On the Framed Singular Instanton Floer Homology From Higher Rank Bundles

On the Framed Singular Instanton Floer Homology from Higher Rank Bundles

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Abstract

In this thesis we study the framed singular instanton Floer homology defined by by Kronheimer and Mrowka in [12]. Given a 3-manifold Y with a link K and $\delta \in H^2(Y, \mathbb{Z})$ satisfying a non-integral condition, they define the singular instanton Floer homology group $I^N(Y, K, \delta)$ by counting singular flat $PSU(N)$-connections with fixed holonomy around K. Take a point $x \in Y \backslash K$, classical point class operators $\mu_i(x)$ of degree $2i$ on $I^N(Y, K, \delta)$ can be defined as in the original Floer theory defined by smooth connections. In the singular instanton Floer homology group $I^N_*(Y, K, \delta)$, there is a special degree 2 operator $\mu(\sigma)$ for $\sigma \in K$. We study this new operator and obtain a universal relation between this operator and the point class operators $\mu_i(x)$. After restricted to the reduced framed Floer homology $F\bar{I}^N_*(Y, K)$, these point classes operators $\mu_i(x)$ become constant numbers related to the $PSU(N)$-Donaldson invariants of four-torus T^4. Then the universal relation becomes a characteristic polynomial for the operator $\mu(\sigma)$ so that we can understand the eigenvalues of $\mu(\sigma)$ and decompose the Floer homology as eigenspaces.

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1 Introduction

The classical Donaldson theory is based the $SU(2)$-bundles (or more generally $U(2)$-bundles) . By studying the moduli spaces of anti-self-dual connections of $SU(2)$ bundles, Donaldson defines his polynomial invariants for closed 4-manifolds. There is also the corresponding Floer theory which be thought as a generalization of Donaldson theory when we are working with manifolds with boundaries. The $SU(N)$-Donaldson invariants are defined by Kronheimer in [7] and the corresponding Floer theory is developed by Kronheimer and Mrowka in [12]. In [12] they develop the Floer theory based on singular connections with fixed holonomy around a codimension 2 submanifold. We generalize some properties of the $SU(2)$-Donaldson invariants in [8] to the higher rank case and also study the effect on the corresponding Floer theory.

Take a triple (Y, K, δ) we denote the singular instanton Floer homology group by $I^N(Y, K, \delta)$. It carries a relative $\mathbb{Z}/(2N)$ grading. For any point $x \in Y \backslash K$ we have the point class operator $\mu_i(x)$ of degree $2i$ which does not depend on the choice of the point x. We also have an operator $\mu(\sigma)$ (or just denote it by σ) of degree 2 for any $\sigma \in K$. This operator only depends on the choice of component of K where σ lies. We obtain a universal relation between these operators. When we are working with the reduced framed singular Floer homology $F\bar{I}(Y, K)$, it has the following simple form

$$\sigma^N + m_{N,2}\sigma^{N-2} + \cdots + m_{N,N} = (1 + (-1)^N)\,\mathrm{id} \tag{1.1}$$

where $m_{N,i}$ are rational numbers. Based on this formula we can understand the eigenvalues of σ and obtain a decomposition of the Floer homology group.

Our motivation to study this operator comes from the Khovanov-Rozansky homology. Given a knot or link K in S^3, there is the Khovanov-Rozansky homology $KR^N(K)$ [6] which

is a bi-graded $\mathbb{Q}$ vector space. For each component e of K, there is a degree 2 operator X_e on $KR^N(K)$ that satisfies $X_e^N = 0$. This structure is similar to the σ-operator in the singular instanton Floer homology of K. The definition of Khovanov-Rozansky homology is based on the technique of matrix factorization. Khovanov-Rozansky homology is defined not only for knots or links but also for graphs with trivalent singularities. For a planar trivalent diagram (we will call it planar web) Γ we have the dimension of $KR^N(^\Gamma) = \mathbb{Q}\{MOY states\}$ is equal to the number of MOY states, where MOY states [18] are certain labeling of the edges of the diagram.

In Khovanov and Rozansky's definition, a potential is chosen in order to define the matrix factorizations. The potential used is x^{N+1} (so the derivative is $(N+1)x^N$) and this choice of potential leads to the vanishing of X_e^N. By perturbing the potential, Gornik [4] defines a variant of Khovanov-Rozansky homology. The potential used by Gornik is $x^{N+1} - (N+1)\beta^N x$ (the derivative is $(N+1)(x^N - \beta^N)$ with roots $\beta\xi^l$ where $\xi = e^{2\pi i/N}$). Gornik's homology for planar webs is isomorphic to Khovanov-Rozansky's homology. But for links it only depends on the number of componens. For each component e we have an operator X_e on Gornik's homology $G(L)$ and all these operators commute and satisfy $X_e^N - \beta^N = 0$. $G(K)$ can be decomposed as the direct sum of all the common eigenspaces of these operators with each eigenspace of exactly dimension 1. So $G(K) \cong \mathbb{C}^{N^{|L|}}$. By filtering the chain complex used to define $G(K)$, the lower degree term $(N+1)\beta^N x$ of the potential is killed so the original $KR(K)$ appears. In this way Gornik shows that there is a spectral sequence whose E_2 page is $KR(K)$ and abuts to $G(K)$. This is generalized by Wu [20] to any potential whose derivative has N distinct roots.

After being generalized to the case of webs, conjecturally the reduced singular instanton Floer homology for planar web is also isomorhphic to the Khovanov-Rozansky homology with a perturbed potential ($\int x^N + m_{N,2}x^{N-2} + \cdots + m_{N,N} - (1 + (-1)^N)$). This should be

the first step the obtain a similar spectral sequence from Khovanov-Rozansky homology.

This thesis is organized as follows. In Sections 2 and 3 we review some background on the Donaldson invariants from higher rank bundles and the moduli space of singular anti-self-dual connections. In Section 4 we obtain the universal relation over closed 4-manifolds but with undetermined universal constants. In order to determine these constants, in Section 5 we calculate concrete examples of Donaldson invariants from singualr instantons by the correspondence between singular instantons and stable parabolic bundles. In Section 6 we adapt the unversal relation from the closed manifolds to the relative case: a relation between operators on the Floer homology. After restricted to the reduced framed Floer homology $F\bar{I}^N_*(Y,K)$, these point classes operators $\mu_i(x)$ become constant numbers related to the $SU(N)$-Donaldson invariants of four-torus T^4. In Section 7 we construct the moduli space of stable bundles over an abelian surface as well as the universal bundles based some work of Mukai. By Donaldson's theorem this is the same as the moduli space of instantons. So we can use these moduli spaces to calculate the Donaldson invariants. More precisely we can reduce the calculation to some calculation over Hilbert scheme of points in an abelian surface. We are able to obtain a complete answer when $N = 3$. Finally in Section 8 we do some calculation for unknots and unlinks.

2 Construct the invariants

Let P be a $U(N)$ bundle over a 4-manifold X and $\mathscr{R}^*$ be the configuration space of all irreducible connections with a fixed determinant connection. The universal $U(N)$ bundle $\mathbb{P}$ over $X \times \mathscr{R}^*$ may not always exit, but the universal $PU(N)$ bundle $\operatorname{ad}\mathbb{P}$ always exists. Taking the pontryagin classes of the adjoint bundle $\mathfrak{su}_{\mathbb{P}}$ and use the slant product we can

obtain homology classes over the configuration space

$$\mu_i : H_j(X, \mathbb{Q}) \to H^{4i-j}(\mathscr{R}^*, \mathbb{Q})$$

$$\alpha \qquad \mapsto p_i(\mathfrak{su}_{\mathbb{P}})/\alpha$$

If the moduli space of anti-self-dual connections is regular, compact and contains no reducibles, we can take the cup product of these classes and pair it with the fundamental class of the moduli space to define the polynomial invariants. In general the moduli space may not be compact. So we need to take the geometrical representatives of some mutiples of these classes and deal with the bubbles very carefully to obtain a compact intersection with the moduli space. This is contained in [3] and [7]. But there is some defect in this definition: it does not work for p_i when $i \geq N$. Here we try to modify this definition to make it work for our purpose. For simplicity we only want to focus on the point classes.

First notice that we have the fibration

$$B\mathbb{Z}_N \to BSU(N) \to BPU(N)$$

Since the rational cohomology of $B\mathbb{Z}_N$ is same as a point, the rational cohomologies of $BSU(N)$ and $BPU(N)$ are isomorphic. This means if we are working with cohomology with rational coefficients the any characteristic classes we can define for $PSU(N)$ bundles must be a polynomial in $\bar{c}_2, \cdots, \bar{c}_N$ which correspond to the Chern classes of $SU(N)$ bundles. In particular the Pontryagin classes of $\mathfrak{su}_{\mathbb{P}}$ are combinations of $\bar{c}_i$'s.

If we want to define the polynomial invariants we also need to construct the geometrical representatives. We know how to construct the geometrical representatives of Chern classes of a complex vector bundle. If our $PU(N)$ bundle comes from a $SU(N)$ bundle, then we can

take the associated bundle E of the standard representation of $SU(N)$. We have $c_i(E) = \bar{c}_i$. In general the $PU(N)$ bundle may not come from a $SU(N)$ bundle so E may not exist. But we can consider the tensor product of N copies of the standard representation of $SU(N)$ which descends to a representation of $PU(N)$. We denote the associated bundle of this representation by H. We have

Proposition 2.1. *The classes $\bar{c}_2, \cdots, \bar{c}_N$ can be expressed as rational polynomials in $c_2(H), \cdots c_N(H)$*

.

Proof. Consider the universal case: let $\mathbb{F}$ and $\mathbb{G}$ be the universal principal bundle over $BSU(N)$ and $BPU(N)$ respectively.

$$\begin{array}{ccc} \mathbb{F} & \longrightarrow & \mathbb{G} \\ \downarrow & & \downarrow \\ BSU(N) & \longrightarrow & BPU(N) \end{array}$$

Take the standard representation of $SU(N)$ and denote the associated bundle by $\mathbb{E}$. Since tensor product of N copies of the standard representation of $SU(N)$ descends to a representation of $PU(N)$, denote the associated bundle by $\mathbb{H}$ we have the pullback diagram

$$\begin{array}{ccc} \mathbb{E}^{\otimes N} & \longrightarrow & H \\ \downarrow & & \downarrow \\ BSU(N) & \longrightarrow & BPU(N) \end{array}$$

From

$$\mathrm{ch}(\mathbb{E}^{\otimes N}) = \mathrm{ch}(\mathbb{E})^N$$

we know the Chern characters of $\mathbb{E}^{\otimes N}$ and $\mathbb{E}$ can determine each other. The first N Chern classes and the first N terms of Chern characters determine each other. So we have the first N Chern classes of $\mathbb{E}^{\otimes N}$ determine the Chern classes of $\mathbb{E}$. Now by the pullback diagram

and the naturality of Chern classes $c_i(H)$ $(0 < i \leq N)$ can generate the cohomology ring of $BPU(N)$. $\square$

Now we can use the geometrical representatives of $c_i(H)$ $(0 < i \leq N)$ to define the polynomial invariants. The good thing is that the degree of $c_N(H)$ is smaller than $4N$ so the counting argument still works for this case which guarantees the compactness of the intersection.

3 Moduli space of singular instantons

Let X be a 4-manifold and Σ be an embedded surface in X. Let E be a $SU(N)$ bundle over X. $E|_\Sigma$ will be reduced to a $S(U(1) \times \cdots \times U(N-1)$ We want to consider singular $SU(N)$-connections which is smooth over $X \backslash \Sigma$ and have asymptotic holonomy

$$\exp(-2\pi i \operatorname{diag}(\lambda, -\frac{\lambda}{N-1}, \cdots, -\frac{\lambda}{N-1})) \tag{3.1}$$

around Σ where $0 < \lambda < (N-1)/N$. When $\lambda = (N-1)/N$ the holonomy is $e^{2\pi i/N}$ id. In this case, passing to $PU(N)$ we will obtain smooth connections.

The complete theory is developed in [12]. Here we restate some useful results in [12]. Let $E_\Sigma = L \oplus F$ be the $S(U(1) \times U(N))$ reduction where L is a $U(1)$ bundle. We call $k = c_2(E)[X]$ the instanton number and $l = -c_1(L)[\Sigma]$ the monopole number. The formal dimension of the moduli space of gauge-equivalence classes of ASD connections is

$$\dim M^\lambda_{k.l} = 4Nk + 2Nl - (N^2-1)(1 - b^1 + b^+) + (N-1)\chi(\Sigma) \tag{3.2}$$

We also have the energy formula

$$\begin{aligned}\text{Energy}(A) &= 2\int_{X\setminus\Sigma} -\operatorname{tr}(*\operatorname{ad} F_A \wedge \operatorname{ad} F_A) \\ &= 32\pi^2 N(k + \frac{N}{N-1}(\lambda l - \frac{1}{2}\lambda^2 \Sigma.\Sigma))\end{aligned}$$

We may also need to consider the $U(N)$ bundle case. In this case let E be a $U(N)$ bundle over X and $E|_\Sigma = L \oplus F$ be a $U(1) \times U(N-1)$ reduction. Now the instanton number is defined as

$$\begin{aligned}k &= -\frac{1}{2N} p_1(\operatorname{ad} E) \\ &= c_2(E) - \frac{N-1}{2N} c_1(E)^2\end{aligned}$$

where $\operatorname{ad} E$ is the adjoint $PU(N)$ bundle associated with E. And the monopole number is defined as

$$l = \frac{1}{N} c_1(E)[\Sigma] - c_1(L)[\Sigma] \tag{3.3}$$

In the $U(N)$ bundle case we need to fix the holonomy of the conections around Σ to be some h such that the projection of h into $PU(N)$ is the same as the projection into $PU(N)$ of the holonomy in the $SU(N)$ case. The gauge group $\mathscr{G}$ we use consists of gauge transformations of determinant 1 and respects the decomposition $E|_\Sigma = L \oplus F$ along Σ. The moduli space of singular instantons consist of gauge equivalent classes of projective ASD connections with fixed holonomy around Σ and fixed determinant. The dimension formula and the energy formula still works in this case.

