to \mathbb{R}^n_+$, the first-price auction with an investment game achieves full efficiency in any perfect Bayesian Nash equilibrium.*

Proof. Take any profile of cost functions $c : [0, \alpha]^n \to \mathbb{R}^n_+$. Let $b^{c_i}(a_i)$ be the valuation of the item in the auction stage when a_i is drawn *ex ante*. Since the *ex ante* cost is sunk and the *ex post* investment strategy is always optimal, it is calculated as

$$b^{c_i}(a_i) = \max_{a \in [a_i, \alpha]} \Big\{ a - \big(c_i(a) - c_i(a_i) \big) \Big\} = \max_{a \in [a_i, \alpha]} \Big\{ a - c_i(a) \Big\} + c_i(a_i).$$

First, show that $b^{c_i}(\cdot)$ is strictly increasing. Take any $a_i, \hat{a}_i \in [0, \alpha]$ such that $a_i < \hat{a}_i$. If $\max_{a \in [a_i, \alpha]}\{a - c_i(a)\} = \max_{a \in [\hat{a}_i, \alpha]}\{a - c_i(a)\}$ holds, it is clear that we have $b^{c_i}(a_i) < b^{c_i}(\hat{a}_i)$ because $c_i(\cdot)$ is strictly increasing. If $\max_{a \in [a_i, \alpha]}\{a - c_i(a)\} > \max_{a \in [\hat{a}_i, \alpha]}\{a - c_i(a)\}$ holds, we can define $a_i^* \equiv \arg\max_{a \in [a_i, \alpha]}\{a - c_i(a)\}$ and this should satisfy $a_i \le a_i^* < \hat{a}_i$. Thus,

$$\max_{a \in [a_i, \alpha]} \Big\{a - c_i(a)\Big\} + c_i(a_i) = a_i^* - c_i(a_i^*) + c_i(a_i) \le a_i^* < \hat{a}_i \le \max_{a \in [\hat{a}_i, \alpha]} \Big\{a - c_i(a)\Big\} + c_i(\hat{a}_i),$$

and $b^{c_i}(a_i) < b^{c_i}(\hat{a}_i)$ still holds.

Let $G_i(\cdot)$ be the c.d.f. of the valuation in the auction stage, which is defined by $G_i(b) = F_i\big((b^{c_i})^{-1}(b)\big)$ for any $b \in [b^{c_i}(0), \alpha]$. This is well-defined because $b^{c_i}(a_i)$ is strictly increasing in a_i.

Since we assume $b^{c_k}(0) \neq b^{c_l}(0)$ for any $k, l \in I$, let $i \in I$ be the unique agent who has the highest net maximum value of the item, i.e., $i \equiv \arg\max_{j \in I}\{b^{c_j}(0)\}$.

[1] agent j such that $j \neq i$: By the construction of $G_k(\cdot)$ for each $k \in I$, the minimum value of the support of $G_k(\cdot)$ is at least $b^{c_k}(0)$. Therefore, since $G_k(\cdot)$ is continuous, by the logic of Proposition 3 of Maskin and Riley (2000) and Lemma 3 of Maskin and Riley (2003), the minimum value of the winning bid is at least $b^{c_i}(0)$ for any choice of *ex ante* distributions by other agents. Thus, for any equilibrium strategy $(\beta^F)_{F \in \mathcal{F}}$ and any $F_{-j} \in \mathcal{F}_{-j}$, the *ex ante* utility of agent j from choosing $\tilde{F}_j \in \mathcal{F}_j$ such that $\gamma_j^{c_j}(\tilde{F}_j) > 0$ is

$$\begin{aligned}
& U_j^{FPA}((\tilde{F}_j, F_{-j}), (\beta^F)_{F \in \mathcal{F}}, \tilde{a}) \\
\le \;& -\gamma_j^{c_j}(\tilde{F}_j) + \int_0^\alpha \big[b^{c_j}(a_j) - b^{c_i}(0)\big]\text{Prob}\{j \text{ wins the auction}|a_j\} d\tilde{F}_j(a_j) \\
= \;& -\int_0^\alpha c_j(a_j)\text{Prob}\{j \text{ loses the auction}|a_j\} d\tilde{F}_j(a_j) \\
& + \int_0^\alpha \big[b^{c_j}(a_j) - c_j(a_j) - b^{c_i}(0)\big]\text{Prob}\{j \text{ wins the auction}|a_j\} d\tilde{F}_j(a_j) \\
\le \;& -\int_0^\alpha c_j(a_j)\text{Prob}\{j \text{ loses the auction}|a_j\} d\tilde{F}_j(a_j) \\
& + \int_0^\alpha \big[b^{c_j}(0) - b^{c_i}(0)\big]\text{Prob}\{j \text{ wins the auction}|a_j\} d\tilde{F}_j(a_j) \\
< \;& 0.
\end{aligned}$$

The strict inequality holds because $c_j(a_j) > 0$ for any $a_j \in (0, \alpha]$ and $b^{c_i}(0) > b^{c_j}(0)$. Therefore, agent j's unique best choice of *ex ante* investment is $F_j(0) = 1$.

[2] agent i: By [1], any $j \neq i$ chooses $F_j(0) = 1$ in any perfect Bayesian Nash equilib-

rium. Therefore, since the tie is broken by the Vickrey auction tie-breaking rule, agent i always wins by bidding $\max_{j \in I \setminus \{i\}} \{b^{c_j}(0)\}$. Then i's best response is any $\tilde{F}_i \in \mathcal{F}_i$ such that $\max_{a \in [a_i, \alpha]} \{a - c_i(a)\} = \max_{a \in [0,\alpha]} \{a - c_i(a)\}$ for any a_i in the support of $\tilde{F}_i$ because otherwise it is suboptimal.

By [1] and [2], in any equilibrium of this game, only agent i may choose a costly *ex ante* investment and wins the auction with probability one. Other agents do not make any costly *ex ante* investments. Since the optimal choice of agent i's *ex ante* investment always maximizes the social welfare, the first-price auction with an investment game achieves full efficiency in any perfect Bayesian Nash equilibrium. □

2.4 English Auction

Next, we consider the English auction instead of the first-price auction. Here, we employ trembling-hand perfect equilibrium as a solution concept to ensure the uniqueness of the equilibrium in the auction stage. Although the formal definition involves the sequences of mixed strategies and beliefs, we use the following reduced version in this game.

Definition 11. Given a profile of cost functions $c : [0,\alpha]^n \to \mathbb{R}^n_+$, a *trembling-hand perfect equilibrium* of the English auction with an investment game is a profile of strategies $(F, \tilde{a})$ that satisfies the following conditions:

1. For each agent $i \in I$, the *ex post* investment strategy $\tilde{a}_i : [0,\alpha] \times \Omega \to [0,\alpha]$ satisfies

$$\tilde{a}_i(a_i, \omega_i) \in \arg\max_{a \in [a_i, \alpha]} \Big[a - c_i(a) \Big] \text{ and}$$

$$\tilde{a}_i(a_i, \omega) = a_i \text{ for any } \omega \neq \omega_i$$

   for each original valuation $a_i \in [0, \alpha]$.

2. For each agent $i \in I$, at any price $p \in [0, \alpha]$, i drops out of the auction if

$$p > \tilde{a}_i(a_i, \omega_i) - \big\{ c_i(\tilde{a}_i(a_i, \omega_i)) - c_i(a_i) \big\},$$

and i stays if

$$p < \tilde{a}_i(a_i, \omega_i) - \left\{c_i(\tilde{a}_i(a_i, \omega_i)) - c_i(a_i)\right\}.$$

3. For each agent $i \in I$, the choice of cumulative distribution function $F_i \in \mathcal{F}_i$ satisfies

$$\begin{aligned} F_i \in \arg\max_{\tilde{F}_i \in \mathcal{F}_i} \Big[&- \gamma_i^{c_i}(\tilde{F}_i) + \int_0^\alpha ... \int_0^\alpha \\ &\max\Big\{0, \left[\tilde{a}_i(a_i, \omega_i) - c_i(\tilde{a}_i(a_i, \omega_i)) + c_i(a_i)\right] - \max_{j \in I \setminus \{i\}} \left[\tilde{a}_j(a_j, \omega_j) - c_j(\tilde{a}_j(a_j, \omega_j)) + c_j(a_j)\right]\Big\} \\ &dF_1^{(k)}(a_1)...dF_{i-1}^{(k)}(a_{i-1})d\tilde{F}_i(a_i)dF_{i+1}^{(k)}(a_{i+1})...dF_n^{(k)}(a_n)\Big]. \end{aligned}$$

for some sequence $\{F_{-i}^{(k)}\}_{k \in \mathbb{N}}$ of the distributions of other agents' valuations such that $F_{-i}^{(k)} \to F_{-i}$ as $k \to \infty$.

The equilibrium *ex post* investment strategy is defined in a straightforward way because the beliefs and the strategies of other agents do not matter in the last stage. In the auction stage, if every action is taken with a positive probability, agent i with valuation a_i is strictly worse off if they stay in the auction for prices higher than $b^{c_i}(a_i)$. In the same way, she would be strictly better off if they stay in the auction for prices lower than $b^{c_i}(a_i)$. Therefore, the equilibrium strategy in the English auction is summarized by the second condition.

Since the allocative efficiency is always satisfied by the English auction, full efficiency requires only investment efficiency.

Definition 12. Given a profile of cost functions $c : [0, \alpha]^n \to \mathbb{R}_+^n$, *the English auction with an investment game achieves full efficiency in a trembling-hand perfect equilibrium* $(F, \tilde{a})$ if the following condition is satisfied:

$$F \in \arg\max_{\tilde{F} \in \mathcal{F}} \Big[- \sum_{i \in I} \gamma_i^{c_i}(\tilde{F}_i) + \int_0^\alpha ... \int_0^\alpha \max_{i \in I} \left[\tilde{a}_i(a_i, \omega_i) - \left\{c_i(\tilde{a}_i(a_i, \omega_i)) - c_i(a_i)\right\}\right] d\tilde{F}_1(a_1)...d\tilde{F}_n(a_n)\Big].$$

As in the first-price auction case, we show a positive result for the English auction: the English auction with an investment game achieves full efficiency in any trembling-hand perfect equilibrium. Given the equilibrium price of the auction, the logic is similar to Theorem 3. Also, this is a direct generalization of Theorem 2 in Chapter 1 as the English auction implements an allocatively efficient social choice function in equilibrium.

Theorem 4. *For any profile of cost functions $c : [0, \alpha]^n \to \mathbb{R}^n_+$, the English auction with an investment game achieves full efficiency in any trembling-hand perfect equilibrium.*

Proof. In any trembling-hand perfect equilibrium of the English auction, no agent $k \in I$ drops out until $b^{c_k}(0)$ is reached for any choice of *ex ante* investment. Therefore, the price of the item is at least $\max_{k \in I}\{b^{c_k}(0)\}$. Again, let $i \in I$ be the unique agent who has the highest net maximum value of the item, i.e., $i \equiv \arg\max_{j \in I}\{b^{c_j}(0)\}$. Consider the incentive of agent $j \in I$ such that $j \neq i$. For j, choosing $\tilde{F}_j \in \mathcal{F}_j$ such that $\gamma_j^{c_j}(\tilde{F}_j) > 0$ is not optimal because the *ex ante* utility is at most

$$\begin{aligned}
& -\gamma_j^{c_j}(\tilde{F}_j) + \int_0^\alpha \left[b^{c_j}(a_j) - b^{c_i}(0)\right] \mathbb{1}_{\{j \in \arg\max_{k \in I}\{b^{c_k}(a_k)\}\}} d\tilde{F}_j(a_j) \\
\leq \; & -\int_0^\alpha \left\{ c_j(a_j) \mathbb{1}_{\{j \notin \arg\max_{k \in I}\{b^{c_k}(a_k)\}\}} + \left[b^{c_j}(0) - b^{c_i}(0)\right] \mathbb{1}_{\{j \in \arg\max_{k \in I}\{b^{c_k}(a_k)\}\}} \right\} d\tilde{F}_j(a_j) \\
< \; & 0.
\end{aligned}$$

Therefore, in any trembling-hand perfect equilibrium of this game, only agent i may choose a costly *ex ante* investment and wins the auction with probability one. Again, this achieves full efficiency. $\square$

2.5 Concluding Remarks

This chapter provides a framework in which both the first-price auction and the English auction fully implement efficient investments in equilibrium. Contrary to the argument that these two auctions may induce different investment incentives, we show that both of them achieve investment efficiency. The important assumption is that agents can make investments after participating in the auction under the same cost structure. This flexibility of investment timing is a natural assumption, but has not been studied well in the literature. Therefore, it would be a fruitful direction to consider what real-life situations fit to our model and investigate how agents respond to the structure of the auction mechanism.

