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Universal Signatures of Quantum Critical Points from Finite-Size Torus Spectra: A Window into the Operator Content of Higher-Dimensional Conformal Field Theories

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The low-energy spectra of many body systems on a torus, of finite size L , are well understood in magnetically ordered and gapped topological phases. However, the spectra at quantum critical points separating such phases are largely unexplored for 2+1D systems. Using a combination of analytical and numerical techniques, we accurately calculate and analyse the low-energy torus spectrum at an Ising critical point which provides a universal fingerprint of the underlying quantum field theory, with the energy levels given by universal numbers times $1/L$. We highlight the implications of a neighboring topological phase on the spectrum by studying the Ising* transition (i.e. the transition between a Z_2 topological phase and a trivial paramagnet), in the example of the toric code in a longitudinal field, and advocate a phenomenological picture that provides qualitative insight into the operator content of the critical field theory.

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Introduction — Quantum critical points continue to attract tremendous attention in condensed matter, statistical mechanics and quantum field theory alike. Recent highlights include the discovery of quantum critical points which lie beyond the Ginzburg-Landau paradigm [1, 2], the striking success of the conformal bootstrap program for Wilson-Fisher fixed points [3], and the intimate connection between entanglement quantities and universal data of the critical quantum field theory [4–8].

A surprisingly little explored aspect in this regard is the finite (spatial) volume spectrum on numerically easily accessible geometries, such as the Hamiltonian spectrum on a 2D spatial torus at the quantum critical point [9]. In the realm of 1+1D conformal critical points there exists a celebrated mapping between the spectrum of scaling dimensions of the field theory in \mathbb{R}^2 and the Hamiltonian spectrum on a circle (space-time cylinder: $S^1 \times \mathbb{R}$) [10]. This result is routinely used to perform accurate numerical spectroscopy of conformal critical points using a variety of numerical methods [11, 12]. In higher dimensions the situation is less favorable: Cardy has shown [13] that the corresponding conformal map can be generalized to a map between \mathbb{R}^d and $S^{d-1} \times \mathbb{R}$. While numerical simulations in this so-called *radial quantization* geometry have been attempted at several occasions [14–18], this numerical approach remains very challenging due to the curved geometry, which is inherently difficult to regularize in numerical simulations.

Although low-energy spectra on different toroidal configurations have been discussed in the context of some specific field theories (in Euclidean spacetime) [19–23], our understanding of critical energy spectra is rather limited beyond free theories [24–28]. This is due to the absence of a known relation between the scaling dimensions of the field theory and the torus energy spectra.

In this Letter we present a combined numerical and analytical study of the Hamiltonian torus energy spectrum of the

3D Ising conformal field theory (CFT), and show that it is accessible with finite lattice studies and proper finite-size scaling. Torus energy spectra provide a universal fingerprint of the quantum field theory governing the critical point and depend only on the universality class of the transition and on the shape and boundary conditions of the torus, which acts as an infrared (IR) cutoff (but not on the lattice discretisation, i.e. the ultraviolet cutoff). We will explicitly demonstrate this here for the Ising CFT. This approach will also be valuable as a new numerical tool to investigate and discriminate quantum critical points.

We provide a quantitative analysis of many low-lying energy levels of the standard Z_2 -symmetry breaking phase transition in the 3D Ising universality class. We also advocate a phenomenological picture that provides qualitative insight into the operator content of the critical point. As an application we reveal that the torus energy spectrum of the confinement transition between the Z_2 topological ordered phase and the trivial (confined) phase of the Toric code (TC) in a longitudinal magnetic field can be understood as a specific combination of a subset of the fields and several boundary conditions of the standard 3D Ising universality class. Since the operator content of the partition function at criticality obviously differs from the standard 3D Ising universality class we term this transition a 3D *Ising** transition [29–31].