3.1 Compactness and bubbles

Now we want to restate the compactness result in [12] we need. Let $[A_n]$ be a sequence of gauge-equivalence classes of ASD connections in $M^{\lambda}_{k,l}$. According to Propositon 2.9 in [12] there is an element $[A_\infty]$ in $M^{\lambda}_{k',l'}$ and a finite set of points $\mathbf{x} \subset X$ such that over $X \backslash \mathbf{x}$ we can find isomorphisms $g_n : E'|_{X\backslash \mathbf{x}} \to E$ so that $g_n^*(A_n)$ converges to A_∞ on compact subsets of $X\backslash \mathbf{x}$. For each $x \in \mathbf{x}$ we can assign numbers k_x and l_x which satisfies

- $k = k' + \sum_{x\in\mathbf{x}} k_x$
- $l = l' + \sum_{x\in\mathbf{x}} l_x$
- If $x \notin \Sigma$, then $k_x > 0$ and $l_x = 0$. If $x \in \Sigma$, then $M^{\lambda}_{k_x,l_x}(S^4, S^2)$ is not empty.

By Proposition 2.10 in [12] we must have $k_x \geq 0$ and $k_x + l_x \geq 0$.

From this dimension formula we can deduce the difference of the formal dimension between $M^{\lambda}_{k,l}$ and $M^{\lambda}_{k',l'}$ is as least $2N$. The extreme case only happens when there is only one bundle point $x \in \Sigma$ with $k_x = 1$, $l_x = -1$ or $k_x = 0, l_x = 1$.

4 A relation between point classes

4.1 Technical assumptions

Now we need to talk about the metric we use to define the moduli spaces. To define the invariants we can use the smooth metric over X. We also want to introduce the cone-like metrics which is modeled on

$$du^2 + dv^2 + dr^2 + (\frac{1}{\nu^2})r^2 d\theta^2$$

near Σ. This metric has a cone angle $2\pi/\nu$ where $\nu \geq 1$ is a real number. When ν is an integer then the metric is a orbifold metric. By a standard cobordism argument [[10], Theorem 2.13], it can be shown that the invariants do not depend on the choice of the holonomy parameter λ. Later we also need to use the gluing theorem to study the boundary of the moduli space. For this purpose we can take some integer v and also some special holonomy parameter λ so that the singular connections are orbifold connections: locally lifted to a branched ν cover the connections become smooth connections. In this case the usual gluing theorem still works. Another case is that we take ν to be a large enough integer so that we still have the "Fredholm package" so that we can still use the gluing result. Notice that in this case there is no restriction on λ: we don't require the connections to be orbifold connections [[12], Section 2.8]. In this case by a standard cobordism argument [[10], Theorem 2.13], it can be shown that the invariants do not depend on the choice of the holonomy parameter λ.

In order to define the invariant, we need to avoid the reducibles. We will assume $b^+(X) \geq 2$. We can achieve the non-integral condition in Proposition 2.19 in [12] by choosing some special parameter λ so that for a generic path of metrics there are no reducibles in the moduli space. Another way to avoid the reducibles is to use the trick in Section 7(iv) in [7]: blow up X at a point $p \notin \Sigma$ to obtain a new manifold

$$\tilde{X} = X \sharp \bar{\mathbb{C}P}^2$$

and replace E by $\tilde{E}$ such that $c_1(\tilde{E}) = c_1(E) + e$ and $c_2(\tilde{E}) = c_2(E)$ where e is the Poincaré dual of the exceptional class. In this case the non-integral condition is always satisfied so we can define the invariants for $\tilde{X}$ and use the blow-up formula to obtain a definition for invariants for X.

By a generic holonomy perturbation [7] [12], all the moduli spaces will become regular. We will keep assuming this when defining invariants.

4.2 Defining the invariants

By taking points away from Σ we can still define the point classes $\bar{c}_i/x$ in the singular connection case. In this case we have a 2-dimensional point class which does not appear in the non-singular case. The gauge group we use fixes the determinant of E and respects the decomposition $E|_\Sigma = L \oplus F$. So along Σ we have a $PS(U(1) \times U(N))$ reduction $\mathbb{S}$ of $\operatorname{ad}\mathbb{P}$. If in some good case the $U(N)$ bundle exists and the $U(1) \times U(N-1)$ reduction $\mathbb{P}'$ also exists, we can use $\mathbb{L}$ to denote the $U(1)$ component of $\mathbb{P}'$, then define a 2-dimensional point class

$$\epsilon = -c_1(\mathbb{L})/[\sigma] \in H^2(\mathscr{R}^*)$$

In the general case, we can still define this characteristic class for $PS(U(1) \times U(N))$ bundle in the same way as in Section 2. Take a special representation of $U(1) \times U(N-1)$: the tensor product of N copies of the standard action of the $U(1)$ factor. We can use this to representation to construct a associated line bundle of K and use the first Chern class of this line bundle to obtain the geometrical representative.

4.3 A universal relation

If we have a $SU(N)$ bundle M which decomposes as $V \oplus N$ where V is a line bundle, then we have

$$\begin{aligned} c_N(V^* \otimes M) &= c_N(M) + c_1(V^*)c_{N-1}(M) + \cdots + c_1(V^*)^N & (4.1) \\ &= 0 & (4.2) \end{aligned}$$

According to the definition of $\bar{c}_i$ and ϵ we have

$$\bar{c}_N(\mathrm{ad}\,\mathbb{P}_x) + \epsilon c_{N-1}(\mathrm{ad}\,\mathbb{P}_x) + \cdots + \epsilon^N = 0 \tag{4.3}$$

in $H^*(\mathscr{R}^*)$

Let $\sigma \in \Sigma$ and (B, D) be a standard neighbourhood pair in (X, Σ). Let $\mathscr{R}^{*,\lambda}_\sigma$ be the configuration space of irreducible singular connections in (B, D). We can use the representation described above to obtain a line bundle $\mathbb{K}$. Let

$$M^\lambda_{k,l} \to \mathscr{R}^{*,\lambda}_\sigma$$

be the restriction map. Take a generic section of $\mathbb{K}$ and denote the zeros by V_σ then $M^\lambda_{k,l} \cap V_\sigma$ will be the Poincaré dual of $N\epsilon$.

Let $\{\sigma_i\}$ be different points on Σ, $\{u_i\}$ be embedded surfaces in generic positions (for simplicity we also suppose they are away from Σ). Suppose the total degree of the classes and the dimension of $M_{k,l}$ coincide then we can define the invariants

$$q_{k,l}(\sigma_1, \cdots, \sigma_m, x_{\bar{c}_p}, \cdots, x_{\bar{c}_q}, u_1, \cdots, u_s) \in \mathbb{Q} \tag{4.4}$$

by counting the number of points in $M_{k,l} \cap V_{\sigma_1} \cap \cdots \cap V_{u_s}$ where V_{u_i} represents $\bar{c}_2/u_i$, then divide the it by some factor associated to the representatives since when we define the representatives we may need to take some multiple of the original classes. The geometrical representatives are chosen to make all the possible intersections transversal. We should also notice that we don't construct the geometrical representatives $V_{x,\bar{c}_j}$ for $\bar{c}_j/x$ actually. Instead we should represent $\bar{c}_j/x$ by a rational polynomial of some actual representatives of $c_j(\mathbb{H})/x_i$ in Section 2(we take diffrent point x_i for every appearance of $c_j(\mathbb{H})$. In the

definition we should think $V_{x_i,\bar{c}_i}$ as a rational polynomial of $V_{x_i,c_j(\mathbb{H})}$ and use $V_{x_i,c_j(\mathbb{H})}$ to get the actual intersection. (4.4) only depends on the homology classes not on the choices of points and surfaces, sometimes we will just write σ^m without specifying m different points on Σ. The standard counting argument guarantees the compactness of the intersection.

Let $z = u_1 u_2 \cdots u_s$ be the formal product of embedded surfaces and V_z denote the intersection of $V_{u_i} s$. If $M \cap V_z$ is compact and has dimension $2N$, then by (4.3) we have

$$\begin{aligned} q_{k,l}((x_{\bar{c}_N} + \sigma x_{\bar{c}_{N-1}} + \cdots \sigma^N)z) &= \langle \bar{c}_N(\mathrm{ad}\,\mathbb{P}_x) + \epsilon c_{N-1}(\mathrm{ad}\,\mathbb{P}_x) + \cdots + \epsilon^N, [M \cap V_z] \rangle \\ &= 0 \end{aligned}$$

In general this formula is not true since the moduli space may not be compact. We need to consider the contribution from the bubbles carefully. The $N = 2$ case is already studied in [8]. We focus on the $N > 2$ case

Proposition 4.1. *Under the above assumption we have a formula*

$$q_{k,l}((x_{\bar{c}_N} + \sigma x_{\bar{c}_{N-1}} + \cdots \sigma^N)z) = m_1 q_{k,l-1}(z) + m_2 q_{k-1,l+1}(z) \tag{4.5}$$

where m_1 and m_2 are some constants which only depend on N.

Proof. We want to know what happens when V_z is not compact. Let $[A_n]$ be a sequence of gauge-equivalence classes of connections in $M \cap V_z$. Suppose $[A_n]$ weakly converge to a connection $A_\infty \in M_{k',l'} \cap V_z$. There is at most 1 bundle point: otherwise it's easy to show A_∞ lie in an intersection a negative dimensional by dimension-counting which contradicts our transversality assumption.

Now we denote the bubble point by x_∞. If x_∞ lie in $X \backslash \Sigma$ then it touches at most 2 surfaces u_i, u_j. $A_\infty \in M_{k',l'} \cap V_{z'}$ where z' is the formal product of surfaces in z except

u_i, u_j. $M_{k',l'} \cap V_{z'}$ has negative dimension hence empty. So the bubble x_∞ must lie on Σ.

There are only 2 possible cases: $(k', l') = (k, l-1)$ or $(k-1, l+1)$. And in both cases we have $A_\infty \in M_{k',l'} \cap V_z$ whose dimension is zero therefore consists of finitely many points. By adding these possible pairs (A_∞, x_∞) we can obtain a compactified space $\overline{M_{k,l} \cap V_z}$. If $x \in X \backslash \Sigma$ the point class $\bar{c}_i/x$ can be extend to this compactified space because we have well-defined restriction map

$$\overline{M_{k,l} \cap V_z} \to \mathscr{R}^*_x$$

Take a point $\sigma_i \in \Sigma$. Then we can get a extended restriction map

$$\overline{M_{k,l} \cap V_z} \backslash \{(A_\infty, \sigma) | \sigma \notin B(\sigma)\} \to \mathscr{R}^*_{\sigma_i}$$

where $B(\sigma)$ is the neighborhood of σ_i we use to define $\mathscr{R}^*_{\sigma_i}$. We can pull back the line bundle $\mathbb{K}$ which is used to define the 2-dimensional point class and try to extend it to $\overline{M_{k,l} \cap V_z}$. To do this we need to understand the neighborhood of (A_∞, σ_i).

Lemma 4.2. *Let $\lambda = (N-1)/2N$ and take a cone-line metric with cone-angle $2\pi/\nu$ over (S^4, S^2). Then we have $M^\lambda_{0,1}(S^4, S^2)$ and $M^\lambda_{1,-1}(S^4, S^2)$ are isomorphic to $\mathbb{R}^2 \times \mathbb{R}^+$ where the $\mathbb{R}^2$-factor means the "center" and the $\mathbb{R}^+$-factor means the "scaling". Any two elements in $M^\lambda_{0,1}(S^4, S^2)$ (or $M^\lambda_{1,-1}(S^4, S^2)$) differ by translations and re-scaling.*

If we take an $SU(2)$ singular instanton $A \in M^{\lambda'}_{0,1}(S^4, S^2)_{\text{rank}=2}$ where $\lambda' = \frac{N}{2(N-1)}\lambda$, a rank $SU(N)$ singular instanton in $M^\lambda_{0,1}(S^4, S^2)$ can be constructed by the following steps:

- Firstly twist A by a flat complex line bundle over $S^4 \backslash S^2$ with holonomy around S^2

$$\exp(-2\pi i \lambda')$$

to obtain a $U(2)$ singular connection A' with asymptotic holonomy around S^2

$$\mathrm{diag}(\exp(-4\pi i\lambda'), 1).$$

- Then use the standard inclusion $U(2) \to U(N)$ to obtain a $U(N)$ singular connection A'' with asymptotic holonomy around S^2

$$\mathrm{diag}(\exp(-4\pi i\lambda'), 1, \cdots, 1) = \mathrm{diag}(\exp(-2\pi i\frac{N}{N-1}\lambda), 1, \cdots, 1).$$

- Finally twist A'' by the flat complex line bundle with honolomy

$$\exp 2\pi i\frac{\lambda}{(N-1)}.$$

Then we obtain a $SU(N)$ singular instanton $\tilde{A}$ in $M^{\lambda}_{0,1}(S^4, S^2)$ around S^2.

Based on the lemma all the other singular instantons can be obtained by translation and re-scaling of $\tilde{A}$. From the construction it is easy to see that the stabilizer of $\tilde{A}$ is

$$\mathrm{Stab}_{\tilde{A}} \cong \{\mathrm{diag}(\alpha, \alpha, B) \in S(U(1) \times U(N-1)) | \alpha \in \mathbb{C}, B \in U(N-2)\}.$$

There is a similar construction for $M^{\lambda}_{1,-1}(S^4, S^2)$.

A neighborhood of (A_∞, σ_i) comes from grafting connections in $M^{\lambda}_{0,1}(S^4, S^2)$ or $M^{\lambda}_{1,-1}(S^4, S^2)$ to A_∞. The gluing parameter $S(U(1) \times U(N-1))/\mathrm{Stab}_A$ is isomorphic to S^{2n-3}.