Chapter 3

Controlled School Choice with Hard Bounds: (Non-)Existence of Fair and Non-wasteful Assignments

3.1 Introduction

Among many school choice programs, *controlled school choice programs* have been widely adopted across the United States in order to maintain gender, racial or socioeconomic balance at schools. For instance, New York City requires Educational Option (EdOpt) schools to keep the diversity of ability levels by accepting students from different ranges of test scores. For each EdOpt school, 16 percent of students must score below the grade level on the standardized test, 68 percent must score at the grade level, and 16 percent must score above the grade level (Abdulkadiroğlu et al., 2005). Another example is the public school choice program in Cambridge (in the US). They classify students into high and low socioeconomic groups, and require that each school should admit certain percentages of students from each group (Fragiadakis and Troyan, forthcoming). As we see in these examples, controlled school choice programs often require not only the upper bounds but also the *lower bounds* of the type-specific quotas.

The controlled school choice problem with both lower and upper bounds of type-specific quotas has been studied by Ehlers et al. (2014). However, they found a problem which is intrinsic to the model: there may not exist a feasible assignment that is fair and non-wasteful, which are the central concepts in the two-sided matching literature. This non-existence result is in contrast with the standard school choice problem with only maximum type-specific quotas, where we can always find such an assignment by the deferred acceptance algorithm (Abdulkadiroğlu and Sönmez, 2003).

In this chapter, we identify a condition on a schools' priority profile which is sufficient and necessary (in a weak sense explained below) for the existence of a feasible, fair and non-wasteful assignment. The condition is that for any students of the same type, the priority order among them is consistent across all schools. We call this condition the *common priority condition*. The first theorem shows that this common priority condition is necessary: if the schools' priority profile does not have a common priority order for some types, we can find a profile of school capacities, type-specific constraints and student preferences for which a feasible assignment exists but no feasible assignment is both fair for same types and non-wasteful (Theorem 5). This implies that in order to achieve a fair and non-wasteful assignment when the common priority condition is violated, the school district needs to carefully set the capacities and type-specific constraints. On the other hand, we also show that this common priority condition is sufficient for ensuring the existence of a fair and non-wasteful assignment (Theorem 6). The proof is constructive. We introduce a novel algorithm called the type-proposing deferred acceptance (TDA) algorithm to find such an assignment for any controlled school choice problem under the common priority condition.

In the TDA algorithm, students are allowed to propose to schools sequentially. As in the standard deferred acceptance algorithm, when a student proposes to a school, she is temporarily assigned to that school if she justifiably claims an empty slot at the school or she justifiably envies someone who is assigned a slot at the school. An important feature of TDA is that whenever student s proposes to schools, all students of the same type who have lower common priority than s also propose to schools following s. Moreover, they propose from their first choices. Owing to this structure, we successfully eliminate justifiable claims for empty slots and justifiable envy for other students. We also show that the TDA algorithm terminates in finite steps in spite of this complicated structure.

In the real-world school choice problems, the common priority condition is satisfied in some cases while it is not in others. Suppose gender is the only type information: any student is classified as either a male or a female. Consider Chicago's selective high schools, which prioritize all students using a unique measure of the compound test score. Since they use a single priority order for any students, they have common priority over students of the same gender as well. Moreover, even if schools prioritize males and females differently, the condition will still be satisfied as long as the same test score is used by all schools within each gender. On the other hand, as in Boston Public Schools, if schools prioritize students according to walk-zones, then in general, we cannot guarantee that the common priority condition is satisfied. This is because two male (female) students can live in different walk-zones, and hence the priority order of these students may be reversed between different schools.

In the literature of matching theory, Ehlers et al. (2014) is the first paper that analyzed the controlled school choice problem with minimum type-specific quotas, and proposed several solutions to it.[1] First, they weakened both the fairness and the non-wastefulness conditions, and showed that there exists a feasible assignment that is fair for same types and constrained non-wasteful.[2] Second, they also proposed constraints to be interpreted as soft bounds, and showed that the student-proposing deferred acceptance algorithm finds an assignment that minimizes violations of controlled choice constraints among fair assignments. In our model, we stick to the hard-bound interpretation, and do not relax any requirements for the solution concept. We rather focus on the condition on a schools' prior-

[1]Other important forms of capacity constraints in matching markets have also been studied. For example, see Biró et al. (2010) and Westkamp (2013) for complex constraints in college admission problems, and Kamada and Kojima (2015a,b, 2016) for distributional constraints in matching markets.

[2]An assignment is constrained non-wasteful if, when a student justifiably claims an empty slot at a school, the resulting assignment in which the student is assigned to that school is not fair for same types. See Section 3.2 for the definitions of non-wastefulness (justifiable claim for an empty slot) and fairness.

ity profile, and show how restrictive it must be in order to ensure the existence of a feasible assignment that is fair and non-wasteful.

There is also a growing literature on the minimum quotas and affirmative action in matching problems. Fragiadakis and Troyan (forthcoming) introduce a new dynamic quotas mechanism that respects all minimum quotas and satisfies fairness and incentive properties. The difference between their approach and ours is that they do not require non-wastefulness as the maximum quotas are endogenously set within their mechanism. Fragiadakis et al. (2015) and Goto et al. (2015) consider minimum quotas in matching problems, and propose new strategy-proof mechanisms which satisfy weak versions of fairness and non-wastefulness. But since their models do not have types of students, they do not allow a school to have a separate floor for each type of students. Other related papers studied the welfare consequence of affirmative action in school choice. Kojima (2012) shows that affirmative action can hurt every minority student under any stable mechanisms and the top trading cycles mechanism. Hafalir et al. (2013) suggest the use of minority reserves instead of majority quotas, and show that the minority reserve policy improves the welfare of minority students compared to the majority quota policy. This analysis has been further generalized by Kominers and Sönmez (forthcoming), who introduced slot-specific priorities.

The rest of Chapter 3 proceeds as follows. Section 3.2 introduces the model. Section 3.3 presents the impossibility theorem when the schools' priority profile does not have a common priority order for some type of students. In Section 3.4, we introduce the type-proposing deferred acceptance algorithm, and show that it produces a feasible assignment that is fair and non-wasteful when the common priority condition is satisfied. Section 3.5 concludes.

3.2 Model: Controlled School Choice

We consider a controlled school choice problem studied by Ehlers et al. (2014). A controlled school choice problem $(S, C, (q_c)_{c \in C}, P_S, \succ_C, T, \tau, (\underline{q}_c^T, \overline{q}_c^T)_{c \in C})$ consists of the following elements:

1. a finite set of students $S = \{s_1, ..., s_n\}$;

2. a finite set of schools $C = \{c_1, ..., c_m\}$;

3. a capacity vector $q = (q_{c_1}, ..., q_{c_m})$, where q_c is the capacity of school $c \in C$ or the number of seats in $c \in C$;

4. a students' preference profile $P_S = (P_{s_1}, ..., P_{s_n})$, where P_s is the strict preference relation of student $s \in S$ over C, i.e. cP_sc' means that student s strictly prefers school c to school c';

5. a schools' priority profile $\succ_C = (\succ_{c_1}, ..., \succ_{c_m})$, where $\succ_c$ is the strict priority ranking of school $c \in C$ over S; $s \succ_c s'$ means that student s has higher priority than student s' to be enrolled at school c;

6. a type space $T = \{t_1, ..., t_k\}$;

7. a type function $\tau : S \to T$, where $\tau(s)$ is the type of student s;

8. for each school c, two vectors of type-specific constraints $\underline{q}_c^T = (\underline{q}_c^{t_1}, ..., \underline{q}_c^{t_k})$ and $\overline{q}_c^T = (\overline{q}_c^{t_1}, ..., \overline{q}_c^{t_k})$ such that $\underline{q}_c^t \leq \overline{q}_c^t \leq q_c$ for all $t \in T$, and $\sum_{t \in T} \underline{q}_c^t \leq q_c \leq \sum_{t \in T} \overline{q}_c^t$.

Note that we require students to rank all schools and assume that they are acceptable to any school. $\underline{q}_c^t$ is the minimal number of slots that school c must allocate to students of type t and $\overline{q}_c^t$ is the maximal number of slots that school c is allowed to allocate to students of type t. We call $(\underline{q}_c^t, \overline{q}_c^t)$ the floor and the ceiling for type t at school c. We assume $q_c \leq \sum_{t \in T} \overline{q}_c^t$, but this is without loss of generality because we cannot assign more students than $\sum_{t \in T} \overline{q}_c^t$ to school c even if q_c is larger than that.

The first five elements consist of a standard school choice problem considered by Abdulkadiroğlu and Sönmez (2003). They also introduced types of students as an extension of their original model and considered maximum quotas for each type of students. In a separate paper, Abdulkadiroğlu (2005) considers the college admission model with type-specific quotas. However, the major difference between their models and this model is that we have minimum quotas for each type of students in addition to maximum quotas. As we see in the following sections, the standard discussion based on the deferred acceptance algorithm in Abdulkadiroğlu and Sönmez (2003) and Abdulkadiroğlu (2005) does not go through in our model as we have minimum quotas.

An assignment μ is a function from S to $C \cup S$ such that

1. $\mu(s) \notin C \Rightarrow \mu(s) = s$ for every student $s \in S$;

2. $|\mu^{-1}(c)| \leq q_c$ for every school $c \in C$.

Under an assignment μ, we allow students to be unassigned. $\mu(s) = s$ means that student s is not assigned to any school under μ.[3]

A set of students $S' \subseteq S$ respects (controlled choice) constraints at school c if for every type $t \in T$, $\underline{q}_c^t \leq |\{s \in S' | \tau(s) = t\}| \leq \overline{q}_c^t$. An assignment μ respects constraints if for every school c, $\mu^{-1}(c)$ respects constraints at c, i.e.,

$$\underline{q}_c^t \leq |\{s \in \mu^{-1}(c) | \tau(s) = t\}| \leq \overline{q}_c^t$$

for every type $t \in T$. An assignment μ is legally feasible if μ respects constraints and every student is assigned to a school. In addition, in any feasible assignment, we require every student to be assigned to a school because we assume that students rank all schools and they are acceptable to all schools.

Given the impossibility theorem under hard-bound interpretation, Ehlers et al. (2014) propose type-specific constraints to be interpreted as soft bounds: schools can admit fewer students than their floors or more students than their ceilings, but they give highest priority to types who do not fill their floors, medium priority to types who fill their floors but not ceilings, and lowest priorities to types who fill their ceilings. In this chapter, however, we

[3] Ehlers et al. (2014) define fairness and non-wastefulness for feasible assignments, i.e., assignments such that constraints are satisfied and every student is assigned to a school. Here, we define these properties for assignments where students may be unassigned in order to define the type-proposing deferred acceptance algorithm for Theorem 6. Therefore, we allow students to be unassigned in an assignment.

stick to the hard-bound interpretation, and further analyze how restrictive the condition must be in order to ensure the existence of fair and non-wasteful assignments under this feasibility requirement.

