Hodge Theory and Unitary Representations of Reductive Lie Groups

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Hodge theory and unitary representations of reductive Lie groups

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ABSTRACT. We present an application of Hodge theory towards the study of irreducible unitary representations of reductive Lie groups. We describe a conjecture about such representations and discuss some progress towards its proof.

1. Introduction

In this paper we state a conjecture on the unitary dual of reductive Lie groups and formulate certain foundational results towards it. We are currently working towards a proof. The conjecture does not give an explicit description of the unitary dual. Rather, it provides a strong functorial framework for studying unitary representations. Our main tool is M. Saito's theory of mixed Hodge modules [19], applied to the Beilinson-Bernstein realization of Harish Chandra modules [4]

Our paper represents an elaboration of the first named author's lecture at the Sanya Inauguration conference. Like that lecture, it aims to provide an expository account of the present state of our work.

We consider a linear reductive Lie group $G_{\mathbb{R}}$ with maximal compact subgroup $K_{\mathbb{R}}$. The complexification G of $G_{\mathbb{R}}$ contains a unique compact real form $U_{\mathbb{R}}$ such that $K_{\mathbb{R}} = G_{\mathbb{R}} \cap U_{\mathbb{R}}$. We denote the Lie algebras of $G_{\mathbb{R}}$, $U_{\mathbb{R}}$, $K_{\mathbb{R}}$ by the subscripted Gothic letters $\mathfrak{g}_{\mathbb{R}}$, $\mathfrak{u}_{\mathbb{R}}$, and $\mathfrak{k}_{\mathbb{R}}$; the first two of these lie as real forms in $\mathfrak{g}$, the Lie algebra of the complex group G, and similarly $\mathfrak{k}_{\mathbb{R}}$ is a real form of $\mathfrak{k}$, the Lie algebra of the complexification K of the group $K_{\mathbb{R}}$. To each irreducible unitary representation π of $G_{\mathbb{R}}$, one associates its Harish Chandra module: a finitely generated module V over the universal enveloping algebra $U(\mathfrak{g})$, equipped with an algebraic action of K which is both compatible with the $U(\mathfrak{g})$ module structure and admissible, in the sense that $\dim \operatorname{Hom}_K(W, V) < \infty$ for each irreducible K-module W. The irreducibility of π implies the irreducibility of its Harish Chandra module V, and

the unitary nature of π is reflected by a positive definite hermitian form $(\ ,\ )$ on V that is $\mathfrak{g}_{\mathbb{R}}$-invariant, i.e.,

$$(1.1) \qquad (\zeta u, v) + (u, \zeta v) = 0 \quad \text{for all } u, v \in V, \ \zeta \in \mathfrak{g}_{\mathbb{R}}.$$

Conversely, the completion of any irreducible Harish Chandra module with a positive definite $\mathfrak{g}_{\mathbb{R}}$-invariant hermitian form is the representation space of an irreducible unitary representation π of $G_{\mathbb{R}}$ [6]. It should be noted that the $\mathfrak{g}_{\mathbb{R}}$-invariant hermitian form is unique up to scaling, if it exists at all.

The correspondence between irreducible unitary representations and irreducible Harish Chandra modules with $\mathfrak{g}_{\mathbb{R}}$-invariant hermitian form makes it possible to break up the problem of describing the unitary dual $\widehat{G}_{\mathbb{R}}$ into three sub-problems:

$$(1.2) \qquad
\begin{array}{ll}
\text{a)} & \text{classify the irreducible Harish Chandra modules;} \\
\text{b)} & \text{single out those carrying a non-zero } \mathfrak{g}_{\mathbb{R}}\text{-invariant hermitian form;} \\
\text{c)} & \text{determine if the } \mathfrak{g}_{\mathbb{R}}\text{-invariant hermitian form has a definite sign.}
\end{array}$$

Problem a) has been solved in several different, yet fundamentally equivalent ways [3, 7, 8, 12, 13, 24], and b) can be answered by elementary considerations. That leaves c) as the remaining – and difficult – open problem. This problem has been solved for certain groups, e.g. [22, 25], and certain classes of representations, e.g. [1], but these solutions do not at all suggest a general answer.

In any irreducible Harish Chandra module V, the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ acts by a character, the so-called infinitesimal character of V. When addressing the problem (1.2c), we may and shall assume that

$$(1.3) \qquad\qquad V \text{ has a real infinitesimal character;}$$

for the definition of this notion see (2.5) below. Indeed, any irreducible unitary Harish Chandra module is irreducibly and unitarily induced from one that satisfies this hypothesis [11, theorem 16.10]. Vogan and his coworkers have pointed out that the hypothesis (1.3) on an irreducible Harish Chandra module V implies the existence of a non-zero $\mathfrak{u}_{\mathbb{R}}$-invariant hermitian form $(\ ,\ )_{\mathfrak{u}_R}$; as in the $\mathfrak{g}_{\mathbb{R}}$-invariant case, the hermitian form is then unique up to scaling. When also a non-zero $\mathfrak{g}_{\mathbb{R}}$-invariant hermitian form $(\ ,\ )$ exists, then the two are very directly related [27]. If, for example, $G_{\mathbb{R}}$ and $K_{\mathbb{R}}$ have the same rank, there exist a root of unity c and a character ψ from the semigroup[1] $\widehat{K}_{\mathbb{R}}$ to the group of roots of unity such that, after a suitable rescaling of one of the hermitian forms,

$$(1.4) \qquad (v, v) = c\,\psi(j)(v, v)_{\mathfrak{u}_{\mathbb{R}}} \quad \text{for } j \in \widehat{K}_{\mathbb{R}} \text{ and any } j\text{-isotopic vector } v \in V.$$

Both c and ψ can be easily and explicitly described. In this identity $c\,\psi(j) = \pm 1$, of course, unless V contains no j-isotypic vectors $v \neq 0$. When $G_{\mathbb{R}}$ and $K_{\mathbb{R}}$ have unequal rank, the relation between the two hermitian forms is only slightly less direct.

The close connection between the two hermitian forms means that the problem (1.2c) can be answered if one has sufficiently precise information about the $\mathfrak{u}_{\mathbb{R}}$-invariant hermitian form.

Our conjecture is best stated in the context of $\mathcal{D}_{\lambda}$-modules, and we shall do so later in this note. But for applications, the case of Harish Chandra modules is

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[1]with respect to the semigroup structure obtained by identifying each class in $\widehat{K}_{\mathbb{R}}$ with its highest weight.

most important. For this reason, in the introduction we shall concentrate on the Harish Chandra case.

Let V be a Harish Chandra module with real infinitesimal character. We do not assume that V is irreducible, but V should be "functorially constructible". In particular, this includes – but is not limited to – so-called standard Harish Chandra modules [7], their duals, and extensions of the former by the latter, as considered by Beilinson-Bernstein in their proof of the Jantzen conjectures [4]. We shall show that in these cases V carries two canonical, functorial filtrations: an increasing filtration by finite dimensional K-invariant subspaces,

$$(1.5) \qquad 0 \subset F_a V \subset F_{a+1} V \subset \ldots \subset F_p V \subset \ldots \subset \cup_{a \le p < \infty} F_p V \;=\; V \,,$$

which we call the *Hodge filtration*, and a finite filtration by Harish Chandra submodules,

$$(1.6) \qquad 0 \subset W_b V \subset W_{b+1} V \subset \ldots \subset W_k V \subset \ldots \subset W_c V \;=\; V \,,$$

the *weight filtration*. If V is functorially constructible, then so are the $W_k V$ and the quotients $W_k V / W_{k-1} V$. The Hodge filtration is a *good filtration*, in the sense that

$$(1.7) \quad \mathfrak{g}(F_p V) \subset F_{p+1} V \ \text{ for all } p \,, \quad \text{and} \quad F_p V + \mathfrak{g}(F_p V) = F_{p+1} V \ \text{ for } p \gg 0 \,.$$

Functorality means not only that the filtrations are preserved[2] by morphisms of Harish Chandra modules, i.e., by simultaneously $U(\mathfrak{g})$- and K-invariant linear maps, provided they are functorially constructible in the same sense as the Harish Chandra modules in question. The filtrations are induced by geometrically defined Hodge and weight filtrations of the $\mathcal{D}_\lambda$-module realization of V, and therefore inherit all the functorial properties of these geometric filtrations. In particular, like all filtrations in Hodge theory, both $F.V$ and $W.V$ are strictly preserved by morphisms:

$$(1.8) \qquad T(F_p V_1) = (T V_1) \cap F_p V_2 \quad \text{and} \quad T(W_k V_1) = (T V_1) \cap W_k V_2$$

whenever $T : V_1 \to V_2$ is a functorially constructible morphism of Harish Chandra modules. Moreover,

$$(1.9) \qquad\qquad W_k V / W_{k-1} V \ \text{ is completely reducible for each } k.$$

In the case of standard modules, this latter assertion can be strengthened. Indeed, the weight filtration then coincides with the Jantzen filtration [4], and hence with the socle filtration of V.

Still assuming that V has a real infinitesimal character, we now also suppose that V is irreducible. The $\mathfrak{u}_\mathbb{R}$-invariant hermitian form $(\ ,\ )_{\mathfrak{u}_\mathbb{R}}$ on V is determined only up to scaling. Vogan and his coworkers show that it has the same sign on all the lowest K-types, and use this fact to fix a choice of sign. We shall give a geometric construction of $(\ ,\ )_{\mathfrak{u}_\mathbb{R}}$, which also results in a preferred choice of sign. Our choice, it turns out, agrees with that of Vogan et al. After these preparations, we can state:

Conjecture 1.10. $(-1)^{p-a}(v,v)_{\mathfrak{u}_\mathbb{R}} > 0 \quad \text{if} \quad v \in F_p V \cap (F_{p+1} V)^\perp \,, \quad v \ne 0 \,.$

Here a denotes the lowest index in the Hodge filtration, as in (1.5). The Hodge filtration is typically very difficult to compute. On the other hand, it has excellent functorial properties. Our conjecture, if proved, would help make the unitary dual accessible to geometric and functorial methods.

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[2]Up to a shift of indices that can be described explicitly.

Vogan and his coworkers are aiming at an algorithm, one that could be implemented on a large scale computer, to determine whether a given irreducible Harish Chandra module is unitarizable. In effect, they want to compute the signature character of the hermitian form $(\ ,\ )_{\mathfrak{u}_\mathbb{R}}$, i.e., the sum

$$(1.11) \qquad \sum_{j \in \widehat{K_\mathbb{R}}} (p_j - q_j)\, \phi_j \ ,$$

with $\phi_j =$ character of $j \in \widehat{K_\mathbb{R}}$, and p_j, q_j denoting the multiplicity with which j occurs in $(\ ,\ )_{\mathfrak{u}_\mathbb{R}}$ with positive and negative sign, respectively. This depends on knowing the change of the signature character of standard modules as the inducing parameter crosses the various reduction hyperplanes; they have a conjectured formula for this change [27]. Our conjecture would transparently describe the change of signature across reduction points, and would thereby imply their conjecture. We should emphasize that the Hodge filtration is potentially a much finer invariant than the signature character. One might hope that our conjecture, if proved, will lead to a much more efficient unitarizability algorithm. It would have other consequences as well.

At various periods during our collaboration on this project, we were guests of the MFO Oberwolfach, the MPI Bonn, and the University of Essen. We thank all three institutions for their hospitality. We are also indebted to Dragan Miličić for helpful discussions.

2. Invariant hermitian forms

Let us be more precise about the class of groups we are considering than we were in the introduction[3]. We start with G, a connected, complex, reductive algebraic group, defined over $\mathbb{R}$. The group $G_\mathbb{R}$ whose unitary representations are of interest to us can be any group between the group of all real points in G and the identity component of the group of real points. Then $G_\mathbb{R}$ has a finite number of connected components, and $\operatorname{Ad} g$, for $g \in G_\mathbb{R}$, defines an inner automorphisms of the Lie algebra of G; however $\operatorname{Ad} g$ may not be an inner automorphism of the Lie algebra of $G_\mathbb{R}$.

We fix a maximal compact subgroup $K_\mathbb{R} \subset G_\mathbb{R}$, whose complexification we denote by K. The choice of $K_\mathbb{R}$ is unique up to an inner automorphism, and is therefore not an essential choice. There exists a unique compact real form $U_\mathbb{R} \subset G$ such that $U_\mathbb{R} \cap G_\mathbb{R} = U_\mathbb{R} \cap K = K_\mathbb{R}$. As always in this paper, we denote the Lie algebras by the corresponding lower case German letters. The conjugate linear automorphisms

$$(2.1) \qquad \sigma, \tau\ :\ \mathfrak{g} \to \mathfrak{g}\,, \qquad \sigma|_{\mathfrak{g}_\mathbb{R}} = 1\,, \qquad \tau|_{\mathfrak{u}_\mathbb{R}} = 1\,,$$

commute, and

$$(2.2) \qquad \theta\ =\ \sigma\tau\ =\ \tau\sigma$$

is the Cartan involution [9]. As usual, $\mathfrak{p}$ and $\mathfrak{p}_\mathbb{R}$ shall denote the (-1)-eigenspaces of θ in $\mathfrak{g}$ and $\mathfrak{g}_\mathbb{R}$. Then

$$(2.3) \qquad \mathfrak{g}\ =\ \mathfrak{k} \oplus \mathfrak{p}\,, \quad \mathfrak{g}_\mathbb{R}\ =\ \mathfrak{k}_\mathbb{R} \oplus \mathfrak{p}_\mathbb{R}\,, \quad \mathfrak{u}_\mathbb{R}\ =\ \mathfrak{k}_\mathbb{R} \oplus i\,\mathfrak{p}_\mathbb{R}\,,$$

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[3]Our hypotheses on $G_\mathbb{R}$ are the most natural from an expository point of view, but certainly not the most general under which our results are valid. To extend these results, along the lines sketched in the appendix of [7], is mostly a technical exercise.

are the Cartan decompositions of $\mathfrak{g}$, $\mathfrak{g}_{\mathbb{R}}$, and $\mathfrak{u}_{\mathbb{R}}$.