So we have

$$Nbh(A_\infty, \sigma_i) \cap M_{k,l} \cap V_z \cong S^{2N-3} \times (0,1) \times D^2$$

where S^{2N-3} is the gluing parameter, $(0,1)$ is the scaling and $D^2 \subset \Sigma$ is the center.

$$Nbh(A_\infty, \sigma_i) \cong cone(S^{2N-3}) \times D^2 \cong D^{2N}$$

The boundary of the neighborhood is S^{2N-1}. Any line bundle over S^{2N-1} must be trivial so $\mathbb{K}$ can be extended to $\overline{M_{k,l} \cap V_z}$. This extended line bundle does not depend on the choice of σ_i since for any 2 points on Σ we can connect them by a path and get a one-parameter family of line bundles.

Now we can use the first Chern class of the extended $\mathbb{K}$ and $\bar{c_i}/x$ over $\overline{M_{k,l} \cap V_z}$ to calculate the left-hand side of (4.5).

Over $M_{k,l} \cap V_z$ the universal bundle $\operatorname{ad} \mathbb{P}_\sigma$ is reduced to a $P(U(1) \times U(N-1)$ bundle $\mathbb{S}$. If this reduction can be extended to the whole $\overline{M_{k,l} \cap V_z}$ then by (4.1) the left-hand side of (4.5) is zero. The obstruction for the extension is the homotopy class of

$$\phi : \partial Nbh(A_\infty, \sigma_i) \to \mathbb{O}|_{\partial Nbh(A_\infty, \sigma_i)}$$

where $\mathbb{O}$ is the associated $PU(N)/(P(U(1) \times U(N-1))) \cong \mathbb{C}P^{N-1}$ bundle of $\operatorname{ad} \mathbb{P}$. ϕ is determined by the reduction $\mathbb{S}$. This obstruction can be described by an integer (since $\pi_{2N-1}(\mathbb{C}P^{N-1}) = \mathbb{Z}$) and the contribution m from different $A_\infty \in M_{k,l-1}$ (or $M_{k-1,l+1}$) should be the same since the neighborhood comes from the grafting $M^\lambda_{0,1}(S^4, S^2)$ (or $M^\lambda_{1,-1}(S^4, S^2)$) to A_∞. □

Proof. Proof of the lemma. This can be read directly from the description of the moduli space $M_{0,1}$ by Murari [17] in or see the remark after Theorem 2.13 in [12]. Alternatively we can argue as the following.

To prove the lemma we first need to understand the smooth instantons over S^4.Let

E be a $SU(2)$ bundle over S^4 with $c_2[S^4] = 1$. Then the moduli space of anti-self-dual connections is isomorphic to $\mathbb{R}^+ \times \mathbb{R}^4$ where $\mathbb{R}^4$ means the center of the curvature in $\mathbb{R}^4$ and $\mathbb{R}^+$ means the scaling of the connection. Because of Unlenbeck's theorem of removable singularity there is no difference between ASD connections over S^4 and $\mathbb{R}^4$. Fix one ASD connection over $\mathbb{R}^4$, we can get any other ASD connection up to a gauge transform of instanton number 1 by translate and re-scale the fixed connection. If we are working with a $SU(N)$ bundle, any ASD connection of instanton number 1 is reducible and reduced to a $SU(2)$ bundle by Theorem 8.4 in [1].

Take $\lambda = (N-1)/(2N\nu)$. After lifting the singular instantons to a branched ν-cover $(\mathbb{R}^4, \mathbb{R}^2) \to (\mathbb{R}^4, \mathbb{R}^2)$ (branched along $\mathbb{R}^2$), it becomes a regular 1-instanton centered in $\mathbb{R}^2$. The instanton number is calculated by the energy formula (3.3) So the instanton comes from an $SU(2)$-instanton centered in $\mathbb{R}^2$. Therefore any two of them only differ by translation and re-scaling. For a general λ because moduli space with different parameters are isomorphic (Corollary 2.14 in [12]), the same result holds. □

Remarks. The assumption that z is a product of surface classes defined by $\bar{c}_2/u$ for u away from Σ is not necessary. We can take z as the product of surface classes defined by $\bar{c}_i/u$, point classes for $x \in X \backslash \Sigma$, points classes for $\sigma \in \Sigma$ and also curve classes defined by $c_1(\mathbb{L})/\alpha$ where $\alpha \subset \Sigma$.

5 Calculation of the universal coefficients

Since the two universal coefficients m_1, m_2 do not depend on the choice of the manifold X, we can calculate concrete examples to determine m_1, m_2.

We will show that $m_1 = 1$ and $m_2 = (-1)^N$. The basic technique is the correspondence between singular instantons and stable parabolic bundles over a Kähler pair (X, Σ)

described in the appendix. In our case stable parabolic bundle is a pair (E, L) where E is a holomorphic bundle over X, L is a subline bundle of $E|_\Sigma$. When we take the holonomy parameter small enough then the stability of the parabolic bundle (E, L) is same as the stability of the holomorphic bundle E. In general the moduli space of parabolic bundles is more complicated than the moduli space of stable bundles. But we can take some special topological data so that the stable bundle E lies in a zero dimensional moduli space. Then the only problem is to classifying the sunline bundle of $E|_\Sigma$. Based on this strategy, we use the pairs $(\mathbb{C}P^2, \mathbb{C}P^1)$ and $(\Sigma_1 \times \Sigma_2, \{p\} \times \Sigma_2)$ to get the answer.

5.1 First attempt: using the pair $(\mathbb{C}P^2, \mathbb{C}P^1)$

Our first attempt to mimic the proof in [8]. Even though $\mathbb{C}P^2$ does not satisfies our requirement $b^+ \geq 2$ to avoid reducibles for a generic path of metrics, it is still safe for this special case: there are no reducibles for any choice of metric in this case. Equip $\mathbb{C}P^2$ with a Kähelr metric with a cone-like singularity along $\mathbb{C}P^1$ with cone angle $2\pi/\nu$. We also take the holonomy parameter λ to be a rational number a/b where $b(N-1)$ divides ν, then we have the moduli space of irreducible singular ASD connections is the same as the moduli space of stable parabolic bundles of the form $(E, L, -a/(b(N-1)), a/b)$. We take a small enough λ so that the stability of $(E, L, -a/(b(N-1)), a/b)$ is the same as the stability of E. In this case, to understand the moduli space of stable bundles is just to understand the the moduli space of stable bundles and classify the subline bundles along $\mathbb{C}P^1$.

In general, the moduli space of stable bundles is not easy to understand. For our parabolic bundles we also need to understand the subline bunles over $\mathbb{C}P^1$, so it should be even more complicated. But the basic idea is to choose some particular topoloical data so that the moduli space of stable bundles is a 0-dimensional space: just finite many

points. Then the only thing we need to do is to understand the subline bundle. In [8], the tangent bundle of $\mathbb{C}P^2$ is used. We need to use higher rank stable bundles which lie in zero-dimensional moduli spaces. Unfortunately, we may not find such kind of bundles for all ranks.

There is a criterion in Chapter 16 of [13] which can be used to determine whether there is a stable bundle E with given $(c_1, c_2, \text{rank} = N)$ lying in a zero-dimensional moduli space. From the dimension formula

$$\dim_{\mathbb{R}} M_{c_1,c_2,N} = 4Nc_2 - 2(N-1)c_1^2 + 2(N^2-1)$$

we obtain $N(2c_2 + N) - (N-1)c_1^2 = 1$ so the rank N and c_1 must be coprime. If there is such a bundle $\mathscr{E}$, it muse be unique: if not, suppose we can find another stable bundle $\mathscr{F}$, then we have

$$\chi(E^\vee \otimes F) = \chi(E^\vee \otimes E) = h^0(E^\vee \otimes E) - h^1((E^\vee \otimes E)) + h^2(E^\vee \otimes E) = 1$$

where $h^0(E^\vee \otimes E) = 1$ because E is stable, $h^1(E^\vee \otimes E) = \dim(M_{c_1,c_2,N}) = 0$, $h^2(E^\vee \otimes E) = h^0(E^\vee \otimes E(-3)) = 0$ by Serre duality.

Next we need to understand the restriction $E|_{\mathbb{C}P^1}$ in order to understand the subline bundle. According to Proposition 17.2.1 in [13], $E|_{\mathbb{C}P^1}$ is rigid: $E|_{\mathbb{C}P^1} \cong \bigoplus_{i=1}^N \mathcal{O}_{\mathbb{C}P^1}(a_i)$ where $|a_i - a_j| \leq 1$ for any $1 \leq i, j \leq N$. By tensoring with a line bundle, we may assume $0 < m = c_1(E)[\mathbb{C}P^1] < N$. We have

$$E|_{\mathbb{C}P^1} = \mathcal{O}(1)^{\oplus m} \oplus \mathcal{O}^{\oplus N-k}$$

Now we take $c_1(L) = 1$ (so the monopole number is $c_1(E)/N - c_1(L) = k/N - 1$), we need

to understand the subline bundle of $E|_{\mathbb{C}P^1}$ with degree 0. This is the same as understanding the embedding of $\mathcal{O}(1)$ into $E|_{\mathbb{C}P^1}$. We have

$$\mathrm{Hom}(\mathcal{O}(1), \mathcal{O}(1)^{\oplus m} \oplus \mathcal{O}^{\oplus N-m}) \cong \mathbb{C}^m$$

So the moduli space $M_{k,m/N-1}$ is just $\mathbb{C}P^{m-1}$.

Pick a point $\sigma \in \mathbb{C}P^1$, then we have $M_{k,m/N-1} \to \mathbb{P}(E_\sigma)$ is an isomorphism. And the universal line bundle $\mathbb{L}_\sigma$ coincides with the pullback of the tautological line bundle over $\mathbb{P}(E_\sigma)$. So the point class $\epsilon = c_1(\mathbb{L}_\sigma^*)$ is the hyperplane class in $M_{k,m/N-1} \cong \mathbb{C}P^{m-1}$. We conclude

$$q_{k,m/N-1}(\sigma^{m-1}) = \langle \epsilon^{m-1}, [\mathbb{C}P^{m-1}] \rangle = 1$$

Next we want to increase the monopole number by 1 and understand the moduli space $M_{m/N}$. In this case, the subline bundle L has $c_1 = 0$. It suffices to consider the embeddding of $\mathcal{O}$ into $\mathcal{O}(1)^{\oplus m} \oplus \mathcal{O}^{\oplus N-k}$.

$$\mathrm{Hom}(\mathcal{O}, \mathcal{O}(1)^{\oplus m} \oplus \mathcal{O}^{\oplus N-m}) \cong \mathbb{C}^{N+m}$$

However, the situation here is different from the previous case because not every non-zero map will give us a subline bundle: some non-zero maps in the hom set may have zeros. A map from $\mathcal{O}$ to $\mathcal{O}(1)^{\oplus m} \oplus \mathcal{O}^{\oplus N-m}$ can be denoted by $(f_1, \cdots, f_m, g_1, \cdots, g_{N-m})$ where $f_i' s$ are sections of $\mathcal{O}(1)$ and $g_j' s$ are sections of $\mathcal{O}$. Such a map has a zero if and only if $g_i = 0$ for all i and $f_i' s$ only differ by scalars. If we fix the zero $p \in \mathbb{C}P^1$, then to obtain the map with a (unique) zero p is just to obtain a non-zero map $\mathcal{O}(p) \to \mathcal{O}(1)^{\oplus m}$. This means these

maps are parameterized by $\mathbb{C}P^{m-1} \times \mathbb{C}P^1$. So we have

$$M_{k,m/N} \cong \mathbb{C}P^{N+m-1} \backslash \mathbb{C}P^{m-1} \times \mathbb{C}P^1$$

Actually $\mathbb{C}P^{N+m-1}$ coincides with the Uhlenbeck compactification of $M_{k,m/N}$. Take a point $(a, p) \in \mathbb{C}P^{m-1} \times \mathbb{C}P^1$, then a corresponds to a point in the lower dimensional moduli space $M_{k,m/N-1}$ and p is the bubble point.

Take $M + m - 1$ points $p_1, \cdots, p_{N+m-1} \in \mathbb{C}P^1$, the universal subline bundle restricted to point p_i is denoted by $\mathbb{L}_{p_i}$. $\mathbb{L}_{p_i} \to M_{k,m/N}$ is the restriction of $\mathbb{O}(-1)$ over $\mathbb{C}P^{N+m-1}$. It can be extend to the compactified moduli away from ideal connections with a bubble at p_i: $\mathbb{C}P^{N+m-1} - \mathbb{C}P^{m-1} \times \{p_i\}$. This is also done in the proof of Proposition 4.1. Finally this line bundle can be extended uniquely to $\mathbb{C}P^{N+m-1}$: just as $\mathcal{O}(-1)$. Based on this, the polynomial invariant is just

$$q_{k,m/N}(\sigma^{N+m-1}) = \langle c_1^{M+m-1}(\mathbb{L}^*), [\mathbb{C}P^{N+m-1}] \rangle = 1$$

Here we only talk about the point classes defined by points in $\mathbb{C}P^1$. For a point $x \in \mathbb{C}P^2 \backslash \mathbb{C}P^1$, the point classes defined by x must be zero since the bundle E is fixed in our case. The moduli space $M_{k-1,1+m/N}$ is empty because E already lie in a zero-dimensional moduli space M_k so that M_{k-1} must have negative dimension hence empty. In summary, we obtain

- $q_{k,m/N}(\sigma^{m-1}(x_{\bar{c}_N} + \sigma x_{\bar{c}_{N-1}} + \cdots \sigma^N)) = 1.$
- $q_{k,m/N-1}(\sigma^{m-1}) = 1.$
- $q_{k-1,1+m/N}(\sigma^{m-1}) = 0$

Now we can conclude that $m_1 = 1$ for any rank N for which we can find non-empty zero dimensional moduli space of stable bundles.