Even when there are enough seats for students in total, a controlled school choice problem with hard bounds may not have a feasible solution. This is because there may be less (more) students of a certain type than the sum of all floors (ceilings) for that type. Therefore, as in Ehlers et al. (2014), we will only consider problems where the constraints ensure the existence of a legally feasible assignment.

Next, let us introduce desirable properties of assignments in the controlled school choice problem. The first property is basically the non-wastefulness condition considered by Balinski and Sönmez (1999). But since we have type-specific constraints in our model, students are allowed to claim an empty slot only when we can achieve a feasible assignment by allowing the student to take the slot in the current assignment.

Definition 13. *Student s justifiably claims an empty slot at school c under the assignment ν* if

(nw1) $cP_s\nu(s)$,

(nw2) there exists a feasible assignment μ such that $\mu(s) = c$ and $\mu(s') = \nu(s')$ for any student $s' \in S \setminus \{s\}$ such that $\nu(s') \neq s'$.

An assignment ν is *non-wasteful* if there is no student who justifiably claims an empty slot at any school.

(nw1) means that student s prefers an empty slot at school c to her assignment under ν. (nw2) means that we can achieve a feasible assignment by assigning student s a slot at school c, keeping all other assigned students under ν unchanged, and assigning other students vacant slots of schools.

The second property is the no-envy condition, which plays a significant role in the school choice literature. As in the case of non-wastefulness, Ehlers et al. (2014) modified the standard definition due to the structure of controlled school choice, and we follow their approach. Even when a student prefers some school to her own assignment and there is another student who has lower priority in the school, her envy is not justified if the student's move would violate the legal constraints. Definition 14 formally states the conditions for a student to have justified envy.

Definition 14. *Student s justifiably envies student s' at school c under the assignment ν* if

(f1) $\nu(s') = c$, $cP_s\nu(s)$ and $s \succ_c s'$,

(f2) there exists a feasible assignment μ such that $\mu(s) = c$, $\mu(s') \neq c$, and $\mu(\hat{s}) = \nu(\hat{s})$ for any student $\hat{s} \in S \setminus \{s, s'\}$ such that $\nu(\hat{s}) \neq \hat{s}$.

A assignment ν is *fair across types (or fair)* if there is no student who justifiably envies any other student.

(f1) is the standard condition that student s prefers school c to her assignment under ν, but there is a student s' who is assigned to school c and has lower priority than s at school c. (f2) means that we can achieve a feasible assignment by assigning student s a slot at school c, student s' a slot at some school other than c, keeping all other assigned students under ν unchanged and assigning other students vacant slots of schools.

Finally, we introduce a weaker version of fairness where envy is justified only if both the envying student and the envied student are of the same type.

Definition 15. *Student s justifiably envies student s' of the same type at school c under the assignment ν* if

(f1) $\nu(s') = c$, $cP_s\nu(s)$ and $s \succ_c s'$,

(f2) there exists a feasible assignment μ such that $\mu(s) = c$, $\mu(s') \neq c$, and $\mu(\hat{s}) = \nu(\hat{s})$ for any student $\hat{s} \in S \setminus \{s, s'\}$ such that $\nu(\hat{s}) \neq \hat{s}$.

(f3) $\tau(s) = \tau(s')$.

A assignment ν is *fair for same types* if there is no student who justifiably envies any other student of the same type.

(f1) and (f2) are the same conditions as in Definition 14. (f3) means that student s and s' are of the same type.

3.3 Non-Existence of Fair and Non-wasteful Assignments

The main contribution of this chapter is to provide a condition on a schools' priority profile which is sufficient and also necessary (in a weak sense shown in Theorem 5) for the existence of a fair and non-wasteful assignment. First, we define the key property, which we call the *common priority condition*, of a schools' priority profile $\succ_C$.

Definition 16. A schools' priority profile $\succ_C$ has a *common priority order* for type $t \in T$ if

$$s \succ_c s' \;\Leftrightarrow\; s \succ_{c'} s'$$

for any $c, c' \in C$ and $s, s' \in S$ such that $\tau(s) = \tau(s') = t$.

The common priority condition says that for any two students of the same type, the priority order should be consistent across all schools. In the real-life school choice programs, the common priority condition is satisfied in some programs, but it is difficult to be satisfied in others. Suppose gender is the only type information: any student is classified as either a male or a female. If a single priority order is used for any students as in Chicago's selective high schools, this condition would be satisfied. However, there are many school boards which introduce other priority such as walk-zone priority or siblings priority. In such cases, the common priority condition is hard to be met because two male (female) students can live in different walk-zones or have siblings in different schools, and hence the priority order of these students is reversed between different schools.

Although this condition is somewhat restrictive, we can show that this is essentially necessary for ensuring the existence of fair and non-wasteful assignments. More precisely, our first theorem shows that when the schools' priority profile does not have a common priority order for some type $t \in T$, we can find out a profile of school capacities, type-specific constraints and student preferences for which a feasible assignment exists but no feasible assignment is both fair for same types and non-wasteful. This is a straightforward extension of Theorem 1 of Ehlers et al. (2014), which finds a controlled school choice problem where the set of feasible assignments that are fair for same types and non-wasteful may be empty.

Theorem 5. *Suppose $|C| \geq 3$ and $|S| \geq 2$. Suppose that the schools' priority profile $\succ_C$ does not have a common priority order for some type $t \in T$. Then there exist a profile q of school capacities, type-specific constraints $(\underline{q}_c^T)_{c \in C}$ and $(\overline{q}_c^T)_{c \in C}$, and a students' preference profile P_S for which a feasible assignment exists but there is no feasible assignment that is fair for same types and non-wasteful.*

Proof: Since the schools' priority profile $\succ_C$ does not have a common priority order for some type $t \in T$, there are two schools $c_1, c_2 \in C$ and two students $s_1, s_2 \in S$ such that

$$\tau(s_1) = \tau(s_2) = t,$$

$$c_1 : s_1 \succ_{c_1} s_2,$$

$$c_2 : s_2 \succ_{c_2} s_1.$$

Because $|C| \geq 3$, there is another school c_3. We set the capacities of school c_1 and c_2 greater than zero, and the capacity of school c_3 to $|S|$. For any school, the ceiling for each type is set to be equal to the capacity of the school. The floor of c_3 for type t is $|S_t| - 1$, and all other floors are zero. We consider the following preferences of students. Student s_1's first choice is c_2, her second choice is c_1, her third choice is c_3, and she prefers these three schools to any other schools. Student s_2 also prefers these three schools to any other schools, but the order of the first and the second choices are reversed. Other students prefer c_3 the best. These are summarized in the following table. (s denotes any student in $S \setminus \{s_1, s_2\}$.)

	$\succ_{c_1}$	$\succ_{c_2}$	$\succ_{c_3}$	P_{s_1}	P_{s_2}	P_s
	s_1	s_2		c_2	c_1	c_3
	s_2	s_1		c_1	c_2	$\vdots$
				c_3	c_3	
				$\vdots$	$\vdots$	
capacities	> 0	> 0	$\lvert S\rvert$			
ceiling for any type	> 0	> 0	$\lvert S\rvert$			
floor for type t	0	0	$\lvert S_t\rvert - 1$			
floor for other types	0	0	0			

In any non-wasteful assignment, any student $s \in S \setminus \{s_1, s_2\}$ should be assigned to school c_3. Thus, we only need to consider the following four assignments.

$$\mu_1 = \begin{pmatrix} c_1 & c_2 & c_3 \\ \emptyset & s_1 & S \setminus \{s_1\} \end{pmatrix}, \; \mu_2 = \begin{pmatrix} c_1 & c_2 & c_3 \\ \emptyset & s_2 & S \setminus \{s_2\} \end{pmatrix},$$

$$\mu_3 = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_2 & \emptyset & S \setminus \{s_2\} \end{pmatrix}, \; \mu_4 = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_1 & \emptyset & S \setminus \{s_1\} \end{pmatrix}.$$

The problem is that because of the floor constraint of school c_3 for type t, at least one of s_1 and s_2 has to be assigned to c_3. Under μ_1, s_2 justifiably envies s_1 at c_2 because s_2 has higher priority than s_1 at c_2. If s_1 is replaced by s_2, we obtain μ_2. However, since s_2 prefers c_1 to c_2, she justifiably claims an empty slot of c_1 under μ_2. If s_2 moves to c_1, then under μ_3, now s_1 justifiably envies s_2 at c_1 because s_1 has higher priority than s_2 at c_1. We obtain μ_4 by replacing s_2 by s_1, but under μ_4, s_1 justifiably claims an empty slot of c_2. Therefore, none of these candidates are both fair for same types and non-wasteful. $\square$

3.4 Existence of Fair and Non-wasteful Assignments

In the previous section, we demonstrated that we cannot guarantee the existence of fair and non-wasteful assignments without the common priority condition. Here, we show that the common priority condition is actually sufficient for any controlled school choice problem to have such an assignment. Namely, the next theorem shows that a feasible assignment that is both fair across types and non-wasteful always exists when the schools' priority profile has a common priority order for every type $t \in T$. We prove the existence in a constructive way. We propose an algorithm called the *type-proposing deferred acceptance (TDA) algorithm*.

When the common priority condition is satisfied, TDA works as follows. Students are allowed to propose to schools sequentially. Students are first arbitrarily ordered by types, and among each type of students, they are ordered by the common priority order.[4] At the beginning of each step, we find student s who is the first one among all unassigned students according to the order. If there is any student of the same type who has lower priority than s and is assigned a slot at some school, then we *release her from the school*. In this step, student s and all students of the same type who have lower priority than s propose to schools sequentially. When a student proposes to a school, she is successfully (but temporarily) assigned to that school if she justifiably claims an empty slot at the school or she justifiably envies someone who is assigned a slot at the school. Otherwise, she proposes to the next school according to her preference ranking. Once she is assigned a slot at some school, then the next student starts proposing to schools. As in the standard deferred acceptance algorithm, the assignment made in this step is temporary in the sense that in later steps, a student may be justifiably envied by another student and may be rejected by the school which she is assigned to in this step.

An interesting part of this TDA algorithm is that when s proposes to schools, all other students who do so in the same step propose *from their first choices*. Since the feasibility of assignments concerns whether the minimum quotas can be satisfied by assigning vacant slots to currently unassigned students, s's choice in this step affects whether other students of the same type are allowed to get a slot of a school that they prefer. For example, if s is rejected by a school with no minimum quota and now fills one of the minimum quotas of

[4]The TDA algorithm itself is defined for any order of students within a same type, but we only explain the one where the order is set by the common priority.

another school which is unpopular among other students of the same type, one of them might find it feasible to get a slot at a school which rejected her before s moves. In order to successfully eliminate such justifiable claims and envy, we allow a student to propose to all schools whenever some student of the same type who has higher priority becomes unassigned. Note that the common priority condition is essential for eliminating justifiable claims and envy within the same type. If we run TDA with some arbitrary order within a same type of students when there is no common priority among them, there can be a student who justifiably envies another student of the same type since the order within TDA and the priority order of some school are inconsistent.

We call this algorithm the *type-proposing deferred acceptance algorithm* because in each step, only one of the types is chosen and a subset of the students of the type are allowed to propose within the step. Here, the formal definition of the TDA algorithm is as follows.

Type Proposing Deferred Acceptance Algorithm

Start: First, fix an arbitrary order of types, say $t_1 - t_2 - ... - t_k$. Within each type of students, also fix an order of students. For example, we denote the set of any type $t \in T$ of students by $\{s_1^t, s_2^t, ..., s_{|S_t|}^t\}$, and order them by the index, i.e., $s_1^t - s_2^t - ... - s_{|S_t|}^t$. In each step, only a subset of one type of students are allowed to make proposals to schools. We always define a tentative assignment ν in which students may be unassigned. Any tentative assignment must be such that we can allocate the unassigned students to schools and make the resulting assignment feasible. Let $\mathcal{F}$ denote the set of all feasible assignments and ν_0 be the empty assignment, i.e. $\nu_0(s) = s$ for all $s \in S$.