3D Ising universality class — In order to demonstrate the universal nature of the low-energy spectrum we study the 2+1D transverse field Ising (TFI) model

$$H_{\text{TFI}} = -J \sum_{\langle i,j \rangle} \sigma_i^z \sigma_j^z - h \sum_i \sigma_i^x \quad (1)$$

on five different two-dimensional Archimedean lattices [32] at their respective quantum critical point [33][34]. In our finite size simulations the spatial setup is a torus whose linear extents are determined by two spanning vectors ω_1 and ω_2 (c.f. left part of Fig. 1). The finite area leads to a discrete momen-

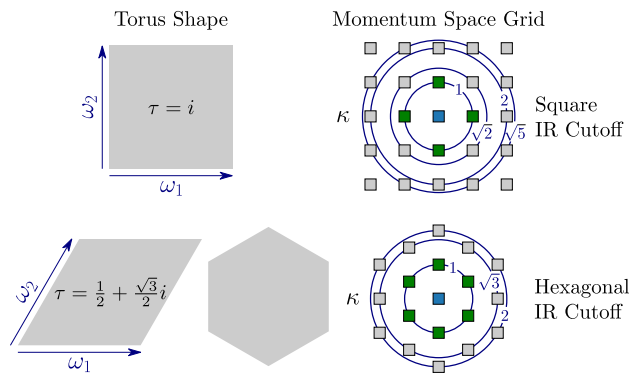


FIG. 1. The two torus geometries with 4-fold and 6-fold rotation symmetry and their momentum-space grid in the vicinity of the $\Gamma = (0, 0)$ point. In the center of the lower row we display the Wigner-Seitz cell of the torus, highlighting the 6-fold symmetry. The momentum space variable κ is defined as $\kappa = \frac{L}{2\pi} |\mathbf{k}| \tau_2$ with $\tau = \tau_1 + i\tau_2$, $L = |\omega_1| = |\omega_2|$ and \mathbf{k} a momentum of the finite-size cluster.

tum space (c.f. right part of Fig. 1) and is equivalent to an infrared cutoff in the field theory. The use of a lattice model on the other hand leads to an ultraviolet (UV) cutoff in the form of a Brillouin zone. In the following we will only consider tori with $L = |\omega_1| = |\omega_2|$ and two different choices of the modular parameter $\tau = \omega_2/\omega_1$: $\tau = i$ ($\tau = 1/2 + \sqrt{3}/2i$) corresponding to a square (hexagonal) symmetry. The square and square-octagon (triangular, honeycomb and kagome) lattices are simulated using a square (hexagonal) IR-cutoff geometry to preserve the microscopic C_4 (C_6) point group symmetry in the IR.

In a first step we have calculated the low-energy spectrum of the Hamiltonian Eq. (1) using exact diagonalization (ED) in all symmetry sectors on finite samples with up to $N = 40$ spins in total. The spectrum can be divided into \mathbb{Z}_2 even and odd sectors (spin-flip symmetry), combined with irreducible representations of the lattice space group. In the paramagnetic phase at large h/J one finds a unique \mathbb{Z}_2 even ground state in the fully symmetric spatial representation, with a finite gap above the ground state. At small h/J one finds two quasi-degenerate ground states in the \mathbb{Z}_2 even and odd sector respectively (both in the symmetric spatial representation), again with a finite gap above the ground state. At the quantum critical point $(h/J)_c$ however the low-lying spectrum collapses as $1/\sqrt{N} \sim 1/L$, i.e. it exhibits a mass spectrum with the mass scale set by the IR cutoff. To eliminate this scaling we will multiply the excitation gaps with \sqrt{N} in the following and will call that the spectrum. In Fig. 2 we display the finite size spectra at the Ising critical point for all five different lattices in the zero momentum sector $\Gamma = (0, 0)$, as well as the first momentum away from the Γ point ($\kappa = 1$ in the right part of Fig. 1). Since the speed of light is not known at this stage, the spectrum for each lattice has been globally rescaled such that the extrapolated energy of the first excited level (which is \mathbb{Z}_2 odd and spatially symmetric) is set to one.