5.2 Flip symmetry and the calculation of m_2

Now suppose we are working with a general Kähler pair (X, Σ) (not only the pair $(\mathbb{C}P^2, \mathbb{C}P^1)$. In the rank 2 case, a flip symmetry between different moduli spaces is used to obtain m_2 [8]. More precisely, there is an isomorphism

$$\Phi : M^{\lambda}_{k,l} \to M^{1/2-\lambda}_{k',l'}$$

where

$$k' = k + l - \frac{1}{4}\Sigma.\Sigma,\ l' = \frac{1}{2}\Sigma.\Sigma - l.$$

Suppose $(E, L) \in M_{k,l}$ and $(E', L') \in M_{k',l'}$, we have

$$c_1(E') = c_1(E) - [\Sigma]$$

From the parabolic bundle viewpoint, this isomorphism can be described as the following: E' is defined by the short exact sequence

$$0 \to E' \to E \to i_*(E|_\Sigma/L) \to 0$$

where $i : \Sigma \to X$ is the inclusion map. Using i to pull back this short exact sequence, we lose the left exactness and obtain L' by the long exact sequence

$$0 \to L' \to E'|_\Sigma \to E|_\Sigma \to i^*i_*(E|_\Sigma/L) \to 0$$

This definition also works for the higher rank case. If we start from a stable parabolic bundle (E, L, a, b) where $\operatorname{rank} E = N$ and $\operatorname{rank} L = m$. Then it is easy to check that $(E', L', b - m/N, a + (N - m)/N)$ is also a stable parabolic bundle. We have

$$\operatorname{rank} L' = N - m$$

$$c_1(E') = c_1(E) - (N - m)[\Sigma]$$

$$ch_2(E') = ch_2(E) - c_1(E|_\Sigma/L)[\Sigma] + \frac{N - m}{2}\Sigma.\Sigma$$

$$c_1(L) + c_1(L') + (N - m)\Sigma.\Sigma = c_1(E)[\Sigma]$$

The monopole numbers are

$$l = \frac{\operatorname{rank} L}{N}c_1(E)[\Sigma] - c_1(L)[\Sigma],\ l' = \frac{\operatorname{rank} L'}{N}c_1(E')[\Sigma] - c_1(L')[\Sigma]$$

Based on these formulas, we obtain

$$k' = k + l - \frac{m(N - m)}{2N}\Sigma.\Sigma,\ l + l' = \frac{m(N - m)}{N}\Sigma.\Sigma$$

Notice that the formulas for instanton numbers and monopole numbers are symmetric: if we switch (k, l) with (k', l'), we obtain the same formula. This is because the construction is symmetrical: if we apply the construction to (E', L') we will obtain (E, L) up to a twist by a line bundle.

Even though we focus on the case $\operatorname{rank} L = 1$ as we have discussed in Secion 3, but in this calculation we have to take into consideration another case: $\operatorname{rank} L = N - 1$. The only

difference is just to choose a different holonomy around Σ:

$$\exp(-2\pi i \operatorname{diag}(\frac{\lambda}{N-1}, \cdots, \frac{\lambda}{N-1}, -\lambda)) \tag{5.1}$$

We also have the 1-1 correspondence between singular ASD connections and stable parabolic bundles. But now we have $\operatorname{rank} L = N-1$ instead of 1. If we take the holonomy parameter small enough, then a stable parabolic bundle is just a stable bundle E' and a rank $N-1$ subbundle L of $E|_\Sigma$. Or equivalently, a quotient line bundle F of $E|_\Sigma$. If we apply the flip symmetry to (E, F), we can get a parabolic bundle (E', L'). From the exact sequences used to define the flip symmetry, it can be seen that we have a canonical isomorphism between F and $L \otimes \mathcal{O}_X[-\Sigma]$.

Now we come back to the $(\mathbb{C}P^2, \mathbb{C}P^1)$ case. A unique stable bundle E is taken as in the previous subsection. This time we want to understand the quotient line bundle instead of subline bundle. The discussion is almost the same. Now we are going to use the same notation as in the previous subsection. Take $F = \mathcal{O}$ we get a moduli space isomorphic to $\mathbb{C}P^{N-m}$. Denote this space by $\tilde{M}_{k',l'-1}$ Take $F = \mathcal{O}(1)$ we get an non-compact moduli space and a compactification defined by $\mathbb{P}(\operatorname{Hom}(E|_{\mathbb{C}P^1}, \mathcal{O})) \cong \mathbb{C}P^{2N-m-1}$. This space can be identified to the Uhlenbeck compactification as before. Denote this space by $\bar{\tilde{M}}_{k',l'}$. After applying the flip symmetry, we obtain two moduli spaces $\bar{M}_{k,l}$ and $M_{k'-1,l'+1}$. We also have $M_{k',l'-1} \cong \tilde{M}_{k'-1,l+1} =$. Now we can do the calculation of Donaldson invariants. The only difference to the calculation before is that F is a quotient line bundle. Therefore σ is the negative hyperplane class. In summary, we obtain

- $q_{k',l'}(\sigma^{m-1}(x_{\bar{c}_N} + \sigma x_{\bar{c}_{N-1}} + \cdots \sigma^N)) = (-1)^{2N-m-1}$.
- $q_{k'-1,l'+1}(\sigma^{N-m-1}) = (-1)^{N-m-1}$.

- $q_{k',l'-1}(\sigma^{N-m-1}) = 0$

So we conclude that $m_2 = (-1)^N$ for any rank N for which we can find non-empty zero dimensional moduli space of stable bundles.

5.3 Calculation of m_1 and m_2 for all ranks

The defect of the previous argument is that it does not work for all ranks. Now we will use a different example to do the calculation. Let C_1 and C_2 be two elliptic curves. Now we take $(X, \Sigma) = (C_1 \times C_2, \{p\} \times C_2)$. Take $c_2(E) = 0, c_1(E) = [C_1 \times \{p_2\}]$ where $p_2 \in C_2$. The moduli space of ASD connections with these c_1 and c_2 is easy to understand: since the instanton number $k = 0$ in this case, the moduli space consist of (projectively) flat connections. From the stable bundle point of view, it can be described as the following: Let $E \to C_2$ be the unique stable bundle of rank N and determinant $\mathcal{O}_{C_2}(p_2)$ over C_2 (see [A]), then we have

$$M_0 = \{L \boxtimes E | L \in \mathrm{Pic}^0(C_1), L^{\otimes N} = 0\}$$

So we have N^2 points in this moduli space. In order to understand the moduli space of parabolic bundles, we need to study the subline bundle of E over $\Sigma \cong C_2$. Now we denote the unique stable bundle (with fixed determinant) over Σ of rank i by E_i so $E = E_N$. We have a recursive construction of these bundles. We set $E_1 = (p_2)$. E_2 is the unique non-trivial extension

$$0 \to \mathcal{O} \to E_2 \to E_1 \to 0$$

Since $\dim \mathrm{Ext}^1(\mathcal{O}(p_2), \mathcal{O}) = \dim \mathrm{Ext}^0(\mathcal{O}, \mathcal{O}(p_2))$, the extension is unique. If E_i is the unique stable bundle of rank i, then we have

$$H^1(\Sigma, E_i) \cong \mathrm{Ext}^1(\mathcal{O}, E_i) \cong (\mathrm{Ext}^0(E_i, \mathcal{O}))^\vee = 0$$

where the last equality comes from the stability of E_i. Now $\dim H^0(\Sigma, E_i) = \chi(E_i) = 1$ by Riemann-Roch formula, E_{i+1} is the non-trvial extension

$$0 \to \mathcal{O} \to E_{i+1} \to E_i \to 0$$

From the construction we see there is a unique subbundle of E isomorphic to $\mathcal{O}$ since $\dim \mathrm{Hom}(\mathcal{O}, E) = 1$. Actually for any line bundle $\xi \in \mathrm{Pic}^0(\Sigma)$,

$$\dim \mathrm{Hom}(\xi, E) = \dim \mathrm{Hom}(\mathcal{O}, E \otimes \xi^{-1}) = 1$$

because $E \otimes \xi^{-1}$ is also a stable bundle (with a different determinant). From this discussion, we conclude

$$M_{0,1/N} \cong \mathrm{Pic}^0(\Sigma) \times M_0$$

It is enough for us to focus on one component of the moduli space, which is just a copy of $\mathrm{Pic}^0(\Sigma)$.

We also want to understand the moduli space $M_{0,1+1/N}$. By Riemann-Roch formula and the stability of E, we have

$$\dim \mathrm{Hom}(\mathcal{O}(-p), E) = N + 1$$

If a map $\mathcal{O}(-p) \to E$ has no zero then it gives a subbundle of E. It may have at most 1 zero because of the stability of E. Suppose the zero is point $q \in \Sigma$, then a map $\mathcal{O}(p) \to E$ with z zero at q is the same as a bundle map $\mathcal{O}(q-p) \to E$. Notice that the map $\mathcal{O}(q-p) \to E$ is unique up to scalar and this subline bundle is an element in $M_{0,1/N}$. Let $\mathscr{P}_{-1}$ be the

Poincaré line bundle over $\mathrm{Pic}^{-1}(\Sigma) \times \Sigma$, one component of $M_{0,1+1/N}$ is an open subset of

$$Y = \mathbb{P}(\pi_{\Sigma,*}\mathscr{H}om(\mathscr{P}_{-1}, \pi_\Sigma^* E))$$

where $\pi_\Sigma : \mathrm{Pic}^{-1}(\Sigma) \times \Sigma \to \mathrm{Pic}^{-1}(\Sigma)$ is the projection and $\mathscr{H}om$ is the sheaf hom fuctor. Y is a $\mathbb{C}P^N$ bundle over $\mathrm{Pic}^{-1}(\Sigma)$. By cohomology and base change, it is easy to see the fiber over a line bundle $\mathcal{O}(-p) \in \mathrm{Pic}^{-1}(\Sigma)$ is $\mathbb{P}(\mathrm{Hom}(\mathcal{O}(-p), E))$. From the description above we see a point in $Y - M_{0,1+1/N}$ corresponds to a pair $(\xi, q) \in M_{0,1/N} \times \Sigma$. So Y can be identified with the Uhlenbeck compactification of $M_{0,1+1/N}$.

Now we can calculate the Donaldson invariants. This time we want to use curve classes in the moduli spaces defined by embedded curves in Σ. The universal subline bundle $\mathbb{L}$ over $\mathrm{Pic}^0(\Sigma) \times \Sigma$ restricted to each slice $\{\xi\} \times \Sigma$ is isomorphic to ξ (here we view $\mathrm{Pic}^0(\Sigma)$ as a component of $M_{0,1/N}$). So there is a line bundle F over $\mathrm{Pic}^0(\Sigma)$ so that

$$\mathbb{L} \cong \pi^*_{\mathrm{Pic}^0(\Sigma)} F \otimes \mathscr{P}$$

Actually F can be calculated very precisely by cohomology and base change and projection formula:

$$F \cong \pi_{\mathrm{Pic}^0(\Sigma),*}\mathscr{H}om(\mathscr{P}, \pi_\Sigma^* E)$$

Let α, β be two embedded curves in Σ which intersect transversally at exactly 1 point. Then we have

$$\begin{aligned} q_{0,1/N}(\alpha\beta) &= N^2\langle c_1(\mathbb{L})/\alpha \cdot c_1(\mathbb{L})/\beta, \mathrm{Pic}^0(\Sigma)\rangle \\ &= N^2\langle c_1(\mathscr{P})/\alpha \cdot c_1(\mathscr{P})/\beta, \mathrm{Pic}^0(\Sigma)\rangle \\ &= N^2 \end{aligned}$$

Now we want to calculate $q_{0,1+1/N}(\sigma^N\alpha\beta)$. We focus on the case $N \geq 3$. Then there is no need to choose geometrical representatives for c_1/α and c_1/β, because after cut down the moduli by $\sigma_1, \cdots, \sigma_N$, the 2-dimensional space is already compact by the standard counting argument. So we can take the integration of $c_1/\alpha \cdot c_1/\beta$ over this 2-dimensional space honestly without using geometrical representatives to doing the intersection. Let π be the composition of $M_{0,1+1/N} \to Y \to \mathrm{Pic}^{-1}(\Sigma)$. Let $\mathbb{L} \to M_{0,1+1/N}$ be the universal subline bundle. We have the restriction of $\mathbb{L}$ to each slice $\{s\} \times \Sigma$ is isomorphic to $\pi(x) \in \mathrm{Pic}^{-1}(\Sigma)$. Therefore

$$\mathbb{L} \cong F \otimes (\pi \times \mathrm{id}_\Sigma)^* \mathscr{P}_{-1}$$

where F is a line bundle over $M_{0,1+1/N}$. When we are calculating the slant product of $c_1(\mathbb{L})$ and a curve in Σ we can ignore F. So $c_1(\mathbb{L})/\alpha \cdot c_1(\mathbb{L})/\beta$ is the pullback of $c_1(\mathscr{P}_{-1})/\alpha \cdot c_1(\mathscr{P}_{-1})/\beta$, which is the pullback of a single point, say $\mathcal{O}(-p)$. Now we obtain a subspace of $M_{0,1+1/N}$ which is isomorphic to

$$\mathbb{P}(\mathrm{Hom}(\mathcal{O}(-p), E))\backslash\{\text{maps with a zero}\} \cong \mathbb{C}P^N\backslash\Sigma$$

The compactification of this space is just $\mathbb{C}P^N$. Now the situation is the almost same as

that in the previous subsections. σ will give the hyperplane class in $\mathbb{C}P^N$. We obtain

$$q_{0,1+1/N}(\sigma^N \alpha\beta) = N^2$$

Notice that $M_{0,1+1/N}$ has N^2 components. Compare this with $q_{0,1/N}(\alpha\beta)$ we obtain $m_1 = 1$. Like what we do in the previous subsection, we can study the quotient line bundle and then take the flip symmetry to study the dual case. The argument can be done with almost no change, we obtain $m_2 = (-1)^N$.