Step 1: All of type t_1 students propose to schools in the order $s_1^{t_1} - s_2^{t_1} - ... - s_{|S_{t_1}|}^{t_1}$.

Step 1.1: Student $s_1^{t_1}$ proposes to her first choice school under $P_{s_1^{t_1}}$, say c_1^1. If there is some $\mu \in \mathcal{F}$ such that $\mu(s_1^{t_1}) = c_1^1$, then set $\nu_{1.1}(s_1^{t_1}) = c_1^1$ and $\nu_{1.1}(s) = \nu_0(s) = s$ for all $s \in S \setminus \{s_1^{t_1}\}$.

If there is not such an assignment, student $s_1^{t_1}$ is rejected by c_1^1. She proposes to her second choice school under $P_{s_1^{t_1}}$, say c_1^2. If there is some $\mu \in \mathcal{F}$ such that

$\mu(s_1^{t_1}) = c_1^2$, then set $\nu_{1.1}(s_1^{t_1}) = c_1^2$ and $\nu_{1.1}(s) = \nu_0(s) = s$ for all $s \in S \setminus \{s_1^{t_1}\}$.

In this way, continue this process until $s_1^{t_1}$ is assigned to some school. This substep terminates when $s_1^{t_1}$ is assigned to some school.

In general, for $j = 2, 3, ...,$

Step 1.j: Student $s_j^{t_1}$ proposes to her first choice school under $P_{s_j^{t_1}}$, say c_j^1. If there is some $\mu \in \mathcal{F}$ such that $\mu(s_j^{t_1}) = c_j^1$ and $\mu(s') = \nu_{1.j-1}(s')$ for all students s' such that $\nu_{1.j-1}(s') \neq s'$, then set $\nu_{1.j}(s_j^{t_1}) = c_j^1$ and $\nu_{1.j}(s) = \nu_{1.j-1}(s)$ for all $s \in S \setminus \{s_j^{t_1}\}$.

If there is not such an assignment, student $s_j^{t_1}$ is rejected by c_j^1. She proposes to her second choice school under $P_{s_j^{t_1}}$, say c_j^2. If there is some $\mu \in \mathcal{F}$ such that $\mu(s_j^{t_1}) = c_j^2$ and $\mu(s') = \nu_{1.j-1}(s')$ for all students s' such that $\nu_{1.j-1}(s') \neq s'$, then set $\nu_{1.j}(s_j^{t_1}) = c_j^2$ and $\nu_{1.j}(s) = \nu_{1.j-1}(s)$ for all $s \in S \setminus \{s_j^{t_1}\}$.

In this way, continue this process until $s_j^{t_1}$ is assigned to some school. This substep terminates when $s_j^{t_1}$ is assigned to some school.

Step 1 terminates when all students of type t_1 are assigned to schools. Let ν_1 be the final tentative assignment when all the substeps of Step 1 terminate.

$\vdots$

Step p: If there is a student s who was unassigned when Step $p-1$ terminated, i.e., $\nu_{p-1}(s) = s$, let s_k^t be the student whose type (t) is the minimal index among such students and who has the minimal index (k) among such students of the same type. Now construct a new tentative assignment $\nu_{p.0}$ *by releasing all the students of type t with larger index than* s_k^t *from schools under* ν_{p-1}, i.e., set $\nu_{p.0}(s_l^t) = s_l^t$ for any s_l^t such that $l > k$, and $\nu_{p.0}(s) = \nu_{p-1}(s)$ for any other students $s \in S \setminus \{s_l^t | l > k\}$.

Step p.1: If s_k^t was assigned to some school before Step $p-1$, then let c be the school to which she was assigned in the most recent step. Let c' be the most preferred school under $P_{s_k^t}$ among schools that are less preferred than c. If s_k^t was not assigned to any school before Step $p-1$, then take her first choice school as c'. s_k^t proposes to school c'.

(1) If there is $\mu \in \mathcal{F}$ such that $\mu(s_k^t) = c'$ and $\mu(s') = \nu_{p.0}(s')$ for all students s' such that $\nu_{p.0}(s') \neq s'$, then student s_k^t justifiably claims an empty slot at school c' under $\nu_{p.0}$. Then we set $\nu_{p.1}(s_k^t) = c'$ and $\nu_{p.1}(s') = \nu_{p.0}(s')$ for all $s' \in S \setminus \{s_k^t\}$.

(2) If (1) is not true but there is a student such that s_k^t justifiably envies that student at school c' under $\nu_{p.0}$, then let s be the student who has the lowest priority under $\succ_{c'}$ among those students. Then we set $\nu_{p.1}(s_k^t) = c'$, $\nu_{p.1}(s) = s$ and $\nu_{p.1}(s') = \nu_{p.0}(s')$ for all $s' \in S \setminus \{s_k^t, s\}$, i.e., school c' rejects s and tentatively admits s_k^t.

If (1) and (2) are not true, student s_k^t is rejected by school c'. She proposes to a school c'' which is most preferred under $P_{s_k^t}$ among schools that are less preferred than c'. Check if (1) or (2) applies to c''. Continue this process until s_k^t is assigned to some school. This substep terminates when s_k^t is assigned to some school.

In general, for $q = 1, 2, ...,$

Step p.q+1: Student s_{k+q}^t proposes to her *first choice school* under $P_{s_{k+q}^t}$, say c_q^1

(1) If there is $\mu \in \mathcal{F}$ such that $\mu(s_{k+q}^t) = c_q^1$ and $\mu(s') = \nu_{p.q}(s')$ for all students s' such that $\nu_{p.q}(s') \neq s'$, then student s_{k+q}^t justifiably claims an empty slot at school c_q^1 under $\nu_{p.q}$. Then we set $\nu_{p.q+1}(s_{k+q}^t) = c_q^1$ and $\nu_{p.q+1}(s') = \nu_{p.q}(s')$ for all $s' \in S \setminus \{s_{k+q}^t\}$.

(2) If (1) is not true but there is a student such that s_{k+q}^t justifiably envies that student at school c_q^1 under $\nu_{p.q}$, then let s be the student who has the lowest priority under $\succ_{c_q^1}$ among those students. Then we set $\nu_{p.q+1}(s_{k+q}^t) = c_q^1$, $\nu_{p.q+1}(s) = s$ and $\nu_{p.q+1}(s') = \nu_{p.q}(s')$ for all $s' \in S \setminus \{s_{k+q}^t, s\}$, i.e., school c_q^1 rejects s and tentatively admits s_{k+q}^t.

If (1) and (2) are not true, student s^t_{k+q} is rejected by school c^1_q. She proposes to her second choice school c^2_q under $P_{s^t_{k+q}}$. Check if (1) or (2) applies to c^2_q. Continue this process until s^t_{k+q} is assigned to some school. This substep terminates when s^t_{k+q} is assigned to some school.

Step p terminates when all students in $\{s^t_l | l > k\}$ are assigned to schools. Let ν_p be the final tentative assignment when all the substeps of Step p terminate.

End: The algorithm terminates when every student is assigned to some school. The tentative assignment becomes final.

Since steps are nested in each step, we call Step 1, Step 2, ... as just "steps" and Step 1.1, Step 1.2, ..., Step p.1, Step p.2, ... as "substeps" to distinguish them.

One of the important differences between TDA and the standard deferred acceptance algorithm (see Abdulkadiroğlu and Sönmez, 2003) is that in TDA, a student can propose to and be assigned to a school which rejected her once. Consider the following story. Suppose a student s^t_k, whose type is t and index within type t is k, is rejected by school c at some substep. Suppose this is not the first substep of the first step, and let ν be the tentative matching achieved just before this substep. Under ν, the sum of unfilled minimum quotas for type t for all schools must be less than or equal to the number of unassigned students of type t including s^t_k, and suppose that they are equal to each other. Then it is possible that school c has an empty slot for type t under ν, but s^t_k is rejected by c because she has to fill one of the minimum quotas of some different school. Suppose in some later step, a student s^t_l of the same type t with $l < k$ becomes unassigned. By the definition of TDA, when s^t_l proposes in this step, every student of the same type whose index is larger than l becomes unassigned and proposes to schools starting from *her first choice*. If s^t_l proposes to a school whose minimum quota for type t was not filled under ν and fills one of the slots for type t, then s^t_k may be assigned to school c because it is now feasible, although it was not possible when she proposed last time. Thus in TDA, a student can be assigned to a school which once rejected her in a previous step.

Beucase of this complicated structure, it is not obvious that TDA always terminates in finite steps. The next theorem however, shows that this is actually true. In addition, we show that TDA eliminates any justifiable envy and justifiable claim for an empty slot when the order of students is set by the common priority order of $\succ_C$ for every type $t \in T$.

Theorem 6. *Suppose that the schools' priority profile $\succ_C$ has a common priority order for every type $t \in T$. Then the set of feasible assignments that are fair across types and non-wasteful is nonempty in any controlled school choice problem.*

Proof: We prove this using TDA. For every type $t \in T$, we set the order of students in TDA by the common priority order of $\succ_C$ for type t. First, we show that TDA terminates in finite steps. And then we show that the assignment produced by this TDA is non-wasteful and fair across types.

[1] TDA terminates in finite steps.

To show this, we extend the concept of the rank of schools from individuals to types. For each assignment, define *the aggregate rank of schools* R_t for each type $t \in T$. Given an assignment, let $r_t \in \{1, ..., m\}^{|S_t|}$ be the vector of ranks of schools for students of type t where $r_t(1)$ denotes the rank of the school to which s_1^t is assigned, $r_t(2)$ denotes the rank of the school to which s_2^t is assigned, and so on. Define R_t by

$$R_t \equiv r_t(1)m^{|S_t|-1} + r_t(2)m^{|S_t|-2} + ... + r_t(|S_t| - 1)m + r_t(|S_t|).$$

For each type $t \in T$, consider the steps where type t students propose to schools. Since no students of type t end up being unassigned in the steps they propose, we can define this aggregate rank of schools R_t for the assignment achieved at the end of each of these steps. Then we can see that R_t is strictly increasing over these steps for the following reason. In each step, the student with the minimal index among those who propose is assigned to a school that is strictly less preferred than the one she was assigned to in the most recent step. Therefore, by the definition of R_t, even when the ranks of all other students with larger indices go up from m to 1, R_t strictly increases. As the maximum of R_t is finite, it should stop increasing at some step, and hence TDA terminates in finite steps.

[2] Non-wastefulness and fairness across types.

Consider the assignment ν_p produced by TDA at the end of each step $p = 1, 2, \ldots$. We show that (i) no student *who is assigned a slot at some school under* ν_p justifiably claims an empty slot or justifiably envies another student, and (ii) no student s *who is unassigned under* ν_p *but was assigned a slot at some school before Step* p justifiably claims an empty slot or justifiably envies another student at $\nu_{p'}(s)$ or schools that are more preferred than $\nu_{p'}(s)$ where $p'(< p)$ is the most recent step s was assigned a slot. We show this by mathematical induction.

[ν_1] First, it is easy to show that (i) is true for ν_1. In Step 1, only type t_1 students propose to schools, and they propose in the order of the common priority for t_1. Therefore, no type t_1 student justifiably envies another student either of the same type or a different type. In addition, since each student proposes to schools from her first choice in the descending order and is assigned to a school if it is feasible, she does not justifiably claim an empty slot of any school under ν_1. (ii) is also true because none of the unassigned students under ν_1 have been assigned a slot of schools.

