Models of Matching Markets

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Models of Matching Markets

A dissertation presented
by
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to
The Department of Economics
in partial fulfillment of the requirements
for the degree of
Doctor of Philosophy
in the subject of
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Models of Matching Markets

Abstract

The structure, length, and characteristics of matching markets affect the outcomes for their participants. This dissertation attempts to fill the lacuna in our understanding about matching markets on three dimensions through three essays. The first essay highlights the role of constraints at the interviewing stage of matching markets where participants have to make choices even before they discover their own preferences entirely. Two results stand out from this setting. When preferences are ex ante aligned, relaxing the interviewing constraints for one side of the market improves the welfare for everyone on the other side. Moreover, such interventions can lead to a decrease in the number of matched agents. The second essay elucidates the importance of rematching opportunities when relationships last over multiple periods. It identifies sufficient conditions for existence of a stable matching which accommodates the form of preferences we expect to see in multi-period environments. Preferences with inter-temporal complementarities, desire for variety and a status-quo bias are included in this setting. The third essay furthers our understanding while connecting two of the sufficient conditions in a specialized matching with contracts setting. It provides a novel linkage by providing a constructive way of arriving at a preference condition starting from another and thus proving that the later implies the former.
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To Akanksha and our parents
Introduction

There are many interactions which fall outside the scope of a layman’s perception of a market. The interactions of men and women in the context of marriage, the assignment of school kids to schools, and matching of kidney donors to terminally-ill patients being just a few of such interactions. There is a burgeoning literature which continues to contribute to the betterment of many what we economists call as markets. More precisely, they are known as matching markets, as one simply can not choose in these markets but also needs to be chosen. It is a match between two or more participants of the market. The study and applications of matching markets has expanded the scope and our understanding of the numerous unconventional interactions with a rich theory, and innovative empirical methods. Through this dissertation, three aspects of matching market are discussed at length, which have still not received adequate attention from academics and practitioners alike. Some of the current work was created as part of a joint project with Maciej H. Kotowski.

The first essay discusses interactions in matching markets when the preferences are not completely formed. It sheds light on one such process, interviewing, in the context of job-matching process. Here the two sides of the market are firms and graduating students.

Interviewing constraints in these markets shape labor market outcomes. The first essay proposes a framework for interviewing in a many-to-one matching market where firms and graduating students have capacity constraints on the number of interviews. While intuition suggests that relaxing interviewing constraints should improve the market’s operation, this is not always the case. As participants are strategic in their interviewing decisions, interviewing constraints carry subtle implications for aggregate surplus, employment, and the distribution of
welfare. We relate five insights obtained in the essay to various features of matching markets. First, due to constraints firms frequently pass over even stellar candidates at the market’s interviewing stage and as a result, some highly-skilled students may “fall through the cracks.” Second, relaxing students’ interviewing constraints benefits all firms and only the best students, but it adversely impacts the lower-ranked students. Third, this increase in students’ capacities improves the social surplus but may decrease the number of matched agents. This may be undesirable if a social planner cares about the number of matched agents along with or as compared to, the social surplus. Fourth, in some cases a higher-ranked student may be worse off than a lower-ranked student due to firms’ interviewing constraints. We show how credible signaling can ameliorate such inefficiencies. Lastly, interviewing in the presence of a low capacity acts as a sorting mechanism and an increase in the students’ interviewing capacity may even lead to a decrease in social welfare due to reduced sorting.

The second essay deals with time, an essential component of many interactions. We examine a dynamic, multi-period, bilateral matching market, such as a labor market where workers are long-lived and production occurs over a period of time. We define and identify sufficient conditions for the existence of a dynamically stable matching. Our framework accommodates many forms of inter-temporal preference complementarities, including a taste for variety and a status-quo bias. We propose extensions of our model incorporating imperfect information and financial transfers. We relate our analysis to market unraveling and to common market design applications, including the medical residency match.

Third, matching with contracts is an interesting model of many matching markets and the studies to find the necessary and sufficient conditions for existence of stability are many. In the third essay, we clarify a relationship between two sufficient conditions for existence of stability. We prove that the unilateral substitutability property introduced in Hatfield and Kojima (2010) implies the substitutable completability property from Hatfield and Kominers (2014). A substitutable completion of a preference is a substitutable preference created by adding some sets of contracts to the original preference order. We provide an algorithm which when operated on the unilaterally substitutable preferences produces such a substitutable completion. Thus it provides a constructive
proof of the connection between the two properties.

Overall, through this dissertation, many aspects of matching markets are explored that bring forth interesting perspectives and applications. This study aspires to fuel the interest in the field for many questions that are related to the ones being answered and provide the steps in the right direction.
Chapter 1

Interviewing in Matching Markets

Matching markets are characterized by agents who have preferences over whom they interact with, unlike the commodity markets or stock markets. The major focus of studies in matching markets has been on what happens after the preferences are formed.\(^1\) This essay presents a theoretical model of interviewing, a process before preferences are finalized. It answers positive and normative questions related to the interviewing process and the implications of altering interviewing constraints at the margin. It sheds light on the existing design of certain markets and provides a framework to better design some others.

In both centralized and decentralized labor markets, agents’ preferences are formed over a long process of multiple interactions between the two sides. As an example consider the annual matching process of doctors to residency positions organized by the National Resident Matching Program (NRMP). Like any other job matching process, the doctor-residency matching process has multiple stages—application, screening of applications, interviewing, and a final matching or market clearing. The ‘preference formation’ stages before the final matching process, impose constraints due to financial and time costs. For instance, medical fellowship candidates are

\(^1\)As a result, we know about various aspects of these markets, once these final preferences shape up. For instance, among other things we know about the existence of a stable matching (Gale and Shapley, 1962), the incentives to participate in such markets (Roth, 1982), and the dynamics when the relationships last over multiple periods (Kadam and Kotowski, 2015a,b). Also see Roth (1984a), Roth (1990), Roth and Peranson (1999), Abdulkadiroglu and Sönmez (2003), Roth \textit{et al.} (2004), and Roth (2008b) along with the references therein. Important exceptions which study application or interviewing processes are Lee and Schwarz (2012), Chade \textit{et al.} (2014) and Che and Koh (2015) respectively. We discuss the related literature in Section 1.2.
expected to use their vacation days for interviewing and that imposes a clear constraint on the number of interviews a candidate can take up.\textsuperscript{2}

A compelling intuition suggests that if the interviewing constraints were completely relaxed, the first-best outcome would obtain. In practice, however, this is hardly a feasible or realistic policy prescription. Instead, the practical market design quandary is about the consequences of increasing interviewing (or application) capacities at the margin.\textsuperscript{3} Will increasing interviewing capacity move the economy towards greater surplus? What are the distributional welfare consequences? This essay provides answers to these and related questions. One may expect that a rising capacity (of interviewing) will lift all boats. However, in many situations, jobs offered to some candidates may come at a cost of not being able to offer them to some other candidates. Hence the above welfare evaluation may be, and as we show indeed is, imprecise.

We propose a two-sided matching model with non-transferable utility\textsuperscript{4} to study the impact of changing interviewing capacities. There is a finite set of firms and a continuum mass of students. All the students and firms have \emph{almost} identical preferences over each other which are common knowledge.\textsuperscript{5} Preferences, although almost aligned, are crucially dependent on what we call a ‘fitness factor.’\textsuperscript{6} This parameter takes one of the two values—‘fit’ and ‘misfit’—for

\begin{itemize}
\item[\textsuperscript{2}]This is based on personal communication with Dr. Niesen. Niesen \textit{et al.} (2015) find that the median fellowship applicant accepted 10 interviews and canceled 1 interview citing costs associated with interviewing in their survey results.
\item[\textsuperscript{3}]Interviewing constraints matter in many markets. Although interviewing data is hard to get, we report some aggregate numbers from the summary reports of NRMP’s proprietary survey data. This data shows that doctors reject interview invitations, sometimes regrettably as they are left unmatched through the main rounds. The survey indicates that there are some positions that remain unmatched after the main rounds but get filled up in the secondary rounds. We take this coupled with another survey data from Niesen \textit{et al.} (2015) as directional evidence for the relevance of interviewing constraints. We present the detailed evaluations in appendix Section A.1.2.
\item[\textsuperscript{4}]Specifically, we do not consider the models where the salary is a part of the negotiation. This is a good approximation for many entry-level labor markets where many contracts are relatively standardized (Agarwal, 2015; Avery \textit{et al.}, 2007; Roth, 1984a).
\item[\textsuperscript{5}]For instance, consider the matching market of entry-level hiring for college graduates. The students are ranked by firms as per their credentials (almost) identically by all firms. There is also a ranking over various firms (within an industry) from the students’ perspectives, which is roughly identical. Students may consider the ranking over various consulting or law firms published by some companies (Vault Rankings & Reviews) to guide their preferences.
\item[\textsuperscript{6}]In the real world, the preference of firms is determined by how well the candidates performs on the interviews and is perceived to fit with a firm’s culture in various rounds of interviewing. A recent opinion piece discussed this issue about the potential candidate’s cultural fit (Rivera, 2015, 2015). See also Chatman (1991) and Rivera (2012).
\end{itemize}
each firm–student pair. A firm learns the fitness factors only through interviewing the students. Moreover, a firm learns this information only for those students whom they interview, specific to itself. Particularly, a firm does not see the fitness values for any student with any other firm. For any firm, the misfit and uninterviewed applicants remain unacceptable. Among those students who are interviewed and found fit, a firm prefers hiring students with higher ability and all firms evaluate the students’ abilities identically. This keeps the interviewing process relevant while keeping the problem tractable. Firms and students choose their interviews optimally subject to their constraints and knowing that the fitness factor will be discovered during the interviewing process. Based on the actual interviews that take place, firms and students form preferences and match. We focus on a stable matching outcome as it has been recognized as paramount for a market’s successful operation (Roth, 2002).

We nest the above environment into a multi-stage game. In the game’s first stage, the applicants apply to (all) firms (as applications are assumed to be costless). In the next stage, the firms strategically extend interview invitations to some students. The students accept some of the interview offers given their capacity. All interviews take place and then the final matching is realized.\textsuperscript{7} Note that both firms and students are limited in the number of interviews they can conduct.

In our model, since fitness is the only information discovered through the interviewing process, a firm extends a final job offer to all students who are found fit through the interviewing process.\textsuperscript{8} Moreover, a misfit student is never extended a job offer as the firm strictly prefers to leave the position empty or offer it to some other fit student. If this is the case, a student only needs to choose the best interview offers when the fitness factors across firms is identically and independently drawn. The firms are, however, more strategic about their choices and have a mix of (some good, some average and some safe) candidates to whom they extend their interview

\textsuperscript{7}Specifically, we rule out the issues related to timing in these markets. We assume that it is prohibitively expensive for a firm to wait for the firms at the top to finish their hiring because hiring is a time-consuming process. We talk briefly about the timing of markets in Appendix Section A.1.3.

\textsuperscript{8}If a firm were to not extend a final job offer to a student despite finding her fit, the firm could choose to not extend an interview invitation at the first place. This plays out at equilibrium of the game and discussed in detail in the main model.
offers (Proposition 1). This suggests that the firms expend their scarce interviewing resource by evaluating a student’s ability and also the probability of being able to successfully hire her. We establish the existence of an (essentially) unique equilibrium of the application, interviewing, and matching game (Theorem 1). We compare the different equilibrium outcomes of these games as the students’ interviewing capacity changes. When the student interviewing capacities increase, the total surplus generated from the matching also increases (Proposition 2). However, this is not a Pareto improvement and some students at the bottom are worse off (Proposition 3).

In the setting analyzed so far if a social planner has to choose the levels for interviewing capacities, increasing them as much as possible seems optimal. However, in many real-world settings the interviewing capacity for students is restricted and, sometimes, chosen to be so. A case in point is that of the job placement process for graduates of Indian management and engineering institutes. Students at these institutions choose the list of firms they want to interview with, as many firms are concurrently invited to interview by the institutions. Although there is no explicit capacity on interviewing for the students, implicitly there is a constraint as is the case in many other settings. The placement processes at these institutes are crafted to ensure that the students get good job offers and a maximum number of students are placed. Although an increase in interviewing capacity increases utilitarian welfare, it may decrease the number of students who get matched (Proposition 4). This suggests that the capacity constraints on students may be aligned with the placement process objective.

The intuition for the above results can be easily obtained by considering a simple setting with two firms, A and B and two students i and j. It is common knowledge that firm A is better than firm B and student i is better than student j from an ex-ante perspective. However, there are fitness factors specific to every firm-student pair which can be discovered only through interviewing. Suppose the fitness factors are independent and each firm-student pair is equally likely to be a fit or a misfit. The values of surplus accrued to the firm in various matches are shown in Table 1.1.

---

9This is particularly true in cases where there is congestion at the interviewing stage (Roth, 2008b; Avery et al., 2001).

10This is based on personal communication with placement directors, alumni and current students. In addition to the number of students placed, the time it takes for each institute to complete the placement process also gets media attention (Economic Times Bureau; Financial Express Bureau).
Table 1.1: An example of two firms A and B and two students i and j. The positive numbers represent the match surplus accrued to the firm when the student is found fit in the interviewing process by the firm and the negative number is the firm’s share of the surplus if a firm hires a misfit candidate.

The (negative) positive value is accrued to the firm if the student is a (mis)fit for the firm. The student also gets the exact same values as the firm from a given match.

Under the absence of interviewing constraints, both firms and students interview with each other. The expected sum of surplus generated from the first-best outcome is $2 \times 164.375 = 328.75$ and the expected number of matches is 1.325.\(^{11}\) Now suppose that interviewing is so costly that a firm and a student can only sign up for a maximum of one interview each. With the interviewing constraints in place, firm A and student i interview with each other. Note that firm B would have preferred to interview student i although it knows that it stands a chance to hire student i only if she is found misfit for firm A and a fit for itself. This is the case as the value generated by student i (120) and the value generated by student j (30) are such that $0.25 \times 120 > 0.5 \times 30$. However, given the interviewing constraints, firm B chooses to interview student j. The expected surplus from these interviewing choices under constraints falls to $0.5 \times 200 + 0.5 \times 30 = 115$. The expected number of matches is now $0.5 + 0.5 = 1$.

Consider the case where the constraints on the students are relaxed but those on the firms are still in place. Firm B prefers to interview student i over spending the only interview spot on

\(^{11}\)This considers all the different possibilities of fitness factors for the four pairs between the firms (A and B) and students (i and j). The following table summarizes these outcomes where 1 stands for the firm-student pair being fit, × represents a misfit, and . stands for any realization of the fitness value for the pair.

\[\begin{array}{cccccc}
\text{Ai} & \text{Bj} & \text{Aj} & \text{Bi} & \text{Probability} & \text{Firms’ share of surplus} & \text{No. of matches} \\
1 & 1 & . & . & 1/4 & 230 & 2 \\
1 & \times & . & . & 1/4 & 200 & 1 \\
\times & . & 1 & 1 & 1/8 & 220 & 2 \\
\times & . & 1 & \times & 1/8 & 100 & 1 \\
\times & . & \times & 1 & 1/8 & 120 & 1 \\
\times & 1 & \times & \times & 1/16 & 30 & 1 \\
\times & \times & \times & \times & 1/16 & 0 & 0 \\
\end{array}\]
student $j$. The expected social surplus is $0.5 \times 200 + 0.25 \times 120 = 130$ and the expected number of matches is $0.5 + 0.25 = 0.75$ with a higher interviewing capacity for the students. The expected social surplus has increased in this case. However, the expected number of matches has gone down from 1 to 0.75. The welfare, in terms of the expected utility and probability of getting a job, for student $i$ has increased at the cost of welfare for student $j$. The expected welfare for firm 2 strictly increases.

It is not surprising that the system approaches efficiency and the surplus from matching increases as we relax the interviewing constraints. Interviewing capacities act like frictions and prevent the attainment of the first-best. Roughly speaking, it is expected that when the frictions are reduced, more surplus will be generated.\footnote{This holds if there is enough agreement about preferences before the interviewing stage. We present an example in the next section, which actually highlights that this intuition can be misleading when there is enough heterogeneity.} The surprising feature is that the increase in surplus comes with a reduced number of matched agents. From a social planner’s perspective, if the goal is not only to raise the social surplus but also to ensure that a maximum number of positions get filled up then the two objectives can come directly in conflict with each other. Moreover, if students’ preferences exhibit risk aversion where being unemployed leaves the students with a significantly lower utility (e.g. a large negative value rather than 0 in the context of the above example), this could also reduce the utilitarian surplus.\footnote{This result relies on the assumption that being unemployed or leaving a job unfilled has different implications for the students and firms respectively. This asymmetry drives the wedge between socially optimal first-best and the second-best outcome.} We however, assume risk neutrality of all the agents for most of the discussion.

The baseline model expands upon the preceding example’s conclusions. After evaluating the impact of interviewing capacity change on the overall outcomes, we investigate the effects on individual welfare for different agents in the economy. We show that increasing capacity for the students increases the welfare for all firms and the students at the top. However, the students in the lower ability range are worse off due to the capacity increase. It is clear that since the number of matched agents could be lower, it does not lead to a Pareto improvement.\footnote{This channel is different from the endogenous channel that leads to different kind of jobs being offered by an employer when search frictions reduce as discussed by Acemoglu (1999) and Autor (2001).}

We can precisely
identify the agents who benefit and those who are hurt from a capacity increase.

To check the robustness of our results, we extend our model in various directions. Specifically, we introduce correlation in the information discovered during the interviewing process, fitness, and also different ex-ante preferences. We also show that although basic intuition suggests that interviewing constraints are wasteful and their impact should be minimized, they retain a key benefit. They facilitate sorting of agents among different firms if there are information asymmetries. When interviewing capacities increase, the informativeness of a student choosing to interview with a firm decreases and this may lead to a reduced social surplus.

However, we do not mean to suggest that reducing interviewing constraints is bad for the overall economy but just that a social planner should be aware of the potential winners and losers from such a reduction to evaluate the correct impact. We also relate various insights from our framework to some anecdotal phenomena (e.g. strategic choices by firms, students falling through the cracks, etc.) and some institutional settings (e.g. the hiring processes at the Indian Institutes of Management (IIMs)). We hope the tractable framework we provide will be further used for the study of interviewing processes.

The remainder of this chapter is organized as follows. Section 1.1 presents some examples with 2 firms and a continuum of students to describe our general model’s main results in simple settings. Section 1.2 describes the connection of this essay with the existing literature. In Section 1.3, we present the general model. Sections 1.4 and 1.5 establish the main results. Section 1.6 discusses applications and extensions of the main model where we introduce correlation in the fitness factor, and ex-ante preference heterogeneity amongst students and firms. Section 1.7 concludes.

1.1 Numerical examples with 2 firms

In this section, we present three numerical examples to describe some of the key insights from this essay. The first example presents the main results about strategic choices by the firms and the impact of an increase in interviewing capacity on the overall welfare, the number of matched agents, and changes in the distribution of welfare. This illustrates our findings in the baseline model and the extensions. Thereafter, we consider two variants where not all of the results in the
main model hold. These variants highlight the crucial assumptions which drive the main results and show some surprising effects of interviewing constraints. In the first variant we introduce heterogeneity on the fitness factors with respect to the firms and show that some higher-ranked students could be worse off ex-ante than lower-ranked students. In the second variant, we consider heterogeneous student preferences and show that an increase in interviewing capacity can lead to a lower welfare.

We start by describing the general setting which is common to all the examples. There are 2 firms and a continuum of students of mass 1. The two firms are labeled 1 and 2. A student’s type is \( \theta = (\succ^\theta, e^\theta, f^\theta) \) drawn from a distribution \( G \) with support \( \Theta = \{1 \succ 2, 2 \succ 1\} \times [0, 1] \times \{-1, 1\}^2 \). The first component is the student’s preference over firms. The next component \( e^\theta \) is the students’ ‘ability’ and the last component is a 2-dimensional vector which has the student’s firm-specific ‘fitness factors.’ We assume that the distribution of the students over \( \Theta \) is such that \( e^\theta \) is uniformly distributed over \([0, 1]\). Moreover, the fitness factor with firm \( k \) is independent of \( \succ^\theta, e^\theta \) and other fitness factors, and takes value 1 (which indicates that the student is a fit for the firm) with probability \( p_k \) and \(-1\) (indicating a misfit) with probability \( 1 - p_k \).

We outline in Figure 1.1 the market’s timeline. We assume that the ability parameter is common knowledge and based on that information firms decide their interview offers. A student accepts some interview offers from the ones she receives. The interviews take place and the firms learn the fitness factor perfectly for the students they interview. The students and firms match based on the preferences formed at the end of the interviewing process.\(^{15}\)

### 1.1.1 Students’ preferences agree about firms’ rankings

In the first example, all students prefer firm 1 over firm 2. The fitness factor with firm \( k \) is 1 with probability \( p_k \). Here we assume that \( p_1 = p_2 = 0.2 \). Each firm has a hiring quota of 0.07 mass of students. There is a maximum number of interviews that the students (firms) can take up (conduct) which represents the capacity constraint. Each firm can interview up to 0.39 mass of students.

\(^{15}\)We specifically assume that firms do not extend offers immediately after they discover a student’s fitness factor. The strategic choice about the timing of the interviews and/or job offers is an important issue, which we abstract away from, in the scope of this discussion. We briefly discuss this issue in appendix Section A.1.3.
students. Similarly, the students have a capacity on the total number of interviews they can take up. In the low-capacity regime, students can interview with up to $k_{LC} = 1$ firm and in the high capacity regime they can interview with up to $k_{HC} = 2$ firms. We keep the firm interviewing capacity fixed at 0.39 in both the regimes for ease of comparison. We also make an assumption that the firms have a slight distaste for interviewing. This assumption will ensure that a firm will not interview more students than it needs to to fill up its hiring quota. A firm will never interview students just to fill up its interviewing capacity.

The first component $e^q$ summarizes the relative desirability of the student, if she is found fit. More precisely, the surplus generated by firm $k$ and student $\theta$ is given by $2U(k, e^\theta)$ and is assumed to be split equally; where

$$U(k, e^\theta) = \begin{cases} e^\theta & \text{if the student is a fit for firm } k, \\ -\infty & \text{if the student is a misfit for firm } k. \end{cases}$$

We describe the strategic decisions of firms and students. As the interactions take place over time, we can start evaluating the decisions from the end of the timeline. Once the preferences are formed based on the interviewing process outcomes, there is a unique stable matching between

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16 The results are not knife-edge and do not depend on the exact choice of these numbers. We use these numbers for ease of calculations.

17 Note that the match utility is such that there will be a positive assortative matching absent any constraints, due to supermodularity.

18 The definition of stability is adapted from Azevedo and Leshno (2016)’s definition of stability to account for
the firms and students as all students’ rank ordering over (acceptable) firms is identical. We have assumed that the firms dislike interviewing more students than ‘required’ and hence they will extend a job offer to all the students who are found fit. Since fitness is the only information revealed in the interviewing process if a firm were to not extend an offer to a student found fit, it will not interview them at the first place or its interview offers will be rejected by the students accordingly.\(^\text{19}\)

We first investigate the firm interviewing strategies in the low-capacity regime. A student always accepts an interview offer from the best firm. The best firm knows that if it interviews some mass of students, it would find 20% of them to be fit or put differently, a fifth of that mass of students will be employable by firm 1. Since firm 1 has a hiring quota of 0.07, it decides to interview the best 0.35 mass of students. Thus it chooses to extend interview offers to all students with ability \(e^q \in [0.65, 1]\). In the low-capacity regime, since each student can only accept a single interview offer all the students who would be interviewed by firm 1 are effectively not available to firm 2 for interviewing. The fitness factor with firm 2 is 1 with probability 0.2. Firm 2 will also find 20% or a fifth of the mass employable from any set of students it chooses to interview. Firm 2 will interview the best students not interviewed by firm 1 as each student can only interview with one firm. Hence it will extend interview offers to all students with \(e^q \in [0.3, 0.65]\). These interviewing choices are summarized in Figure 1.2.

We now continue with the analysis and look at the firms’ interviewing strategies in the high-capacity regime. In this setting, a student can interview with up to 2 firms. The choice for the first firm is unaffected. Firm 1 continues to extend interview offers to all students with ability \(e^q \in [0.65, 1]\).

In the high-capacity regime, firm 2 can extend interview offers to any of the students and they will accept the offers. Let us assume that the firm’s optimal strategy is to continue with its old interviewing strategy of extending interview offers to students with ability \(e^q \in [0.3, 0.65]\). multiplicity introduced by the continuum setting. This will be discussed in detail in the next section.

\(^{19}\)Note that due to the continuum assumption and continuity of firm utility on the ability dimension, a firm will never have to randomize in extending job offers after the student is found fit. For example, firm 1 can choose to interview the top \(5x\) mass if it is looking to hire \(x\) students since the probability of finding a student fit is \(p = 0.2\).
Students’ ability | 0 | 1

Firm 1’s interview offers | 0.65

Firm 2’s interview offers | 0.3

Figure 1.2: Interview offers by firms 1 and 2 when students can accept only one interview offer. Firm 1 interviews the best students available. Firm 2 interviews the best students who can accept its interview offer.

We can check if the firm has a profitable deviation from this conjectured optimal. The lowest ability student is found fit with probability 0.2 and hence generate a utility of 0.06. Note that the students at the top with ability close to 1 are being interviewed by firm 1. These students are found misfit, and hence available for firm 2, with probability 0.8. Firm 2 is only interested in hiring those candidates who are fit for it. Thus, firm 2 will find the student at the top employable with probability of $0.8 \times 0.2 = 0.16$ and expected utility of $0.16 \times e^0 = 0.16 \times 1 = 0.16$. This shows that the firm will be better off to interview the students with ability close to 1 rather than interviewing students with ability around 0.3. In fact firm 2 continues to move its interview offers to the students who are also being interviewed by firm 1 till it no longer remains optimal. In this case, we observe that $0.2U(2, 0.56) = \frac{0.2 \times 0.56}{2} = 0.16U(2, 0.7)$. Incidentally, this does not violate the firm’s hiring quota of 0.07 mass of students and its interviewing capacity of 0.39. The interviewing strategies for both firms are summarized in Figure 1.3.

In the low-capacity regime, firm 2 interviewed only 0.35 mass of students and successfully filled up its hiring quota. However, in the high-capacity regime, although firm 2 fills up its interviewing capacity of 0.39, it now has vacant positions following the match outcome.\(^{20}\)

We pause here to highlight the important characteristics of the interview offers and the

\(^{20}\)Firm 2 hires 0.066 mass of students or 5.7% fewer students. It gets $0.2 \times 0.09 = 0.018$ mass from interviewing the students without an interview from the best firm and $0.2(0.8) \times 0.3 = 0.048$ mass from interviewing those with such an offer.
Students’ ability 0 |-----------------------------------------------| 1

Firm 1’s interview offers
0.65

Firm 2’s interview offers
0.56  0.7

Figure 1.3: Interview offers by firms 1 and 2 when students can accept up to two interview offers. Firm 1 interviews the best students available. Firm 2, however, chooses not to and strategically extends interview offers with a gap in the middle.

resulting outcomes across the two regimes.

1. In the low-capacity regime, firm 2 would have preferred to interview the students at the very top but can not as the students’ interview capacity binds. The effective value of expending an interview slot on the students with $e^q = 1$ is greater than that on the students with $e^q = 0.3$.

2. All students in the ability region $[0.65, 0.7)$, who are not hired by firm 1 but would have been found fit for firm 2 (if interviewed) i.e. $f_2^q = 1$, fall through the cracks.

3. Firm 2 does strictly better under the high-capacity regime as it still has the option to extend the same interview offers as in the low-capacity regime but chooses not to.

4. Firm 2 hires a smaller mass of students in the high-capacity regime in spite of interviewing more students (and exhausting its interviewing capacity).

The last two points above highlight the increase in surplus that may be accompanied by a decrease in the number of matched agents. The welfare for the second firm is strictly higher. The expected utility for students above the ability threshold of $e^q \geq 0.7$ increases from $0.2U(1, e^q)$ to $0.2U(1, e^q) + 0.16U(2, e^q)$. The probability of getting a job for these students increases from 0.2 to 0.36. However, the expected utility and probability of getting a job for students with ability
$e^\theta \in [0.3, 0.56)$ decrease from positive values to 0. These results are not particular to this example and we prove through the main results that they hold in a more general setting as well.

### 1.1.2 A better student does worse than a worse student

We consider a slightly different setting to highlight the possibility that a student with a better ex-ante score or evaluation can be worse off. Consider the setting where the students prefer a job from firm 1 over that from firm 2. The probabilities that a student is fit are not the same across firms. Specifically, for firm 1, this value $p_1$ is 0.2 and that for firm 2, $p_2$ is 0.5. The two firms want to hire 0.07 and 0.18 mass of students respectively. They both have an interviewing capacity of 0.39. The students can interview with up to 2 firms. The surplus generated by firm $k$ and student $\theta$ when the student is a fit for the firm is given by $2U(k, e^\theta)$;\(^{21}\) where

$$U(k, e^\theta) = \begin{cases} e^\theta_k & \text{if the student is a fit for firm } k, \\ -\infty & \text{if the student is a misfit for firm } k. \end{cases}$$

As in the example above, the firms make the strategic decisions about their interview offers. Again, it is easy to see that the first firm interviews the top 0.35 mass of students and the choice for firm 2 is exactly the same as described in Figure 1.3.\(^{22}\) We observe that the students with ability $e^\theta \in [0.65, 0.7)$ fall through the cracks if they do not get a job offer from firm 1. However in this case, these students are worse off even in an ex-ante sense as compared to students with ability $e^\theta \in [0.56, 0.65)$. Consider the ex-ante expected utility of a student with ability $e^\theta$ who gets an interview offer from firm 1. It is $p_1 \times U(1, e^\theta) = 0.2e^\theta$. This is smaller than the ex-ante expected utility for this student, if she got an interview offer from firm 2 instead. This expected utility is $p_2 \times U(2, e^\theta) = 0.5 \times e^\theta/2 = 0.25e^\theta$. Counter-intuitively, the students with ability in $[0.65, 0.7)$ regret having a greater score on the ability dimension. Note that it is not an equilibrium for firm 2 to extend an interview offer to students in this range.

\(^{21}\)Note that the match utility is such that there will be a positive assortative matching absent any constraints.

\(^{22}\)Note that $p_1 = 0.2$ and $p_2 = 0.5$. Firm 2 wants to hire 0.18 mass of students and its interviewing capacity is 0.39. It chooses to interview the students above ability 0.7 and those in the ability range of $[0.56, 0.65)$ as $p_2 U(2, 0.56) = 0.5 \times 0.56/2 = p_2(1 - p_1)U(2, 0.7)$. 

16
1.1.3 Students’ preferences differ about firms’ rankings

Now suppose there is uncertainty about students’ preferences. They are equally likely to be 1 ∼ 2 or 2 ∼ 1. This is perfectly and privately known to the students even before the interviewing stage. The student is a fit for a firm with an equal and independent probability of 0.5, i.e. \( p_1 = p_2 = 0.5 \). Each firm has a hiring need of 0.09 mass of students and an interviewing capacity of 0.18. The students can interview with up to 1 firm in the low-capacity regime and with up to 2 firms in the high-capacity regime. A firm \( k \) and a student with ability \( e^\theta \) generate an expected surplus of \( 2U(k, e^\theta) \). The firm always gets \( U(k, e^\theta) \) given by the following.

\[
U(k, e^\theta) = \begin{cases} 
  e^\theta & \text{if the student is a fit for firm } k, \\
  -\infty & \text{if the student is a misfit for firm } k.
\end{cases}
\]

A student, however, gets an \( \epsilon \) higher utility if the firm is her (ex-ante) first choice firm and an \( \epsilon \) lower payoff if it is not. Thus a student of ability \( e^\theta \) with a preference 1 ∼ 2 gets a utility of \( U(1, e^\theta) + \epsilon = e^\theta + \epsilon \) if she gets a job offer from firm 1 and if she is a fit for firm 1. This student gets a utility of \( e^\theta - \epsilon \) with a job offer from firm 2 if she is a fit for the firm.

In the low-capacity regime, a student simply chooses to interview with her top choice firm if faced with two interview offers. A firm chooses to extend interview offers to the best students knowing that only half of them will accept its interview offers. Thus the firms extend interview offers to the top 0.36 mass of students and will be sure that only 0.18 mass of students will accept each firm’s interview offers. Thus interviewing acts as a sorting mechanism in this case. The interview offers for both firms are summarized in Figure 1.4. The dashed lines indicate that the firms do not interview all students in that ability range but only those who accept their interview offers.

In the high-capacity regime, a student can accept offers from both firms. The firms can no longer extend interview offers and expect the students to sort themselves according to their preferences. Thus, the sorting benefit is lost with increased interviewing capacity. The strategic choices by the firms are shown in Figure 1.5. The solid lines indicate that firms extend interview offers to all students in that ability range. The dotted line indicates that only one of the firms
extends an interview offer to the student with that ability. However, how the firms decide sharing of these students is not uniquely defined.\textsuperscript{23}

To evaluate the optimality of these, we need to find the probability that a student is available for a given firm when the student is interviewed by both firms. Consider a high ability student who gets an interview offer from both firms. Recall that $p_1$ is the probability that a student is fit for firm 1 and $p_2$ is the same for firm 2. The probability that firm 1 is able to hire this candidate $= \text{Probability (student’s preference is 1 > 2)} \times p_1 + \text{Probability (student’s preference is 2 > 1)} \times (1 - p_2) \times p_1 = 0.5 \times 0.5 + 0.5 \times 0.5 \times 0.5 = 0.375$. A low-ability student, who gets an offer from only one of the two firms, is found fit with probability 0.5. The firms’ strategies are optimal if the expected utility from interviewing the lowest-ability students in the two regions, the one where both compete and the one where they divide up the set of students, are equal. Both firms choose to extend interview offers to the students at the top in the ability range $[0.94, 1]$ and the students in the ability range $[0.7, 0.94)$ get an interview offer from only one of the two firms.\textsuperscript{24}

\textsuperscript{23}There are multiple strategies for the two firms which are effectively similar. Firm 1 can choose to extend interview offers to students such that $e^6 \in [0.7, 0.79) \cup [0.85, 0.88) \cup [0.94, 1]$ and firm 2 can choose to extend interview offers to those with ability in $[0.79, 0.85) \cup [0.88, 0.94) \cup [0.94, 1]$. Clearly this is not the only distribution of students which is optimal. The other choices differ by which firm extends an interview offer to a student of a particular ability but they are essentially equivalent as only one of them extends an offer to the students in the ability range of $[0.7, 0.94)$.\textsuperscript{24}

\textsuperscript{24}We can start with a conjecture for the optimal strategy as interview offers similar to the one when the students faced a capacity constraint. Suppose that the firms ‘share’ the top 0.36 mass of students in ‘some’ way to meet their hiring needs while not violating the interviewing constraint and not competing with each other. Consider the firm which interviews students with ability 0.64 and gets an expected utility of 0.32. However, it may choose to extend an
Figure 1.5: Interview offers by firms 1 and 2 when students can accept both interview offers and the students’ preferences over firms vary.

In this case, it is easier to see that each firm hires a smaller mass of students than when there was no overlap in the interview regions. It is less obvious that the social surplus decreases. However, the sum of utilities for the firms decreases by 4% along with a 9% reduction in the number of matched agents when the interviewing costs reduce. The sum of utilities for the students should account for the idiosyncratic components that the students get. It is the case that even the sum of students’ utilities is lower in the high-capacity regime.

Consider \( y \) as the mass of interview offers that are shifted away to the students at the top. This move is optimal if \((0.64 + y) \times 0.5 = (1 - y) \times 0.375\). This leads to \(1 - y = \frac{164}{175} = 0.9371\) which we represent as 0.94 above for simplicity.

The exactly optimal interview offers are such that both firms interview students with ability \([\frac{123}{175}, \frac{164}{175}]\) and ‘share’ the students in the ability range \([\frac{123}{175}, \frac{164}{175}]\). The sum of firms’ utilities in the low-capacity regime is given by the following.

\[
\int_{0.64}^{1} 0.5 dt = \frac{1 - (0.64)^2}{4} = 0.1476
\]

In the high-capacity regime, both firms extend interview offers to students in the ability range \([1 - y, 1]\) where \(y = \frac{11}{175}\) and share the students in the ability range \([0.64 + y, 1 - y]\). The sum of firms’ utilities is the following.

\[
2 \left[ \int_{1-y}^{1} 0.375 dt \right] + \int_{0.64+y}^{1-y} 0.5 dt = 0.14172
\]

We refer an interested reader to Appendix Section A.1.4 for the calculations.
1.2 Related Literature

In this section, we review various strands of literature that are related to this work. We start with papers that have some stages of costly interviewing in the context of matching models. After interviewing models, we review the work related to application stages which are also one of the preference formation processes. We will then connect our work to the matching theory and search theory literature.

In the domain of interviewing, the closest paper is Lien (2013). He provides a model of interviewing with a finite set of firms and a finite number of students who are each looking for one position. In his setup, the firms can interview up to 2 candidates. He focuses on the non-assortative nature of interview offers and the final match in this setting. For certain parameter specifications, some students fall through the cracks due to the non-assortative nature of matching at the interviewing stage. We focus on a general many-to-one matching setting where both sides of the market face interviewing capacities and more importantly, analyze the effect of changing these capacities on welfare, the number of matched agents, and the distributional consequences.

Lee and Schwarz (2012) focus on the network aspect of an interview schedule for multiple firms and multiple agents in a one-to-one matching market. They find that interviewing schedules with maximum overlap are welfare improving as compared to the ones with less overlap. Ely and Siegel (2013) analyze the implications of revelation of intermediate interviewing decisions by firms in a common-value labor market. Their focus is on a setting where firms compete for a single worker or multiple workers who are not substitutes. They show that severe adverse selections shuts off all the firms except the top firm(s) from participation in recruiting.

Josephson and Shapiro (2013) look at information-based unemployment resulting from a schedule of interviews presented to the participants (by a central coordinating organization). Rastegari et al. (2013) solve the problem of centralized interview schedule for partially informed agents with the objective of stability and a minimum number of interviews. Using a one-to-one model they establish a computationally-efficient interview-minimizing policy. Das and Li (2014) study the impact of greater commonality about the candidates’ evaluations available before interviewing using simulations. They find that more commonality in the ex-ante quality signal
can cause firms to focus on the same candidates and reduce the match probabilities.

The process of interviewing attracts attention from applied sociologists and psychologists. Our study is the closest in its premise regarding fitness factor to findings reported by Rivera (2012). She studies the hiring decisions of elite professional service firms and provides evidence that ‘... [Hiring] is also a process of cultural matching between candidates, evaluators, and firms.’ She further suggests that, ‘Concerns about shared culture were highly salient to employers and often outweighed concerns about absolute productivity.’ Chatman (1991) finds a positive impact of person-organization fit on hiring, acclimatization, satisfaction and tenure of individuals in different organizations.

In the college-applications literature, the recent work by Che and Koh (2015) analyzes the strategic choices by colleges in extending admissions decisions when there is aggregate uncertainty about student preferences and students face costless applications. They show that there exist equilibria where colleges give more weight to the idiosyncratic elements of a student’s application to minimize head-on competition and aggregate uncertainty. We focus on the presence of interviewing constraints on both sides of the market and analyze the impact of interviewing capacity changes on the utilitarian welfare as well as welfare of different agents, i.e. top-ranked and medium-ranked students and various firms.

Avery and Levin (2010) study the college admissions problem and focus on the presence of early admissions. They show that early admissions applications being credibly limited to only one college, are a way for a student to express enthusiasm about attending a college especially when the colleges care about it. Avery et al. (2014) study the timing game for admission decisions where the colleges are the strategic players. They create a model where students have different information about their chances of admissions under different regimes. When students have more information when applying, a second ranked college prefers allowing more applications per students to less, or in terms of their model select a different examination date than the best ranked college and thus effectively allows students to apply to multiple colleges. Yenmez (2015) addresses the college admissions offers as a many-to-many matching problem with contracts.

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27 Emphasis is present in the original text.
where the contractual terms represent an early admission or regular decision among other things.

Chade and Smith (2006) focus on the problem of portfolio choice for applications for a single student. They present a greedy algorithm which solves the combinatorial optimization problem. Chade et al. (2014) talk about the equilibrium model of college admissions in a setup with two ranked colleges. With incomplete information about the student quality as seen by the colleges and incomplete information about the portfolio of students by the colleges, they generate interesting results about ‘stretch’ and ‘safety’ application portfolios. Hafalir et al. (2014) undertake a theoretical and experimental investigation about the efforts put by students when they can apply to all, i.e. in their setting both, colleges versus that exerted by the students in a decentralized setting where they can only apply to one college.

On the technical front, this essay borrows from and expands upon the setup of a finite set of firms and a continuum mass of students in Azevedo and Leshno (2016). We adapt and use their definition of stability. We make two significant changes to their setup for the problem we study. First, the preferences of both sides of the market are not completely known to the market participants. The preference of firms over students is dependent on two parameters—student’s ‘ability’ and mutual ‘fitness.’ The second departure from Azevedo and Leshno (2016) setup is that we focus on a specific case of complete agreement on the firm ranking by the students and also a complete agreement on the ‘ability’ parameter of the students by firms for most of our discussion. This keeps the model tractable and still keeps the possibility of idiosyncratic preferences of firm and students open through the fitness factor.

Kadam and Kotowski (2015a) have a section dealing with preference discovery over a multi-period horizon of the agents. Their analysis is restrictive to the cases where the first period can be viewed as interviewing but each side is limited to only one interview. They take the preferences of the agents as given and do not consider the strategic choices that we focus on in this paper. Other papers with dynamic matching models (Akbarpour et al., 2016; Baccara et al., 2015) focus on the uncertainty of the number and nature of agents present in an economy and evaluate the puzzle of when to match an accumulated pool of agents. Arnosti et al. (2014) discuss the impact of congestion as application costs are reduced in a dynamic matching market.
Chakraborty et al. (2010) generalize the study of stable matching markets to include incomplete and interdependent valuations for one side of the market. In their setting, they focus on the ex-post robustness of the match outcomes. However, the approach we take here is that the preferences are revealed before the final match through the preference formation processes. Arnosti (2015) studies the impact of short lists and various types of preferences between the participants in a centralized clearinghouse for a one-to-one matching market. He studies and highlights the preferences of participants that lead to a large number of matching and a greater quality of matching. In his setting, all agents are ex-ante homogenous and hence resolve the strategic decisions purely randomly.

Search-theoretic matching models in the tradition of DMP models (Diamond, 1971, 1982; Mortensen, 1970, 1982; Pissarides, 1979, 1985), have been successfully applied to explain many labor market phenomena. Rogerson et al. (2005) provide a comprehensive survey of the work in this field. The search-theoretic literature seeks an explanation for the equilibrium phenomenon about employment-unemployment spells, layoffs, job switches, wage dispersion, etc. Albrecht et al. (2006) study the impact of multiple applications in a directed search model and find that more than one applications results in inefficient equilibrium with wage dispersion. In this literature, most of the interactions are sequential and the market participants have to decide instantaneously whether or not to exit the market with the current match. However, in many settings especially in the entry-level markets, interviews take place over time and the uncertainty is due to strategic decisions by the agents involved. Through the current paper, we have explored the later phenomenon. There is some recent work in labor economics which use the settings of modern online labor markets which provide a greater transparency about contacts made between employers and potential employees. They analyze experimental and quasi-experimental settings to analyze the impact of reduced search costs on the aggregate outcomes of the market. In an online experiment, Horton (2015) shows that reducing employer search costs increases the number of filled vacancies by 20%.
1.3 General Model

We now describe the general model and also define the metrics—social surplus and the number of matched agents—we use to compare different interviewing capacity regimes. We describe the elements of the general model in the following order: the market participants, their preferences, and the utilities from a match.

We consider a many-to-one matching market of a finite set of $F$ firms and a continuum of students $S$ of mass 1. We denote the set of firms with a minor abuse of notation as $F = \{1, 2, \cdots, F\}$. Each firm $i$ wants to match to a continuum of students and hire only up to $q_i$ mass of students. These hiring capacities for all $F$ firms can be summarized as an $F$-dimensional vector $q = [0, 1]^F$.

All students agree on the ranking of the firms to be $\succ_S: 1 \succ 2 \succ \cdots \succ F$. Students are of different types. The type $\theta = (\succ, e, f^\theta)$ is drawn from a continuous distribution $G$ over $\Theta = \{\succ\} \times \Theta_e \times \{-1, 1\}^F$ where $\Theta_e = [0, 1]$.\footnote{We include the preference $\succ^\theta$ in a student’s type although all students’ preferences agree. We later relax this requirement in Section 1.6.2.} The $F$-dimensional vector, $f^\theta$ summarizes the idiosyncratic component of the firm’s preferences over the students. We assume that firm $i$ wants to hire only those students who are known with certainty to be ‘fit’ for the firm as summarized by the $i$th component of $f^\theta$, i.e. $f^\theta_i = 1$. The fitness factor for any firm is assumed to be independent of the ability scalar $e$, independent across the fitness factors for other firms, and is 1 with probability $p$. The distribution $G$ over $\Theta$ is such that it satisfies the above conditions and that the marginal distribution over $[0, 1]$ is uniform. Let $\eta(S')$ be the mass of a measurable set $S' \subseteq S$. We also assume that the firms are on the short side of the market, precisely $\sum_{i \in F} \frac{q_i}{p} \leq 1$. Intuitively, this assumption says that there are enough candidates to guarantee that all firms will be able to meet their hiring needs.

A firm and a fit student match to generate a surplus of $2U(i, e)$ which is split equally between them.\footnote{Equal sharing of surplus is not required for any of the results. Any fixed surplus splitting arrangement between the firms and students is sufficient.} The function $U: F \times [0, 1] \to \mathbb{R}^+$ monotonically increases with the student’s ability ($e$) and the desirability of the firm. We further assume that
Students know their $e^\theta$ and apply to firms
Each firm learns $e^\theta$ of the students applying and sends interview offers to some of them
Firms learn the firm-specific fitness factor of the students they interview
Students accept some interview offers
Students and firms match

Figure 1.6: The timing of the model.

$U(i, e^\theta) = h(i)V(e^\theta)$

such that $h : F \rightarrow \mathbb{R}_+$ and $V : [0,1] \rightarrow \mathbb{R}_+$ are the parts of surplus due to the firm and the student respectively. For all $i, j \in F$, $h(i) > h(j)$ if $i > j$ and $V(\cdot)$ is an increasing function of the student ability. Essentially, we assume that the total surplus generated will be of increasing differences and separable in the firm and student identities.

We define the economy as $E = [G, q, U]$. In addition to these parameters, the constraints on interviewing determine the market outcome. Each firm $M$ can interview with up to $q_M k_M$ students and each student can interview with up to $k_S$ firms. We can summarize the interviewing constraints for the firms and students as a vector $k$. In our discussion we will focus on two interviewing capacity regimes, a low-capacity regime ($LCr$) and a high-capacity regime ($HCr$). The student interviewing capacity is smaller in the $LCr$ than in the $HCr$ while firm capacities are assumed to be the same.

The timing of the model can be summarized as in Figure 1.6, which is similar to that in the example above in Section 1.1.1. For the purpose of the current discussion, we assume applications are costless and hence all students apply to all firms. Equivalently, we can assume that the firms know $e^\theta$ for all the students. Nevertheless, we place the “application phase” in to the timeline. Our purpose of doing so is twofold. First, it relates a firm’s discovery of the $e^\theta$ values with the realistic phenomenon of students sending their applications including their resumes and letters to the firm. Second, it leaves the model general enough to include application costs, as we plan on
1.3.1 Stable Matching and Equilibrium

We start by defining a nondegenerate set of students to remove among other things, those sets that have zero measure holes.

**Definition 1.** A *nondegenerate set of students* is defined as a set $X \subseteq \Theta_e$ such that the following holds.\(^{30}\)

1. If there is a decreasing sequence $e^{\theta_j} \in X$ that converges to $e^{\theta}$, then $e^{\theta} \in X$. Also if there is an increasing sequence $e^{\theta_j} \in X$ and $e^{\theta_j}$ converges to 1, then $1 \in X$.

2. There does not exist a student with $e^{\theta} \in X$ such that for every decreasing sequences $e^{\theta_j}$ that converge to $e^{\theta}$, we can find a corresponding $J$ such that for all $j' \geq J$, we have $e^{\theta_j} \notin X$.

Condition (1) implies that for every sequence of students, who are of progressively lower ability and have an interview offer from a firm, the limit ability student is also included in the interview offer by the firm. The next part looks at a similar increasing ability sequence converging to ability of 1. Condition (2) implies that for every student who is an element of the set, we can always find a neighborhood on the right, i.e. students with higher ability, who are also included in the set.

We will work with interview offers and match sets which respect the firm interviewing capacities and firm hiring capacities respectively. In a continuum setting, it is always possible to include or exclude countable massless points and change the interview offers (or matches) while still maintaining the capacity constraint. The nondegeneracy requirement rules out such multiplicities at the interviewing and matching stage.

**Definition 2.** A set of *nondegenerate interview offers from firm i* is defined as $\sigma_i : 2^{\Theta_e} \to 2^{\Theta_e}$ such that the following holds.

1. $\sigma_i(X) \subseteq X$ for all $X \subseteq \Theta_e$.\(^{30}\)

\(^{30}\)Technically, we want to restrict attention to finite unions of intervals of the form $(a, b)$ if $b < 1$ and $[a, 1]$ for $a \neq 1$. 26
2. $\sigma_i(X)$ is a nondegenerate set of students.