One explicitly observes that the critical energy spectra of lattices with the same type of IR cutoff τ (the two leftmost panels and the three rightmost panels) agree to rather high precision with each other, when taking $1/N$ finite-size corrections into account [35]. This means that - as is generally expected from a field theory point of view - the obtained critical energy spectra indeed do not depend on the chosen UV discretization. In order to corroborate the extrapolations based on ED we performed extensive Quantum Monte Carlo (QMC) simulations [33] of the transverse field Ising model at the critical point for all five lattices. Based on imaginary time spin-spin correlations it is possible to access the finite size gaps on lattices up to $N = 30 \times 30$ lattice sites [36]. These data points (red small filled circles) in Fig. 2 reproduce the ED data where available, and allow us to confirm and sharpen the precision of the extrapolated energy spectrum. Based on the quantum numbers of the first few low-lying energy levels we choose to label them as torus analogues of the spectrum of scaling dimensions of the 3D Ising CFT: σ_T and σ'_T refer to the first two levels in the \mathbb{Z}_2 odd sector in the spatially symmetric representation, while ϵ_T is the first excited state (above the vacuum 1) in the \mathbb{Z}_2 even and spatially symmetric sector. The "... + $\Delta\kappa$ " label refers to levels at the first momentum away from the Γ point, $\kappa = 1$. These levels are four-fold degenerate on the square torus, while they are six-fold degenerate for the hexagonal torus. Although there is no known relation between the torus spectrum and the scaling dimensions in flat space, this phenomenological approach shows a qualitatively similar structure as the operator content of the quantum field theory.

ϵ -expansion — We also compute the energy levels using ϵ -expansion. Our starting point is ϕ^4 theory, which we define by the Hamiltonian density

$$\mathcal{H} = \int d^d x \left[\frac{1}{2} \Pi^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{s}{2} \phi^2 + \frac{u}{4!} \phi^4 \right] \quad (2)$$

in d dimensions with the equal-time commutator $[\phi(x, t), \Pi(x', t)] = i\delta^d(x - x')$, and specialize to the critical point, $s = s_c$, $u = u^*$. We generalize the two-dimensional torus to arbitrary dimension by taking $d/2$ copies of the desired tori in Fig. 1, so that all spatial point-symmetries are preserved during the calculation and no extra length scales are introduced.

Our approach to the critical theory in a finite volume originated from Lüscher [37], and was extended to deal with finite size criticality in classical systems by others [38, 39]. The key observation is that the zero mode of the field generates incurable infrared divergences in perturbation theory, so it must be separated and treated non-perturbatively. In the context of the finite-size spectrum, this can be understood from Eq. (2) by noticing that the Gaussian theory at $s = 0$ does not contain any potential term for the zero mode, giving a continuous spectrum, whereas any finite u will confine the zero mode producing a discrete spectrum. Therefore, the correct perturbative approach is to treat the momentum of the zero mode at the same order as its interactions.