Any two Cartan subalgebras of $\mathfrak{g}$ are conjugate under an inner automorphism, and this automorphism becomes unique if each of the two Cartans has been "ordered" – i.e., equipped with a choice of positive root system – and if, moreover, the automorphism is required to preserve the order. By definition, the universal Cartan $\mathfrak{h}$ is a representative of the unique isomorphism class of ordered Cartans, under inner automorphisms preserving the order. The abstract Weyl group W of $\mathfrak{g}$ acts on $\mathfrak{h}$, the W-action preserves the root system $\Phi(\mathfrak{h}) \subset \mathfrak{h}^*$ as well as the weight lattice $\Lambda \subset \mathfrak{h}^*$, and $\Phi(\mathfrak{h})$ contains a distinguished positive root system $\Phi^+(\mathfrak{h})$. Note that the real form

$$(2.4) \qquad \mathfrak{h}_{\mathbb{R}}^* =_{\mathrm{def}} \mathbb{R} \otimes_{\mathbb{Z}} \Lambda \subset \mathfrak{h}^*$$

has nothing to do with the real forms $\mathfrak{g}_{\mathbb{R}}$ and $\mathfrak{u}_{\mathbb{R}}$ of $\mathfrak{g}$; in fact, when $\mathfrak{h}$ is identified with the complexification of a Cartan subalgebra of $\mathfrak{u}_{\mathbb{R}}$, then $\mathfrak{h}_{\mathbb{R}}^*$ consists of purely imaginary elements. Once and for all we fix a faithful algebraic representation of G. The trace form of that representation induces a W-invariant inner product $(\ ,\ )$ on $\mathfrak{h}_{\mathbb{R}}^*$.

We write $\mathrm{HC}(\mathfrak{g}, K)$ for the category of Harish Chandra modules of the pair $(\mathfrak{g}, K)$, and $\mathrm{HC}(\mathfrak{g}, K)_\lambda$ for the full subcategory of Harish Chandra modules with infinitesimal character χ_λ, $\lambda \in \Lambda$, in Harish Chandra's notation. Then $\mathrm{HC}(\mathfrak{g}, K)_\lambda = \mathrm{HC}(\mathfrak{g}, K)_\mu$ if and only if $\lambda = w\mu$ for some $w \in W$. It has been known for a long time that isomorphism classes of irreducible unitary representations of $G_{\mathbb{R}}$ correspond bijectively to irreducible Harish Chandra modules with a positive definite $\mathfrak{g}_{\mathbb{R}}$-invariant hermitian form, again up to isomorphism [6, theorem 9]. That reduces the problem of describing the unitary dual $\widehat{G}_{\mathbb{R}}$ to the determination of all irreducible Harish Chandra modules with a positive definite hermitian form. We shall say that

$$(2.5) \qquad \begin{aligned} &V \in \mathrm{HC}(\mathfrak{g}, K) \ \text{ has real infinitesimal character} \\ &\text{if } \ V \in \mathrm{HC}(\mathfrak{g}, K)_\lambda\,, \ \text{ with } \ \lambda \in \mathfrak{h}_{\mathbb{R}}^*\,. \end{aligned}$$

In addressing the unitarity problem for $G_{\mathbb{R}}$, one can restrict attention to irreducible Harish Chandra modules with real infinitesimal character, without essential loss of generality. In fact, if we identify $\mathfrak{h}$ with the complexified Lie algebra of a concrete, maximally split Cartan subgroup $H_{\mathbb{R}} \subset G_{\mathbb{R}}$, any irreducible unitarizable Harish Chandra module $V \in \mathrm{HC}(\mathfrak{g}, K)_\lambda$ arises as the Harish Chandra module of a unitarily, irreducibly induced representation $\mathrm{Ind}_{P_{\mathbb{R}}}^{G_{\mathbb{R}}} \tau$, and the Harish Chandra module of the inducing representation τ is unitarizable, with real infinitesimal character; the Langlands component $M_{\mathbb{R}}$ of $P_{\mathbb{R}}$ coincides with the centralizer of $\mathrm{Im}\,\lambda$, and $\mathrm{Im}\,\lambda$ is the inducing parameter[4] [11, theorem 16.10].

Because of the reduction of the unitarity problem we just described, we may and shall impose the standing hypothesis that

$$(2.6) \qquad \lambda \in \mathfrak{h}_{\mathbb{R}}^*\,, \ \text{ and } \lambda \text{ is dominant,}$$

in the sense that $(\alpha, \lambda) \geq 0$ for all $\alpha \in \Phi^+(\mathfrak{h})$. As subscript in $\mathrm{HC}(\mathfrak{g}, K)_\lambda$, the parameter λ is only determined up to the action of W. Every $\lambda \in \mathfrak{h}_{\mathbb{R}}^*$ has a unique dominant W-translate, so the dominance condition pins down λ uniquely within its W-orbit. Initially the dominance condition does not matter, but it will play a crucial role in the last section.

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[4] Here $\mathrm{Im}\,\lambda$ should be interpreted using the real structure (2.4).

Theorem 2.7. (Vogan et al. [27]) *Suppose that $V \in \mathrm{HC}(\mathfrak{g}, K)_\lambda$ is irreducible, with λ as in* (2.6). *Then V carries a non-zero $\mathfrak{u}_\mathbb{R}$-invariant hermitian form $(\ ,\ )_{\mathfrak{u}_\mathbb{R}}$. It is unique up to scaling, and can be rescaled so that it becomes positive definite on the lowest K-types in the sense of Vogan* [23].

We shall construct the $\mathfrak{u}_\mathbb{R}$-invariant hermitian form geometrically in section 5. Vogan and his collaborators not only established the existence of the $\mathfrak{u}_\mathbb{R}$-invariant hermitian form by algebraic methods, and had the insight to relate it to the $\mathfrak{g}_\mathbb{R}$-invariant form when the latter exists. Their observation can be stated as follows:

Proposition 2.8. *Let $V \in \mathrm{HC}(\mathfrak{g}, K)_\lambda$ be an irreducible Harish Chandra module as in theorem* 2.7, *which in addition carries a non-zero $\mathfrak{g}_\mathbb{R}$-invariant hermitian form $(\ ,\ )$. When $(\ ,\ )$ is suitably rescaled by a positive constant, there exists a K-invariant linear map $T : V \to V$ such that*

    i) $(u, v) = (Tu, v)_{\mathfrak{u}_R}$ *for all $u, v \in V$,*

    ii) $T^2 = 1$, *and*

    iii) $T \zeta v = (\theta \zeta)(T v)$ *for all $v \in V$, $\zeta \in \mathfrak{g}$.*

The linear map T can also be described explicitly, in geometric terms. That will be done in section 5. If G and K have the same rank, the relationship between the two hermitian forms has a simple, completely explicit, combinatorial description; see corollary 2.12 below.

PROOF. Let V^c denote the complex conjugate of the space of K-finite vectors in the algebraic dual V^* of V, or equivalently, the space of conjugate linear maps from V to $\mathbb{C}$ that vanish on a subspace of finite codimension. The two hermitian forms induce two conjugate linear isomorphisms $i_{\mathfrak{u}_\mathbb{R}}, i_{\mathfrak{g}_\mathbb{R}} : V \xrightarrow{\sim} V^c$. Then $T = i_{\mathfrak{u}_\mathbb{R}}^{-1} \circ i_{\mathfrak{g}_\mathbb{R}} : V \xrightarrow{\sim} V$ is a $\mathbb{C}$-linear isomorphism, and T satisfies the condition i) by construction. Both hermitian forms are $K_\mathbb{R}$-invariant, hence T has this property as well. But the complexification K meets every connected component of $K_\mathbb{R}$, so the $\mathbb{C}$-linear map T commutes with the action of all of K. On the infinitesimal level this implies iii), at least for any $\zeta \in \mathfrak{k}$. Any $\zeta \in \mathfrak{p}$ acts in a symmetric manner with respect to $(\ ,\ )_{\mathfrak{u}_\mathbb{R}}$ and skew symmetrically with respect to $(\ ,\ )$; cf. (2.3). Thus iii) also applies to any $\zeta \in \mathfrak{p}_\mathbb{R}$, hence finally to any $\zeta \in \mathfrak{g}$. Using iii) twice, we find that T^2 commutes with the action of both K and $\mathfrak{g}$ – in other words, T^2 is an automorphism of the irreducible Harish Chandra module V, and consequently $T^2 = c\,1$ with $c \in \mathbb{C}^*$. If we scale T by $|c|^{-1/2}$ and $(\ ,\ )$ by $|c|^{1/2}$, we can arrange that $|c| = 1$ after rescaling, without destroying i) and iii). Since the lowest K-types have multiplicity one, the K-invariant linear map T must act as a multiple of the identity on each of the minimal K-types, necessarily by a real multiple of the identity since $(v, v)_{\mathfrak{u}_\mathbb{R}}$ and (v, v) take only real values. Changing the sign of $(\ ,\ )$ if necessary, we can arrange that T^2 acts by a positive constant, hence $c = 1$, implying ii). $\qquad\square$

Let V^+ and V^- denote the $+1$ and -1 eigenspace of T, respectively. Then, as a formal consequence of the proposition,

(2.9)

    a) $V = V^+ \oplus V^-$ (K-invariant direct sum),

    b) $\mathfrak{p}\, V^+ \subset V^-$, $\mathfrak{p}\, V^- \subset V^+$,

    c) V^+ and V^- are orthogonal relative to both $(v, v)_{\mathfrak{u}_\mathbb{R}}$ and (v, v),

    d) $v \in V^\epsilon \implies (v, v)_{\mathfrak{u}_R} = \epsilon\,(v, v)$, for both $\epsilon = +$ and $\epsilon = -$.

If $\operatorname{rk} G = \operatorname{rk} K$, the decomposition (2.9 a) can be described completely explicitly, as we shall see next. In this situatation $G_{\mathbb{R}}$ contains a compact Cartan subgroup $T_{\mathbb{R}}$, which we may and shall assume lies in $K_{\mathbb{R}}$. We temporarily identify $\mathfrak{h}$ with the complexification of the Lie algebra $\mathfrak{t}_{\mathbb{R}}$ of $T_{\mathbb{R}}$. Then $\mathfrak{t}_{\mathbb{R}} = i\mathfrak{h}_{\mathbb{R}}$, since weights take purely imaginary values on $\mathfrak{t}_{\mathbb{R}}$. One calls a root $\alpha \in \Phi(\mathfrak{h}) \cong \Phi(\mathfrak{t}, \mathfrak{g})$ $\mathfrak{t}_{\mathbb{R}}$-compact or $\mathfrak{t}_{\mathbb{R}}$-noncompact depending on whether the α-root space $\mathfrak{g}^{\alpha}$ is contained in $\mathfrak{k}$ or $\mathfrak{p}$. These notions are preserved by the action of the "real Weyl group" $W(T_{\mathbb{R}}, G_{\mathbb{R}}) = N_{G_{\mathbb{R}}}(T_{\mathbb{R}})/T_{\mathbb{R}}$, since $K_{\mathbb{R}}$ contains the normalizer $N_{G_{\mathbb{R}}}(T_{\mathbb{R}})$. It follows that the notions of $\mathfrak{t}_{\mathbb{R}}$-compactness and $\mathfrak{t}_{\mathbb{R}}$-noncompactness do not depend on the particular choice of identification $\mathfrak{h} \cong \mathfrak{t} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{t}_{\mathbb{R}}$. Let $\Lambda_{\Phi} \subset \Lambda$ denote the lattice spanned by the roots. Then

$$(2.10) \qquad \psi : \Lambda_{\Phi} \to \{\pm 1\}, \quad \psi(\alpha) = \begin{cases} 1 & \text{if } \alpha \text{ is } T_{\mathbb{R}}\text{-compact}, \\ -1 & \text{if } \alpha \text{ is } T_{\mathbb{R}}\text{-noncompact}, \end{cases}$$

is a well defined character since the sum of two roots is $T_{\mathbb{R}}$-compact if and only if the summands are both $T_{\mathbb{R}}$-compact or both $T_{\mathbb{R}}$-noncompact. The Harish Chandra module V breaks up as the direct sum of the various K-isotypic subspaces,

$$(2.11) \qquad \begin{aligned} V &= \oplus_{\mu} V(\mu), \ V(\mu) = K\text{-isotypic subspace of highest weight } \mu, \\ &\text{with} \quad \mu \in \{\lambda \in \Lambda \mid (\alpha, \lambda) \geq 0 \ \text{if } \alpha \in \Phi^{+}(\mathfrak{h}) \text{ is } T_{\mathbb{R}}\text{-compact.}\} \end{aligned}$$

We single out μ_0, the highest weight of one of the lowest K-types. Then $V = U(\mathfrak{g})V(\mu_0)$ by irreducibility. Hence $V(\mu)$ can be nonzero only when $\mu - \mu_0 \in \Lambda_{\Phi}$. We know that T acts on $V(\mu_0)$ either as the identity or as -1. Replacing $(\ ,\ )$ by its negative, if necessary, we may as well suppose that $T = 1$ on $V(\mu_0)$. Also, if ζ lies in the α-root space $\mathfrak{g}^{\alpha}$, $T\zeta T^{-1} = (-1)^{\psi(\alpha)}\zeta$ by assertion iii) of proposition 2.8. This implies:

Corollary 2.12. *In the equal rank case, if $V(\mu_0) \subset V^{+}$ as may be assumed, the isotypic subspace $V(\mu)$ lies in V^{+} or V^{-} depending on wether $\psi(\mu - \mu_0)$ equals $+1$ or -1.*

In particular, V^{+} and V^{-} have no K-types in common. That remains true even in the unequal rank case, provided one replaces $G_{\mathbb{R}}$ with the appropriate extension of $G_{\mathbb{R}}$ by the two element group $\{1, T\}$.