6 An operator on the framed instanton Floer homology group

We have studied the point classes for a closed 4-manifold. Now we want to turn to the 3-manifold case. The complete theory is developed in [12]. Given a 3-manifold Y, an embedded knot or link K in Y, a complex line bundle δ, if the non-integral condition in [12] is satisfied then there is a well-defined Floer homology group

$$\mathbb{I}_*(Y, K)_\delta$$

where we still the singular $U(N)$ connections with specified holonomy in Section 3 and δ is the determinant of the $U(N)$ bundle over Y. We should also require the holonomy parameter $\lambda = (N-1)/(2N)$ to achieve the monotone condition. In [12] for any pair (Y, K) the framed instanton Floer homology group is defined as

$$FI^N_*(Y, K) = \mathbb{I}_*(Y \sharp T^3, K)_\delta$$

where $c_1(\delta)$ is the Poincaré dual of $S^1 \times \{point\} \subset S^1 \times T^2 = T^3$. FI^N has a relative grading by $\mathbb{Z}/2N$.

Now the point classes σ and $\bar{c}_i$ in the 4-manifold case become operators on the Floer homology groups of degree 2 and $2i$. The relation (4.5) becomes

$$\sigma^N + x_{\bar{c}_2}\sigma^{N-2} + \cdots + x_{\bar{c}_N} = (1 + (-1)^N)\,\mathrm{id} \tag{6.1}$$

In order to make these operators well-defined we should work with the Floer homology groups with rational coefficients since the operators are not integral.

Notice that we keep using gauge transformations of determinant 1 in our theory. But we may use more general $U(N)$ bundle automorphisms which do not fix the determinant. This leads to non-trivial actions on the Floer homology groups. Such a gauge transformation (mod out determinant 1 gauge group) corresponds uniquely to an element in $H^1(Y\sharp T^3, \mathbb{Z}/N)$. In particular, if we take the subgroup of $H^1(Y\sharp T^3, \mathbb{Z}/N)$ consisting of elements which are only non-zero only on $S^1 \times \{point\} \subset S^1 \times T^2 = T^3$, then we get a $\mathbb{Z}/N$-action on the framed Floer homology groups and the generator $\mathbf{s}$ of $\mathbb{Z}/N$ is a degree 4 map on $FI^N_*(Y, K)$. More precisely we have an action on the Floer chain complex and the homology of the quotient chain complex is defined as the reduced framed Floer homology group $\bar{F}I^N_*(Y, K)$. Since we are working with rational coefficients, it's not hard to see that the quotient chain complex is isomorphic to the invariant sub-chain complex and the reduced homology group is isomorphic to the invariant subgroup.

We want to study action of the point classes on $\bar{F}I^N_*(Y, K)$. We have $\mathbb{I}^N(T^3)_\delta = \mathbb{Z}^N$. In this case we have a relative $\mathbb{Z}/4N$ grading since these is no embedded link. The generators of $C_*(T^3)_\delta$ are flat connections $\alpha_0, \alpha_1, \cdots, \alpha_{N-1}$ of degree $0, 4, \cdots 4(N-1)$. $\mathbb{Z}/N$ gives a transitive action on these generators. From this we know that the action of $\bar{c}_i$ on $\mathbb{I}^N(T^3)_\delta$

is always 0 whenever i is odd since the relative grading of any two generators differ by a multiple of 4. We have

$$\bar{c}_{2j}(\alpha_k) = m_{N,2j}\alpha_{k+j}$$

where $m_{N,2j}$ only depends on j and N not on k because the point class action commutes with the $\mathbb{Z}/N$ action. $m_{N,j}$ can be calculated from the higher rank polynomial invariants of T^4. For example we have $m_{N,2} = N$.

Consider the product cobordism $(Y\sharp T^3, K) \times [-1, 1]$. Take $T^2 \hookrightarrow Y\sharp T^3 \times \{0\}$ which can also be thought as $\{p\} \times T^2 \subset S^1 \times T^2 = T^3$. Now we can consider a new cobordism

$$(Y\sharp T^3, K) \times [-1, 1] \backslash N(T^2)$$

Apply I^N_* to this cobordism we can get a map

$$f : FI^N_*(Y, K) \otimes I^N_*(T^3)_\delta \to FI^N_*(Y, K)$$

We also have

$$f(a \otimes \alpha_0) = a$$

$$f(a \otimes \mathbf{s}b) = \mathbf{s}f(a \otimes b)$$

$$f(a \otimes \bar{c}_i b) = \bar{c}_i f(a \otimes b)$$

where $\mathbf{s}$ denote the generator of $\mathbb{Z}/N$. If we take $b = \sum \alpha_i$ in the last equality we can deduce that $\bar{c}_i|_{\bar{F}I} = m_{N,i}\,\mathrm{id}$.

Now we can state the theorem

Theorem 6.1. *If we identify $\bar{F}I^N_*(Y, K)$ as the $\mathbb{Z}/N$ invariant subgroup of $FI^N_*(Y, K)$,*

then $\bar{c}_i|_{\bar{F}I} = m_{N,i}\,\mathrm{id}$ *for some rational number* $m_{N,i}$*. So the relation between* σ *and* $\bar{c}_i$ *becomes*

$$\sigma^N + m_{N,2}\sigma^{N-2} + \cdots + m_{N,N} = (1 + (-1)^N)\,\mathrm{id} \tag{6.2}$$

where $m_{N,2} = N$ *and* $m_{N,i} = 0$ *whenever* i *is odd. In particular when* $N = 3$ *we have* $\sigma^3 + 3\sigma = 0$*.*

We can give an alternative definition of $\bar{F}I^N_*(Y,K)$ based our understanding of the action of $\bar{c}_2$. We define

$$\bar{F}I^N_*(Y,K) := FI^N_*(Y,K)_{\bar{c}_2,N}$$

where the right hand side means the eigenspace of $\bar{c}_2$ associated to eigenvalue N. Here we have to work in the $\mathbb{C}$ coefficients in order to obtain the eigenspace decomposition.

7 Calculation of the coefficients $m_{N,i}$

Instanton Floer homology has the features of a topological quantum field theory. We can use this to relate $m_{N,i}$ to the Donaldson invariants for a 4-torus. For example, we have

$$\bar{c}_2(\alpha_k) = m_{N,2}\alpha_{k+1}$$

We can use the bundle automorphism $g_{\mathbf{s}}$ corresponds to $\mathbf{s} \in \mathbb{Z}/N$ to glue to two ends of $T^3 \times [-1,1]$ and get a $U(N)$ bundle over a 4-torus. Since $\mathbf{s}$ maps α_i to α_{i+1}, the "shifted" trace ($Nm_{N,2}$) of the operator $\bar{c}_2$ should be equal to the Donaldson invariants over the 4-torus defined by evaluating the point class $\bar{c}_2$ over a 4 dimensional moduli space. But there is a subtle issue on the choice of gauge group in this process which is addressed in Section 5.2 of [11] for the rank $N = 2$ case. The same discussion shows that after we add

a additional factor N in front of the "shifted" trace, we will get

$$q(x_{\bar{c}_2}) = N^2 m_{N,2}$$

According to Donaldson-Uhlenbeck-Yau's result, we have a one-to-one correspondence between anti-self-dual connections and stable bundles over a Kähler manifold. So we can use stable bundles to calculate the Donaldson invariants. Next we will use Fourier-Mukai transform to construct stable bundles over an abelian surface.

7.1 Fourier-Mukai transform

In Mukai's paper [15] he defined the Fourier-Mukai transform which is an equivalence of the derived categories of an abelian variety and its dual abelian varieties. Here we want want to use this tool to construct stabel bundles over a 4-torus and calculate the higher rank Donaldson invariants of a 4-torus. Here we will summarize some results we need in Mukai's paper.

Let X and Y be two varieties, F be a coherent sheaf over $X \times Y$. We have the following functor:

$$\mathscr{S}_F : Coh(X) \to Coh(Y)$$

$$E \qquad \mapsto \pi_{Y*}(F \otimes \pi_X^* E)$$

where $Coh(X)$ and $Coh(Y)$ are the categories of coherent sheaves over X and Y. π_X and π_Y are projections from $X \times Y$ to X and Y.

Now let's take X to be an abelian variety and Y to be the dual abelian variety $\hat{X}$. Let E be the normalized Poincaré line bundle $\mathscr{P}$ over $X \times \hat{X}$. We hope that $\mathscr{S}_F$ gives an equivalence of categories so that we can use it to construct the sheaves on one alebian

variety from the sheaves on the other one. Unfortunately this is not true in general. But it is true if we pass it to the derived categories. Let

$$FM : D(X) \to D(\hat{X})$$

be the derived functor induced by $\mathscr{S}$. Similarly we can define $\hat{\mathscr{S}}$ to be

$$\hat{\mathscr{S}}_F : Coh(\hat{X}) \to Coh(X)$$

$$E \qquad \mapsto \pi_{X*}(\mathscr{P} \otimes \pi^*_{\hat{X}} E)$$

We denote the corresponding derived functor by $\widehat{FM}$.

Theorem 7.1 (Mukai,1981)**.** *We have the following isomorphisms of functors:*

$$FM \circ \widehat{FM} \cong (-id_{\hat{X}})^*[-g]$$

$$\widehat{FM} \circ FM \cong (-id_X)^*[-g]$$

where $g = \dim_{\mathbb{C}} X$ *and* $[-g]$ *means "shift towards the right".*

Now we know $D(X)$ and $D(\hat{X})$ are equivalent. So we can try to use this equivalence to construct sheaves. In the derived category, objects are quasi-isomorphic classes of cochain complexes. Let F be an object in $D(X)$, we use $FM^i(F)$ to denote the i-th cohomology group of $FM(F)$. If we start from a sheaf over X, we will get a cochain complex of sheaves after the Fourier-Mukai transform. But if $FM^i(F) = 0$ for all $i \neq k$, the we say F has weak index k. If $H^i(F \otimes \mathscr{P}_\xi) = 0$ for all $i \neq k$ and all $\xi \in \hat{X}$,then we say F has index k.

If F has index k, then we denote $FM^k(F)$ by $\hat{F}$. This gives us a equivalence of categories:

$$\widehat{\ }: WI(X) \to WI(\hat{X})$$

where $WI(X)$ and $WI(\hat{X})$ denote the subcategories of sheaves with weak indexes. We also call this functor Fourier-Mukai transform. We summarize the properties as follows:

1 $i(F) + i(\hat{F}) = g$ where i means the weak index.

2 $\hat{\hat{F}} \cong (-1_X)^* F$

3 $\mathrm{Ext}^i(F, G) \cong \mathrm{Ext}^{i+i(F)-i(G)}(\hat{F}, \hat{G})$

4 $\chi(\hat{F}) = (-1)^{i(F)} \operatorname{rank}(F)$

$\operatorname{rank}(\hat{F}) = (-1)^{i(F)} \chi(F)$

5 $\widehat{F \otimes \mathscr{P}_\xi} \cong \tau_\xi^* \hat{F}$

$\widehat{\tau_x^* F} \cong \hat{F} \otimes \mathscr{P}_{-x}$ where τ is the translation map.

7.2 Construct stable bundles

Let C be a curve with genus g greater or equal to 2. Let X be the Jacobian of C. Let $j : C \to X$ be the embedding of C into X which is only well-defined up to a translation in X. j also induces an isomorphism on the first homology groups. Take a line bundle L over C with $\deg L > 2g - 2$. We have

$$H^i(X, j_* L \otimes \mathscr{P}_\xi) \cong H^i(C, L \otimes \mathscr{P}_\xi) = 0$$

where $i > 1$.

We also have

$$H^1(X, j_*L \otimes \mathscr{P}_\xi) \cong H^1(C, L \otimes \mathscr{P}_\xi) \cong H^1(C, K_C \otimes (L \otimes \mathscr{P}_\xi)^\vee)^\vee \cong 0$$

The second equality is because of the Riemann-Roch formula. The last equality is because $\deg(L \otimes \mathscr{P}_\xi)^\vee < 0$.

Now we know j_*L has index 0. Its Fourier-Mukai transform $E = \widehat{j_*L}$ is a vector bundle such that

1. $i(E) = g$.
2. $\operatorname{rank} E = \chi(L) = d + 1 - g$.
3. $\chi(E) = (-1)^g \operatorname{rank} j_*L = 0$.

Consider the map $C \times \hat{X} \to X \times \hat{X}$. The pullback of the Poincaré line bundle $\mathscr{P}$ is just the Poincaré line bundle over $C \times \hat{X}$. Here $\hat{X}$ can also be thought as the Jacobian of C (The Jacobian of a curve is principally polarized so it is isomorphic to its dual). We have

$$E = \pi_{\hat{X}*}(\pi_C^* L \otimes \mathscr{P})$$

Let $\{a_i\}_{1 \le i \le 2g}$ be a basis of $H^1(C, \mathbb{Z}) = H^1(X, \mathbb{Z})$ with the standard intersection form ($\langle a_i, a_{i+g} \rangle = 1$, $\langle a_i, a_j \rangle = 0$ if $j \neq i \pm g$). Let $\{\alpha_i\}_{1 \le i \le 2g}$ be the corresponding dual basis for $H^1(\hat{X}, \mathbb{Z})$ so that we have

$$c_1(\mathscr{P}) = \sum_{i=1}^{2g} a_i \smile \alpha_i$$

By Index Theorem, we have

$$\operatorname{ch}(E) = \operatorname{ch}(L \otimes \mathscr{P}) \operatorname{Td}(C)/[C]$$

From this we can get

$$c_1(E) = -\sum_{i=1}^{g} \alpha_i \smile \alpha_{i+g}$$

We will take $[\omega] = \sum_{i=1}^{g} \alpha_i \smile \alpha_{i+g}$ as the Kähler class used to defined the degree and the slope stability. We also assume the Picard number of $\hat{X}$ is 1. Therefore the first Chern class of any coherent sheaf must be a multiple of $c_1(E)$ (or equivalently a multiple of $[\omega]$. We have

Proposition 7.2. *E is stable.*

Proof. Suppose E is not stable. Then we can find a stable (torsion free) sheaf F so that $\operatorname{rank} F < \operatorname{rank} E$, there is a non-zero regular map from F to E and

$$\frac{\deg F}{\operatorname{rank} F} \geq \frac{\deg E}{\operatorname{rank} E}$$

From this we can deduce $\deg F \geq 0$. Now there are two possible cases:

Case 1 : $\operatorname{rank} F = 1$.