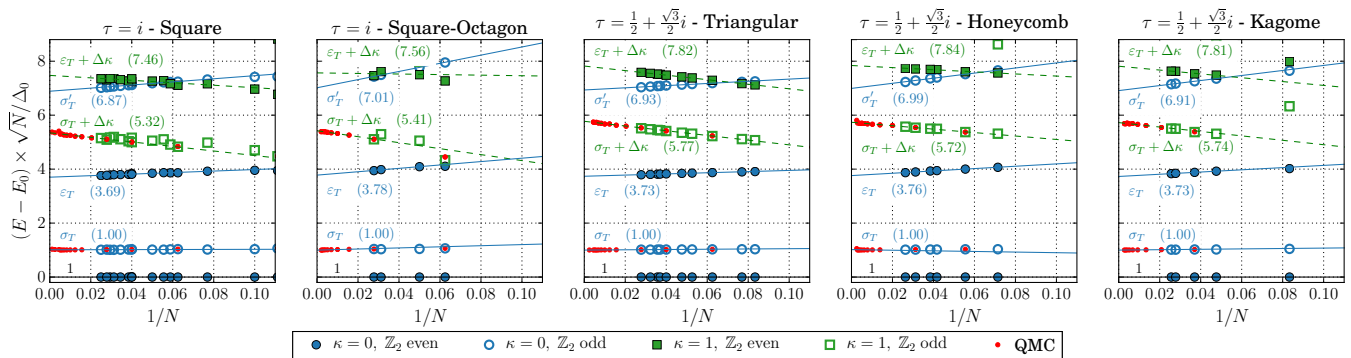


FIG. 2. Normalized low-energy torus spectrum for the Ising QFT for the modular parameters $\tau = i$ and $\tau = 1/2 + \sqrt{3}/2i$ obtained with ED (large symbols) and QMC (small red filled circles). Filled (empty) symbols denote \mathbb{Z}_2 even (odd) levels. Linear fits in $1/N$ for levels with $\kappa = 0$ ($\kappa = 1$) are shown by blue solid (green dashed) lines (cf. color coding in Fig. 1) and the values of the fields after extrapolation to the thermodynamic limit $1/N \rightarrow 0$ are given in parentheses. The normalization constant Δ_0 is chosen such that the first \mathbb{Z}_2 odd level extrapolates to one. We observe a universal torus spectrum for the lattices with the same type of IR cutoff (same τ).

By splitting the fields in Eq. (2) and proper normalization of the zero-mode terms the Hamiltonian can be decomposed into a quadratic part \mathcal{H}_0 describing the Fock spectrum of the finite-momentum modes, and an interaction part V containing all zero-mode contributions and non-linearities.

At zeroth order, our states are given by finite momentum Fock states multiplied by arbitrary functionals of the zero mode, so these states are infinitely degenerate. We then derive an effective Hamiltonian within each degenerate subspace using a perturbation method due to C. Bloch [40]. This effective Hamiltonian acts in a degenerate subspace, but its eigenvalues correspond to the exact eigenvalues of the original Hamiltonian to desired order. It turns out, that the effective Hamiltonians take the form of a strongly-coupled oscillator with coefficients depending on the degenerate subspaces. The coefficients of the more complicated expansion for the energy levels (expansion in $\epsilon^{1/3}$) can be found in [41]. In addition, the effective Hamiltonian will couple different Fock states with the same energy and momentum whenever possible, leading to off-diagonal terms. These off-diagonal terms were computed numerically from the unperturbed wave-function. Further details about the ϵ -expansion approach can be found in the Supplemental Material [42].

In Fig. 3 we show the universal torus spectrum obtained from ϵ -expansion for the two choices of τ and compare it to numerical results from ED and QMC computations [43] normalized by the speed of light c [44][45]. We observe a remarkable agreement between the two different methods. This further illustrates the interpretation of the torus spectra as a universal fingerprint of the critical field theory and their accessibility from numerical finite lattice simulations. The larger discrepancies between numerical and ϵ -expansion data for some higher levels in the spectrum may result from the extrapolation to the thermodynamic limit using only ED data with strong finite-size effects, especially for $\kappa > 0$ [46].

2+1D Ising universality class* — In this section we are investigating the confinement transition of a \mathbb{Z}_2 spin liquid.

Such a topological quantum phase transition is characterized by the lack of any local order parameters. \mathbb{Z}_2 spin liquids are characterized by the presence of two bosons, the e and m particles. These fractionalized particles can only be created in pairs and obey mutual anyonic statistics. The confinement transition can then be driven by condensing either the e or the m particles. Without loss of generality, we will consider the condensation of the m particles and call it's corresponding field ϕ . The critical theory turns out to be that of *Ising**: ϕ can only be created in pairs, so the effective Lagrangian must be even in a real field ϕ , implying we should only include \mathbb{Z}_2 even states in a critical Ising theory. In addition, ϕ and $-\phi$ are physically indistinguishable, and so both periodic and anti-periodic boundary conditions have to be considered. We emphasize that this mapping is independent of any specific microscopic lattice model and should hold generically between universal theories and their topological counterparts.

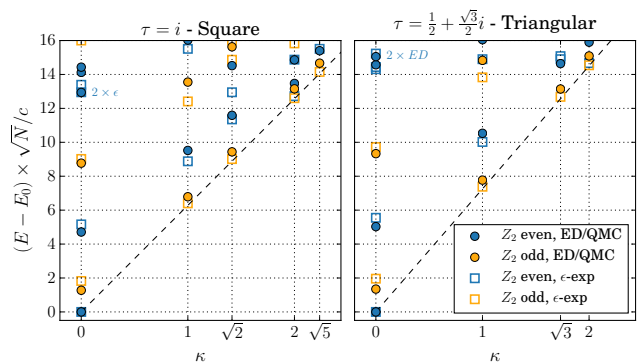


FIG. 3. Universal torus spectra for the Ising QFT for the modular parameters $\tau = i$ (left panel) and $\tau = 1/2 + \sqrt{3}/2i$ (right panel). Full symbols denote numerical results obtained by ED and QMC (the lowest \mathbb{Z}_2 odd levels), while empty symbols denote the ϵ -expansion results. The dashed line shows a dispersion according to the speed of light.