Condition (1) in the above definition is just a translation of what an interview offer would mean (even in a discrete setting). It ensures that the function maps a set of students for each firm to a subset of students to reflect the decision of a firm to extend interview offers to a subset of students who applied.

A firm does not want to hire those students whom it has not interviewed. Hence, the preferences for all firms, which we denote as $P_F$, are defined with respect to a specific interview assignment that results from the interview offers and acceptances from the students. We now define stability for a given set of preferences and denote $\emptyset$ as the outside option for a firm which is preferred over a misfit partner or a partner with unknown fitness.

**Definition 3.** A stable matching with respect to preferences $P_F$ and $P_Q$ for all the firms and students is a function $\mu : F \cup \Theta \to F \cup \emptyset \cup 2^{\Theta}$ such that

1. $\mu(\emptyset) \in F \cup \{\emptyset\}$ for all $\emptyset \in \Theta$.  
2. $\mu(i) \subseteq \Theta$ for all $i \in F$ such that $\eta(\mu(i)) \leq q_i$ and for all $\emptyset \in \mu(i)$, $\emptyset \succ_P \emptyset$.  
3. $i = \mu(\emptyset)$ if and only if $\emptyset \in \mu(i)$.  
4. $\nexists$ a firm $i$ and a student $\emptyset$ such that $\emptyset \succ_P \emptyset$, $i \succ_P \mu(\emptyset)$, and either $\eta(\mu(i)) < q_i$ or $e^\emptyset > e^{\emptyset'}$ for some $\emptyset' \in \mu(f)$.

Our definition is identical to the standard notion of stability. The first three conditions focus on an individually rational many-to-one matching between the firms and the students. Specifically, the first condition ensures that a student is matched to a firm or itself. A firm is matched to a subset of students such that its hiring quota is not violated and it prefers to match with all the students rather than not matching with some and leaving some positions unfilled. The third condition ensures that a firm is matched to only those students who are matched with it. The fourth condition ensures that there is no blocking set consisting of a firm and a student.\textsuperscript{31} We

\textsuperscript{31}Under responsive preferences, pairwise stability is sufficient for a more general stability concept defined with a blocking coalition of a firm and a set of students (Roth, 1985a).
now define a nondegenerate stable matching to avoid multiplicities of the kind that we ruled out when we defined nondegenerate interviewing offers.

**Definition 4.** A *nondegenerate stable matching* with respect to preferences $P_F$ and $P_\Theta$ for all the firms and students is a matching $\mu$ such that it is stable and the set of students matched to each firm is a nondegenerate set.

We use nondegenerate stable matching using Azevedo and Leshno (2016)’s insight to avoid inconsequential zero-mass multiplicites which could be artificially created in a continuum setting.

Note that if the preferences for firms and students are formed after nondegenerate interview offers, the stable matching can still leave out a countable number of acceptable students.\(^{32}\) We rule out such possibilities using the above definition. We implicitly assume that the final matching is generated using a student-proposing deferred acceptance algorithm,\(^{33}\) but will shortly prove that the choice of assignment mechanism (or the implicit presence of a centralized matching mechanism), is inconsequential to the final outcome. Now we are ready to define the equilibrium of the application, interviewing, and matching game.

**Definition 5.** An *equilibrium* of the application, interviewing, and matching game is

1. a strategy of applications for each student, $\sigma_S : \Theta_e \rightarrow 2^F$,
2. a strategy for each firm to extend interview offers, $\sigma_i : 2^\Theta_e \rightarrow 2^\Theta_e$ for all $i \in F$,
3. a strategy of interview acceptances for each student $\sigma_{\theta_e} : 2^F \rightarrow 2^F$ for all $\theta_e \in \Theta_e$, and
4. a set of preference reports $P_\theta$ for all $\theta \in \Theta$ and $P_M$ for all $M \in F$

such that each firm and student find its/her strategies optimal given those of the other firms and students and a nondegenerate stable matching results from the use of student-proposing deferred acceptance algorithm on the reported preferences.

\(^{32}\)This is true because the absence of blocking pair requires that mass of students matched to the firm be strictly less than its hiring quota.

\(^{33}\)See Gale and Shapley (1962) or Roth (1990) for the description of the deferred acceptance algorithm.
An equilibrium comprises of strategies for all the firms and students at each stage of the game–application, interview offers, acceptance of the interviews and the final matching. We ruled out strategies that may differ in zero measure sets using nondegenerate interview offers and matching. There is still some possibility of artificially different looking equilibria in the following sense. Consider an equilibrium where firm $M$ extends interview offers to all students with ability in $[e^q_1, e^q_2]$ and all those students reject firm $M$’s interview offers. Consider a second equilibrium where firm $M$ does not extend interview offers to these students in the first place. This following condition identifies these equilibria as essentially identical. Two equilibria are essentially identical if 1) the interview stage matching is identical, and 2) the final matching is identical.

**Definition 6.** An equilibrium is essentially unique if all equilibria that may exist are essentially identical to each other.

We use the standard measure of welfare, the social surplus generated from a matching. We also keep track of the number of matched agents to measure the aggregate employment.

**Definition 7.** The social surplus from a matching is defined as the sum of the total utilities from a match outcome.

\[
\text{Surplus of a match } \mu = \sum_{i \in F} \int_{\mu(i)} 2U(i, e^q) d\eta(\theta)
\]

**Definition 8.** The number of matched agents is defined as the total number of positions which are filled in the match outcome.

\[
\text{Number of matched agents in a match } \mu = \sum_{i \in F} \eta(\mu(i))
\]

The surplus of a matching measures the overall efficiency of the match and is a useful benchmark to evaluate the impact of the interviewing capacities. The number of matched agents measures the number of positions that get filled up. We also use surplus and the number of matched agents in the context of a single firm to mean the surplus and the number of positions filled for the firm much like the above definitions.
1.4 Equilibrium existence

We know that the firms and students face capacity constraints on the number of interviews they can conduct or participate in. We start by evaluating the strategies from the end of the timeline. We describe the choices made at the preference reporting stage, the interviewing stage, and then combine them into an equilibrium existence result.

1.4.1 Final match

The final match is generated by a student-proposing deferred acceptance algorithm. The following lemmas, which are proved in Appendix Section A.2.1, establish the final outcome and the strategic choices. All students agree over the ranking of the firms and find truth-telling (trivially) optimal.

Lemma 1. All students report their true preference at the preference reporting stage.

For a many-to-one matching, we know that truth-telling need not be optimal for the firms (Roth (1990) Theorem 5.10 and 5.14). The choice for the firms is less obvious but it turns out that the firms also tell the truth.

Lemma 2. Firms’ optimal strategy includes truth-telling, i.e. each firm lists all the acceptable students in the correct order.

We know that a stable matching exists for any set of reported preferences from Gale and Shapley (1962). The uniqueness is related to the uniqueness in Azevedo and Leshno (2016) but we give an independent proof which is applicable in our setting.

Lemma 3. There exists a stable matching and it is unique for the reported preferences by firms and students.

The uniqueness of the stable matching proves that the choice of the stable matching algorithm or even the existence of such a centralized procedure to obtain a matching is inconsequential as long as we focus on the final outcomes which are stable. We nonetheless continue to use the lens of student-proposing deferred acceptance algorithm as it splits the decision for the students and firms into interviewing decisions and preference reporting decisions, making the analysis more manageable and in line with the existing literature. We could have completely abstracted away
from a centralized mechanism and suggested that one of the possibly many stable matchings are randomly chosen after the preferences are formed.

1.4.2 Interviewing stage

With a unique stable matching and no strategic choices for the firms or students at the preference reporting stage, we focus on the strategic choices at the interviewing stage. A student faced with more interview offers than her capacity $k_s$, has to decide which ones to accept. We make the following assumption which is motivated from a distaste for interviewing to simplify students’ strategic choices significantly.\textsuperscript{34}

**Definition 9.** At equilibrium, a firm $M$ is indifferent between interviewing two sets of students $S_1$ and $S_2$, if the final set of students matched is exactly identical across the two choices, all else equal.

**Assumption 1. Distaste for interviewing** If a firm is indifferent between interviewing a set $S_1$ and set $S_2$ of students such that $S_1 \subseteq S_2$ and $\mu(S_1) < \mu(S_2)$, then the firm selects the smaller set $S_1$.

This assumption gives us the following lemma and proves the unique strategies used by the students and firms.

**Lemma 4.** The nondegenerate interview offers by firms are uniquely solvable by iterated elimination of dominated strategies when all firms have a distaste for interviewing. Moreover, the students accept the best interview offers up to their capacities.

Thus, we note that the students always select the best interview offers they receive but the choices for the firms are not as obvious.

**Proposition 1.** All firms other than the best firm, need not always interview the best students when filling up its interview capacity.

\textsuperscript{34}If we do not assume this, we just shift the burden of strategic choice on to the students. For instance, consider the interview offers from firm 1 in the example in Section 1.1.1. Firm 1 could have extended interview offers to students with ability $e^d \in [0.61, 1]$. However all students in the ability range $[0.61, 0.65)$ would choose to reject the interview offers in the low-capacity regimes as they know that inspite of getting interview invitations they will never get the final job offer. Firm 1 will exhaust its needs from interviewing the top 0.35 mass. We abstract away from all these strategic choices by making the above simplifying assumption.
Firm 2’s strategic choice in the high-capacity regime in Section 1.1.1 confirms this conclusion and we skip the proof.

1.4.3 Essentially unique equilibrium

We prove the the existence of an essentially unique equilibrium outcome although there may be many other equilibria which differ only in zero measure sets.

**Theorem 1.** *For the economy \( E = [G, q, U] \) where agents face interviewing constraints described by \( k \), there exists an equilibrium and it is essentially unique.*

We provide a brief sketch of the proof here and relegate the details to Appendix Section A.2.1. The assumption about firms’ interview aversion pins down the students’ strategic choices and in turn those of the firms following the interviews. The interviewing decision of the best firm is not dependent on that of any other firm or the students as per our discussion above and Lemma 4. The best firm, i.e. firm 1, interviews the required number of students at the top to fill up its quota \( q_1 \). It knows that the students are found fit with probability \( p \) and hence the top \( \frac{q_1}{p} \) mass will be interviewed if the interviewing capacity allows (i.e. \( \frac{q_1}{p} < k_1 q_1 \)). The second firm’s choice will account for the fact that some of the top students have an interview offer from the first firm and will be available to it only when they are found fit for itself but are found misfit with firm 1. This can be iteratively continued. Thus, the best response can be found by a firm to the strategies by other better firms. Thus, an equilibrium exists where the firm’s interview offers can be found and the students accept the best interview offers. Moreover, the continuity along the ability dimension and the exact probabilities give us a unique strategy for each firm which will be optimal up to the redundancies of zero mass of students. Thus, the equilibrium is essentially unique.

1.4.4 Equilibrium characterization

We established a firm’s optimal strategy to interview a discontinuous set of students along the ability dimension in Proposition 1. We define such firm strategies as interview offer strategies with ‘gaps.’ More formally, we mean the following.
Definition 10. Firm $i$ has gaps in its interview offers if there exists a student $q_1$ with ability $\theta^q_1$ such that $\theta^q_1 \notin \sigma_i(\Theta_c)$ and there are at least two students $\theta, \theta$ such that $\theta, \theta \in \sigma(i, \Theta_c)$ such that $\theta^q_1 \in (\theta, \theta)$.

Some firms may extend their interview offers with lots of gaps and we define a concept which will be helpful for our analysis.

Definition 11. Firm $i$ has a sufficiently large number of gaps in its interview offers if for every pair of students $\theta, \theta$ with $\theta, \theta \in \sigma_i(\Theta_c)$ such that the two students have different number of interviews from firms better than $i$, there exists a student $q_1$ with ability $\theta^q_1 \in (\theta, \theta)$ such that $\theta^q_1 \notin \sigma_i(\Theta_c)$.

Note that firm 2’s interview offers in the example in Section 1.1.1 had sufficiently large number of gaps. We present the above conditions to present some general sufficient conditions under which we get a decrease in the number of matches.\textsuperscript{35}

1.5 Main Results

We will compare the equilibrium outcomes under two regimes—a low-capacity regime and a high-capacity regime. Consider the students’ interviewing constraint in the low-capacity regime to be $k_S = k_{LC}$ and that in the high-capacity regime to be $k_S = k_{HC}$. The firm interviewing capacities are assumed to remain the same across the two regimes. We also say that all firms do not have excess interview capacity under the low-capacity regime if at equilibrium for each firm the mass of students interviewed is exactly equal to its interviewing capacity. Mathematically, we mean for all $M \in F$, we have $\eta(\sigma_M(\Theta_c) \cap \{\theta | M \in \sigma_k(\cdot)\}) = q_M k_M$. We have the following two propositions about the impact of an increase in the students’ interviewing capacity.

Proposition 2. When the interviewing regime moves from a low-capacity regime to a high-capacity regime, the utilitarian surplus of the match weakly increases. If firms do not have any excess interview capacity and the surplus strictly increases, the number of matched agents strictly decreases for at least one firm.

\textsuperscript{35}Loosely speaking, the firms have a sufficiently large number of gaps in their interview offers when the utility for students is not identical and the firms do not have large interviewing capacities.
Proposition 3. When the interviewing regime shifts from a low-capacity regime to a high-capacity regime and the interviewing offers are different, there exist two threshold abilities $e^{q_1}(<1)$ and $e^{q_2}(>0)$ such that:

1. All students with ability at or above $e^{q_1}$ are weakly better off and there exists a non-zero mass of students who are strictly better off, in terms of the expected utility from a match as well as the ex-ante probability of finding a match.

2. All students with ability strictly below $e^{q_2}$ are weakly worse off and there exists a non-zero mass of students who are strictly worse off, in terms of the expected utility from a match as well as the ex-ante probability of finding a match.

Moreover, all firms are weakly better off and there exists a non-empty set of firms such that all firms in that set are strictly better off.

The broad idea of the proofs of these propositions can be drawn from our discussion in the example in Section 1.1.1 above. When students’ interviewing capacity increases, medium-tier firms see a change in the set of students they can consider. A medium-tier firm can choose to extend interview offers to some students at the top for whom the added interviewing capacity has relaxed their constraints on the number of interviews they can accept. If such a firm has a binding interviewing capacity, it may choose to get the better students, who are available with a lower probability, than the average students, who are available with a higher probability. Moreover, the first such firm to make such a switch has a positive externality on the subsequent firms as they are left with better quality student pools. With the better quality student pools the next firm, too, chooses to interview more in the region of high ‘ability’ students and less in the region of average students. Thus, the surplus for each firm and hence the utilitarian surplus, weakly increases.\textsuperscript{36} When the firm interview offers are different, there will be some students who receive incremental interview offers from firms in the high-capacity regime. These students will be better off. As the firms’ interviewing capacities bind, new offers to the above set of students come at the cost of not extending them to some lower ranked students and these students are worse off.

\textsuperscript{36}A firm can always choose to interview the same set of students it was interviewing in the LCr but possibly may decide not to, which implies that the firm is weakly better off.
We now identify some sufficient conditions under which an increase in the utilitarian surplus will necessarily come along with a reduced number of matched agents.

**Proposition 4.** Suppose that all firms have a sufficiently large number of gaps in their interview offers in the \( \text{LCr} \) and the surplus \( U(\cdot, \cdot) \) is concave with a concave first derivative on the student ability parameter. When the interviewing regime moves from \( \text{LCr} \) to \( \text{HCr} \) then without excess interviewing capacity for the firms, the number of matched agents in the overall match weakly decreases. If the surplus strictly increases, the number of matched agents strictly decreases.

The details of the proofs of all the propositions above are presented in the appendix Section A.2.2.

### 1.6 Applications and Extensions

In this section, we extend the main model along three directions and discuss the impact of correlated fitness factors, diversity in student preferences over firms, and marginal costs of interviewing on social surplus and individual welfare.

#### 1.6.1 Correlated fitness factors

We have maintained the assumption that the firm specific fitness factors are independent of each other and of the ability parameter. In many economic settings, it is possible that a student who is found misfit for a particular firm will be misfit for some other firm with a higher probability. In the first extension, we relax the independence assumption of fitness factors across firms. In the second subsection, we will discuss a few possibilities for the correlations in fitness factors. Fitness factor can be correlated with the student ability or dependent on the firm identity.

**Fitness factor for a firm correlated with fitness factors for other firms**

We assumed in our earlier discussion that the probability that a student is found fit for any firm is equal to \( p \). This was true for both the unconditional probability and the probability conditional,
say, on being fit for some other firm. We maintain the assumption about the unconditional probability. However, we now move towards a correlated setting.

If we assume a positive correlation in the firm fitness factors, a student found fit for a particular firm will be fit for another firm with a probability higher than \( p \). With the same unconditional probabilities of being fit \( (p) \), the probability that a student, who was found misfit by some firm, is found fit by another firm will be lower than \( p \).\(^{37}\) Consider a symmetric setting where the correlation between all firms is identical. The fitness factors are binary variables and we can measure the correlation using the phi coefficient.\(^{38}\) In a symmetric setting, we assume that all firm pairs have the same phi coefficient. The correlation matrix for the fitness factors for all the firms will be represented by a \( F \times F \) matrix where all the off-diagonal elements will be equal.

However, in the current setting we also care about correlations with more than one variable beyond what can be captured in the usual correlation matrix, e.g. the probability that a student is fit for firm 3 knowing that she is found misfit for firms 1 and 2, etc. The correlation in these conditional relationships is relevant for us. Although a fully general setting requires a specification of an exponential number (in \( F \)) of parameters which are internally consistent, the setting simplifies significantly under symmetry. We can summarize the correlation information in a \( F \)-dimensional vector \( \tilde{p} = [p_0, p_{-1}, p_{-2}, \ldots, p_{-(F-1)}]' \) where \( p_{[-i]} \) is the factor by which

\(^{37}\)Consider a simple example where the unconditional probability that a student is found fit for a firm is 0.5. Moreover if the student is found fit for one firm, then the probability that the student is found fit for the second firm is 0.8. The unconditional probability of finding the student fit for firm 2 can also be evaluated as the following. Probability (student is fit for firm 1) \( \times \) Probability (student is fit for firm 2 given that she is fit for firm 1) + Probability (student is misfit for firm 1) \( \times \) Probability (student is fit for firm 2 given that she is misfit for firm 1) = 0.5 \( \times \) 0.8 + 0.5 \( \times \) 0.5 = 0.5. This suggests that if the student is found misfit for firm 1 then she will be found fit for firm 2 only with probability 0.2.

\(^{38}\)Suppose we have two random variables, fitness factors for two firms \( M \) and \( N \) and the different probabilities are summarized below.

<table>
<thead>
<tr>
<th></th>
<th>Fit for ( N )</th>
<th>Misfit for ( N )</th>
<th>Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fit for ( M )</td>
<td>( p_{11} )</td>
<td>( p_{10} )</td>
<td>( p_1 )</td>
</tr>
<tr>
<td>Misfit for ( M )</td>
<td>( p_{01} )</td>
<td>( p_{00} )</td>
<td>( p_0 )</td>
</tr>
<tr>
<td>Sum</td>
<td>( p_1 )</td>
<td>( p_0 )</td>
<td>1</td>
</tr>
</tbody>
</table>

The statistical measure of association, the phi coefficient, for the fitness factors can be calculated as follows.

\[
\phi = \frac{p_{11}p_{00} - p_{10}p_{01}}{\sqrt{p_1p_0p_1p_0}}
\]
the probability of a student being fit is reduced given that the student is misfit for \( i \) other firms.\(^{39}\)

For instance, a student found misfit by 3 firms is found fit for a fourth firm with probability \( p_{[-3]} \times p \).

Consider an example with 3 firms where the correlation vector \( \tilde{p} = [1 \ 0.8 \ 0.64] \). The probability that a student is found fit for firm 2 if she is a misfit for firm 1 is \( 0.8p \) and vice versa. Similarly the probability of finding a student as a fit for any given firm (say firm 3) if she is a misfit for the other two firms is \( 0.64p \).\(^{40}\)

The distribution over student types is again over \( \{ surpassed \} \times [0,1] \times \{-1, -1\}^F \) where \( surpassed \) is the common preference of all students over the firms. This distribution \( G^{corr} \) agrees with the correlation in the fitness factors as we described above and thus our economy now becomes \( E^{corr} = [G^{corr}, q, U] \).

We focus on essentially unique equilibria and maintain all our other assumptions about strategies and utilities for the firms and students. We discuss the extension of the equilibrium existence result, theorem 1, here. We relegate the discussion about the analogues of Proposition 2, Proposition 3 and Proposition 4 to Appendix Sections A.2.3 and A.2.4.

**Theorem 2.** In an economy \( E^{corr} \), where agents face an interviewing constraint \( k \), there exists an equilibrium and it is essentially unique.

We provide the detailed proof in appendix Section A.2.1. The existence of uniqueness is

\(^{39}\)Suppose we are interested in finding the probability that a student is fit for firm \( j \) given that she is a misfit for some \( i \) firms. Due to the symmetry of the setting, the identity of the \( i \) firms is irrelevant and the only parameter that will feature in the analysis is \( p_{[i]} \). Also note that \( p_{[i]} = 1 \) but we include it for the sake of completeness.

\(^{40}\)The **phi coefficient**, for the fitness factors for two firms in this economy will be 0.2. It is calculated as follows.

\[
\phi = \frac{p(1 - p)(1 - p)(1 - p)}{p(1 - p)^2 p_{[-1]}^2} = 1 - p_{[-1]} = 0.2
\]

It agrees with the intuition that smaller the value of \( p_{[-1]} \) greater the correlation between the fitness factors. A value of \( p_{[-1]} < 1 \) indicates that the correlation is positive and \( p_{[-1]} > 1 \) indicates a negative correlation between the fitness factors.
intuitively very similar to Theorem 1. The iterative evaluation of the firm strategies now factors in
the correlation vector that we described above but proceeds as before. The first firm decides its
interview offers so that it interviews just enough students to fill the number of job openings it has
or if that is not possible, until its interviewing capacity is reached. The presence of correlation
does not impact the choice made by the first firm. Firm 2 faces two regions of students. One
where the students have an interview offer from the first firm and another consisting of the
remaining students who do not. The students in the region with an interview from the better firm
are available only when they are found a misfit with the first firm and now further due to the
presence of correlation with smaller probability $p_{p-1}$. Firm 2 can evaluate its interview offers
optimally given the choices made by firm 1. Thus, the strategies for all the firms can be evaluated
to prove the existence of an equilibrium. Moreover, since the argument proceeds as in the earlier
case with iterated elimination of dominated strategies, the equilibrium is essentially unique. This
suggests, and we prove in the appendix Sections A.2.3 and A.2.4, that the original analysis applies
in this case.

Fitness factor of firms dependent on the student ability and the firm’s identity

The fitness factor for a firm was assumed to be independent of the student ability and the firm’s
identity. We now consider situations where the probability of finding a student fit for some firm
depends on the ability of the student and/or the rank of the firm. We consider the following three
cases.

1. The fitness probability is a function of the student ability and is given by a function
   \[ p : \Theta_e \rightarrow [0, 1]. \]

2. The probability depends on the firm identity but not on the student ability and can be
   summarized by an F-dimensional vector. The $i$th component of the vector is $p_i$ which
   corresponds to the probability of being fit for firm $i$.

3. The probability depends on both student ability and firm identity and be presented by $F$
   functions where the $i$th function corresponds to firm $i$’s fitness probability function, given
by \( p_i : \Theta_e \rightarrow [0, 1] \).

The existence result extends very easily when the fitness factor for a firm-student pair remains just dependent on the student ability but still independent across firms, i.e. \( p(e^\theta) \). The student decisions about which interview offers to accept continue to remain straightforward and hence the interviewing strategy for the firms can be solved using iterative reasoning. These arguments hold even when the probability of a student being fit is dependent on the firm identity but decreases for less desirable firms. This assumption still maintains the simplicity of student decisions when faced with many interview offers, i.e. pick the best interview offers from any pool. We get the results about the existence of an essentially unique equilibrium, the increase in surplus and the decrease in the number of matched students for at least one firm following a move from a low-capacity regime to a high-capacity regime in these settings. To avoid repetition, we state and prove these in appendix Section A.2.7. These arguments also naturally extend when the fitness factor depends on both the firm identity and the student ability as long as the probability of being fit for a given student decreases with the decreasing firm desirability, i.e. \( p_M(e^\theta) \) decreases as the firm desirability decreases. Note that, when the fitness factor varies with firm identity in any other way or specifically increases for less desirable firms, the iterative reasoning will not help us for the equilibrium existence.\(^{41}\) We do not investigate this issue any further in this work.

### 1.6.2 More general student preferences

We have maintained the assumptions that students unanimously agree about the ranking over all firms and that the firms agree entirely about the student ability parameter. We relax these assumptions and investigate equilibrium existence. We also present the result about decrease in welfare with increased interviewing capacities for the students.

Student preferences often assume more complex forms than complete agreement over firm rankings. In some cases, there is a broad agreement about the ‘tier’ to which a particular firm belongs and everyone in the market agrees on the ordering of the tiers. However, within a

\(^{41}\)Although the iterative reasoning fails, it is straightforward to analyze the equilibrium existence as we did in the example in Section 1.1.2.
particular tier there might be idiosyncratic preferences among the students. Similarly, firm’s ex-ante evaluations for students can have some correlation but need not be exactly identical. In the current discussion, we focus on student preferences which are ‘tiered.’ We discuss the related case where the students’ ability evaluations differ across firms in Appendix Section A.1.5 as the results are similar. We start by defining block-correlated preferences.

**Definition 12.** The marginal distribution of $G$ over $\Theta^-$, i.e. the distribution of student preferences, is block correlated if there exists a partition $F_1, F_2, \ldots, F_B$ of the firms such that

1. If firm $i \in F_b$, $i' \in F_{b'}$ and $b < b'$ then all students prefer $i$ over $i'$.
2. Each student’s preference within each block are uniform and independent.

Coles et al. (2013) use a variant of block-correlated preferences and demonstrate the value of signaling in a one-to-one matching market. We use these preferences for the students and establish the existence of an equilibrium. The student preferences are not identical. They are drawn from permutations of possible preferences over firms, say $\Theta^\sim$. The distribution $G$ is over $\Theta^\sim \times [0,1] \times \{-1,1\}^{F}$.

Our definition for block correlated preferences differs from that in Coles et al. (2013) as we do not impose the restriction that the firm preferences over students have to be uniform and independent. We continue with our earlier assumption that the preferences of firms are defined by the two components–ability and firm-specific fitness factor, i.e. $e^\theta$ and $f^\theta$. The type of a student also includes her preference over the firms which is not trivially unique as it was in the earlier setting. Each student $\theta$ has a type $(\triangleright^\theta, e^\theta, f^\theta)$ where $\triangleright^\theta$ is drawn from block-correlated preferences over the firms. The distribution of the student types is such that the ability value is drawn over $[0,1]$ uniformly and independently with respect to the other elements of the student type, including the preference over firms. The fitness-factor for any given firm is also independently drawn and is 1 (indicating a fit) with probability $p$ and $-1$ (indicating a misfit) with the complimentary probability $1 - p$. We now define a rank of a firm within a particular block which will be helpful for the exact specification of utilities for the students.
**Definition 13.** The rank within a block of a firm $i$ under preference $\succ$, which we denote as $\rho(\succ, i)$, is defined as the difference between the rank of $i$ on $\succ$ and the sum of the number of firms in better blocks than the block to which $i$ belongs. If $i \in F_b$ then $\rho(i, \succ) = \text{rank of } i \text{ under } \succ - \sum_{i=1}^{b-1} B_i$.

Intuitively, the rank within a block provides the position of a firm under the preference only within its block. Consider an example with 4 firms labeled 1 through 4 where they belong to either of the two blocks $B_1$ and $B_2$. The first block $B_1 = \{1, 2\}$ and $B_2 = \{3, 4\}$. The student preferences are with equal probability any ordering from the following set $\{1234, 2134, 1243, 2143\}$. Consider the preference ordering 2143 which implies that the student’s preference is $2 \succ 1 \succ 4 \succ 3$. The rank within the block for firm 4 is 1, i.e. $\rho(\succ, 4) = 1$.

In our earlier discussion, we had assumed that a firm $M$ and a fit student $q$ generate a surplus of $2U(M, e^q)$. The firms were ordered and all the students agreed on their preferences over these firms. Now the students agree on the tiers. The role of the firms’ rank is taken by the rank of the tier in this more general setting. In other words, in our main model there were $F$ blocks and each block had a single firm. Now we assume that the expected surplus generated by a firm in block $F_b$ and a fit student with ability $e^q$ is given by $2U(b, e^q)$. The firm always gets half of this surplus, $U(b, e^q)$. However, the utility a student gets from a match is slightly different as per the students’ idiosyncratic preference over firms within a block. More precisely, the student utility is the sum of $U(b, e^q)$ and an idiosyncratic element given by $e(\rho(f, \succ), b)$ where $e(\cdot, \cdot) : F^2 \rightarrow \mathbb{R}$ has the following properties.

1. The average of $e(x, b)$ is 0 over all firms in block $F_b$, i.e. if $x \in \{1, 2, \ldots, B_b\}$ where $B_b$ is the number of firms in block $F_b$ then $\sum_{x=1}^{B_b} e(x, b) = 0$.

2. $e(x, b)$ is decreasing in $x$.

We can now describe the economy as $E^{BC} = [G^{BC}, q, U, e]$. The timing of the model is the same as we discussed in our main model in Figure 1.6. We can now prove the following theorem about equilibrium existence.

**Theorem 3.** In a block correlated economy $E^{BC}$, there exist infinitely many equilibria and all of them have the following features.
1. The equilibria are essentially identical to each other or

2. The equilibria differ in the firm interviewing strategies such that any given student always gets the same number of interview offers from a given block.

We provide the detailed proof in Appendix Section A.2.5. The existence follows almost similarly to the existence in the main theorem except that here we evaluate the strategies iteratively for each block at a time rather than iteratively solving them for each firm. The uniqueness is up to the number of firms in a given block that extend their interview offers to a student for a given value of $e^q$. This leads to the multiplicity of equilibria.

When the students have different preferences, interviewing capacities act as a mechanism to sort the students in different interview positions as per their preferences. When the capacity increases, there may be reduced sorting and this could decrease the welfare. The following proposition captures this intuition.

**Proposition 5.** When the student preferences are block-correlated and the interviewing capacity for students is increased, the overall surplus of the match does not always increase.

We only need to provide an example where the overall surplus decreases with an increase in capacity. The example in Section 1.1.3 serves this purpose and we skip the proof. This serves as a caution that increasing capacities for students may not be enough to increase the social welfare in such settings.

However, when the students do not agree about the preferences over firms, there is room for students to signal their preferences to the firms (Coles et al., 2010). If a student can signal the first ranked firm about her preference then the student will benefit from getting an interview offer from her favorite firm, if the firm responds to such signals. Moreover, a firm will find it optimal to respond to such a signal as it will ensure that the firm’s interview spots are used in the most effective manner. This is possibly, a win-win situation. However, due to the potential cheap talk nature of these signals, a student has incentives to send such signals to every firm and that in turn will cause the signal to lose its value.

Limited credible signaling can solve the problems associated with the cheap talk nature of
the signals. It is analyzed in Coles et al. (2013) in a one-to-one matching setting between firms and students where the firms make up to one offer to a particular student and the students have a chance to send up to one signal to a firm before the firms send their offers. However, their analysis is silent about the impact when the firms also agree to some degree on the desirability of the students as we would expect in many situations. This question is pertinent especially given the insight in Kushnir (2013) where introduction of signaling harmfully impacts overall welfare. In his setup the student preferences are identically aligned with very high probability close to 1 and are idiosyncratic with the complementary probability. Our discussion here also elucidates upon the robustness of these insights when firms can extend more than one (interview) offers. When there are more interview offers than the recruiting capacity for each firm, a firm can choose to respond to some ‘signalers’ and some ‘non-signalers.’ We show that signaling improves welfare in our setting under certain conditions and present it in Appendix Section A.2.6

1.6.3 Marginal Interviewing Cost Setup

In the discussion so far, we assumed that the interviewing constraints manifested as capacities for the students and firms on the number of interviews. We now present an example similar to the one in Section 1.1.1 with marginal interviewing costs. It is natural to consider settings where the marginal cost is increasing. We show that our insights hold under convex interviewing costs.

Consider two firms 1 and 2 and a continuum of students. A student’s type is given by \((e^q, f^q)\) where \(e^q\) is the ability of the student drawn uniformly from \([0, 1]\). The fitness factor of a student with firm 1 is 1 with probability 0.5 and that with firm 2 is 1 also with probability 0.5. The two fitness factors are independent for all students. All students prefer firm 1 over firm 2. Each firm has a hiring quota of 0.2 mass of students. The marginal cost for firm \(M\) to interview the \(k\)th mass of students is given by \(c_M(k) = \frac{0.6}{M} k^2\) and aggregate cost for firm \(M\) to interview \(k\) mass of students is \(C_M(k) = \int_0^k c_M(t) dt = \frac{0.2}{M} k^3\). The interviewing costs for the students can be given by a function \(c : \{1, 2\} \rightarrow \mathbb{R}\) where \(c(n)\) represents the total cost of interviewing with \(n\) firms.\(^{42}\) The

\(^{42}\)This can easily be transformed in to a marginal interviewing cost but for the ease of exposition we describe them as aggregate costs for the students. However it is clear that this is not a setting where interviewing constraints appear as capacities.
surplus generated by a student with ability $e^\theta$ and firm $M$ is given by $2U(M,e^\theta)$. However, the net surplus will be the surplus after accounting for the interviewing costs.

$$U(k,e^\theta) = \begin{cases} e^\theta/k & \text{if the student is a fit for firm } k, \\ -\infty & \text{if the student is a misfit for firm } k. \end{cases}$$

We start our analysis by assuming that the student interviewing costs are high. Here, the cost of interviewing with the one firm is 0.04 and that with the two firms is 0.2. Thus, a student considers the marginal cost of $0.2 - 0.04 = 0.16$ when she decides if she wants to interview with 2 firms instead of 1 firm. Note that this is not to say that the student pays different costs based on the firm identity (firm 1 versus firm 2). It only depends on the number of interviews the students decides to take up. It is straightforward to verify that the interview offers in this setting will be as shown in Figure 1.7. Firm 1 extends interview invitation to students with ability in $[0.6, 1]$ and firm 2 extends interviews to those with ability $[0.2, 0.6)$.

Now, we evaluate the impact of reducing the interviewing costs for students. Suppose these costs of interviewing are reduced to a half of their values. The students’ cost of interviewing for one firm is 0.02 and for two firms is 0.1. Thus, a student considers the marginal cost of

\[ U(k,e^\theta) = \begin{cases} e^\theta/k & \text{if the student is a fit for firm } k, \\ -\infty & \text{if the student is a misfit for firm } k. \end{cases} \]

Figure 1.7: Interview offers by firms 1 and 2 when students’ interviewing costs are high.

Note that due to the interviewing costs for students, even the best students will not accept an interview offer from firm 2, if they are already being interviewing with firm 1. The expected share of surplus if a student accepts an interview offer from firm 2 = Probability (misfit for firm 1) $\times$ Probability (fit for firm 2) $\times$ $U(e^\theta, 2) = 0.5 \times 0.5 \times \frac{e^\theta}{2} < 0.16$ for all $e^\theta$. The firms’ optimal decisions are explained in appendix Section A.1.6.
Students’ ability

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Firm 1’s interview offers

- 0.6

Firm 2’s interview offers

- 0.35
- 0.71

Figure 1.8: Interview offers by firms 1 and 2 when students’ interviewing costs are low. Firm 1 interviews the best students available. Firm 2, however, chooses not to and strategically extends interview offers with a gap in the middle.

0.1 – 0.02 = 0.08 when she decides if she wants to interview with 2 firms instead of just 1 firm.

The interviewing strategies for the firms are described in Figure 1.8. Firm 1 continues with the same interviewing strategy and firm 2 has a different interview offers with a gap in the middle where some students with ability greater than 0.6 are not interviewed by firm 2. The optimality of these interview offers is discussed in Section A.1.6. Thus, when the firms’ marginal costs of interviewing are increasing, we get similar results.

### 1.7 Conclusion

Interviewing processes are organized in a variety of ways for the entry-level markets, which are the focus of this work. We established a tractable model of interviewing in a many-to-one matching framework to generate results that are in line with anecdotal evidence. The assumption about a continuum of students gave us a convenient (essentially) unique equilibrium. We found that some firms have gaps in their interview offers. Moreover, when the interviewing capacities increase for one side of the market and there is enough agreement about preferences before the

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44With the reduced interviewing cost for the students at least some students who get an interview offer from firm 1 will take up firm 2’s interview offers. The expected surplus from an interview offer from firm 2 is $0.125e^0$ as explained in the footnote above. All students with $e^0 \geq 0.08 \div 0.125 = 0.64$ can accept the second interview offer from firm 2, should one be made. Firm 2 extends interview offers to students with ability in $[0.353, 0.6) \cup [0.705, 1]$ which are roughly represented above. The number of matched agents to firm 2 is slightly less than its required number 0.2. We explain the optimal decisions by firm 2 and calculation about the reduced number of matched agents in the appendix Section A.1.6.
interviewing process, the utilitarian surplus increases. This, however, does not bring a Pareto improvement if the interviewing constraints matter for the firms and students. An increase in interviewing capacity for one side of the market, e.g. students, improves the welfare for agents on the opposite side and the agents at the top on the same side of the market. However lower ranked students are worse off due to an increase in students’ interviewing capacity.

An increase in interviewing capacity can be accompanied with a lower number of matched agents. If a social planner is concerned about the number of matched agents, she may choose to keep the capacities low for the market participants. This insight helped us contextualize the capacity constraints in place for the graduating students in Indian management schools. This reduction in the number of matched agents also provides a caution for an asymmetric reduction in interviewing costs when the objective is to match as many market participants as possible while improving the overall welfare.\(^{45}\) In the context of residency markets, reducing interviewing constraints with the help of technology or using coordination where it reduces the costs for both sides of the market will be advantageous.

We check the robustness of our results by extending the main model along various dimensions like existence of correlation in the firm fitness factors and diversity of ex-ante student or firm preferences over the opposite side. At the same time, we also qualify the general insight that reduced frictions improves utilitarian surplus and prove that this does not hold if there are information asymmetries. There are situations where increased student interviewing capacities reduced not only the number of matched agents but also the aggregate surplus from matching.

Our analysis offers several policy-relevant guidelines. First, if there is enough agreement about preferences before the interviewing process, we can recommend the social planner or market designer to coordinate efforts to increase the number of contacts made for both sides of the market. This will raise surplus as agents on both sides will be able to interact with more agents and have a more complete evaluation of the market. In some markets such an intervention is feasible as a central organization coordinates at least some parts of the process.\(^{46}\) Second, credible

\(^{45}\)This insight holds even when a social planner cares about the utilitarian welfare if the agents are risk-averse.

\(^{46}\)For example, the American Association for Colleges of Podiatric Medicine (AACPM) organizes a centralized
signaling mechanism can help mitigate the risk of a lower number of matched agents as it will help the coordination aspect of interviewing especially with idiosyncratic preferences. Third, if the lower-ranked students and firms can be nudged towards a greater number of interviews, it will be welfare-enhancing because these are the agents who are left unmatched due to interviewing constraints.

As interviewing constraints change for the participants of various labor markets either due to a centralized interviewing process or due to remote interviews being conducted via a video conference, a market designer needs to improve the benefits for both sides of the market appropriately to bring in Pareto improvements. The details of the market we consider matter and our current exercise underscores this point.

We now focus on aspects of the model which we did not discuss earlier and will represent some of the future directions for this work. First, the fitness factor was a digital signal with either a fit or a misfit value. If the fitness factor is a continuous variable instead, we conjecture that all the results of the model continue to hold. Second, the dependence of firm fitness factors on the firm identity in non-trivial ways is an important extension to bring the model closer to reality. Third, although there is agreement about the introduction of signaling before interviewing stages to improve the overall welfare, there is no guidance about the optimal number and nature of these. The model we analyzed can be used to take up simulations and add insights on this front.

We think that interviewing is an interesting area of matching which needs theoretical and empirical investigation to ensure better outcomes in matching markets. We provide a step in that direction through this work to understand the black-box of the interviewing phenomenon.
Chapter 2

Multi-period Matching

Gale and Shapley (1962) elegantly tackled the problems of “College Admissions and the Stability of Marriage.” By privileging stability, their analysis suggests an immutability to a match’s outcome. But, this is not what we often observe. Consider a few consequences of seemingly, or aspirationally, stable pairings:

1. After freshman year, a student transfers to another college.

2. After ten years of marriage, a couple divorces. Each then marries a new partner.

3. To repay her student loans, an MBA graduate works in management consulting for two years. Once debt free, she joins a start-up for a fraction of her old salary.

As illustrated by the preceding cases, three important characteristics color most economic and social relationships. First, relationships have a temporal component. They last multiple periods and they can be revised with the passage of time. Commitment is limited and intended long-term interactions—four years of college, a lifelong marriage, a committed career—often see interim revisions. Second, preferences are path dependent. Switching costs, a desire for variety, and inter-temporal financial constraints introduce chronological complementarities among outcomes. Finally, an agent is often uncertain about the future and refines his opinions as new information

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1 Co-authored with Maciej H. Kotowski
comes to light. Any analysis of a two-sided market where relationships are not ephemeral, as in the above examples, must address these features. A one-period model misses them all.

A central challenge in extending Gale and Shapley’s (1962) static model beyond a single period is that there is no immediate analog to their stability condition in a dynamic, multi-period framework. As emphasized by Roth (2002) and others, the long-term viability of a market often hinges upon its ability to consistently coordinate upon a stable outcome. In this essay, we propose a new stability definition—dynamic stability—that confronts the richness of a multi-period economy. Addressing the above desiderata, it allows for limited commitment, accommodates history dependence, and builds upon a robust model of agent behavior and beliefs. Drawing on familiar intuitions, a market outcome is dynamically stable if at each moment in time it is individually rational and no pair of agents can arrange a mutually-preferable relationship plan conjecturing that the wider market evolves in an unfavorable manner. The notion is succinct and open to applications and extensions.

Although it draws on classic ideas, dynamic stability provides a level of generality absent from prior studies of multi-period matching economies. Some alternative stability definitions are application specific or confined to economies satisfying stringent properties. For instance, switching costs, status-quo bias, or a taste for variety—common in the real world—are not compatible with the time-separable preference specification common to the literature. The sufficient condition ensuring the existence of a dynamically stable matching, however, accommodates these features. Furthermore, dynamic stability can be easily employed in applications involving monetary transfers or preference uncertainty. For simplicity, our exposition below centers on a two-period setting, but we extend our analysis to more than two periods in Appendix B.2 and in a companion paper (Kadam and Kotowski, 2015b).

Though we adopt Gale and Shapley’s terminology of a matching between men and women, our model’s applications extend beyond the study of interpersonal relations. Labor markets offer a germane application. According to the U.S. Bureau of Labor Statistics (2012), “individuals born from 1957 to 1964 held an average of 11.3 jobs from ages 18 to 46.” Young workers, of course,

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2Kadam and Kotowski (2015b) investigate a special case of the model developed in this study.
move between jobs frequently. But, even middle-aged workers may have volatile employment arrangements, with 32.8 percent of jobs started by those aged 40 to 46 ending in less than a year (U.S. Bureau of Labor Statistics, 2012). Such dynamics have long been recognized by the labor-market search-and-matching literature (Rogerson et al., 2005), but they have been absent from studies of matching following Gale and Shapley (1962). Our model admits such career dynamics while incorporating common labor-market features, including learning-by-doing and non-constant wage profiles.

While our analysis is primarily a positive description of a multi-period economy, much research on matching markets is motivated by normative market design applications. Our model offers an alternative perspective on these applications as well since they are often multi-period problems. For example, a celebrated instance of market design is the National Resident Matching Program (NRMP) medical residency match (Roth and Peranson, 1999). The NRMP is a clearinghouse that matches graduating medical students to hospital residency programs in the United States using an algorithm. While much attention is placed on the initial match of a trainee-doctor to a residency program, a deeper look at this market reveals a rich multi-period structure. This is hardly surprising as medical residency is a long-term engagement. Some programs, for example, only provide introductory instruction (PGY-1) and students must also match with a complementary advanced specialty (PGY-2). Others provide all years of training. Attrition and program switching occur as well (McAlister et al., 2008; Yaghoubian et al., 2012). To quote one resident’s experience:

S.M. matched to a preliminary year in Internal Medicine and an advanced position in Anesthesiology. Surprisingly, he found himself enjoying intern year much more than he had expected. At the end of intern year he moved on to Anesthesiology. Several months later S.M. realized he had been happier with the day-to-day work in Internal Medicine than in Anesthesiology. (…) [H]e arranged to finish the year in Anesthesiology and then return to the Internal Medicine program as a PGY-2. (Losada, 2010)

In light of such outcomes, an assessment of a match’s success and of participants’ welfare requires

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3 “National Resident Matching Program” and “NRMP” are registered trademarks of the National Resident Matching Program.

4 “PGY-1” stands for Post Graduate Year 1. It is the first year of training doctors receive after graduating from medical school.
a longer-term perspective, which our model provides. Its adaptation to other market design exercises, such as school assignment, follows accordingly.

Though our setting is too lean to address the details found in many market design applications, our analysis points to several themes deserving broader attention from scholars and practitioners. First, the multi-period nature of many situations introduces subtle complementarities in agents’ preferences that may undermine an assignment’s long-term stability. Mechanisms sensitive to this concern are therefore critical. Second, agents often participate in markets with limited knowledge about the future. When possible, proposed assignments should be robust to (or accommodate) agents’ path dependent, post-match learning. Finally, as elaborated upon by Kadam and Kotowski (2015b), (re-)matching frequency and assignment length are design variables. Novel solutions to otherwise complex problems may follow from their adjustment.

This chapter is organized as follows. In the following section we briefly review the related literature. In Section 2.2 we introduce our model. We also define and motivate our definition of dynamic stability. Section 2.3 identifies sufficient conditions for the existence of a dynamically stable matching and we propose a multi-period generalization of Gale and Shapley’s (1962) deferred acceptance algorithm to find stable matchings. Importantly, the existence of a dynamically stable matching is not a consequence of a naive repetition of successive one-period matching markets. In fact, consecutive “spot markets” can generate unstable outcomes, kindle regret, and encourage strategic behavior. Section 2.4 examines multi-period matching mechanisms in greater detail and we explain how the NRMP matching algorithm, mentioned above, resolves the chronological complementarities among training programs and how it differs from our procedures. Section 2.5 considers model variants and extensions. If information is imperfect we argue that re-matching after learning new information often fails to generate a Pareto improvement relative to an initial matching derived with imperfect information. We relate this observation to market unraveling, whereby agents commit to matching at earlier and earlier times. We also examine the (possibly) destabilizing consequences of savings and credit in multi-period matching markets.

5Goldacre et al. (2010) conclude that about a quarter of doctors in the United Kingdom change specialty within the first ten years of their careers. We are unaware of analogous statistics for the United States.
with transfers. Section 2.6 concludes. An online supplement collects omitted theorems and proofs (Appendix B.1), generalizes our model to $T$ periods (Appendix B.2), analyzes our model’s core (Appendix B.3), and presents additional examples and discussion (Appendices B.4 and B.5).

2.1 Literature

Our study focuses on an economy where agents interact over multiple periods, forming and revising their relationships with time. Similar two-sided,\footnote{Abdulkadiroğlu and Loertscher (2007), Bloch and Cantala (2013), and Kurino (2014), among others, study dynamic variants of one sided markets (Shapley and Scarf, 1974; Hylland and Zeckhauser, 1979). We do not study this case, though our analysis is complementary.} one-to-one matching markets are studied by Damiano and Lam (2005), Kurino (2009), and Kotowski (2015). These studies propose alternative definitions of stability emphasizing different characteristics of dynamic markets. Among these papers, Kotowski (2015) is the closest to our setting and motivation. He proposes an alternative solution concept, robust prescient stability, which is neither weaker nor stronger than dynamic stability. Self-sustaining stability, proposed by Damiano and Lam (2005), incorporates the idea of coalition proofness (Bernheim et al., 1987). Kurino’s (2009) proposal, credible group stability, draws on the intuition provided by the bargaining set (Zhou, 1994; Klijn and Massó, 2003). Our proposal, dynamic stability, emphasizes alternative elements of multi-period interactions and we discuss these at length in Section 2.2.3. An important precursor to Kurino’s (2009) analysis is Corbae et al. (2003). Corbae et al. (2003) develop a dynamic, multi-period bilateral matching model to investigate questions in monetary economics, which is not the focus of our study.

Several authors have also considered multi-period, many-to-one or many-to-many matching models. Dur (2012), Bando (2012), Pereyra (2013), and Kennes et al. (2014a) propose models in this vein. Though these papers span a family of multi-period applications and environments, some even incorporating overlapping generations of agents, our analysis is not a special case of any of them. Notably, our model allows for richer forms of inter-temporal complementarity, our stability concepts are distinct, and our algorithms for constructing stable assignments are new.