[ν_2] Next, we prove (i) for ν_2. In Step 2, all of type t_2 students propose to schools. When type t_2 students fill empty slots of schools, type t_1 students who are still assigned under ν_2 should neither justifiably claim an empty slot nor justifiably envy another student under ν_2 as well because (i) holds for ν_1. Moreover, since each student of type t_2 proposes to schools from her first choice in the descending order and is assigned to a school if it is feasible, she neither justifiably claims an empty slot nor justifiably envies another student under ν_2.

For (ii), students who are unassigned under ν_2 but assigned under ν_1 must be of type t_1. If a student s of type t_1 is rejected by a school at Step 2, there should be a type t_2 student who has higher priority and is assigned a slot of $\nu_1(s)$ instead of s. Since this is true for any other student assigned to $\nu_1(s)$ at some substep of Step 2, s neither justifiably claims an empty slot nor justifiably envies another student at $\nu_1(s)$. In addition, since (i) holds for ν_1, s neither justifiably claims an empty slot nor justifiably envies another student at schools that are more preferred than $\nu_1(s)$.

[ν_p] Consider Step $p > 2$. Prove (i) and (ii) for ν_p if they are true for ν_{p-1}.

Suppose there is a student s who was unassigned when Step $p-1$ terminated, i.e., $\nu_{p-1}(s) = s$. Otherwise, the proof is done because $\nu_p = \nu_{p-1}$. Let s_k^t be the student whose type (t) is the minimal index among such students and who has the minimal index (k) among such students of the same type. At the beginning of Step p, all the students of type t with larger index than s_k^t are removed from schools, and s_k^t and those students, i.e., students in $\{s_l^t | l \geq k\}$, propose to schools in the order of their indices.

First, we prove (i). Note that all students of type t are assigned under ν_p. Hence, we need to consider the following students; s_k^t, students in $\{s_l^t | l > k\}$, and other students who are assigned the same slots under ν_{p-1} and ν_p.

If s_k^t had not been assigned to any school before Step p, then she neither justifiably claims an empty slot nor justifiably envies another student under ν_p because she proposes to schools from her first choice in the descending order. If s_k^t was assigned to school c in the most recent step before Step p, s_k^t proposes to schools that are less preferred to c in the descending order. Because ν_{p-1} satisfies (ii), s_k^t neither justifiably claims an empty slot nor justifiably envies another student under ν_p as well.

All students in $\{s_l^t | l > k\}$ propose to schools from their first choices in the descending order. Also, the order among these students in TDA is consistent with the common priority for them. Therefore, they neither justifiably claim an empty slot nor justifiably envy another student under ν_p.

Consider other students who are assigned the same slots under ν_{p-1} and ν_p. We show the following statements ($*$): Consider a slot at school c occupied by student s under ν_{p-1}.

- If $\tau(s) \neq t$, then under ν_p, the slot is kept by the same student or it is taken by another student s' whose priority is higher than s at c.

- If $\tau(s) = t$ and the minimum quota of c for type t is not binding under ν_{p-1}, then under ν_p, the slot is kept by the same student or it is taken by another student s' whose priority is higher than s at c.

Suppose s's type is not t. If s is rejected from c at Step p, there must be a student s' of type

t who has higher priority than s and is assigned the slot. Suppose s is of type t and the index is $l > k$. (If the index is $l < k$, then this student will not be rejected in Step p.) Then, under ν_p, s can never be assigned to any school more preferred to c, because (i) is true for ν_{p-1}. Therefore, under ν_p, s's slot at c should be replaced by herself or a student of type t whose index is lower than s. Thus, both statements of $(*)$ are true. Therefore, students who are assigned the same slots under ν_{p-1} and ν_p neither justifiably claim an empty slot nor justifiably envy another student under ν_p because (i) holds for ν_{p-1}.

Second, we prove (ii). Since all students of type t are assigned under ν_p, we only need to consider students of other types who are unassigned under ν_p but were assigned a slot at some school before Step p. Take such student s, and let $p'(< p)$ be the most recent step in which s was assigned a slot at some school. There are two cases: $p' < p - 1$, i.e., student s was also unassigned under ν_{p-1}, or $p' = p - 1$, i.e., she was assigned to some school under ν_{p-1} and rejected at Step p. In the first case, since (ii) holds for ν_{p-1}, under ν_{p-1}, s neither justifiably claims an empty slot nor justifiably envies another student at $\nu_{p'}(s)$ or schools that are more preferred than $\nu_{p'}(s)$. By $(*)$, under ν_p, any slot occupied by student s at school c under ν_{p-1} is kept by the same student or it is taken by another student s' whose priority is higher than s at c unless s is of type t and the minimum quota of c for type t is binding under ν_{p-1}. Therefore, under ν_p, s neither justifiably claims an empty slot nor justifiably envies another student at $\nu_{p'}(s)$ or schools that are more preferred than $\nu_{p'}(s)$ as well. In the second case, since student s was rejected at Step p and (i) holds for ν_{p-1}, s would neither justifiably claim an empty slot nor justifiably envy another student at $\nu_{p-1}(s)$ or schools that are more preferred than $\nu_{p-1}(s)$.

Thus, both (i) and (ii) hold for ν_p, and the proof is done. □

3.5 Concluding Remarks

This chapter showed that the common priority condition on a schools' priority profile is sufficient and necessary for the existence of a feasible assignment that is fair and non-wasteful. The necessity part implies that if the common priority condition is violated, we cannot guarantee the existence of a feasible, fair and non-wasteful assignment in general, and the school district needs to carefully set the capacities and type-specific constraints. The sufficiency part implies that we can use our TDA algorithm whenever the common priority condition is satisfied.

One of the limitations of our results is that we showed the first impossibility theorem by allowing for arbitrary quotas and type-specific constraints. Therefore, it is not clear what will happen if we restrict our attention to a smaller class of realistic quotas and constraints. It is also interesting to empirically see the relationship between a schools' priority profile and type-specific constraints, and examine whether a fair and non-wasteful assignment is likely to exist or not in the real-life school choice problems. Moreover, if there are environments where the common priority condition is likely to hold, it is also a good direction to analyze the incentive problems of the TDA algorithm for a practical use.

Appendix A

Appendix to Chapter 1

A.1 Proof of Theorem 1

Consider any I and Ω with $|I| \geq 2$ and $|\Omega| \geq 2$. Consider any arbitrary allocatively efficient social choice function $h : \mathbb{R}^{\Omega \times I} \to \Omega \times \mathbb{R}^{I}$. We will examine two cases where h is not strategy-proof and h is strategy-proof. In the former case, we construct the sets of valuations and cost functions under which an inefficient Bayesian Nash equilibrium exists in the investment game, exploiting the equation that strategy-proofness of h is violated. In the latter case, we show that a simple auction has multiple equilibria in the investment game and one of them is less efficient than the other for any strategy-proof h.

[1] When h is not strategy-proof. Since the social choice function h is not strategy-proof, there are $i \in I$, $v \in \mathbb{R}^{\Omega \times I}$ and $\tilde{v}^i \in \mathbb{R}^{\Omega}$ such that

$$v^i(h_\omega(\tilde{v}^i, v^{-i})) - h_t^i(\tilde{v}^i, v^{-i}) > v^i(h_\omega(v)) - h_t^i(v). \tag{A.1}$$

Consider the sets of valuations $V \subseteq \mathbb{R}^{\Omega \times I}$ such that

$$V^i = \{v^i, \tilde{v}^i\} \text{ and}$$
$$V^j = \{v^j\} \text{ for all } j \in I \setminus \{i\}.$$

Consider a profile of cost functions $c : V \times \Theta \to C$ such that for any $\theta \in \Theta$ with $p(\theta) > 0$,

$$
\begin{aligned}
& c^i(v^i, \theta^i) = \max\Big\{0, \beta\Big[v^i(h_\omega(v)) - h^i_t(v) - \big(\tilde{v}^i(h_\omega(\tilde{v}^i, v^{-i})) - h^i_t(\tilde{v}^i, v^{-i})\big)\Big]\Big\}, \\
& c^i(\tilde{v}^i, \theta^i) = \max\Big\{0, \beta\Big[\tilde{v}^i(h_\omega(\tilde{v}^i, v^{-i})) - h^i_t(\tilde{v}^i, v^{-i}) - \big(v^i(h_\omega(v)) - h^i_t(v)\big)\Big]\Big\} \text{ and} \\
& c^j(v^j, \theta^j) = 0 \;\text{ for all } j \in I \setminus \{i\}.
\end{aligned}
$$

Note that

$$
c^i(v^i, \theta^i) - c^i(\tilde{v}^i, \theta^i) = \beta\Big[v^i(h_\omega(v)) - h^i_t(v) - \big(\tilde{v}^i(h_\omega(\tilde{v}^i, v^{-i})) - h^i_t(\tilde{v}^i, v^{-i})\big)\Big]
$$

always holds. Here, the only choice of valuations for each $j \in I \setminus \{i\}$ is v^j. Also, since the cost of investment is constant across any type $\theta \in \Theta$ with $p(\theta) > 0$, we can concentrate on the types which occur with a positive probability, and a Bayesian Nash equilibrium reduces to a Nash equilibrium in this case. Thus, we only need to analyze the choice of agent i's valuation for Nash equilibria and efficient choices.

First, consider i's incentive for choosing between v^i and $\tilde{v}^i$. For any cost type $\theta^i \in \Theta^i$ which occurs with a positive probability, the total utility from choosing v^i when the valuations of other agents are v^{-i} is

$$
-c^i(v^i, \theta^i) + \beta\Big[v^i(h_\omega(v)) - h^i_t(v)\Big],
$$

and that from choosing $\tilde{v}^i$ is

$$
-c^i(\tilde{v}^i, \theta^i) + \beta\Big[\tilde{v}^i(h_\omega(\tilde{v}^i, v^{-i})) - h^i_t(\tilde{v}^i, v^{-i})\Big].
$$

The difference is

$$
\begin{aligned}
& -c^i(v^i, \theta^i) + \beta\Big[v^i(h_\omega(v)) - h^i_t(v)\Big] - \Big\{ - c^i(\tilde{v}^i, \theta^i) + \beta\Big[\tilde{v}^i(h_\omega(\tilde{v}^i, v^{-i})) - h^i_t(\tilde{v}^i, v^{-i})\Big]\Big\} \\
= \; & \beta\Big[v^i(h_\omega(v)) - h^i_t(v) - \big(\tilde{v}^i(h_\omega(\tilde{v}^i, v^{-i})) - h^i_t(\tilde{v}^i, v^{-i})\big)\Big] - \Big(c^i(v^i, \theta^i) - c^i(\tilde{v}^i, \theta^i)\Big) \\
= \; & 0.
\end{aligned}
$$

Therefore, v^i and $\tilde{v}^i$ are indifferent for agent i, and both v and $(\tilde{v}^i, v^{-i})$ are Nash equilibria of the investment game.