As a microscopic model illustrating this transition we study the critical energy spectrum of the Toric Code Hamiltonian perturbed by a longitudinal field [47–51]:

$$H_{TC} = -J \sum_s A_s - J \sum_p B_p - h \sum_i \sigma_i^x \quad (3)$$

$$A_s = \prod_{i \in s} \sigma_i^x, \quad B_p = \prod_{i \in p} \sigma_i^z$$

The σ_i describe $S = 1/2$ -spins on the $2N$ edges of a square lattice, p denotes a plaquette and s a star on the lattice. All A_s and B_p commute with each other and so the model can be solved analytically for $h = 0$ by setting all $A_s = 1$ and all $B_p = 1$ [52]. On a torus the ground state manifold is, however, four-fold degenerate and can be characterized by the eigenvalues ± 1 of Wilson loops winding around the torus. An e (m) particle is described by setting $A_s = -1$ ($B_p = -1$) on a star (plaquette). The longitudinal field introduces a dispersion for the m particles which finally condense and drive the phase transition at $h = h_c$ by confinement of the e particles [29–31, 47].

The above considerations regarding the relationship between Ising and Ising* quantum field theories (QFT) can be made very explicit for the Toric Code. The Toric Code Eq. (3) in the sector without e particles ($A_s = 1$) can be exactly mapped to an even TFI model on the dual square lattice with N sites, where only the even spin-flip sector is present [47, 53, 54]. The groundstate manifold, described by the eigenvalues of the Wilson loops, maps to both, periodic and anti-periodic boundary conditions of the Ising model [55]. In the following we will make use of this mapping to compute the finite-size torus spectrum of the Ising* transition for $\tau = i$ using ED.

In the left part of Fig. 4 we present the low-energy finite-size spectrum of the Ising* transition obtained with ED simulations. The spectrum is rescaled with the same factor Δ_0 as in Fig. 2 such that they can be easily compared. The relationship between the critical Ising and Ising* theories results in the fact that the levels called $\varepsilon_T(+\Delta\kappa)$ in Fig. 2 are identically present in the Ising* spectrum (c.f. P/P levels in Fig. 4). The most remarkable feature, however, is the presence of very low-lying levels in the spectrum. They arise from the groundstate manifold in the spin-liquid phase, where their splitting exponentially scales to zero with L . At criticality they, however, scale as $1/\sqrt{N}$ as the entire low-energy spectrum. The small relative splitting of the four lowest levels is surprisingly small. The right panel of Fig. 4 shows a comparison of the universal torus spectrum for an Ising* transition obtained with ED and ϵ -expansion similar to Fig. 3 [56]. A zoom into the conspicuous low-energy levels is shown in the inset. Again we observe a decent agreement of the different methods.

Conclusions — We have computed the universal torus energy spectrum for the Ising and Ising* transitions in 2+1D providing a characteristic fingerprint of the corresponding conformal field theories and have highlighted the implications of a neighbouring \mathbb{Z}_2 spin liquid on the torus spectrum. Additionally, we have highlighted a phenomenological picture based on the quantum numbers of the individual energy levels which shows a structure qualitatively similar to the operator content