A complementary class dynamic matching models examines one-time matchings that arise
over multiple periods. Here focus has ranged from questions of preference formation (Kadam, 2015a) and unraveling (Roth and Xing, 1994) to managing the (stochastic) arrival and departure of agents or objects (Ünver, 2010; Leshno, 2012; Akbarpour et al., 2016; Baccara et al., 2015; Thakral, 2015). Doval (2015) examines an economy of this latter type and independently proposes a definition of “dynamic stability,” which is distinct from our proposal. In our model, agents interact over multiple periods and revise their assignments over time. In Section 2.5.3 we illustrate how economies with one-time matchings can be studied within our model through an appropriate preference restriction.

Though we stress the dynamic and sequential aspects of multi-period interactions, an interesting parallel exists between multi-period, one-to-one matching markets and static, many-to-many matching markets. Over a lifetime, each agent can have many partners. Recently, Hatfield and Kominers (2012b) have examined such markets in the matching with contracts framework (Hatfield and Milgrom, 2005). Dynamic matching problems can be analyzed within this paradigm by allowing agents to encode the date(s) of their relationship(s). While we at times employ parallel reasoning, our model is not subsumed by their analysis. Two differences are paramount. First, we do not presume substitutable preferences, which are key requirement for the existence of stable outcomes in static many-to-many matching markets. And second, as we further explain in Section 2.2.3, our notion of blocking endogenously links successive assignments through agents’ beliefs and conjectures regarding the market’s future development. This results in a definition of stability that differs from the definition favored in the static many-to-many matching literature. The possible absence of a man-optimal stable matching (Example 11) further contrasts our model with classic many-to-many matching models (Roth, 1984b).

2.2 The Model

Mindful of the noted applications, for expositional ease we present our model using Gale and Shapley’s terminology of a matching between men and women. Synonyms for common applica-

7Dimakopoulos and Heller (2014) adopt this approach to model the assignment of trainee lawyers to courts in Germany.
tions would be students and schools, doctors and hospitals, or workers and firms. For brevity, we define some concepts only from the perspective of a typical man. Our model is symmetric and all definitions apply to women with obvious changes in notation.

### 2.2.1 The One-Period Market

To introduce notation and to provide a benchmark, we briefly review Gale and Shapley’s (1962) one-period matching market. There are finite, disjoint sets of men, \( M = \{m_1, \ldots, m_{|M|}\} \), and women, \( W = \{w_1, \ldots, w_{|W|}\} \). Each man (woman) can be matched to one woman (man) or not matched at all. By convention, a man (woman) who is not matched to a woman (man) is treated as matched to himself (herself). Thus, \( W_m := W \cup \{m\} \) is the set of \( m \)'s potential partners and \( w \)'s potential partners are \( M_w := M \cup \{w\} \). As there is only one period, each agent has a strict preference only over potential partners.

A matching is a function that assigns a partner to each agent. More formally, the function \( \mu: M \cup W \rightarrow M \cup W \) is a one-period matching if \( \mu(m) \in W_m \) for all \( m \in M \), \( \mu(w) \in M_w \) for all \( w \in W \), and \( \mu(i) = j \implies \mu(j) = i \) for all \( i \). A stable matching cannot be blocked by any agent or pair. That is, (i) each agent weakly prefers his/her assigned partner to being not matched; and, (ii) no pair prefers to be together in lieu of their assigned partners.

**Theorem 4** (Gale and Shapley (1962)). *There exists a stable one-period matching.*

To prove Theorem 4, Gale and Shapley introduce the (man-proposing) deferred acceptance algorithm, which proceeds as follows:

1. In round 1, each man proposes to his most preferred partner. Given the received proposals, each woman tentatively engages her favorite suitor and rejects the others.

2. In round \( \tau \geq 2 \), each man proposes to his most preferred partner who has not yet rejected him. Each woman evaluates any received proposals and her engaged partner (if any) and tentatively engages her favorite suitor and rejects the others.

The above process continues until no rejections occur. At this point, engaged pairs are matched...
and all others remain unmatched. The resulting assignment is stable. As Roth (2008a) explains, the algorithm has enjoyed wide application and we rely on it in many arguments to follow.\footnote{Appendix B.5 describes the algorithm more formally and includes an example.}

### 2.2.2 A Multi-period Market

Extending the model, suppose agents interact over two periods. In every period, each man (woman) can be matched with one woman (man) or not matched at all. An agent’s partners in periods $t$ and $t'$ may differ. We call this sequence of matchings a \textit{partnership plan}. Thus, $(j,k)$ is a partnership plan for $i$ where he is matched with $j$ in period 1 and with $k$ in period 2. When confusion is unlikely, we write $jk$ for $(j,k)$. The plan $jk$ is \textit{persistent} if $j = k$. Else, it is \textit{volatile}. Each agent has a strict and rational preference over partnership plans. If $i$ prefers plan $(j,k)$ to plan $(j',k')$, we write $jk \succ i j'k'$. As usual, $jk \succeq i j'k'$ if $jk \succ i j'k'$ or $jk = j'k'$. The function $\mu : M \cup W \rightarrow (M \cup W)^2$ is a \textit{multi-period matching} if for all $i$, $\mu(i) = (\mu_1(i), \mu_2(i))$ and $\mu_1$ and $\mu_2$ are one-period matchings. Henceforth, we refer to a multi-period matching simply as a \textit{matching}.

In a one-period market, stability combines (i) an individual-rationality requirement and (ii) a pairwise no-blocking condition. The conditions assert that an agent or a pair cannot benefit by pursuing his/her/their best option outside of the market. A natural translation of this ideas to a multi-period setting proceeds on a period-by-period basis. At the market’s beginning, each agent’s unilateral outside option is to remain unmatched. More formally, we say that agent $i$ can \textit{period-1 block} the matching $\mu$ if $ii \succ_i \mu(i)$. Similar logic guides blocking by a pair, though the set of available outside options is exponentially richer. If a pair blocks the matching $\mu$ in period 1, they pursue their most preferred arrangement among themselves in lieu of following the plan encoded in $\mu$. They may aspire to a two-period partnership or they may decide upon a more unusual timing. Thus, the pair $\{m, w\}$ can \textit{period-1 block} the matching $\mu$ if

1. $ww \succ_m \mu(m)$ and $mm \succ_w \mu(w)$;

2. $wm \succ_m \mu(m)$ and $mw \succ_w \mu(w)$;

3. $mw \succ_m \mu(m)$ and $wm \succ_w \mu(w)$; or,
A matching is *ex ante individually rational* if it cannot be period-1 blocked by any agent. An *ex ante stable* matching cannot be period-1 blocked by any agent or by any pair.

Though ex ante stability may be an appropriate solution concept for some applications, it presumes that agents can unequivocally commit to a matching, which may be rare in practice. Thus, our preferred stability definition, dynamic stability, eschews commitment and allows agents to also block a matching conditional on the market’s history. We say that agent \( i \) can *period-2 block* the matching \( m \) if \( (m_1(i), i) \succ_i \mu(i) \). Similarly, the pair \( \{m, w\} \) can *period-2 block* the matching \( \mu \) if

1. \( (\mu_1(m), w) \succ_m \mu(m) \) and \( (\mu_1(w), m) \succ_w \mu(w) \); or,
2. \( (\mu_1(m), m) \succ_m \mu(m) \) and \( (\mu_1(w), w) \succ_w \mu(w) \).

A matching is *dynamically individually rational* if for all \( t \) it cannot be period-\( t \) blocked by any agent. A matching is *dynamically stable* if for all \( t \) it cannot be period-\( t \) blocked by any agent or pair.\(^{10}\)

**Example 1.** Consider a market with two men and two women. Their preferences are:

\[
\begin{align*}
&\succ_{m_1} : \ w_1w_1, w_1m_1, w_1w_2, m_1w_1, w_2w_1, m_1m_1  \quad \succ_{w_1} : \ w_2m_2, m_2m_1, m_1m_2, m_1m_1, w_1w_1 \\
&\succ_{m_2} : \ w_2w_2, w_2m_2, w_2w_1, m_2w_2, w_1w_2, m_2m_2  \quad \succ_{w_2} : \ m_1m_1, m_1m_2, m_2m_1, m_2m_2, w_2w_2
\end{align*}
\]

Here, \( w_1w_1 \) is \( m_1 \)'s most preferred plan, \( w_1m_1 \) is second best, and so on. Unlisted plans are not individually rational. This market has three ex ante stable matchings, which are listed in Table 2.1. To read the table, under \( \mu^1 \), \( \mu^1(m_1) = w_1w_1 \), \( \mu^1(m_2) = w_2w_2 \), and so on. As clear from the table, both persistent and volatile matchings may be ex ante stable.

If we focus on the matching \( \mu^2 \), we note that \( m_1 \) is paired with \( w_2 \) in period 2. However, \( w_1m_1 \succ_m w_1w_2 = \mu^2(m_1) \). Thus, it seems unlikely \( m_1 \) would agree to a continuation of \( \mu^2 \) after period 1. He prefers to renege on his commitment and the consequences for him of doing so are unambiguously positive. Thus, \( \mu^2 \) is not dynamically stable. The remaining matchings, \( \mu^1 \) and \( \mu^3 \),

\( ^9 \)Condition 4 may seem redundant. We include it for completeness since single-agent and pairwise definitions of blocking are special cases of more general coalition-based definitions (Appendix B.3).

\( ^{10} \)Doval (2015) has contemporaneously and independently proposed an alternative definition of “dynamic stability.” We contrast our model with her’s in the discussion to follow and in Section 2.5.3.
Table 2.1: All ex ante stable matchings in Example 1.

<table>
<thead>
<tr>
<th>Matching</th>
<th>$m_1$</th>
<th>$m_2$</th>
<th>$w_1$</th>
<th>$w_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu^1$</td>
<td>$w_1w_1$</td>
<td>$w_2w_2$</td>
<td>$m_1m_1$</td>
<td>$m_2m_2$</td>
</tr>
<tr>
<td>$\mu^2$</td>
<td>$w_1w_2$</td>
<td>$w_2w_1$</td>
<td>$m_1m_2$</td>
<td>$m_2m_1$</td>
</tr>
<tr>
<td>$\mu^3$</td>
<td>$w_2w_1$</td>
<td>$w_1w_2$</td>
<td>$m_2m_1$</td>
<td>$m_1m_2$</td>
</tr>
</tbody>
</table>

are both dynamically stable. By allowing blocking in multiple periods conditional on the market’s history, dynamic stability refines its ex ante counterpart through its accommodation of limited commitment. Intuitively, it offers the same type of refinement as provided by sub-game perfection to the Nash equilibria of an extensive-form game.

Below we provide additional discussion of our main solution concept. We highlight its key features and we identify sources of divergence from other proposals in the literature. We address the existence of stable matchings and propose mechanisms to identify such assignments in Sections 2.3 and 2.4, respectively.

2.2.3 Discussion

We consider dynamic stability to be the simplest and the most natural multi-period generalization of Gale and Shapley’s original idea. However, as Damiano and Lam (2005) explain, defining stability in a multi-period economy involves resolving many new issues that are absent from the static case. Naturally, this leads to many plausible variations of stability, each with associated costs and benefits.

There are two key differences between static and dynamic markets captured by the contrast between ex ante and dynamic stability. First, commitment—or its absence—plays an important role when interaction occurs over multiple periods.\textsuperscript{11} When commitment by all parties is perfect, ex ante contracting almost trivially leads to an efficient outcome and is congruent with classic

\textsuperscript{11} Diamantoudi et al. (2015) study the role of commitment in a dynamic matching market. Assuming additively-separable, time-invariant, and history-independent preferences, they show that repetitions of a single-period stable matching is a stationary sequential equilibrium when commitment is not possible. An analogous conclusion applies in our model when preferences are similarly restricted (see Section 2.4).
definitions of an economy’s core. However, classic definitions of the core insufficiently capture the lack of trust and the change in incentives agents often experience as relationships are encountered sequentially. Gale (1978) recognized an analogous phenomenon in an exchange economy when consumption occurs sequentially. Dynamic stability is a natural counterpart in our model to Gale’s (1978) notion of the sequential core in his economy.\textsuperscript{12} It allows blocking in the classic sense and further improvements conditional on the passage of time. Therefore, a stable outcome must be robust to a wider class of deviations than expected in the static case.

The second distinguishing feature of a multi-period economy is that agents must subscribe to some model concerning the economy’s subsequent evolution, especially after a departure from a proposed plan. In principle, this counterfactual reasoning may lead to a complex chain of conjectures. Favoring spareness, dynamic stability models future developments through an implicit robustness criterion embedded in the definition of blocking, whereby agents anticipate unfavorable future developments. This approach contributes to the model’s comparative simplicity, despite the possibility of history-dependent preferences, and is well-illustrated by the following example.

\textbf{Example 2.} Consider a market with two men and one woman. Their preferences are:

\begin{align*}
\succ_{m_1} & : m_1 w_1, w_1 w_1, m_1 m_1 \\
\succ_{m_2} & : w_1 w_1, m_2 w_1, m_2 m_2 \\
\succ_{w_1} & : m_1 m_1, m_2 m_2, w_1 m_2, w_1 m_1, w_1 w_1
\end{align*}

This economy’s only dynamically stable matching is underlined. Interestingly, \( m_1 w_1 \succ_{m_1} w_1 w_1 \).

This raises the question: \textit{Should }m_1\textit{ leave }w_1\textit{ for period 1 only and then return?} Though a seemingly promising idea, and a frequently entertained possibility (Damiano and Lam, 2005; Bando, 2012; Kennes et al., 2014a), its reasonableness depends on the market’s contemporaneous and subsequent development. Since preferences are history dependent and the within-period interaction among agents is modeled with minimal details, the possibilities and stories are many.

1. If \( w_1 \) is passive and is unmatched given \( m_1 \)’s period 1 absence, then the proposition is

\textsuperscript{12}Unlike the sequential core, dynamic stability considers only pairwise blocking actions. We discuss the rationale for this restriction below. Appendix B.3 provides coalition-based extensions of our key results.
promising only if \( m_2 \) is equally unassertive and does not pursue a period 2 relationship with \( w_1 \). In this case, \( w_1 \) would accept \( m_1 \)'s return in period 2. If \( m_2 \) is alert and matches with \( w_1 \) in period 2, \( m_1 \) remains unmatched. In either case, \( w_1 \) feels regret as a matching with \( m_2 \) from the outset would have been preferable.

2. Anticipating regret, suppose \( w_1 \) reacts to \( m_1 \)'s period 1 absence from the market and pairs with \( m_2 \) in period 1 instead. Now, \( m_1 \) will lose-out in period 2 because \( w_1 \) would rather maintain her partnership with \( m_2 \).

3. If there is uncertainty, matters are complicated further. For instance, if \( m_1 \) conjectures that

\[ \succ_{w_1}: m_1 m_1, m_2 m_2, w_1 m_2, w_1 w_1, w_1 m_1, \ldots \]

he risks being rejected by \( w_1 \) in period 2 even if he believes \( m_2 \) is timid and resigned to remaining unmatched in both periods.

In practice, all of the preceding cases—and many more—are plausible models of this market’s development. A knotty selection problem follows. Dynamic stability posits that \( m_1 \) resolves this ambiguity conservatively. He assumes that his period 1 absence will precipitate unfavorable developments and he should not count on a subsequent chance to pair with \( w_1 \).

Generalizing the example’s intuition, when agents block a matching, our definitions assume that each agent believes the market will evolve in the most unfavorable manner (to them) in response to their deviation. In practice, of course, this implies exclusion from the wider market.\(^{13}\) Excluded agents anticipate implementing the best continuation plan given their conjectured restricted circumstances.\(^{14}\) The result is a robust, detail-free model rationalizing agents’ behavior adaptable to many applications. It avoids elaborately prophetic counterfactual reasoning concerning the market’s evolution, which becomes taxing even in modestly-sized markets with short time horizons.\(^{15}\) Moreover, it ensures that dynamic stability can be easily applied when imperfect

\(^{13}\)In a market design application, the return of “non-cooperative” agents is controllable by the designer and market exclusion may be real rather than conjectured. For example, in the NRMP matching process, applicants/residency programs that fail to honor the prescribed outcome, without securing a waiver, may be barred from accepting alternative positions/applicants or participating in future NRMP Matches (NRMP, 2014b).

\(^{14}\)Similarly, in a repeated game players anticipate best responding to the planned punishment should they deviate from a proposed equilibrium.

\(^{15}\)Corbae et al. (2003), Kurino (2009), and Doval (2015) propose stability definitions whereby agents coordinate on
information further impedes such assessments. We hinted at this application in the example above and we elaborate upon it in Section 2.5.1.

Following Gale and Shapley (1962) and Corbae et al. (2003), we define stability with pairwise blocking. Others have emphasized the importance of blocking actions by larger groups (Damiano and Lam, 2005; Kurino, 2009; Doval, 2015). For completeness, we present coalition-based analogues of our key concepts and results in Appendix B.3 where we study our economy’s core. However, we prefer the pairwise specification for three reasons. First, we wish to maintain a minimal departure from classic bilateral matching models, which motivate our analysis. Second, our definitions capture the practical nature of interaction in markets where large-scale coordination among agents is not possible, rarely observed, or illegal. The difficulties of large-scale collusion commonly acknowledged in static models continue to apply. Finally, stronger definitions may preclude meaningful applied analysis. Conditions ensuring the core’s non-emptiness, for example, may not apply to an application of interest.

To conclude this section, we wish to acknowledge that many further differences among proposed stability definitions are due to the institutional characteristics of the markets under study. For example, typically schools guarantee a student a seat for many years after enrollment. Likewise priorities are commonly defined in terms of next-period assignments rather than lifetime “enrollment plans.” Such features, and others, have been incorporated into alternative definitions of stability or fairness (Dur, 2012; Pereyra, 2013; Kennes et al., 2014a). Dynamic stability seeks to draw on such features to a minimal degree. Instead, we prefer to introduce application-specific constraints through restrictions on agents’ preferences. For instance, a strong bias in favor of persistent plans can model a school’s commitment to not “bump” currently-enrolled students in favor of higher-priority students in future years. In Section 2.5.3 we illustrate how similar restrictions can model irrevocable assignments.

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16 In Appendix B.3 we provide conditions ensuring the core’s non-emptiness and coincidence with the set of dynamically stable matchings.

17 We discuss such preferences in Appendix B.3. They satisfy a property we term strong inertia.
2.3 Existence of Stable Matchings

As every dynamically stable matching is also an ex ante stable matching, we first verify the latter’s existence. To do so, we employ an algorithm that will reappear below. In the plan deferred acceptance procedure, each man proposes to one woman at a time and specifies their exclusive relationship’s timing. A man and woman may be together for both periods or together in one period and single otherwise. Such proposals are made just like in the usual deferred acceptance algorithm with the woman tentatively accepting her best available option. This process can be interpreted as men proposing from a restricted set of “contracts,” as in the setting of Hatfield and Milgrom (2005). Once no proposals are rejected, the final matching is set.

Algorithm 1 (PDA). The (man-proposing) plan deferred acceptance procedure identifies a matching \( \mu^* \) as follows. For each \( m \) let \( X^0_m = \bigcup_{w \in W} \{ww, wm, mw\} \). At \( \tau = 0 \), no plans in \( X^0_m \) have been rejected. In round \( \tau \geq 1 \):

1. Let \( X^\tau_m \subset X^0_m \) be the subset of plans that have not been rejected in some round \( \tau' < \tau \). If \( X^\tau_m = \emptyset \) or \( mm \succ_m x \) for all \( x \in X^\tau_m \), then \( m \) does not make any proposals. Otherwise, \( m \) proposes to the woman identified in his most preferred plan in \( X^\tau_m \). If \( ww \) is his most preferred plan, he proposes a two-period relationship to \( w \). If \( wm \) (\( mw \)) is his most preferred plan, he proposes a one-period partnership with \( w \) for period 1 (2). In period 2 (1), both \( m \) and \( w \) are to be unmatched.

2. Let \( X^\tau_w \) be the set of plans made available to \( w \). If \( ww \succ_w x \) for all \( x \in X^\tau_w \), \( w \) rejects all proposals. Otherwise, \( w \) (tentatively) accepts her most preferred plan in \( X^\tau_w \) and rejects the others. A woman may accept at most one plan at a time.

The above process continues until no rejections occur. If \( w \) accepts \( m \)'s proposal in the final round, define \( \mu^*(m) \) and \( \mu^*(w) \) accordingly. If \( i \) does not make or receive any proposals in the final round, set \( \mu^*(i) = ii \).

We will illustrate the PDA’s operation as part of Example 4 below. In the interim, we note that the procedure’s outcome enjoys the following property.
Theorem 5. The PDA matching is ex ante stable.

As with all formal results, we prove Theorem 5 in Appendix B.1.\textsuperscript{18} The argument proceeds along familiar lines. Any incipient blocking pair must meet in some round in the PDA’s operation, which implies that the woman must have rejected the man’s proposal. As the woman’s tentative matching only improves in later rounds of the PDA, the woman must prefer the final outcome to the blocking arrangement.

The PDA outcome is ex ante stable and may also be dynamically stable, but not always. At times, a dynamically stable matching may not exist.

Example 3. Consider a market with one man and one woman.\textsuperscript{19} Their preferences are:

\[ m : \text{wm, ww, mm} \quad w : \text{mm, ww} \]

There are only two candidate stable matchings. The matching where \( \mu(m) = \text{mm} \) is not ex ante stable as the couple can period-1 block it. The matching where \( \mu'(m) = \text{ww} \) is ex ante stable. However, it is not dynamically stable since \( m \) will renge after period 1.

2.3.1 Inertia and Sequential Complementarity

Example 3 suggests that preferences must exhibit more structure to ensure the existence of a dynamically stable matching. Ideally, this structure should be compatible with common behavioral characteristics or economic situations. We develop this structure progressively, starting with separable preferences and successively incorporating status quo bias, switching costs, and complementarities among successive assignments.

A common way to define a multi-period preference involves aggregating a single-period preference. The workhorse discounted utility model is one instance of this approach. Drawing on this benchmark, let \( P_i \) be agent \( i \)'s ranking of potential partners abstracting from temporal

\textsuperscript{18}As suggested above, one can prove Theorem 5 as a corollary to Hatfield and Milgrom (2005). We provide an independent argument as we employ and modify the PDA procedure in subsequent analysis.

\textsuperscript{19}Hatfield and Kominers (2012a) consider a similar example of a doctor and a hospital contracting morning and afternoon shifts. They suggest the doctor and the hospital should sign a unified contract covering both shifts. In our model, that suggestion corresponds to the ex ante stable outcome.
considerations. We call $P_i$ a spot ranking. As usual, $jP_i k$ if $j$ is superior to $k$ and $jR_i k$ if $jP_i k$ or $j = k$. It is natural to expect that partnership plans with superior partners are preferable. We say that $\succ_i$ reflects $P_i$ if $j\succ_i j'k'$ whenever $jR_i j'$, $kR_i k'$, and at least one ranking is strict. Examples of preferences that reflect a spot ranking include lexicographic preferences or those represented by a standard discounted utility function.

Though analytically convenient, preferences that reflect a spot ranking fail to capture many features of inter-temporal decision-making, such as switching costs and status-quo bias (Samuelson and Zeckhauser, 1988). For example, workers often prefer longer-term employment with the same employer instead of frequent job changes. Similarly, families often want their younger child (the period 2 match) to attend the same school as his older sibling (the period 1 match) (Dur, 2012). Both cases illustrate a form of complementarity where successive assignments to one partner are more valuable. To accommodate these cases, we say that the preference $\succ_i$ exhibits inertia if

$$
jk \succ_i jj \implies kk \succ_i jk \text{ and } jk \succ_i kk \implies jj \succ_i jj.
$$

All preferences that reflect a spot ranking satisfy (I); however, the condition is much more inclusive and it offers a natural starting point for applied studies of multi-period matching. For example, inertia accommodates status quo bias by allowing persistent plans to “rise” in rank. To illustrate, take a preference satisfying (I), like the lexicographic preference

$$
\succ_i: jj, jk, ji, kj, kk, ki, ij, ik, ii.
$$

Next, shuffle $kk$ to the left to yield, for example,

$$
\succ_i': jj, jk, kk, ji, kj, ki, ij, ik, ii.
$$

The new preference $\succ_i'$ also satisfies (I) and models an agent more willing to maintain an initial

---

20Reflection resembles Roth’s (1985a) responsiveness. It differs since the timing of assignments matters.

21That is, $jk \succ_i j'k' \iff u_i(j) + \delta u_i(k) > u_i(j') + \delta u_i(k')$ where $u_i(j) > u_i(k) \iff jP_i k$.

22In their model of daycare assignment, Kennes et al. (2014a) assume that children’s preferences satisfy a condition called “rankability.” Inertia is a weaker condition.
assignment with $k$ despite $j$ appearing to be a superior one-period match.

Inertia captures a basic form of chronological complementarity among successive assignments. However, complementarity sometimes involves change for its own sake. For example, a student may prefer to attend different universities for undergraduate and graduate education. The diversity of experience itself adds value. A further weakening of inertia can capture such sentiment. The preference $\succ_i$ satisfies *sequential improvement complementarity* if

$$jk \succ_i jj \succ_i ii \implies kk \succ_i jj.$$  \tag{SIC}

The economic content of SIC can be phrased as follows. If an agent prefers to switch assignments after period 1, then the change ought to be toward an ex ante better option. This conviction is common. Many people happily move from an unpaid internship to a regular job. The reverse change is typically unwelcome. Intuitively, the prospect of a better future assignment incentivizes an agent to follow through on a particular plan.

Though the intuition behind SIC is compelling, SIC is too weak to assure the existence of a dynamically stable matching (Example 12). Particular complications arise (only) when agents are sometimes unmatched. Thus, we propose two independent conditions that impose additional structure on preferences. The first posits that agents are averse to remaining unmatched in one period. The preference $\succ_i$ exhibits *singlehood aversion* if

$$ji \succ_i ii \implies jj \succ_i ji \text{ and } ij \succ_i ii \implies jj \succ_i ij.$$ \tag{SA}

Singlehood aversion is satisfied in many applications. For instance, below we discuss the multi-period nature of medical residency. Primary training (the period 1 assignment) and specialty training (the period 2 assignment) are both necessary to qualify as a physician. Temporary unemployment spells can be similarly undesirable for a worker.

The second restriction is a mild strengthening of SIC applicable only when an agent is unmatched in a one period. The preference $\succ_i$ satisfies *revealed dominance of singlehood* if

$$jk \succ_i ji \succ_i ii \implies kk \succ_i ji \text{ and } ik \succ_i ij \succ_i ii \implies kk \succ_i ij.$$ \tag{RDS}
If agent $i$’s preference reveals a desire to match with agent $k$ in period 2, he would accept that assignment in both periods in lieu of singlehood.

### 2.3.2 The Plan Deferred Acceptance Procedure with Adjustment

Above we have presented several classes of preferences that span many situations and applications. They are also sufficient for the existence of a dynamically stable matching.

**Theorem 6.** There exists a dynamically stable matching if (a) $\succ_i$ satisfies SIC and SA for all $i$; or, (b) $\succ_i$ satisfies SIC and RDS for all $i$.

The conditions invoked by Theorem 6 are independent. They cannot in general be substituted or mixed together while ensuring stability (Example 12). Preferences with inertia satisfy SIC and SA. Hence, there exists a dynamically stable matching when $\succ_i$ exhibits inertia for all $i$.

To prove Theorem 6, we generalize the PDA procedure by incorporating an adjustment phase after the initial assignment is determined.

**Algorithm 2 (PDAA).** The (two-period, man-proposing) plan deferred acceptance procedure with adjustment identifies a matching $\mu^*$ as follows:

**Step 1.** Implement the PDA procedure and call the resulting matching $\tilde{\mu}^1 = (\tilde{\mu}^1_1, \tilde{\mu}^1_2)$. For each agent $i$ who is assigned a partner in period 2 ($\tilde{\mu}^1_2(i) \neq i$), set $\mu^*(i) = \tilde{\mu}^1(i)$ and exclude the agent from further consideration.

**Step 2.** For each remaining man, define a preference among the remaining women conditional on his interim assignment: $w \succ_m^1 (m) w' \iff (\tilde{\mu}^1_1(m), w) \succ_m (\tilde{\mu}^1_1(m), w')$. Define the remaining women’s conditional preferences analogously. Next, implement Gale and Shapley’s (1962) (man-proposing, one-period) deferred acceptance algorithm where each agent makes/accepts proposals according to his/her conditional preference, $\succ_1^1$. If $\tilde{\mu}^2_2(\cdot)$ is the resulting one-period matching, for all agents involved set $\mu^*(i) = (\tilde{\mu}^1_1(i), \tilde{\mu}^2_2(i))$.

---

23Inertia means that $jk \succ_i jj \implies kk \succ_i jk$; hence, SIC holds because $kk \succ_i jj$. Setting $k = i$ and $j = i$ in the definition of inertia confirms SA. Preferences with inertia do not satisfy RDS. We are grateful to Fanqi Shi for alerting us to this fact.

24Kennes et al. (2014a) present a similar definition when introducing their “isolated preference relation.”
Remark 1. A $T$-period generalization of Theorem 6 and Algorithm 2 is found in Appendix B.2. The intuition extends by induction.

The PDAA operates in two steps, and both are required to ensure stability. Step 1 corresponds to the PDA and secures the matching’s ex ante stability. Step 2 improves upon the PDA matching and ensures that the final outcome cannot be period-2 blocked. The following example illustrates the importance of this second step. The economy’s only dynamically stable matching features a cyclic assignment among two men and two women.

Example 4. There are three men and three women with preferences

$\succ_{m_1}: w_1w_2, w_2w_2, w_1m_1, m_1m_1$  
$\succ_{m_2}: w_2w_1, w_1w_1, w_2m_2, m_2m_2$  
$\succ_{m_3}: w_1w_1, w_2w_2, w_3w_3, m_3m_3$  
$\succ_{w_1}: m_1m_1, m_1m_2, m_1w_1, w_1w_1$  
$\succ_{w_2}: m_2m_2, m_2m_1, m_2w_2, w_2w_2$  
$\succ_{w_3}: m_3m_3, w_3w_3$

Table 2.2 summarizes the PDAA’s operation. The first step, coinciding with the PDA, terminates in three rounds. In round 1, $m_1$ proposes a two-period partnership to $w_2$ and is rejected. The proposals of $m_2$ and $m_3$ are similarly rejected. By the third round, each man’s proposal is accepted. The resulting interim matching is ex ante stable.

Since $m_3$ and $w_3$ are matched in period 2, their final matching is set. The others go on to the PDAA’s second step. Their conditional preferences at $\hat{\mu}_1^1(\cdot)$ are:

$\succ_{m_1}^1: w_2, m_1$  
$\succ_{m_2}^1: w_1, m_2$  
$\succ_{m_3}^1: m_2, w_1$  
$\succ_{w_1}^1: m_1, w_2$

Conditional on $\hat{\mu}_1^1(\cdot)$, $m_1$ and $w_2$ wish to match together for period 2. Similarly, $m_2$ and $w_1$ wish to match together. Of course, the deferred acceptance algorithm leads to this outcome. The final assignment is this economy’s only dynamically stable matching.

Conditional on the period 1 match, the adjustment step in the PDAA refines the assignments of agents who are unmatched in period 2. A tempting suggestion is to implement a similar adjustment for agents who are unmatched in period 1, holding fixed the period-2 assignment. This adjustment is not part of the PDAA as it may unwittingly introduce instability (Example 14).

Several properties of dynamically stable matching are straightforward to derive. First, a bias
toward persistent plans need not preclude volatile outcomes. For instance, all preferences in Example 1 exhibit inertia yet both women prefer the volatile dynamically stable matching \( m^3 \) to its persistent counterpart. Therefore, assignment volatility is not entirely driven by a preference for variety. Instead, it may emerge as a compromise among competing interests.

Second, if preferences exhibit inertia, then every persistent dynamically stable matching is Pareto optimal (Theorem 13). More generally, however, mis-coordination of assignment timing may lead to Pareto inferior outcomes (Example 15). This sensitivity to timing is common in dynamic markets. Analogously, and for example, the repetition of an inefficient stage-game Nash equilibrium is an equilibrium of the dynamic game as a whole.

### 2.4 Matching Mechanisms and Strategic Considerations

It is reassuring that a generalization of the deferred acceptance algorithm, the PDAA, leads to a dynamically stable outcome. Like the deferred acceptance algorithm, the PDAA may serve as a centralized assignment mechanism or it may metaphorically describe a decentralized market where the proposing side enjoys market power. Of course, many multi-period generalizations of the deferred acceptance algorithm are possible. Some alternatives coincide with the PDAA in special cases while others generally fail to identify a dynamically stable outcome. The PDAA also enjoys several appealing strategic qualities, such as strategyproofness, which other natural mechanisms may lack.

---

**Table 2.2: Operation of the PDAA Procedure in Example 4.**

<table>
<thead>
<tr>
<th>Step</th>
<th>Proposal Extended</th>
<th>Proposal(s) Received</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 (PDA)</td>
<td>( w_2 w_2 ) ( w_1 w_1 ) ( w_1 w_1 ) ( { m_2 m_2, m_3 m_3 } ) ( m_1 m_1 ) ( - )</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>( w_1 m_1 ) ( w_2 m_2 ) ( w_2 w_2 ) ( m_1 w_1 ) ( { m_2 w_2, m_3 m_3 } ) ( - )</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>( w_1 m_1 ) ( w_2 m_2 ) ( w_3 w_3 ) ( m_1 w_1 ) ( m_2 w_2 ) ( m_3 m_3 )</td>
<td></td>
</tr>
</tbody>
</table>

2 (Adjustment) 1 \( w_2 \) \( w_1 \) \( - \) \( m_2 \) \( m_1 \) \( - \)

Final Assignment \( w_1 w_2 \) \( w_2 w_1 \) \( w_3 w_3 \) \( m_1 m_2 \) \( m_2 m_1 \) \( m_3 m_3 \)
Repeating Single-Period Assignments

Likely the simplest multi-period generalization of the deferred acceptance procedure involves repeating one-period matchings derived using Gale and Shapley’s (1962) original algorithm. Translating this idea to our setting requires care. When preferences are defined over partnership plans, there need not exist a stand-alone, single-period preference that can be used to identify a stable one-period matching. A simple way to infer a single-period preference involves eliciting an agent’s preference for persistent plans. We call $P_{\succ_i}$ agent $i$’s *ex ante* spot ranking induced by $\succ_i$. It is defined as $jP_{\succ_i} k \iff j j \succ_i k k$. $P_{\succ_i}$ can be readily employed to define a multi-period matching using Gale and Shapley’s (1962) algorithm.

**Algorithm 3** (EDA). The *(man-proposing) ex ante deferred acceptance procedure* assigns the one-period matching identified by the *(man-proposing, one-period) deferred acceptance algorithm* where each agent makes/accepts proposals according to $P_{\succ_i}$ in each period.

By design, the EDA constructs a persistent matching. When preferences satisfy SIC and SA this assignment coincides with the PDAA matching and is dynamically stable (Lemma 13). Moreover, the EDA draws on relatively little information, which may be helpful in applications. Beyond $P_{\succ_i}$, an agent need not know or communicate $\succ_i$ in its entirety.

A natural further generalization of the EDA captures the intuition of successive spot markets. It repeats Gale and Shapley’s deferred acceptance procedure conditional on the period 1 assignment. At a high level, Damiano and Lam (2005), Kurino (2009), Dur (2012), and Pereyra (2013) rely on mechanisms implementing stable assignments from successive single-period markets. The following operationalization of this idea is a specialization of a mechanism studied by Kennes et al. (2014a,b).

**Algorithm 4** (SDA). The *(man-proposing) spot-market deferred acceptance procedure* defines the matching $\tilde{\mu} = (\tilde{\mu}_1, \tilde{\mu}_2)$ on a period-by-period basis as follows:

*Period 1*. Define $\tilde{\mu}_1$ as the one-period matching identified by the *(man-proposing) deferred acceptance algorithm* where each agent $i$ makes/accepts proposals according to his/her *ex ante* spot ranking $P_{\succ_i}$.

---

25Kennes et al. (2014a) present a similar definition when introducing their “isolated preference relation.”
spot ranking, $P_{\succ_i}$.

**Period 2.** Define agent $i$’s conditional spot ranking at $j$ as $kP_{\succ_i}^{j,l} \iff (j,k) \succ_i (j,l)$. Set $\tilde{\mu}_2$ to be the one-period matching identified by the (man-proposing) deferred acceptance algorithm where each agent $i$ makes/accepts proposals according to his/her conditional spot ranking at $\tilde{\mu}_1(i), P_{\succ_i}^{\tilde{\mu}_1(i)}$.

When preferences reflect a spot ranking, the SDA matching coincides with the EDA matching and is dynamically stable. More generally, these properties are not maintained.

**Example 5.** Consider the following market where agents’ preferences exhibit inertia.

\[
\succ_{m_1} : w_2w_2, w_1w_2, w_1w_1, \ldots \quad \succ_{m_1} : m_1m_1, m_2m_2, m_3m_3, m_1m_2, m_1m_3, \ldots \\
\succ_{m_2} : w_1w_1, w_3w_5, w_3w_1, \ldots \quad \succ_{w_2} : m_3m_3, m_1m_1, m_3m_1, \ldots \\
\succ_{m_3} : w_1w_1, w_2w_1, w_2w_2, \ldots \quad \succ_{w_3} : m_2m_2, \ldots
\]

As confirmed in Appendix B.5, the SDA matching is

\[
\tilde{\mu}(m_1) = w_1w_2 \quad \tilde{\mu}(m_2) = w_3w_3 \quad \tilde{\mu}(m_3) = w_2w_1 \\
\tilde{\mu}(w_1) = m_1m_3 \quad \tilde{\mu}(w_2) = m_3m_1 \quad \tilde{\mu}(w_3) = m_2m_2
\]

This matching is neither ex ante nor dynamically stable.\(^{26}\) For example, $m_1$ and $w_2$ can period-1 block $\tilde{\mu}$. Instead, the dynamically stable matching is

\[
\mu^*(m_1) = w_1w_1 \quad \mu^*(m_2) = w_3w_3 \quad \mu^*(m_3) = w_2w_2 \\
\mu^*(w_1) = m_1m_1 \quad \mu^*(w_2) = m_3m_3 \quad \mu^*(w_3) = m_2m_2
\]

This example also illustrates why the PDAA procedure does not allow all agents to re-match conditional on their period 1 assignment. Here, that exercise’s outcome coincides with the unstable SDA matching.

Dynamic matching procedures that ignore complementarities between successive periods, like

\(^{26}\)The SDA procedure is a specialization of the DA-IP mechanism proposed by Kennes et al. (2014a) to assign children to daycares. To nest Example 5 in their framework, call men “children” and women “daycares” with unit capacity. All preferences satisfy their assumptions. If $\{P_{\succ_{m_1}}, P_{\succ_{m_2}}, P_{\succ_{m_3}}\}$ is the initial priority structure, the matching $\hat{\mu}$ is “stable” in their sense of the term (Kennes et al., 2014a, Definition 8). Hence, our definitions of ex ante and dynamic stability are distinct from, and not weaker than, their proposal.
the SDA, introduce an “exposure problem” into the matching process. The same issue arises in a multi-item auction where complementary goods are sold through multi-round procedures (Milgrom, 2000; Bulow et al., 2009). In Example 5, \( w_1 \) is vulnerable if she pursues a relationship with \( m_1 \), her favorite partner. In period 1 she is able to match with \( m_1 \), seemingly making progress toward her most preferred outcome, \( m_1m_1 \). Nevertheless, she faces risk concerning the durability of others’ preferences. Others’ changing opinions impose an externality on \( w_1 \), ultimately leading to disappointment. The PDAA mitigates this problem by allowing some agents to match for period 2 from the outset.

**Backward Induction and the NRMP**

The PDAA determines assignments chronologically. First, matchings for period 1, and possibly period 2, are specified. And then, period 2 adjustments are made. This operation contrasts with the backward induction reasoning common in multi-period scenarios (Doval, 2015). As a practical illustration of a mechanism using this latter approach, consider again the NRMP. Four program types participate in the NRMP’s Main Residency Match\(^\text{®} \) (Table 2.3).\(^{27}\) Students can enter Categorical and Primary programs immediately after medical school (PGY-1) and these programs lead to certification in their specialty, usually after 3–6 years. Preliminary programs provide one or two years of training, are open to students immediately after medical school (PGY-1), but do not lead to certification. Instead, they are prerequisites for Advanced programs, which students enter subsequently (PGY-2) for an additional 3–5 years of training. Physician positions are advanced positions (PGY-2) that start in the current year but are available only to students who have completed graduate medical education.\(^{28}\) As noted in Table 2.3, non-Categorical positions constitute about 24 percent of available positions.

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\(^{27}\)“Main Residency Match” is a registered trademark of the National Resident Matching Program.

\(^{28}\)In their account of unraveling in the medical resident matching process, Roth and Xing (1994) note how some advanced specialties would match with students far in advance. “In [the matches for neurological surgery (PGY-2), otolaryngology (PGY-2 and PGY-3) and urology (PGY-3)], medical-school seniors obtain their second- and third-year employment from 18 to 30 months before they will begin work, and also before they will be matched to their PGY1 positions” (Roth and Xing, 1994, p. 1021, original emphasis). Though the institutional context now differs, such phenomena highlight this market’s multi-period nature.
Table 2.3: Available and Filled Positions in the 2014 Main Residency Match.∗

<table>
<thead>
<tr>
<th>Program Type</th>
<th>Categorical / Primary</th>
<th>Preliminary†</th>
<th>Advanced</th>
<th>Physician</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Program Year</td>
<td>PGY-1</td>
<td>PGY-1</td>
<td>PGY-2</td>
<td>PGY-2</td>
<td>–</td>
</tr>
<tr>
<td>Starting Calendar Year</td>
<td>2014</td>
<td>2014</td>
<td>2015</td>
<td>2014</td>
<td>–</td>
</tr>
<tr>
<td>Positions Available</td>
<td>22,557</td>
<td>4,121</td>
<td>2,719</td>
<td>274</td>
<td>29,671</td>
</tr>
<tr>
<td>Positions Filled</td>
<td>22,149</td>
<td>3,538</td>
<td>2,592</td>
<td>211</td>
<td>28,490</td>
</tr>
</tbody>
</table>

∗ Adapted from NRMP (2014a, Table 7).
† Sum of Medicine-Preliminary, OB/GYN-Preliminary, Pediatrics-Preliminary, Surgery-Preliminary, and Transitional programs.

There is a clear sequential complementarity between Preliminary and Advanced training. The NRMP algorithm addresses this complementarity anti-chronologically. For each Advanced program in his rank order list, a student may also submit a supplemental ranking of Preliminary programs. (Submitting supplemental lists is not required.) If the algorithm matches an applicant to an Advanced program, it then attempts to match him to a Preliminary program from the associated supplemental ranking. Each of these steps is carried out using a variation of the deferred acceptance algorithm.29 Successfully matching to a Preliminary program is not assured and a student may be matched only to the Advanced program at the algorithm’s conclusion.

This procedure may lead to outcomes reminiscent of S.M.’s experience recounted in the introduction. For example, consider a student applying to advanced programs $a_2$ and $b_2$ in cities $a$ and $b$. A Preliminary program in city $a$, call it $a_1$, complements the co-located Advanced program. Likewise, Preliminary program $b_1$ complements its co-located Advanced program in city $b$. For concreteness, suppose the student’s (true) preference is

$$a_1a_2 \succ_i b_1b_2 \succ_i b_1a_2 \succ_i a_1b_2$$

(2.1)

where the period 1 assignment corresponds to preliminary training and the period 2 assignment corresponds to advanced training. Within the NRMP, the student can submit supplemental

29The NRMP algorithm is often equated with the deferred acceptance algorithm, but there are several application-specific differences. See Roth (1996) and Roth and Peranson (1999).
rankings accompanying each advanced program. For example, a plausible submission given his preferences may be
\[ a_2 \{ a_1 \succ_i b_1 \} \succ_i b_2 \{ b_1 \succ_i a_1 \} \succ_i \cdots. \]
(2.2)
Since \( a_1 \sim a_2 \sim_i b_1 \), \( \{ a_1 \succ_i b_1 \} \) is the supplemental list accompanying advanced program \( a_2 \). Suppose the NRMP algorithm assigns the student to his most preferred advanced program, \( a_2 \), and then to his second-choice preliminary program, \( b_1 \). The \( b_1 a_2 \) outcome creates a risk for dynamic instability. Like S.M. from the introduction, the student may seek to transfer at the preliminary program’s conclusion since \( b_1 b_2 \succ_i b_1 a_2 \). If he succeeds in doing so, he is better off.\(^{30}\)

But, this revision may mask a broader welfare loss. The student’s initial match to \( a_2 \) may have displaced another student from that position, resigning her to a less-preferred program. A chain of further displacements may affect others as well.

The complementarity among Preliminary and Advanced programs is but one of many practical challenges that the NRMP matching process must address. Others include couples wishing to match together, programs requiring even or odd numbers of residents, and the transfer of unfilled positions among programs within the same hospital. Theoretically, any of these features can compromise a matching’s stability (Roth, 1996). In practice, however, the algorithm appears to successfully navigate the resulting portfolio of concerns.

**Strategic Issues**

We have thus far abstracted from the strategic concerns that emerge in dynamic markets. However, it is reasonable to assume that an agent may wish to improve his outcome by mis-representing his interests. A matching procedure is called *strategyproof* if it is a dominant strategy for each agent to truthfully reveal his preferences. Otherwise, some agent can manipulate the outcome.

In the one-period case, there does not exist a strategyproof matching mechanism that always yields a stable outcome (Roth, 1982). For example, in the man-proposing deferred acceptance algorithm it is a dominant strategy for each man to truthfully reveal his preferences. For the

\[^{30}\text{In the NRMP algorithm, students are the “proposers.” Therefore, program } b_2 \text{ may find student } i \text{ acceptable since it did not necessarily reject him during the initial NRMP match when } (2.2) \text{ was the student’s report.}\]
women, however, it is often worthwhile to “truncate” preferences (Roth and Rothblum, 1999). This involves pretending that the least-desirable acceptable partners are unacceptable.

In a multi-period market, the scope for manipulation is considerably richer. Sometimes, an agent receiving proposals may wish to claim her most preferred partner is not acceptable. Consider again the SDA procedure and Example 5. Had \( w_1 \) shunned the period-1 proposal of \( m_1 \), her favorite partner, the SDA would have matched her with \( m_2 \) in both periods, which she prefers.\(^{31}\) Curiously, sometimes a proposing agent may also wish to strategize. In the NRMP example above, if program \( b_2 \) considered the student acceptable, the student could have improved his match by claiming \( b_2 \{b_1 \succ_i a_1\} \succ_i a_2 \{a_1 \succ_i b_1\} \) instead of (2.2).

More positive conclusions apply to the mechanisms we have proposed. Unqualified strategyproofness is impossible (Roth, 1982), but the PDAA outcome can be implemented in a strategyproof manner for the proposing side when preferences satisfy our assumptions. If preferences satisfy inertia, or SIC and SA, then implementing the EDA suffices since the one-period deferred acceptance algorithm is strategyproof for the proposing side. Alternatively, we can conclude the following.

**Theorem 7.** Suppose agents’ preferences satisfy SIC and RDS. The PDAA is strategyproof for the proposing side if announcements are restricted to preferences satisfying SIC and RDS.

The domain restriction on announcements is necessary. Otherwise, there exist economies where every mechanism that identifies a dynamically stable matching can be manipulated by at least one man and by at least one woman (Example 16).

### 2.5 Variants and Extensions

We conclude by sketching three extensions of our model. These cover limited information, financial transfers with credit and savings, and irrevocable assignments.

\(^{31}\)Kennes et al. (2014b) show that the scope for a beneficial strategic manipulation of their dynamic matching mechanism becomes small with increasing market size.
2.5.1 Limited Information and Learning

Most agents enter into multi-period relationships with limited knowledge about future preferences. Marriages are announced and dissolved, employees change jobs, and students transfer schools as new facts emerge. Learning justifies relationship revisions, especially when change improves upon an initial assignment. Though we are sympathetic to this intuition, our analysis qualifies it considerably once a matching’s overall stability is accounted for.

To investigate some of the implications, we amend our model as follows. Each agent has a preference over partnership plans \( \succ_i \), but does not know the complete ranking. Each agent’s partial knowledge will improve with time. Specifically, assume that at period 1 agent \( i \) knows the following:

(L1) His preferences have inertia.

(L2) His ex ante spot ranking is \( P_{\succ i} \).

Given this limited information, there are many ex post preferences that the agent may actually hold. For example, if \( jP_{\succ}, kP_{\succ}, l \) then the agent knows that \( jj \succ_i kk \succ_i ll \), but the relative ranking of \( kj \) is unknown. As time passes, agent \( i \) learns more.

(L3) If in period 1 agent \( i \) is assigned to \( k \), he learns his preferences for plans of the form \( kl' \), for all \( l' \).

Continuing the above illustration, after being matched with \( k \), agent \( i \) could discover that \( kj \succ_i kk \succ_i kl \).

Together, (L1)–(L3) outline a simple model of path-dependent learning. The situation is consistent with agent \( i \) learning about switching costs or the strength of preference inertia. In the above illustration, agent \( i \) knows that \( jj \succ_i kk \), but is initially unsure whether switching to \( j \) in period 2 after being matched with \( k \) in period 1 is worthwhile. Given his period 1 knowledge, \( kj \succ_i kk \) and \( kk \succ_i kj \) are both plausible. He recognizes the true case only after a period-1 match to \( k \).