In this case, F and $\det F$ are isomorphic away from a codimension 2 subvariety of $\hat{X}$ since F is torsionfree. So we have a non-zero regular map from $\det F$ to E on the open subset where F and $\det F$ are isomorphic. Since the complement has codimension 2, this map can be extended to the whole $\hat{X}$.

If $\deg \det F = \deg F > 0$, then $c_1(F)$ must be a positive multiple of $[\omega]$. $\det F$ is a positive line bundle so by Kodaira vanishing's theorem we have $H^i(\hat{X}, \det F \otimes \mathscr{P}_x) = 0$ for all $i > 0$. This means $\det F$ has index 0. Now we can apply Property 3 of Fourier-Mukai

transfrom and get

$$\begin{aligned}
\mathrm{Hom}(\det F, E) &= \mathrm{Ext}^0(\det F, E) \\
&\cong \mathrm{Ext}^{i(\det F)-i(E)}(\widehat{\det F}, \hat{E}) \\
&\cong \mathrm{Ext}^{-g}(\widehat{\det F}, \hat{E}) \\
&\cong 0
\end{aligned}$$

So we get a contradiction: there is no non-zero map from $\det F$ to E.

If $\deg \det F = \deg F = 0$, then we must have $\det F = \mathscr{P}_x$ for some $x \in X$. $\mathscr{P}_x$ has index g and the Fourier-Mukai transform is $\mathbb{C}(x)$ for some $x \in X$(the skyscraper sheaf at x). Apply Property 3 of Fourier-Mukai transform we get

$$\begin{aligned}
\mathrm{Hom}_{\hat{X}}(\mathscr{P}_x, E) &\cong \mathrm{Hom}_X(\mathbb{C}(x), j_*L) \\
&\cong \mathrm{Hom}_C(j^*\mathbb{C}(x), L) \\
&\cong 0
\end{aligned}$$

The last equality is correct no matter whether $x \in C$.

Case 2 : $\mathrm{rank}\, F > 1$.

In this case, we have

$$\begin{aligned}
H^g(F \otimes \mathscr{P}_x) &\cong \mathrm{Ext}^g(\mathscr{P}_{-x}, F) \\
&\cong \mathrm{Ext}^0(F, \mathscr{P}_{-x})\check{} \\
&\cong \mathrm{Hom}(F, \mathscr{P}_{-x})\check{} \\
&\cong 0
\end{aligned}$$

where the second equality is the Serre duality and the last equality is because if the stability of F. From this we can conclude that $H^i(\widehat{FM}(F)) = R^i\pi_{X*}(F \otimes \mathscr{P}) = 0$ for all $i \geq g$. So $\widehat{FM}(F)$ is isomorphic to a cochain complex

$$0 \to F_0 \to F_1 \to \cdots \to F_{g-1} \to 0$$

in $D(X)$. We also have $\widehat{FM}(E) \cong (-1)_X^*(j_*L)[-g]$. Now we have

$$\mathrm{Hom}(F, E) \cong \mathrm{Hom}_{D(X)}(FM(F), FM(E)) \cong 0$$

which is a contradiction. □

7.3 Connectedness of moduli spaces

We have the following result

Proposition 7.3. *Suppose X is a K3 surface, an abelian surface, or a projective plane. Fix Chern classes $c_1 = \alpha, c_2 = m$ and rank N, we use M to denote the the moduli space of stable bundles with these fixed topological data. We have M is always regular. If M contains a connected compact component M_0 such that the universal stable bundle over M_0 exists, then we have $M = M_0$.*

Proof. Let E be a stable bundle in the moduli space M. By Serre's duality, we have

$$H^0(X, K \otimes \mathrm{End}_0(E)) \cong (H^2(X, \mathrm{End}_0(E)))^\vee$$

where $\mathrm{End}_0(E)$ means the traceless endomorphism bundle. If X is a K3 surface or an abelian surface, then K is trivial. Therefore $H^0(X, K \otimes \mathrm{End}_0(E)) = H^0(X, \mathrm{End}_0(E))$

because of the stability of E. If X is $\mathbb{C}P^2$, then $K = \mathcal{O}(-3)$. In this case the injection $\mathcal{O}(-3) \otimes \mathrm{End}_0(E) \to \mathcal{O} \otimes \mathrm{End}_0(E)$ induces an injection $H^0(X, K \otimes \mathrm{End}_0(E)) \to H^0(X, \mathrm{End}_0(E)) = 0$. In call these cases the obstruction space $H^2(X, \mathrm{End}_0(E))$ is zero. So M is always regular. The dimension of M is

$$\dim_{\mathbb{C}} M = H^1(X, \mathrm{End}(E)) = 2 - \chi(\mathrm{End}(E))$$

Now suppose M is not connected. Let M_0 be a connected component of M and take $F \in M \backslash M_0$. We use $\mathbb{V}$ to denote the universal stable bundle over $M_0 \times X$. Because of stability, a homomorphism between two stable bundles is either 0 or an isomorphism. So we have $H^0(X, F^\vee \otimes \mathbb{V}_m) = 0$ for any $m \in M_0$. We also have $H^2(X, F^\vee \otimes \mathbb{V}_m) = 0$ by Serre's duality. Therefore by cohomology and base change we obtain a vector bundle of rank $-\chi(\mathrm{End}(E)) \geq 0$ over M_0:

$$U = R^1 \pi_{M_0,*}(F^\vee \otimes \mathbb{V}).$$

Because the rank is smaller than $\dim_{\mathbb{C}} M = l$, we have $c_l(U) = 0$. On the other hand, we can take a bundle E_0 parameterized by a point $m_0 \in M_0$ (i.e $E_0 \cong \mathbb{V}_{m_0}$. We consider the family $E_0^\vee \otimes \mathbb{V}$. We have $H^0(X, E_0^\vee \otimes \mathbb{V}_m)) = H^2(X, E_0^\vee \otimes \mathbb{V}_m) = 0$ for all $m \neq m_0$. $\dim H^1(X, E_0^\vee \otimes \mathbb{V}_m)) = l - 2$ when $m \neq m_0$ but the dimension jumps at $m = m_0$. Another way to understand this is to use the the family of elliptic operators D over X

$$D_m = \bar{\partial}_{\mathbb{V}_m} \oplus \bar{\partial}^*_{\mathbb{V}_m} : \Omega^{0,0}(E_0^\vee \otimes \mathbb{V}_m) \oplus \Omega^{0,2}(E_0^\vee \otimes \mathbb{V}_m) \to \Omega^{0,1}(E_0^\vee \otimes \mathbb{V}_m)$$

parameterized by M_0. We have $\ker D_m = H^0(X, E_0^\vee \otimes \mathbb{V}_m) \oplus H^2(X, E_0^\vee \otimes \mathbb{V}_m)$ and $\mathrm{coker}\, D_m = H^1(X, E_0^\vee \otimes \mathbb{V}_m)$. The Chern classes of the index bundle of D can be cal-

culated by Atiyah-Singer index theorem, which only depends on the topology of E_0 and $\mathbb{V}_m$ not the holomorphic structure. Therefore if we replace E_0 by E in the definition of D to obtain another family of operators D', we have $c(-\operatorname{ind} D) = c(-\operatorname{ind} D') = c(U)$. We will show that $c_l(-\operatorname{ind} D) = \pm 1$ to obtain a contradiction by an argument used in Lemma 6.11 of [9].

We can patch up the vector spaces $\Omega^{0,0}(E_0^\vee \otimes \mathbb{V}_m) \oplus \Omega^{0,2}(E_0^\vee \otimes \mathbb{V}_m)$ and $\Omega^{0,1}(E_0^\vee \otimes \mathbb{V}_m)$ to obtain two vector bundles A and B of infinite rank over M_0 respectively. Then $D : A \to B$ is a fiber-wise linear map. D_m is injective whenever $m \neq m_0$ and $\ker D_{m_0} = H^0(X, E_0^\vee \otimes E_0) \oplus H^2(X, E_0^\vee \otimes E_0) \cong \mathbb{C} \oplus \mathbb{C}$. There is no obstruction to extend the two 1-dimensional subspace to two trivial subline bundles $\underline{\mathbb{C}_1}$ and $\underline{\mathbb{C}_2}$ and make $A = \underline{\mathbb{C}_1} \oplus \underline{\mathbb{C}_2} \oplus C$ where C is a complement subbundle of infinite rank. Now $D : C \to B$ is injective so that $N = B/D(C)$ is a bundle of rank l and $N|_{m_0} = H^1(X, E_0^\vee \otimes E_0)$. We have

$$\operatorname{ind} D = -(N - \underline{\mathbb{C}_1} - \underline{\mathbb{C}_2}).$$

In particular $c_l(-\operatorname{ind} D) = c_l(N)$. On the other hand $D : \underline{\mathbb{C}_1} \to N$ gives us a global section of N which vanishes at s single point m_0. The only thing left is to show that this zero is transversal to the zero section. It is not hard to see that the tangent map of this section in the fiber direction is

$$\begin{aligned} T_{m_0} M_0 &\to \operatorname{Hom}(\underline{\mathbb{C}_1}|_{m_0}, N_{m_0}) \\ H^1(X, \operatorname{End} E_0) &\to \operatorname{Hom}(H^0(X, \operatorname{End} E_0), H^1(X, \operatorname{End} E_0)) \\ a &\mapsto (b \mapsto ba) \end{aligned}$$

This is an isomorphism. So m_0 is a transversal zero. We have the top Chern class $c_l(N) =$

± 1.

□

Some remarks:

- The stability of E is first proved by Umemura [19]. Here we give a different proof based on Fourier-Mukai transform.

- By translation and tensoring with line bundles we can get a family of stable bundles. When $g = 2$ this gives us a component of the moduli space. Because of the connectedness of the moduli space we just prove, we conclude that we obtain all the stable bundles by translation and tensoring with line bundles. Actually it is shown directly by Mukai [15] that all the stable bundles are from this construction.

- When we prove the stability of E, we only use the fact that the $-c_1(E)$ is the smallest positive integral class and j_*L has index 0. $c_1(E)$ only depends on the homology class of C. So we can also take another $C' \subset X$ which may be singular but has the same homology class as C. To guarantee j_*L has index 0, it suffices to take an ample enough line bundle L' over C'. So we can get a family of stable bundles parameterized by pairs (C', L').

- Instead of working on the Picard variety of a curve, we can start with an abelian surface of Picard number 1. Choose any curve (possibly singular) C which generates the Neron-Severi group and an ample enough line bundle over C, we can obtain a stable bundle over the dual surface by Fourier-Mukai transform. Based on this we can also obtain a family of stable bundle and the dimension of this family is equal to the dimension of the moduli space of stable bundles with fixed topological data.

7.4 Caculation of the Donaldson invariants

By Donaldson's theorem, we know we have a one-to-one correspondence between rank N stable bundles with fixed determinant and $PSU(U)$ anti-self-dual connections. We can use this to calculate the Donaldson invariants of a 4-torus. The Donaldson invariants only depend on the differentiable structure, so we can take any abelian variety we want. We will take the torus to be dual $\hat{X}$ of the Jacobian variety X of a genus 2 complex algebraic curve and require that the Picard number of X is 1. In this case, we have proved that the Fourier-Mukai transform of j_*L is a stable bundle and any other stable bundle with the same topological data as E must differ from E by a translation and tensoring with a line bundle. Let's use $\mathscr{M}$ to denote the moduli space of stable bundles. Now we have an isomorphism

$$F : X \times \hat{X} \to \mathscr{M}$$

$$(x, \xi) \mapsto \tau_\xi^* E \otimes \mathscr{P}_x$$

We also have the determinant map $\det(\tau_\xi^* E \otimes \mathscr{P}_x) = \tau_\xi^* \det E \otimes \mathscr{P}_x^{\otimes N}$ where $N = d+1-g$. The general dimension formula for $PSU(N)$ ASD connection is

$$\dim = 4N(c_2(E) - \frac{N-1}{2N} c_1^2(E)) - (N^2-1)(b_0 - b_1 + b_+)$$

In our case,

$$\begin{aligned} \dim &= 4N(c_2(E) - \frac{N-1}{2N} c_1^2(E))[\hat{X}] \\ &= 4N(-ch_2(E) + \frac{1}{N} c_1^2(E))[\hat{X}] \\ &= 4N(-\chi(E)) + 2c_1^2(E)[\hat{X}] \\ &= 4 \end{aligned}$$

where the third equality comes from Hirzebruch-Riemann-Roch formula.

After we fix the determinant, we will get a 4-dimensional subspace which coincides with our dimension formula. But this is not enough, we still need to check the regularity of our moduli space:

$$H^2(\hat{X}, \operatorname{End}_0 E) \cong H^0(\hat{X}, \operatorname{End}_0 E) \cong 0$$

where the first isomorphism follows from the Serre duality and the second isomorhism follows from the stabiliy of E.

By [5], we have the following isomrphism

$$\phi : \hat{X} \to \hat{\hat{X}} = X$$

$$\xi \mapsto \tau_\xi^* \det E \otimes (\det E)^{-1}$$

To fix the determinant, we can require $\phi(\xi) + Nx = 0$ where we use the group structure of X. We have the following short exact sequence

$$0 \longrightarrow X \xrightarrow{f} \hat{X} \times X \xrightarrow{g} X \longrightarrow 0$$

where $f(x) = (-N\phi^{-1}(x), x)$ and $g(\xi, x) = \phi(\xi) + Nx$ is the determinant map. So we know the moduli space $\mathscr{M}^0$ of ASD connections is isomorphic to X.