Next, compare the social welfare between v and $(\tilde{v}^i, v^{-i})$. For v, the sum of utility of all agents is

$$\sum_{j\in I}\Big\{-c^j(v^j,\theta^j)+\beta v^j(h_\omega(v))\Big\} = -c^i(v^i,\theta^i)+\beta\sum_{j\in I}v^j(h_\omega(v)).$$

And for $(\tilde{v}^i, v^{-i})$, the sum of utility of all agents is

$$-c^i(\tilde{v}^i,\theta^i)+\beta\Big[\tilde{v}^i(h_\omega(\tilde{v}^i,v^{-i}))+\sum_{j\in I\setminus\{i\}}v^j(h_\omega(\tilde{v}^i,v^{-i}))\Big].$$

The difference of these two is:

$$
\begin{aligned}
& -c^i(v^i,\theta^i)+\beta\sum_{j\in I}v^j(h_\omega(v))+c^i(\tilde{v}^i,\theta^i)-\beta\Big[\tilde{v}^i(h_\omega(\tilde{v}^i,v^{-i}))+\sum_{j\in I\setminus\{i\}}v^j(h_\omega(\tilde{v}^i,v^{-i}))\Big] && \text{(A.2)}\\
\geq\ & \beta\Big[\sum_{j\in I}v^j(h_\omega(\tilde{v}^i,v^{-i}))-\tilde{v}^i(h_\omega(\tilde{v}^i,v^{-i}))-\sum_{j\in I\setminus\{i\}}v^j(h_\omega(\tilde{v}^i,v^{-i}))\Big] && \text{(A.3)}\\
& -(c^i(v^i,\theta^i)-c^i(\tilde{v}^i,\theta^i)) && \text{(A.4)}\\
=\ & \beta\Big[v^i(h_\omega(\tilde{v}^i,v^{-i}))-\tilde{v}^i(h_\omega(\tilde{v}^i,v^{-i}))\Big]-(c^i(v^i,\theta^i)-c^i(\tilde{v}^i,\theta^i)) && \text{(A.5)}\\
>\ & \beta\Big[v^i(h_\omega(v))-h_t^i(v)+h_t^i(\tilde{v}^i,v^{-i})-\tilde{v}^i(h_\omega(\tilde{v}^i,v^{-i}))\Big]-(c^i(v^i,\theta^i)-c^i(\tilde{v}^i,\theta^i)) && \text{(A.6)}\\
=\ & 0, && \text{(A.7)}
\end{aligned}
$$

in which the inequality in (A.3) follows from the allocative efficiency of h; the inequality in (A.6) follows from equation (A.1). Therefore, $(\tilde{v}^i, v^{-i})$ is not an efficient investment profile although it is supported by a Nash equilibrium. Hence, there is an inefficient equilibrium of the investment game, and efficient *ex ante* investments are not Bayesian Nash implementable given h.

[2] When h is strategy-proof. We consider a slight modification of Example 4 of Hatfield et al. (2015); an auction where two agents bid for a single good. Consider any social choice function h which is allocatively efficient and strategy-proof. Suppose $\{i,j\}\subseteq I$ and $\{\omega^i,\omega^j\}\subseteq\Omega$. Since $|I|$ and $|\Omega|$ may be more than two, we choose the sets of valuation

functions in the following way:

$$V^i = \{a\mathbb{1}_{\{\omega=\omega^i\}} : a \in [0,10]\},$$
$$V^j = \{a\mathbb{1}_{\{\omega=\omega^j\}} : a \in [0,10]\},$$
$$V^k = \{0\} \text{ for any } k \in I \setminus \{i,j\}.$$

Here ω^i and ω^j each represent the alternatives where i and j obtain the item respectively. Consider the following cost functions: for any $\theta \in \Theta$ with $p(\theta) > 0$,

$$c^i(a\mathbb{1}_{\{\omega=\omega^i\}}, \theta^i) = \frac{1}{6}\beta a^2,$$
$$c^j(a\mathbb{1}_{\{\omega=\omega^j\}}, \theta^j) = \frac{1}{4}\beta a^2,$$
$$c^k(0, \theta^k) = 0 \text{ for any } k \in I \setminus \{i,j\}.$$

Since the utility of agents other than i and j is always zero, focus on the investment choices of agents i and j. Also, since they have the same cost of investment for any cost types which occur with a positive probability, we can concentrate on such types, and a Bayesian Nash equilibrium reduces to a Nash equilibrium.

First, consider efficient investment profiles under this allocatively efficient h. It is clear that only one of agents i and j should make a positive investment. If agent i obtains the item, the optimal choice of valuation should be

$$\arg\max_{a\in[0,10]} \beta\Big\{-\frac{1}{6}a^2 + a\Big\} = 3.$$

If agent j obtains it, the optimal choice of valuation should be

$$\arg\max_{a\in[0,10]} \beta\Big\{-\frac{1}{4}a^2 + a\Big\} = 2.$$

The social welfare achieved by $(3\mathbb{1}_{\{\omega=\omega^i\}}, 0)$ is

$$\beta\Big\{-\frac{3}{2} + 3\Big\} = \frac{3}{2}\beta$$

and the social welfare achieved by $(0, 2\mathbb{1}_{\{\omega=\omega^i\}})$ is

$$\beta\Big\{-1 + 2\Big\} = \beta.$$

Thus, $(3\mathbb{1}_{\{\omega=\omega^i\}}, 0)$ is the unique investment profile of i and j which maximizes the social welfare.

Then consider the other investment profile $(0, 2\mathbb{1}_{\{\omega=\omega^j\}})$, and show that it is a Nash equilibrium of the investment game. First, it is clear that the valuation of agent j is a best response to i's choice 0 because it maximizes the value of the item. Next, given $\bar{v}^j \equiv 2\mathbb{1}_{\{\omega=\omega^j\}}$,

$$\begin{aligned} & \arg\max_{v^i \in V^i} \Big\{ -c^i(v^i, \theta^i) + \beta v^i(h_\omega(v^i, \bar{v}^j)) + \beta \bar{v}^j(h_\omega(v^i, \bar{v}^j)) \Big\} \\ = \; & \arg\max_{v^i \in V^i} \Big\{ -\frac{1}{\beta} c^i(v^i, \theta^i) + v^i(h_\omega(v^i, \bar{v}^j)) + \bar{v}^j(h_\omega(v^i, \bar{v}^j)) \Big\} \\ = \; & 0 \end{aligned}$$

holds. This is because given agent j's valuation $\bar{v}^j = 2\mathbb{1}_{\{\omega=\omega^j\}}$, the equation is maximized when agent j obtains the item and agent i does not make any investments (the value of the second equation becomes 2, which cannot be achieved by any positive valuation of agent i since the sum of the first two terms do not exceed $\frac{3}{2}$). Since h is allocatively efficient and strategy-proof, $h_t^i(\cdot, \bar{v}^j)$ is written as a Groves function (Green and Laffont, 1977):

$$h_t^i(v^i, \bar{v}^j) = g(\bar{v}^j) - \bar{v}^j(h_\omega(v^i, \bar{v}^j)).$$

Hence,

$$\begin{aligned} & \arg\max_{v^i \in V^i} \Big\{ -\bar{c}^i(v^i, \theta^i) + v^i(h_\omega(v^i, \bar{v}^j)) - h_t^i(v^i, \bar{v}^j) \Big\} \\ = \; & \arg\max_{v^i \in V^i} \Big\{ -\bar{c}^i(v^i, \theta^i) + v^i(h_\omega(v^i, \bar{v}^j)) - g(\bar{v}^j) + \bar{v}^j(h_\omega(v^i, \bar{v}^j)) \Big\} \\ = \; & \arg\max_{v^i \in V^i} \Big\{ -\bar{c}^i(v^i, \theta^i) + v^i(h_\omega(v^i, \bar{v}^j)) + \bar{v}^j(h_\omega(v^i, \bar{v}^j)) \Big\} \end{aligned}$$

should hold for any cost function $\bar{c}^i : V^i \times \Theta^i \to C^i$. Thus, we have

$$\begin{aligned}
& \arg\max_{v^i \in V^i} \Big\{ - c^i(v^i, \theta^i) + \beta v^i(h_\omega(v^i, \bar{v}^j)) - \beta h_t^i(v^i, \bar{v}^j) \Big\} \\
= \;& \arg\max_{v^i \in V^i} \Big\{ - \frac{1}{\beta} c^i(v^i, \theta^i) + v^i(h_\omega(v^i, \bar{v}^j)) - h_t^i(v^i, \bar{v}^j) \Big\} \\
= \;& \arg\max_{v^i \in V^i} \Big\{ - \frac{1}{\beta} c^i(v^i, \theta^i) + v^i(h_\omega(v^i, \bar{v}^j)) + \bar{v}^j(h_\omega(v^i, \bar{v}^j)) \Big\} \\
= \;& 0.
\end{aligned}$$

This means that 0 is the best response for agent i, and hence $(0, 2\mathbb{1}_{\{\omega=\omega^j\}})$ is a Nash equilibrium of the investment game. However, this does not achieve investment efficiency given h because it is less efficient than $(3\mathbb{1}_{\{\omega=\omega^i\}}, 0)$. Therefore, there is an inefficient equilibrium of the investment game, which means that efficient *ex ante* investments are not Bayesian Nash implementable given h.

A.2 Proof of Theorem 2

For the sufficiency of commitment-proofness, first we characterize the set of PBNE of the investment game given a commitment-proof social choice function h. Whenever h is commitment-proof, the set of PBNE is characterized by the following two properties: (i) no agent chooses a costly *ex ante* investment given her cost type, and (ii) the *ex post* investment is optimal for any information set. Next, we show that any PBNE maximizes the expected social welfare when h is allocatively constrained-efficient for some $\Omega' \subseteq \Omega$ with $\Omega' \neq \emptyset$.

For the necessity of commitment-proofness, we show that whenever h is allocatively constrained-efficient but is not commitment-proof, we can construct a profile of the sets of valuations and associated cost functions for which there exists a PBNE of the investment game that does not maximize the expected social welfare.

[1] <u>Sufficiency of commitment-proofness.</u> Take any $\beta \in (0,1)$, $V \subseteq \mathbb{R}^{\Omega \times I}$ and $c : V \times \Theta \to C$, and fix them. We show that when h is commitment-proof, the set of PBNE of the

investment game given h and β is characterized by $\Sigma^* \times \mathcal{M}^*$ such that

$$\Sigma^* \equiv \left\{ \sigma \in \Sigma \;\middle|\; \text{for each } i \in I, c^i(\sigma^i(\theta^i), \theta^i) = 0 \text{ for any } \theta^i \in \Theta^i \right\}, \text{ and}$$

$$\mathcal{M}^* \equiv \left\{ \mu \in \mathcal{M} \;\middle|\; \text{for each } i \in I, \right.$$

$$\left. \mu^i(v^i, \omega, \theta^i) \in \operatorname*{arg\,max}_{\bar{v}^i \in \{\bar{v}^i \in V^i | c^i(\bar{v}^i, \theta^i) \geq c^i(v^i, \theta^i)\}} \left\{ \bar{v}^i(\omega) - c^i(\bar{v}^i, \theta^i) \right\} \text{ for any } (v^i, \omega, \theta^i) \in V^i \times \Omega \times \Theta^i, \right\}.$$

First, by the definition of a PBNE of the investment game, it is obvious that the equilibrium *ex post* investment strategies are written as $\mathcal{M}^*$.

Next, we analyze the *ex ante* investment game given the optimal *ex post* investment strategies $\mu^* \in \mathcal{M}^*$. Take any agent $i \in I$, and consider i's incentive for the *ex ante* investment when her cost type is $\theta^i \in \Theta^i$. Consider two arbitrary *ex ante* investments with the following properties: $v^{0i} \in V^i$ such that $c^i(v^{0i}, \theta^i) = 0$, and $v^i \in V^i$ such that $c^i(v^i, \theta^i) > 0$.