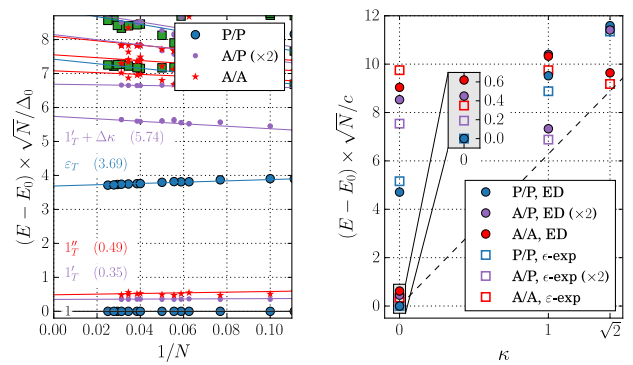


FIG. 4. Universal torus spectra for the Ising* QFT and the modular parameter $\tau = i$. The labels A/P etc. denote the boundary conditions along the two directions of the torus, where P(A) means (anti-)periodic. Left: Normalized low-energy spectrum from ED with the same normalization constant Δ_0 as in Fig. 2. The levels in the P/P sector are the $\varepsilon_T(+\Delta\kappa)$ levels from the TFI spectrum. A very remarkable feature is the presence of the four very low-lying levels which govern the four-fold degenerate groundstate manifold in the deconfined phase. See Fig. 2 for further details. Right: Full symbols denote numerical results obtained by ED, while empty symbols denote ϵ -expansion results. The dashed line shows a dispersion with the speed of light. The inset is a zoom into the four lowest levels. See Fig. 3 for further details.

of the field theory in flat space. Using the numerical and analytical technology presented in this paper it will be possible to inspect and chart the characteristic spectrum of more complex quantum critical points, such as $O(N)$ Wilson-Fisher fixed points, Gross-Neveu-Yukawa type phase transitions in interacting Dirac fermion models [57, 58] or designer Hamiltonians displaying deconfined criticality [2].

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Let us finally apply the mapping on the different ground-state sectors characterized by the eigenvalues of $t_{1,2}$. Using Eq. (8) and Eq. (9) it follows that

$$t_1 = \prod_{p=0}^{L-1} \mu_{(p,j)}^x \mu_{(p+1,j)}^x = \mu_{(0,j)}^x \mu_{(L,j)}^x \quad (12)$$

where the index (p, j) labels the position $p\hat{x} + j\hat{y}$ on the dual lattice and L is the linear extend of the torus. An equivalent relation can be computed for t_2 . The different ground-state sectors of the Toric Code therefore map onto periodic and antiperiodic boundary conditions of the transverse field Ising model for both directions around the torus.

Finite size gap estimation with QMC

To estimate the gaps in a larger range of system sizes than reachable by ED, we use a continuous-time world-line Monte-Carlo scheme, supplemented with a cluster update [33] to overcome critical slowing down at the quantum phase transition.

For our computations, we used system linear sizes ranging up to $L = 30$ (for the simplest lattices). For each system, the average energy was computed from several (from 16 to 256) independent runs of 10^4 to 10^6 measurements. Between two measurements, the number of cluster updates n_c was chosen so that $n_c \langle s_c \rangle \gtrsim \beta L^2$, where $\langle s_c \rangle$ is the average cluster size. This leads to an autocorrelation-time of order one Monte Carlo step or less.

The world-line Monte-Carlo allows to extract the excitation spectrum of the system through the evaluation of the imaginary-time spin-spin correlation function. Indeed, the spin-spin correlation function at momentum \mathbf{q} is given by

$$S^{zz}(\mathbf{q}, \tau) = \frac{1}{\beta N} \left\langle \sum_{i,j} \int_0^\beta d\tau' e^{-i\mathbf{q}\cdot(\mathbf{r}_i - \mathbf{r}_j)} s_i^z(\tau') s_j^z(\tau' + \tau) \right\rangle \approx e^{-\Delta_{\mathbf{q}} \tau}, \quad (13)$$

where $\Delta_{\mathbf{q}}$ is the excitation gap at momentum \mathbf{q} . To optimize the estimation of the gap, we fit an exponential decay to the spin-spin correlation function for imaginary times $\tau > \tau_{\min}$ (Fig. 7, top panel) and find the value τ_{\min}^* that minimizes the relative fitting error on the value of the gap (Fig. 7, bottom panel).