Now we can construct the universal bundle over $\mathscr{M}^0 \times \hat{X} = X \times \hat{X}$. Consider the map

$$h : X \times \hat{X} \to \hat{X}$$

$$(x, \xi) \mapsto \xi - N\phi^{-1}(x)$$

Let $\mathbb{E} = h^*E \otimes \mathscr{P}$. We have

$$\mathbb{E}|_{\{x\}\times\hat{X}} \cong \tau^*_{-N\phi(x)}E \otimes \mathscr{P}_x \cong F \circ f(x)$$

So $\mathbb{E}$ is the universal bundle. Now we want to evaluate the point class over the moduli space, which is defined as

$$q(\xi) = -\frac{1}{2N}p_1(\mathbb{E}|_{X\times\{\xi\}})[X]$$

$h|_{X\times\{\xi\}}$ is a map of deg N^4, so we have

$$\begin{aligned} q(\xi) &= N^4(-\frac{1}{2N}p_1(E)[\hat{X}]) \\ &= N^4(-ch_2(E) + \frac{1}{2N}c_1^2(E))[\hat{X}]) \\ &= N^3 \end{aligned}$$

Now we can conclude that $m_{N,2} = N$.

7.5 Higher dimensional moduli space

We can use the same idea to calculate $m_{N,i}$ for a general i. But now we need a higher dimensional moduli space to do the calculation. In [16] Mukai constructs a $4N + 8$ (real) dimensional moduli space of stable sheaves. The moduli space is isomorphic to

$$\hat{X} \times X^{[N+1]}$$

where $X^{[N+1]}$ is the Hilbert scheme of $N+1$ points and X is a principally polarized abelian surface. More precisely, given a ideal sheaf I of $N + 1$ point, take a ample line bundle L

with holomorphic Euler number 1. the

$$\widehat{I \otimes L}$$

is a stable sheaf and all the stable sheaves with the same topological data come from this construction. We denote this moduli space of stable sheaves by $\bar{\mathscr{M}}_1$. Here $\mathscr{M}_1$ means the moduli space of stable bundles which is a open subset of $\bar{\mathscr{M}}_1$. The universal stable sheaf can be constructed in this case. We can try to use this to calculate the Donaldson invariants for the point classes $\bar{c}_i$. One issue is that this is the moduli space of stable sheaves: there are some non-locally free stable sheaves. This is a compactification of the moduli space of stable bundles which is different with the Uhlenbeck compactification. But we can still compare the two different compactifications. I will prove that when N is large this moduli space can still be used to calculate the Donaldson invariants.

7.5.1 Construction of the universal sheaf

Let L be a fixed principal polarization of X, $I \in X^{[N+1]}$ be an ideal sheaf. Then we have

1. $i(I \otimes L) = 1$ hence $i(\widehat{I \otimes L}) = 1$.
2. $\chi(I \otimes L) = -N$ hence $\mathrm{rank}(\widehat{I \otimes L}) = N$ by Property 4 of Fourier-Mukai transform.
3. $c_1(\widehat{I \otimes L}) = \alpha_1 \smile \alpha_3 + \alpha_2 \smile \alpha_4$ by Index Theorem.

We have the following pullback diagram

$$\begin{array}{ccc} X^{[N+1]} \times \hat{X} \times X \times \hat{X} & \xrightarrow{m_{13}} & X^{[N+1]} \times X \times \hat{X} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ X^{[N+1]} \times \hat{X} \times \hat{X} & \xrightarrow{m} & X^{[N+1]} \times \hat{X} \end{array}$$

where m_{13} is the product of the addition map $\hat{X} \times \hat{X} \to \hat{X}$ and the identity map on $X^{[N+1]} \times X$, m is the product of the addition map $\hat{X} \times \hat{X} \to \hat{X}$ and the identity map on $X^{[N+1]}$, π and π' are projections.

Let $\mathscr{P}_{21}$ be the Poincaré line bundle over the product of X and the first copy of $\hat{X}$, $\mathscr{P}_{23}$ be the Poincaré line bundle over the product of X and the second copy of $\hat{X}$, $\underline{I}_{N+1}$ be the universal ideal sheaf over $X^{[N+1]} \times X$. We can obtain a sheaf $L \otimes \underline{I}_{N+1} \otimes \mathscr{P}_{21} \otimes \mathscr{P}_{23}$ over $X^{[N+1]} \times \hat{X} \times X \times \hat{X}$. By cohomology and base change, $R^i\pi_*(L \otimes \underline{I}_{N+1} \otimes \mathscr{P}_{21} \otimes \mathscr{P}_{23}) = 0$ when $i > 1$ and $R^1\pi_*(L \otimes \underline{I}_{N+1} \otimes \mathscr{P}_{21} \otimes \mathscr{P}_{23}) = 0$ gives the universal stable sheaf.

Notice that we have

$$L \otimes \underline{I}_{N+1} \otimes \mathscr{P}_{21} \otimes \mathscr{P}_{23} \cong m_{13}^*(L \otimes \underline{I}_{N+1} \otimes \mathscr{P})$$

m is a smooth map hence flat, so we have

$$R^i\pi_*(m_{13}^*(L \otimes \underline{I}_{N+1} \otimes \mathscr{P})) = m^* R^i\pi'_*(L \otimes \underline{I}_{N+1} \otimes \mathscr{P})$$

Let $\underline{\mathcal{O}}_{N+1}$ be the structure sheaf of the universal subscheme of $X^{N+1} \times X$, we have the following exact sequence of sheaves over $X^{[N+1]} \times X$

$$0 \to \underline{I}_{N+1} \to \ \mathcal{O}_{X^{[N+1]} \times X} \to \underline{\mathcal{O}}_{N+1} \to 0$$

Pullback to $X^{[N+1]} \times X \times \hat{X}$ and tensor with $L \otimes \mathscr{P}$, we get

$$0 \to \underline{I}_{N+1} \otimes L \otimes \mathscr{P} \to L \otimes \mathscr{P} \to \underline{\mathcal{O}}_{N+1} \otimes L \otimes \mathscr{P} \to 0$$

Apply the pushforward π'_*, we get

$$0 \to \pi'_*(\underline{I}_{N+1} \otimes L \otimes \mathscr{P}) \to \pi'_*(L \otimes \mathscr{P}) \to \pi'_*(\underline{I}_{N+1} \otimes L \otimes \mathscr{P}) \to R^1\pi'_*(L \otimes \underline{I}_{N+1} \otimes \mathscr{P}) \to 0$$

For a generic choice of ideal sheaf $I \in X^{[N+1]}$, we get $H^0(X, I \otimes L \otimes \xi) = 0$, so by cohomology and base change, $\pi'_*(\underline{I}_{N+1} \otimes L \otimes \mathscr{P})$ is a torsion sheaf (generically zero). $\pi'_*(L \otimes \mathscr{P})$ is a line bundle, so we conclude that $\pi'_*(\underline{I}_{N+1} \otimes L \otimes \mathscr{P})$ is 0. We can use Grothendieck-Riemann-Roch theorem is calculate the Chern character of $R^1\pi'_*(L \otimes \underline{I}_{N+1} \otimes \mathscr{P})$. From now on we will denote this universal sheaf by $\mathbb{F}$.

7.5.2 Comparison between the Gieseker compactification and Uhlenbeck compactification

For a generic choice ideal sheaf $I \in X^{[N+1]}$, $\widehat{I \otimes L}$ is a stable bundle. But it may be the case that $\widehat{I \otimes L}$ is not locally free (but still torsion-free and stable). Let $E = \widehat{I \otimes L}$ be such a sheaf. E is locally free away from finitely many points in X since it is torsion-free. The double dual $\check{\check{E}}$ is self-reflexive hence locally free over away from a codimension 3 subvariety. Since we are working on an abelian surface, $\check{\check{E}}$ is locally free over X. Since $\check{\check{E}}$ and E are isomorphic away from finitely many points, they have the same c_1 and E has a smaller c_2 (because $\check{\check{E}}/E$ is supported a zero dimensional subspace). According to Proposition 5.4 in [16], $\check{\check{E}}/E$ must have length 1 hence is a skyscraper sheaf $\mathcal{O}_p$ for some point in $\hat{X}$. We have a short exact sequence

$$0 \to E \to \check{\check{E}} \to \mathcal{O}_p \to 0 \tag{7.1}$$

It is easy to see that $\check{\check{E}}$ is also stable. Therefore the dual $\check{\check{E}}$ lie in the (complex) 4-dimensional moduli space $\mathscr{M}$ of stable bundles in Section 7.4. On the other hand, start with a stable bundle $F \in \mathscr{M}$, take a point $p \in \hat{X}$ and a non-zero map f from F to $\mathcal{O}_p$, then kernel of F

is a stable non-locally free sheaf with the same topology as E. Now we can conclude that the complement of locally free sheaves in $\bar{\mathscr{M}}_1$ is a $\mathbb{C}P^{N-1}$ bundle over $\mathscr{M} \times \hat{X}$.

To obtain the moduli space of instantons, we need to fix the determinant line bundle of the stable bundles. After fix the determinant, we obtain a condimension 1 submanifold $\bar{\mathscr{M}}_1^0$ of $\bar{\mathscr{M}}_1$. For the non-locally free part, we have $\det(E) = \det(\check{\check{E}})$. After fix the determinant, we get a $\mathbb{C}P^{N-1}$ bundle over $\mathscr{M}_0^0 \times \hat{X}$.

$\mathscr{M}_1^0$ is isomorphic to the moduli space of instantons and we have a compactification $\bar{\mathscr{M}}_1^0$ of it by adding non-locally free stable sheaves. Notice that because $c_1(E)$ is a prime cohomology class, there is no strictly semi-stable sheaf in our case. On the other hand, we have the Uhlenbeck compactification of $\mathscr{M}_1^0$ which is used to define the Donaldson invariants. We denote the Uhlenbeck compactification by $UC(\mathscr{M}_1^0)$. We have

$$UC(\mathscr{M}_1^0) = \mathscr{M}_o^0 \sqcup \mathscr{M}_0^0 \times \hat{X}$$

where $\hat{X}$ denotes the bubbles. There is a canonical map from $\bar{\mathscr{M}}_1^0$ to $UC(\mathscr{M}_1^0)$. It maps a non-locally free sheaf E with the exact sequence (7.1) to $(E^{\vee\vee\vee}, p)$

We will show that even though the two compactifications are not exactly the same, under certain conditions we can still use $\bar{\mathscr{M}}_1^0$ and the universal sheaf over it to calculate the Donaldson invariants.

Lemma 7.4. *Let $d_1, d_2, \cdots, d_l$ be a sequence of point classes defined by normalized Chern classes $\bar{c}_{i_1}, \cdots, \bar{c}_{i_2}, \cdots, \bar{c}_{i_l}$. Let V_i be geometric representatives for d_i used in Section 4.2. Suppose $\sum_i \deg d_i = 4N + 4$ and we can find a subset $z \subset \{1, 2, \cdots, l\}$ such that*

1. *$\mathscr{M}_1^0 \cap V_z$ is compact where $V_z = \bigcap_{i \in z} V_i$.*

2. $\sum_{i\in z}\deg d_i < 2(N-1)$ *then we have*

$$q(d_1d_2\cdots d_l) = \sharp\mathscr{M}_1^0\cap V_1\cap\cdots\cap V_l = \langle\bar{c}_{i_1}(\mathbb{F})\bar{c}_{i_2}(\mathbb{F})\cdots\bar{c}_{i_l}(\mathbb{F}), [\bar{\mathscr{M}}_1^0]\rangle$$

Remarks Here we only focus on point classes but the equality holds for any classes which come from the slant product of Chern classes of the universal bundle and any homology class in $\hat{X}$. This proposition allow us to calculate the Donaldson invariants by $\mathscr{M}_1^0$ even though this is not the ordinary compactification used in the definiton.

Proof. The proof is merely an elementary cohomology argument. Since we only care about point classes, we take an arbitrary point $p\in\hat{X}$ and restrict the universal sheaf $\mathbb{F}$ to $\bar{\mathscr{M}}_1^0\times\{p\}$. For simplicity we will just write $\bar{\mathscr{M}}_1^0$. We have the following long exact sequence

$$\cdots\to H^i(\bar{\mathscr{M}}_1^0,\mathscr{M}_1^0)\to H^i(\bar{\mathscr{M}}_1^0)\to H^i(\mathscr{M}_1^0)\to H^{i+1}(\bar{\mathscr{M}}_1^0,\mathscr{M}_1^0)\to\cdots$$

Denote the complement of $\mathscr{M}_1^0$ by S, it is a submanifold of real codimension $2(N-1)$. Then by excision, we have

$$H^i(\bar{\mathscr{M}}_1^0,\mathscr{M}_1^0)\cong H^i(N(S),\partial N(S))$$

By Thom isomorphism, $H^i(N(S),\partial N(S)) = H^{i-2(N-1)}(S) = 0$ whenever $i<2(N-1)$. Combine this with the long exact sequence we obtain $H^i(\bar{\mathscr{M}}_1^0,\mathscr{M}_1^0)\to H^i(\bar{\mathscr{M}}_1^0)$ is injective when $i<2(N-1)$. $\mathscr{M}_1^0\cap V_z$ is the Poincaré dual of $d_z=\sum_{j\in z}d_j$ in $\mathscr{M}_1^0$. Because of the injectiveness we just obtain, it is also the Poincaré dual of $\smile_{j\in z}\bar{c}_j(\mathbb{F})$ in $\bar{\mathscr{M}}_1^0$. So we have

$$\begin{aligned} q(d_1 d_2 \cdots d_l) &= \langle \smile_{j \notin z} d_j, [\mathscr{M}_1^0 \cap V_z] \rangle \\ &= \langle \smile_{j \notin z} \bar{c}_j(\mathbb{F}), [\mathscr{M}_1^0 \cap V_z] \rangle \\ &= \langle \bar{c}_{i_1}(\mathbb{F}) \bar{c}_{i_2}(\mathbb{F}) \cdots \bar{c}_{i_l}(\mathbb{F}), [\bar{\mathscr{M}}_1^0] \rangle \end{aligned}$$

□

By a standard counting argument, when d_1, d_2, d_3 are 4-dimensional point classes, $\mathscr{M}_1^0 \cap V_{\{1,2,3\}}$ is compact. To achieve Condition 2 we take $N \geq 8$. Now we conclude that when $N \geq 8$

$$\langle \bar{c}_2{}^{N+1-i}(\mathbb{F}) \bar{c}_{2i}(\mathbb{F}), [\mathscr{M}_1^0] \rangle = N^{N+3-i} m_{N,2i}$$

This allows us the calculate $m_{N,2i}$ by working on the Hilbert scheme of points. A lot of work has been done by algebraic geometricians on the Chern characters of the universal sheaves. Meanwhile there is a complete description for the cohomology ring of the Hilbert schemes of points in [14]. Even though I have not obtained a complete answer for the coefficients. Based on these know results, the calculation is a realistic task.