We can show that for any strategies $\sigma^{-i} \in \Sigma^{-i}$ of other agents, v^{0i} gives a strictly higher expected utility than v^i for agent i. To see this, take any cost types $\theta^{-i} \in \Theta^{-i}$ of other agents and let $b^{-i} \equiv b^{c^{-i}, \theta^{-i}, \sigma^{-i}(\theta^{-i})}$. Agent i's *ex ante* utility from choosing v^i for $\theta^{-i} \in \Theta^{-i}$

is written as:

$$
\begin{aligned}
& -c^i(v^i,\theta^i)+\beta\Big[\mu^{*i}(v^i,h_\omega(b^{c^i,\theta^i,v^i},b^{-i}),\theta^i)(h_\omega(b^{c^i,\theta^i,v^i},b^{-i}))-h^i_t(b^{c^i,\theta^i,v^i},b^{-i}) && \text{(A.8)}\\
& -c^i(\mu^{*i}(v^i,h_\omega(b^{c^i,\theta^i,v^i},b^{-i}),\theta^i),\theta^i)+c^i(v^i,\theta^i)\Big] && \text{(A.9)}\\
= \;& \beta\Big[\mu^{*i}(v^i,h_\omega(b^{c^i,\theta^i,v^i},b^{-i}),\theta^i)(h_\omega(b^{c^i,\theta^i,v^i},b^{-i}))-h^i_t(b^{c^i,\theta^i,v^i},b^{-i}) && \text{(A.10)}\\
& -c^i(\mu^{*i}(v^i,h_\omega(b^{c^i,\theta^i,v^i},b^{-i}),\theta^i),\theta^i)\Big]-(1-\beta)c^i(v^i,\theta^i) && \text{(A.11)}\\
< \;& \beta\Big[\mu^{*i}(v^i,h_\omega(b^{c^i,\theta^i,v^i},b^{-i}),\theta^i)(h_\omega(b^{c^i,\theta^i,v^i},b^{-i}))-h^i_t(b^{c^i,\theta^i,v^i},b^{-i}) && \text{(A.12)}\\
& -c^i(\mu^{*i}(v^i,h_\omega(b^{c^i,\theta^i,v^i},b^{-i}),\theta^i),\theta^i)\Big] && \text{(A.13)}\\
= \;& \beta\Big[b^{c^i,\theta^i,v^i}(h_\omega(b^{c^i,\theta^i,v^i},b^{-i}))-h^i_t(b^{c^i,\theta^i,v^i},b^{-i})-c^i(v^i,\theta^i)\Big] && \text{(A.14)}\\
\leq \;& \beta\Big[b^{c^i,\theta^i,v^i}(h_\omega(b^{c^i,\theta^i,v^i},b^{-i}))-h^i_t(b^{c^i,\theta^i,v^i},b^{-i}) && \text{(A.15)}\\
& -\max\Big\{0,\max_{\omega\in\Omega}\big\{b^{c^i,\theta^i,v^i}(\omega)-b^{c^i,\theta^i,v^{0i}}(\omega)\big\}\Big\}\Big] && \text{(A.16)}\\
\leq \;& \beta\Big[b^{c^i,\theta^i,v^{0i}}(h_\omega(b^{c^i,\theta^i,v^{0i}},b^{-i}))-h^i_t(b^{c^i,\theta^i,v^{0i}},b^{-i})\Big] && \text{(A.17)}\\
= \;& \beta\Big[\mu^{*i}(v^{0i},h_\omega(b^{c^i,\theta^i,v^{0i}},b^{-i}),\theta^i)(h_\omega(b^{c^i,\theta^i,v^{0i}},b^{-i}))-h^i_t(b^{c^i,\theta^i,v^{0i}},b^{-i}) && \text{(A.18)}\\
& -c^i(\mu^{*i}(v^{0i},h_\omega(b^{c^i,\theta^i,v^{0i}},b^{-i}),\theta^i))\Big], && \text{(A.19)}
\end{aligned}
$$

in which the last equation (A.18)-(A.19) is agent i's *ex ante* utility from choosing v^{0i} for $\theta^{-i}\in\Theta^{-i}$. The inequality in (A.12) holds because $c^i(v^i,\theta^i)>0$ and $\beta<1$; the equality in (A.14) follows from the definition of b^{c^i,θ^i,v^i}; the inequality in (A.15) follows from Lemma 1; the inequality in (A.17) follows from the fact that h is commitment-proof; and the equality in (A.18) follows from the definition of $b^{c^i,\theta^i,v^{0i}}$. Note that when there are more than one valuations $v^{0i},\tilde{v}^{0i}\in V^i$ such that $c^i(v^{0i},\theta^i)=c^i(\tilde{v}^{0i},\theta^i)=0$, they give exactly the same utility. Since the above inequality holds for any cost types $\theta^{-i}\in\Theta^{-i}$ of other agents, taking the expectation over Θ^{-i}, we have

$$
\begin{aligned}
v^{0i}\in\arg\max_{v^i\in V^i}\Big\{&-c^i(v^i,\theta^i)+\beta\sum_{\theta^{-i}\in\Theta^{-i}}p(\theta^{-i}|\theta^i)\\
&\Big[\mu^{*i}(v^i,h_\omega(b^{c^i,\theta^i,v^i},b^{-i}),\theta^i)(h_\omega(b^{c^i,\theta^i,v^i},b^{-i}))-h^i_t(b^{c^i,\theta^i,v^i},b^{-i})\\
&-c^i(\mu^{*i}(v^i,h_\omega(b^{c^i,\theta^i,v^i},b^{-i}),\theta^i),\theta^i)+c^i(v^i,\theta^i)\Big]\Big\}.
\end{aligned}
$$

Thus, for any strategies $\sigma^{-i} \in \Sigma^{-i}$ of other agents, the best response for agent i with cost type θ^i is to choose a least costly investment $v^{0i} \in V^i$ such that $c^i(v^{0i}, \theta^i) = 0$. As this is true for any cost type and any other agent, the set of equilibrium *ex ante* investment strategies is represented by Σ^*. Therefore, we can characterize the set of PBNE by $\Sigma^* \times \mathcal{M}^*$.

Finally, we show that the expected social welfare given h is maximized under any PBNE $(\sigma^*, \mu^*) \in \Sigma^* \times \mathcal{M}^*$. For any cost type profile $\theta \in \Theta$, the social welfare given h under any investment strategies $(\sigma, \mu) \in \Sigma \times \mathcal{M}$ is written as:

$$
\begin{aligned}
& \sum_{i \in I} \Big\{ - c^i(\sigma^i(\theta^i), \theta^i) + \beta \Big[\mu^i(\sigma^i(\theta^i), h_\omega(b^{c,\theta,\sigma(\theta)}), \theta^i)(h_\omega(b^{c,\theta,\sigma(\theta)})) && \text{(A.20)} \\
& -c^i(\mu^i(\sigma^i(\theta^i), h_\omega(b^{c,\theta,\sigma(\theta)}), \theta^i) + c^i(\sigma^i(\theta^i), \theta^i) \Big] \Big\} && \text{(A.21)} \\
= \; & \sum_{i \in I} \Big\{ \beta \Big[\mu^i(\sigma^i(\theta^i), h_\omega(b^{c,\theta,\sigma(\theta)}), \theta^i)(h_\omega(b^{c,\theta,\sigma(\theta)})) - c^i(\mu^i(\sigma^i(\theta^i), h_\omega(b^{c,\theta,\sigma(\theta)}), \theta^i), \theta^i) \Big] && \text{(A.22)} \\
& -(1-\beta) c^i(\sigma^i(\theta^i), \theta^i) \Big\} && \text{(A.23)} \\
\leq \; & \sum_{i \in I} \beta \Big[\mu^i(\sigma^i(\theta^i), h_\omega(b^{c,\theta,\sigma(\theta)}), \theta^i)(h_\omega(b^{c,\theta,\sigma(\theta)})) - c^i(\mu^i(\sigma^i(\theta^i), h_\omega(b^{c,\theta,\sigma(\theta)}), \theta^i), \theta^i) \Big] && \text{(A.24)} \\
\leq \; & \sum_{i \in I} \beta b^{c^i, \theta^i, \sigma^{*i}(\theta^i)}(h_\omega(b^{c,\theta,\sigma(\theta)})) && \text{(A.25)} \\
\leq \; & \sum_{i \in I} \beta b^{c^i, \theta^i, \sigma^{*i}(\theta^i)}(h_\omega(b^{c,\theta,\sigma^*(\theta)})) && \text{(A.26)} \\
= \; & \sum_{i \in I} \beta \Big[\mu^{*i}(\sigma^{*i}(\theta^i), h_\omega(b^{c,\theta,\sigma^*(\theta)}), \theta^i)(h_\omega(b^{c,\theta,\sigma^*(\theta)})) && \text{(A.27)} \\
& -c^i(\mu^{*i}(\sigma^{*i}(\theta^i), h_\omega(b^{c,\theta,\sigma^*(\theta)}), \theta^i), \theta^i) \Big]. && \text{(A.28)}
\end{aligned}
$$

The last equation (A.27)-(A.28) is the social welfare given h and θ under strategies (σ^*, μ^*). The inequality in (A.24) holds because $c^i(\sigma^i(\theta^i), \theta^i) \geq 0$ and $\beta < 1$; the inequality in (A.25) follows from the definition of $b^{c^i, \theta^i, \sigma^{*i}(\theta^i)}$; the inequality in (A.26) follows from the fact that h is allocatively constrained-efficient; the equality of (A.27) follows from the definitions of $b^{c^i, \theta^i, \sigma^{*i}(\theta^i)}$ and μ^{*i}. Since this holds for any cost type profile $\theta \in \Theta$, taking the expectation over Θ, a PBNE (σ^*, μ^*) maximizes the expected social welfare.

Therefore, for any $V \subseteq \mathbb{R}^{\Omega \times I}$ and $c : V \times \Theta \to C$, any PBNE of the investment game given h and $\beta \in (0, 1)$ maximizes the expected social welfare, and hence efficient investments are implemented in PBNE.

[2] Necessity of commitment-proofness. Consider a social choice function h which is allocatively constrained-efficient for some $\Omega' \subseteq \Omega$ with $\Omega' \neq \emptyset$ but is not commitment-proof. We show that for some $V \subseteq \mathbb{R}^{\Omega \times I}$, $c : V \times \Theta \to C$ and $\beta \in (0,1)$, there is a PBNE which does not maximize the expected social welfare.

First, since h is not commitment-proof, there are $i \in I$, $b \in \mathbb{R}^{\Omega \times I}$ and $\tilde{b}^i \in \mathbb{R}^{\Omega}$ such that

$$\tilde{b}^i(h_\omega(\tilde{b}^i, b^{-i})) - h_t^i(\tilde{b}^i, b^{-i}) - \left(b^i(h_\omega(b)) - h_t^i(b)\right) > \max\left\{0, \max_{\omega \in \Omega}\left\{\tilde{b}^i(\omega) - b^i(\omega)\right\}\right\}. \quad \text{(A.29)}$$

Consider the following profile of the set of valuations:

$$V^i = \{b^i, \tilde{b}^i\},$$
$$V^j = \{b^j\} \text{ for all } j \in I \setminus \{i\}.$$

Consider a profile of cost functions $c : V \times \Theta \to C$ such that for any $\theta \in \Theta$ with $p(\theta) > 0$,

$$c^i(b^i, \theta^i) = 0,$$
$$c^i(\tilde{b}^i, \theta^i) = \begin{cases} \max_{\omega \in \Omega}\left\{\tilde{b}^i(\omega) - b^i(\omega)\right\} & \text{if } \max_{\omega \in \Omega}\left\{\tilde{b}^i(\omega) - b^i(\omega)\right\} > 0, \\ \delta & \text{otherwise,} \end{cases}$$
$$c^j(b^j, \theta^j) = 0 \text{ for all } j \in I \setminus \{i\},$$

where $\delta > 0$. Any agent $j \in I \setminus \{i\}$ always chooses $b^j \in V^j$ in the investment game because there is only one choice in V^j.