Beyond (L1)–(L3), we do not introduce further beliefs or priors. Forgoing additional ad hoc micro-level assumptions, we model the market’s operation in reduced form by assuming that its
outcome is the result of some matching mechanism. Though the term “matching mechanism” has
the connotation of a centralized process, our intended meaning is broader. It should be interpreted
as a black-box encompassing a pattern of regularized interaction leading to a matching in any
economy. More formally, the function $A(\cdot)$ is a matching mechanism if it assigns a matching to each
economy. An economy is a tuple $e = (M, W, (\succ_i))$ encompassing sets of men and women along
with their preferences. Thus, $A(e) = (A_1(e), A_2(e))$ is a matching among agents in $e$ consistent
with the interaction summarized by $A(\cdot)$. The PDA, the PDAA, the EDA, and the SDA are examples of matching mechanisms.

Whereas matching mechanisms may differ along many dimensions, our restricted information
structure draws attention to those with two pertinent properties. First, a reasonable mechanism
should not leverage information that agents themselves do not know. In period 1 agents know
only their ex ante spot rankings. A non-prophetic mechanism bases its period-1 assignment only
on this information. Formally, matching mechanism $A(\cdot)$ is non-prophetic if for all economies $e = (M, W, (\succ_i))$ and $e' = (M, W, (\succ'_i))$ such that $P_{\succ_i} = P_{\succ'_i}$ for each $i$, $A_1(e) = A_1(e')$. The EDA
and the SDA are both non-prophetic mechanisms.

Second, the mechanism should lead to a dynamically stable matching, at least when preferences
exhibit inertia. Normatively, dynamic stability is a desirable benchmark in this setting since it
subsumes an appealing no-regret property. To illustrate, observe that dynamic stability has both
prospective and retrospective interpretations. In its forward-looking form, an agent threatens
to veto a matching that has not yet occurred. In its backward-looking form, dynamic stability
captures how an agent feels ex post. While an agent cannot “turn back the clock” to period-1
block once on period 2’s threshold, by (L3) he can assess a matching’s continuation relative to
persistent alternatives. If $m$ discovers that $ww \succ_m \mu(m)$ and $w$ learns that $mm \succ_w \mu(w)$, both
will feel regretful not pairing together at an earlier opportunity. A dynamically stable matching
insulates agents from such regret.

We call a matching mechanism dynamically stable on (preference domain) $\mathcal{P}$ if it identifies a
dynamically stable matching whenever each agents’ preference $\succ_i$ belongs to $\mathcal{P}$. Let $\mathcal{I}$ be the

---

32For simplicity, we do not consider random mechanisms.
domain of preferences with inertia. Many matching mechanisms are both non-prophetic and dynamically stable on $I$. The EDA is an example, but there are others as well. Some, in fact, may result in volatile outcomes by leveraging newly available information when setting the period 2 assignment.\footnote{A simple example is the following. First, in the following economy

\[
\succsim_{m_1} : w_1w_1, w_1w_2, w_2w_2, w_2w_1, m_1m_1 \\
\succsim_{m_2} : w_2w_2, w_2w_1, w_1w_1, w_1w_2, m_2m_2 \\
\succsim_{w_1} : m_2m_2, m_2m_1, m_1m_2, m_1m_1, w_1w_1 \\
\succsim_{w_2} : m_1m_1, m_1m_2, m_2m_1, m_2m_2, w_2w_2
\]

the mechanism assigns the dynamically stable matching where $\mu(m_1) = w_1w_2$ and $\mu(m_2) = w_2w_1$. In all other economies, it assigns the EDA matching.}

At this point, two natural questions arise. First, what are the properties of a stable matching in such market? And second, when is re-matching after period 2 preferences are known a welfare-enhancing process? Two theorems help answer these questions. Theorem 8 begins with the dynamically stable matching $\mu = (\mu_1, \mu_2)$ generated by a non-prophetic mechanism. It shows that $\check{\mu} = (\mu_1, \mu_1)$ is also dynamically stable. Thus, in markets operating as we have assumed, forgoing a re-matching for period 2 is innocuous if stability is the sole concern.

**Theorem 8.** Let $A$ be a non-prophetic matching mechanism that is dynamically stable on $I$. Suppose that in economy $e$ agents’ preferences exhibit inertia and $A(e) = \mu = (\mu_1, \mu_2)$. Then $\check{\mu} = (\mu_1, \mu_1)$ is a dynamically stable matching in economy $e$.

**Remark 2.** In general, prolonging the initial assignment of a dynamically stable matching does not preserve dynamic stability, even when agent’s preferences exhibit inertia. Prolonging the period 1 assignment of $\mu^3$ in Example 4 is not a dynamically stable outcome. A similar conclusion applies to swapping assignment order.

Since $\mu$ and $\check{\mu}$ are both dynamically stable, they can be compared on an equal footing when answering the second question posed above. While re-matching between periods offers an opportunity to improve welfare in light of newly available information, Theorem 9 shows that the scope for improvement is actually very small. In fact, re-matching cannot be Pareto improving relative to maintaining the interim status-quo. Furthermore, if an agent gains from re-matching, then his or her initial partner must necessarily be harmed by the ordeal—even when he or she finds a new period 2 partner.
Theorem 9. Assume each agent’s preference exhibits inertia and $\mu = (\mu_1, \mu_2)$ is dynamically stable. Let $\bar{\mu} = (\mu_1, \mu_1)$. If $\mu(i) \succ_i \bar{\mu}(i)$, then $\mu_2(i) = j$ and $\bar{\mu}(j) \succ j \mu(j)$.

An application unifying the preceding analysis considers market unraveling. When markets unravel, parties commit to relationships far in the future, often before valuable information becomes known. It is known that uncertainty contributes to unraveling as early contracting provides insurance (Roth and Xing, 1994; Li and Rosen, 1998; Halaburda, 2010; Ostrovsky and Schwarz, 2010; Echenique and Pereyra, 2013). Our model reinforces this intuition in a new way. Suppose agents can interact on two occasions, in periods 1 and 2, and they learn new information before period 2. If a non-prophetic and dynamically stable (on $I$) mechanism describes this market’s operation, agents will generally be averse to the prospect of revising their initial matching. Prolonging the initial period-1 matching is dynamically stable (Theorem 8); hence, they will not feel regretful ex post. Furthermore, half of agents who re-match between periods will be harmed by the change (Theorem 9). The high incidence of loss relative to the interim status-quo renders the existence of a vibrant period-2 (re-)matching market quite precarious, despite the arrival of new information. Hence, the period-2 market has a natural inclination to thin out and to fold into the period-1 interaction.

The job market for entry-level lawyers in the United States approximates our two-period setting and illustrates the above phenomenon. This market de facto operates through the market for summer law interns in the preceding year (the period 1 matching). Most firms have a summer program and extend job offers for the following year (the period 2 matching) to a high fraction (more than 90 percent) of interns (NALP, 2014). Thus, most firms and students elect to prolong their initial matching rather than waiting for additional information to arrive during the student’s final year of law school.

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34Roth and Xing (1994) and Ginsburg and Wolf (2004) describe this market in detail. Avery et al. (2001) examine the closely-related market for judicial law clerks.
2.5.2 Financial Transfers

An important generalization of Gale and Shapley’s model involves financial transfers (Crawford and Knoer, 1981; Kelso and Crawford, 1982). Considering this extension in a multi-period setting introduces concerns absent from the one-period case, such as savings or credit. The ability to shift income across time affects welfare directly through consumption smoothing. More subtly, however, this ability also has indirect consequences through its implications for incentives and, ultimately, market stability.\(^{35}\) Our model lets us disentangle both direct and indirect effects.

For a unified exposition and notation, we continue to consider a market with men and women.\(^{36}\) Extending our original model, a single-period outcome now involves an assigned partner, \(j_t\), and a transfer (received) of a numeraire commodity, \(y_t\), in that period. Preferences are now defined over sequences of such pairs:

\[
((j_1, y_1), (j_2, y_2)) \succ_i ((j'_1, y'_1), (j'_2, y'_2)) \succ_i \cdots
\]

To make notation more compact, we sometimes write \(((i_1, y_1), (i_2, y_2)) \equiv (i_y y_z)\). We continue to call \(\mu_t: M \cup W \rightarrow M \cup W\) a single-period matching. We assume that period-\(t\) transfers between agents belong to the finite set \(Y \subset \mathbb{Z}\) and are specified by the function \(\sigma_t: M \cup W \rightarrow Y\).\(^{37}\) We assume that \(0 \in Y\) and \(y_t \in Y \iff -y_t \in Y\). The functions \(\mu_t\) and \(\sigma_t\) are compatible if \(\mu_t(i) = j \implies \sigma_t(i) = -\sigma_t(j)\). Thus, when together, a credit for \(i\) is a debit for \(j\). A (multi-period)

outcome, \(\rho = (\rho_1, \rho_2) \equiv (\frac{y_1}{y_1} \frac{y_2}{y_2}, \frac{y_2}{y_2})\), is a sequence of compatible single-period outcomes, \(\rho_t = (\mu_t, \sigma_t)\).

In multi-period economies, access to credit or savings affects behavior. To illustrate, suppose \((j_2, y_2) \succ_i (j_2, y_2) \succ_i (j_1, y_1)\). Absent further embellishments, if given a choice between \((j_2, y_2)\) and \((j_1, y_1)\), agent \(i\) should opt for the former. Suppose, however, that the agent can save part of his period-1 allocation of the transferable good. Now, if given a choice between \((\frac{y_2}{y_2})\) and \((\frac{y_1}{y_1})\), the

\(^{35}\)These conclusions echo Rogerson’s (1985) from his analysis of a multi-period, principal-agent relationship.

\(^{36}\)Though intrahousehold transfers feature in many domestic arrangements, we hope this nomenclature does not shroud this extension’s broader applicability. Instead, we may consider a matching between firms and workers, with transfers interpreted as wages, or a matching between students and schools, with transfers being tuition charges or scholarships.

\(^{37}\)The discrete set of feasible transfers may be denominated in the smallest practical unit, such as dollars.
latter is superior. Access to savings lets him independently transform the transfer stream \((3, 1)\) into \((2, 2)\) thereby replicating \((\frac{1}{2} \frac{1}{2})\), which he prefers. From \(i\)'s point of view, the plans \((\frac{1}{2} \frac{1}{2})\) and \((\frac{1}{3} \frac{1}{3})\) become “equivalent” once saving is possible.

While financial access can be subsumed into preferences a priori, we introduce it separately to allows for welfare comparisons as financial capabilities change. As a first step, endow each agent with a financial technology, \(f_i(\cdot): Y \times Y \rightarrow 2^{Y \times Y}\). The set \(f_i(y)\) consists of all independently attainable profiles at \(y = (y_1, y_2)\). Intuitively, \(f_i(y)\) can be interpreted as agent \(i\)'s intertemporal budget set. For example, if agent \(i\) can save without interest, we might define \(f_i(y) = \{(y_1, y_2), (y_1 - 1, y_2 + 1), \ldots\}\). By saving, an agent can shift consumption of the transferable good into the future, as in the discussion above. For simplicity, we assume that \(y \in f_i(y)\) and we do not consider financial technologies with stochastic returns. Abusing notation, let \(f_i((\frac{1}{j} \frac{1}{j})) = \{((\frac{1}{j} \frac{1}{j})): (y_1, y_2) \in f_i((1, 2))\}\) be the set of attainable plans at \((\frac{1}{j} \frac{1}{j})\).

To model decision making, define the \(f\)-adaptation of \(\succ_i\) as

\[
(\frac{1}{j} \frac{1}{j}) \succ_f (\frac{1}{j'} \frac{1}{j'}) \iff \exists (\frac{1}{j} \frac{1}{j}%',) \in f_i\left((\frac{1}{j} \frac{1}{j})\right) \text{ such that } \forall (\frac{1}{j} \frac{1}{j}%) \in f_i\left((\frac{1}{j'} \frac{1}{j'})\right), (\frac{1}{j} \frac{1}{j}) \succ_i (\frac{1}{j} \frac{1}{j}%).
\]

If \(x \succ_i^f x'\), then \(x\) gives agent \(i\) access to a better plan than \(x'\). If \(x \succ_i^f x'\) and \(x' \succ_i^f x\), then \(x\) and \(x'\) are \(f\)-equivalent and we write \(x \sim_i^f x'\).\(^{38}\) This equivalence formalizes the observation from the motivating discussion above.

Despite the expanded domain, the definitions of blocking and stability retain their prior form with \(f\)-adapted preferences guiding behavior. Agent \(i\) can period-1 block \(\rho\) if \((\frac{1}{0} \frac{1}{0}) \succ_i^f \rho(i)\) and he can period-2 block \(\rho\) if \((\frac{1}{r_i} \frac{1}{r_i}) \succ_i^f \rho(i)\). Blocking by a pair generalizes similarly. The pair \((m, w)\) can period-1 block \(\rho\) if there exist \(y_1, y_2 \in Y\) such that

1. \((\frac{m}{y_1} \frac{m}{y_2}) \succ_w^f \rho(w)\) and \((\frac{w}{y_1 - y_2} \frac{w}{y_2}) \succ_m^f \rho(m)\);
2. \((\frac{w}{y_1} \frac{m}{y_2}) \succ_w^f \rho(w)\) and \((\frac{w}{y_1} \frac{w}{y_2 - y_1}) \succ_m^f \rho(m)\);
3. \((\frac{m}{y_1} \frac{w}{y_2}) \succ_w^f \rho(w)\) and \((\frac{w}{y_1} \frac{m}{y_2}) \succ_m^f \rho(m)\) or,

\(^{38}\)Lemma 14 shows that \(\sim_i^f\) is an equivalence relation. As usual, \(x \succ_i^f x'\) if \(x \succ_i^f x'\) or \(x \sim_i^f x'\).

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4. \((\frac{w}{0} \frac{w}{0}) \succeq_w \rho(w)\) and \((\frac{m}{0} \frac{m}{0}) \succeq_m \rho(m)\).

They can period-2 block \(\rho\) if there exists \(y_2 \in Y\) such that

1. \((\frac{\nu_i(w)}{\nu_i(w)} \frac{m}{0}) \succeq_w \rho(w)\) and \((\frac{\nu_i(m)}{\nu_i(m)} \frac{w}{-y_2}) \succeq_m \rho(m)\); or,

2. \((\frac{\nu_i(w)}{\nu_i(w)} 0) \succeq_w \rho(w)\) and \((\frac{\nu_i(m)}{\nu_i(m)} 0) \succeq_m \rho(m)\).

The outcome \(\rho\) is ex ante stable if it cannot be period-1 blocked by any agent or pair. It is dynamically stable if it cannot be period-t blocked by any agent or pair in any period.

Like in our original model, ex ante stable outcomes exist without further qualifications (Lemma 15), but a restriction is required to ensure the existence of a dynamically stable outcome.

The \(f\)-adapted preference \(\succeq_f^i\) satisfies generalized sequential improvement complementarity if

\[
(\frac{i}{y_0}) \succeq_f^i \left(\frac{j}{y_0} \right) \preceq_i \left(\frac{i}{0} \right) \implies \left(\frac{k}{y_0} \right) \succeq_i \left(\frac{j}{y_0} \right).
\]

(G-SIC)

G-SIC and SIC share a common underlying intuition and are equivalent when transfers are impossible, i.e. \(Y = \{0\}\). The remaining sufficient conditions generalize analogously. The \(f\)-adapted preference \(\succeq_f^i\) satisfies generalized singlehood aversion if

\[
(\frac{i}{y_0}) \succeq_f^i \left(\frac{i}{i} \right) \implies \left(\frac{i}{y_0} \right) \succeq_i \left(\frac{i}{i} \right) \implies \left(\frac{i}{i} \right) \succeq_i \left(\frac{i}{i} \right).
\]

(G-SA)

The \(f\)-adapted preference \(\succeq_f^i\) satisfies generalized revealed dominance of singlehood if

\[
\left(\frac{i}{y_0} \right) \succeq_f^i \left(\frac{i}{i} \right) \implies \left(\frac{k}{y_0} \right) \succeq_f^i \left(\frac{i}{i} \right) \implies \left(\frac{k}{y_0} \right) \succeq_i \left(\frac{i}{y_0} \right) \implies \left(\frac{k}{y_0} \right) \succeq_i \left(\frac{i}{y_0} \right) \implies \left(\frac{i}{i} \right) \succeq_i \left(\frac{i}{i} \right).
\]

(G-RDS)

**Theorem 10.** There exists a dynamically stable outcome if (a) each agent’s \(f\)-adapted preference satisfies G-SIC and SA; or, (b) each agent’s \(f\)-adapted preference satisfies G-SIC and G-RDS.

To prove Theorem 10 we generalize of the PDAA procedure. We define this procedure in Appendix B.1 and its operation parallels the original PDAA with transfers incorporated into proposals (Crawford and Knoer, 1981; Kelso and Crawford, 1982). Verifying stability mimics the proof of Theorem 6.
The following example illustrates some of the implications of changing financial capabilities for stability and welfare.

**Example 6.** Consider an economy with one employer \((m)\) and one worker \((w)\) with preferences given by (2.3) and (2.4), respectively.

\[
\cdots \succ_m \left( \begin{array}{c} w \\ -1 & -2 & -2 \end{array} \right) \succ_m \left( \begin{array}{c} w \\ -2 & -2 & -2 \end{array} \right) \succ_m \left( \begin{array}{c} m \\ 0 & 0 & 0 \end{array} \right) \succ_m \cdots \succ_m \left( \begin{array}{c} w \\ -1 & -3 & -3 \end{array} \right) \succ_m \cdots \tag{2.3}
\]

\[
\cdots \succ_w \left( \begin{array}{c} m \\ 1 & 1 \end{array} \right) \succ_w \left( \begin{array}{c} m \\ 2 & 2 \end{array} \right) \succ_w \left( \begin{array}{c} m \\ 3 & 3 \end{array} \right) \succ_w \left( \begin{array}{c} m \\ 1 & 1 \end{array} \right) \succ_w \left( \begin{array}{c} w \\ 0 & 0 \end{array} \right) \succ_w \cdots \succ_w \left( \begin{array}{c} w \\ 2 & 0 \end{array} \right) \succ_w \cdots \tag{2.4}
\]

In this situation, the employer must hire the worker for two periods to produce output, but is unable to pay more than 2 dollars in wages per period. To work in both periods, the worker demands a lifetime income of 4 dollars. However, her most preferred outcome is to work for one period and then retire, provided she gets some income in retirement. Consider the following three possibilities:

1. Suppose neither party can save or borrow. In this case there is a unique dynamically stable outcome. The worker is employed for both periods at a wage of 2 per period: 
   \[\rho^*(m) = \left( \begin{array}{c} w \\ -2 & -2 \end{array} \right) \text{ and } \rho^*(w) = \left( \begin{array}{c} m \\ 2 & 2 \end{array} \right).\]

2. Suppose \(w\) gains access to savings, \(f_w(y) = \{(y_1, y_2), (y_1 - 1, y_2 + 1), \ldots\}\), but \(m\) does not. Thus, \(\left( \begin{array}{c} m \\ 1 & 1 \end{array} \right) \succ_w \left( \begin{array}{c} m \\ 2 & 0 \end{array} \right) \succ_w \left( \begin{array}{c} m \\ 3 & 2 \end{array} \right) \text{ and } w\) will period-2 block any plan where she receives 2 in period 1. Thus, there is no dynamically stable outcome. (G-SA is not satisfied.)

3. Suppose, additionally, that \(f_m(y) = \{(y_1, y_2), (y_1 - 1, y_2 + 1), \ldots\}\) and \(m\) can delay payments to \(w\). Therefore, \(\left( \begin{array}{c} w \\ -1 & -3 \end{array} \right) \sim_m \left( \begin{array}{c} w \\ -2 & -2 \end{array} \right)\). Now the outcome where \(\rho^*(m) = \left( \begin{array}{c} w \\ -1 & -3 \end{array} \right)\) and \(\rho^*(w) = \left( \begin{array}{c} m \\ 1 & 3 \end{array} \right)\) is dynamically stable. The employer prevents the worker from quitting prematurely by backloading her compensation.

Example 6 shows that transfers and finance have cross-cutting implications for market stability and welfare. Generally, credit or savings should improve welfare as they expand an agent’s consumption possibilities. However, once we focus on stable outcomes, matters may be different. Stable outcomes may fail to exist. And, even if they do exist, welfare may decline. The dynamically
stable outcome in the absence of savings (case 1) Pareto-dominates the outcome when saving is possible (case 3).

2.5.3 Irrevocable Assignments

A central point of our analysis is that agents’ assignments may change with time. However, our model can also be employed to study economies where matchings are irrevocable and may be formed in either period 1 or period 2. Doval (2015) has recently analyzed a dynamic matching market of this form. Developed contemporaneously and independently of our analysis, she proposes a definition of dynamic stability that differs from our proposal. Given her definition, she observes that stable matchings often fail to exist. She proposes restrictions on agents’ preferences ensuring that they do.

A preference restriction lets us embed an economy with permanent assignments into our model. Call $\succ_i$ amendment averse if

$$jk \succ_i ii \implies k = j \text{ or } j = i.$$  

(AA)

Amendment aversion places no restrictions on agents’ preferences beyond assignment immutability. Even time-inconsistent preferences, such as $jj \succ_i ik \succ_i kk \succ_i ij$, satisfy amendment aversion. When preferences satisfy amendment aversion, the PDAA assignment is dynamically stable and the PDAA procedure is strategyproof for the proposing side (Theorems 14 and 15).

2.6 Concluding Remarks

We have proposed a conservative, portable, multi-period generalization of the classic model of one-to-one matching. Our analysis shifts focus away from a one-shot interaction toward a long-term appraisal of a market’s operation and of agents’ welfare. Such a focus is the natural

39 Doval (2015) also considers one-sided matching and situations where agents arrive stochastically. We confine our discussion to the instance of her model closest to our setting.

40 AA is neither weaker nor stronger than SIC. Example 13 shows that AA and SIC cannot be readily substituted for one another.
one since many bilateral interactions, such as marriage, employment, or schooling, are far from fleeting, though often impermanent. The conditions supporting stable outcomes are behaviorally-plausible and, we contend, quite common. Sequential improvement complementarity allows for complementarities among distinct partners and status-quo bias. While common attitudes, such as those featuring inertia, seemingly tilt preferences toward persistent plans, preferred stable outcomes may in fact be volatile. Though our discussion focuses on two periods, these conclusions generalize. Our model readily accommodates monetary transfers with credit and finance, and the common case where agents are uncertain about their future preferences. These extensions provide subtle qualifications of our primary analysis.

For brevity we have suppressed many natural embellishments. For example, we have not directly addressed the arrival or departure of agents nor the many-to-one nature of some matching problems. While such extensions introduce their own complications, the intuition of our stability definitions carry over naturally. Similarly, we have only sketched the strategic nuances of multi-period markets. We consider these questions as promising areas for future research and we hope that our analysis provides a foundation for their investigation. Of course, many of these features will also be central when the time-component of relationships is incorporated into market design exercises. In that context, time can serve as a design variable rather than a constraint. New solutions to previously challenging problems may emerge.
Chapter 3

Unilateral substitutability implies substitutable completability

The literature on many-to-one matching markets with contracts started with the seminal contributions by Kelso and Crawford (1982)\(^1\) and Hatfield and Milgrom (2005).\(^2\) The practical applications of these markets with contracts have recently been investigated in some interesting contexts like cadet branching (Sönmez (2013); Sönmez and Switzer (2013)), matching with regional caps (Kamada and Kojima (2012, 2015a)), and diversity design in school choice (Kominers and Sönmez (2013)).

Roth (1990) described the importance of stability for practical matching markets. He observed that the markets which generated a stable outcome continued to operate over longer periods of time than the ones which did not guarantee this property. For many-to-one matching with contracts, the literature has provided many conditions on the agents’ preferences which are sufficient for stability, e.g. substitutability (Kelso and Crawford (1982); Roth (1984a)), unilateral substitutability

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\(^2\)This generalized the many-to-one matching market from Gale and Shapley (1962). That has in turn been further extended and generalized to interesting domains, viz. supply chain networks (Ostrovsky (2008); Hatfield and Kominers (2012b)) and many-to-many matching markets with contracts (Hatfield and Kominers (2012a)).

\(^3\)Echenique and Pereyra (2013) has shown the surprising isomorphism between Kelso and Crawford (1982) and Hatfield and Milgrom (2005).
(Hatfield and Kojima (2010)), bilateral substitutability (Hatfield and Kojima (2010)), and substitutable completability (Hatfield and Kominers (2014)). However, the literature has not fully explored the connections between these sufficient conditions, which might be useful for practical applications. This essay shows that unilateral substitutability implies substitutable completability.

The preference of an agent on the many-side of the market satisfies the substitutability condition when the agent does not have any complementarities between contracts.\(^4\) In other words, the agent views each contract independently and never finds a contract that is rejected from some set of contracts to be acceptable only in the presence of another contract. A many-to-one preference of an agent satisfies the substitutable completability condition if there is a substitutable completion, i.e. a certain ‘related’ substitutable preference in the many-to-many setting for that agent. The preference of an agent has the unilateral substitutability property when the preference exhibits complementarities, if any, of only a certain kind; put differently, there may be certain ‘permissible’ violations of the substitutability condition.

Allowing for a broader class of complementarities gives the bilateral substitutability condition, and further expanding the ‘allowed’ set of violations of the substitutability condition yields weak substitutability condition (abbreviated as W.Sub.). The weak substitutability condition is the strongest known necessary condition\(^5\) for stability. The following description and the two Venn diagrams summarize the relationships between these conditions that are known from the existing literature.

\(^4\)All the relevant conditions are defined in section 3.1.2.

\(^5\)The necessary condition implies that if there is an agent with a preference not satisfying the weak substitutability condition then there exists a setup of weak substitutable preferences for other agents such that no stable matching exists as proved in Hatfield and Kojima (2008).
substitutability [Sub.] \Rightarrow unilateral substitutability [U.Sub.]

\Rightarrow bilateral substitutability [B.Sub.]

substitutability \Rightarrow substitutable completability [Sub.Comp.]

bilateral substitutability \Leftrightarrow substitutable completability

substitutable completability \Leftrightarrow bilateral substitutability

By proving that unilateral substitutability implies substitutable completability, we are able to provide the following unified Venn Diagram.

To fix ideas, consider the many-to-one matching setting of hospitals and doctors where a hospital can sign at most one contract with each of (possibly) many doctors. Each doctor can sign at most one contract. A set of contracts which involves multiple contracts with at least one
doctor is termed an infeasible set. A substitutable completion of a preference, more precisely, is defined as a substitutable preference, i.e. a preference without any violations of the substitutability condition, created by the addition of infeasible sets to the original preference order. A preference that has a substitutable completion is defined as substitutably completable. Hatfield and Kominers (2014) give this definition involving the addition of infeasible sets to preference orders and provide techniques that work for some class of preferences, namely, slot-specific preferences and task-specific preferences.

In our proof, we provide an algorithm which for any unilaterally substitutable preference arrives at a substitutable completion. The algorithm ensures that the additions of infeasible sets, as allowed by the definition of completion, are at the ‘right’ places in the preference order, ensuring that the existing violations of the substitutability property are sequentially eliminated and that no new violations are created. Thus with the algorithm, we provide a constructive proof that unilateral substitutability implies substitutable completability.

The unilateral substitutability property was found to be relevant in the case of the cadet branching market (Sönmez and Switzer (2013)). It guaranteed the existence of a stable matching in that setting although the strongest of the sufficient conditions above, i.e. substitutability, was not satisfied. The substitutable completable property gives the intuitive understanding that in the setting of many-to-one matching market with contracts, there exists a stable matching which is a projection from the many-to-many matching market with contracts setting to the many-to-one case.

Moreover, in the case of matching with regional caps, Kamada and Kojima (2015a) (essentially) use the substitutable completable property while connecting their setting to the matching with contracts framework. They view regions and doctors as the two sides of the matching market instead of hospitals and doctors. The hospital in a particular region being assigned to the doctor is specified in the contract and a region may choose a particular doctor for two separate hospitals.

The two most important practical applications of the matching with contracts framework have

\[\text{More precisely, Kamada and Kojima (2015a) Theorem 2 follows directly from Kamada and Kojima (2015b) Theorem 1 which they prove using (many-to-many) matching with contracts framework.}\]
used two different sufficient conditions. This essay provides a synthesis for the two conditions being used by proving that one is implied by the other.

The rest of this chapter is organized as follows. Section introduces the preliminaries of the many-to-one matching market with contracts. Section 3.2 gives the theorem, and a constructive proof using an algorithm. Section 3.3 discusses the properties of stable matches. In the last section, I summarize the contribution of this paper.

3.1 Preliminaries

We use the model from Hatfield and Milgrom (2005), and Hatfield and Kojima (2010) and extend the notation from Kominers and Sönmez (2013). A many-to-one matching market with contracts is parametrized by a finite set of doctors, $D$, a finite set of hospitals, $H$, and a finite set of contracts, $X$.

Each contract $x \in X$ is associated with a doctor $d(x) \in D$ and a hospital $h(x) \in H$. We extend this notation to sets of contracts by defining $d(X') \equiv \bigcup_{x \in X'} \{d(x)\}$ and $h(X') \equiv \bigcup_{x \in X'} \{h(x)\}$ for $X' \subseteq X$.

**Definition 14.** For a doctor $d$, a hospital $h$ and a set of contracts $X'$, the subset of contracts with only a specific agent are defined as follows.

$$X'_d \equiv \{x \in X'|d(x) = d\}$$

$$X'_h \equiv \{x \in X'|h(x) = h\}$$

Each doctor $d$ has access to a $\emptyset_d$ contract which represents being not matched to anybody. We say that $X' \subseteq X$ is **feasible** for a hospital $h$ if $X' \subseteq X_h$ and for all $x, x' \in X'$, we have $d(x) = d(x')$ implies $x = x'$. A set of contracts $X' \subseteq X$ is called an **allocation** if $x, x' \in X'$ and $d(x) = d(x')$ implies $x = x'$. These conditions ensure that each doctor can sign at most one contract.
3.1.1 Preferences of hospitals and doctors

For each \(d \in D\), \(P_d\) is a strict preference relation on contracts in \(X_d \cup \{\emptyset\}\). For each \(h \in H\), \(P_h\) is a strict preference relation on all feasible subsets of contracts of \(X\) (including the null set). The primitives of this model are preferences over contracts or sets of contracts (and not choice functions over subsets). Hence, the Aygün and Sönmez (2013a,b)’s irrelevance of rejected contracts (IRC) condition holds in our setting.\(^7\) The IRC condition requires that the removal of rejected contracts should not affect the chosen set of contracts.

**Definition 15.** Given a set of contracts \(X\), a choice function \(C : 2^{X} \rightarrow 2^{X}\) satisfies the irrelevance of rejected contracts (IRC) condition if for all \(Y \subset X\) and for all \(z \in X \setminus Y\), we have \(z \notin C(Y \cup \{z\})\) implies \(C(Y) = C(Y \cup \{z\})\).

We define a choice function \(C_d(X')\) (and \(C_h(X')\)) based on the strict preference ordering \(P_d\) (and \(P_h\) respectively). The choice function \(C_d(X')\) will be a singleton that represents the chosen contract (possibly \(\emptyset\)) by the doctor \(d\) from \(X' \cup \{\emptyset\}\). Similarly, \(C_h(X')\) will represent the chosen set of contracts (possibly \(\emptyset\)) by the hospital \(h\) from \(X'\) based on its strict preference ordering \(P_h\) over its feasible subsets of contracts. Formally, for any set \(X' \subseteq X\), \(d \in D\), and \(h \in H\) let

\[
C_d(X') = \max_{P_d} \{x \in X'|d = d(x)\} \cup \{\emptyset\} \quad (3.1)
\]

\[
C_h(X') = \max_{P_h} \{Z \subseteq X'|x \in Z \Rightarrow h = h(x)\}. \quad (3.2)
\]

The rejection sets are defined as the contracts not chosen by the agents, i.e. \(R_d(X') = X' \setminus \{C_d(X')\}\) and \(R_h(X') = X' \setminus \{C_h(X')\}\). Further, the chosen set for all doctors is defined as the union of the chosen sets for each of the doctors, i.e. \(C_D(X') = \cup_{d \in D} C_d(X')\). Likewise, we define the chosen sets for all hospitals, i.e. \(C_H(X') = \cup_{h \in H} C_h(X')\).

3.1.2 Stability and previous results

**Definition 16.** An allocation \(X' \subseteq X\) is a stable allocation (or a stable set of contracts) if

\(^7\)When the primitives are choice functions instead of preference orders over sets of contracts, then this condition is not automatically guaranteed. See Aygün and Sönmez (2013b) for an illuminating discussion.
1. **(Individual Rationality)** \( C_D(X') = C_H(X') = X' \), and

2. **(No Blocking)** there does not exist a hospital \( h \) and a set of contracts \( X'' \neq C_h(X') \) such that \( X'' = C_h(X' \cup X'') \subset C_D(X' \cup X'') \).

Kelso and Crawford (1982) and Hatfield and Milgrom (2005) showed that the *substitutability* condition is sufficient for stability.

**Definition 17.** Contracts are *substitutes* for a hospital \( h \in H \) if \( \forall X'' \subseteq X' \subseteq X \), the rejection sets are isotone, i.e. \( R_h(X'') \subseteq R_h(X') \). In other words, contracts are *substitutes* for a hospital \( h \in H \) if for all contracts \( x, z \in X \) and all sets \( Y \subseteq X \), we have \( z \in C_h(Y \cup \{ z, x \}) \) implies \( z \in C_h(Y \cup \{ z \}) \).

**Definition 18.** A tuple \((x, z, Y)\) where \( x, z \in X \) and \( Y \subseteq X \), that fails the substitutes requirement, i.e. \( z \in C_h(Y \cup \{ z, x \}) \) but \( z \notin C_h(Y \cup \{ z \}) \), is called a *substitutability violation*. The contract \( x \) in this case is called an *alterer contract* and the contract \( z \) is termed as a *recalled contract*.

The hospital preferences satisfy the substitutability condition if there are no substitutability violations. By permitting some specific violations, Hatfield and Kojima (2010) arrived at the following weaker conditions, which they proved to be sufficient for stability. Preferences are unilaterally substitutable if the only substitutability violations, \((x, z, Y)\), that exist are with sets \( Y \) such that \( d(z) \in d(Y) \), i.e. the doctor involved in the recalled contract has at least some contract in the set \( Y \). The bilateral substitutability condition is weaker than the unilateral substitutability condition and the permissible substitutability violations, \((x, z, Y)\), are with sets \( Y \) such that \( d(x) \in d(Y) \) or \( d(z) \in d(Y) \), i.e. either the doctor involved in the alterer contract or the recalled contract has some contract in the set \( Y \).

**Definition 19.** Contracts are *unilateral substitutes* for a hospital \( h \in H \) if for all contracts \( x, z \in X \)

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8 The requirement specified has a bite only when \( x \in C_h(Y \cup \{ z, x \}) \) otherwise the requirement of \( [z \in C_h(Y \cup \{ z, x \}) \Rightarrow z \in C_h(Y \cup \{ z \})] \) trivially holds due to IRC (definition 15).

9 The definition was introduced and named as such in Hatfield, Immorlica and Kominers (2012).

10 The following preference \( P_h \) fails the substitutability condition where \( d(z) = d(z') \neq d(x) \) and all contracts are with hospital \( h \). \( P_h : \{ x, z \} \succ \{ x \} \succ \{ z' \} \succ \{ z \} \) as \((x, z, \{ z' \})\) is a substitutability violation. \( x \) is the alterer contract and \( z \) is the recalled contract.
and all sets $Y \subseteq X$ such that $d(z) \not\in d(Y)$, we have $z \in C_h(Y \cup \{z, x\})$ implies $z \in C_h(Y \cup \{z\})$.\footnote{The requirement specified has a bite only when $x \in C_h(Y \cup \{z, x\})$. See footnote 8}

**Remark 3.** If $(x, z, Y)$ is a substitutability violation in a preference which has the unilateral substitutability property then $d(z) \in d(Y \setminus \{z\})$.

Note that the definition of unilateral substitutability explicitly requires that for all the substitutability violations we should have $d(z) \in d(Y)$. The above remark which is stronger than the explicit requirement implies that for any substitutability violation $(x, z, Y)$ in unilaterally substitutable preferences, the doctor involved with the recalled contract has at least some contract other than $z$ in $Y$. If this were not true, i.e. suppose $d(z) \not\in d(Y \setminus \{z\})$ then we will get a contradiction to the unilateral substitutability property. Define $\bar{Y} \equiv Y \setminus \{z\}$ and it follows that $z \in C_h(Y \cup \{z, x\}) = C_h(\bar{Y} \cup \{z, x\})$ and $z \not\in C_h(Y \cup \{z\}) = C_h(\bar{Y} \cup \{z\})$. Thus $(x, z, \bar{Y})$ is a substitutability violation and by assumption $d(z) \not\in d(\bar{Y})$. This would imply the contradiction.

**Definition 20.** Contracts are **bilateral substitutes** for a hospital $h \in H$ if for all contracts $x, z \in X$ and all sets $Y \subseteq X$ such that $d(x)$, $d(z) \not\in d(Y)$, we have $z \in C_h(Y \cup \{z, x\})$ implies $z \in C_h(Y \cup \{z\})$.\footnote{The requirement specified has a bite only when $x \in C_h(Y \cup \{z, x\})$. See footnote 8}

**Remark 4.** If $(x, z, Y)$ is a substitutability violation in a preference which satisfies bilateral substitutability property then $d(z) \in d(Y \setminus \{z\})$ or $d(x) \in d(Y \setminus \{x\})$.

The above remark which is stronger than the explicit requirement of the definition implies that for any substitutability violation, $(x, z, Y)$, in bilaterally substitutable preferences, the doctor involved with the recalled contract has at least some contract other than $z$ in $Y$ or the doctor involved with the alterer contract has at least some contract other than $x$ in $Y$. This holds for exactly the same reasons as in Remark 3 above.

Hatfield and Kominers (2014) provided a condition termed substitutable completestability as a sufficient condition for the existence of a stable matching. They use the many-to-many matching markets setup and preferences to give this sufficient condition.
Definition 21. A many-to-many preference for a hospital $h$ is a preference relation over all subsets $X' \subseteq X$ such that $X'_h = X'$.\(^{13}\)

The process of preference completion involves adding infeasible sets that include multiple contracts with some doctors, to a many-to-one preference relation (which is only over feasible sets of contracts).

Definition 22. A completion of a many-to-one choice function $C_h$, corresponding to the preference $P_h$ for a hospital $h \in H$, is a choice function $\bar{C}_h$ such that for all $Y \subseteq X$ either

1. $\bar{C}_h(Y) = C_h(Y)$ or

2. $\exists$ distinct $z, \hat{z} \in \bar{C}_h(Y)$ such that $d(z) = d(\hat{z})$

If a choice function $C_h$ has a completion that is substitutable and satisfies the IRC condition, then we say that the $C_h$ is substitutably completable.

In our setting where we take the preference $P_h$ as the primitive for the hospitals, the equivalent definition of substitutable completability is as follows.

Definition 23. A preference $P_h$ is substitutably completable for hospital $h \in H$ if there exists a substitutable many-to-many preference $\bar{P}_h$ such that $\bar{P}_h$ satisfies the following conditions:

- For feasible $X', X'' \subseteq X$, if $X' \succ_h X'' \succ_h \emptyset$ under $P_h$ then $X' \succ_h X'' \succ_h \emptyset$ under $\bar{P}_h$.

- For feasible $X' \subseteq X$, if $\emptyset \succ_h X'$ under $P_h$ then it remains unacceptable, i.e. $\emptyset \succ_h X'$ under $\bar{P}_h$.

$\bar{P}_h$ is defined as a substitutable completion of the preference $P_h$.

Since the resulting completion is substitutable, there exists a stable matching in the many-to-many setting due to the sufficiency of the substitutability condition (Hatfield and Kominers (2012b,a)). This stable matching also respects the restrictions of the many-to-one setting, as

\(^{13}\)A preference profile in the many-to-one matching market with contracts was over all feasible subsets $X' \subseteq X_h$ where as the many-to-many preference is over all subsets $X' \subseteq X_h$. 

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the preferences for the doctors were not changed and they still find only singleton contracts acceptable.\textsuperscript{14}

3.2 Main Result

We now present the main result of this paper. We provide the intuition behind the constructive proof and describe the algorithm before proceeding to the proof of this theorem.

**Theorem 11.** If the hospital preferences are unilaterally substitutable then they are substitutably completable, i.e. for any $P_\text{h}$ with the unilateral substitutability property, there exists a completion $\bar{P}_\text{h}$ which satisfies the substitutability property.

We now present the main idea behind how the constructive proof achieves a substitutable completion of preferences. Recall that a preference has the substitutability property if and only if

- there are no substitutability violations.

Further recall that a preference is unilaterally substitutable if and only if

- there are no substitutability violations OR

- there are only ‘permissible’ substitutability violations.

Specifically, all violations $(x, z, Y)$ that exist are the permissible ones where the doctor of the recalled contract, i.e. $d(z)$, has some contract in $Y$ (other than $z$).\textsuperscript{15} For example, consider the following preferences $P_\text{h}$ for a hospital $h$. Here we have, $d(z) = d(z') \neq d(x)$ and all the contracts listed are with hospital $h$.

\[
\begin{align*}
P_\text{h} : \quad & \{x, z\} \succ \{x\} \succ \{z'\} \succ \{z\} \succ \emptyset \\
(x, z, \{z'\}) & \quad \text{is the only substitutability violation} \\
& \quad \text{and it is permissible.}
\end{align*}
\]

\textsuperscript{14}A stable match in the many-to-many setting should be individually rational for each doctor which rules out a subset of contracts that is not an allocation.

\textsuperscript{15}See remark 3.

The algorithm uses the recalled contract and the existing contract(s) with \(d(z)\) in set \(Y\) and adds them to the chosen set of contracts under the original preferences from \(Y \cup \{z\}\) to create the new chosen set under the new preference. The change in the chosen set is achieved by making the new chosen set (just) better than the older chosen set. The presence of at least two contracts with \(d(z)\) ensures that the new chosen set being added in the preference relation is an infeasible set. This ensures that the resulting new preference is a completion. To ensure that the completion process does not create any new substitutability violations while correcting the existing ones, the algorithm may also change a chosen set that includes a contract involving \(d(z)\) under the original preference. This further ensures that the final completion created by the algorithm is substitutable. The following example elucidates this process.

\[
C_h(\{z, z'\}) = \{z'\} \text{ as per the preference relation } P_h
\]

\[
\hat{C}_h(\{z, z'\}) = \{z, z'\} \text{ by completing the preference to } \bar{P}_h
\]

\[
\bar{P}_h : \{x, z\} \succ \{x\} \succ \{z', z\} \succ \{z'\} \succ \{z\} \succ \emptyset
\]

\[
(x, z, \{z'\}) \text{ is not a substitutability violation}
\]

Before proceeding to the proof, we introduce the following concept which will be used in one of the steps of the algorithm, and provide an example which clarifies the relevance of this concept.

**Definition 24.** The maximal subset of \(\tilde{Y}\) with only unrejected contracts with a doctor, say \(d_1\), under a preference \(P_{\tilde{h}}\), is defined as the set \(\hat{Y} \subseteq \tilde{Y}\) such that

1. All non-\(d_1\) contracts of \(\tilde{Y}\) are present in \(\hat{Y}\), i.e. \(\hat{Y} \setminus \hat{Y}_{d_1} = \hat{Y} \setminus \tilde{Y}_{d_1}\)

2. \(\forall Y'' \subseteq \hat{Y}, d_1 \in d(Y'') \implies d_1 \in d(C_h(Y''))\)

and \(\hat{Y}\) is the maximal such subset under the partial order of set inclusion.\(^{16}\)

Note the maximal set defined above is unique due to the many-to-one nature of the setting.\(^{17}\)

\(^{16}\)Maximality of \(\hat{Y}\) implies that \(\not\exists\) any other set \(\hat{Y} \subseteq \tilde{Y}\) but not a subset of \(\hat{Y}\) for which the following conditions holds, \(\hat{Y} \setminus \hat{Y}_{d_1} = \hat{Y} \setminus \tilde{Y}_{d_1}\) and \(\forall Y'' \subseteq \hat{Y}, d_1 \in d(Y'') \implies d_1 \in d(C_h(Y''))\)

\(^{17}\)See Appendix section Claim 1 for the proof.
This maximal subset only has contracts which are, loosely speaking, not always rejected in the original preference. Consider the following preference which will explain the above concept.

**Example 7.**

\[
P_h : \{x, z\} \succ \{x\} \succ \{z\} \succ \emptyset \succ \{\hat{z}\}
\]

\[
d(z) = d(z') = d(\hat{z}) \neq d(x)
\]

Let \( \hat{Y} = \{z', z, \hat{z}\} \). The maximal subset of \( \hat{Y} \) with only unrejected contracts with doctor \( d_1 = d(\hat{z}) \) under \( P_h \) is \( \hat{Y} = \{z', z\} \) because \( d(\hat{z}) \notin d(C_h(\{\hat{z}\})) \).

Now we are ready to present the algorithm with the understanding of the underlying intuition presented above.

### 3.2.1 Algorithm

**Step 0** Define \( P_0^h \equiv P_h \) and \( C_0^h(\cdot) \equiv C_h(\cdot) \) for the hospital \( h \). Go to step 1.

**Step i** Does there exist a substitutability violation \((x_i, z_i, Y_i)\) for \( P_{i-1}^h \)?

- If yes define \( C_i^h(\hat{Y}) \) as follows for all \( \hat{Y} \subseteq X_h \).
  
  * If \( d(z_i) \notin d(C_{i-1}^h(\hat{Y})) \) then \( C_i^h(\hat{Y}) \equiv C_{i-1}^h(\hat{Y}) \).
  
  * If \( d(z_i) \in d(C_{i-1}^h(\hat{Y})) \) then find the maximal subset \( \hat{Y} \) of the set under consideration, \( \hat{Y} \) with only unrejected contracts with doctor \( d(z_i) \) under \( P_{i-1}^h \). Define the choice function as \( C_i^h(\hat{Y}) \equiv C_{i-1}^h(\hat{Y}) \cup [\hat{Y} \setminus d(z_i)] \).

The choice function \( C_i^h \) has the underlying preference relation \( P_i^h \). The existence of such a preference relation over subsets of \( X_h \) is proved in Claim 2 of the Appendix with a constructive proof.

Go to step \( i + 1 \).

- If not then \( I \equiv i - 1 \) and \( \bar{P}_h \equiv P_i^h \) and terminate the algorithm.\(^{19}\)

---

\(^{18}\)Without loss of generality, we can restrict our attention to \( x_i, z_i \in X_h \) and \( Y_i \subseteq X_h \).

\(^{19}\)Kadam (2015b) has an illustrative example worked out in Section 4.
We start with the following four observations about the algorithm which will be useful at various steps of the proofs that will follow.

1. During step $i$, preferences are changed to $P_i^h$ only if a substitutability violation $(x_i, z_i, Y_i)$ is identified in $P_{i-1}^h$.

2. The chosen sets are weakly increasing in the order of set inclusion at all steps of the algorithm. Specifically, for all $\tilde{Y}$, we have $C_{i-1}^i(\tilde{Y}) \subseteq C_i^i(\tilde{Y})$. We refer to this property as the property of a weak order over the chosen sets.

3. The chosen set $C_i^i(\tilde{Y})$ could be different from $C_{i-1}^i(\tilde{Y})$ only if $d(z_i) \not\in d(C_{i-1}^i(\tilde{Y}))$.

4. Moreover, $C_i^i(\tilde{Y})$ could be different from $C_{i-1}^i(\tilde{Y})$ only in contracts with doctor $d(z_i)$.

We use $C_i^i(\cdot)$ and $P_i^h$ interchangeably, above and for the rest of the proof, as may be appropriate for the context. The existence of $P_i^h$ is guaranteed in the algorithm as proved in Claim 2 of Appendix section 17.

3.2.2 Lemmas

We define a new violation as a substitutability violation that exists in $P_i^h$ but not in $P_{i-1}^h$ for some $i \geq 1$ in the above algorithm. We now prove that no such new violations are created by the algorithm.

**Lemma 5.** No new violations are created in the algorithm above.

**Proof** We prove this by contradiction. Suppose there are new violations introduced in the algorithm above. Let us consider the first step ‘$i’ where a new violation, $(x, z, Y)$ is introduced. We have $z \in C_i^i(Y \cup \{x, z\})$ but $z \notin C_{i-1}^i(Y \cup \{z\})$. Recall from observation 1 that $(x_i, z_i, Y_i)$ is the substitutability violation in preferences $P_{i-1}^h$ identified in step $i$. We now prove the following.

(I) The contract $z$ is added to the chosen set of $Y \cup \{x, z\}$ in step $i$, i.e. $z \notin C_i^i(Y \cup \{z\})$ and $z \in C_i^i(Y \cup \{x, z\})$.

(II) The doctor $d(z)$ has some contract in the chosen set from $Y \cup \{z\}$ under $P_{i-1}^h$. 

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(III) The contract $z$ is a part of the chosen set under $P^i_h$ of $Y \cup \{z\}$ at step $i$ which contradicts the assumption that $(x, z, Y)$ is a substitutability violation.

(I) To prove the first claim recall that $z \in C^i_h(Y \cup \{x, z\})$ but $z \notin C^i_h(Y \cup \{z\})$. Along with observation 2 about the weak order over the chosen sets, this implies that $z \notin C^i_h(Y \cup \{z\})$.