Back in the equal rank case, one can extend the character ψ of (2.10) to a homomorphism from the full weight lattice to the group of roots of unity,

$$(2.13) \qquad \psi : \Lambda \to \{z \in \mathbb{C} \mid z^{N} = 1 \ \text{for some } N \in \mathbb{Z}_{>0}\}.$$

In terms of this notation, with $c = \psi(-\mu_0)$, the relation between the two hermitian forms can be restated as in (1.4), in the introduction.

3. Brief review of the theory of mixed Hodge modules

In the sequence of papers [17–20] Morihiko Saito developed his theory of mixed Hodge modules. We shall need a slightly extended version of the theory: Saito requires the existence of a rational structure, which implies that the local monodromy transformations for the underlying local systems have roots of unity as eigenvalues. For our applications we need to relax this hypothesis; all we can assume is that the eigenvalues have absolute value one, and this follows from the presence of a polarization. On the other hand, we shall only consider complex algebraically constructible

sheaves and, correspondingly, $\mathcal{D}$-modules[5] in the complex algebraic setting. Not all of the theory of mixed Hodge modules depend on this latter restriction, but it does simplify various statements.

We begin with the notions of complex Hodge structure and complex mixed Hodge structure. Let H be a finite dimensional complex vector space. A complex Hodge structure of weight n on H consists of the datum of a direct sum decomposition

$$(3.1) \qquad\qquad H \;=\; \oplus_{p+q=n} H^{p,q}\,.$$

The cohomology group in degree n of a compact Kähler manifold carries a natural Hodge structure of weight n. In that case, $n \geq 0$ and the summation extends only over non-negative integers p, q. In general, p and q are allowed to be arbitrary integers adding up to n, which need not be positive, either. The Hodge structure (3.1) determines, and is determined by, the two increasing filtrations $F.\,H$ and $\overline{F}.\,H$,

$$(3.2) \qquad F_p\,H \;=\; \oplus_{r\geq -p}\,H^{r,n-r}, \quad \overline{F}_q\,H \;=\; \oplus_{s\geq -q}\,\overline{H}^{n-s,s}\,;$$

indeed, the identity

$$(3.3) \qquad\qquad H^{p,q} \;=\; F_{-p}\,H \cap \overline{F}_{-q}\,H$$

recovers the summands $H^{p,q}$ from the two filtrations. One calls $F.\,H$ the Hodge filtration, and $\overline{F}.\,H$ is the conjugate Hodge filtration. Traditionally one uses the decreasing filtrations $\{F^p H\} = \{F_{-p}\,H\}$ and $\{\overline{F}^q H\} = \{\overline{F}_{-q}\,H\}$ instead. However, in the context of $\mathcal{D}$-modules, which crucially enter the theory of mixed Hodge modules, increasing filtrations are more natural. For this reason we work with increasing filtrations right from the beginning. Also, in Hodge theory one customarily assumes that the underlying vector space H is defined over $\mathbb{Q}$, and that $H^{q,p}$ is the complex conjugate of $H^{p,q}$. In that case, the complex conjugate vector space $\overline{H}$ coincides with H, and $\overline{F}.\,H$ is the complex conjugate of the filtration $F.\,H$, as the notation suggests. We do not require the existence of a rational structure, or even a real structure, and the notation $\overline{F}.\,H$ has merely symbolic significance. Of course one can extend a complex Hodge structure (3.2) on H to one on $H \oplus \overline{H}$, which has at least a real structure, by formally taking the direct sum of $H = \oplus H^{p,q}$ and $\overline{H} = \oplus \overline{H}^{p,q}$, with $\overline{H}^{p,q} = \overline{H^{q,p}}$.

One can take not only the direct sum of two complex Hodge structures, provided they have the same weight, but also the tensor product of any two, of possible different weights, as well as symmetric and tensor powers and the dual of a single complex Hodge structure. The weights behave in the obvious way. A morphism of complex Hodge structures is a linear map between the underlying vector spaces which preserves the Hodge types, or equivalently, the two filtrations. By definition, a polarization of the Hodge structure (3.1) is a – necessarily nondegenerate – hermitian pairing $Q : H \times \overline{H} \to \mathbb{C}$ such that

$$(3.4) \qquad \begin{aligned} &\text{a)} \ \ Q(H^{p,q}, H^{r,s}) = 0 \ \text{ if } \ (r,s) \neq (p,q)\,, \ \text{ and} \\ &\text{b)} \ \ (-1)^p\,(2\pi i)^n\,Q(v,\overline{v}) > 0 \ \text{ for all } \ v \in H^{p,q}, \ \ v \neq 0\,. \end{aligned}$$

We call the datum $(H,\,F.\,H,\,Q\,)$ a polarized complex Hodge structure of weight n. Note that $H^{p,q} = F_{-p}H \cap (F_{-p-1}\,H)^{\perp}$, so that

$$(3.5) \qquad \overline{F}_q\,H \;=\; \oplus_{s\geq -q}\,H^{n-s,s} \;=\; \oplus_{s\geq -q}\,\left(F_{s-n}\,H \cap (F_{s-n+1}\,H)^{\perp}\right)\,.$$

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[5]For us, "$\mathcal{D}$-module" shall always mean left $\mathcal{D}$-module. That is not the case in Saito's papers.

The conjugate Hodge filtration $\overline{F}. H$ can therefore be reconstructed from the Hodge filtration, in conjunction with the polarization.

A complex mixed Hodge structure on the finite dimensional complex vector space H consists of three increasing filtrations $F. H$, $\overline{F}. H$, $W. H$, such that for all integers k,

$$\text{the induced filtrations } F_p(W_k H) = F_p H \cap W_k H \,,$$

(3.6)
$$\overline{F}_q(W_k H) = \overline{F}_q H \cap W_k H \,, \text{ determine a complex}$$

$$\text{Hodge structure of weight } k \text{ on } W_k H / W_{k-1} H \,.$$

One calls $W. H$ the weight filtration. If each of the quotients $W_k H / W_{k-1} H$ comes equipped with a specific polarization, one calls that a graded polarized complex mixed Hodge structure. As in the case of Hodge structures of pure weight, one customarily assumes that H is defined over $\mathbb{Q}$, that $\overline{F}. H$ is the complex conjugate of $F. H$, and in addition, that the weight filtration is defined over $\mathbb{Q}$. As before, one can embed any complex mixed Hodge structure into one with underlying real structure on $H \oplus \overline{H}$. The linear algebra of mixed Hodge structures – which is not entirely trivial – depends not on the rational structure, but only on the real structure. All conclusions therefore apply in the complex case. In particular morphisms of complex mixed Hodge structures – i.e., linear maps between the underlying vector spaces that preserve the three types of filtration – preserve the filtrations strictly:

(3.7)
$$T(F_p H_1) = (T H_1) \cap F_p H_2 \,, \quad T(\overline{F}_q H_1) = (T H_1) \cap \overline{F}_q H_2 \,,$$
$$\text{and} \quad T(W_k V_1) = (T V_1) \cap W_k V_2$$

whenever $T : H_1 \to H_2$ is a morphism of complex mixed Hodge structures.

A polarized complex variation of Hodge structure of weight n, on a complex smooth quasi-projective variety Z, is specified by the datum of a vector bundle $\mathbb{H} \to Z$, a flat connection ∇ on $\mathbb{H}$, a flat hermitian form Q on the fibers of $\mathbb{H}$, and an increasing filtration of the sheaf $\mathcal{O}_Z(\mathbb{H})$ by locally free sheaves of $\mathcal{O}_Z$-modules

(3.8)
$$0 \subset F_a \mathcal{O}_Z(\mathbb{H}) \subset F_{a+1} \mathcal{O}_Z(\mathbb{H}) \subset \cdots \subset F_p \mathcal{O}_Z(\mathbb{H}) \subset \cdots \subset \mathcal{O}_Z(\mathbb{H}) \,,$$

subject to the following two conditions:

(3.9a)
$$\nabla F_p \mathcal{O}_Z(\mathbb{H}) \subset F_{p+1} \mathcal{O}_Z(\mathbb{H}) \text{ for all } p \,,$$

and

(3.9b)
$$\text{for each } z \in Z, \ \left(\mathbb{H}_z, (F. \mathcal{O}_Z(\mathbb{H})) / (\mathcal{I}_z F. \mathcal{O}_Z(\mathbb{H})), Q_z \right) \text{ is a}$$
$$\text{polarized Hodge structure on } \mathbb{H}_z, \text{ of weight } n \,;$$

here $\mathcal{I}_z$ denotes the ideal sheaf of $\{z\} \subset Z$, $\mathbb{H}_z$ the fiber of $\mathbb{H}$ at z, and Q_z the restriction of Q to $\mathbb{H}_z$. One refers to the condition (3.9a) as Griffiths transversality. In (3.9b) the conjugate Hodge filtration is not specified; as mentioned earlier, in the presence of a polarization the Hodge filtration completely determines the conjugate Hodge filtration.

As usual, $\mathcal{D}_Z$ shall denote the sheaf of linear differential operators on the smooth quasi-projective variety Z, with algebraic coefficients. It is a sheaf of algebras over $\mathcal{O}_Z$, and is naturally filtered by the degree of differential operators. We let $\mathrm{Mod}(\mathcal{D}_Z)_{\mathrm{coh}}$ denote the category of (sheaves of) $\mathcal{D}_Z$-modules which are coherent

over $\mathcal{D}_Z$. A good filtration of $\mathcal{M} \in \mathrm{Mod}(\mathcal{D}_Z)_{\mathrm{coh}}$ is an increasing filtration by $\mathcal{O}_Z$-submodules $F_p\mathcal{M} \subset \mathcal{M}$, coherent over $\mathcal{O}_Z$, such that

(3.10) $(\mathcal{D}_Z)_k F_p\mathcal{M} \subseteq F_{p+k}\mathcal{M}$ for all k, p, with equality holding for $p \gg 0$;

$(\mathcal{D}_Z)_k = F_k\mathcal{D}_Z$ refers to the filtrants of $\mathcal{D}_Z$, of course. Good filtrations exist, at least locally. The associated graded $\mathrm{gr}_{.}\mathcal{M}$ with respect to a good filtration $F_{.}\mathcal{M}$ can be regarded as a coherent sheaf of $\mathcal{O}_{T^*Z}$-modules[6] on the cotangent bundle T^*Z. As is easy to see, the support of $\mathrm{gr}_{.}\mathcal{M}$ in T^*Z – the so-called singular support of $\mathcal{M}$ – does not depend on the particular choice of a good filtration. We recall that $\mathcal{M}$ is said to be holonomic if its singular support has the minimal possible dimension, or equivalently, if the singular support is Lagrangian. Alternatively and equivalently one can characterize the property of being holonomic by the cohomological condition

(3.11) $\mathcal{E}xt^k_{\mathcal{D}_Z}(\mathcal{M}, \mathcal{D}_Z) \;=\; 0 \quad \text{if} \;\; \mathrm{k} \neq \dim Z$.

In other words, $\mathcal{M}$ is holonomic if and only if it is Cohen-Macaulay.

   Mixed Hodge modules involve filtered $\mathcal{D}$-modules that are are regular holonomic. Regular holonomic $\mathcal{D}$-modules are most easily defined in a functorial fashion: one can describe them as constituting the smallest full subcategory of the category of holonomic modules invariant under the usual operations on $\mathcal{D}$-modules, invariant under extensions, and containing all $\mathcal{D}$-modules generated by flat connections with regular singularities. It is not obvious that such a category exists. That can be established by characterizing regular holonomic modules directly, for example by reduction to the case of a one dimensional base manifold Z. It is then a non-trivial matter to show that the the resulting category is indeed the smallest full subcategory of the category of holonomic modules possessing the invariance properties spelled out before. Of course, "invariance under the usual operations" requires passage to the derived category, since some of them only exist on the derived level. The definition of the category of mixed Hodge modules is entirely analogous, again with a functorial working definition, and a direct characterization involving a reduction to the one dimensional case.