First, let's find a PBNE of this investment game. Agent i has two choices b^i and $\tilde{b}^i$. Consider her optimal choice in the second investment stage. When i chooses $\tilde{b}^i$ prior to the mechanism, since $c^i(\tilde{b}^i, \theta^i) > c^i(b^i, \theta^i)$ for any cost type $\theta^i \in \Theta^i$ which occurs with a positive probability, the optimal choice of a valuation function in the *ex post* stage is $\tilde{b}^i$ for any $\omega \in \Omega$ because it is the unique choice for her. Thus, the valuation at the time of the mechanism is

$$b^{c^i, \theta^i, \tilde{b}^i}(\omega) = \left\{\tilde{b}^i(\omega) - c^i(\tilde{b}^i, \theta^i)\right\} + c^i(\tilde{b}^i, \theta^i) = \tilde{b}^i(\omega)$$

for each $\omega \in \Omega$. On the other hand, when i chooses b^i prior to the mechanism, in the *ex post* stage, she can still choose from $\{b^i, \tilde{b}^i\}$ because b^i is a costless valuation. However, by the construction of the cost function, we can see that

$$b^i(\omega) \geq \tilde{b}^i(\omega) - c^i(\tilde{b}^i, \theta^i)$$

for any $\omega \in \Omega$ and $\theta^i \in \Theta^i$ which occurs with a positive probability. Thus, the valuation at the time of the mechanism is

$$b^{c^i,\theta^i,b^i}(\omega) = \max_{\bar{v}^i \in \{b^i, \tilde{b}^i\}} \left\{ \bar{v}^i(\omega) - c^i(\bar{v}^i, \theta^i) \right\} = b^i(\omega)$$

for each $\omega \in \Omega$. To summarize, for any $\theta^i \in \Theta^i$ which occurs with a positive probability, agent i's optimal investment strategy and the valuation at the time of the mechanism is as follows:

Ex Ante Valuation	Valuation at the Mechanism	Optimal *Ex Post* Valuation
b^i	b^i	for any ω: b^i (or $\tilde{b}^i$ if $b^i(\omega) = \tilde{b}^i(\omega) - c^i(\tilde{b}^i)$)
$\tilde{b}^i$	$\tilde{b}^i$	for any ω: $\tilde{b}^i$

Given this optimal strategy in the second stage, we consider the choice of agent i in the first investment stage. Other agents' choices are fixed to b^{-i}. For any $\theta^i \in \Theta^i$ which occurs with a positive probability, the utility of agent i when choosing an investment $\tilde{b}^i$ is

$$-c^i(\tilde{b}^i, \theta^i) + \beta\Big[\tilde{b}^i(h_\omega(\tilde{b}^i, b^{-i})) - h^i_t(\tilde{b}^i, b^{-i})\Big]$$

and when choosing an investment b^i, it is

$$\beta\Big[b^i(h_\omega(b)) - h^i_t(b)\Big].$$

The difference of these two is calculated as:

$$\begin{aligned}
& -c^i(\tilde{b}^i, \theta^i) + \beta\Big[\tilde{b}^i(h_\omega(\tilde{b}^i, b^{-i})) - h^i_t(\tilde{b}^i, b^{-i})\Big] - \beta\Big[b^i(h_\omega(b)) - h^i_t(b)\Big] \\
= \ & -(1-\beta)c^i(\tilde{b}^i, \theta^i) + \beta\Big[\tilde{b}^i(h_\omega(\tilde{b}^i, b^{-i})) - h^i_t(\tilde{b}^i, b^{-i}) - c^i(\tilde{b}^i, \theta^i)\Big] - \beta\Big[b^i(h_\omega(b)) - h^i_t(b)\Big] \\
= \ & -(1-\beta)c^i(\tilde{b}^i, \theta^i) \\
& +\beta\Big[\tilde{b}^i(h_\omega(\tilde{b}^i, b^{-i})) - h^i_t(\tilde{b}^i, b^{-i}) - \Big(b^i(h_\omega(b)) - h^i_t(b)\Big) - \max\Big\{\delta, \max_{\omega \in \Omega}\{\tilde{b}^i(\omega) - b^i(\omega)\}\Big\}\Big] \\
> \ & 0,
\end{aligned}$$

in which $c^i(\tilde{b}^i, \theta^i) = \max\left\{\delta, \max_{\omega\in\Omega}\left\{\tilde{b}^i(\omega) - b^i(\omega)\right\}\right\}$ holds for sufficiently small $\delta > 0$, and the final inequality holds from equation (A.29) when we take β sufficiently close to 1 and $\delta > 0$ sufficiently small. Therefore, for any $\theta^i \in \Theta^i$ which occurs with a positive probability, agent i chooses $\tilde{b}^i$ in a PBNE, i.e., there is a PBNE $(\sigma^*, \mu^*) \in \Sigma \times \mathcal{M}$ such that for any $\theta \in \Theta$ with $p(\theta) > 0$,

$$\begin{aligned}
&\sigma^{*i}(\theta^i) = \tilde{b}^i,\\
&\sigma^{*j}(\theta^j) = b^j \text{ for any } j \in I \setminus \{i\},\\
&\mu^{*i}(b^i, \omega, \theta^i) = b^i \text{ for any } \omega \in \Omega,\\
&\mu^{*i}(\tilde{b}^i, \omega, \theta^i) = \tilde{b}^i \text{ for any } \omega \in \Omega, \text{ and}\\
&\mu^{*j}(b^j, \omega, \theta^j) = b^j \text{ for any } \omega \in \Omega.
\end{aligned}$$

However, this PBNE (σ^*, μ^*) does not maximize the expected social welfare. Consider another profile of strategies $(\sigma, \mu^*) \in \Sigma \times \mathcal{M}$ such that for any $\theta \in \Theta$ with $p(\theta) > 0$,

$$\begin{aligned}
&\sigma^{i}(\theta^i) = b^i,\\
&\sigma^{j}(\theta^j) = b^j \text{ for any } j \in I \setminus \{i\},\\
&\mu^{*i}(b^i, \omega, \theta^i) = b^i \text{ for any } \omega \in \Omega,\\
&\mu^{*i}(\tilde{b}^i, \omega, \theta^i) = \tilde{b}^i \text{ for any } \omega \in \Omega, \text{ and}\\
&\mu^{*j}(b^j, \omega, \theta^j) = b^j \text{ for any } \omega \in \Omega.
\end{aligned}$$

The only difference between σ^* and σ is that agent i chooses $\tilde{b}^i$ under σ^{*i}, but she chooses b^i under σ^i. For any $\theta \in \Theta$ with $p(\theta) > 0$, the social welfare from (σ^*, μ^*) is written as:

$$\begin{aligned}
& -c^i(\tilde{b}^i, \theta^i) + \beta\tilde{b}^i(h_\omega(\tilde{b}^i, b^{-i})) + \sum_{j\in I\setminus\{i\}} \beta\left\{b^j(h_\omega(\tilde{b}^i, b^{-i}))\right\} && \text{(A.30)}\\
< \quad & \beta\left\{\tilde{b}^i(h_\omega(\tilde{b}^i, b^{-i})) - c^i(\tilde{b}^i, \theta^i) + \sum_{j\in I\setminus\{i\}} b^j(h_\omega(\tilde{b}^i, b^{-i}))\right\} && \text{(A.31)}\\
\leq \quad & \beta\left\{b^i(h_\omega(\tilde{b}^i, b^{-i})) + \sum_{j\in I\setminus\{i\}} b^j(h_\omega(\tilde{b}^i, b^{-i}))\right\} && \text{(A.32)}\\
\leq \quad & \beta\sum_{j\in I} b^j(h_\omega(b)) && \text{(A.33)}
\end{aligned}$$

in which the last equation (A.33) is the social welfare from strategies (σ, μ^*). The inequality in (A.31) holds because $c^i(\tilde{b}^i, \theta^i) > 0$ and $\beta < 1$; the inequality in (A.32) holds because $b^i(\omega) \geq \tilde{b}^i(\omega) - c^i(\tilde{b}^i, \theta^i)$ for any $\omega \in \Omega$; the inequality in (A.33) follows from the fact that h is allocatively constrained-efficient. Therefore, there is a PBNE (σ^*, μ^*) which does not maximize the expected social welfare, and hence efficient investments are not implementable in PBNE given this h and β.

A.3 Proof of Proposition 1

For any efficient allocation rule h_ω, consider the following transfer rule h_t which divides the maximum sum of valuations equally among all agents:

$$h_t^i(b) = b^i(h_\omega(b)) - \frac{1}{n}\sum_{i \in I} b^i(h_\omega(b)). \tag{A.34}$$

It is clear that h is budget-balanced. It suffices to show that h is commitment-proof. Consider any $i \in I$, $b \in \mathbb{R}^{\Omega \times I}$, $\tilde{b}^i \in \mathbb{R}^\Omega$ and $x \geq 0$ such that $\tilde{b}^i(\omega) \leq b^i(\omega) + x$ for all $\omega \in \Omega$. We will show:

$$\tilde{b}^i(h_\omega(\tilde{b}^i, b^{-i})) - h_t^i(\tilde{b}^i, b^{-i}) - x \leq b^i(h_\omega(b)) - h_t^i(b)$$

for this transfer rule (A.34). Since $x \geq \max\left\{0, \max_{\omega\in\Omega}\left\{\tilde{b}^i(\omega) - b^i(\omega)\right\}\right\}$ holds,

$$
\begin{aligned}
& \text{(RHS) - (LHS)} \\
\geq\ & \left[b^i(h_\omega(b)) - h_t^i(b)\right] - \left[\tilde{b}^i(h_\omega(\tilde{b}^i, b^{-i})) - h_t^i(\tilde{b}^i, b^{-i})\right] + \max\left\{0, \max_{\omega\in\Omega}\left\{\tilde{b}^i(\omega) - b^i(\omega)\right\}\right\} \\
=\ & \frac{1}{n}\sum_{i\in I} b^i(h_\omega(b)) - \frac{1}{n}\left\{\tilde{b}^i(h_\omega(\tilde{b}^i, b^{-i})) + \sum_{j\in I\setminus\{i\}} b^j(h_\omega(\tilde{b}^i, b^{-i}))\right\} + \max\left\{0, \max_{\omega\in\Omega}\left\{\tilde{b}^i(\omega) - b^i(\omega)\right\}\right\} \\
=\ & -\frac{1}{n}\left\{\tilde{b}^i(h_\omega(\tilde{b}^i, b^{-i})) + \sum_{j\in I\setminus\{i\}} b^j(h_\omega(\tilde{b}^i, b^{-i})) - \sum_{i\in I} b^i(h_\omega(b))\right\} + \max\left\{0, \max_{\omega\in\Omega}\left\{\tilde{b}^i(\omega) - b^i(\omega)\right\}\right\} \\
=\ & -\frac{1}{n}\left\{\tilde{b}^i(h_\omega(\tilde{b}^i, b^{-i})) - b^i(h_\omega(\tilde{b}^i, b^{-i})) + \sum_{i\in I} b^i(h_\omega(\tilde{b}^i, b^{-i})) - \sum_{i\in I} b^i(h_\omega(b))\right\} \\
& + \max\left\{0, \max_{\omega\in\Omega}\left\{\tilde{b}^i(\omega) - b^i(\omega)\right\}\right\} \\
\geq\ & -\frac{1}{n}\left\{\tilde{b}^i(h_\omega(\tilde{b}^i, b^{-i})) - b^i(h_\omega(\tilde{b}^i, b^{-i}))\right\} + \max\left\{0, \max_{\omega\in\Omega}\left\{\tilde{b}^i(\omega) - b^i(\omega)\right\}\right\} \\
\geq\ & -\frac{1}{n}\max\left\{0, \max_{\omega\in\Omega}\left\{\tilde{b}^i(\omega) - b^i(\omega)\right\}\right\} + \max\left\{0, \max_{\omega\in\Omega}\left\{\tilde{b}^i(\omega) - b^i(\omega)\right\}\right\} \\
=\ & \frac{n-1}{n}\max\left\{0, \max_{\omega\in\Omega}\left\{\tilde{b}^i(\omega) - b^i(\omega)\right\}\right\} \\
\geq\ & 0.
\end{aligned}
$$

The second inequality holds from the allocative efficiency of h. Therefore, this h is commitment-proof and the proof is done.

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