As $(x, z, Y)$ is not a violation in step $i-1$, we necessarily have $z \notin C^i_h(Y \cup \{x, z\})$. This shows that $z$ is added to the chosen set from $Y \cup \{x, z\}$ which is modified during step $i$. This proves claim (I).

Before we move to the proof of claim (II), let us denote by $\hat{Y}_1$ the maximal subset of $Y \cup \{x, z\}$ with only unrejected contracts with doctor $d(z_i)$ under preference $P^i_{h-1}$ as evaluated in step $i$ of the algorithm above.\(^{21}\) Using claim (I), the algorithm, and observation 4, we have the following.

$$d(z) = d(z_i) \text{ and } z \in \hat{Y}_1 \quad (3.3)$$

(II) We prove the second claim by contradiction. Suppose not, i.e. $d(z) \notin d[C^{i-1}_h(Y \cup \{z\})]$. We have maintained the assumption that the primitives in this setting are preferences over contracts and not choice correspondences, which guarantees the IRC condition.

Since $d(z)$’s contract is not in the chosen set of $Y \cup \{z\}$ (by assumption), we can remove any or all of the $d(z)$ contracts and still leave the chosen set unaffected. Specifically, we remove all but $z$ of the $d(z)$ contracts from set $Y$ and create $\hat{Y} = (Y \setminus Y_d(z)) \cup \{z\}$. We then appeal to IRC to have $C^{i-1}_h(\hat{Y}) = C^{i-1}_h(Y \cup \{z\})$ which also implies the following.

$$d(z) \notin d[C^{i-1}_h(\hat{Y})] \quad (3.4)$$

Recall that the maximal subset $\hat{Y}_1$ (introduced above) of $Y \cup \{x, z\}$ with only unrejected contracts with doctor $d(z) = d(z_i)$ under preference $P^i_{h-1}$ satisfies the following properties.

(a) $\hat{Y}_1 \setminus [\hat{Y}_1]_{d(z)} = Y \cup \{x, z\} \setminus [Y \cup \{x, z\}]_{d(z)} = Y \cup \{x\} \setminus Y_{d(z)}$

\(^{20}\)We know that $z \in C^{i-1}_h(Y \cup \{x, z\}) \Rightarrow z \in C^{i-1}_h(Y \cup \{z\})$ is true.

\(^{21}\)Refer to the definition 24 above for a description of a maximal subset of a set with only unrejected contracts with a specific doctor under a specific preference.

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(b) \( \forall Y''_i \subseteq \hat{Y}_i, d(z) \in d(Y''_i) \Rightarrow d(z) \in d[C^{-1}_h(Y''_i)] \)

The set \( \hat{Y} \setminus \{z\} \) includes all (and only) the non-\( d(z) \) contracts from the set \( Y \) whereas the set \( \hat{Y}_1 \setminus [\hat{Y}_1]_{d(z)} \) includes all (and only) the non-\( d(z) \) contracts from the set \( Y \cup \{x, z\} \). Hence we have \( \hat{Y} \setminus \{z\} \subseteq \hat{Y}_1 \setminus [\hat{Y}_1]_{d(z)} \). From observation (3), we know that \( z \in \hat{Y}_1 \) which implies \( \hat{Y} \subseteq \hat{Y}_1 \). Furthermore, using the property (b) of the set \( \hat{Y}_1 \), we have the following.

Since \( \hat{Y} \subseteq \hat{Y}_1 \)

We have \( d(z) \in d(\hat{Y}) \Rightarrow d(z) \in d[C_h(\hat{Y})] \)

However, this is at odds with condition (3.4) above. This leads to the required contradiction. Hence we have \( d(z) \in d[C^{-1}_h(Y \cup \{z\})] \).

(III) Let the maximal subset of \( Y \cup \{z\} \) with unrejected contracts with doctor \( d(z) \) under \( C^{-1}_h \) as per definition 24, evaluated at step \( i \) of the algorithm be \( \hat{Y}_2 \). We know that similar to \( \hat{Y}_1 \) above, the set \( \hat{Y}_2 \) satisfies the following properties.

(a) \( \hat{Y}_2 \setminus [\hat{Y}_2]_{d(z)} = Y \cup \{z\} \setminus [Y \cup \{z\}]_{d(z)} = Y \setminus Y_{d(z)} \)

(b) \( \forall Y''_i \subseteq \hat{Y}_2, d(z) \in d(Y''_i) \Rightarrow d(z) \in d[C^{-1}_h(Y''_i)] \).

(c) \( \hat{Y}_2 \) is a maximal subset of \( Y \cup \{z\} \) which satisfies properties (a) and (b).

From (II), we know that \( d(z) \in d[C^{-1}_h(Y \cup \{z\})] \), the chosen set is (possibly) altered at step \( i \) using \( \hat{Y}_2 \). We want to prove that \( z \in \hat{Y}_2 \). Suppose not. Then \( \hat{Y}_2 \cup \{z\} \) is a larger set than \( \hat{Y}_2 \) but is not the maximal subset with unrejected contracts with doctor \( d(z) \) under \( C^{-1}_h \), evaluated at step \( i \) of the algorithm which was required for constructing the new preference. Let us observe that all non-\( d(z) \) contracts of \( Y \cup \{z\} \) are present in \( \hat{Y}_2 \) and consequently in \( \hat{Y}_2 \cup \{z\} \) as well. Hence \( \hat{Y}_2 \cup \{z\} \) must fail requirement (b) of the maximal subset. There must exist a set \( Y''_2 \subseteq \hat{Y}_2 \cup \{z\} \) such that the following is true.

\[ d(z) \in d(Y''_2) \text{ but } d(z) \notin d[C^{-1}_h(Y''_2)] \quad (3.5) \]
However, $Y_2'' \not\subseteq \hat{Y}_2$ otherwise that would be a violation of the second requirement for the maximal subset, $\hat{Y}_2$. This is possible only if $z \in Y_2''$.

Noting that $Y_2''$ contains $z$, we will define $\hat{Y}_2$ as the subset of $Y_2''$ which removes all but \{z\} of the $d(z)$ contracts (and keeps all the non-$d(z)$ contracts), i.e. $\hat{Y}_2 = (Y_2'' \setminus [Y_2'']_{d(z)}) \cup \{z\} \subseteq Y_2''$.

Since $d(z) \not\in d[C_h^{-1}(Y_2'')]$, by repeatedly applying IRC on condition (3.5) to remove all but the \{z\} contract from $Y_2''$, we get the following.

$$d(z) \not\in d[Y_2'' \setminus [Y_2'']_{d(z)}] \cup \{z\} = d[C_h(\hat{Y}_2)] \quad (3.6)$$

Recall that $\hat{Y}_1$ is the maximal subset of $Y \cup \{x, z\}$. We will now show that $\hat{Y}_2$ is a subset of $\hat{Y}_1$ by comparing it with $\hat{Y}_1$ and $\hat{Y}_2$ more closely.

$$\hat{Y}_2 \setminus \{z\} = Y_2'' \setminus [Y_2'']_{d(z)}$$
$$\subseteq \hat{Y}_2 \setminus [\hat{Y}_2]_{d(z)} = Y \setminus Y_{d(z)}$$
$$\subseteq (Y \cup \{x\}) \setminus Y_{d(z)} = \hat{Y}_1 \setminus [\hat{Y}_1]_{d(z)} \subseteq \hat{Y}_1$$

We have $\hat{Y}_2 \subseteq \hat{Y}_1 \cup \{z\} = \hat{Y}_1$ (from condition (3.3)).

Since we have $z \in \hat{Y}_2$ by condition (b) of the maximal subset, $\hat{Y}_1$ we have $d(z) \in d[C_h(\hat{Y}_2)]$ which contradicts the conclusion above in condition (3.6). This implies our initial assumption that $z \not\in \hat{Y}_2$ is incorrect.

Hence we have $z \in \hat{Y}_2$ and thus $z$ is a part of the chosen set from $Y \cup \{z\}$ at step $i$, i.e. $z \in C_h[i][Y \cup \{z\}]$.

This essentially means that $(x, z, Y)$ can not be a new substitutability violation at the end of step $i$, a contradiction.

An immediate corollary is that the unilateral substitutability property is preserved at each step in the algorithm above. Otherwise, there would be a new violation in $P_i$ that was not present in the original preference $P_h$. This would contradict the lemma.

**Corollary 1.** The preference $P_h^i$ created in the algorithm above at each step $i$ has the unilateral substitutability property if $P_h$ is unilaterally substitutable.
The next lemma shows that the algorithm eliminates the substitutability violations one by one.

**Lemma 6.** Suppose preferences satisfy the unilateral substitutability condition. For each step $i$, the substitutability violation $(x_i, z_i, Y_i)$ under $P_{h}^{i-1}$ is corrected and is not a violation under $P_{h}^{i}$.

**Proof** We prove this in two steps. In the first part, we establish that for the substitutability violation $(x_i, z_i, Y_i)$, $\exists z_i \in C_{h}^{i-1}(Y_i \cup \{z_i\})$ such that $d(z_i) = d(z_i)$. In the second part, we prove that $z_i$ is added to the chosen set from $Y_i \cup \{z_i\}$ in step $i$ and thus the violation is fixed.\(^{22}\)

**Part I** By Lemma 5, we know that the violation considered in step $i$ also existed in the original preferences $P_h$. Further note that $z_i \in C_h(Y_i \cup \{z_i, x_i\})$ and hence there does not exist any other $d(z_i)$ contract in $C_h(Y_i \cup \{z_i, x_i\})$ as the original preferences were defined only over feasible subsets of contracts. Using irrelevance of rejected contracts, we will have $C_h(Y_i \cup \{z_i, x_i\}) = C_h(Y_i \setminus [Y_i]_{d(z)} \cup \{z_i, x_i\})$ where $[Y_i]_{d(z)}$ is the set of contracts in $Y_i$ involving doctor $d(z)$.

If $\not\exists z_i \in C_h(Y_i \cup \{z_i\})$ such that $d(z_i) = d(z_i)$ then $C_h(Y_i \cup \{z_i\}) = C_h(Y_i \setminus [Y_i]_{d(z)} \cup \{z_i\})$.

Define $\tilde{Y} = Y_i \setminus [Y_i]_{d(z)}$ and we have

\[
\begin{align*}
    z_i & \notin C_h(\tilde{Y} \cup \{z_i\}) \\
    z_i & \in C_h(\tilde{Y} \cup \{z_i, x_i\})
\end{align*}
\]

$(x_i, z_i, \tilde{Y})$ is a substitutability violation of $P_h$ and $d(z_i) \notin d(\tilde{Y})$ which implies the preferences do not satisfy unilateral substitutability condition. Hence, our initial assumption must be incorrect and in fact, $d(z_i) \in d(C_h(Y_i \cup \{z_i\}))$. Moreover, by the weak order over the chosen sets there exists a $\hat{z}_i \in C_{h}^{i-1}(Y_i \cup \{z_i\})$ such that $d(\hat{z}_i) = d(z_i)$.

**Part II** Consider the maximal subset\(^{23}\) $\hat{Y}$ of $Y_i \cup \{z_i\}$ that includes only the unrejected contracts with $d(z_i)$. We prove that $z_i \in \hat{Y}$ by contradiction. Suppose not and we have $z_i \notin \hat{Y}$. Consider $\hat{Y} \cup \{z_i\}$ which contains all the non-$d(z_i)$ contracts of $Y_i \cup \{z_i\}$ but is not the required maximal

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\(^{22}\)The claims in these two parts are very similar to those of steps (II) and (III) of Lemma 5 but the proof is very different because for the ‘new’ violation we knew that $z \in \hat{Y}_1$. Instead, here we have $z_i \in C_h(Y_i \cup \{x_i, z_i\})$ but the unilateral substitutability condition helps us prove these claims.

\(^{23}\)See Definition 24
subset. Hence it fails the second condition of the maximal subset.

$$\exists \tilde{Y} \subseteq \hat{Y} \cup \{z\} \text{ such that }$$

$$d(z_i) \in \tilde{Y} \text{ but } d(z_i) \not\in d[C_{h}^{-1}(\tilde{Y})]$$

We can conclude that $$z_i \in \hat{Y}$$. If not, $$\tilde{Y} \subseteq \hat{Y}$$ and $$\hat{Y}$$ will not be a (maximal) subset with only unrejected contracts with $$d(z_i)$$. By the irrelevance of rejected contracts condition, there also exists $$\tilde{Y}_1 \cup \{z_i\} \subseteq \hat{Y}$$ with $$[\tilde{Y}_1]_{d(z_i)} = \emptyset$$, i.e. the only $$d(z_i)$$ contract in $$\tilde{Y}_1 \cup \{z_i\}$$ is $$z_i$$, satisfying $$d(z_i) \in d(\tilde{Y}_1 \cup \{z_i\})$$ but $$d(z_i) \not\in d[C_{h}^{-1}(\tilde{Y}_1 \cup \{z_i\})]$$.

Thus we have $$\tilde{Y}_1 \cup \{z_i\} \subseteq \tilde{Y} \subseteq \hat{Y} \cup \{z_i\} \subseteq Y_i \cup \{z_i\} \subseteq Y_i \cup \{z_i, x_i\}$$. We also have the following.

$$z_i \in \tilde{Y}_1 \cup \{z_i\} \quad \text{and} \quad z_i \not\in C_{h}^{-1}(\tilde{Y}_1 \cup \{z_i\})$$

$$z_i \in Y_i \cup \{z_i, x_i\} \quad \text{and} \quad z_i \in C_{h}^{-1}(Y_i \cup \{z_i, x_i\})$$

By the weak order over the chosen sets, we also know that $$z_i \not\in C_{h}(\tilde{Y}_1 \cup \{z_i\})$$ and since the violation existed in the original preferences $$P_{h}$$, $$z_i \in C_{h}(Y_i \cup \{z_i, x_i\})$$. Moreover, we have that $$[\tilde{Y}_1]_{d(z_i)} = \emptyset$$ and hence $$\tilde{Y}_1 \cup \{z_i\} \subseteq Y_i \setminus [\tilde{Y}_1]_{d(z_i)} \cup \{z_i, x_i\}$$. Thus we have the following.

$$z_i \in \tilde{Y}_1 \cup \{z_i\} \quad \text{and} \quad z_i \not\in C_{h}(\tilde{Y}_1 \cup \{z_i\})$$

$$z_i \in Y_i \setminus [\tilde{Y}_1]_{d(z_i)} \cup \{z_i, x_i\} \quad \text{and} \quad z_i \in C_{h}(Y_i \setminus [\tilde{Y}_1]_{d(z_i)} \cup \{z_i, x_i\})$$

Define $$\tilde{Y} \equiv Y_i \setminus [\tilde{Y}_1]_{d(z_i)} \cup \{z_i, x_i\}$$. Label all the elements in $$\tilde{Y} \setminus [\tilde{Y}_1 \cup \{z_i\}]$$ as $$q_1, q_2, q_3, \ldots, q_J$$. It is clear that $$d(q_j) \neq d(z_i) \forall j \in \{1, 2, \ldots, J\}$$. Moreover, when we add one contract at a time to the set $$\tilde{Y}_1 \cup \{z_i\}$$ to arrive at newer chosen sets, $$z_i$$ will become acceptable (eventually).

There would be a first $$j$$ such that we have the following $$z_i \in C_{h}(\tilde{Y}_1 \cup \{q_1, q_2, \ldots, q_j\} \cup \{z_i\})$$. We found a substitutability violation $$(x, z, Y)$$ where $$x = q_j$$, $$z = z_i$$, and $$Y = \tilde{Y}_1 \cup \{q_1, q_2, \ldots, q_{j-1}\}$$ where $$d(z) \not\in d(Y)$$ and thus the original preference is not unilaterally substitutable. This is a contradiction and in fact we always have $$z_i \in \hat{Y}$$.

If $$z_i \in \hat{Y}$$ then $$z_i \in C_{h}(Y_i \cup \{z_i\})$$ and the violation does not exist in $$P_{h}^i$$.

\[\Box\]

**Lemma 7.** $$P_{h}^i$$ is a completion of $$P_{h}$$ \(\forall i\).
Proof For $i = 0$ this is trivially true as each preference is a trivial completion of itself. For $i \geq 1$, we need to prove that in the algorithm, $P^i_h$ is modified from $P^{i-1}_h$ by including infeasible sets and not changing the order of feasible sets. If we use the choice function definition of the completion instead, then we need to prove that the chosen contracts by the hospital are either the same or have more than one (distinct) contracts with the same doctor. For any $\tilde{Y}$ such that $d(z_i) \notin d[C^{i-1}_h(\tilde{Y})]$, $C^i_h(\tilde{Y}) = C^{i-1}_h(\tilde{Y})$. If $\tilde{Y}$ is such that $d(z_i) \in d[C^{i-1}_h(\tilde{Y})]$, then the corresponding maximal subset $\hat{Y}$ contains either one contract or more than one contracts involving $d(z_i)$ and the hospital $h$. If the maximal subset contains only one contract, that contract will be a part of the chosen set in step $i - 1$. Then essentially we have $C^i_h(\hat{Y}) = C^{i-1}_h(\hat{Y})$. Moreover, if the maximal subset contains two or more contracts involving $d(z_i)$ and the hospital $h$, all those contracts would be included in the new definition of $C^i_h(\hat{Y})$. This ensures that there are at least two distinct contracts with the same doctor $d(z_i)$ when $C^i_h(\tilde{Y}) \neq C^{i-1}_h(\tilde{Y})$.

Both these conditions imply that $C^i_h$ is a completion of $C^{i-1}_h$ and by induction on $i$, we can say that $C^i_h$ is a completion for all $i$. The existence of a unique $P^i_h$ as guaranteed by Claim 2 in the appendix for each $C^i_h$ completes the proof of the claim above.

Now we are ready to give the proof of the main result.

Proof of Theorem 11 Since there is a finite number of contracts, for a given preference ordering for a hospital there is a finite number of (possible) substitutability violations. Since at each step no new violations are created (by Lemma 5) and at least one violation is reduced (by Lemma 6), the number of violations at each step is strictly less than the number of violations in the previous step. Since we had a finite number to begin with and it strictly decreases progressively, the algorithm is bound to end in finitely many steps. After the algorithm terminates, we have $P^i_h$ defined as $\bar{P}_h$. This preference has no violations and hence satisfies the substitutability condition. It is a completion of the original preference $P_h$ by Lemma 7. Hence it is a substitutable completion of the preferences $P_h$ with the unilateral substitutability property. Note that this property was needed to guarantee that Lemma 6 goes through. This ensured that the number of violations strictly decrease at each step which was crucial for this algorithm to work.

In the algorithm presented above at each step $i$, all the substitutability violations involving
the recalled contract doctor, i.e. \( d(z) \), are fixed. This gives the following proposition about the number of steps in the algorithm above.

**Proposition 6.** If the hospital preferences are unilaterally substitutable then the completion algorithm identifies at most \( |D| \) substitutability violations and thus completes in at most \( |D| \) steps.

There are at most \( |D| \) doctors who would have a substitutability violation in a given hospital preference and hence the number of steps in the algorithm 3.2.1 is capped at the cardinality of the set of doctors.\(^{24}\) We present the proof of this in the appendix.

### 3.3 Properties of stable matchings

The set of stable matchings satisfy certain properties, like the existence of a doctor-optimal stable matching, a lattice structure, etc. under suitable restrictions on the preferences of the hospitals. We can summarize a few key properties in various settings.

<table>
<thead>
<tr>
<th>Setting</th>
<th>Preferences of all agents</th>
<th>Properties</th>
</tr>
</thead>
<tbody>
<tr>
<td>Many-to-many</td>
<td>Substitutable</td>
<td>( \Rightarrow ) Existence of doctor-optimal stable matching</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \Rightarrow ) Existence of doctor-pessimal stable matching</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \Rightarrow ) Existence of a Lattice</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( + ) Law of Aggregate Demand ( \Rightarrow ) Rural Hospital Theorem</td>
</tr>
<tr>
<td>Many-to-one</td>
<td>Substitutably completable</td>
<td>( \exists ) a completion in the many-to-many setting which is substitutable</td>
</tr>
<tr>
<td>Many-to-one</td>
<td>Unilaterally Substitutable</td>
<td>( \Rightarrow ) Existence of doctor-optimal stable matching</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \Rightarrow ) Existence of doctor-pessimal stable matching</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \Rightarrow ) Existence of a Lattice</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( + ) Law of Aggregate Demand ( \Rightarrow ) Rural Hospital Theorem</td>
</tr>
</tbody>
</table>

\(^{24}\)This is different from the computation time for running this algorithm. It has exponential time complexity in the number of contracts. Each step in the algorithm above searches through \( 2^{|X|} - 1 \) subsets of the set of contracts and alters the preferences to arrive at the preference \( P_h' \) at the end of the step.
We now know that a unilaterally substitutable preference has a related substitutable preference in the many-to-many setting. Moreover, all the stable matchings in the corresponding many-to-many setting with the substitutable completions of preferences are stable under the original many-to-one setting. However, with unilaterally substitutable preferences, the existence of a lattice structure or a doctor-pessimal matching is not guaranteed. This might appear puzzling given that these results hold in the many-to-many setting when the preferences for all agents had the substitutability property.

The resolution to this puzzle lies in the fact that under substitutably completed (many-to-many) preferences, only a subset of stable matchings under the original (many-to-one) preferences continue to remain stable.\(^{25}\)

Consider the following example\(^{26}\) where the unilateral substitutability condition is satisfied but not the substitutability condition.

\[
P_h : \{x, y''\} \succ \{x, y\} \succ \{x', y'\} \\
\succ \{x'', y''\} \succ \{x'', y'\} \succ \{x', y''\} \\
\succ \{x', y\} \succ \{x', y'\} \succ \{x, y\} \\
\succ \{x''\} \succ \{y''\} \succ \{x'\} \succ \{y'\} \\
\succ \{x\} \succ \{y\} \succ \emptyset
\]

Hatfield and Kojima (2010) show that there is a doctor-optimal stable matching \{x', y'\} and two other stable matchings, \{x'', y\} and \{x, y''\}, none of which is doctor-pessimal. However, under the completed preferences (using the above algorithm) we would have \{x', y'\} as the one and only stable matching. It is the doctor optimal stable matching and also the doctor pessimal stable matching among the set of stable matchings under the completed preference.

\(^{25}\)The notion of stability used here is the one for many-to-many preferences as defined in Hatfield and Kominers (2012a) Definition 2. An allocation \(A \subseteq X\) is stable (with respect to \(X\)) if it is (i) Individually Rational for all \(f \in D \cup H\) \(C_f(A) = A_f\) and (ii) Unblocked there does not exist a nonempty blocking set \(Z \subseteq X\) such that \(Z \cap A = \emptyset\) and for all \(f \in d(A) \cup h(A), A_f \subseteq C_f(A \cup Z)\). Also, it is under this notion of stability that the set of stable matchings in many-to-many preferences have a lattice structure.

\(^{26}\)From page 1717 of Hatfield and Kojima (2010)
3.4 Conclusion

In these closing remarks, we highlight two points that have not been discussed so far. First, on a technical note, there could be more than one completion of preferences which are substitutable, and the algorithm described above arrives at just one of the possible completions. Second, the substitutably completable preferences are puzzling and are different from other known necessary and various sufficient conditions for stability. This is because substitutable completability is defined implicitly and not described in terms of choices made by the hospital. Through this work, we provide a connection with the unilateral substitutability property, which is defined explicitly.

Hatfield and Kominers (2014) use techniques to arrive at a completion for specific preferences, namely, slot-based preferences and task-based preferences. We provide an algorithm for new sub-domains of the substitutably completable preferences. In doing so, we further clarify the connection between known concepts. A generalized algorithm for reaching a completion of preferences remains elusive as does a general characterization of the substitutably completable preferences. The hidden structure in various conditions which guarantee the existence of stability, which is yet to be uncovered, will close the lacuna in our understanding. We present a small step in this direction and hope that further research will provide us the necessary and sufficient condition for the existence of stable matchings.
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Appendix A

Appendix to Chapter 1

A.1 Discussion

A.1.1 Interactions in Matching Markets

The two sides of the matching market interact over an extended period of time. The process is broadly the same across different settings and is summarized in Table A.1.

A.1.2 Costly Interviewing affects the market outcomes

We investigate whether interviewing constraints at all affect market wide outcomes. To that end, we compare the NRMP survey data (National Resident Matching Program ®, Data Release and Research Committee, b) from the residency directors and applicants. Residency directors indicated that on an average, of the 856 applications received for 7 advertised positions, 119 candidates were sent an interview invitation and 96 were actually interviewed. Similar data from the survey of doctors (National Resident Matching Program ®, Data Release and Research Committee, a) indicates that of the 15 interview offers, the candidates (who were matched at the end of the process) attended only 11 of them. This suggests that the impact of interviewing capacities are non-trivial. The programs do not interview all applicants and candidates do not accept all the interview invitations. Moreover, each year through the secondary match process organized as the Supplemental Offer Acceptance Procedure (SOAP), most of the positions that remain unfilled in
<table>
<thead>
<tr>
<th>Positions advertised</th>
<th>Podiatry Match</th>
<th>College Admissions</th>
<th>IIT-B placements</th>
</tr>
</thead>
<tbody>
<tr>
<td>Program registration on CASPR website (April - July)</td>
<td>Implicit (and selective)</td>
<td>Companies send Job Application Forms (JAF)</td>
<td></td>
</tr>
<tr>
<td>Applications</td>
<td>Future residents apply (September, October)</td>
<td>Applications sent via Common App /Universal App</td>
<td>Student apply on the placement website</td>
</tr>
<tr>
<td>Interview Offers</td>
<td>Interview lists due from the programs (late November)</td>
<td>Not Applicable</td>
<td>Companies inform the placement cell about the interview lists</td>
</tr>
<tr>
<td>The interviewing process</td>
<td>Centralized Residency Interviewing Program (CRIP) (Early January)</td>
<td>Implicitly decided by the colleges</td>
<td>Companies visit campus on assigned days to conduct interviews</td>
</tr>
<tr>
<td>Job offers</td>
<td>Programs list their preferences in an order</td>
<td>Colleges send admissions/waitlist/reject decisions</td>
<td>Final list of candidates</td>
</tr>
<tr>
<td>Final matching</td>
<td>Candidates submit their rank order lists to the centralized system and a match is announced</td>
<td>Students accept one offer</td>
<td>Candidates decide to accept and compete only for better firms, if any</td>
</tr>
</tbody>
</table>

Table A.1: Various interactions comparable to the National Residency Match Program process in the National Podiatry match, the college admissions process and the engineering graduate placements process in Indian Institute of Technology-Bombay

the main process, match with a resident who is unmatched in the main process. In the NRMP main Match 2015, of the 30212 positions, 1306 remained unfilled. Out of the 34905 applicants, 8025 applicants were left completely unmatched. Through the Supplemental Offer Acceptance Procedure, 1129 positions were filled (National Resident Matching Program ®, Results and Data). This provides some directional evidence that interviewing constraints matter.

A.1.3 Timing of the market

In many markets, the timing of various transactions is coordinated and the participants comply with these required timelines. The residency and fellowship matchings that take place through the SF match is yet another example of coordinated timings by various specialties.¹ The repeated nature of these interactions and advantages a centralized application and rank ordering system provides to the programs is sufficient for their continued participation. As a result even the

participants follow the timeline and the guidelines.

In some markets, the timing is not particularly easy to coordinate and in fact repeated efforts to get these in place have failed. A notable example of this is the clerkship market where graduating law students in the U.S seek clerkship positions with various Judges in the different Circuits. We refer an interested reader to Avery et al. (2001, 2007) for a detailed account of this market. We do not expect to capture this market through our discussion in the example and the main model.

In some markets the timing of interviewing is crucial even when there is coordination. To consider an example, closer to home, the economics job market for graduating economists has centralized interviewing which represents only the first stage of the process. The later stages which include fly-outs and closer interviews that take place in a decentralized manner and only a very few candidates are invited for this process. Arguably the reason is the immense cost on getting the entire faculty to attend the job market candidates seminars and form preferences over them. This market clears top-down where the departments lower in the rank order, make their decisions after the top departments have already conducted their fly-outs and extended their interview offers. We abstract away from such considerations in our model.

A.1.4 Example in Section 1.3: Surplus calculation

Recall that the sum of students’ utilities is the same along with the sum of the idiosyncratic components related to the students’ preferences over the firms. In the low-capacity regime, all the students who match get a utility of \( e \) as they match with their first preference firm, if they do. Of the 0.36 mass of students who are interviewed by the two firms combined, 0.18 mass gets an offer and hence the aggregate idiosyncratic component is 0.36\( e \). In the high-capacity regime, the top 0.06 mass of students get interviewed by both firms and 50% of them get an offer from their first preference firm and a further 25% of them get an offer from their second choice firm, while the remaining 25% are found misfit for both firms. The sum of idiosyncratic components for these top students is 0.06\( [0.5e - 0.25e] = 0.015e \). However, the remaining students are randomly shared by the two firms and there is no sorting along the student preferences and hence on average the matched students get a 0 idiosyncratic utility. Thus, the total idiosyncratic utility for the students
is $0.015 < 0.36e$, obtained in the low-capacity regimes. Thus, even the sum of student welfare is lower under the high-capacity regime. However, note that the students at the top in the ability range $[0.94, 1]$ get a higher expected utility and are matched with greater probability while the students in the ability range $[0.64, 0.7]$ are worse off in terms of the expected utility. Moreover, the sum of gains is lower than the sum of losses in this case unlike the first example.

A.1.5 Analysis of an economy where firms’ evaluations are not identical

In this discussion, we will just focus on the result about reduction in overall welfare when the students’ interviewing costs are reduced when firms’ evaluations of student abilities are correlated but not identical. We do not discuss the equilibrium existence result as it follows very closely from the proof of Theorem 1.

**Proposition 7.** When the ability of students is evaluated differently by each firm, a reduction in student interviewing costs does not necessarily increase the overall quality of matching.

**Proof** We present an example to prove this.

Consider an economy with 2 firms and a continuum of students of mass 1. The two firms are labeled 1 and 2. Each student’s type $\theta = (e^\theta, f^\theta)$ is drawn from a cumulative distribution $F$ over $\Theta = [0, 1]^2 \times \{-1, 1\}^2$. The first component $e^\theta = (e_1^\theta, e_2^\theta)$ represents the ability vector. Each firm $i$ only sees the $i$–th component of the ability for the students who apply. More precisely, it can not see the value for $e_j^\theta$ for $j \neq i$. The fitness factor is similar to the fitness factor in the main model. We assume that the distribution of $e_1^\theta$ is uniform over $[0, 1]$ and we assume a specific form of correlation between $e_1^\theta$ and $e_2^\theta$ as shown in figure A.1. We assume that a $\rho = 2/3$ mass of students is uniformly distributed along the ‘correlated’ diagonal, i.e. $e_1^\theta = e_2^\theta$ and the remaining $1/3$ is distributed along the $e_1^\theta = 1 - e_2^\theta$ diagonal. Each student is fit for a firm with probability 0.5 independent of everything else. A fit firm-student pair generates a utility given by $2U(i, e_i^\theta) = \frac{1}{2} V(e_i^\theta)$ where $V(x) = x - \frac{x^3}{3}$. 

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In the low-capacity regime, a student can interview with only 1 firm and this changes to 2 in the high-capacity regime. As is always the case, the choice of regime does not impact the firms ranked \( \leq k_{LC} \). The first firm always interviews the best 0.24 mass according to its own criterion, i.e. all students with ability, \( e_1^d \in [0.76, 1] \) as shown in figure A.2. In the low-capacity regime, \( 2/3rd \) of the top 0.24 mass according to firm 2, is in firm 1’s interview offer region. These students lying on the correlated diagonal of the distribution will reject an interview offer from firm 2. Hence, firm 2 extends interview offers to all students with ability \( e_2^d \geq 0.6 \). Only a third of the top 0.24 and all the remaining 0.16 mass students will accept firm 2’s interview offers. Note that firm 2 is successful in hiring some of the top students with ability \( e_2^d \geq 0.76 \).

In the high-capacity regime all students can accept firm 2’s interview offers. However a fraction (specifically \( 2/3rd \) fraction) of the top 0.24 mass students are also interviewed by firm 1 and hence available only if they are found a misfit with it. Firm 2 evaluates the effective probability of being able to hire a student in the top 0.24 ability spectrum as follows.

**Figure A.1:** The student distribution along the two ability parameters, \( e_1^d \) and \( e_2^d \).
Fraction of students hired by firm 2 in the overlap region
\[\frac{2}{3} \times \text{Prob(a student is a misfit with firm 1)} \times \text{Prob(a student is a fit for firm 2)} + \frac{1}{3} \times \text{Prob(a student is a fit for firm 2)}\]
\[= \frac{2}{3} \times 0.5 \times 0.5 + \frac{1}{3} \times 0.5 = \frac{1}{3}\]

If firm 2 were to interview students such that \(q_2^d \in (0.24, 0.76)\) then it will be able to hire each of these students with probability 0.5 as none of them have an interview offer from firm 1. Firm 2 in fact chooses to interview students between \(q_2^d \in [0.52, 0.76)\) as \(0.5 U(2, 0.52) > \frac{1}{3} U(2, 1)\).\(^2\) Note that firm 2 chooses not to interview any student in the top ability region and hence does not hire any student with ability \(q_2^d \geq 0.76\). It is clear that when firm 2 hires a strictly dominated distribution on the ability parameter of the students and hence the utility for firm 2 is strictly lower.

In this case the student surplus is exactly equal to the firm surplus and hence the overall quality of matching goes down when the students’ interviewing costs are reduced. \(\square\)

A.1.6 Example in Section 6.3: Surplus calculation for firm 2

In the high interviewing cost regime, we know that the students who have an interview offer from firm 1 will not accept firm 2 interview offers. To check for the optimality of firms’ interviewing decisions, we take a two step procedure. First, we can verify if the firms’ interviewing decisions justify the cost they incur. Second, we verify if there is a profitable deviation by interviewing some other set of students. Both firms decide to interview 0.4 mass of students. The marginal cost of interviewing for firm 2 when it interview 0.4 mass of students is \(\frac{0.6}{2}(0.4)^2 = 0.048\). The expected surplus from interviewing the student at ability \(q^d = 0.2\) is \(0.5 \times \frac{0.2^2}{2} = 0.05\). The marginal decision at the smallest ability student by firm 2 is positive. At all other students the surplus received is

\[^2\text{Recall that } U(i, q^d) = \frac{1}{i} V(q^d) \text{ where } V(x) = x - \frac{x^3}{x} \text{ for firm } i \text{ and a student with ability } q_2^d\]
The interview offers by the firms in the low-capacity regime.

The interview offers by the firms in the high-capacity regime.

Figure A.2: Firm 1 interview offers are in red and those by firm 2 are in blue.

higher and the costs are lower for both firms. This proves that the all the interviewing decisions justify the cost incurred by the firms. Since both firms are able to recruit the required number of candidates and are doing the best they can given the interviewing constraints faced by the students.

We also want to evaluate the firms’ optimal decisions when the interviewing costs for students are reduced so that the cost is 0.02 to interview with 1 firm and 0.1 to interview with 2 firms. We explained above that any student with ability $e^\theta \geq 0.64$ will find it worthwhile to take up the second interview offer from firm 2, if one is extended.\(^3\)

Let us verify that firm 2’s decisions are optimal. If firm 2 can interview all the students with ability $[0.35, 0.6) \cup [0.7, 1]$,\(^4\) the firms’ hiring quota will be met. We will rather prove that firm 2 actually chooses to interview a smaller region than this because the marginal cost of interviewing the students when they are interviewing 0.55 mass of students is greater than the surplus share from interviewing students with ability 0.35 or 0.7. This will be sufficient to prove that the optimal

\(^3\)The expected surplus for a student with ability of 0.64 is $\text{Probability}($misfit for firm 1$) \times \text{Probability}($fit for firm 2$) \times \frac{0.64}{2} = 0.08$.

\(^4\)The number of matches when the interview offers are as specified above will be $0.25 \times 0.5$ from the students in the lower ability region and $0.3 \times 0.25$ from the students with interview offers from firm 1. This adds up to 0.2, the hiring quota for firm 2.
regions will be a strict subset of the region specified above and optimally so. The marginal cost of interviewing 0.55 mass of students for firm 2 = $\frac{0.6}{2} \times (0.55)^2 = 0.9075$. However, the expected share of surplus for firm 2 from interviewing the marginal student of ability 0.35 or 0.7 is given by $\text{Prob}(\text{fit}) \times V(2, 0.35) = 0.5 \times \frac{0.35}{2}$ and $\text{Prob}(\text{misfit for firm 1}) \times \text{Prob}(\text{fit for firm 2}) \times V(2, 0.7) = 0.5 \times 0.5 \times \frac{0.7}{2}$. These values are $0.0875 < \text{the marginal cost we inferred above}$. This proves that firm 2 will actually reduce its interview offers in both regions till the marginal cost of interviewing is just equal to the surplus share from the marginal student. This will necessarily reduce the number of matched agents with firm 2.

A.2 Proofs

A.2.1 Proofs of Theorem 1 and Theorem 2

We prove the more general case, Theorem 2 and observe that our economy in the main model, $E$ is the special case of the correlated economy $E^{corr}$ where the correlation is 0. More precisely, if we set the $F$-dimensional vector $\tilde{p} = [1, 1, 1, \cdots, 1]$, the correlated economy boils down to our economy in the main model. Thus, Theorem 1’s proof follows from the proof of Theorem 2.

Recall that in this setting, the students agree about their preferences over the firms. The marginal distribution over ability is uniform over $[0, 1]$. The fitness factors are correlated and it is summarized by $\tilde{p} = [p_0, p_{-1}, p_{-2}, \cdots, p_{-(F-1)}]$. We prove theorem 2 with a series of lemmas in the following steps.

Step i) For all preferences, there exists a unique stable matching and truth-telling is optimal for both firms and students.

Step ii) Each firm has a unique nondegenerate interview offer strategy solvable by iterated elimination of dominated strategies under assumption 1 for all firms.

Step iii) The firm and student preferences after nondegenerate interview offers result in a unique nondegenerate stable matching.

The next two lemmas will prove step i) above and also prove Lemmas 1, 2, and 3.
Lemma 8. The firms and students find truth-telling optimal to the mechanism which implements a student-proposing deferred acceptance algorithm.

**Proof** All students agree on the preference ranking of the firms. We know from Dubins and Freedman (1981) that students have a dominant strategy of truth-telling under student-proposing deferred acceptance algorithm.\(^5\)

Consider the strategic choice for a firm. Note that when students completely agree on their preferences over firms, the student-proposing deferred acceptance algorithm boils down to a serial dictatorship algorithm where the firms make their choice in the order of their ranking. We know that in a serial dictatorship the only time the preferences of a firm matter is when it has to choose a set of students and hence truth-telling is optimal.

Lemma 9. For any preference reporting by the firms where students report the preferences truthfully, there exists a unique stable matching.

**Proof** The existence of a unique nondegenerate stable matching follows very closely from the existence and uniqueness result due to Azevedo and Leshno (2016) for the true complete preferences. However we define stability in terms of the preferences that firms and students realize at the end of the interviewing process. Hence, we present a simple proof in our context.

From above we know that the outcome of the student-proposing deferred acceptance algorithm coincides with the serial dictatorship outcome where firms go in the order of their desirability from the students’ perspective. The outcome is necessarily a stable matching with respect to the reported preference which follows from Gale and Shapley (1962).

To prove the uniqueness, suppose there is another stable matching which differs from the match outcome above. Consider the best firm \(i\) which has a different assignment in this stable matching as compared to the SD outcome above. Clearly this firm can not be assigned to a student who was assigned to a better firm in the SD outcome.\(^6\) Either firm \(i\) has a new student who was assigned to a lower ranked firm \(i'\) than itself in the SD outcome or the firm has an empty spot.

\(^5\)This is true even in more general settings than the one we consider of complete agreement over firm ranking.

\(^6\)If this is the case then firm \(i\) will not be the best firm which has a different assignment.
Serial dictatorship gave firm $i$ a chance to act before firm $i'$. This is inconsistent with stability of the Serial dictatorship outcome, which we know from above is, in fact stable. This gives us the required contradiction and there is no firm which has a different match as compared to the SD outcome above. Due to the property of stable matchings, there is no student who has a different assignment. Thus, there does not exist another stable matching outcome proving the required uniqueness.

We now present the proof of step ii) and Lemma 4.

**Proof of Lemma 4** Consider the interviewing strategy by firm 1. Firm 1 with a distaste for interviewing will extend interview offers to all those students whom it can hire following a desirable outcome in the interviewing process. A student facing an interview offer from firm 1 knows that under assumption 1 she will get a final job offer with probability $p$. Any other firm’s interview offer will result in an offer with equal or lesser probability. The student’s best response to this is to always accept an interview offer from firm 1. Thus, the best firm extends interview offers just enough to fill its hiring capacity unless the interviewing capacity is met before that. The students never regret accepting an interview offer from firm 1 and possibly rejecting some other firm’s interview offer later in the process because before the interviewing process the unconditional fitness probability across firms is identical and independent of ability of the student, i.e. $p$.

Firm 2 knows the interview offers of firm 1 which are uniquely pinned down above. Firm 2 can choose to interview any set of students but just enough to ensure that it will be willing to extend an offer to each and every student found fit, knowing that some of them will be hired away by firm 1. Under the distaste for interviewing property, firm 2’s interview offers result in a final offer with probability $pp_{-1}$. Any other firm’s offers will convert in a final offer with the same probability if it is the next best firm and with lower probability if it is not. Hence, a student facing an interview offer and having excess interviewing capacity will always choose to accept the interview offer.

This argument can be made for any firm $i$ knowing the interview offers for all firms better than itself and knowing that the students indeed accept the best offers up to their interviewing
capacity. Thus, there is a unique nondegenerate interview offer strategy for the firms.

The exact interview offers from each firm can be iteratively found as done in the above proof to ensure that the interview offers meet the following three conditions.

1. Firm $i$’s interview offers are optimal given the interviews offers by firms 1 through $i - 1$.

2. None of the students will have to reject the interview from firm $i$ due to their capacity constraints.

3. The interview offer set is nondegenerate and respects the capacity firm $i$ has.

Note that there are possible redundancies in the interview offers by firms, where a firm can extend interview offers to students who do not have any interviewing capacity and those students in turn reject the offers made by such firms. We abstract away from such multiplicities. We know that a student will never regret having accepted a ‘better’ interview offer as per our discussion above.

The first step accounts for an existence of a stable matching. However, the equilibrium requires a nondegenerate stable matching outcome. From step ii) we know that there are unique nondegenerate interview offers from the firms. In the last step we indeed prove the connection that when the firm strategy includes only nondegenerate interview offers the resulting preferences lead to a nondegenerate stable matching. Suppose not and consider the best firm $i$ for whom the interviewing strategy is not nondegenerate. Of the two possibilities suppose that there is a student $\theta$ who is not assigned to firm $i$ although all her right neighbors are assigned to $i$ and the student still does not form a blocking pair with that firm. This can essentially happen if that student was not interviewed by the firm. However, we can conclude that the student had interviewing capacity as the students to her right accepted an interview offer from $i$.\footnote{If this were not true, there exists at least one firm better than $i$ which extended its interview offers to the current student but not to her right neighbors. The students on the right will not form a blocking pair with this better firm only if they were not interviewed. This contradicts the initial assumption that firm $i$ was the best such firm.} Thus, it must be the case that she did not receive an interview offer from firm $i$. This is ruled out by the first requirement of nondegenerate interview offers and hence our assumption is incorrect. Suppose that there is a student $\theta$ who is matched to a firm but none of her right neighbors are. The right neighbors will
not form a blocking pair with this firm only if they were not interviewed. Similar argument leads us to the conclusion that the preferences that result after nondegenerate interview offers form nondegenerate stable matchings.

Now we can combine all the results above to present the proof for Theorem 2. We have assumed the application decision to be trivial by making applications costless. Thus, the following strategies comprise an equilibrium.

1. All students apply to all firms.
2. Each firm follows the interview offer strategy as described in the proof of step ii).
3. All students accept all the interview offers they receive.
4. A firm finds all the ‘fit’ students acceptable and lists them in the true order and all students list the firms in the true order.

The above strategies constitute an equilibrium follows from step i) and ii) and the stable matching that results is a nondegenerate stable matching due to step iii). This is an ‘essentially’ unique equilibrium because the only other strategies that survive the preference reporting stages are the ones that do not materially impact the equilibrium outcome or the ones where the firms add or remove finite number of degenerate mass sets of students at the interview offer or preference reporting stage. This later possibility will still give us an ‘essentially’ unique equilibrium.

A.2.2 Proofs of Proposition 2 and 4

The proofs of Propositions 2 and 4 follow from the proofs of Propositions 8 and 9 which establishes the more general case with correlated fitness factors in Section A.2.3. The first 4 steps outlined in the proof prove Proposition 2 and 8. This proof does not rely on any of the extra assumptions, specifically maximum diversity of interview offers, which are used in the proof of Proposition 4 and 9.

A.2.3 Proof of Propositions 8 and 9

We first state all the propositions and then present their proofs.
**Proposition 8.** When the interviewing regime moves from LCr to HCr, the quality of the match weakly increases. If firms do not have any excess interview capacity and the quality strictly increases, the quantity strictly decreases at least for one firm.

**Proposition 9.** Suppose that all firms have maximum diversity in their interview offers in the LCr regime and the component of the student surplus $V(\cdot)$ is concave with a concave first derivative. When the interviewing regime moves from LCr to HCr, without excess interviewing capacity for the firms, the quantity of the overall match weakly decreases. If the quality strictly increases, the quantity of the matching strictly decreases.

We present the proofs of Propositions 8 and 9 together. A reader interested only in the proof of Propositions 2 and 4 can read the proof using the assumption that the correlation vector $\mathbf{\hat{p}} = [p_0 \, p_{-1} \, p_{-2} \cdots p_{-(F-1)}] = [1 \, 1 \cdots 11]$ for the general case we do not make any such assumption and state the proof below for the correlated economy. The results about an increase in quality and the existence of a firm with lower quantity follows from the first four steps of the proof and we do not use any of the extra assumptions used in Proposition 4 until after that.

Consider the low-capacity (LCr) and high-capacity (HCr) to be with student interviewing capacities of $k_{LC}$ and $k_{HC}$ respectively. We know that $k_{LC} < k_{HC}$ and the students can interview with more firms in the HCr. We keep the firm interviewing capacity fixed for simplicity and to make the comparisons across regimes stark.

We prove the following Lemmas.

**Lemma 10.** If the interview offers are different between the HC and LC regimes, the best firm to extend different interview offers is ranked worse than $k_{LC}$. For this firm, the utility of the match outcome strictly increases and the number of positions filled strictly decreases in the HC regime.

We call this firm, if it exists as firm $i$.

**Proof of Lemma 10** We prove this in two steps.

Step 1) Compare the equilibrium under HC regime to that under LC regime and identify the best firm which has a different interview offer.
Step 2) For the best firm identified above, say $i$, identify the regions of students as it sees in different regimes and its choices.

**Step 1** From Theorem 2, we know that for every interviewing cost regime, there is an (essentially) unique equilibrium and hence, a unique match outcome. The equilibrium outcomes can be identified simply by the optimal interviewing strategy for each firm which respects the capacity for the firms and the students. Consider the best firm $i$ which has different interview offers across the two regimes. Note that $i \geq k_{LC} + 1$. It is clear that for all firms weakly better than the $k_{LC}$th firm the student interviewing capacity definitely does not matter in both the regimes. Thus, their choices are identical in both the regimes.

**Step 2** Identify the regions of students as seen by firm $i$, $I_i(0), I_i(1), I_i(2), \ldots, I_i(k_{LC} - 1), I_i(\infty)$ under the $LC$ regime. Each $I_i(j)$ for finite $j$ stands for the set of students with $j$ interview offers from firms better than firm $i$ and $I_i(\infty)$ is the set of students who have no excess interviewing capacity as they already have $k_{LC}$ interview offers. Note the following properties about these sets and their proofs below.

**Sets are ordered.** Any $\theta_i \in I_i(i)$ and $\theta_j \in I_i(j)$ such that $i < j$, we have $e^{\theta_i} < e^{\theta_j}$. **Sets $I_i(j)$ for all finite $j$ have open right boundary and closed left boundary.** Set $I_i(\infty)$ is a closed set. **Sets for all finite $i$ are nondegenerate.** Except possibly $I_i(\infty)$, all other sets have non-zero mass.

A reader not interested in the specifics of the proof of the above properties can skip this paragraph. Suppose the first property does not hold. Consider a specific $i$ and $j$ such that $i < j$ but there exists a $\theta_i \in I_i(i)$ and $\theta_j \in I_i(j)$ such that $e^{\theta_i} > e^{\theta_j}$. Student $\theta_i$ has fewer interview offers than student $\theta_j$. Consider the lowest ranked firm $i' \succ i$ which does not interview student $\theta_i$ but interviews student $\theta_j$ and such a firm exists. We can replace the equilibrium strategy of $i'$ to interview student $\theta_i$ and a small neighborhood to the right instead of $\theta_j$ and a small neighborhood on the right. The said firm will be strictly better off. We found a nondegenerate deviation for the said firm which gives us the required contradiction. Hence, the above sets are ordered as described above. The second property about the structure of the sets follows from the fact that the interview offers have a similar structure. The structure of the interviews is guaranteed due to  

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8for a reader interested in the proofs of Proposition 2 and 4 alone, this can read from Theorem 1 in stead.
the requirement on interview offers that they must be nondegenerate. The last property follows immediately from the previous observation.