   Let us write $\mathrm{Mod}(\mathcal{D}_Z)_{\mathrm{rh}}$ for the category of regular holonomic $\mathcal{D}_Z$-modules, and $\mathrm{Mod}(\mathcal{D}_Z)_{\mathrm{frh}}$ for the category of all $\mathcal{M} \in \mathrm{Mod}(\mathcal{D}_Z)_{\mathrm{rh}}$ which come equipped with a specific good filtration. Forgetting the filtration, one can regard each $\mathcal{M} \in \mathrm{Mod}(\mathcal{D}_Z)_{\mathrm{frh}}$ as defining an object in $\mathrm{Mod}(\mathcal{D}_Z)_{\mathrm{rh}}$. Morphisms in $\mathrm{Mod}(\mathcal{D}_Z)_{\mathrm{frh}}$ are required to preserve the filtration, of course; a morphism $T : \mathcal{M} \to \mathcal{N}$ is said to be strict if $T(F_k\mathcal{M}) = T(\mathcal{M}) \cap F_k\mathcal{N}$ for all k. It was mentioned already that $\mathrm{Mod}(\mathcal{D}_Z)_{\mathrm{rh}}$ is stable under all the usual operations and constructions. On the other hand, $\mathrm{Mod}(\mathcal{D}_Z)_{\mathrm{frh}}$ has very limited functorial properties in general. One way to view the mixed Hodge modules is, roughly speaking, as specifying a subcategory of $\mathrm{Mod}(\mathcal{D}_Z)_{\mathrm{frh}}$ with funtorial properties as extensive as those of $\mathrm{Mod}(\mathcal{D}_Z)_{\mathrm{rh}}$.

   For our purposes, the abelian category of complex mixed Hodge modules, $\mathrm{CMHM}(Z)$, is particularly relevant. Its objects are pairs $(\mathcal{M}, W_{.}\mathcal{M})$, consisting of some $\mathcal{M} \in \mathrm{Mod}(\mathcal{D}_Z)_{\mathrm{frh}}$ and a finite increasing filtration $W_{.}\mathcal{M}$ of $\mathcal{M}$ by submodules $\mathcal{N} \in \mathrm{Mod}(\mathcal{D}_Z)_{\mathrm{frh}}$ of $\mathcal{M}$, each equipped with the good filtration induced by $\mathcal{M}$. Not every such pair describes an object in $\mathrm{CMHM}(Z)$, of course. For reasons that shall become apparent, we call $W_{.}\mathcal{M}$ the weight filtration and the intrinsic good

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[6]more precisely, $\mathrm{gr}\,\mathcal{M} = \pi_*\mathcal{F}$ is the sheaf direct image, under the natural projection $\pi : T^*Z \to Z$, of a graded coherent $\mathcal{O}_{T^*Z}$-module $\mathcal{F}$ on T^*Z, and $\mathrm{gr}_{.}\mathcal{D}_Z = \pi_*\mathcal{O}_{T^*Z}$.

filtration of $\mathcal{M}$ the Hodge filtration. A morphism in CMHM(Z) is a morphism of the underlying $\mathcal{D}_Z$-modules which preserves both filtrations. At first glance this appears to be a weak requirement, but it turns out that any such morphism is strict with respect to both filtrations. An object $(\mathcal{M}, W.\mathcal{M})$ of CMHM(Z) is said to be pure of weight k if $W_k\mathcal{M} = \mathcal{M}$ and $W_{k-1}\mathcal{M} = 0$. The full subcategory of pure objects is CHM(Z), the category of complex mixed Hodge modules on Z. We denote the bounded derived categories of, respectively, CMHM(Z) and Mod($\mathcal{D}_Z$)$_{\mathrm{rh}}$ by $D^b(\mathrm{CMHM}(Z))$ and $D^b(\mathrm{Mod}(\mathcal{D}_Z)_{\mathrm{rh}})$. We extend the notion of weights to $D^b(\mathrm{CMHM}(Z))$ by the convention that

(3.12)
$$\mathcal{M} \text{ has weight } \leq k \text{ if } H^\ell(\mathcal{M}) \text{ has weight } \leq k + \ell \text{ for all } \ell, \text{ and}$$
$$\mathcal{M} \text{ has weight } \geq k \text{ if } H^\ell(\mathcal{M}) \text{ has weight } \geq k + \ell \text{ for all } \ell.$$

Thus, if $\mathcal{M} \in D^b(\mathrm{CMHM}(Z))$ is pure of weight k, then $H^\ell(\mathcal{M})$ is pure of weight $k + \ell$. The passage to the underlying filtered $\mathcal{D}_Z$-module, followed by forgetting the filtration, defines a forgetful functor from CMHM(Z) to Mod($\mathcal{D}_Z$)$_{\mathrm{rh}}$, and also on the level of the derived categories. We shall say that a functor on Mod($\mathcal{D}_.$)$_{\mathrm{rh}}$ lifts to CMHM($\cdot$) if there exists a commutative square incorporating the original functor and the two forgetful functors. We use the same terminology for the derived categories, of course.

We can now state the properties which characterize CMHM(Z), as the smallest category satisfying them all. First of all,

(3.13a)   every $\mathcal{M} \in \mathrm{CHM}(Z)$ is completely reducible,

and for every $\mathcal{N} \in \mathrm{CMHM}(Z)$ and every k,

(3.13b) $\mathrm{gr}_{W,k} \mathcal{N} \in \mathrm{Mod}(\mathcal{D}_Z)_{\mathrm{frh}}$ exists as object in CHM(Z), of weight k.

If $f : Z \to X$ is an algebraic map between smooth complex quasi-projective varieties,

(3.13c)
$$\text{the } \mathcal{D}\text{-module functors } f_+, f_! : D^b(\mathrm{Mod}(\mathcal{D}_Z)_{\mathrm{rh}}) \to D^b(\mathrm{Mod}(\mathcal{D}_X)_{\mathrm{rh}})$$
$$\text{and } f^+, f^! : D^b(\mathrm{Mod}(\mathcal{D}_X)_{\mathrm{rh}}) \to D^b(\mathrm{Mod}(\mathcal{D}_Z)_{\mathrm{rh}}) \text{ lift}$$
$$\text{to functors } f_*, f_! : D^b(\mathrm{CMHM}(Z)) \to D^b(\mathrm{CMHM}(X)),$$
$$\text{respectively } f^*, f^! : D^b(\mathrm{CMHM}(X)) \to D^b(\mathrm{CMHM}(Z)).$$

Because of the Cohen-Macaulay property (3.11), the definition of the duality functor $\mathbb{D}$ does not require passage to the derived category:

(3.13d)
i)   the functor $\mathbb{D}$ lifts from Mod($\mathcal{D}_Z$)$_{\mathrm{rh}}$ to CMHM(Z), and
ii) if $\mathcal{N} \in \mathrm{CHM}(Z)$ has weight k, $\mathbb{D}\mathcal{N}$ is pure of weight $-k$.

Recall that the $\mathcal{D}$-module dual $\mathbb{D}\mathcal{M}$ of any $\mathcal{M} \in \mathrm{Mod}(\mathcal{D}_Z)_{\mathrm{rh}}$ can be identified with $\mathcal{E}xt^{\dim Z}_{\mathcal{D}_Z}(\mathcal{M}, \mathcal{D} \otimes (\Omega^{\dim Z})^{-1})$. For singleton spaces,

(3.13e)
CMHM($\{pt\}$) is isomomorphic to the category
of complex mixed Hodge structures.

Complex mixed Hodge modules involve two filtrations, the Hodge and weight fitrations. Complex mixed Hodge structures – in particular, CMHM($\{pt\}$) – involve the

conjugate Hodge filtration in addition. This apparent inconsistency will be resolved in the appendix. Finally,

(3.13f)
$$\text{every polarized variation of Hodge structure of weight } n$$
$$\text{defines an object in CHM}(Z), \text{ of weight } n + \dim Z.$$

Of course this forces us define the weight filtration on $\mathcal{O}_Z(\mathbb{H})$ by decreeing it to be pure of weight $n + \dim X$. The Hodge filtration of the resulting object in CHM(Z) coincides with the Hodge filtration of the original variation of Hodge structure.

Let us mention two important properties of CMHM(Z) that follow from its definition. Recall the convention (3.12). If $f : Z \to X$ is an algebraic map,

(3.14)   the functors f^* and $f_!$ lower weights, and f_* and $f^!$ raise them;

here "lower" and "raise" are to be taken in the sense of non-increasing and non-deceasing, respectively. Then (3.14) has clear meaning when the functors are applied to a pure object. In general, the meaning is clarified by the strictness property

(3.15)
$$\text{every morphism of complex mixed Hodge modules preserves both}$$
$$\text{the Hodge filtration and the weight filtration strictly.}$$

In particular, there exist no nontrivial morphisms in CHM(Z) between objects of different weights.

Saito's categories of mixed Hodge modules MHM(Z) and of Hodge modules HM(Z) carry also a rational structure. This rational structure, for an object $(\mathcal{M}, W.\mathcal{M})$ of CMHM(Z), consist of a perverse sheaf of $\mathbb{Q}$-vector spaces $\mathcal{S}$ and a concrete isomorphism between DR($\mathcal{M}$) – the de Rham functor, applied to $\mathcal{M} \in$ Mod($\mathcal{D}_Z$)$_{\text{rh}}$ – and the complexification $\mathbb{C} \otimes_{\mathbb{Q}} \mathcal{S}$ of $\mathcal{S}$. In effect, Saito establishes the properties listed above in the presence of rational structures. In the appendix we shall sketch the modifications of his arguments that are necessary to prove these properties also in the absence of rational structures.

The category Mod($\mathcal{D}_Z$)$_{\text{frh}}$ of filtered regular holonomic $\mathcal{D}_Z$ was studied even before Saito's theory of mixed Hodge modules [14]. This category behaves well under the direct image under projective morphisms, and inverse image under smooth morphisms. In those cases, the effect of the induced morphisms in the category CMHM(Z) on the Hodge filtration is consistent with its effect in the category Mod($\mathcal{D}_Z$)$_{\text{frh}}$. An algebraic map $f : Z \to X$ between smooth complex quasi-projective varieties can be decomposed into a closed embedding via its graph, followed by the projection $Z \times X \to X$. Thus, to describe the functorial behavior of Mod($\mathcal{D}_Z$)$_{\text{frh}}$ under projective direct image, it suffices to treat the two special cases of closed smooth embedding and projection in a product with projective fiber. These two cases are easy to describe explicitly, and they are relevant for our application.

First we consider a smooth closed embedding $i : Z \to X$. We can choose local algebraic coordinates $z_1, \ldots, z_m, z_{m+1}, \ldots, z_{m+n}$, such that $Z \subset X$ is cut out by the equations $z_1 = z_2 = \cdots = z_m = 0$. The direct image $i_+\mathcal{M}$ of any $\mathcal{M} \in$ Mod($\mathcal{D}_Z$)$_{\text{frh}}$ can be identified as $\mathcal{D}_X$-module, locally in terms of the chosen coordinates, with

(3.16)
$$i_+\mathcal{M} \cong \mathcal{M}[\tfrac{\partial}{\partial z_1}, \tfrac{\partial}{\partial z_2}, \ldots, \tfrac{\partial}{\partial z_m}],$$

i.e., polynomials in the normal derivatives $\frac{\partial}{\partial z_j}$, $1 \leq j \leq m$, having local sections of $\mathcal{M}$ as coefficients. The filtration of $i_+\mathcal{M}$ can then described in terms of the

filtration of $\mathcal{M}$ by the identity

$$(3.17) \qquad F_p \, i_+ \mathcal{M} \; \cong \; \sum_{q+|\alpha|+m\leq p} (F_q \, \mathcal{M}) \, (\tfrac{\partial}{\partial z_1})^{\alpha_1} (\tfrac{\partial}{\partial z_2})^{\alpha_2} \cdots (\tfrac{\partial}{\partial z_m})^{\alpha_m} \, ,$$

with $|\alpha| = \alpha_1 + \cdots + \alpha_m$. As was mentioned already, this describes the effect on the Hodge filtration when $\mathcal{M}$ arises as the underlying filtered $\mathcal{D}_Z$-module of a object in CMHM(Z). In that situation,

$$(3.18) \qquad W_k \, i_+ \mathcal{M} \; \cong \; (W_k \, \mathcal{M})[\tfrac{\partial}{\partial z_1}, \tfrac{\partial}{\partial z_2}, \dots, \tfrac{\partial}{\partial z_m}] \, ,$$

describes the weight filtration. Globally the passage from $\mathcal{M}$ to $i_+ \mathcal{M}$ involves a twist by the top exterior power of $T_Z^* X$, the dual normal bundle, but that does not affect the descriptions (3.17–3.18) of the Hodge and weight filtrations.