8 Calculation for unknots and unlinks

We know that the operator σ satisfies a universal formula. But it may still have 0 as an eigenvalue. We will show that σ is not a nilpotent operator for unknot in S^3. Here we want to use the notation and construction in Section 5.3. Since we have a complete understanding of the moduli space $M_{0,1/N}$ and the universal subline bundle $\mathbb{L}$, it is easy to

calculate

$$q_{0,1/N}(\sigma) = -N^2$$

Apply Proposition 4.1, we obtain

$$q_{0,1+1/N}(\sigma^{N+1}) = -N^2$$

Consider $T^3 \cong \{x\} \times S^1 \times C_2 \hookrightarrow S^1 \times S^1 \times C_2 = X$. Also assume the S^1 in the definition of T^3 does not contain p_1 where $p_1 \times C_2 = \Sigma$. We may assume T^3 is close to Σ so that we can deform a small part of Σ a little bit to make the intersection T^3 and Σ be an unknot. Locally the intersection looks like $\mathbb{R}^3 \cong \{0\} \times \mathbb{R}^3 \subset \mathbb{R}^4$ intersects a cone in $\mathbb{R}^3 \times \{0\} \subset \mathbb{R}^4$.

If we cut X along this T^3, we will obtain a cobordism $(T^3 \times I, \check{\Sigma})$ from an unknot to another unknot. Here $\check{\Sigma}$ means we cut Σ along a small null-homotopic circle and obtain a union of a closed disk and a genus 1 surface with boundary. If we glue the product cobordism $(T^3 \times I, U \times I)$ (U is the unknot) to $(T^3 \times I, \check{\Sigma})$, we will obtain the original pair (X, Σ). The calculation of Donaldson invariants $q_{0,1+1/N}(\sigma^{N+1})$ is equivalent to calculate the action of σ^{N+1} on $FI_*^N(U)$ (for knots and links in S^3 we will omit S^3 when writing the Floer homology group for simplicity) and $I_*^N(T^3 \times I, \check{\Sigma})$ then apply a pairing formula.

From this we conclude that σ^{N+1} on $FI_*^N(U)$ is non-zero. $FI_*^N(U) \cong \mathbb{C}^{N^2}$ is calculated in [12]. We have the following decomposition

$$FI_*^N(U) \cong \bigoplus_{i=0}^{N-1} FI_*^N(U)_{\bar{c}_2, N\xi^i}$$

where $\xi = e^{2\pi i/N}$. Each copy $FI_*^N(U)_{\bar{c}_2, N\xi^i}$ is isomorphic to $\mathbb{C}^N$ and carries a action by σ since $\bar{c_2}$ and σ commute. Therefore from $\sigma^N \neq 0$ we obtain σ is not nilpotent as an operator on $FI_*^N(U)$.

Now suppose N is odd. $FI_*^N(U)$ is a $\mathbb{Z}/2N$ graded vector space with a set of generators all of even degrees. Suppose α is a generator and $\sigma^N(\alpha) \neq 0$, then we have

$$\sigma^N(\sum_{i=0}^{N-1}(\bar{c}_2/N)^i\alpha) = \sigma^N(\sum_{i=0}^{N-1}\mathbf{s}^i\alpha) = \sum_{i=0}^{N-1}\mathbf{s}^i\sigma^N(\alpha) \neq 0$$

The sum is non-zero because all terms have different degrees. We have $\sum_{i=0}^{N-1}(\bar{c}_2/N)^i\alpha \in \bar{F}I_*^N(U) = FI_*^N(U)_{\bar{c}_2,N} \subset FI_*^N(U)$. Therefore σ on $\bar{F}I_*^N(U)$ is not nilpotent. To summarize,

Proposition 8.1. *Let U be the unknot in S^3, then we have σ is not a nilpotent operator on $FI_*^N(U)$. When N is odd, σ is not a nilpotent operator on $\bar{F}I_*^N(U)$. When $N = 3$, we have decomposition*

$$\bar{F}I_*^N(U) = \bar{F}I_*^N(U)_{\sigma,-\sqrt{3}i} \oplus \bar{F}I_*^N(U)_{\sigma,0} \oplus \bar{F}I_*^N(U)_{\sigma,\sqrt{3}i}$$

where the three spaces are the three 1 dimensional eigenspaces of σ.

Proof. The only thing left is the discussion for the $N = 3$ case. When $N = 3$, then $\sigma^3+3\sigma = 0$. Since σ is not nilpotent, it must have non-zero eigenvalues. Our Floer homology can be defined over $\mathbb{Q}$, so the characteristic polynomial of σ is a rational polynomial. This means $\sqrt{3}i$ and $-\sqrt{3}i$ must appear as σ's eigenvalues in pairs. We conclude that σ is diagonalizable and the three eigenvalues for σ are $\sqrt{3}i, 0, -\sqrt{3}i$. □

Let (Y_1, L_1) and (Y_2, L_2) be two pair of links, we have the following result

Proposition 8.2. *For each component of a link, there is a σ operator. We have the following isomorphism*

$$\bar{F}I_*^N(Y_1\sharp Y_2, L_1 \sqcup L_2) \cong \bar{F}I_*^N(Y_1, L_1) \otimes \bar{F}I_*^N(Y_2, L_2)$$

Moreover, the σ operators from different component of the links respect this isomorphism.

Proof. The proof is an application of the excision formula due to Floer [2]. Take $T_1 = T^2 \subset S^1 \times T^2 = T^3$ be a torus away from L_1 in $Y_1 \sharp T^3$, take $T_2 = T^2 \subset S^1 \times T^2 = T^3$ be a torus away from L_2 in $Y_2 \sharp T^3$. Cut (T^3, L_1) along T_1 and (T^3, L_2) along T_2 and glue them with the correct orientation, we will obtain $(Y_1 \sharp Y_2 \sharp T^3, L_1 \sqcup L_2)$. In [2], they need the fact that the $SO(3)$ instanton Floer homology of T^3 is 1-dimensional to do the proof. The only new ingredient we need to apply the proof in [2] is that

$$I_*^N(T^3)_{\delta,\bar{c}_2,N} = \mathbb{C}$$

There is no change in the other part of the proof in [2]. The isomorphism is given by certain cobordism, so the σ operators' action respects the isomorphism. □

As an application, we can obtain

Proposition 8.3. *Let L be an unlink in S^3 with k components. So we have k operators $\sigma_1, \cdots \sigma_k$. Then $\bar{F} I_*^3(L) = \mathbb{C}^{3^k}$ is a direct sum of the common eigenspace of these operators. For each choice of eigenvalues, the dimension of the common eigenspace is* 1.

APPENDIX

A Parabolic bundles and singular instantons

Here we are trying to generalize the discussion in Section 8 of [9] to bundles of rank greater than 2. More precisely, we want to discuss the relation among singular connections, orbifold bundles and parabolic bundles.

A.1 Parabolic bundles

Let (X, Σ) be a pair such that X is a Kähler manifold and Σ is an embedded (complex) curve. A parabolic bundle over such a pair has the following data:

- A holomorphic vector bundle G over X.
- A filtration of $G|_\Sigma$: $G_1 \supseteq G_2 \supseteq \cdots \supseteq G_l$ where $G_1 = G|_\Sigma$ and G_i is a proper subbundle of G_{i-1}.
- An increasing sequence of weights $-1 < i_1 < i_2 < \cdots < i_l < 1$ with $i_l - i_1 < 1$. $l_j = \operatorname{rank} G_j - \operatorname{rank} G_{j+1}$ is called the multiplicity of i_j.

Let ω be the Kähler form, we can define the degree of such a parabolic bundle as

$$\deg G = \langle c_1(G), [\omega] \rangle + (\sum_k l_k i_k) \langle [\Sigma], [\omega] \rangle$$

Let $(G, \{G_s\}, (i_1, \cdots, i_l))$ and $(F, \{F_d\}, (j_1, \cdots, j_m))$ be two parabolic bundles, a parabolic map f is a map from G to F and whenever $f(G_s) \subseteq F_d$ and $f(G_s) \nsubseteq F_{d+1}$ we must have $i_s \leq j_d$.

We call a parabolic bundle G stable if for any parabolic map from F to G where F has rank smaller than G and f is injective as a sheaf map we always have $\deg F / \operatorname{rank} F < \deg G / \operatorname{rank} G$.

A.2 Orbifold bundles

We want to use the same definition of complex *orbifold* (X, Σ) as in [9]. For each point of Σ there is a neighbourhood U such that the orbifold structure is given by

$$\phi : \tilde{U}/\mathbb{Z}_\nu \cong U$$

where Z_ν acts on $\tilde{U}$ with fixed points homeomorphic to $U \cap \Sigma$ under the map ϕ. For any two such charts U_1 and U_2 we have transition function only well-defined up to the action of $\mathbb{Z}_\nu$ on the overlap. For any 3 charts the transition functions satisfies the cocycle condition up to the action of $\mathbb{Z}_\nu$. Similarly we can also define the orbifold sheaf which is $\mathbb{Z}_\nu$-equivariant over $\tilde{U}$ and the transition isomorphism is only well-defined up to the action of $\mathbb{Z}_\nu$. $\mathcal{O}_X$ denote the structure sheaf of X.

For any orbifold sheaf F we can take the $\mathbb{Z}_\nu$-invariant sections to get a ordinary sheaf $\mu(F)$. In particular $\mathcal{O}_{\bar{X}} = \mu(\mathcal{O}_X)$ gives us a complex structure to the underlying topological space of X. We denote this complex manifold by $\bar{X}$.

Another example is the orbifold sheaf $\mathcal{O}^{(b)}$ of meromorphic functions with poles only along Σ. We may think this orbifold sheaf as "$\mathcal{O}(b\tilde{\Sigma})$". We have

$$\mu(\mathcal{O}^{(b)}) \cong \mathcal{O}_{\bar{X}}(m\Sigma)$$

where $a - \nu \leq m\nu < a$.

Locally free orbifold sheaves is locally characterized by weights (mod ν) of the action of the generator of $\mathbb{Z}_\nu$ on the fiber. $\mathcal{O}^{(a)}$ has weight a.

We define a functor

$$E \to \bar{E} = \mu(E \otimes \mathcal{O}^{(\nu)})$$

Given a orbifold bundle E over X of weights $(a_1, a_2, \cdots, a_l)$ ($-\nu < a_1 \leq \cdots \leq a_l < \nu$ and $a_l - a_1 < \nu$) and rank l we can define a parabolic bundle over $(\bar{X}, \Sigma)$ as follows:

- $\bar{E}$ is the bundle over $\bar{X}$.
- First we define a filtration $F_1 \supseteq F_2 \supseteq \cdots \supseteq F_l$ by $F_1 = G|_\Sigma$ and $F_i = \mathrm{Im}(\mu(E \otimes \mathcal{O}^{(-a_i)}) \to \bar{E})$. Notice that $\mathcal{O}^{(-a_i)}$ is a subsheaf of $\mathcal{O}^{(\nu)}$ so we have a map $\mu(E \otimes$

$\mathcal{O}^{(-a_i)}) \to \bar{E}$. This is not a strictly decreasing filtration, but we can always extract a strictly decreasing filtration $E_1 \supseteq \cdots \supseteq E_m$ from $F_1 \supseteq \cdots \supseteq F_l$ and a strictly increasing sequence of weights $(b_1, \cdots, b_m)$ from $(a_1, \cdots, a_l)$.

- The weights for this parabolic bundle is defined as $(b_1/\nu, b_2/\nu, \cdots, b_m/\nu)$.

On the other hand, the inverse construction also exits. Given a parabolic bundle of weights $(a_1/\nu, a_2/\nu, \cdots, a_l/\nu)$, we can reconstruct an orbifold bundle following the argument in [10]. It is also easy to see that our construction and its inverse is functorial. The same argument in [9] can also be used to show that this construction preserves the degree. So it also preserves the stability.

Now we can come back to the singular connections. Let $\bar{X}$ be a Kähler manifold with cone-like singularity along Σ with cone angle $2\pi/\nu$.

Proposition A.1. *we have one-to-one correspondence between any two of the following three*

- *Irreducible (projectively) anti-self-dual connections with the following holonomy around Σ:*

$$\exp(-2\pi i \operatorname{diag}(\lambda_1, \cdots, \lambda_1, \lambda_2, \cdots, \lambda_2, \cdots, \lambda_m, \cdots, \lambda_m))$$

  where $1 > \lambda_1 > \lambda_2 > \cdots > \lambda_m > -1$ and $\lambda_1 - \lambda_m < 1$. We also assume $\lambda_i = c_i/\nu$ where $c_i \in \mathbb{Z}$.

- *Stable orbifold bundles with weights $(c_m, \cdots, c_m, c_{m-1} \cdots, c_{m-1}, \cdots, c_1, \cdots, c_1)$.*

- *Stable parabolic bundles $(E, E_1 \supset E_2 \supseteq \cdots \supseteq E_m)$ with weights $(c_m/\nu, c_{m-1}/\nu, \cdots, c_1/\nu)$ where $\operatorname{rank} E_i - \operatorname{rank} E_{i+1}$ is equal to the multiplicity of λ_{m+1-i}.*

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