Compare similar regions for the HC regime and let us call them \( \bar{I}_i(0), \bar{I}_i(1), \bar{I}_i(2), \ldots, \bar{I}_i(k_{HC} - 1), I_i(\infty) \). Since the interviewing strategies of all the better firms are unchanged, we essentially have

\[ \bar{I}_i(i) = I_i(i) \quad \forall i \leq k_{LC} - 1 \]

Moreover the region \( \bar{I}_i(\infty) \subset I_i(\infty) \) and the new additions are sets \( \bar{I}_i(k_{LC}), \bar{I}_i(k_{LC} + 1), \ldots, \bar{I}_i(k_{HC} - 1) \). Since firm \( i \) chooses to extend interview offers to a new set of students it does strictly better than in the LC regime.\(^9\) We have assumed that the firms have no excess interviewing capacity so the firm chooses to replace the students in the regions \( I_i(i) \) with lower \( i \) by the ones in the high ability region, i.e. \( \bar{I}_i(k_{LC}+) \), who can now take up the interview offers from this firm. Note that since firm \( i \) moves some of its interview offers from students with lesser interviews from better firms to students with more interview offers from better firms, the quantity of the match for firm \( i \) will be necessarily lower. \( \square \)

We now define a concept about a better set of students and use it in the next Lemma.

**Definition 25.** Compared to the LC regime, a firm sees a strictly better set of students in the HC regime, if the following holds.

- For all students \( \theta \) with \( \theta \in I_i(j) \) and \( \theta \in \bar{I}_i(\bar{j}) \), we have \( \bar{j} \leq j \)

- There exists a student \( \theta \) with \( \theta \in I_i(j) \) and \( \theta \in \bar{I}_i(\bar{j}) \), such that \( \bar{j} < j \)

**Lemma 11.** If the interview offers are different between the LC and HC regime, all firms worse than firm \( i \) see strictly better set of students and get weakly better utility from matching in the HC regime.

The above two Lemmas 10 and 11 taken together prove Propositions 8 and thus also Proposition 2.

**Proof of Lemma 11** We prove this in two steps.

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\(^9\)As the interview offers have open right boundaries, the firm will move non-zero mass of interview offers to the new region and hence will do strictly better.
Step 1) Identify the impact on the subsequent firm(s) due to a different optimal strategy by firm \( i \) as the set of students it sees is strictly better.

Step 2) Identify the quality improvement for all firms.

**Step 1** Consider the impact of firm \( i \)'s strategy in the HC regime on the next firm's evaluations. Note that as compared to the LC regime in the HC regime, some students in the region \( I_i(0) \) do not receive an interview offer from firm \( i \). These students were a part of \( I_{i+1}(1) \) in the LC regime but are a part of \( I_{i+1}(0) \) in the HC regime. This argument holds for a nondegenerate set of students in each of the regions (by continuity). Thus, the lower ranked firm sees a strictly better set of students.

**Step 2** Whenever a firm sees a strictly better set of students, the firm will do at least as well as it was doing earlier even if it continues with the same strategy as it followed in the LC regime. This can be further strictly improved if the firm re-optimizes its interview offers. This argument can be continued for all subsequent firms and thus essentially all firms lower than \( i \) do strictly better under the HC regime. □

From the above Lemmas, we know that if one firm has a different interviewing strategy then it will have a strictly better outcome. All the other firms will also have a weakly better match outcome. Thus, if the match outcomes are different under the two regimes then the quality of the match strictly improves. However, it is possible that no firm follows a different interviewing strategy and hence we get the weaker result that the match quality weakly improves when the regime changes from LC to HC. This proves Propositions 8 and Proposition 2. Note that this result only relied on the assumption that \( U(i, e^\theta) \) is an increasing function of ability. This agrees with our intuition that a system moves towards a more efficient outcome if the frictions are reduced. □

**Lemma 12.** If the interview offers are different between the LC and HC regimes and all firms have maximum diversity in their interview offers in the LC regime, then the utility of matching strictly increases for firm \( i \) and all firms worse than \( i \). This is accompanied by a strict decrease in the number of positions filled for each of these firms.
Figure A.3: The optimal choices by firms $i$ and $i + 1$ under different regimes.

All the above Lemmas 10, 11, and 12 taken together essentially prove Propositions 9 and thus also Proposition 4.

Proof of Lemma 12 Consider the actual choices made by firm $i$ from the different regions under different regimes. Let us call $x_i(0), x_i(1), x_i(2), \cdots, x_i(k)$ such that $x_i(j)$ is the ability of the worst student it interviews in region $I_j$. Such a value exists because each set is open on the right but closed on the left. Similarly let us define $x_{i+1}(0), x_{i+1}(1), x_{i+1}(2), \cdots, x_{i+1}(k)$.

Given the actual choices by the firm and the fact that each firm’s interview offers had maximum diversity, we know the following.\(^{10}\)

$$U(i, x_i(0)) = \prod_{j=0}^{i} (1 - pp_j) \times U(i, x_i(i)) \quad \forall i \in 1, 2, 3, \cdots, k$$

$$U(i + 1, x_{i+1}(0)) = \prod_{j=0}^{i} (1 - pp_j) \times U(i + 1, x_{i+1}(i)) \quad \forall i \in 1, 2, 3, \cdots, k$$

Since the firms decide their interview offers as best response to better firm’s decisions, we can further say that $x_{i+1}(i) \leq x_i(i) \quad \forall i \in 1, 2, 3, \cdots, k$ as can be seen in Figure A.3. We start our investigation of the impact of the change for firm $i$ on the subsequent firms. Consider the relationship between $x_i(0) - x_{i+1}(0)$ and $x_i(1) - x_{i+1}(1)$. Note that these points are optimal choices by firm $f$ and $i + 1$. Due to the assumption on the functional form of $U(i, e^d) = h(i) V(e^d)$, the following holds too.

$$U(i + 1, x_{i+1}(0)) = (1 - pp_{-1}) \times U(i + 1, x_{i+1}(1))$$

$$V(x_{i+1}(0)) = (1 - pp_{-1}) \times V(x_{i+1}(1))$$

\(^{10}\)We need the maximum diversity assumption to ensure that these condition holds as equalities. If not, we would have $U(i, x_i(0)) \leq (1 - p)^i \times U(i, x_i(i)) \quad \forall i \in 1, 2, 3, \cdots, k.$
Consider an $\epsilon_0$ move in $x_{i+1}(0)$ and an $\epsilon_1$ move in $x_{i+1}(1)$ such that

$$V(x_{i+1}(0) + \epsilon_0) = (1 - pp_{-1}) \times V(x_{i+1}(1) + \epsilon_1)$$

For $\epsilon_0$ sufficiently close to 0, $\epsilon_1$ will also be close to 0 and we will get the following.

$$\epsilon_0 = (1 - pp_{-1}) \frac{V'(x_{i+1}(1))}{V'(x_{i+1}(0))} \epsilon_1$$

We know that $x_{i+1}(1) > x_{i+1}(0)$ and $V(\cdot)$ is a concave function, so $V'(x_{i+1}(1)) < V'(x_{i+1}(0))$. This proves that at every point $x_{i+1}(0), x_{i+1}(1)$, the optimal points corresponding to $x_{i+1}(0) + \epsilon_0, x_{i+1}(1) + \epsilon_1$ will be such that $\epsilon_0 < \epsilon_1$. This argument will lead us to the following relationship.

$$x_i(0) - x_{i+1}(0) < x_i(1) - x_{i+1}(1)$$

To evaluate the impact of the move in optimal choices by firm $i$ on the immediately next firm, consider the $\epsilon_i^0$ change at $x_i(0)$ and $\epsilon_{i+1}^0$ change at $x_{i+1}(0)$ corresponding to the ‘same’ $\epsilon_1$ move at $x_i(1)$ and $x_{i+1}(1)$. This comparison will help us evaluate the optimal decision of firm $i + 1$ following the changes for firm $i$ in the following way. Suppose firm $i$ moves the lowest ability interview offer to a $\Delta_1$ higher point. There is an extra set of students in the region $\bar{I}_{i+1}(0)$ due to this move. If this firm decides to naively just shift the interview regions to account for the newer students available, then the present comparison can shed some light on whether firm $i + 1$ should move even further in no interview region or should it replace its interview offers from that region with the students in the region with one interview (from better firms).

From the above analysis, we know that

$$\epsilon_i^0 = (1 - pp_{-1}) \frac{V'(x_i(1))}{V'(x_i(0))} \epsilon_1$$

$$\epsilon_{i+1}^0 = (1 - pp_{-1}) \frac{V'(x_{i+1}(1))}{V'(x_{i+1}(0))} \epsilon_1$$

$$\frac{\epsilon_i^0}{\epsilon_{i+1}^0} = \frac{V'(x_i(1))}{V'(x_{i+1}(0))}$$
Let us call $V'(\cdot)$ as $g(\cdot)$. We know that $g(x) > 0, g'(x) \leq 0, g''(x) \leq 0$. We also know $0 < x_i(0) - x_{i+1}(0) < x_i(1) - x_{i+1}(1)$

By Mean Value Theorem, we know that

$$g(x_{i+1}(0)) - g(x_i(0)) = |g'(x_0)| \left( x_i(0) - x_{i+1}(0) \right)$$
$$g(x_{i+1}(1)) - g(x_i(1)) = |g'(x_1)| \left( x_i(1) - x_{i+1}(1) \right)$$

where $x_0 \in (x_{i+1}(0), x_i(0))$ and $x_1 \in (x_{i+1}(1), x_i(1))$. Moreover, $x_i(0) \leq x_{i+1}(1)$ implies that $x_0 < x_1$. We know that $g'(\cdot) \leq 0$ throughout and $g''(\cdot) \leq 0$. The following follows from these properties of $g(\cdot)$

$$g'(x_0) \geq g'(x_1)$$
$$-g'(x_0) \leq -g'(x_1)$$
$$|g'(x_0)| \leq |g'(x_1)|$$
$$|g'(x_0)|(x_i(0) - x_{i+1}(0)) < |g'(x_1)|(x_i(1) - x_{i+1}(1))$$
$$g(x_{i+1}(0)) - g(x_i(0)) < g(x_{i+1}(1)) - g(x_i(1))$$

We further know that since $x_i(0) \leq x_i(1), g'(\cdot) \leq 0, g > 0$, we have $\frac{1}{g(x_i(0))} \leq \frac{1}{g(x_i(1))}$

The above two inequalities give us the following.

$$\frac{g(x_{i+1}(0)) - g(x_i(0))}{g(x_i(0))} < \frac{g(x_{i+1}(1)) - g(x_i(1))}{g(x_i(1))}$$
$$\Rightarrow \frac{g(x_{i+1}(0))}{g(x_i(0))} < \frac{g(x_{i+1}(1))}{g(x_i(1))}$$
$$\Rightarrow \frac{g(x_{i+1}(1))}{g(x_i(0))} < 1 \Rightarrow \frac{V'(x_i(1))}{V'(x_{i+1}(0))} < 1$$

$$\frac{\epsilon_i^0}{\epsilon_{i+1}^0} < 1$$

We have thus proved that an $\epsilon_1$ move (to the right) in both $x_i(1)$ and $x_{i+1}(1)$ corresponds to a larger change in $x_{i+1}(0)$ than in $x_i(0)$. By aggregating the small moves all throughout, we can say
that if \( x_i(1) - x_{i+1}(1) = \bar{x}_i(1) - \bar{x}_{i+1}(1) = \Delta_1 \) then

\[
\bar{x}_i(0) - x_i(0) < \bar{x}_{i+1}(0) - x_{i+1}(0)
\]

\[
\bar{x}_i(0) - \bar{x}_{i+1}(0) < x_i(0) - x_{i+1}(0)
\]

We now explain the exact implication of our finding. If firm \( i+1 \) were to naively keep the ‘same’ mass in different regions of students, i.e. the mass of students with interviews in region \( \bar{I}_{i+1}(0) \) (under the HC regime) is the same as the mass in region \( I_{i+1}(0) \) (under the LC regime), then it will interview students \( \theta \) with ability such that \( \bar{x}_i(0) - [x_i(0) - x_{i+1}(0)] \leq e^\theta < \bar{x}_i(0) \).

From above we know that

\[
\bar{x}_i(0) - [x_i(0) - x_{i+1}(0)] < \bar{x}_{i+1}(0)
\]

\[
V(\bar{x}_i(0) - [x_i(0) - x_{i+1}(0)]) < V(\bar{x}_{i+1}(0)) = (1 - p)V(\bar{x}_{i+1}(1)).
\]

This shows that firm \( i+1 \) will choose to shift some interview offers from the \( \bar{I}_{i+1}(0) \) region to \( I_{i+1}(1) \) region. This will not only further increase the quality but it will decrease the quantity of the matching. This argument was not crucially dependent on region with 0 and 1 interviews, it is equally applicable to the regions with 1 and 2 interviews. So the firm will choose to shift more of its mass to the region with 2 interviews. This continues all the way and we get the result that the firm has weakly better quality and weakly lower quantity of matching. \( \square \)

**A.2.4 Proof of Proposition 3**

We know from Lemma 10 that if the interview offers differ, the best firm, say firm \( i \), to extend a different set of interviews will get a higher utility from matching (and will have a lower number of positions filled up). We also know that all firms worse than firm \( i \) see a strictly better set of students and get a weakly higher utility from Lemma 11. This essentially proves the firm welfare result in the above proposition. The existence of firm \( i \) proves that there is a non-empty set of firms which is strictly better off.

Consider firm \( i \) and the region \( I_i(\infty) \) as explained in Step 1 of the proof of Lemma 10. Recall that \( I_i(\infty) \) is the set of students who have no excess interviewing capacity as they already have
$k_{LC}$ interview offers. Consider the student with lowest ability who is a part of $I_i(\infty)$ and call her student $\hat{\theta}$. All students with ability above $e^{\hat{\theta}}$ have $k_{LC}$ interview offers from firms better than $i$. By assumption, these firms do not change their interview offers. Moreover firm $i$ (and possibly many subsequent firms) may extend interview offers to some of these students. Thus, these students do weakly better in terms of the expected utility from the match outcome and the probability of finding a match. The existence of a different nondegenerate interviewing strategy by firm $i$ ensures that there is a non-zero mass of these students. Thus, all these students with interview offers from firms $i$ and worse do strictly better as they get more interview offers than in the $HC$ regime while keeping all the offers they received in the $LC$ regime. Their welfare increases in terms of the expected utility from the match as well as the probability of being matched.

Now consider the firm, call it firm $j$, which has different interview offers between the two regimes, extends interview offers to students in $I_j(0)$, and is the worst firm to be so. All firms better than firm $j$ who do not extend an interview offer to the students in $I_j(0)$ region students in the $LC$ regime, face a strictly better set of students and hence will continue to not extend any interview offers to all the students in $I_j(0)$.

All firms worse than firm $j$ who have a different interview offer between the two regimes, do not extend interview offers to the students in their $I_j(0)$ region by definition of firm $j$. Consider a student $\tilde{\theta}_1$ which belongs to $I_j(0)$, gets an interview offer from one of the worse firms with different interview offers, and is such a student with the lowest ability. It is clear that $e^{\tilde{\theta}_1} \neq x_j(0)$ otherwise this firm would be better off by extending some interview offers to the students in $I_j(0)$ region by continuity. However, we have assumed that firm $j$ is the lowest ranked firm to do so.

Consider the interviewing strategy for firm $j$ in the $HC$ regime which is different than that in the $LC$ regime. The effective value of expending an interview slot is the lowest for students with

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$^{11}$Recall that the set $I_j(0)$ is the set of students who do not have an interview offer from any of the firms better than $j$. Moreover such a region always exists for all firms as the students are on the long side of the market.

$^{12}$If no such firm exists, then we iteratively look for a firm which has different interview offers between the two regimes, extends interview offers to students in $I_j(m)$, and is the worst firm to be so, for different values of $m \in \{1,2,3, \cdots \}$ in the increasing order of numbers. Since firm $i$ exists with different interview offers between the two regimes, we will certainly be able to find the firm $j$ corresponding to the smallest $m$ possible. We will provide the proof for $m = 0$ as it is easier to follow the intuition. The arguments stay valid should the smallest $m$ in the above steps be larger than 0 merely by replacing the $(0)$ with $(m)$.
lowest ability in $I_j(0)$ region, i.e. for the student with ability $x_j(0)$. Recall that a firm chooses to allocate its interview offers across different regions so that the effective values for the interview slots are equal to the student with ability $x_j(0)$ or larger. When faced with strictly better students firm $j$ will first choose to eliminate the students with lowest effective value so as to extend interview offers to students with better effective value of interviewing. Hence there will be a non-zero mass of students with ability greater than $x_j(0)$ who will no get an interview offer in the $HC$ regime although they did receive an offer in the $LC$ regime.

Consider a student $\tilde{q}_2$ such that this student belongs to $I_j(0)$, gets an interview offer in both the regimes from firm $j$ and there exists a non-empty set of students with ability $\in [x_j(0), e^{\hat{\theta}_2})$, who do not have an interview offer in the $HC$ regime although they were interviewed by firm $j$ in the $LC$ regime.

Label the student with the lower ability among $\tilde{q}_1$ and $\tilde{q}_2$ as $\tilde{q}$. Consider the set of students in $I_j(0)$ with ability strictly lower than $e^{\hat{\theta}}$, call this set as $X$. The students in this set were necessarily not extended any interview offers from those worse firms who have different interview offers even in the $LC$ regime. They will continue to not receive an offer even in the $HC$ regime. All other firms worse than firm $j$ continue to extend the same interview offers across the two regimes and hence are not considered in the welfare comparisons. We have already proved that all firms better than $j$ will not extend any interview offers to students in $I_j(0)$ and $X \subseteq I_j(0)$. All students with ability in $[x_j(0), e^{\hat{\theta}})$ are extended an interview offer from firm $j$ in the $LC$ regime but not in the $HC$ regime.

Thus, we have proved that these students who belong to set $X$, get a weakly lower expected utility and also a weakly lower probability of finding a match. There also exists a non-zero mass of students who are strictly worse off.

Hence we have $e^{\theta_1} = e^{\hat{\theta}}$ and $e^{\theta_2} = e^{\hat{\theta}}$ such that the following claims in proposition 3 hold.

1. All students with ability at or above $e^{\theta_1}$ are weakly better off and there exists a non-zero mass of students who are strictly better off, in terms of both–the expected utility from a match and the ex-ante probability of finding a match.

2. All students with ability strictly below $e^{\theta_2}$ are weakly worse off and there exists a non-zero
mass of students who are strictly worse off, in terms of both—the expected utility from a
match and the ex-ante probability of finding a match.

The same arguments work exactly for the correlated setting as well and we just state the
relevant proposition and skip the proof.

**Proposition 10.** When the interviewing regime shifts from \( LC \) to \( HC \) and the interviewing offers are
different, there exist two threshold abilities \( e^{\theta_1} \) and \( e^{\theta_2} \) such that the following holds.

1. All students with ability at or above \( e^{\theta_1} \) are weakly better off and there exists a non-zero mass of
students who are strictly better off, in terms of the expected utility from a match as well as the ex-ante
probability of finding a match.

2. All students with ability strictly below \( e^{\theta_2} \) are weakly worse off and there exists a non-zero mass of
students who are strictly worse off, in terms of both the metrics—the expected utility from a match and
the ex-ante probability of finding a match.

Moreover, all firms are weakly better off and there exists a non-empty set of firms which are strictly better
off.

**A.2.5 Proof of Theorem 3**

The proof of Theorem 3 follows very closely to the proof of Theorems 1 and 2. However, the
economy has block correlated preferences for the students and we present the relevant arguments
here. Recall that the only difference in the current case from the main model is that the students
preferences do not agree entirely but only over ‘blocks’ of firms.

We prove this in the following three steps as in Section A.2.1.

Step i) For all preferences, there exists a unique stable matching and truth-telling is optimal for
both firms and students.

Step ii) Each block has a unique nondegenerate interview offer strategy solvable by iterated
elimination of dominated strategies under assumption 1 for all firms. However the offers
are not unique for each firm within the block.
Step iii) The firm and student preferences after nondegenerate interview offers result in a unique nondegenerate stable matching.

In the current block correlated economy, we need to evaluate if the firms have incentives to tell the truth to a student-proposing deferred acceptance algorithm. Note that the preferences of the firms over students whom they find acceptable are exactly the same. There are differences about which students a firm might find acceptable (based on the fitness factors). We know that a firm with responsive preferences might possibly alter the outcomes of the matching process if it initiates a rejection chain with a student who would have been otherwise acceptable. If such a rejection chain comes back to the firm with a better student, then we say that the firm has an incentive to initiate the rejection chain. Suppose such a firm exists, say firm $i$, and it rejects an acceptable student to initiate such a rejection chain. The student may apply to the next firm on her preference list, if such a firm exists. Any firm will tentatively accept such a student and will only reject a student with lower ability, if at all. Thus the rejection chain strictly goes in the direction of lower ability students. Even if the rejection chain comes back to firm $i$ it would be with students who have worse ability. Thus, truth-telling is in fact a dominant strategy for the firms.

The uniqueness for the proof of step i) comes from the fact that any other stable matching that exists can be found using a rejection chain initiated by some firm. Immorlica and Mahdian (2005). From the above discussion, we know that no firm would be willing to initiate a rejection chain. Thus, there is no other stable matching.

The interviewing strategy can be solved sequentially for each block and is similar to our discussion for the main model where the strategies were found for each firm one by one. Consider the first block of firms. Each firm $i$ wants to hire $q_i$ mass of students and has an interviewing capacity of $k_i q_i$. If $t$ of the $B_1$ firms\(^{13}\) extend interview offers to some student $\theta$, we can find the effective value of interviewing her as follows.

- A fraction $\frac{1}{t}$ will find firm $i$ as the best amongst the ones who have extended interview offers. Such students will be found fit with probability $p$ and be effectively available for $i$ to

\(^{13}\)We denote the cardinality of set $B_1$ with a slight abuse of notation as $B_1$. 

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hire with probability $p$.

- A fraction $\frac{1}{t}$ will find firm $i$ to be the second best amongst the ones who have extended interview offers. Such students will be available only if found misfits by their respective best choice firms, i.e. $1 - p$ and hence are effectively available with probability $(1 - p)p$.

- $\ldots$

- A fraction $\frac{1}{t}$ of the students will find firm $i$ to be the $\hat{t}$th best firm and hence will be effectively available for firm $i$ with probability $(1 - p)^{\hat{t} - 1}p$.

- $\ldots$

Thus the effective value for firm $i$ of interviewing student $\theta$ who accepts $t$ interview offers from $B_1$ will be given by $\frac{1}{t}(p + (1 - p)p + (1 - p)^2p + \cdots + (1 - p)^{\hat{t} - 1}p)V(\theta)$$^\text{e}$

Suppose $x_{B_1}, x_{B_1 - 1}, \ldots, x_2$, and $x_1$ are the abilities of the lowest ability students who get an interview offer from $B_1, B_1 - 1, \ldots, 2$, and 1 firm(s) respectively from block 1. Optimality requires that the following conditions hold.

$$\frac{1}{B_1}(p + (1 - p)p + (1 - p)^2p + \cdots + (1 - p)^{B_1 - 1}p)V(x_{B_1})$$

$$= \frac{1}{B_1 - 1}(p + (1 - p)p + (1 - p)^2p + \cdots + (1 - p)^{B_1 - 2}p)V(x_{B_1 - 1}) =$$

$$\ldots$$

$$= \frac{1}{2}(p + (1 - p)p)V(x_2)$$

$$= pV(x_1)$$

It is important to note that the choice of the above ability points is uniquely identifiable subject to the following constraints.

1. All firms in the block either meet their interviewing capacity or hiring quota.

2. None of the firms interview more students than required to fill their capacity.

In the discussion above, we have implicitly assumed that the student interviewing capacity $k_S$ is has played no role, i.e. $k_S \geq B_1$. If this is not true, the blocks with $> k_S$ interviews for
the students can not exist. All firms will extend interview offers to this set of students and the students will pick the best $k_5$ firms as per their preference order.\footnote{Note that it will not be an equilibrium where the firms randomly extend interview offers so that only $k_5$ offers are extended to the students. To find a profitable deviation, we only need to consider the firm who is getting the lowest ability students and knows that if it deviates to this very top regions, it will be better off as only those students will accept its interview offers who find it amongst the $k_5$ best firms of the interview offers they have.}

We can continue with the strategic choices of the next block given the choices by the first block. The ability thresholds for $B_2, B_2 - 1, \ldots, 1$ offers from this block can be found using similar method. We also need to ensure that the students can accept those many interview offers, a condition which can possibly be relevant in some cases. The interview offers for each block can thus be iteratively found.

We did not specify the interview offers for each firm individually within any block. The exact strategies can be any of the (infinite number of) possibilities. As long as the specified number of firms, say $t$ are extending the interview offers to those students who belong to a specific region, i.e. the students with ability between $x_t$ and $x_{t+1}$, it will be an equilibrium. If less number of firms extend an interview offer, one of the firms will have a profitable deviation to instead extend interviews in that region. If more than $t$ firms extend interview offers to the said region, one of the firms will prefer to expend its interview slots elsewhere.

\subsection*{A.2.6 Introduction of a signaling stage}

We continue with the block correlated economy $E^{BC}$ described above in Section 1.6.2 and add a signaling stage to it. We allow for the students to send at most one signal. The timing of the game has more stages to account for the presence of signaling. It is as follows.

1. Student preferences over firms are realized and the ability parameter is revealed to the students.

2. Students choose whether to send a signal and if so, to which firm along with the applications.

3. Firms see the signals and applications from all the students who sent them.
4. Each firm decides to send interview offers to some students based on the applications and signals it received and the belief it forms based on the signals.

5. Students accept some interview offers and the accepted interviews take place.

6. Students and firms report their preferences to a central clearing authority and matching takes place as per the student-proposing deferred acceptance algorithm.

We now define the equilibrium of the modified game as follows. We denote \( \mathcal{N} \) as the option for the students to not send a signal and with a slight abuse of notation also refer \( \mathcal{N} \) as the possibility where a firm does not get any signals from any of the students.

**Definition 26.** An equilibrium of the signaling, interviewing and matching game is

1. a strategy of applications and signaling for each student, \( \sigma_S : \Theta_e \times \Theta^e \rightarrow 2^F \times (F \cup \mathcal{N}) \),

2. a belief for each firm about the preferences of the students who sent it a signal, \( \mu_i(\cdot | S_i) \) for all \( i \in F \) where \( S_i \subseteq S \cup \mathcal{N} \) is the set of students who sent it a signal.

3. a strategy for each firm to extend interview offers, \( \sigma_i : 2^{\Theta_e} \times S_i \rightarrow 2^{\Theta_e} \times S_i \) for all \( i \in F \),

4. a strategy of interview acceptances for each student \( \sigma_{q_\theta} : 2^F \rightarrow 2^F \) for all \( \theta \in \Theta_e \), and

5. a set of preferences \( P_{\theta} \) for all \( \theta \in \Theta \) and \( P_i \) for all \( i \in F \)

such that each firm and student find its/her strategies optimal given those of the other firms and students and a nondegenerate stable matching results.

Note that the equilibrium we focus on is in pure strategies for the students at the application and signaling stages. This implies that all students with a given ability and preference follow the same strategies. Moreover, we will restrict attention to symmetric strategies for students, i.e. \( \sigma_S(\theta, \eta(\cdot)) = \eta(\sigma_S(\theta, \cdot)) \) where \( \eta(\cdot) \) is a permutation of the rank ordering over firms (which is consistent with block-symmetric preferences).
We now also add the following technical assumption on the idiosyncratic component of the students’ utilities.\textsuperscript{15}

\begin{assumption}
\frac{(1-p)^{B_i-1}}{B_i} [U(i,e^i) - U(j,e^j)] \geq [c(1,i) - c(B_i,i)] + [c(1,j) - c(B_j,j)] \ \forall e^i
\end{assumption}

This assumption ensures that the idiosyncratic utility for the students from any firm is a very small portion of the total utility. We will explain the exact role of this assumption when we use it.\textsuperscript{16}

Bets-in-block (pure) strategies refer to pure strategies where students send a signal to the best firm in a block if the student decides to send a signal to a particular block. If the firms have best-in-block beliefs, i.e. a firm concludes that it is the best firm in its block for the student who sends a signal, the optimal response by the students is to send a signal to its best-in-block firm. Getting an interview offer from the best ranked firm is the best thing a signal can do in this case. The other possibilities that a firm think it is the second-best firm if a student send a signal can be ruled out using Cho and Kreps (1987) Intuitive Criterion. We only focus on best-in-block strategies in conjunction with best-in-block firms.

\textit{Remark 5.} For all non-babbling symmetric equilibria of the signaling, interviewing and matching game, we have the following. Each student with ability $e^\theta$ sends a signal to the best firm in the block which meets the following two conditions.

1. The firms in the block respond to the signals from students with ability $e^\theta$, and
2. It is the best block where all firms do not extend interview offers to the specific student

We focus on best-in-block pure strategies for the students and best-in-block beliefs for the firms.

Consider a student $\theta$. This student has an option to send a signal to one of the blocks as we focus on pure strategies for the students. It will be wasteful to send a signal to the (best firm in

\textsuperscript{15}Recall that $U(b,x)$ is the utility that any firm in block $b$ gets when matched with a fit student of ability $x$. It is also the expected utility that the student gets from being matched with a random firm in block $b$. The idiosyncratic component of the student’s utility is given by $c(t,b)$–corresponding to the firm which has a rank $t$ within block $b$–which adds to the utility $U(b,x)$. See definition 13 for the exact meaning of rank within a block.

\textsuperscript{16}Consider the example of 4 firms and 2 blocks. If the idiosyncratic utility is a much smaller portion as compared to the common value for each firm. In this case the assumption is satisfied.
the block which comprises of firms who all send an interview offer to this student. Even without signaling the student can be sure of getting an interview offer from the best firm in such a block (at equilibrium). Consider the best block $i$ such that all firms in the block do not send an interview offer to the student (at equilibrium). Also consider another such block $j$ where all firms in that block do not send interview offers to the student with certainty. If such a $j$ does not exist, by weak optimality of sending a signal only to such blocks, the student will send a signal to the best firm in block $i$ and the proposition will be hold trivially.

We only need to consider the following three cases where we compare the option of sending a signal to block $i$ best firm (option 1) with the option of sending a signal to block $j$ best firm (option 2).\(^{17}\)

I) It is an equilibrium where the student sends a signal to block $i$ and not to block $j$.

II) There does not exist an equilibrium where the student sends a signal to block $j$.

III) It can not be the case that the student does not send a signal to block $i$ or block $j$.

**Case I** If all students of ability $e^{q}$ send a signal to their respective best firms in block $i$, then each firm expects to get a signal from $\frac{1}{B_{i}}$ of these students. Each firm has the following coordinated strategy\(^{18}\) for the students of this ability.

(i) Send an interview offer to all students who have sent a signal.

(ii) Send an interview offer to $\frac{t_{i}-1}{B_{i}}$ of those students from whom it has not received a signal.

Due to the presence of continuum of students, this results in the following equilibrium offers for a student who sends a signal to this block $i$.

1. An interview offer from the firm it sent a signal.

\(^{17}\)The remaining case where the student sends a signal to both block $i$ and block $j$ is impossible as we focus on symmetric pure strategies and hence not considered.

\(^{18}\)We call this strategy coordinated as the firms ‘virtually’ coordinate on the equilibrium to ensure that each student has $t_{i}$ and exactly $t_{i}$ offers from the firms in this block. The equilibration process is of great interest but out of scope for the current discussion. If anything, the presence of signaling will ease the equilibration process.
2. Interview offers from $t_i - 1$ firms whom it did not send a signal.

The student with ability $e^0$ can choose to send a signal to the best firm in block $i$, i.e. take the equilibrium path of action. It can otherwise decide to send a signal to the best firm in block $j$. Let us evaluate the two options in terms of the expected utility the student can get. Suppose that there are $t_x$ interview offers from firms which lie between blocks $i$ and $j$.

Option 1 results in an interview offer from the best firm in block $i$, say $i_1$, with probability 1, $t_i$ offers from other firms in $B_i \setminus \{i_1\}$ and interview offers from any $t_j$ firms in block $j$. The relevant parts of the expected utility ($Util(\text{opt1})$)\textsuperscript{19} from the above offers can be expressed as follows.

$$[p + (1 - p)p + \cdots + (1 - p)^{t_i-1}p]U(i,e^0)$$

$$+(1 - p)^{t_i}t_x(p + (1 - p)p + \cdots + (1 - p)^{t_j-1}p)U(j,e^0)$$

$$+ [p\epsilon(1,i) + p(1 - p)e(\cdot,i) \cdots]$$

$$+(1 - p)^{t_i + t_x}(p\epsilon(\cdot,j) + p(1 - p)e(\cdot,j) + \cdots)$$

Option 2 results in $t_i - 1$ offers from any of the firms in $B_i$, $t_j$ offers from any of the firms in block $j$, and also an offer from the best firm in block $j$ (if it has not already sent an interview offer). The relevant parts of the expected utility ($Util(\text{opt2})$)\textsuperscript{20} from the above offers can be expressed as follows.

$$[p + (1 - p)p + \cdots + (1 - p)^{t_i-2}p]U(i,e^0)$$

$$+(1 - p)^{t_i-1}t_x(p + (1 - p)p + \cdots + (1 - p)^{t_j-1}p)U(j,e^0)$$

$$+ \frac{B_j - t_j}{B_j} (1 - p)^{t_i-1+t_x+t_j}pU(j,e^0)$$

$$+ [p\epsilon(\cdot,i) + p(1 - p)e(\cdot,i) \cdots]$$

$$+(1 - p)^{t_i+t_x}(p\epsilon(1,j) + p(1 - p)e(\cdot,j) + \cdots)$$

$$+ \frac{B_j - t_j}{B_j} (1 - p)^{t_i-1+t_x+t_j}p\epsilon(\cdot,j)$$

\textsuperscript{19}The explanation for these expressions is provided below.

\textsuperscript{20}The explanation for these expressions is provided below.
The expected utility expressions corresponding to each of the options above have four components. The first component has a series of probabilities multiplying $U(i, e^\theta)$. For option 1, the best in block firm $i_1$ extends an interview offer to this student and finds her fit with probability $p$. If she is found misfit for the first firm, she has a chance $p$ of being found fit for the next firm which extends an interview offer to her. A total of $t_i$ such firms extend an interview offer to this student. We know that the best firm $i_1$ extends an interview offer to the student. However, the identity of the remaining $t_i - 1$ firms is not known for sure. This also explains the third component of the expected utility expression. It has $pe(1, i)$ coming from firm $i_1$ and the remaining $e(\cdot, i)$ corresponding to the other firms. Similarly the second and fourth components correspond to the block-specific utility for interview offers from block $j$ and the firm-specific components where the identity of none of the $t_j$ firms is known for sure. It is important to note that the expected utilities corresponding to the $j$th block kick in only after all the better firms, i.e. $t_i + t_x$ in number, have found the student misfit. This explains the leading $(1 - p)^{t_i + t_x}$ multiplying the block-specific and firm-specific expected utility components from block $j$.

The expected utility expression for the second option also has four components. The first and the third components have only $t_i - 1$ terms as the student gets only $t_i - 1$ offers from the firms who have not seen a signal from this student. The second component has the regular $t_j$ terms corresponding to the $t_j$ interview offers from the firms in block $j$. However, there is a probability $\frac{B_j - t_j}{B_j}$ that the best firm in this block, say $i_1$, was not going to make an offer and hence the other $t_j$ firms were the ones making an interview offer. This leads to the case where the student gets $t_j + 1$ interview offers from this block.

We are now ready to put a bound on the difference between the expected utility of sending a
signal to block $i$ and that to block $j$. 

\[
Util(opt1) - Util(opt2) > (1 - p)^{t_i-1}pU(i, e^\theta) + (1 - p)^{t_i+t_x+t_j-1}pU(j, e^\theta)
\]
\[
+ p\epsilon(1, i) + (1 - p)\epsilon(B_i, i) + (1 - p)^{t_i+t_x}\epsilon(B_j, j)
\]
\[
- \left[ (1 - p)^{t_i-1}pU(j, e^\theta) + \frac{B_j - t_j}{B_j} (1 - p)^{t_i+t_x+t_j-1}U(j, e^\theta) \right]
\]
\[
+ \epsilon(1, i) + (1 - p)^{t_i+t_x}\epsilon(1, j)
\]
\[
\geq (1 - p)^{t_i-1}p[U(i, e^\theta) - U(j, e^\theta)]
\]
\[
- \left[ (1 - p)(\epsilon(1, i) - \epsilon(B_i, i)) + (1 - p)^{t_i+t_x}(\epsilon(1, j) - \epsilon(B_j, j)) \right]
\]

We know the following from our technical assumption about the utilities from different blocks.

\[
\frac{1}{B_i}(1 - p)^{t_i}U(i, e^\theta) - U(j, e^\theta) \geq [\epsilon(1, i) - \epsilon(B_i, i)] + [\epsilon(1, j) - \epsilon(B_j, j)]
\]

\[
(1 - p)^{t_i-1}(U(i, e^\theta) - U(j, e^\theta)) \geq (1 - p)[\epsilon(1, i) - \epsilon(B_i, i)] + (1 - p)^{t_i+t_x}[\epsilon(1, j) - \epsilon(B_j, j)]
\]

\[
Util((opt1) - Util(opt2) > 0
\]

Thus, there is no profitable deviation for the student from the equilibrium course of action.

**Case II** If all students of ability $e^\theta$ send a signal to their respective best firms in block $j$, then each firm in block $j$ expects to get a signal from $\frac{1}{B_i}$ of these students. Each firm in block $j$ has the following coordinated strategy\(^{21}\) for the students of this ability.

1. Send an interview offer to all students who have sent a signal.

2. Send an interview offer to $\frac{t_{j-1}}{B_i}$ of those students from whom it has not received a signal.

Block $i$ firms send an interview offer to $\frac{t_{i}}{B_i}$ of the students in a coordinated manner. We now evaluate the two options that the student faces – sending a signal to the best firm from either block $i$ or block $j$.

Option 1 results in interview offers from $t_i$ random firms in block $i$ and a possibly additional ($t_i + 1$th) interview offer (from this block) from the best firm, say $i_1$, (if it has not already sent an interview offer).

\[^{21}\text{We call this strategy coordinated as the firms ‘virtually’ coordinate on the equilibrium to ensure that each student has } t_i \text{ and exactly } t_i \text{ offers from the firms in this block.} \]
offer) and \( t_j - 1 \) interview offers from a random set of firms in block \( j \). The relevant parts of the expected utility \((\text{Util}(\text{opt1}))\) from the above offers can be expressed as follows.

\[
\begin{align*}
&[p + (1 - p)p + \cdots + (1 - p)^{t_i - 1}p]U(i, e^\theta) + \frac{B_i - t_i}{B_i} (1 - p)^{t_i}pU(i, e^\theta) \\
+ &\left[\frac{t_i}{B_i} (1 - p)^{t_i + t_s} + \frac{B_i - t_i}{B_i} (1 - p)^{t_i + t_s + 1}\right](p + (1 - p)p + \cdots + (1 - p)^{t_i - 2}p)U(j, e^\theta) \\
+ &\left[p(1 - i) + p(1 - p)e(\cdot, i) \cdots (1 - p)^{t_i - 1}p(1 - i)e(\cdot, i)\right] + \frac{B_i - t_i}{B_i} p(1 - p)^{t_i}e(\cdot, i) \\
+ &\left[\frac{t_i}{B_i} (1 - p)^{t_i + t_s} + \frac{B_i - t_i}{B_i} (1 - p)^{t_i + t_s + 1}\right](p(\cdot, j) + (1 - p)p(\cdot, i) + \cdots + (1 - p)^{t_i - 2}pe(\cdot, i))
\end{align*}
\]

Option 2 results in \( t_i \) interview offers from any of the firms in block \( i \), an interview offer from the best firm in block \( j \), say \( j_1 \), and \( t_j - 1 \) interview offers from any of the firms from the set \( B_j \setminus \{j_1\} \). The relevant parts of the expected utility \((\text{Util}(\text{opt2}))\) from the above offers can be expressed as follows.

\[
\begin{align*}
&[p + (1 - p)p + \cdots + (1 - p)^{t_i - 1}p]U(i, e^\theta) \\
+ &\left(p(1 - i) + p(1 - p)e(\cdot, i) \cdots (1 - p)^{t_i - 1}p(1 - i)e(\cdot, i)\right) \\
+ &\left(1 - p\right)^{t_i + t_s}(p(1, j) + (1 - p)p(\cdot, i) + \cdots + (1 - p)^{t_i - 1}pe(\cdot, i))
\end{align*}
\]

The expected utility expressions corresponding to each of the options above can be broken down in four components like we did in case I. The first component has a series of probabilities multiplying \( U(i, e^\theta) \) corresponding to the \( t_i \) offers which the student gets. There is a possibility of a \( t_i + 1 \)th interview offer from this block if the \( t_i \) firms that were meant to extend the interview offers to her did not include the best in block firm, \( i_1 \). Note that this also explains the third component of the expected utility expression which accounts for the fact that \( i_1 \) will definitely interview the student and the identities of the other \( t_i - 1 \) or \( t_1 \) firms is not known for sure. The \( t_j - 1 \) interview offers from the \( j \)th block will result in the series of probabilities multiplying \( U(j, e^\theta) \). However, the expected utilities corresponding to the \( j \)th block matter only after the \( t_i \) or \( t_i + 1 \) firms from block \( i \) and the \( t_s \) firms from blocks between \( i \) and \( j \) find the student a misfit. Due to the uncertainty about the exact number of offers from block \( i \) due to the off-equilibrium
action from this student, we account for the different possibilities when evaluating the worth of interview offers from block \( j \).

The expected utility expression for the second option also has four components. The first and the third components correspond to the \( t_i \) random offers from block \( i \). The second component all the \( t_j \) terms corresponding to the \( t_j \) interview offers from the firms in block \( j \) including the best firm \( j_1 \). The fourth component accounts for the firm-specific utility from being matched to a firm within block \( j \).

We are now ready to put a bound on the difference between the expected utility of sending a signal to block \( i \) and that to block \( j \).

\[
\text{Util}(\text{opt}1) - \text{Util}(\text{opt}2) > \frac{B_i - t_i}{B_i} (1 - p)^{t_i} p U(i, e^\theta) + p \epsilon(1, i) + (1 - p) \epsilon(B_i, i) + (1 - p)^{t_i+t_x} \epsilon(B_j, j) - \frac{B_i - t_i}{B_i} (1 - p)^{t_i+t_x} p U(j, e^\theta) + p \epsilon(1, i) + (1 - p)^{t_i+t_x} \epsilon(1, j) \\
\geq \frac{B_i - t_i}{B_i} (1 - p)^{t_i} p [U(i, e^\theta) - U(j, e^\theta)] - [(1 - p) \epsilon(1, i) - \epsilon(B_i, i)) + (1 - p)^{t_i+t_x} (\epsilon(1, j) - \epsilon(B_j, j))] 
\]

We know the following from our technical assumption about the utilities from different blocks.

\[
\frac{1}{B_i} (1 - p)^{t_i} (U(i, e^\theta) - U(j, e^\theta)) \geq [\epsilon(1, i) - \epsilon(B_i, i)] + [\epsilon(1, j) - \epsilon(B_j, j)] \\
\frac{B_i - t_i}{B_i} (1 - p)^{t_i} (U(i, e^\theta) - U(j, e^\theta)) \geq (1 - p) [\epsilon(1, i) - \epsilon(B_i, i)] + (1 - p)^{t_i+t_x} [\epsilon(1, j) - \epsilon(B_j, j)] 
\]

\[
\text{Util}(\text{opt}1) - \text{Util}(\text{opt}2) > 0 
\]

Thus, there is a profitable deviation for the student to send a signal to the best block firm \( i_1 \) instead of the conjectured equilibrium strategy of sending it to block \( j \). This leads to a contradiction and hence it is not an equilibrium to send a signal to this block.

**Case III** We want to prove that it can not be an equilibrium to waste the signal for a student and not send it to any firm. We will prove that the student has a profitable deviation of sending it
to the best block firm. We will proceed with similar steps and compare the option of sending the signal to block \( i \) to that of not sending it at all.

Option 1 of sending a signal to the best firm in block \( i \) results in \( t_i \) interview offers from any of the firms in block \( i \) and a possibly additional \((t_i + 1)\) interview offer (from this block) from the best firm, say \( i_1 \), (if it has not already sent an offer).

Option 2 of not sending the signal at all results in \( t_i \) interview offers from any of the firms in block \( i \).

It is clear to see that there is only a possible extra interview offer from a \( t_i + 1 \)th firm and the expected utility of sending a signal to the \( i \)th block firm is greater than not sending it to any firm. This is true even if the interview capacity of the student binds because with probability \( \frac{B_i= t_i}{B_i} \) the student’s best choice firm does not send an interview offer to her without the signal.

We have thus proved that of the three possible scenarios, the only equilibrium is that of sending a signal to the best firm in block \( i \) which was the best block where all firms did not send an interview offer to the student. \( \square \)

Now with the student signaling strategies in our hand, we can focus on the equilibrium characterization and the impact of signaling on these markets. We will focus on non-babbling symmetric equilibria where at least some signals from some students are not ignored by the firms. This provides the result that signaling can achieve the sorting mechanism that was achieved under small interviewing capacity.

**Proposition 11.** There exists at least one non-babbling symmetric signaling equilibrium. There exists a symmetric non-babbling signaling equilibrium such that the sum of students’ utilities goes up, the sum of firms’ utilities and the number of matched agents stay the same.

**Proof** We know from Theorem 2’s proof above that for all preferences that result from any equilibrium interviewing results in a unique stable matching and truth-telling is optimal. We will take that result and use it in our setting here with signaling as after the interviews have taken place the two settings are not different.

We will find the firm interview offers by solving them for each block iteratively. We will also solve for the signaling strategies of the students as we know that they send them to the best block
firm of the best block for which all firms do not extend an offer to her and also respond to signals sent by students of her ability. We can come up with the equilibrium offers by iterated elimination of dominated strategies. From theorem 2’s proof we know that the first block firms extend at least \(1, 2, 3, \ldots, B_1\) interview offers to students whose ability is greater than \(x_1, x_2, \ldots, x_{B_1}\) respectively. Now consider the strategy of these students such that they send the best firm in this block a signal if they do not get an offer from all the firms.

A signal sent from a student has two impacts.

1. There is a direct effect of signaling that each firm recognizes that a student finds it best, expects a signal from \(\frac{1}{B_1}\) of the students and send an interview offer to these students

2. There is an indirect effect of signaling on those firms whom she did not send a signal as those firms recognize that such a student has sent a signal to her best firm and will be interviewed by that firm.

The indirect effect manifests itself in a possibly different strategy for the firm about the cut-offs on the ability dimension if the students send signals to all the top firms. Consider the region where the students received two interview offers from the first block. Earlier a firms that interviews such a student and will be able to hire her with probability \((0.5)(p + (1 - p)p)\). The firms chose to extend 2 interview offers to all students with ability between \(x_2\) and \(x_3\). However, now consider the same region of students where the students send a signal to their respective best firms. A firm that does not receive a signal recognizes that the probability of hiring this student is \((1 - p)p\) which is lower than the probability of hiring for such a student when there was no signaling. The firm would optimally decide to extend its interview offers only to the students with ability \(x_{2.1}(> x_2)\). Similarly there will be different ability thresholds for extending interview offers for students who have not sent a signal, i.e. \(x_{3.1}\).

The following is an equilibrium.

1. Students with ability in the range of \(x_k\) and \(x_{k-1}\) do not send signals to block 1 firms and all firms in this block ignore any signals from these students.
2. Other students who do not receive $B_1$ offers from block $B_1$ send a signal to their best firm in this block.

3. Students extend interview offers as per the above strategy of sending an offer to everyone who sends a signal and also those above the thresholds even when they do not send a signal.

The match increases the utility of the students as they get weakly better matches. The firms continue to get the same number of students in all the regions and are not affected. Hence the welfare for the students goes up and that of the firms remains unchanged. The equilibrium number of matches also stays the same.

The equilibrium strategies are uniquely identified up to multiplicities within a block. The solution can be obtained iteratively while being mindful of the signaling strategy by the students of best-in-block firm from the best block which does not extend an interview offer for sure and responds to the signals. When compared with the results from Coles et al. (2013), the result about firm welfare might appear at odds. They prove that in all non-babbling signaling equilibria in their setting, the student welfare improves but the firm welfare is ambiguously affected. The ambiguous effect of signaling on the firms’ welfare in Coles et al. (2013) was due to the competition effect when a particular firm pays attention to some student who was not very high up on its ranking but sent it a signal. However, the alignment of firm preferences on the ability parameter of the students rules out this effect and ensures that the overall welfare for the firms does not decrease. The number of matches stay the same as at this equilibrium the firms use the interview offers similar to those in the case of no-signaling but just align the offers more towards the students who signal.

The student welfare result follows in similar spirit because if anything the students are going to gain more from getting an offer from their best ranked firm within a block. However, the comparison is subtle because for some other symmetric non-babbling equilibria, the welfare for some students may go down with signaling. Although our current model does not have any student-specific idiosyncratic utility to the firms from employing students who rank it at the top of their list, it is an easy extension to include and we summarize this in the following Remark without providing its proof.
Remark 6. If the firm receive a part of the idiosyncratic component of the utility, there exists a non-babbling equilibrium where the sum of firms’ utilities weakly increases when they pay attention to the signals from the students.