To describe the effect of a projection $\pi : X \times Z \to Z$ with projective fiber X, we consider a filtered module $\mathcal{M} \in \mathrm{Mod}(\mathcal{D}_{X \times Z})_{\mathrm{frh}}$. Recall that if one disregards the filtration,

$$(3.19) \qquad \pi_+ \mathcal{M} \; = \; R\pi_*(\mathcal{M} \otimes \Omega_X^{\cdot}[\dim X]) \, ,$$

as an object of the derived category $D^b(\mathrm{Mod}(\mathcal{D}_{X \times Z})_{\mathrm{rh}})$. Here $(\Omega_X^{\cdot}[n])^p = \Omega_X^{n+p}$, since the the index p occurs as a superscript, in contrast to $(F_{\cdot}\mathcal{M}[m])_p = F_{p-m}\mathcal{M}$ in the case of a subscript; this is consistent with the convention of "lowering the index" $F^p \leftrightarrow F_{-p}$. The shift by n has the effect of putting the complex in degrees $-n, 1-n, \dots, -1, 0$. The filtration

$$(3.20) \qquad F_p \pi_+ \mathcal{M} \; = \; R\pi_*(F_p\mathcal{M} \to F_{p+1}\mathcal{M} \otimes \Omega_X^1 \to \cdots \to F_{p+n}\mathcal{M} \otimes \Omega_X^n)[n] \, ,$$

with $n = \dim X$, specifies $\pi_+ \mathcal{M}$ as object in $D^b(\mathrm{Mod}(\mathcal{D}_{X \times Z})_{\mathrm{frh}})$. When $\mathcal{M}$ underlies an object in CMHM($X \times Z$),

$$(3.21) \qquad \mathcal{M} \text{ has weight } k \quad \Longrightarrow \quad \pi_+ \mathcal{M} \text{ has weight } k \, .$$

In this connection one should recall (3.12).

To understand the behavior of the Hodge filtration under general direct images one needs to treat also the case of an open embedding. That is far more delicate and can only be done in the context of mixed Hodge modules, not for general filtered $\mathcal{D}$-modules; see the final remarks in section 5 below.

4. The Beilinson-Bernstein construction

We shall apply the theory of complex mixed Hodge modules to $U(\mathfrak{g})$-modules with real infinitesimal character. This involves Beilinson-Bernstein's notion of localization of $U(\mathfrak{g})$-modules on the flag variety X, which we shall review in this section. For the present purposes our "standing hypothesis" (2.6) is irrelevant, but we shall re-impose it after this section.

Recall that X, the variety of Borel subalgebras of $\mathfrak{g}$, is a smooth, complex projective variety, equipped with a transitive algebraic action of G. If $\mathfrak{b} \subset \mathfrak{g}$ is a Borel subalgebra with unipotent radical $\mathfrak{n}$, the universal Cartan algebra $\mathfrak{h}$ can be uniquely identified with any particular concrete subalgebra of $\mathfrak{b}$ so that $\mathfrak{n}$ becomes the direct sum of the root spaces corresponding to the negative roots,

$$(4.1) \qquad \mathfrak{b} \; = \; \mathfrak{h} \oplus \mathfrak{n} \, , \quad \mathfrak{n} \; = \; \oplus_{\alpha \in \Phi^+} \mathfrak{g}^{-\alpha} \, .$$

Essentially by definition, any λ in the weight lattice $\Lambda \subset \mathfrak{h}_{\mathbb{R}}^*$ lifts to an algebraic character e^λ of the Borel subgroup $B \subset G$ with Lie algebra $\mathfrak{b}$. There exists a unique algebraic, G-equivariant line bundle $\mathcal{L}_\lambda \to X$ on whose fiber over $\mathfrak{b}$ the isotropy

group B acts via e^λ, and every G-equivariant algebraic line bundle on X arises in this manner,

$$(4.2) \qquad \Lambda \;\cong\; \text{group of } G\text{-equivariant line bundles}\,, \quad \lambda \;\mapsto\; \mathcal{L}_\lambda \,.$$

The line bundle $\mathcal{L}_\lambda$ is ample in the sense of algebraic geometry precisely when λ is dominant and regular, i.e., when $(\alpha, \lambda) > 0$ for all $\alpha \in \Phi^+$.

It will be convenient to assume temporarily that the algebraic group G is simply connected; this hypothesis will be relevant only in the next two paragraphs. One defines

$$(4.3) \qquad\qquad \rho \;=\; \tfrac{1}{2} \sum\nolimits_{\alpha \in \Phi^+} \alpha \,.$$

Then 2ρ is a sum of roots, hence an element of the weight lattice. The fiber at $\mathfrak{b}$ of the tangent bundle of X is naturally isomorphic to $\mathfrak{g}/\mathfrak{b}$, hence

$$(4.4) \qquad \mathcal{L}_{-2\rho} \;\cong\; \wedge^n T^* X \;=\; \text{canonical bundle on } X \qquad (\, n = \dim X \,).$$

According to a standard fact on algebraic groups,

$$(4.5) \qquad\qquad G \text{ simply connected} \quad \Longrightarrow \quad \rho \in \Lambda \,,$$

in which case the canonical bundle $\mathcal{L}_{-2\rho}$ has a well defined square root $\mathcal{L}_{-\rho}$.

We define $\mathcal{D}_\lambda = \mathcal{O}(\mathcal{L}_{\lambda-\rho}) \otimes_{\mathcal{O}_X} \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{O}(\mathcal{L}_{\rho-\lambda})$. It is a G-equivariant sheaf of associative algebras over $\mathcal{O}_X$, locally isomorphic to $\mathcal{D}_X$, the sheaf of linear differential operators on X with algebraic coefficients. By definition, it acts on sections of $\mathcal{L}_{\lambda-\rho}$. The Lie algebra $\mathfrak{g}$ also acts, by infinitesimal translation, on sections of $\mathcal{L}_{\lambda-\rho}$. Thus $\mathfrak{g}$ embeds into the space of global sections of $\mathcal{D}_\lambda$, $\mathfrak{g} \hookrightarrow \Gamma \mathcal{D}_\lambda$, and this inclusion induces a morphism of associative algebras

$$(4.6) \qquad\qquad\qquad U(\mathfrak{g}) \;\longrightarrow\; \Gamma \mathcal{D}_\lambda$$

which is compatible with the degree filtrations on both sides. One can show easily that the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ acts on $\mathcal{O}(\mathcal{L}_{\lambda-\rho})$ via the character χ_λ, as defined in section 2. It follows that the homomorphism (4.6) drops from $U(\mathfrak{g})$ to the quotient

$$(4.7) \qquad U_\lambda(\mathfrak{g}) \;\;=_{\mathrm{def}}\;\; U(\mathfrak{g})/\operatorname{Ker}\{\chi_\lambda : Z(\mathfrak{g}) \to Z(\mathfrak{g})\} \cdot U(\mathfrak{g}) \,,$$

as a homomorphism

$$(4.8) \qquad\qquad\qquad U_\lambda(\mathfrak{g}) \;\longrightarrow\; \Gamma \mathcal{D}_\lambda$$

which again is compatible with the degree filtrations.

Like any algebraic line bundle, the G-equivariant line bundle $\mathcal{O}(\mathcal{L}_{\lambda-\rho})$ has local trivializations. Over the intersection of any two trivializing neighborhoods, these are related by a nowhere vanishing, algebraic transition function. Any local trivialization of $\mathcal{O}(\mathcal{L}_{\lambda-\rho})$ simultaneously trivializes the sheaf of algebras $\mathcal{D}_\lambda$, and local trivializations of the latter are related via conjugation by the transition functions for $\mathcal{O}(\mathcal{L}_{\lambda-\rho})$. Conjugation of a differential operator by a nowhere vanishing function f involves taking logarithmic derivatives of f. Thus conjugation of differential operators not only by f has meaning, but even conjugation by any complex power f^a. Since $\mathfrak{h}^* = \mathbb{C} \otimes_{\mathbb{Z}} \Lambda$, one can use this reasoning to show that

$$(4.9) \qquad\qquad \mathcal{D}_\lambda \text{ is well defined for any } \lambda \in \mathfrak{h}^* \,,$$

as infinitesimally $\mathfrak{g}$-equivariant, filtered, associative sheaf of algebras over $\mathcal{O}_X$, a so-called "twisted sheaf of differential operators". Unless $\lambda \in \Lambda$, $\mathcal{D}_\lambda$ cannot be realized as the sheaf of linear differential operators acting on sections of a globally defined line bundle, but (4.8) remains valid in this more general context. Now that

$\mathcal{D}_\lambda$ has been defined for any $\lambda \in \mathfrak{h}^*$, the assumption of the simple connectivity of G has served its purpose and can be dropped.

Theorem 4.10 (Beilinson-Bernstein [3])**.** *For any $\lambda \in \mathfrak{h}^*$, the morphism (4.8) is an isomorphism of filtered algebras; thus $H^0(X, \mathcal{D}_\lambda) = U_\lambda(\mathfrak{g})$. Also, $H^p(X, \mathcal{D}_\lambda) = 0$ for all $p > 0$ and any $\lambda \in \mathfrak{h}^*$.*

This theorem is the starting point of Beilinson-Bernstein localization. We let $\mathrm{Mod}(U_\lambda(\mathfrak{g}))_{\mathrm{fg}}$ denote the category of finitely generated $U_\lambda(\mathfrak{g})$-modules – or equivalently, the category of finitely generated $U(\mathfrak{g})$-modules on which $Z(\mathfrak{g})$ acts via the character χ_λ. Analogously $\mathrm{Mod}(\mathcal{D}_\lambda)_{\mathrm{coh}}$ shall refer to the category of $\mathcal{D}_\lambda$-coherent (sheaves of) $\mathcal{D}_\lambda$-modules. Since X is projective, it can be alternatively characterized as the category of finitely generated $\mathcal{D}_\lambda$-modules. In view of the theorem,

$$(4.11) \qquad \begin{aligned} \Delta \;:\; \mathrm{Mod}(U_\lambda(\mathfrak{g}))_{\mathrm{fg}} &\longrightarrow \mathrm{Mod}(\mathcal{D}_\lambda)_{\mathrm{coh}} \,, \qquad \Delta\, M \,=\, M \otimes_{U_\lambda(\mathfrak{g})} \mathcal{D}_\lambda \,, \\ \Gamma \;:\; \mathrm{Mod}(\mathcal{D}_\lambda)_{\mathrm{coh}} &\longrightarrow \mathrm{Mod}(U_\lambda(\mathfrak{g}))_{\mathrm{fg}} \,, \qquad \Gamma\, \mathcal{M} \,=\, H^0(X, \mathcal{M}) \,, \end{aligned}$$

are well defined covariant functors. The former is called "localization", and the latter is simply the global section functor, of course.

Recall that $\lambda \in \mathfrak{h}^*_{\mathbb{R}}$ is dominant if $2\frac{(\lambda, \alpha)}{(\alpha, \alpha)} \geq 0$ for all $\alpha \in \Phi^+$. More generally one calls $\lambda \in \mathfrak{h}^*$ dominant if its real part $\mathrm{Re}\,\lambda$, relative to the real structure $\mathfrak{h}^*_{\mathbb{R}}$, is dominant. There is a related notion,

$$(4.12) \qquad \lambda \in \mathfrak{h}^* \text{ is integrally dominant if } 2\frac{(\lambda, \alpha)}{(\alpha, \alpha)} \notin \mathbb{Z}_{<0} \text{ for any } \alpha \in \Phi^+,$$

which is less restrictive than dominance. As usual, we call $\lambda \in \mathfrak{h}^*$ regular if $(\lambda, \alpha) \neq 0$ for all $\alpha \in \Phi$.

Theorem 4.13 (Beilinson-Bernstein [3])**.** A) *If λ is regular and integrally dominant, the stalks of any $\mathcal{M} \in \mathrm{Mod}(\mathcal{D}_\lambda)_{\mathrm{coh}}$ are generated over $\mathcal{O}_X$ by the global sections of $\mathcal{M}$.*
B) *If λ is integrally dominant, $H^p(X, \mathcal{M}) = 0$ for all $\mathcal{M} \in \mathrm{Mod}(\mathcal{D}_\lambda)_{\mathrm{coh}}$ and any $p > 0$.*

From this, Beilinson-Bernstein deduce, by an essentially formal argument:

Corollary 4.14 (Beilinson-Bernstein [3])**.** *If λ is regular and integrally dominant, the functor Δ defines an equivalence of categories $\mathrm{Mod}(U_\lambda(\mathfrak{g}))_{\mathrm{fg}} \cong \mathrm{Mod}(\mathcal{D}_\lambda)_{\mathrm{coh}}$, with inverse Γ.*

In particular, in the case of a regular integrally dominant $\lambda \in \mathfrak{h}^*$, the irreducible objects in $\mathrm{Mod}(U_\lambda(\mathfrak{g}))_{\mathrm{fg}}$ correspond bijectively to irreducible objects in $\mathrm{Mod}(\mathcal{D}_\lambda)_{\mathrm{coh}}$. That much is still true in the singular – i.e., non-regular – but still integrally dominant case, although the general equivalence of categories fails without the regularity assumption: there may exists $\mathcal{M} \in \mathrm{Mod}(\mathcal{D}_\lambda)_{\mathrm{coh}}$ which have no cohomology in any degree. One can then define a quotient category of $\mathrm{Mod}(\mathcal{D}_\lambda)_{\mathrm{coh}}$ by setting all cohomologically trivial $\mathcal{M}$ equal to zero; the quotient category is equivalent to $\mathrm{Mod}(U_\lambda(\mathfrak{g}))_{\mathrm{fg}}$. If one is interested primarily in irreducible modules – as we are in the study of irreducible unitary representations of $G_{\mathbb{R}}$ – one can deal with the singular case more simply by realizing any irreducible $M \in \mathrm{Mod}(U_\lambda(\mathfrak{g}))_{\mathrm{fg}}$ as the space of global sections of a unique $\mathcal{M} \in \mathrm{Mod}(\mathcal{D}_\lambda)_{\mathrm{coh}}$; as was just mentioned, one can do so provided λ is at least integrally dominant.