A.2.7 Fitness factor correlated with ability and firm identity

Let us call the economy where the fitness factor for students depend on the ability and firm identity as $E_{fitab}$. Specifically the economy is characterized by a function $p : F \times \Theta \rightarrow [0, 1]$ where $p(i, e^\theta)$ is the probability of finding a student with ability $e^\theta$ fit for firm $i$. We assume that $p(i, e^\theta)$ is decreasing in $i$.

Theorem 12. In an economy $E_{fitab}$ where the agents face an interviewing constraint $[k_F, k_S]$, there exists an equilibrium and it is essentially unique.

The proof of this theorem will follow exactly the same way as proof of Theorem 2 presented in Section A.2.1. We draw attention to the differences and avoid presenting the entire proof.

Our results for decrease in quantity of matching and the quality and quantity tradeoff also follow almost identically with the same steps.

Step i) For all preferences, there exists a unique stable matching and truth-telling is optimal for both firms and students.

Step ii) Each firm has a unique nondegenerate interview offer strategy solvable by iterated elimination of dominated strategies under assumption 1 for all firms.

Step iii) The firm and student preferences after nondegenerate interview offers result in a unique nondegenerate stable matching.

In this setting the fitness factor is independent across firms but is possibly dependent on the identity of the firm and the student ability. Thus, the probability of finding a student $\theta$ fit for a particular firm $i$ is given by $p(i, e^\theta)$. The information about these functions is common knowledge just as the value of $p$ was known to all the market participants. Given the characteristics of $p(\cdot, \cdot)$ function, the student’s choice remains the same when faced with more interview offers than they
can accept, i.e. pick the best interview offers. This in turn simplifies the interview offer strategy decisions for the firms. The problem for each firm remains to find out the optimal region of student abilities which maximizes the expected value of spending those interview slots given the decision by better firms. In the discussion so far it always included the best students available who have a certain number of interviews. However, now the fitness factor is also dependent on the student ability and the student at the very top might be hired away with much higher probability and hence each firm needs to find the exact region where the value of its interview slots is maximized. This does not affect the process by which the firms decide their interview offers. The iterative procedure of deciding the interview offers continues to hold. The rest of the proof applies in this case too.

**Proposition 12.** When the interviewing regime moves from LC to HC, the quality of the match weakly increases. If firms do not have any excess interview capacity and the quality strictly increases, the quantity strictly decreases at least for one firm.

The move from LC to HC regimes also naturally extends itself to this more general setting. The quality improvement holds as the firms make strategic choices in the HC regime while declining the options available even under the LC regime. The decrease in quantity for the firm which extends different interview offers also follows in very similar spirit from our proof of Proposition 2
Appendix B

Appendix to Chapter 2

B.1 Proofs and Omitted Lemmas

Proof of Theorem 5. Let $\mu^*$ be the matching identified by the PDA procedure. According to the procedure, no agent will be assigned to a plan that is worse than being unmatched in both periods. Thus, $i
\not\succ_i \mu^*(i)$. Also, no pair $\{m, w\}$ can block $\mu^*$. If not, then there exists a plan $x \in \{wm, mw, ww\}$ for $m$ and a corresponding plan for $w$ that both prefer. If $x \succ_m \mu^*(m)$, then $m$ must have proposed that arrangement to $w$ at some round before he made his proposal defined in $\mu^*(m)$. $w$ must have rejected that original proposal; thus, she must prefer $\mu^*(w)$. However, this is a contradiction.

Proof of Theorem 6. Let $\mu^*$ be the matching identified by the PDAA procedure. Let $\tilde{\mu}^1$ be the matching identified by the PDAA’s first step. As a preliminary observation, note that $\mu^*(i) \succ_i \tilde{\mu}^1(i)$ for all $i$. Hence, $\mu^*$ cannot be period-1 blocked. If this was not the case, then the same blocking agent/pair can period-1 block $\tilde{\mu}^1$, which is not possible. (The PDA procedure identifies an ex ante stable matching.) Next we consider the Theorem’s two cases.

(a) **Suppose $\succ_i$ satisfies SIC and SA.** Suppose agent $i$ can period-2 block $\mu^*$, i.e. $(\mu^*_1(i), i) \succ_i \mu^*(i)$.

\[\text{If } \mu^*(i) = (\hat{\mu}^1_1(i), \hat{\mu}^2_1(i)) \neq \hat{\mu}^1(i), \text{ then it must be that } \hat{\mu}^1_1(i) = i. \text{ Since the Gale and Shapley (1962) deferred acceptance procedure generates an individually rational (one-period) matching, } \hat{\mu}^2_1(i) \succ_i \hat{\mu}^1_1(i) \implies \mu^*(i) = (\hat{\mu}^1_1(i), \hat{\mu}^2_1(i)) \succ_i (\hat{\mu}^1_1(i), i) = \hat{\mu}^1(i).\]
If $\mu_1^*(i) = i$, then $ii \succ_i \mu^*(i)$, which is a contradiction. Therefore, $\mu_1^*(i) = j \neq i$. There are two cases. If $\tilde{\mu}_1^*(i) = jj$, then by SIC $ji \succ_i \mu^*(i) \succeq_i \tilde{\mu}_1^*(i) = jj \implies ii \succ_i jj = \tilde{\mu}_1^*(i)$, which is a contradiction. If instead $\tilde{\mu}_1^*(i) = ji$, then $ji \succ_i \mu^*(i) \succeq_i \tilde{\mu}_1^*(i) = ji$, which is also contradiction. Therefore, agent $i$ cannot period-2 block $\mu^*$.

If $m$ and $w$ can period-2 block $\mu^*$, then $(\mu_1^*(m), w) \succ_m \mu^*(m) \succeq_m \tilde{\mu}_1^*(m) \succeq_m mm$ and $(\mu_1^*(w), m) \succ_w \mu^*(w) \succeq_w \tilde{\mu}_1^*(w) \succeq_w ww$. If $\mu_1^*(m) = w$ and $\mu_1^*(w) = m$, then $m$ and $w$ can period-1 block $\tilde{\mu}_1^*$, which is not possible. The same applies when $\mu_1^*(m) = m$ and $\mu_1^*(w) = w$.

Without loss of generality, there are two remaining cases: (i) $\mu_1^*(m) = m$ and $\mu_1^*(w) = m' \neq m$ and (ii) $\mu_1^*(m) = w' \neq w$ and $\mu_1^*(w) = m' \neq m$. We consider each separately.

(i) First, suppose $\mu_1^*(m) = m$ and $\mu_1^*(w) = m' \neq m$. In this case, $mw \succ_m \mu^*(m) \succeq_m mj = \tilde{\mu}_1^*(m) \succeq_m mm$ and $m'm \succ_w \mu^*(w) \succeq_w m'k = \tilde{\mu}_1^*(w) \succeq_w ww$ for some $j \in W_m$ and $k \in M_w$. If $j = w'$, then $mw' = \tilde{\mu}_1^*(m)$ and, correspondingly, $w'm = \tilde{\mu}_1^*(w')$. SA implies $w'w' \succ_m mw'$ and $mm \succ_w w'm$. Thus, $m$ and $w'$ can period-1 block $\tilde{\mu}_1^*$, which is a contradiction. If instead $j = m$, then SA implies $ww \succ_m mm = \tilde{\mu}_1^*(m)$. Similarly, if $k = w$ then SA implies $m'm' \succ_w m'w = \tilde{\mu}_1^*(w)$ and $ww \succ_m wm' = \tilde{\mu}_1^*(m')$. Thus, $w$ and $m'$ can period-1 block $\tilde{\mu}_1^*$, which is a contradiction. If instead $k = m'$, then $m'm \succ_w m'm' = \tilde{\mu}_1^*(w) \succeq_w ww$. SIC implies that $mm \succ m'm' = \tilde{\mu}_1^*(w)$. But now $m$ and $w$ can period-1 block $\tilde{\mu}_1^*$, which is a contradiction.

(ii) Suppose $\mu_1^*(m) = w' \neq w$ and $\mu_1^*(w) = m' \neq m$. In this case, $w'w \succ_m \mu^*(m) \succeq_m w'j = \tilde{\mu}_1^*(m) \succ_m mm$ and $m'm \succ_w \mu^*(w) \succeq_w m'k = \tilde{\mu}_1^*(w) \succeq_w ww$ for some $j \in W_m$ and $k \in M_w$. If $j = m$, then SA implies that $w'w' \succ_m w'm = \tilde{\mu}_1^*(m)$ and, correspondingly, $mm \succ_w mw' = \tilde{\mu}_1^*(w')$. Thus, $m$ and $w'$ can period-1 block $\tilde{\mu}_1^*$, which is a contradiction. If instead $j = w'$, then $w'w \succ_m w'w' = \tilde{\mu}_1^*(m) \succeq_m mm$. SIC implies that $ww \succ_m w'w' = \tilde{\mu}_1^*(m)$. A parallel argument implies that $mm \succ_w m'm' \tilde{\mu}_1^*(w)$. Thus $m$ and $w$ can period-1 block $\tilde{\mu}_1^*$, which is a contradiction.

(b) Suppose $\succ_i$ satisfies SIC and RDS. The same argument as in the preceding case applies, except for cases (i) and (ii), which we replace with the following:
(i) First, suppose $\mu^*_1(m) = m$ and $\mu^*_1(w) = m' \neq m$. In this case, $mw \succ_m \mu^*(m) \succ_m mj = \check{\mu}_1(m) \succ_m mm$ and $m'm \succ_w \mu^*(w) \succ_w m'k = \check{\mu}_1(w) \succ_w ww$ for some $j \in W_m$ and $k \in M_w$. If $j = m$, SIC implies that $ww \succ_m mm = \check{\mu}_1(m)$. If $j = w' \neq m$, RDS implies $ww \succ_m mw' = \check{\mu}_1(m)$. Similarly, if $k = w$, then by RDS $m'm \succ_w m'w \succ_w ww \implies mm \succ_w m'w = \check{\mu}_1(w)$. Finally, if $k = m'$, SIC implies $mm \succ_w m'm' = \check{\mu}_1(w)$. Whatever the case, $m$ and $w$ can period-1 block $\check{\mu}_1$—a contradiction.

(ii) Instead, and second, suppose $\mu^*_1(m) = w' \neq w$ and $\mu^*_1(w) = m' \neq m$. In this case, $w'w \succ_m \mu^*(m) \succ_m w'j = \check{\mu}_1(m) \succ_m mm$ and $m'm \succ_w \mu^*(w) \succ_w m'k = \check{\mu}_1(w) \succ_w ww$ for some $j \in W_m$ and $k \in M_w$. If $j = m$, RDS implies that $ww \succ_m w'm = \check{\mu}_1(m)$. If $j = w' \neq m$, SIC implies that $ww \succ_m w'w' = \check{\mu}_1(m)$. By the same reasoning, we conclude that $mm \succ_w m'k = \check{\mu}_1(w)$ for $k \in \{w, m'\}$. Thus, $m$ and $w$ can period-1 block $\check{\mu}_1$, which is a contradiction.

\[ \square \]

**Proof of Theorem 7.** Fix a profile of reported preferences for all agents $i \neq m_1$. Let $\mu^*$ be the PDAA matching when $m_1$ truthfully reveals his preference $\succ_{m_1}$. Let $\check{\mu}$ be the PDAA matching when $m_1$ states his preference as $\check{\succ}_{m_1} \neq \succ_{m_1}$. Assume that $\check{\mu}(m_1) \succ_{m_1} \mu^*(m_1)$.

Suppose that all other agents report preferences $\succ_i$, $i \neq m_1$, that satisfy SIC. First, the same argument verifying that the one-period deferred acceptance algorithm is strategyproof for $m_1$ leads us to conclude that $\check{\mu}(m_1) \notin \{w_1 w_1, w_1 m_1, m_1 w_1, m_1 m_1\}$ for some $w_1 \in W$. Thus, $\check{\mu}(m_1) = w_1 w_2$ for $w_1 \neq w_2$. To attain this volatile final matching in the PDAA, $m_1$ must have announced a preference where $w_1 w_2 \check{\succ}_{m_1} w_1 m_1 \check{\succ}_{m_1} m_1 m_1$. Since the announced preference must satisfy RDS, $w_2 w_2 \check{\succ}_{m_1} w_1 m_1$. Note that $w_1 m_1$ is $m_1$’s (interim) matching after step 1 of the PDAA procedure given his report of $\check{\succ}_{m_1}$.

Now consider $w_2$. Since she participates in step 2 of the PDAA, $\check{\mu}(w_2) = j m_1 \succ_{w_2} j w_2 \check{\succ}_{w_2} w_2 w_2$ where $j w_2$ is her (interim) matching after the procedure’s first step and $j \neq m_1$. Since $\succ_{w_2}$ satisfies SIC and RDS, $m_1 m_1 \succ_{w_2} j w_2$. Since $w_2 w_2 \check{\succ}_{m_1} w_1 m_1$ and $m_1 m_1 \succ_{w_2} j w_2$, the interim matching following step 1 of the PDAA procedure is not ex ante stable with respect to the stated preferences,
which is a contradiction.

Lemma 13. Suppose each agent’s preferences satisfy SA. Then the EDA and the PDAA assignments coincide.

Proof of Lemma 13. Let \( \tilde{\mu}^1 \) be the interim matching identified in the PDAA procedure’s first step. Let \( \mu^* \) be the final PDAA matching. Three facts are of note:

1. If \( i \neq j \), the \( \tilde{\mu}^1(i) \neq ji \). SA implies that \( jj \succ_i ji = \tilde{\mu}^1(i) \succ_i ii \) and, likewise \( ii \succ_j ij = \tilde{\mu}^1(j) \succ_j jj \). Hence, \( i \) and \( j \) can period-1 block \( \tilde{\mu}^1 \), which is a contradiction.

2. If \( i \neq j \), the \( \tilde{\mu}^1(i) \neq ij \). The argument parallels the preceding case.

3. If \( \tilde{\mu}^1(i) = ii \), then \( \mu^*(i) = ii \). Suppose that \( \mu^*(i) \neq \tilde{\mu}^1(i) \). Then, there exists a \( j \) such that \( \mu^*(i) = ij \succ_i \tilde{\mu}^1(i) \). Likewise, \( \mu^*(j) = ji \succ_j \tilde{\mu}^1(j) \). But this implies \( i \) and \( j \) can period-1 block \( \tilde{\mu}^1 \), which is a contradiction.

Facts (1)–(3) imply that \( \tilde{\mu}^1(i) = ii \) or \( \tilde{\mu}^1(i) = jj \) and no adjustment occurs in step 2 of the PDAA procedure. Thus, \( \tilde{\mu}^1 = \mu^* \). Clearly, the same matching obtains if in phase 1 men were restricted to proposing persistent plans. This corresponds to men proposing according to \( P_{>m} \) in the EDA mechanism.

Theorem 13. If each agents’ preferences exhibit inertia, then every persistent dynamically stable matching is Pareto optimal.

Proof of Theorem 13. Let \( \mu^* \) be a persistent dynamically stable matching. Suppose \( \mu^* \) is not Pareto optimal. If it is Pareto-dominated by a persistent matching \( \tilde{\mu} \), then there must exist \( m \in M \) and \( w \in W \) such that \( \tilde{\mu}(m) = w w \succ_m \mu^*(m) \) and \( \tilde{\mu}(w) = m m \succ_w \mu^*(w) \). However, this contradicts \( \mu^* \) being dynamically stable.

Thus, \( \mu^* \) must be Pareto-dominated by a matching \( \tilde{\mu} \) which is volatile for some man, \( m_1 \). This implies \( \tilde{\mu}(m_1) = m_1 w_1 \) or \( w_1 m_1 \) or \( w_1 w_2 \). If \( \tilde{\mu}(m_1) = m_1 w_1 \succ_{m_1} \mu^*(m_1) \succ_{m_1} m_1 m_1 \) or \( \tilde{\mu}(m_1) = w_1 w_2 \succ_{w_1 w_2} \mu^*(w_1 w_2) \succ_{w_1 w_2} w_1 w_1 w_2 \).

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$w_1 m_1 \succ m_1 \mu^*(m_1) \succ m_1 m_1$, inertia implies that $w_1 w_1 \succ m_1 \tilde{\mu}(m_1)$. If $\tilde{\mu}(m_1) = w_1 w_2 \succ m_1 \mu^*(m_1)$, inertia implies that $w_1 w_1 \succ m_1 w_1 w_2$ or $w_2 w_2 \succ m_1 w_1 w_2$. Thus, $\tilde{\mu}(m_1)$ is always dominated by a persistent plan involving one of the women matched to under $\tilde{\mu}(m_1)$. Without loss of generality, suppose $w_1 w_1 \succ m_1 \tilde{\mu}(m_1) \succ m_1 \mu^*(m_1)$. As $\succ w_1$ exhibits inertia,

$$m_2 m_2 \succ w_1 m_1 m_2 = \tilde{\mu}(w_1) \succ w_1 \mu^*(w_1) \succ w_1 m_1 m_1$$

for some $m_2 \in M$, $m_2 \neq m_1$. Otherwise, $m_1$ and $w_1$ would be able to block $\mu^*$. Since $\mu^*$ is dynamically stable and $\succ m_2$ exhibits inertia,

$$w_3 w_3 \succ m_2 w_3 w_1 = \tilde{\mu}(m_2) \succ m_2 \mu^*(m_2) \succ m_2 w_1 w_1$$

for some $w_3 \in W$, $w_3 \neq w_1$. Continuing in this fashion we can define a sequence of distinct men $m_3, m_4, \ldots$ and a sequence of distinct women $w_1, w_3, w_4, \ldots$. However, this is a contradiction as there is a finite number of agents in the economy.

**Theorem 14.** If each agent’s preferences are amendment averse, the PDAA assignment is dynamically stable.

**Proof of Theorem 14.** Let $\mu^*$ be the PDAA assignment. Let $\tilde{\mu}^1$ be the interim assignment after step 1 of the PDAA procedure. It is sufficient to show that $\mu^*$ cannot be period-2 blocked. There are three possibilities.

1. If $\tilde{\mu}^1(i) = ji$, then AA implies that $ii \succ ji$. But now agent $i$ can period 1 block $\tilde{\mu}^1$, which is a contradiction.

2. If $\tilde{\mu}^1(i) = jj$, then $\mu^*(i) = jj$. If agent $i$ can period-2 block, it follows that $jk \succ ji$, but such a preference would violate AA.

3. Suppose $\tilde{\mu}^1(i) = ij$. Now, if $i$ can period-2 block $\tilde{\mu}^1$ with $k$, then $ik \succ ij = \tilde{\mu}^1(i)$. Correspondingly, for agent $k$ it must be the case that $kj \succ k \tilde{\mu}^1(k)$. But this implies that $j$ and $k$ can period-1 block $\tilde{\mu}^1$ which is not possible.

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The preceding cases exhaust all possibility. Therefore, $\mu^*$ cannot be period 2 blocked.

**Theorem 15.** Suppose agents’ preferences satisfy AA. The PDAA is strategyproof for the proposing side.

**Proof of Theorem 15.** Recall that AA implies that if $jk \succ_i ii$, then either $j = k$ or $j = i$. Let $\mu^*$ be the PDAA matching when all agents truthfully reveal their preferences, $\succ_i$. If $\hat{\mu}^1$ is the PDA interim matching identified in step 1 of the PDAA, it is straightforward to verify that $\mu^*(i) = \hat{\mu}^1(i)$ for all $i$ when preferences satisfy AA. Let $\hat{\mu}$ be the PDAA matching when agent $m$ misreports his preference as $\succ_m$ and others continue to report truthfully. Let $\hat{\mu}^1$ be the corresponding interim matching identified in step 1 of the PDAA algorithm. Suppose $\hat{\mu}(m) \succ_m \mu^*(m) \succ_m mm$. There are two cases.

First, suppose $\hat{\mu}(m) = \hat{\mu}^1(m)$. This implies that $m$ did not participate in the adjustment phase of the PDAA. He proposed $\hat{\mu}^1(m)$ in the PDA phase and it was never rejected. However, if $\hat{\mu}(m) \succ_m \mu^*(m)$, then he must have also proposed that matching when $\succ_m$ was his reported preference. This proposal was rejected given the proposals of others. As these are unchanged, it will also be rejected given $m$’s manipulation.

Suppose instead that $\hat{\mu}(m) \succ_m \hat{\mu}^1(m)$. In this case $m$ must participate in the adjustment phase of the PDAA. Given that others’ preferences satisfy AA, the only way $m$ may advance to the adjustment step is if $\hat{\mu}^1(m) = mm$. Moreover, all agents who participate in the adjustment phase are also unmatched in both periods. In particular, there exists some $w$ such that $\hat{\mu}(m) = mw \succ_m \hat{\mu}^1(m) = mm$ and $\hat{\mu}(w) = wm \succ_w \hat{\mu}^1(w) = ww$. As the assignment of $m$ improved in the adjustment phase, his reported preference must have set $mw \succ_m mm$. But this implies he proposed $mw$ as part of PDA step and it was rejected by $w$. Thus, $\hat{\mu}^1(w) \succ_w ww$, which is a contradiction.

**Proof of Theorem 8.** Let $A(e) = \mu = (\mu_1, \mu_2)$ and $\bar{\mu} = (\mu_1, \mu_1)$. Suppose $\mu_1 \neq \mu_2$. First, we verify that $\bar{\mu}$ is dynamically individually rational. Suppose $mm \succ_m \bar{\mu}(m)$ for some $m \in M$. This implies $\mu_1(m) = w_1 \in W$. If $\mu_2(m) = m$, then $w_1m \succ_m mm \succ_m w_1w_1 = \bar{\mu}(m)$, which contradicts inertia.
Therefore, $\mu_2(m) = w_2 \neq w_1$ and

$$w_2w_2 \succ_m w_1w_2 = \mu(m) \succ_m mm \succ_m w_1w_1.$$ 

Now consider an alternative economy, $e'$, with the same agents and where the preferences of all $i \neq m$ are exactly as in $e$, i.e. $\succ_i = \succ_i'$ for all $i \neq m$. However, the preferences of agent $m$, $\succ_m'$, are defined as follows: (i) $\succ_m'$ has the same ex ante spot ranking as $\succ_m$, i.e. $P_{\succ_m'} = P_{\succ_m}$; (ii) for all $j \neq k, ii \succ_m' jk$; and, (iii) if $i \neq j$ and $k \neq l$, $ij \succ_m kl \iff ij' \succ_m kl$. Clearly, $\succ_m'$ exhibits inertia and $mm \succ_m' w_1w_1 \succ_m w_1i$ for all $i \in W_m \setminus \{w_1\}$.

As the procedure $A$ is non-prophetic, $A_1(e) = A_1(e')$. This implies that in the matching $A_1(e')$, agent $m$ is matched to $w_1$ in period 1. But this contradicts $A$ always generating a dynamically stable matching when agents’ preferences exhibit inertia. Thus, $\bar{\mu}(m) \succeq_m mm$. And hence, $\bar{\mu}(m) \succeq_m (\mu_1(m), m)$ as well. So, $\bar{\mu}$ is dynamically individually rational.

Suppose some pair, $m$ and $w$, can period-1 block $\bar{\mu}$. There are three cases:

1. Suppose $ww \succ_m (\mu_1(m), \mu_1(m)) = \bar{\mu}(m)$ and $mm \succ_w (\mu_1(w), \mu_1(w)) = \bar{\mu}(w)$. Clearly, $w \neq \mu_1(m)$ and $m \neq \mu_1(w)$. Now consider an alternative economy $e'$ where the preferences of all agents other than $m$ and $w$ are identical to those in $e$. However, the preference of $m$, $\succ_m'$, is identical to $\succ_m$ except that all persistent partnership plans are shuffled to the very top of the preference ranking and $P_{\succ_m'} = P_{\succ_w}$. (This is analogous to the definition of $\succ_m'$ above.) Define $\succ_w'$ similarly. In this alternative economy, matching mechanism $A$ must assign agent $m$ to $\mu_1(m)$ in period 1. However $ww \succ_m' (\mu_1(m), i)$ for all $i \in W_m$. Likewise, $w$ must be assigned to $\mu_1(w)$, but $mm \succ_w' (\mu_1(w), j)$ for all $j \in M_w$. Hence, $m$ and $w$ would be able to period-1 block the matching generated by $A$ in the economy $e'$. But $A$ always identifies a dynamically stable matching when preference have inertia—a contradiction.

2. Suppose $mw \succ_m \bar{\mu}(m)$ and $wm \succ_w \bar{\mu}(w)$. Since preferences have inertia, $\succ_i \in \mathcal{S}$, $ww \succ_m mw \succ_m \bar{\mu}(m)$ and $mm \succ_w wm \succ_w \bar{\mu}(w)$. Thus, case (1) applies.

3. Suppose $wm \succ_m \bar{\mu}(m)$ and $mw \succ_w \bar{\mu}(w)$. The same reasoning as case (2) applies.

Finally, suppose $m$ and $w$ can period-2 block $\bar{\mu}$. Then $(\mu_1(m), w) \succ_m \bar{\mu}(m) \implies ww \succ_m
\((\mu_1(m), w) \succeq_m \bar{\mu}(m)\) and \((\mu_1(w), m) \succ_w \bar{\mu}(w)\) \implies mm \succ_w (\mu_1(w), m) \succ_w \bar{\mu}(w_1)\). But this implies \(m\) and \(w\) can period-1 block \(\bar{\mu}\), which is not possible. \(\square\)

**Proof of Theorem 9.** Suppose \(\mu(i) = (\mu_1(i), \mu_2(i)) \succ_i (\mu_1(i), \mu_1(i)) = \bar{\mu}(i)\). By inertia, \((\mu_2(i), \mu_2(i)) \succ_i \mu(i) \succ_i \bar{\mu}(i)\). This implies \(\mu_2(i) = j \neq i\). Thus,\(jj \succ_i \mu(i) \succ_i \bar{\mu}(i)\). Since \(\mu\) is dynamically stable, \(\mu(j) = (\mu_1(j), i) \succ_j ii\). However, inertia implies \(\bar{\mu}(j) = (\mu_1(j), \mu_1(j)) \succ_j (\mu_1(j), i) = \mu(j)\). \(\square\)

**Lemma 14.** \(\sim_i^f\) is an equivalence relation.

**Proof.** We must verify three properties: reflexivity, symmetry, and transitivity.

1. \(\sim_i^f\) is reflexive. \(x \sim_i^f x' \implies [x \not\sim_i^f x' \& x' \not\sim_i^f x] \implies [x' \not\sim_i^f x \& x \not\sim_i^f x'] \implies x' \sim_i^f x\).

2. \(\sim_i^f\) is symmetric. Suppose \(x \not\sim_i^f x\). Then \(x \not\sim_i^f x\). Hence, \(\exists \bar{x} \in f_i(x)\) such that \(\forall x' \in f_i(x), \bar{x} \succ_i \bar{x}'\). However, \(\bar{x} \in f_i(x)\). Thus, \(\bar{x} \succ_i \bar{x}\)—a contradiction.

3. \(\sim_i^f\) is transitive. Suppose \(x \sim_i^f x'\) and \(x' \sim_i^f x''\). To arrive at a contradiction, suppose \(x \not\sim_i^f x''\). Thus, \(\exists \bar{x} \in f_i(x)\) such that \(\forall x'' \in f_i(x''), \bar{x} \succ_i \bar{x}''\). Since \(x \sim_i^f x'\), \(\exists \bar{x}' \in f_i(x')\) such that \(\bar{x} \not\succ_i \bar{x}'\). This implies \(\bar{x}' \succeq_i \bar{x}\). But then \(\bar{x}' \succeq_i \bar{x} \succ_i \bar{x}''\) for all \(\bar{x}'' \in f_i(x'')\), which implies \(x' \not\succ_i x''\)—a contradiction. The same argument applies if instead we assume \(x'' \not\succ_i x\). Hence, \(x \sim_i^f x''\). \(\square\)

**Remark 7.** The proof of Theorem 10 proceeds similarly to the proof of Theorem 6. The argument is constructive using a generalization of the PDAA procedure. An analogous generalization of Gale and Shapley’s (1962) deferred acceptance algorithm incorporating transfers is the salary-adjustment process of Crawford and Knoer (1981). We apply their intuition to our problem. Therefore, the argument should be familiar.

**Definition 27.** The conditional preference induced by \(\succ_i^f\) at \((j, y_1)\) is defined as \((k, y_2) \succ_i^{(f)(j,y_1)} (l, y'_2) \iff (j, k) \succ_i (j, l)\).
Algorithm 5 (GPDA). The (man-proposing) generalized plan deferred acceptance procedure with adjustment identifies an outcome $\rho^*$ as follows. First, for each $m$ define

$$X_m^0 = \{(w \ y_1 \ y_2), (w \ m \ 0), (m \ w \ 0, y_2) : w \in W; y_1, y_2 \in Y\}.$$ 

Initially, no element of $X_m^0$ has been rejected. In round $t \geq 1$:

1. Let $X_m^r \subset X_m^0$ be the set of plans that have not been rejected in some round $\tau' < \tau$. If $X_m^r = \emptyset$ or $(m \ m) \succ_m x$ for all $x \in X_m^{r'}$ then $m$ does not make any proposals. In this case, set $\hat{\rho}_1^1(m) = (m \ 0 \ 0)$. Otherwise, let $(i \ y_1 \ y_2)$ be $\succ^f_m$-maximal in $X_m^r$.\footnote{All $\succ^f_m$-maximal elements involve the same potential partner, but may differ in the profile of transfer payments. They are $f$-equivalent.} If there are several $\succ^f_m$-maximal elements in $X_m^r$, choose a fixed ordering of these elements and propose the first one. (This ordering is maintained through all subsequent rounds, if necessary.) Let $(i \ y_1 \ y_2)$ be that plan. It can assume one of three forms:

   (a) If $(i \ y_1 \ y_2) = (w \ w \ y_1 \ y_2)$ then $m$ proposes to $w$ a two-period partnership along with the transfer payments $(y_1, y_2)$. From $w$’s perspective, this corresponds to $(m \ w \ y_1 \ y_2)$.

   (b) If $(i \ y_1 \ y_2) = (m \ y_1 \ 0 \ 0)$ then $m$ proposes to $w$ a period-1 partnership along with the transfer payments $(y_1, 0)$. From $w$’s perspective, this plan corresponds to $(m \ y_1 \ 0 \ 0)$.

   (c) If $(i \ y_1 \ y_2) = (m \ w \ 0 \ y_2)$ then $m$ proposes to $w$ a period-2 partnership along with the transfer payments $(0, y_2)$. From $w$’s perspective, this plan corresponds to $(w \ m \ 0 \ y_2)$.

2. Let $X_w^r$ be the set of plans made available to $w$. If $(w \ 0 \ 0 \ 0 \ 0) \succ^f_w x$ for all $x \in X_m^r$, $w$ rejects all proposals. Otherwise, if $m$ made the $\succ^f_w$-maximal in $X_m^r$ proposal to $w$, $w$ (tentatively) accepts his proposal and rejects the others.

The above process continues until no further rejections occur. Let $\hat{\rho}_1^1$ be the resulting (interim) matching. If $\hat{\rho}_2^1(i) \neq (i, 0)$, set $\rho^*(i) = \hat{\rho}_1^1(i)$, and remove $i$ from the market.

Let $\hat{M}_2$ and $\hat{W}_2$ be the sets of men and women who were not removed from the market in the preceding step. Thus, $\hat{\rho}_1^1(i) = (i, 0)$ for all $i \in \hat{M}_2 \cup \hat{W}_2$. For these agents define a new period-2
outcome, $\hat{\rho}^2_2$, as follows. For all $m \in \hat{M}_2$, let $\hat{X}^0_m = \{(w, y_2) : w \in \hat{W}_2, y_2 \in Y\}$. Initially no element in $\hat{X}^0_m$ has been rejected. In round $\tau \geq 1$:

1. Let $\hat{X}^\tau_m \subseteq \hat{X}^0_m$ be the set of period-2 outcomes that have not been rejected in any round $\tau' < \tau$. If $\hat{X}^\tau_m = \emptyset$ or $(m, 0) \succ_m^{(f)(\hat{\rho}^1_1(m))} x$ for all $x \in \hat{X}^\tau_m$, then $m$ does not make any proposals. In this case, set $\hat{\rho}^2_2(m) = (m, 0)$.

Otherwise, let $(w, y_2)$ be $(f)(\hat{\rho}^1_1(m))$-maximal in $\hat{X}^\tau_m$. If there are several $(f)(\hat{\rho}^1_1(m))$-maximal elements in $\hat{X}^\tau_m$, choose a fixed ordering of these elements and propose the first one. (This ordering is maintained through all subsequent rounds, if necessary.) In this case, $m$ proposes to $w$ a partnership for period 2 along with a period-2 transfer payment of $y_2$. From $w$’s perspective, that period-2 outcome corresponds to $(m, -y_2)$.

2. Let $\hat{X}^\tau_w$ be the set of period-2 outcomes made available to $w$. If $(w, 0) \succ_w^{(f)(\hat{\rho}^1_1(w))} x$ for all $x \in \hat{X}^\tau_w$, $w$ rejects all proposals. Otherwise, if $m$ made the $(f)(\hat{\rho}^1_1(w))$-maximal in $\hat{X}^\tau_w$ proposal to $w$, $w$ (tentatively) accepts his proposal and rejects the others.

The above process continues until no further rejections occur. At this point all tentatively-accepted proposals are confirmed. Let $\hat{\rho}^2_2$ be the resulting (one-period) matching among the agents in $\hat{M}_2 \cup \hat{W}_2$. For all $i \in \hat{M}_2 \cup \hat{W}_2$, set $\rho^*(i) = (\hat{\rho}^1_1(i), \hat{\rho}^2_2(i))$.

**Lemma 15.** The interim GPDAA outcome, $\hat{\rho}^1$, is ex ante stable.

**Proof of Lemma 15.** This lemma generalizes Theorem 5. From Definition 5, it is clear that $\hat{\rho}^1(i) \succ_i^{\hat{\rho}^1(\hat{\rho}^1_1(i))} \emptyset$. Hence, no agent can period-1 block $\hat{\rho}^1$. If a pair $m$ and $w$ can period-1 block $\hat{\rho}^1$ then there must exist a plan for $m$ and a corresponding plan for $w$, that both $m$ and $w$ strictly prefer (given $f$) to their assignment under $\hat{\rho}^1$. However, this implies that $m$ must have proposed that plan to $w$ at some stage of the procedure and she must have rejected it. Hence, she must prefer her assignment under $\hat{\rho}^1$ to that proposal, which is a contradiction. □

**Proof of Theorem 10.** We show that the GPDAA outcome, $\rho^*$, is dynamically stable. Let $\hat{\rho}^1$ be the interim outcome from the procedure’s first step. By standard reasoning, $\rho^*(i) \succ_i \hat{\rho}^1(i)$ for all $i$. 162
Step 2 in the GPDAA weakly improves agents’ assignment. Since \( \bar{\rho}^1 \) is ex ante stable (Lemma 15), \( \rho^* \) must also be ex ante stable. Therefore, it cannot be period-1 blocked by any agent or by any pair. Next we consider the theorem’s two cases.

(i) \( \succ^f_i \) satisfies G-SIC and G-SA. Suppose agent \( i \) can period-2 block \( \rho^* \), i.e. \( (\rho^i_1(i), (i, 0)) \succ^f_i \rho^*(i) \). Thus, \( \rho^i_1(i) \neq (i, 0) \) and \( \rho^*_2(i) \neq (i, 0) \); else, we would arrive at a contradiction. Let \( \rho^i_1(i) = (j, y_1) \) for some \( j \neq i \). There are two cases. First, if \( \bar{\rho}^i_1(i) = (j, y_1, y_2) \), then by G-SIC,

\[
\left( \begin{array}{c}
\frac{1}{y_1} \\
0
\end{array} \right) \succ_m^f \rho^*(i) \succ_m^f \bar{\rho}^i_1(i) = \left( \begin{array}{c}
\frac{1}{y_1} \\
0
\end{array} \right) \implies \left( \begin{array}{c}
\frac{1}{y_1} \\
0
\end{array} \right) \succ^f_i \bar{\rho}^i_1(i),
\]

which is also a contradiction. If instead, and second, \( \bar{\rho}^i_1(i) = (j, i, y_1) \), then \( \left( \begin{array}{c}
\frac{1}{y_1} \\
0
\end{array} \right) \succ^f_i \rho^*(i) \succ^f_i \bar{\rho}^i_1(i) = (j, i, y_1) \), which is a contradiction. Therefore, agent \( i \) cannot period-2 block \( \rho^* \).

Suppose \( m \) and \( w \) can period-2 block \( \rho^* \). This implies that there exists \( y_2' \in Y \) such that \( \left( \frac{\mu^m_1(m) y_2'}{\mu^m_1(m) y_2'} \right) \succ_m^f \rho^*(m) \succ_m^f \bar{\rho}^1(m) \) and \( \left( \frac{\mu^w_1(w) m}{\mu^w_1(w) - y_2'} \right) \succ_w^f \rho^*(w) \succ_w^f \bar{\rho}^1(w) \). If \( \mu^m_1(m) = w \) or \( \mu^w_1(w) = m \), then \( m \) and \( w \) can period-1 block \( \bar{\rho}^1 \), which is not possible. Thus, without loss of generality, there are two remaining cases.

(a) Suppose \( \rho^m_1(m) = (m, 0) \) and \( \rho^w_1(w) = (m', y_1'), m' \neq m \). Thus, \( \left( \frac{m^w_1 w}{0 y_2'} \right) \succ_m^f \rho^*(m) \succ_m^f \bar{\rho}^1(m) \)

\[
\left( \begin{array}{c}
m w \\
0 y_2'
\end{array} \right) = \bar{\rho}^1(m) \quad \text{for some} \quad (j, y_2), \quad \left( \begin{array}{c}
m y_1' \\
0 y_2'
\end{array} \right) \succ_w^f \rho^*(w) \succ_w^f \bar{\rho}^1(w) \quad \text{for some} \quad (k, y_2').
\]

If \( (j, y_2) = (w', y_2) \) then \( \bar{\rho}^1(m) = \left( \begin{array}{c}
m w' \\
0 y_2'
\end{array} \right) \) and, correspondingly, \( \bar{\rho}^1(w) = \left( \begin{array}{c}
w' m \\
0 y_2'
\end{array} \right) \).

G-SA implies that \( \left( \frac{w y_2' w'}{0 y_2'} \right) \succ_m^f \left( \begin{array}{c}
m w' \\
0 y_2'
\end{array} \right) = \bar{\rho}^1(m) \)
\( \text{and} \quad \left( \frac{m y_1' - y_2'}{w y_2'} \right) \succ_w^f \left( \begin{array}{c}
m w' \\
0 y_2'
\end{array} \right) = \bar{\rho}^1(w) \).

Thus, \( m \) and \( w' \) can period-1 block \( \bar{\rho}^1 \), which is a contradiction. If instead \( (j, y_2) = (m, 0) \), then G-SA implies \( \left( \frac{w y_2' y_2'}{0 y_2'} \right) \succ_m^f \left( \begin{array}{c}
m y_2' \\
0 y_2'
\end{array} \right) \succ_m^f \bar{\rho}^1(m) \). Now consider agent \( w \). If \( (k, y_2') = (w, 0) \), then G-SA implies \( \left( \frac{w m'}{y_1' y_1} \right) \succ_w^f \bar{\rho}^1(w) \). Correspondingly, \( \frac{w w'}{0 y_2'} \succ_m^f \bar{\rho}^1(m') \)
\( \text{and} \quad m \) and \( m' \) can period-1 block \( \bar{\rho}^1 \), which is a contradiction. Thus, necessarily,

\( (k, y_2') = (m', y_2') \). But G-SIC implies that \( \left( \frac{m y_1' - y_2'}{w y_2'} \right) \succ_w^f \left( \begin{array}{c}
m w' \\
0 y_2'
\end{array} \right) = \bar{\rho}^1(w) \). But now \( m \) and \( w \) can period-1 block \( \bar{\rho}^1 \), which is a contradiction.

(b) Suppose \( \rho^m_1(m) = (w', y_1), w \neq w' \), and \( \rho^w_1(w) = (m', y_1'), m' \neq m \). The same argument as for \( w \) in the preceding paragraph shows that either \( m \) and period-1 block \( \bar{\rho}^1 \) with \( w' \), \( w \) and period-1 block \( \bar{\rho}^1 \) with \( m' \), or \( m \) and \( w \) can period-1 block together.

(ii) \( \succ^f_i \) satisfies G-SIC and G-RDS. The same argument as in the preceding case applies
This appendix presents partnership plans.

\[ x \in (x_1, x_2, \ldots, x_T) \in W_m^T. \]

A plan for \( w \in W \) is defined analogously. When confusion is unlikely, we will write \( x = x_1x_2\cdots x_T \). The truncation of plan \( x \) to the first \( t-1 \) periods is \( x_{<t} := x_1 \cdots x_{t-1} \). Its continuation from period \( t \) is \( x_{\geq t} := x_tx_{t+1} \cdots x_T \). Definitions of \( x_{<t} \) and \( x_{\geq t} \) follow analogously. When we write \( x = (x_{<t-1}, j, k, x_{>t}) \), then \( j \) is the period-\((t-1)\) partner and \( k \) is the period-\( t \) partner. We let \( x^j_k \) mean that plan \( x \) is such that \( x_t \in \{j, k\} \) for all \( t \). Of course, \( x^i = i \cdots i \) is a constant plan. As beforehand, preferences \( \succ_i \) are defined over partnership plans.

\[ x^i = i \cdots i \]

### B.2 A T-Period Market

This appendix presents \( T \)-period extension of Theorem 6 and of the PDAA procedure.

Let \( M \) and \( W \) be finite sets of men and women, respectively. A partnership plan for \( m \in M \) is a sequence of partners \( x = (x_1, x_2, \ldots, x_T) \in W_m^T. \) A plan for \( w \in W \) is defined analogously. When confusion is unlikely, we will write \( x = x_1x_2\cdots x_T \). The truncation of plan \( x \) to the first \( t-1 \) periods is \( x_{<t} := x_1 \cdots x_{t-1} \). Its continuation from period \( t \) is \( x_{\geq t} := x_tx_{t+1} \cdots x_T \). Definitions of \( x_{<t} \) and \( x_{\geq t} \) follow analogously. When we write \( x = (x_{<t-1}, j, k, x_{>t}) \), then \( j \) is the period-\((t-1)\) partner and \( k \) is the period-\( t \) partner. We let \( x^j_k \) mean that plan \( x \) is such that \( x_t \in \{j, k\} \) for all \( t \). Of course, \( x^i = i \cdots i \) is a constant plan. As beforehand, preferences \( \succ_i \) are defined over partnership plans.

\[ x^i = i \cdots i \]
Definition 28. The function $\mu_t: M \cup W \to M \cup W$ is a one-period matching if

1. For all $m \in M$, $\mu_t(m) \in W_m$;

2. For all $w \in W$, $\mu_t(w) \in M_w$; and,

3. For all $i$, $\mu_t(i) = j \implies \mu_t(j) = i$.

A matching, $\mu: M \cup W \to (M \cup W)^T$ is a sequence of one-period matchings, i.e. $\mu = (\mu_1, \ldots, \mu_T)$.

To define blocking and stability, we adopt a nomenclature that we reuse in our analysis of the core (Appendix B.3). A coalition $C \subset M \cup W$ is a non-empty set of agents.

Definition 29. The function $\mu^C_t: C \to C$ is a one-period matching for coalition $C$ if

1. For all $m \in M \cap C$, $\mu_t^C(m) \in W_m \cap C$;

2. For all $w \in W \cap C$, $\mu_t^C(w) \in M_w \cap C$; and,

3. For all $i \in C$, $\mu_t^C(i) = j \implies \mu_t^C(j) = i$.

Definition 30. A coalition $C$ can period-t block the matching $\mu$ if there exists a sequence of one-period matchings for the coalition, $\mu^{C}_{\leq t} = (\mu_t^C, \mu_{t+1}^C, \ldots, \mu_T^C)$, such that $(\mu_{<t}(i), \mu^{C}_{\leq t}(i)) \succ_i \mu(i)$ for all $i \in C$.

Definition 31. A coalition $C$ is admissible if (i) $C = \{i\}$ for some $i \in M \cup W$ or (ii) $C = \{m, w\}$ for some $m \in M$ and $w \in W$.

Definition 32. The matching $\mu$ is ex ante stable if it cannot be period-1 blocked by any admissible coalition.

Definition 33. The matching $\mu$ is dynamically stable if for all $t$ it cannot be period-t blocked by any admissible coalition.

The following mechanism reduces to the Gale and Shapley (1962) deferred acceptance algorithm when $T = 1$ and to the PDAA procedure when $T = 2$. We rely on it to prove Theorem 16 below.
Algorithm 6 (T-PDAA). The $T$-period (man-proposing) plan deferred acceptance procedure with adjustment identifies a matching $\mu^*$ in a series of steps as follows:

**Step 1.** For each $m$ let

$$X^0_m = \bigcup_{w \in W} \{(i_1, \ldots, i_T): i_t \in \{m, w\}\} \setminus \{x^m\}.$$  

At $\tau = 0$, no plans in $X^0_m$ have been rejected. In round $\tau \geq 1$:

1. Let $X^\tau_m \subset X^0_m$ be the subset of plans that have not been rejected in some round $\tau' < \tau$. If $X^\tau_m = \emptyset$ or $x^m \succ_m x$ for all $x \in X^\tau_m$, then $m$ does not make any proposals. Otherwise, $m$ proposes to the woman identified in his most preferred plan in $X^\tau_m$. Each such plan involves at most one distinct woman. If $x$ is his most preferred plan and it involves $w$, he proposes to $w$ the corresponding plan. For example, if $x = \text{wmmww} \cdots$, then $m$ proposes to $w$ a plan where they are paired for periods 1 and 2, unmatched in period 3, matched together in period 4, and so on.

2. Let $X^\tau_w$ be the set of plans made available to $w$. If $x^w \succ_w x$ for all $x \in X^\tau_w$, $w$ rejects all proposals. Otherwise, $w$ (tentatively) accepts her most preferred plan in $X^\tau_w$ and rejects the others.

The above process continues until no rejections occur. If $w$ accepts $m$'s proposal in the final round, define the interim matchings $\tilde{\mu}^1(m)$ and $\tilde{\mu}^1(w)$ accordingly. If $i$ does not make or receive any proposals in the final round, set $\hat{\mu}^1(i) = i \cdots i$.

**Step $s \geq 2$.** If $\hat{\mu}^{s-1}(i) \neq x^{i}_{\geq s}$, set $\hat{\mu}^s(i) = \hat{\mu}^{s-1}(i)$. Else, let $\hat{M}_s = \{m \in M: \hat{\mu}^{s-1}(m) = x^m_{\geq s}\}$ and $\hat{W}_s = \{w \in W: \hat{\mu}^{s-1}(w) = x^w_{\geq s}\}$ be the sets of the remaining men and women, respectively. Each man (woman) in $\hat{M}_s$ ($\hat{W}_s$) is unmatched in each period $t \geq s$ under the interim matching $\hat{\mu}^{s-1}$.

For each $m \in \hat{M}_s$ let

$$X^0_m = \bigcup_{w \in \hat{W}_s} \{(i_{s}, i_{s+1}, \ldots, i_T): i_t \in \{m, w\}\} \setminus \{x^m_{\geq s}\}.$$  

be a set of candidate continuation plans for $m$. At $\tau = 0$, no continuation plans in $X^0_m$ have been rejected. In round $\tau \geq 1$:  

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1. Let $X^*_m \subset X^0_m$ be the subset of plans that have not been rejected in some round $\tau' < \tau$. If $X^*_m = \emptyset$ or $(\tilde{\mu}_{<s}^{-1}(m), x^m_{s}) \succ_m (\tilde{\mu}_{<s}^{-1}(m), x_{s})$ for all $x_{s} \in X^*_m$, then $m$ does not make any proposals. Otherwise, let $x^{mw}_{s} \in X^*_m$ be $m$'s most preferred, not yet rejected continuation plan, i.e. $(\tilde{\mu}_{<s}^{-1}(m), x^{mw}_{s}) \succ_m (\tilde{\mu}_{<s}^{-1}(m), x_{s})$ for all $x_{s} \in X^*_m \setminus \{x^{mw}_{s}\}$. This plan involves exactly one woman, in this case $w$. He proposes to her a compatible continuation plan. For example, if $x^{mw}_{s} = (w, w, m, w, \ldots)$ then $m$ proposes to $w$ a continuation plan where they are paired in periods $s$ and $s + 1$, unmatched in period $s + 2$, matched together in period $s + 3$, and so on.

2. Let $X^*_w = \{x^{mw}_{s}, x^{m'w}_{s}, \ldots\}$ be the set of continuation plans made available to $w$. If $(\tilde{\mu}_{<s}^{-1}(m), x^w_{s}) \succ_w (\tilde{\mu}_{<s}^{-1}(w), x_{s})$ for all $x_{s} \in X^*_w$, $w$ rejects all proposals. Otherwise, $w$ (tentatively) accepts her most preferred continuation plan in $X^*_w$ and rejects the others.

The above process continues until no rejections occur. If $w$ accepts $m$'s proposal in the final round, say $x^{mw}_{s}$, set $\tilde{\mu}^s(w) = (\tilde{\mu}_{<s}^{-1}(w), x^{mw}_{s})$. Define $\tilde{\mu}^s(m)$ in a corresponding fashion accordingly. If $i$ does not make or receive any proposals in the final round, set $\tilde{\mu}^s(i) = (\tilde{\mu}_{<s}^{-1}(i), x^j_{s})$.

**Final Assignment.** Let $\tilde{\mu}^T$ be the matching identified at the conclusion of step $s = T$. Let this be the final matching, i.e. $\mu^* = \tilde{\mu}^T$.

**Lemma 16.** There exists an ex ante stable matching.$\Box$

**Proof of Lemma 16.** Step 1 of the T-PDAA procedure provides a $T$-period generalization of the plan deferred acceptance procedure. It is straightforward to verify that the matching identified by step 1 of the T-PDAA procedure, $\tilde{\mu}^1$, is ex ante stable. The argument mirrors the reasoning of the two-period case. $\Box$

Restrictions on agents’ preferences were required to ensure the existence of a dynamically stable matching in the two-period case. Those restrictions generalize in a natural way. The preference $\succ_i$ satisfies $T$-period sequential improvement complementarity if for all $t \geq 2$, all $x = x_1 \cdots x_T$, all $x^k$, and all $x^j$,

$$ (x_{<t-1}, j, x^{jk}_{\geq t}) \succ_i (x_{<t-1}, j, x^{ij}_{\geq t}) \succ_i x^i \implies (x_{<t-1}, k, x^{jk}_{\geq t}) \succ_i (x_{<t-1}, j, x^{ij}_{\geq t}). \quad \text{(T-SIC)} $$
Theorem 16. There exists a dynamically stable matching if (i) $\succ_i$ satisfies T-SIC and T-SA for all $i$; or, (ii) $\succ_i$ satisfies T-SIC and T-SA for all $i$.

Proof of Theorem 16. Let $\mu^*$ be the matching identified by the T-PDAA procedure. Let $(\tilde{\mu}^1, \ldots, \tilde{\mu}^T)$ be the sequence of interim matchings identified during the procedure’s operation. Each step of the T-PDAA weakly improves the interim matching, i.e. $\mu^*(i) \succeq_i \tilde{\mu}^{t+1}(i) \succeq_i \tilde{\mu}^t(i)$. Hence, if $\mu^*$ can be period-$t$ blocked by an agent or a pair, then the interim matching $\tilde{\mu}^t$ can be period-$t$ blocked by the same agent or pair. This is because the T-PDAA ensures that $\mu^*_t = \tilde{\mu}^t_t$. Therefore, to verify that $\mu^*$ is dynamically stable, it is sufficient to show that $\tilde{\mu}^t$ cannot be period-$t$ blocked for each $t$.

By Lemma 16, $\tilde{\mu}^1$ cannot be period-1 blocked by any agent or by any pair. Proceeding by induction, suppose $\tilde{\mu}^{t-1}$ cannot be blocked by any agent or pair in any period $t' \leq t - 1$. We will verify that $\tilde{\mu}^t$ cannot be blocked in period $t$. (And thus it cannot be blocked in any period $t' \leq t$.)

To set notation, from the T-PDAA procedure we know that

$$\tilde{\mu}^t(i) = (\tilde{\mu}_{<t}^{t-1}(i), x_{\geq t}^{ij}).$$

for some $j$ and $j'$. We consider the theorem’s two cases separately.

(a) Suppose each agent’s preferences satisfy T-SIC and T-SA. Suppose agent $i$ can period-$t$ block $\tilde{\mu}^t$.

Then

$$(\tilde{\mu}_{<t-1}^{t-1}(i), j, x_{\geq t}^{ij}) \succ_i \tilde{\mu}^t(i) \succeq_i \tilde{\mu}^{t-1}(i) \succeq_i x^i.$$

First, note that $j \neq i$. Otherwise, $(\tilde{\mu}_{<t}^{t-1}(i), i, x_{\geq t}^{ij}) \succ_i \tilde{\mu}^t(i) \succeq_i \tilde{\mu}^{t-1}(i)$. This implies that $\tilde{\mu}^{t-1}$ can
be period-\((t - 1)\) blocked by \(i\), contradicting the induction hypothesis. Since \(j \neq i\), it follows that \(\hat{\mu}^{t-1}(i) = (\hat{\mu}^{t-1}_{<t-1}(i), x^{ij}_{\geq t}) = (\hat{\mu}^{t-1}_{<t-1}(i), j, x^{ij}_{\geq t})\) for some \(j\). Applying T-SIC,

\[
(\hat{\mu}^{t-1}_{<t-1}(i), j, x^{ij}_{\geq t}) \succ_i (\hat{\mu}^{t-1}_{<t-1}(i), j, x^{ij}_{\geq t}) \succ_i x^i
\]

\[\Rightarrow (\hat{\mu}^{t-1}_{<t-1}(i), i, x^{i}_{\geq t}) \succ_i (\hat{\mu}^{t-1}_{<t-1}(i), j, x^{ij}_{\geq t}) = \hat{\mu}^{t-1}(i).\]

Hence, \(\hat{\mu}^{t-1}\) can be period-\((t - 1)\) blocked by \(i\), which is a contradiction.

Suppose instead that \(m\) and \(w\) can period-\(t\) block \(\hat{\mu}^t\). Then there exists a sequence of matchings among \(m\) and \(w\),

\[
\mu^{mw}_{\geq t} = (\mu^{mw}_t, \ldots, \mu^{mw}_T)
\]

such that

\[
(\hat{\mu}^{t-1}_{<t-1}(m), \hat{\mu}^{t-1}_{t-1}(m), \mu^{mw}_{\geq t}(m)) \succ_m \hat{\mu}^t(m) \succ_m \hat{\mu}^{t-1}(m)
\]

and

\[
(\hat{\mu}^{t-1}_{<t-1}(w), \hat{\mu}^{t-1}_{t-1}(w), \mu^{mw}_{\geq t}(w)) \succ_w \hat{\mu}^t(w) \succ_w \hat{\mu}^{t-1}(w).
\]

If \(\hat{\mu}^{t-1}_{t-1}(m) = m\) and \(\hat{\mu}^{t-1}_{t-1}(w) = w\), then \(m\) and \(w\) can period-\((t - 1)\) block \(\hat{\mu}^{t-1}\), which is a contradiction. The same conclusion applies if \(\hat{\mu}^{t-1}_{t-1}(m) = w\) and \(\hat{\mu}^{t-1}_{t-1}(m) = m\). Thus, without loss of generality, there are two cases.

(i) First, suppose \(\hat{\mu}^{t-1}_{t-1}(m) = m\) and \(\hat{\mu}^{t-1}_{t-1}(w) = w' \neq w\). Consider first agent \(m\). It follows that

\[
(\hat{\mu}^{t-1}_{<t-1}(m), m^{mw}_{\geq t}(m)) \succ_m \hat{\mu}^{t-1}(m) = (\hat{\mu}^{t-1}_{<t-1}(m), m^{mj}_{\geq t}) \succ_i x^m
\]

for some \(j \in \{m, w'\}\). If \(j = w' \in W_m\), then \(m^{mj}_{\geq t} \neq x^m\). Hence, T-SA implies that

\[
(\hat{\mu}^{t-1}_{<t-1}(m), w', m^{nw'}_{\geq t}) \succ_m \hat{\mu}^{t-1}(m) = (\hat{\mu}^{t-1}_{<t-1}(m), m^{nw'}_{\geq t}).
\]

And by a symmetric argument for \(w'\),

\[
(\hat{\mu}^{t-1}_{<t-1}(w'), m, x^{m}_{\geq t}) \succ_w \hat{\mu}^{t-1}(w') = (\hat{\mu}^{t-1}_{<t-1}(w'), w', m^{nw'}_{\geq t}).
\]

But this implies that \(m\) and \(w'\) can period \(t - 1\) block \(\hat{\mu}^{t-1}\), which is a contradiction.
Instead, if \( j = m \), then \( x_{\geq t}^{mj} = x_{\geq t}^m \). T-SIC implies that

\[
(\tilde{\mu}_{<t-1}^{t-1}(m), w, \mu_{\geq t}^{m_0}(m)) \succ_m \tilde{\mu}_{<t-1}^{t-1}(m) = (\tilde{\mu}_{<t-1}^{t-1}(m), m, x_{\geq t}^m). \tag{B.2}
\]

Now consider agent \( w \). It follows that

\[
(\tilde{\mu}_{<t-1}^{t-1}(w), m', \mu_{\geq t}^{m_0}(w)) \succ_w \tilde{\mu}_{<t-1}^{t-1}(w) = (\tilde{\mu}_{<t-1}^{t-1}(w), m', x_{\geq t}^w) \succeq_i x^w
\]

for \( k \in \{w, m'\} \). If \( k = w \), then T-SA implies that

\[
(\tilde{\mu}_{<t-1}^{t-1}(w), m', \mu_{\geq t}^{m_0}(w)) \succ_w \tilde{\mu}_{<t-1}^{t-1}(w) = (\tilde{\mu}_{<t-1}^{t-1}(w), m', x_{\geq t}^w).
\]

But this implies that \( w \) and \( m' \) can period \( t - 1 \) block \( \tilde{\mu}^{-1} \), which is a contradiction.

Instead, if \( k = m' \), then T-SIC implies that

\[
(\tilde{\mu}_{<t-1}^{t-1}(w), m', \mu_{\geq t}^{m_0}(w)) \succ_w \tilde{\mu}_{<t-1}^{t-1}(w) = (\tilde{\mu}_{<t-1}^{t-1}(w), m', x_{\geq t}^{wm'}). \tag{B.3}
\]

Together, (B.2) and (B.3) imply that \( m \) and \( w \) can period \( t - 1 \) block \( \tilde{\mu}^{-1} \), which is a contradiction.

(ii) And second, \( \tilde{\mu}_{<t-1}^{t-1}(m) = w' \neq w \) and \( \tilde{\mu}_{<t-1}^{t-1}(w) = m' \neq m \). Arguments parallel to the case for agent \( w \) in the preceding case again show that \( m \) and \( w \) will be able to define a continuation plan that blocks \( \tilde{\mu}^{-1} \) in period \( t - 1 \), which is a contradiction.

(b) Suppose each agent’s preferences satisfy T-SIC and T-RDS. The argument proceeds in parallel to the preceding case. The only difference is in the final two cases, labeled (i) and (ii) above.

(i) Suppose \( \tilde{\mu}_{<t-1}^{t-1}(m) = m \) and \( \tilde{\mu}_{<t-1}^{t-1}(w) = m' \neq w \). It follows that for agent \( m \),

\[
(\tilde{\mu}_{<t-1}^{t-1}(m), m, \mu_{\geq t}^{m_0}(m)) \succ_m \tilde{\mu}_{<t-1}^{t-1}(m) = (\tilde{\mu}_{<t-1}^{t-1}(m), m, x_{\geq t}^{mj}) \succeq_i x^m
\]

for some \( j \in \{m, w'\} \). If \( j = m \), then \( x_{\geq t}^{mj} = x_{\geq t}^m \). T-SIC implies that

\[
(\tilde{\mu}_{<t-1}^{t-1}(m), w, \mu_{\geq t}^{m_0}(m)) \succ_m \tilde{\mu}_{<t-1}^{t-1}(m) = (\tilde{\mu}_{<t-1}^{t-1}(m), m, x_{\geq t}^m). \tag{B.4}
\]
If \( j = w' \in W_m \), then \( x^m_{j=1} \neq x^m_{j\geq t} \). T-RDS implies that

\[
(p^t_{\leq t-1}(m), w, \mu^w_{\geq t}(m)) \succ_m \tilde{p}^t_{\leq t-1}(m) = (p^t_{\leq t-1}(m), m, x^w_{\geq t}). \tag{B.5}
\]

For \( w \), \( (p^t_{\leq t-1}(w), m', \mu^w_{\geq t}(w)) \succ_w \tilde{p}^t_{\leq t-1}(w) = (p^t_{\leq t-1}(w), m', x^{w}_{\geq t}) \succ_i x^w \) where \( k \in \{w, m'\} \). If \( k = w \), then T-RDS implies that

\[
(p^t_{\leq t-1}(w), w, \mu^w_{\geq t}(w)) \succ_w \tilde{p}^t_{\leq t-1}(w) = (p^t_{\leq t-1}(w), m', x^w_{\geq t}). \tag{B.6}
\]

If instead \( k = m' \), then T-SIC implies that

\[
(p^t_{\leq t-1}(w), w, \mu^w_{\geq t}(w)) \succ_w \tilde{p}^t_{\leq t-1}(w) = (p^t_{\leq t-1}(w), m', x^{wm'}_{\geq t}). \tag{B.7}
\]

Together, (B.4)–(B.7) imply that \( m \) and \( w \) can period-(\( t - 1 \)) block the matching \( \tilde{p}^t_{\leq t-1} \) with the continuation plan where \( m \) and \( w \) are matched together in period \( t - 1 \) and \( \mu^w_{\geq t} \) as defined in (B.1) is implemented in periods \( t' \geq t \). This is a contradiction since cannot be period-(\( t - 1 \)) blocked.

(ii) Suppose \( p^t_{\leq t-1}(m) = w' \neq w \) and \( p^t_{\leq t-1}(w) = m' \neq m \). Arguments parallel to cases (B.6) and (B.7) above show that \( m \) and \( w \) will be able to define a continuation plan to block \( \tilde{p}^t_{\leq t-1} \) in period \( t - 1 \), which is a contradiction.

\[\square\]

\textbf{Remark 8}. Though the intuition is suggestive, the above analysis does not apply when \( T = \infty \). Notably, the T-PDAA procedure may not to terminate. The \( T = \infty \) case can be accommodated with further restrictions on preferences.
B.3 The Core

In this appendix we investigate stronger versions of our stability concepts by allowing collective blocking actions. A coalition $C$ is a non-empty subset of agents, $C \subset M \cup W$. A coalition can block a matching if it can define a within-coalition matching that its members prefer. More formally, we have the following analogues of previously introduced definitions.

**Definition 34.** The function $\mu_C^t : C \to C$ is a one-period matching for coalition $C$ if (i) for all $m \in M \cap C$, $\mu_C^t(m) \in W_m \cap C$; (ii) for all $w \in W \cap C$, $\mu_C^t(w) \in M_w \cap C$; and, (iii) for all $i \in C$, $\mu_C^t(i) = j \implies \mu_C^t(j) = i$.

**Definition 35.** A coalition $C$ can period-1 block the matching $\mu$ if there exist one-period matchings for the coalition $C$, $\mu_C^1$ and $\mu_C^2$, such that $(\mu_C^1(i), \mu_C^2(i)) \succ_i \mu(i)$ for all $i \in C$.

**Definition 36.** A coalition $C$ can period-2 block the matching $\mu$ if there exists a one-period matching for coalition $C$, $\mu_C^2$, such that $(\mu_1(i), \mu_C^2(i)) \succ_i \mu(i)$ for all $i \in C$.

The matching $\mu$ is in the ex ante core if it cannot be period-1 blocked by any coalition. The matching $\mu$ is in the dynamic core if for all $t$ it cannot be period-$t$ blocked by any coalition. The preceding definitions reduce to those of ex ante and dynamic stability, respectively, when only one-agent or couple coalitions are allowed. The ex ante core corresponds to the “core” in Damiano and Lam (2005, Definition 3). What we call the dynamic core is sometimes called the sequential core (Gale, 1978) or the recursive core (Damiano and Lam, 2005, Definition 4; Becker and Chakrabarti, 1995)

In a one-period market, the core is not empty and corresponds to the set of pairwise stable matchings (Gale and Shapley, 1962). Several studies of dynamic matching markets have noted the core’s emptiness (Damiano and Lam, 2005; Kurino, 2009). The core may also be empty in static, many-to-many matching markets (Blair, 1988). In our setting, both the ex ante core and the dynamic core can be empty, even when preferences exhibit inertia.
Example 8. Consider the following economy with three men and three women:

$\succ_{m_1} : w_2w_2, w_3w_3, w_1w_2, w_1w_3, m_1m_1$

$\succ_{m_2} : w_3w_3, w_1w_1, w_1w_3, m_2w_3, w_2w_1, m_2m_2$

$\succ_{m_3} : w_1w_1, w_2w_2, w_2w_1, m_3w_1, w_3w_1, w_3w_2, m_3m_3$

$\succ_{w_1} : m_1m_1, m_1w_1, m_1m_2, m_1m_3, w_1w_1$

$\succ_{w_2} : m_2m_2, m_2w_2, m_2m_3, m_2m_1, w_2w_2$

$\succ_{w_3} : m_3m_3, m_3w_3, m_3m_1, m_3m_2, w_3w_3$

All preferences above exhibit inertia. There are four ex ante stable matchings (Table B.1). Each can be blocked by some coalition. Since all matchings in the ex ante and the dynamic cores must be ex ante stable, the ex ante and dynamic cores are empty.

Table B.1: All ex ante stable matchings in Example 8 and blocking coalitions.

<table>
<thead>
<tr>
<th>Matching</th>
<th>$m_1$</th>
<th>$m_2$</th>
<th>$m_3$</th>
<th>$w_1$</th>
<th>$w_2$</th>
<th>$w_3$</th>
<th>Blocking Coalition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu^1$</td>
<td>$w_1w_2$</td>
<td>$w_2w_1$</td>
<td>$m_3m_3$</td>
<td>$m_1m_2$</td>
<td>$m_2m_1$</td>
<td>$w_3w_3$</td>
<td>${m_2, m_3, w_2, w_3}$</td>
</tr>
<tr>
<td>$\mu^2$</td>
<td>$m_1m_1$</td>
<td>$w_2w_3$</td>
<td>$w_3w_2$</td>
<td>$w_1w_1$</td>
<td>$m_2m_3$</td>
<td>$m_3m_2$</td>
<td>${m_1, m_3, w_1, w_3}$</td>
</tr>
<tr>
<td>$\mu^3$</td>
<td>$w_1w_3$</td>
<td>$m_2m_2$</td>
<td>$w_3w_1$</td>
<td>$m_1m_3$</td>
<td>$w_2w_2$</td>
<td>$w_3w_1$</td>
<td>${m_1, m_2, w_1, w_2}$</td>
</tr>
<tr>
<td>$\mu^4$</td>
<td>$m_1m_1$</td>
<td>$m_2m_2$</td>
<td>$m_3m_3$</td>
<td>$w_1w_1$</td>
<td>$w_2w_2$</td>
<td>$w_3w_3$</td>
<td>${m_2, m_3, w_2, w_3}$</td>
</tr>
</tbody>
</table>

To ensure the core’s non-emptiness, it is necessary to impose stronger requirements on preferences than inertia alone. The preference $\succ_i$ exhibits strong inertia if

$$j \neq k \implies jj \succ_i jk \text{ and } kk \succ_i jk.$$  \hspace{1cm} (SI)

When all preferences exhibit strong inertia, all stable matchings are persistent and the set of dynamically stable matchings coincides with the core.\(^3\) Regrettably, strong inertia may be too strong for some applications. At times it may be desirable to allow the preferences of some agents

\(^3\)The economy essentially reduces to a static economy and the conclusions of Gale and Shapley (1962) apply.
in the market to satisfy weaker restrictions. For example, in a school-assignment application, it may be appropriate to assume that the priority structure (schools’ “preferences”) satisfies the strong inertia requirement. Once enrolled at a school, the student can stay enrolled in future years without fear of being “bumped” by a new student. For students, however, it may be desirable to allow preferences to be more flexible.

We say that the preference $\succ_i$ satisfies sequential local improvement complementarity if

$$jk \succ_i l' \succ_i jj \implies kk \succ_i ll'.$$

It is easy to verify that $SI \implies I \implies SLIC \implies SIC$.

**Theorem 17.** Suppose each man’s preference satisfies SLIC and SA and each woman’s preference satisfies SI. The dynamic core is not empty and coincides with the set of dynamically stable matchings.

**Proof of Theorem 17.** Since SI implies SIC and SA, there exists a dynamically stable matching, say $\mu^*$. Suppose that coalition $C$ can period-1 block $\mu^*$ with the within-coalition matching $\mu^C$. Let $w \in C \cap W$ and suppose $\mu^C_i(w) = w$ for some $i$. SI implies that $ww \succ_w \mu^C_i(w) \succ_w \mu^*(w)$, which is a contradiction. Therefore $\mu^C(w) = mm'$ where $m,m' \in C \cap M$. If $m = m'$ then $w$ and $m$ can period-1 block $\mu^*$ as a pair, which contradicts dynamic stability. SI implies that $mm \succ_w \mu^C(m) = mm' \succ_w \mu^*(w)$ and $m'm' \succ_w \mu^C(w) = mm' \succ_w \mu^*(w)$. An analogous conclusion applies to all women in $C$.

Now consider $m \in C \cap M$ and suppose $\mu^C(m) = wm$ for some $w \neq m$. SA implies that $ww \succ_m wm = \mu^C(m) \succ_m \mu^*(m)$. Likewise, SI implies that $mm \succ_w \mu^C(m) \succ_w \mu^*(w)$. But then $m$ and $w$ can block $\mu^*$ together, which is a contradiction. A similar conclusion applies if instead $\mu^C(m) = mw$. Thus, $\mu^C(m) = ww'$ and $w,w' \in C$, $w \neq w'$. If $ww \succ_m \mu^*(m)$, then $m$ and $w$ can (pairwise) block $\mu^*$ in period 1, which is a contradiction. Therefore, suppose $\mu^*(m) \succ_m ww$. Hence, $ww' \succ_m \mu^*(m) \succ_m ww$. SLIC implies that $ww' \succ_m \mu^*(m)$. If $\mu^*(m) = w'w'$, then $\mu^*(w') = mm$. However, noting the argument in the preceding paragraph, $mm \succ_w \mu^*(w') = mm$—a contradiction. Therefore, $w'w' \succ_m \mu^*(m)$. But now $m$ and $w'$ can block $\mu^*$, which is also a contradiction. Therefore, $\mu^*$ cannot be period-1 blocked by the coalition $C$.

Suppose $\mu^*$ can be blocked by coalition $C$ in period 2. Let $w \in C \cap W$. If $\mu^C_2(w) = w$, then
The preferences of the men satisfy SLIC and SA. The preferences of the women satisfy SI. All
consider a market with two men and two women whose preferences are

Example 9. Consider a market with two men and two women whose preferences are

<table>
<thead>
<tr>
<th>Matching</th>
<th>$m_1$</th>
<th>$m_2$</th>
<th>$m_3$</th>
<th>$w_1$</th>
<th>$w_2$</th>
<th>$w_3$</th>
<th>Blocking Coalition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_1$</td>
<td>$w_1w_2$</td>
<td>$w_2w_1$</td>
<td>$m_3m_3$</td>
<td>$m_1m_2$</td>
<td>$m_2m_1$</td>
<td>$w_3w_3$</td>
<td>{m_2, m_3, w_2, w_3}</td>
</tr>
<tr>
<td>$\mu_2$</td>
<td>$m_1m_1$</td>
<td>$w_3w_2$</td>
<td>$w_2w_3$</td>
<td>$w_1w_1$</td>
<td>$m_3m_2$</td>
<td>$m_2m_3$</td>
<td>{m_1, m_3, w_1, w_3}</td>
</tr>
<tr>
<td>$\mu_3$</td>
<td>$w_3w_1$</td>
<td>$m_2m_2$</td>
<td>$w_1w_3$</td>
<td>$m_3m_2$</td>
<td>$w_2w_2$</td>
<td>$m_1m_3$</td>
<td>{m_1, m_2, w_1, w_2}</td>
</tr>
<tr>
<td>$\mu_4$</td>
<td>$m_1m_1$</td>
<td>$m_2m_2$</td>
<td>$m_3m_3$</td>
<td>$w_1w_1$</td>
<td>$w_2w_2$</td>
<td>$w_3w_3$</td>
<td>{m_1, m_2, w_1, w_2}</td>
</tr>
</tbody>
</table>

The following example shows that SLIC cannot be relaxed to SIC in Theorem 17.

Example 10. Consider a market with two men and two women whose preferences are

The preferences of the men satisfy SLIC and SA. The preferences of the women satisfy SI. All
three dynamically stable matchings (Table B.3) are in the dynamic core.
Table B.3: All dynamically stable and core matchings in Example 10.

<table>
<thead>
<tr>
<th>Matching</th>
<th>$m_1$</th>
<th>$m_2$</th>
<th>$w_1$</th>
<th>$w_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu^1$</td>
<td>$w_1w_2$</td>
<td>$w_2w_1$</td>
<td>$m_1m_2$</td>
<td>$m_2m_1$</td>
</tr>
<tr>
<td>$\mu^2$</td>
<td>$w_2w_2$</td>
<td>$w_1w_1$</td>
<td>$m_2m_2$</td>
<td>$m_1m_1$</td>
</tr>
<tr>
<td>$\mu^3$</td>
<td>$w_1w_1$</td>
<td>$w_2w_2$</td>
<td>$m_1m_1$</td>
<td>$m_2m_2$</td>
</tr>
</tbody>
</table>

B.4 Examples

Example 11 (Kadam and Kotowski (2015b)). This example illustrates that there may not exist an optimal dynamically stable matching for all agents on one side of the market. A stable matching is said to be optimal for the men if all men prefer their assignment in that matching to every other stable matching. A woman-optimal stable matching is defined analogously. In Example 14 there does not exist a woman-optimal stable matching. This example employs a more restrictive class of preferences.

There are three men and three women. Their preferences are:

$\succ^m_1$: $w_2w_2, w_2w_1, w_2w_3, w_1w_1, w_3w_3, m_1m_1$

$\succ^m_2$: $w_3w_3, w_3w_2, w_3w_1, w_2w_2, w_1w_1, m_2m_2$

$\succ^m_3$: $w_1w_1, w_1w_3, w_3w_3, w_1w_2, w_2w_2, m_3m_3$

$\succ^w_1$: $m_2m_2, m_1m_1, m_2m_1, m_1m_2, w_1m_2, m_3m_2, w_1w_1, m_3m_3$

$\succ^w_2$: $m_3m_3, m_2m_2, m_3m_2, m_2m_3, w_2m_3, m_1m_3, m_2w_2, m_1m_1$

$\succ^w_3$: $m_1m_1, m_1m_3, m_3m_1, m_3m_1, m_2m_1, m_3m_3, w_3w_3, m_2m_2$

All preferences exhibit inertia. There are three dynamically stable matchings (Table B.4). $m_1$ and $m_2$ like their assigned plans in $\mu^3$ the most. $m_3$ prefers his assignment under $\mu^1$.

This example also shows that dynamically stable matchings do not form a lattice under the “common-preference” partial order (cf. Knuth (1997), Roth (1985b), and Blair (1988)). Kadam and Kotowski (2015b) identify sufficient conditions for the set of dynamically stable matchings to
Table B.4: All dynamically stable matchings in Example 11.

<table>
<thead>
<tr>
<th>Matching</th>
<th>$m_1$</th>
<th>$m_2$</th>
<th>$m_3$</th>
<th>$w_1$</th>
<th>$w_2$</th>
<th>$w_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu^1$</td>
<td>$w_1w_1$</td>
<td>$w_2w_2$</td>
<td>$w_3w_3$</td>
<td>$m_1m_1$</td>
<td>$m_2m_2$</td>
<td>$m_3m_3$</td>
</tr>
<tr>
<td>$\mu^2$</td>
<td>$w_3w_3$</td>
<td>$w_1w_1$</td>
<td>$w_2w_2$</td>
<td>$m_2m_2$</td>
<td>$m_3m_3$</td>
<td>$m_1m_1$</td>
</tr>
<tr>
<td>$\mu^3$</td>
<td>$w_2w_3$</td>
<td>$w_3w_1$</td>
<td>$w_1w_2$</td>
<td>$m_3m_2$</td>
<td>$m_1m_3$</td>
<td>$m_2m_1$</td>
</tr>
</tbody>
</table>

exhibit a lattice structure, albeit under an alternative ordering.

Example 12 (SA and RDS). This example shows that SA and RDS cannot be readily substituted for one another while ensuring a dynamically stable matching.

Consider the economy summarized in Table B.5. All preferences satisfy SIC, but there does not exist a dynamically stable matching. The preferences of $m_1$, $m_2$, and $w_2$ satisfy SA. The preferences of $m_2$, $w_1$, and $w_2$ satisfy RDS.

Table B.5: Preferences in Example 12.

<table>
<thead>
<tr>
<th>Agent</th>
<th>Preference ($\succ_i$)</th>
<th>Inertia</th>
<th>SIC</th>
<th>SA</th>
<th>RDS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_1$</td>
<td>$w_1w_1, w_1w_2, w_1m_1, w_2w_2, m_1m_1$</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td></td>
</tr>
<tr>
<td>$m_2$</td>
<td>$w_2w_2, m_2m_2$</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>$w_1$</td>
<td>$m_1w_1, w_1w_1$</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td></td>
</tr>
<tr>
<td>$w_2$</td>
<td>$m_1m_1, m_2m_1, m_2m_2, w_2w_2$</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
</tbody>
</table>

Example 13 (SIC and AA). This example shows that SIC and AA cannot be readily substituted for one another while ensuring a dynamically stable matching.

Consider the economy summarized in Table B.6. This economy does not have dynamically stable matching. The preferences of $m_1$ and $w_1$ satisfy SIC. The preferences of $w_1$ and $w_2$ satisfy AA. All preferences satisfy RDS.
Table B.6: Preferences in Example 13.

<table>
<thead>
<tr>
<th>Agent</th>
<th>Preference ($\succ_i$)</th>
<th>SIC</th>
<th>SA</th>
<th>RDS</th>
<th>AA</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_1$</td>
<td>$w_1w_2, w_2w_2, w_1w_1, m_1m_1$</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>$w_1$</td>
<td>$m_1m_1, w_1w_1$</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>$w_2$</td>
<td>$w_2m_1, w_2w_2$</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
</tbody>
</table>

Example 14. There is one man and three women whose preferences satisfy SIC and RDS.

\[
\succ_{m_1} : \frac{w_1w_3}{\mu^2}, w_1w_2, m_1w_2, w_2w_2, m_1m_1 \\
\succ_{w_1} : m_1w_1, w_1w_1 \\
\succ_{w_2} : w_2m_1, m_1m_1, \frac{w_2w_2}{\mu^2} \\
\succ_{w_3} : w_3m_1, m_1m_1, w_3w_3
\]

There are two dynamically stable matchings, marked in the preference list above. Neither matching Pareto dominates the other. The PDAA identifies $\mu^1$, which is underlined. It matches $m_1$ and $w_2$ only for period 2. This outcome suggests an obvious adjustment: $m_1$ and $w_1$ should partner for period 1 holding fixed their period 2 pairings. Following this adjustment, $m_1$’s incentives for period 2 change and the resulting assignment is unstable.

Example 15. This example exhibits ex ante and dynamically stable matchings that are not Pareto optimal. Consider the following economy with two men and women whose preferences are

\[
\succ_{m_1} : w_1w_2, w_2w_2, w_1w_1, m_1m_1 \\
\succ_{m_2} : w_2w_1, w_1w_1, w_2w_2, m_2m_2 \\
\succ_{w_1} : m_1m_2, m_2m_2, m_1m_1, w_1w_1 \\
\succ_{w_2} : m_2m_1, m_1m_1, m_2m_2, w_2w_2
\]

The agents’ preferences satisfy SIC, SA, and RDS. There are exactly two ex ante and dynamically stable matchings (Table B.7). The matching $\mu^1$ Pareto-dominates $\mu^2$. The PDA and the PDAA procedure identify the $\mu^2$ matching.

Example 16. This example shows that if agents’ preferences satisfy SIC and RDS, then every matching mechanism that identifies a dynamically stable outcome (if one exists) can be manipulated by at least one man and at least one woman when preference reports are unrestricted.
Table B.7: All ex ante and dynamically stable matchings in Example 15.

<table>
<thead>
<tr>
<th>Matching</th>
<th>$m_1$</th>
<th>$m_2$</th>
<th>$w_1$</th>
<th>$w_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu^1$</td>
<td>$w_1w_2$</td>
<td>$w_2w_1$</td>
<td>$m_1m_2$</td>
<td>$m_2m_1$</td>
</tr>
<tr>
<td>$\mu^2$</td>
<td>$w_2w_2$</td>
<td>$w_1w_1$</td>
<td>$m_2m_2$</td>
<td>$m_1m_1$</td>
</tr>
</tbody>
</table>

Consider the following market with three men and three women:

$\succ_{m_1}: [w_1w_2], w_2w_2, m_1m_1$

$\succ_{m_2}: w_2w_2, w_1w_3, [w_2w_3], m_2m_2$

$\succ_{m_3}: w_2w_2, [w_3w_1], w_1w_1, m_3m_3$

$\succ_{w_1}: m_1m_1, m_2m_2, [m_1m_3], w_1w_1$

$\succ_{w_2}: [m_2m_1], m_1m_1, w_2w_2$

$\succ_{w_3}: m_1m_1, [m_3m_2], m_2m_2, w_3w_3$

Table B.8: All dynamically stable matchings in Example 16.

<table>
<thead>
<tr>
<th>Matching</th>
<th>$m_1$</th>
<th>$m_2$</th>
<th>$m_3$</th>
<th>$w_1$</th>
<th>$w_2$</th>
<th>$w_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu^1$</td>
<td>$w_1w_2$</td>
<td>$w_2w_3$</td>
<td>$w_3w_1$</td>
<td>$m_1m_3$</td>
<td>$m_2m_1$</td>
<td>$m_3m_2$</td>
</tr>
<tr>
<td>$\mu^2$</td>
<td>$w_2w_2$</td>
<td>$w_1w_3$</td>
<td>$w_3w_1$</td>
<td>$m_2m_3$</td>
<td>$m_1m_1$</td>
<td>$m_3m_2$</td>
</tr>
<tr>
<td>$\mu^3$</td>
<td>$w_2w_2$</td>
<td>$m_2m_2$</td>
<td>$m_3m_3$</td>
<td>$w_1w_1$</td>
<td>$m_1m_1$</td>
<td>$w_3w_3$</td>
</tr>
</tbody>
</table>

There are three dynamically stable matchings as summarized in Table B.8. Matching $\mu^1$ is boxed in the preference list. Matching $\mu^2$ is underlined. Matching $\mu^3$, which corresponds to the PDAA matching, is not highlighted. Thus, there are three cases to consider.

1. Consider a matching mechanism that selects the matching $\mu^1$ if all agents truthfully report their preferences.
(a) If $m_2$ claims the preference profile

\[ \hat{\succ} \_{m_2} : w_1 w_3, w_3 m_2, w_3 w_3, m_2 m_2 \]

and all others truthfully report their preferences, then the economy's only dynamically stable matching coincides with $\mu^2$, which $m_2$ prefers.

(b) If $w_1$ claims the preference profile

\[ \hat{\succ} \_{w_1} : m_2 m_3, m_1 w_1, m_1 m_1, w_1 w_1 \]

and all others truthfully report their preferences, then the economy's only dynamically stable matching coincides with $\mu^2$, which $w_1$ prefers.

2. Consider a matching mechanism that selects the matching $\mu^2$ if all agents truthfully report their preferences.

(a) If $m_1$ claims the preference profile

\[ \hat{\succ} \_{m_1} : w_1 w_2, w_1 m_1, w_1 w_1, w_2 m_1, w_2 w_2, w_3 m_1, w_3 w_3, m_1 m_1 \]

and all others truthfully report their preferences, then the economy's only dynamically stable matching coincides with $\mu^1$, which $m_1$ prefers.

(b) If $w_2$ claims the preference profile

\[ \hat{\succ} \_{w_2} : m_2 m_1, m_1 w_2, m_1 m_1, m_2 w_2, m_2 m_2, m_3 w_2, m_3 m_3, w_2 w_2 \]

and all others truthfully report their preferences, then the economy's only dynamically stable matching coincides with $\mu^1$, which $w_2$ prefers.

3. Consider a matching mechanism that selects the matching $\mu^3$ if all agents truthfully report their preferences. In this case, any of the above manipulations benefit the responsible agent.

Every matching mechanism that selects a dynamically stable matching must select one of the above three matchings in the economy above. In every case there exists one man and one woman who
can successfully manipulate the mechanism for their advantage by communicating a preference that does not satisfy SIC and RDS.
B.5 The Deferred Acceptance Algorithm

We often reference the deferred acceptance algorithm of Gale and Shapley (1962). Though well-known, we review this procedure below as it applies to a one-period market. Each man $m$ (woman $w$) has a strict preference ranking, $P_m$ ($P_w$), over potential partners in $W_m$ ($M_w$). If $iP_mj$, then $m$ strictly prefers $i$ to $j$.

**Definition 37.** The (man-proposing) deferred acceptance algorithm constructs a (one-period) matching $\mu$ as follows:

1. In round 1, each man proposes to his most preferred partner as defined by $P_m$. (If $mP_mw$ for all $w \in W$, he does not make any proposals.) Given all received proposals, each woman engages her most preferred partner as defined by $P_w$ and rejects the others. All proposals from unacceptable partners (i.e. ranked below $w$ by $P_w$) are rejected.

2. More generally, in round $t$, each man whose proposal was rejected in the previous round proposes to his most preferred partner who has not yet rejected him. If all such partners are unacceptable, he does not make any proposals. Out of the set of new proposals and her current engagement (if any), each woman engages her most preferred partner and rejects the others. If all proposals are unacceptable, she rejects them all.

The above process stops once no further rejections occur. At that time all engaged pairs are matched and agents without a partner remain single (i.e. are matched to themselves).

The woman-proposing deferred acceptance algorithm is identical to the procedure described above with the roles of men and women reversed. The next example illustrates the algorithm’s operation.

**Example 17.** Let $M = \{m_1, m_2, m_3\}$ and $W = \{w_1, w_2, w_3\}$. The agents’ preferences are:

- $P_{m_1} : w_2, w_1, m_1$
- $P_{m_2} : w_1, w_3, m_2$
- $P_{m_3} : w_1, w_2, m_3$
- $P_{w_1} : m_1, m_2, m_3, w_1$
- $P_{w_2} : m_3, m_1, w_2$
- $P_{w_3} : m_2, w_3$
That is, \( m_1 \) prefers \( w_2 \) to \( w_1 \). He prefers either to being single. \( w_3 \) is not acceptable.

Table B.9 summarizes the round-by-round operation of the man-proposing deferred acceptance algorithm. To read the table, in round 1, \( m_2 \) and \( m_3 \) propose to \( w_1 \). She engages \( m_2 \), who is underlined, and \( m_3 \) is rejected. \( m_1 \) proposes to \( w_2 \) and is engaged. No one proposes to \( w_3 \). Eventually we arrive at the final matching:

\[
\mu(m_1) = w_1 \quad \mu(m_2) = w_3 \quad \mu(m_3) = w_2 \\
\mu(w_1) = m_1 \quad \mu(w_2) = m_3 \quad \mu(w_3) = m_2
\]

**Table B.9:** Round-by-round operation of the deferred acceptance algorithm in Example 17. Engaged partners are underlined.

<table>
<thead>
<tr>
<th>Round</th>
<th>Proposals Received</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( m_2, m_3 )</td>
</tr>
<tr>
<td>2</td>
<td>( m_2 )</td>
</tr>
<tr>
<td>3</td>
<td>( m_1, m_3 )</td>
</tr>
<tr>
<td>4</td>
<td>( m_1 )</td>
</tr>
</tbody>
</table>

B.5.1 The Spot-Market Deferred Acceptance Procedure (Example 5)

Example 5 identifies a case where the spot-market deferred acceptance procedure generates an unstable outcome. Here we provide details concerning that market’s operation.

Let \( M = \{m_1, m_2, m_3\} \) and \( W = \{w_1, w_2, w_3\} \). The agents’ preferences are:

\[ \succ_m^1 : w_2 w_2, w_1 w_2, w_1 w_1, \ldots \quad \succ_{w_1} : m_1 m_1, m_2 m_2, m_3 m_3, m_1 m_2, m_1 m_3, \ldots \]

\[ \succ_m^2 : w_1 w_1, w_3 w_3, w_2 w_1, \ldots \quad \succ_{w_2} : m_3 m_3, m_1 m_1, m_3 m_1, \ldots \]

\[ \succ_m^3 : w_1 w_1, w_2 w_1, w_2 w_2, \ldots \quad \succ_{w_3} : m_2 m_2, \ldots \]

Given \( \succ_i \) we can define each agent’s ex ante spot ranking:

\[ P_{\succ_{w_1}} : w_2, w_1, \ldots \quad P_{\succ_{w_1}} : m_1, m_2, m_3, \ldots \]

\[ P_{\succ_{w_2}} : w_1, w_3, \ldots \quad P_{\succ_{w_2}} : m_3, m_1, \ldots \]

\[ P_{\succ_{w_3}} : w_1, w_2, \ldots \quad P_{\succ_{w_3}} : m_2, \ldots \]
These same preferences appear in Example 17. Constructing $\tilde{\mu}_1$ via the man-proposing deferred acceptance algorithm gives:

\[
\begin{align*}
\tilde{\mu}_1(m_1) &= w_1 & \tilde{\mu}_1(m_2) &= w_3 & \tilde{\mu}_1(m_3) &= w_2 \\
\tilde{\mu}_1(w_1) &= m_1 & \tilde{\mu}_1(w_2) &= m_3 & \tilde{\mu}_1(w_3) &= m_2
\end{align*}
\]

At $\tilde{\mu}_1(\cdot)$, agents’ conditional spot rankings are:

\[
\begin{align*}
p^w_{w_1} &: w_2, w_1, \ldots & p^m_{w_1} &: m_1, m_2, m_3, \ldots \\
p^w_{w_2} &: w_3, w_1, \ldots & p^m_{w_2} &: m_3, m_1, \ldots \\
p^w_{w_3} &: w_1, w_2, \ldots & p^m_{w_3} &: m_2, \ldots
\end{align*}
\]

Using the above spot rankings we can construct $\tilde{\mu}_2$ via the man-proposing deferred acceptance algorithm. In this case, the resulting period 2 assignment is:

\[
\begin{align*}
\tilde{\mu}_2(m_1) &= w_2 & \tilde{\mu}_2(m_2) &= w_3 & \tilde{\mu}_2(m_3) &= w_1 \\
\tilde{\mu}_2(w_1) &= m_3 & \tilde{\mu}_2(w_2) &= m_1 & \tilde{\mu}_2(w_3) &= m_2
\end{align*}
\]

And the resulting multi-period matching is:

\[
\begin{align*}
\tilde{\mu}(m_1) &= w_1 w_2 & \tilde{\mu}(m_2) &= w_3 w_3 & \tilde{\mu}(m_3) &= w_2 w_1 \\
\tilde{\mu}(w_1) &= m_1 m_3 & \tilde{\mu}(w_2) &= m_3 m_1 & \tilde{\mu}(w_3) &= m_2 m_2
\end{align*}
\]

This matching is neither ex ante nor dynamically stable. For example, $m_1$ and $w_2$ can period-1 block $\tilde{\mu}$ since $w_2w_2 \succ_{m_1} \tilde{\mu}(m_1)$ and $m_1 m_1 \succ_{w_2} \tilde{\mu}(w_2)$. 

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Appendix C

Appendix to Chapter 3

Claim 1 The maximal subset of \( \tilde{Y} \) with only unrejected contracts with a doctor, say \( d_1 \) is uniquely defined.

Proof Consider two sets \( \hat{Y}_1 \) and \( \hat{Y}_2 \) both being the maximal subsets as defined above in definition 24. We also have \( \hat{Y}_1 \neq \hat{Y}_2 \) and they can not ordered by set inclusion. Define \( \hat{Y} = \hat{Y}_1 \cup \hat{Y}_2 \). From the definition above we know that \( \hat{Y}_1 \setminus [\hat{Y}_1]_{d_1} = \hat{Y} \setminus \hat{Y}_{d_1} = \hat{Y}_2 \setminus [\hat{Y}_2]_{d_1} = \hat{Y} \setminus \hat{Y}_{d_1} \). Moreover \( \hat{Y} \) also satisfies the second requirement which would mean that neither \( \hat{Y}_1 \) nor \( \hat{Y}_2 \) are maximal. To prove a contradiction suppose not. There exists a \( Y'' \subseteq \hat{Y} \) such that \( d_1 \in d(Y'') \) but \( d_1 \notin d[C_h(Y'')] \). Consider any contract with doctor \( d_1 \) in such a \( Y'' \) and let us call it \( z \). It is clear that \( z \in \hat{Y}_1 \) or \( z \in \hat{Y}_2 \). Consider the set \( Y_1 = Y'' \setminus [Y'']_{d_1} \cup \{z\} \). By IRC, we have \( d_1 \notin d[C_h(Y_1)] \). However, since \( Y'' \setminus [Y'']_{d_1} \subseteq \hat{Y} \setminus \hat{Y}_{d_1} \) we would have \( Y_1 \subseteq \hat{Y}_1 \) or \( Y_1 \subseteq \hat{Y}_2 \). This would violate the second requirement for the above definition for either \( \hat{Y}_1 \) or \( \hat{Y}_2 \) and this would contradict our assumption.

Claim 2 There exists a preference relation \( P^i_h \) over the subsets of \( X_h \) such that \( C^i_h \) defined in step \( i \) in the completion algorithm 3.2.1 corresponds to \( P^i_h \).

Proof We provide a proof by induction. This statement is trivially true by assumption for \( i = 0 \). Assume that it holds for \( i - 1 \) for \( i \geq 1 \). We prove that it holds for \( i \) constructively.

i.1 Define \( \hat{P} \equiv P^{i-1}_h \) and \( \hat{C}(\cdot) \equiv C^{i-1}_h(\cdot) \) for the hospital \( h \) corresponding to the preference relation \( P^{i-1}_h \). Also define \( \hat{X} \equiv 2^{X_h} \setminus \{\emptyset\} \).
i.2 If $\hat{X} = \emptyset$ then $P_i^j \equiv \hat{P}$ and Exit this step $i$ in the completion algorithm 3.2.1 and go to next step $i + 1$. Else choose $\tilde{Y} \in \hat{X}$ and go to step $i.3$.

i.3 Redefine $\hat{X} \equiv \hat{X} \setminus \{\tilde{Y}\}$.

i.4 If $d(z_i) \notin d[\hat{C}(\tilde{Y})]$ then go to step $i.2$. Else go to step $i.5$.

i.5 Find the required maximal subset $\hat{Y}$ for $\tilde{Y}$ and define $Y'' = \hat{C}(\hat{Y}) \cup \hat{Y}_{d(z_i)}$ and go to step $i.6$.

i.6 If $\hat{C}(\hat{Y}) = Y''$ then go to step $i.2$. Else go to step $i.7$.

i.7 Define a preference relation $\hat{P}$ by moving $Y''$ so that it is (just) better than $\hat{C}(\hat{Y})$, i.e. $\forall \hat{Y} \neq Y''$ and $\subseteq X_h$ we should have the following.

$$\hat{Y} \succ Y'' \text{ under } \hat{P} \iff \hat{Y} \succ \hat{C}(\hat{Y}) \text{ under } \hat{P} \quad \text{(C.1)}$$

Clearly $\hat{C}(\hat{Y}) = Y''$ under $\hat{P}$. Go to next step $i.8$.

i.8 Define $\hat{P} \equiv \hat{P}$ and $\hat{C}(\cdot)$ corresponds to the new preference relation $\hat{P}$. Go to step $i.2$.

The above construction takes each non-empty subset of $X_h$ one at a time and modifies the preference relationship iteratively.

In a given step $i$ where the preferences are being modified to fix the substitutability violation concerning the doctor $d(z_i)$ the following statements follow immediately from the algorithm:

- Preference is modified only for sets where a $d(z_i)$ contract is in the chosen set.

- When preference is modified for such sets $\hat{Y}$ all $d(z_i)$ contracts in $\hat{Y}$ are added to the new chosen set by moving this set as being better than the old chosen set.

- Once the set is moved in the preference relation to be better than the old chosen set then in the subsequent iterations of the algorithm above it continues to be better than the old chosen set, which in turn was better than all the feasible and infeasible subsets of the set, in a given step.
Thus at the end of the algorithm above, we have a preference relation \( P^i_h \) and a choice function \( C^i_h(\cdot) \) which agrees with the description in step \( i \) of the completion algorithm.

**Proof of Proposition 6** We first prove that at step \( i \), with a substitutability violation \((x_i, z_i, Y_i)\), there exist no substitutability violation \((\tilde{x}_i, \tilde{z}_i, \tilde{Y}_i)\) under \( P^i_h \) such that \( d(\tilde{z}_i) = d(z_i) \). Consider such a substitutability violation involving the doctor of the recalled contract under \( P^i_h \). Using the same technique as in proof of Lemma 6, using part I we can claim that \( \exists \tilde{z}_i \in C^{i-1}_h(\tilde{Y}_i \cup \{z_i\}) \) such that \( d(\tilde{z}_i) = d(z_i) \). Proceeding along the same lines of part II, we would establish that for the maximal subset \( \tilde{Y}'_i \) of \( \tilde{Y}_i \), \( \tilde{z}_i \in \tilde{Y}'_i \) and hence is added in step \( i \). The substitutability violation no longer exists after step \( i \). Thus \((\tilde{x}_i, \tilde{z}_i, \tilde{Y}_i)\) is not a substitutability violation under \( P^i_h \). Thus at each step all the violations involving a given doctor are fixed. By Lemma 5, we know that no new violations are created and hence there will be no new violations involving the doctor \( d(z_i) \) in the subsequent steps. This proves that the algorithm above identifies up to \(|D|\) violations and completes in \(|D|\) steps.