We shall now describe a general construction of irreducible objects in the category $\mathrm{Mod}(\mathcal{D}_\lambda)_{\mathrm{coh}}$. Since we are not assuming that $\lambda - \rho \in \Lambda$, $\mathcal{L}_{\lambda-\rho}$ need not exist as

global G-equivariant line bundle on X. However, it is not difficult to see that $\mathcal{L}_{\lambda-\rho}$ does exist as an infinitesimally $\mathfrak{g}$-equivariant line bundle on various (Zariski) open subsets $U \subset X$. Its sheaf of sections $\mathcal{O}(\mathcal{L}_{\lambda-\rho})$ is then irreducible as $\mathcal{D}_\lambda$-module on U. We now suppose that $Q \subset X$ is a smooth subvariety, which is contained in one of the Zariski open subsets U on which $\mathcal{L}_{\lambda-\rho}$ has meaning. The second ingredient of the construction is $\mathcal{S} \to Q$, an irreducible algebraic vector bundle on Q with flat connection. Then $\mathcal{O}_Q(\mathcal{S})$ is an irreducible $\mathcal{D}_Q$-module, which is moreover locally free over $\mathcal{O}_Q$. Locally $\mathcal{D}_\lambda$ is isomorphic to $\mathcal{D}_X$, and $\mathcal{O}(\mathcal{L}_{\lambda-\rho}|_Q)$ is locally isomorphic to $\mathcal{O}_X$, so we can take the $\mathcal{D}_\lambda$-module direct image of $\mathcal{O}_Q(\mathcal{L}_{\lambda-\rho}|_Q \otimes_{\mathbb{C}} \mathcal{S})$ under the inclusion

$$(4.15) \qquad\qquad\qquad j \; : \; Q \;\; \hookrightarrow \;\; X \; .$$

This requires passage to the derived category, and the operation of $\mathcal{D}$-module direct image does not preserve coherence in general. However, $\mathcal{O}_Q(\mathcal{L}_{\lambda-\rho}|_Q \otimes_{\mathbb{C}} \mathcal{S})$ is locally free over $\mathcal{O}_Q$, hence holonomic. The $\mathcal{D}$-module direct image does preserve holonomicity, so

$$(4.16) \qquad\qquad j_+\mathcal{O}_Q(\mathcal{L}_{\lambda-\rho}|_Q \otimes_{\mathbb{C}}\mathcal{S}) \; \in \; D^b(\mathrm{Mod}(\mathcal{D}_\lambda)_{\mathrm{hol}}) \; ;$$

here $D^b(\mathrm{Mod}(\mathcal{D}_\lambda)_{\mathrm{hol}})$ refers to the bounded derived category, consisting of complexes with holonomic cohomology.

We let $\mathcal{H}^p(j_+\mathcal{O}_Q(\mathcal{L}_{\lambda-\rho}|_Q \otimes_{\mathbb{C}}\mathcal{S}))$ denote the cohomology sheaves of the direct image. When restricted to Q, the higher direct images – i.e., those indexed by $p > 0$ – vanish, and the restriction of $\mathcal{H}^0(j_+\mathcal{O}_Q(\mathcal{L}_{\lambda-\rho}|_Q \otimes_{\mathbb{C}}\mathcal{S}))$ coincides with $\mathcal{O}_Q(\mathcal{L}_{\lambda-\rho}|_Q \otimes_{\mathbb{C}}\mathcal{S})$. One calls

$$(4.17) \qquad \mathcal{M}(Q,\lambda,\mathcal{S}) \;\; =_{\mathrm{def}} \;\; \mathcal{H}^0(j_+\mathcal{O}_Q(\mathcal{L}_{\lambda-\rho}|_Q \otimes_{\mathbb{C}}\mathcal{S})) \; \in \; \mathrm{Mod}(\mathcal{D}_\lambda)_{\mathrm{hol}}$$

the standard sheaf corresponding to the given set of data. A proof of the following statement can be found in [15], for example.

Proposition 4.18. *The standard module $\mathcal{M}(Q,\lambda,\mathcal{S})$ has a unique irreducible submodule $\mathcal{I}(Q,\lambda,\mathcal{S})$. The quotient $\mathcal{M}(Q,\lambda,\mathcal{S})/\mathcal{I}(Q,\lambda,\mathcal{S})$ has support on the boundary ∂Q; in particular, $\mathcal{M}(Q,\lambda,\mathcal{S})$ is irreducible and coincides with $\mathcal{I}(Q,\lambda,\mathcal{S})$ when $Q \subset X$ is closed. Every irreducible holonomic $\mathcal{D}_\lambda$-module can be realized as $\mathcal{I}(Q,\lambda,\mathcal{S})$ for some suitable choice of Q, λ, $\mathcal{S}$.*

We should remark that the correspondence between irreducible holonomic and standard $\mathcal{D}_\lambda$-modules is essentially bijective: given an irreducible $\mathcal{M} \in \mathrm{Mod}(\mathcal{D}_\lambda)_{\mathrm{hol}}$, one can choose as Q any (Zariski) open subset of the regular set of the support of $\mathcal{M}$ such that the restriction $\mathcal{M}|_Q$ is locally free over $\mathcal{D}_\lambda$; choosing Q maximally makes it unique. Combining the proposition with the equivalence of categories 4.14, one finds:

Corollary 4.19. *If $\lambda \in \mathfrak{h}^*$ is integrally dominant, $H^0(X, \mathcal{I}(Q,\lambda,\mathcal{S}))$ is an irreducible $U_\lambda(\mathfrak{g})$-module or is zero[7]; if non-zero, it is also the unique irreducible submodule of the $U_\lambda(\mathfrak{g})$-module $H^0(X, \mathcal{M}(Q,\lambda,\mathcal{S}))$.*

Holonomicity is not automatic for irreducible coherent $\mathcal{D}_\lambda$-modules in general. However, as we shall discuss at the end of this section, it does become automatic if certain additional structures are imposed – for example, in the Harish Chandra setting. In settings when holonomicity does become automatic, proposition 4.18

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[7]It can vanish only when λ is singular, of course.

and its corollary almost provide a classification of the irreducible $U_\lambda(\mathfrak{g})$-modules. What is missing is information of when an irreducible sheaf fails to have global sections. That missing piece is provided by a result of Beilinson-Bernstein; see [8], for example.

The usual definition of the $\mathcal{D}$-module direct image, e.g. in [2], requires passage from left to right $\mathcal{D}$-modules. One can convert left to right $\mathcal{D}$-modules by twisting with the canonical bundle. If one wants to stay within the universe of left $\mathcal{D}$-modules, as we have chosen to do, the definition of the direct image requires a global twist by the ratio of the canonical bundles of X and Q, i.e., a twist by the top exterior power of the normal bundle $T_Q^* X$. The description of the polarization in the next section depends on an explicit description of $j_+ \mathcal{O}_Q(\mathcal{L}_{\lambda-\rho}|_Q \otimes_\mathbb{C} \mathcal{S})$, which we shall now give.

The inclusion $Q \subset X$ can be expressed as the composition of a smooth closed embedding and an open embedding:

$$(4.20) \qquad j = j_2 \circ j_1, \quad j_1 : Q \hookrightarrow X - \partial Q, \quad j_2 : X - \partial Q \hookrightarrow X.$$

Thus $j_+ \mathcal{O}_Q(\mathcal{L}_{\lambda-\rho}|_Q \otimes_\mathbb{C} \mathcal{S}) = j_{2+}\big(j_{1+}\mathcal{O}_Q(\mathcal{L}_{\lambda-\rho}|_Q \otimes_\mathbb{C} \mathcal{S})\big)$. Let c denote the codimension of Q in X, and

$$(4.21) \quad \mathcal{D}_{Q,\lambda} =_{\text{def}} \mathcal{O}_Q(\mathcal{L}_{\lambda-\rho}|_Q \otimes_\mathbb{C} \wedge^c T_Q^* X) \otimes_{\mathcal{O}_Q} \mathcal{D}_Q \otimes_{\mathcal{O}_Q} \mathcal{O}_Q(\mathcal{L}_{\rho-\lambda}|_Q \otimes_\mathbb{C} \wedge^c T_Q X) .$$

Then $\mathcal{O}_Q(\mathcal{L}_{\lambda-\rho} |_Q \otimes_\mathbb{C} \mathcal{S} \otimes_\mathbb{C} \wedge^c T_Q^* X)$ is tautologically a left $\mathcal{D}_{Q,\lambda}$-module. Less obviously, the sheaf restriction of $\mathcal{D}_\lambda$ to Q is a right $\mathcal{D}_{Q,\lambda}$-module, as can be seen by converting from right to left $\mathcal{D}$-modules and back again, or equivalently by twisting with the canonical bundle of X and the inverse of the canonical bundle of Q. The $\mathcal{D}$-module direct image under the closed smooth embedding j_1 is given by

$$(4.22) \qquad j_{1+}\mathcal{O}_Q(\mathcal{L}_{\lambda-\rho}|_Q \otimes_\mathbb{C} \mathcal{S}) = \mathcal{D}_\lambda \otimes_{\mathcal{D}_{Q,\lambda}} \mathcal{O}_Q(\mathcal{L}_{\lambda-\rho}|_Q \otimes_\mathbb{C} \mathcal{S} \otimes_\mathbb{C} \wedge^c T_Q^* X) .$$

This step does not require going to the derived category, i.e., $j_{1+}\mathcal{O}_Q(\mathcal{L}_{\lambda-\rho}|_Q \otimes_\mathbb{C} \mathcal{S})$ exists as a coherent $\mathcal{D}_\lambda$-module. At the next step, the $\mathcal{D}$-module direct image j_{2+} under the open embedding j_2 coincides with the derived sheaf direct image $R^\bullet j_{2*}$,

$$(4.23) \qquad \begin{aligned} j_+\mathcal{O}_Q(\mathcal{L}_{\lambda-\rho}|_Q \otimes_\mathbb{C} \mathcal{S}) &= j_{2+}\big(j_{1+}\mathcal{O}_Q(\mathcal{L}_{\lambda-\rho}|_Q \otimes_\mathbb{C} \mathcal{S})\big) \\ &= R^\bullet j_{2*}\big(j_{1+}\mathcal{O}_Q(\mathcal{L}_{\lambda-\rho}|_Q \otimes_\mathbb{C} \mathcal{S})\big) . \end{aligned}$$

It follows that the standard $\mathcal{D}_\lambda$-module $\mathcal{M}(Q, \lambda, \mathcal{S})$,

$$(4.24) \quad \mathcal{M}(Q, \lambda, \mathcal{S}) = \mathcal{H}^0(j_+\mathcal{O}_Q(\mathcal{L}_{\lambda-\rho}|_Q \otimes_\mathbb{C} \mathcal{S})) = R^0 j_{2*}\big(j_{1+}\mathcal{O}_Q(\mathcal{L}_{\lambda-\rho}|_Q \otimes_\mathbb{C} \mathcal{S})\big) ,$$

is the ordinary, underived sheaf direct image of the sheaf (4.22) under the open embedding j_2. This is the description we shall need.

As was mentioned in section 2, a Harish Chandra module with infinitesimal character χ_λ is a finitely generated $U_\lambda(\mathfrak{g})$-module, with a compatible structure of algebraic K-module. The analogous notion on the level of $\mathcal{D}$-modules is that of a Harish Chandra sheaf: a finitely generated $\mathcal{D}_\lambda$-module on X, equipped with the structure of K-equivariant quasi-projective $\mathcal{O}_X$-module, such that the two structures are compatible. In this context, compatibility means that the infinitesimal action of $\mathfrak{k}$ on the sheaf, by differentiation of the K-action, coincides with the action of $\mathfrak{k}$ via its inclusion in $\mathfrak{g}$ and the homomorphism (4.8). We let $\text{HC}(\mathcal{D}_\lambda, K)$

denote the category of Harish Chandra sheaves and $(\mathcal{D}_\lambda, K)$-homomorphisms be-tween them[8]. Feeding the K-action into the equivalence of categories 4.14, one obtains:

Corollary 4.25. *If λ is regular and integrally dominant, the functor Δ defines an equivalence of categories $\mathrm{HC}(\mathfrak{g}, K)_\lambda \cong \mathrm{HC}(\mathcal{D}_\lambda, K)$, with inverse functor Γ. If λ is integrally dominant, every irreducible Harish Chandra module can be realized as the space of global sections $H^0(X, \mathcal{M})$ of a unique irreducible Harish Chandra sheaf $\mathcal{M}$.*

By definition, a standard Harish Chandra sheaf is one constructed as in the discussion leading up to (4.16), but with ingredients Q, $\mathcal{S}$ that make the resulting sheaf K-equivariant: we require $Q \subset X$ to be one of the – finitely many – K-orbits in X, and $\mathcal{S}$ an irreducible K-equivariant flat vector bundle over Q. In other words, $\mathcal{S} \to Q$ is a K-homogeneous vector bundle associated to an irreducible represen-tation of the component group of the isotropy subgroup of K at any particular point of Q; the component group is finite abelian, so $\mathcal{S}$ necessarily has rank one. Implicit in the construction is the assumption that $\mathcal{L}_{\lambda-\rho}$ exists as infinitesimally $\mathfrak{g}$-equivariant line bundle on some neighborhood of Q, which turns out to be equiva-lent to a partial integrality condition on $\lambda - \rho$; for details see [7]. The K-equivariant ingredients Q and $\mathcal{S}$ make the sheaf $\mathcal{O}_Q(\mathcal{L}_{\lambda-\rho}|_Q \otimes_\mathbb{C} \mathcal{S})$ K-equivariant, compatibly with its structure as module over the appropriate K-equivariant twisted sheaf of dif-ferential operators on Q. The $\mathcal{D}_\lambda$-direct image inherits a compatible K-equivariant structure by functorality.

Standard Harish Chandra sheaves are all those which arise in this manner, from a K-orbit $Q \subset X$ and an irreducible K-equivariant flat vector bundle $\mathcal{S}$. Coherent sheaves of $\mathcal{D}_\lambda$-modules, equivariant with respect to a subgroup of G which acts on X with finitely many orbits, are automatically regular holonomic. In particular this applies to Harish Chandra sheaves. Thus, as a special case of proposition 4.18, one obtains:

Proposition 4.26. *Irreducible Harish Chandra sheaves correspond bijectively to standard Harish Chandra sheaves, via inclusion of the former in the latter, as the unique irreducible subsheaf. Standard sheaves associated to closed K-orbits are irreducible.*

We mentioned earlier that a criterion of Beilinson-Bernstein characterizes those irreducible sheaves which have no cohomology. In combination with this criterion, corollary 4.25 and proposition 4.26 provide a classification of the irreducible Harish Chandra modules. For details see [8], which also connections the various different classification schemes and relates algebraic properties of Harish Chandra sheaves to analytic properties – such as square integrability and temperedness – of the corresponding $G_\mathbb{R}$-representations.

The flag variety X is the universal complex projective variety with a transitive algebraic G-action – universal in the sense that any other such variety arise as a G-equivariant image of G. These G-equivariant images are called generalized flag varieties. Everything that has been said so far can be adapted to the setting of generalized flag varieties. However, for a generalized flag variety, the sheaves of

---

[8]If $G_\mathbb{R}$ and hence also K are connected, the K-action on a Harish Chandra sheaf is completely determined by the $\mathfrak{k}$-action, and hence by the $\mathcal{D}_\lambda$-module structure. Any $\mathcal{D}_\lambda$-morphism between Harish Chandra sheaves is then a morphism in the category $\mathrm{HC}(\mathcal{D}_\lambda, K)$.

twisted differential operators are defined not for all $\lambda \in \mathfrak{h}^*$, but only for λ in a suitable subspace of $\mathfrak{h}^*$. For that reason, only certain Harish Chandra modules can be constructed from Harish Chandra sheaves on any particular generalized flag variety. Also, K-orbits in X affinely embedded, as was observed by Beilinson-Bernstein – see [7] – but this is not true in the context of generalized flag varieties. In the general construction of standard modules (4.17), the higher direct images $\mathcal{H}^p(j_+\mathcal{O}_Q(\mathcal{L}_{\lambda-\rho}|_Q \otimes_{\mathbb{C}}\mathcal{S}))$, $p > 0$, vanish when $Q \subset X$ is affinely embedded. In that case, then,

$$(4.27) \qquad \mathcal{M}(Q,\lambda,\mathcal{S}) \;=\; j_+\mathcal{O}_Q(\mathcal{L}_{\lambda-\rho}|_Q \otimes_{\mathbb{C}}\mathcal{S}) \;\in\; \mathrm{Mod}(\mathcal{D}_\lambda)_{\mathrm{hol}} \;.$$

This is a significant property of standard Harish Chandra sheaves on the flag variety, which cannot be extended to generalized flag varieties.

5. Mixed Hodge modules on the flag variety

The discussion of the previous section applies to any $\lambda \in \mathfrak{h}^*$. When we put the structure of complex mixed Hodge module on standard Harish Chandra sheaves, it becomes necessary to assume $\lambda \in \mathfrak{h}_{\mathbb{R}}^*$. The passage from the Hodge filtration on a standard Harish Chandra sheaf to the Hodge filtration on the corresponding Harish Chandra module involves a vanishing theorem, which depends on the dominance of λ – integral dominance is not enough. For that reason we reimpose the condition (2.6), i.e., λ is required both to lie in $\mathfrak{h}_{\mathbb{R}}^*$ and to be dominant.

Our application of Hodge modules involves mixed Hodge modules built from filtered $\mathcal{D}_\lambda$-modules, rather than from filtered $\mathcal{D}$-modules as in section 3. Twisting the sheaf $\mathcal{D}$ by a nowhere vanishing function does not affect the degree filtration, and locally $\mathcal{D}_\lambda \cong \mathcal{D}$ by means of such a twist. That makes the passage from $\mathcal{D}$ to $\mathcal{D}_\lambda$ harmless as far as the filtrations are concerned.

The other crucial ingredient of the construction of complex Hodge modules is the polarization. We shall first describe the polarization in the untwisted case, and then discuss the necessary modification for the twisted case. We let $\overline{X}$ denote X equipped with the conjugate algebraic structure. Thus $f \mapsto \bar{f}$ defines a conjugate linear isomorphism $\mathcal{O}_X \cong \mathcal{O}_{\overline{X}}$. Similarly $\mathcal{M} \mapsto \overline{\mathcal{M}}$ describes a bijection between regular holonomic $\mathcal{D}_X$-modules and regular holonomic $\mathcal{D}_{\overline{X}}$-modules. Saito defines the polarization on an irreducible holonomic $\mathcal{D}$-module $\mathcal{M}$, when it exists, as an isomorphism between $\mathcal{M}$ and its conjugate dual. We shall use a more concrete definition, due to Sabbah [16], which is based on ideas of Kashiwara [10]: a nondegenerate hermitian pairing

$$(5.1) \qquad P \,:\, \mathcal{M} \times \overline{\mathcal{M}} \;\longrightarrow\; \mathcal{C}^{-\infty}(X_{\mathbb{R}}) \,, \quad \text{bilinear over } \mathcal{D}_X \times \mathcal{D}_{\overline{X}} \;;$$

here $X_{\mathbb{R}}$ denotes X considered as C^∞ manifold, and $\mathcal{C}^{-\infty}(X_{\mathbb{R}})$ the sheaf of distributions on $X_{\mathbb{R}}$. The important point is that this notion behaves well with respect to the standard operations on regular holonomic $\mathcal{D}$-modules.

In the setting of $\mathcal{D}_\lambda$-modules, $\mathcal{M} \mapsto \overline{\mathcal{M}}$ defines a bijection $\mathrm{Mod}(\mathcal{D}_{X,\lambda})_{\mathrm{rh}} \cong \mathrm{Mod}(\mathcal{D}_{\overline{X},-\lambda})_{\mathrm{rh}}$; the switch from λ to $-\lambda$ is explained by the fact that the character $e^{-\lambda}$ is the complex conjugate of e^λ. The subscripts X for $\mathcal{D}_{X,\lambda}$ and $\overline{X}$ for $\mathcal{D}_{\overline{X},-\lambda}$ were necessary to make the meaning unambiguous. For simplicity, from now on, we shall return to the notation $\mathcal{D}_\lambda$ for $\mathcal{D}_{X,\lambda}$ and write $\overline{\mathcal{D}}_{-\lambda}$ instead of $\mathcal{D}_{\overline{X},-\lambda}$. The analogue of (5.1) in the current situation is a hermitian pairing

$$(5.2) \qquad P \,:\, \mathcal{M} \times \overline{\mathcal{M}} \;\longrightarrow\; \mathcal{C}^{-\infty}(X_{\mathbb{R}}) \,, \quad \text{bilinear over } \mathcal{D}_\lambda \times \overline{\mathcal{D}}_{-\lambda} \,,$$

for $\mathcal{M} \in \mathrm{Mod}(\mathcal{D}_\lambda)_{\mathrm{rh}}$.

The definition of the categories $\mathrm{CHM}(Z)$, $\mathrm{CMHM}(Z)$, $D^b(\mathrm{CMHM}(Z))$ in section 3 carries over to the twisted setting. If $Q \subset X$ is a smooth subvariety such that $\mathcal{L}_{\lambda-\rho}$ exists as an infinitesimally $\mathfrak{g}$-equivariant line bundle on some neighborhood of Q, it makes sense to consider the analogous categories $\mathrm{CHM}(Q)_\lambda$, $\mathrm{CMHM}(Q)_\lambda$, $D^b(\mathrm{CMHM}(Q)_\lambda)$ whose objects have underlying filtered regular holonomic $\mathcal{D}_{Q,\lambda}$-modules; cf. (4.21). If one trivializes $\mathcal{L}_{\lambda-\rho}|_Q$ locally on Q, these three categories become locally isomorphic to their untwisted counterparts.

Going back to the definition (4.17) of the standard $\mathcal{D}_\lambda$-module $\mathcal{M}(Q, \lambda, \mathcal{S})$, we now require the irreducible flat vector bundle $\mathcal{S} \to Q$ to come equipped with a flat hermitian metric. In that case,

$$
\begin{aligned}
(5.3) \qquad P \, &: \, \mathcal{O}_Q(\mathcal{L}_{\lambda-\rho}|_Q \otimes_{\mathbb{C}} \mathcal{S}) \times \overline{\mathcal{O}_Q(\mathcal{L}_{\lambda-\rho}|_Q \otimes_{\mathbb{C}} \mathcal{S})} \; \longrightarrow \; \mathcal{C}^{-\infty}(Q_{\mathbb{R}}) \, , \\
P(\sigma, \overline{\tau}) \, &= \, \langle \, \sigma \, , \, \overline{\tau} \, \rangle \, \in \mathcal{C}^\infty(Q_{\mathbb{R}}) \subset \mathcal{C}^{-\infty}(Q_{\mathbb{R}}) \, ,
\end{aligned}
$$

defines a polarization for the irreducible $\mathcal{D}_{Q,\lambda}$-module $\mathcal{O}_Q(\mathcal{L}_{\lambda-\rho}\,|_Q \otimes_{\mathbb{C}} \mathcal{S})$; here $\langle \, \cdot \, , \, \cdot \, \rangle$ denotes the pointwise hermitian form determined by the flat hermitian metric on $\mathcal{S}$ and the intrinsic metric[9] on $\mathcal{L}_{\lambda-\rho}\,|_Q$. We can then turn $\mathcal{O}_Q(\mathcal{L}_{\lambda-\rho}\,|_Q \otimes_{\mathbb{C}} \mathcal{S})$ into an object in $\mathrm{CHM}(Q)_\lambda$ by imposing the "trivial" Hodge and weight filtrations

$$
\begin{aligned}
(5.4) \qquad F_p \, \mathcal{O}_Q(\mathcal{L}_{\lambda-\rho}|_Q \otimes_{\mathbb{C}} \mathcal{S}) \, &= \, \begin{cases} 0 \, , & p < 0 \, , \\ \mathcal{O}_Q(\mathcal{L}_{\lambda-\rho}|_Q \otimes_{\mathbb{C}} \mathcal{S}) \, , & p \geq 0 \, , \end{cases} \\
W_k \, \mathcal{O}_Q(\mathcal{L}_{\lambda-\rho}|_Q \otimes_{\mathbb{C}} \mathcal{S}) \, &= \, \begin{cases} 0 \, , & k < \dim Q \, , \\ \mathcal{O}_Q(\mathcal{L}_{\lambda-\rho}|_Q \otimes_{\mathbb{C}} \mathcal{S}) \, , & k \geq \dim Q \, . \end{cases}
\end{aligned}
$$

With a slight abuse of notation, we shall write $\mathcal{O}_Q(\mathcal{L}_{\lambda-\rho}\,|_Q \otimes_{\mathbb{C}} \mathcal{S}) \in \mathrm{CHM}(Q)_\lambda$. Here, as in the following, it shall be understood that the specified Hodge and weight filtration constitute part of the structure.

Recall the definition of the morphisms j_1 and j_2 in (4.20). The direct image of $\mathcal{O}_Q(\mathcal{L}_{\lambda-\rho}\,|_Q \otimes_{\mathbb{C}} \mathcal{S})$ under the smooth closed embedding j_1 remains irreducible, hence pure,

$$
(5.5) \qquad\qquad j_{1+} \mathcal{O}_Q(\mathcal{L}_{\lambda-\rho}|_Q \otimes_{\mathbb{C}} \mathcal{S}) \, \in \, \mathrm{CHM}(X - \partial Q)_\lambda \, ,
$$

but the open embedding j_2 may not preserve purity and requires passage to the derived category, so

$$
(5.6) \qquad\qquad j_{2+}\big(j_{1+}\mathcal{O}_Q(\mathcal{L}_{\lambda-\rho}|_Q \otimes_{\mathbb{C}} \mathcal{S})\big) \, \in \, D^b(\mathrm{CMHM}(X)_\lambda) \, .
$$

The cohomology sheaf in degree zero inherits this structure:

$$
(5.7) \qquad \mathcal{M}(Q, \lambda, \mathcal{S}) \, = \, \mathcal{H}^0\big(j_{2+} \circ j_{1+}\mathcal{O}_Q(\mathcal{L}_{\lambda-\rho}|_Q \otimes_{\mathbb{C}} \mathcal{S})\big) \, \in \, \mathrm{CMHM}(X)_\lambda \, .
$$

The unique irreducible subsheaf $\mathcal{I}(Q, \lambda, \mathcal{S})$ of $cM(Q, \lambda, \mathcal{S})$ must coincide with the lowest graded piece of the weigh filtration, which implies

$$
(5.8) \qquad\qquad\qquad \mathcal{I}(Q, \lambda, \mathcal{S}) \, \in \, \mathrm{CHM}(X)_\lambda \, .
$$

The same functorial arguments apply to Beilinson-Bernstein's maximal extension of $\mathcal{O}_Q(\mathcal{L}_{\lambda-\rho}|_Q \otimes_{\mathbb{C}} \mathcal{S})$ [4]. At this point we have put the structure of, respectively, Hodge module and mixed Hodge module on the irreducible $\mathcal{D}_\lambda$-module $\mathcal{I}(Q, \lambda, \mathcal{S})$, the standard $\mathcal{D}_\lambda$-module $\mathcal{M}(Q, \lambda, \mathcal{S})$, its dual, and the maximal extension. The

---

[9]The hypothesis $\lambda \in \mathfrak{h}_{\mathbb{R}}^*$ implies that $\overline{e^{\lambda-\rho}} = e^{\rho-\lambda}$, so the fiber $\overline{\mathcal{L}_{\lambda-\rho}}\,|_x$ at any $x \in Q$ is canonically dual to $\mathcal{L}_{\lambda-\rho}|_x$.

natural inclusion among these are morphism of mixed Hodge modules by construction.

It is entirely possible that the flat vector bundle $\mathcal{S}$ admits a non-trivial polarized variation of Hodge structure. If so, we could have carried out the constructions above starting with the resulting Hodge filtration instead of (5.4). However, only the trivial structure exists universally and is canonical. As was pointed out before, $\mathcal{S}$ has rank one in the K-equivariant Harish Chandra setting. In that case, the structure (5.4) is truly unique. Recall that the hypothesis (2.6) remains in force.

Theorem 5.9. *The global section functor* $\Gamma : \mathrm{CMHM}(X)_\lambda \to U_\lambda(\mathfrak{g})$ *is exact.*

We shall prove this result in a future paper. As an immediate consequence, the spaces of global sections of $\mathcal{I}(Q, \lambda, \mathcal{S})$, of $\mathcal{M}(Q, \lambda, \mathcal{S})$, of the dual of the latter, and of the maximal extension carry canonical, functorial weight and Hodge filtrations.

By functorality, the irreducible $\mathcal{D}_\lambda$-module $\mathcal{I}(Q, \lambda, \mathcal{S})$ has a polarization, with a distinguished choice of sign fixed by the polarization (5.3) for $\mathcal{O}_Q(\mathcal{L}_{\lambda-\rho}|_Q \otimes \mathcal{S})$. Since $U_\mathbb{R}$ is compact, there exists an essentially unique positive $U_\mathbb{R}$-invariant measure dm on X. If σ, τ are global sections of $\mathcal{I}(Q, \lambda, \mathcal{S})$, we can integrate the distribution $P(\sigma, \tau)$ against the smooth measure dm, which results in a hermitian form on $\mathcal{I}(Q, \lambda, \mathcal{S})$.

Proposition 5.10. *The hermitian form* $(\sigma, \tau)_{\mathfrak{u}_\mathbb{R}} = \int_X P(\sigma, \overline{\tau}) \, dm$ *is* $\mathfrak{u}_R$-*invariant.*

PROOF. We identify $\mathfrak{u}_\mathbb{R} \subset \mathfrak{u}_\mathbb{R} \otimes \mathbb{C} = \mathfrak{g} \subset \mathfrak{g} \oplus \overline{\mathfrak{g}}$ with a Lie algebra of real vector fields on X by infinitesimal translation, and correspondingly $\mathfrak{g}$ and $\overline{\mathfrak{g}}$ on the right of this inclusion[10] with Lie algebras of, respectively, holomorphic and antiholomorphic vector fields. We can then express any $\zeta \in \mathfrak{u}_\mathbb{R}$ as a sum $\zeta = \xi + \overline{\xi}$, with $\xi \in \mathfrak{g}$. Since $P(\ ,\ )$ is $\mathcal{D}_\lambda \times \overline{\mathcal{D}}_{-\lambda}$-bilinear, and since $\overline{\mathfrak{g}}$ and $\mathfrak{g}$ annihilate, respectively, $\mathcal{I}(Q, \lambda, \mathcal{S})$ and $\overline{\mathcal{I}(Q, \lambda, \mathcal{S})}$,

$$
\begin{aligned}
(\zeta\sigma, \tau)_{\mathfrak{u}_\mathbb{R}} + (\sigma, \zeta\tau)_{\mathfrak{u}_\mathbb{R}} &= \int_X P(\xi\sigma, \overline{\tau}) \, dm \ + \ \int_X P(\sigma, \overline{\xi}\overline{\tau}) \, dm \\
&= \int_X (\xi + \overline{\xi}) \, (P(\sigma, \overline{\tau})) \, dm \ = \ \int_X \zeta(P(\sigma, \overline{\tau})) \, dm \\
&= \int_X \zeta \, (P(\sigma, \overline{\tau}) \, dm) \ = \ 0 \ .
\end{aligned}
$$

(5.11)

At the final step we have used the $\mathfrak{u}_\mathbb{R}$-invariance of dm and the fact that an infinitesimal translate of a differential form of top degree is exact. $\square$

In the Harish Chandra setting, the Cartan involution θ acts on the flag variety. The irreducible Harish Chandra module $M = \Gamma \, \mathcal{I}(Q, \lambda, \mathcal{S})$ carries a nontrivial $\mathfrak{g}_\mathbb{R}$-invariant hermition form only when θ fixes the geometric data $(Q, \lambda, \mathcal{S})$. In that case θ acts on the sheaf $\mathcal{I}(Q, \lambda, \mathcal{S})$ and its space of global sections M. It is not difficult to show that action of θ coincides with the operator T of proposition 2.8, which relates the $\mathfrak{g}_\mathbb{R}$-invariant hermitian form to the $\mathfrak{u}_\mathbb{R}$-invariant one.

We can now state our conjecture more generally, not only for Harish Chandra modules as in (1.10):

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[10]Via $\mathfrak{g} \ni \zeta \mapsto (\zeta, \overline{\zeta}) \in \mathfrak{g} \oplus \overline{\mathfrak{g}}$.

Conjecture 5.12. *Suppose $M = \Gamma \mathcal{I}(Q, \lambda, \mathcal{S})$ is an irreducible $U_\lambda(\mathfrak{g})$-module corresponding to the geometric data Q, λ, $\mathcal{S}$, as described earlier in this section. Then $v \in F_p M \cap (F_{p+1}M)^\perp$, $v \neq 0$, implies $(-1)^{p-c}(v, v)_{\mathfrak{u}_R} > 0$ for all p.*

The integer $c = \operatorname{codim}_X Q$ turns out to be the lowest index of the Hodge filtration, which was denoted by a in conjecture 1.10.

We have verified the conjecture for irreducible Harish Chandra modules in quite a few cases. The simplest example of an irreducible $\mathcal{D}_\lambda$-module not in Harish Chandra's category is

$$(5.13) \qquad\qquad V_\lambda \;=\; \mathcal{I}(\{\mathrm{pt}\}, \lambda, \mathbb{C}) \;=\; \mathcal{M}(\{\mathrm{pt}\}, \lambda, \mathbb{C}),$$

the $\mathcal{D}_\lambda$-module direct image of the "flat vector bundle" $\mathbb{C}$ over a point – in other words, the irreducible Verma module of lowest weight $\lambda + \rho$. In that case, the hermitian form $(\ ,\ )_{\mathfrak{u}_\mathbb{R}}$ coincides with the Shapolvalov form. Its signature character was computed by Wallach [28], whose formula is compatible with conjecture 5.12 for V_λ but does not imply it. We did verify the conjecture in this particular case. It seems likely that this fact will be relevant for the proof of the general conjecture.

We should mention that the extension of the polarization (5.3), via j_{1+}, from $\mathcal{O}_Q(\mathcal{L}_{\lambda-\rho}|_Q \otimes_\mathbb{C} \mathcal{S})$ to $j_{1+}(\mathcal{O}_Q(\mathcal{L}_{\lambda-\rho}|_Q \otimes_\mathbb{C} \mathcal{S}))$ is easy to describe. Recall the identity (4.22). If σ, τ are local sections of $\mathcal{O}_Q(\mathcal{L}_{\lambda-\rho}|_Q \otimes_\mathbb{C} \mathcal{S} \otimes \wedge^c T_Q^* X)$ and D_1, D_2 local sections of $\mathcal{D}_\lambda$, all with overlapping domains, then $D_1\sigma$, $D_2\tau$ can be regarded as as local sections of $\mathcal{D}_\lambda \otimes_{\mathcal{D}_{Q,\lambda}} \mathcal{O}_Q(\mathcal{L}_{\lambda-\rho}|_Q \otimes_\mathbb{C} \mathcal{S} \otimes \wedge^c T_Q^* X) = j_{1+}(\mathcal{O}_Q(\mathcal{L}_{\lambda-\rho}|_Q \otimes_\mathbb{C} \mathcal{S}))$, and

$$(5.14) \quad \begin{aligned} &P \,:\, j_{1+}(\mathcal{O}_Q(\mathcal{L}_{\lambda-\rho}|_Q \otimes_\mathbb{C} \mathcal{S})) \times \overline{j_{1+}(\mathcal{O}_Q(\mathcal{L}_{\lambda-\rho}|_Q \otimes_\mathbb{C} \mathcal{S}))} \to \mathcal{C}^{-\infty}(X_\mathbb{R} - \partial Q_\mathbb{R})\,, \\ &P(D_1\sigma, \overline{D_2\tau}) = D_1\,\overline{D_2}\,\langle\, \sigma\,, \overline{\tau}\,\rangle \,\in\, \mathcal{C}^\infty(Q_\mathbb{R}, \wedge^{2c} T_{Q_\mathbb{R}}^* X_\mathbb{R}) \subset \mathcal{C}^{-\infty}(X_\mathbb{R} - \partial Q_\mathbb{R})\,, \end{aligned}$$

describes the polarization for $j_{1+}(\mathcal{O}_Q(\mathcal{L}_{\lambda-\rho}|_Q \otimes_\mathbb{C} \mathcal{S}))$. Note that in this identity, σ and $\overline{\tau}$ contract to a section of the top exterior power of the conormal bundle $T_{Q_\mathbb{R}}^* X_\mathbb{R}$ of the real submanifold $Q_\mathbb{R} \subset X_\mathbb{R}$. Any such section represents a distribution on $X_\mathbb{R} - \partial Q_\mathbb{R}$ with support in $Q_\mathbb{R}$, since it can be paired against any smooth measure on $X_\mathbb{R} - \partial Q_\mathbb{R}$, resulting in a form of top degree on $Q_\mathbb{R}$ with distribution coefficients – a smooth form of top degree, in fact – which could be integrated over $Q_\mathbb{R}$ if it were compactly supported. As a distribution on $X_\mathbb{R} - \partial Q_\mathbb{R}$, $\langle\, \sigma\,, \overline{\tau}\,\rangle$ can be acted upon by the differential operators D_1, $\overline{D_2}$.

The direct image of the polarizarion (5.14) under the open embedding j_2 becomes the integrand in the definition (5.10) of the hermitian form $(\ ,\ )_{\mathfrak{u}_\mathbb{R}}$. In a future paper we shall give an explicit description of the $\mathcal{D}$-module direct image under an open embedding. That description provides a handle on the hermitian form $(\ ,\ )_{\mathfrak{u}_\mathbb{R}}$.

A. Appendix

We shall briefly comment on the extension of Saito's theory we need for our applications. In one way, the development of the theory of mixed Hodge modules is formally analogous to that of the theory of regular holonomic $\mathcal{D}$-modules, though the former is far more intricate, of course: in both cases the existence of a category satisfying certain formal properties is reduced to the case of a base space of dimension one. For the one dimensional case, Saito relies on the nilpotent orbit theorem

and SL_2-orbit theorem of [21], though in later pages he refers only to the multi-variable generalization [5]. The extension of Saito's theory depends on a relaxation of the hypotheses of [5, 21].

The two papers describe, in very precise terms, the degeration of a polarized variation of Hodge structure – over a one dinensional base in [21], and over the complement of a divisor with normal crossings, near a point of the divisor, in [5]. In both cases the existence of an underlying rational structure implies that the monodromy eigenvalues are roots of unity. After [21] was written, Deligne observed that the arguments go through as long as the eigenvalues have absolute value one. The same is true for the results of [5], which are based on those of [21]. In the terminology of section 3, this means that [5, 21], with minor modifications, describe also the degeneration of complex polarized variations of Hodge structure; the polarization forces the eigenvalue one property.

Taking the complex conjugate of a complex mixed Hodge structure with its complex conjugate results in a mixed Hodge structure with underlying real structure – rather than an underlying rational structure, as is commonly assumed. This resolves the seeming contradiction commented upon in section 3, following (3.13e). The only critical use of the underlying rational structure in Saito's theory involves the orbit theorems; for other purposes an underlying real structure suffices. In this way one can avoid relying on an underlying rational structure, and via the trick of taking the direct sum with the complex conjugate, even on a real structure.

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