Finite-size extrapolation of the energy gaps

To motivate the dominant $1/N$ scaling used in our extrapolations of the finite-size energy gaps $L \Delta$ we have calculated the dispersion relation $\epsilon(\mathbf{k})$ at criticality within linear spin-wave theory (LSWT). We can use this dispersion to compute the finite-size scaling of a $\kappa \neq 0$ level within this approach by

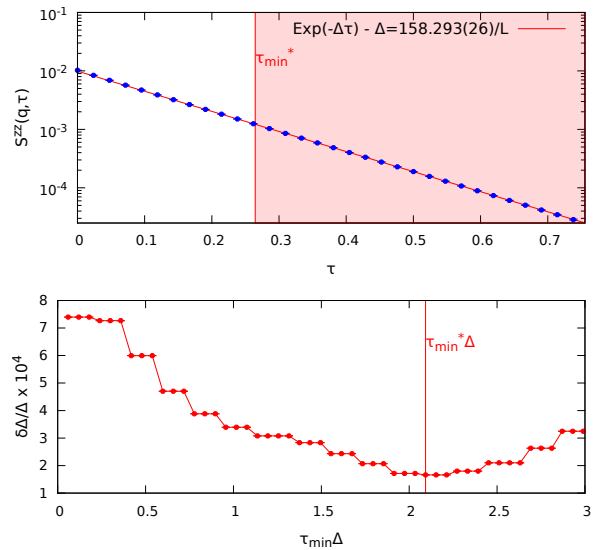


FIG. 7. Extraction of the excitation gaps from QMC data (here we display the example of a triangular lattice system of size $L = 20$, at the smallest non-zero momentum). *top panel* Spin-spin correlation function as a function of imaginary time compared to the fitted exponential. *bottom panel* Relative fitting error on the evaluation of the gap as a function of the minimal imaginary time used in the fit.

considering a momentum $\mathbf{k} = \frac{1}{L}(u, v)$ (on the square lattice) and expanding the resulting expression in powers of $1/L$

$$\epsilon(\mathbf{k}) = \frac{c}{L} \left(1 - \frac{a}{L^2} + \mathcal{O}(1/L^4) \right) \quad (14)$$

$$c = \sqrt{(u^2 + v^2)/2} \quad (15)$$

$$a = \frac{u^4 + v^4}{24(u^2 + v^2)} \quad (16)$$

The dominant corrections to the spectral levels $L \Delta$ within the LSWT approach are thus $1/L^2 = 1/N$. In Fig. 8 we compare the finite-size extrapolation of a spectral level in the full LSWT approach and in the power-series expansion up to $1/N$. The effect of higher order corrections is small already for intermediate size systems.

We have also considered fitting approaches with additional $1/L$ terms for the extrapolation of the spectral levels to the thermodynamic limit $N \rightarrow \infty$. In Fig. 9 we compare different fitting approaches of the QMC gaps $L \Delta$ for σ_T levels in different κ sectors on the triangular lattice. While a correction in purely $1/L$ (red fits) gives poor results, correction terms in $1/L^2$ with and without an additional $1/L$ term fit the data very well leading to similar extrapolated gaps. The fitting procedure with both terms is, however, much more instable when very small systems are included in the fit, often leading to strong dips close to $1/N = 0$.

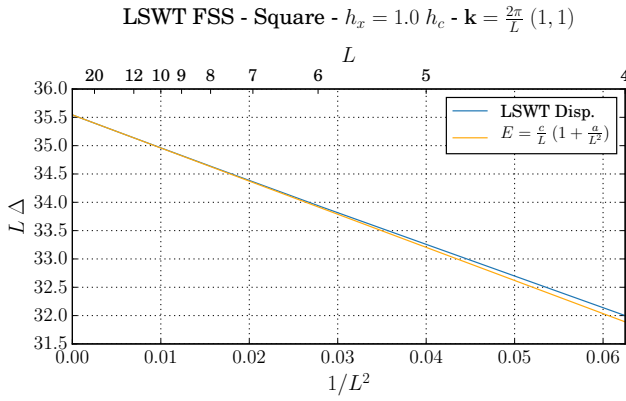


FIG. 8. Finite size scaling of a $\kappa = \sqrt{2}$ spectral level at criticality within a linear spin-wave theory (LSWT) approach. The blue curve is according to the LSWT dispersion, the yellow curve is a power-series of this dispersion up to the second non-vanishing order.

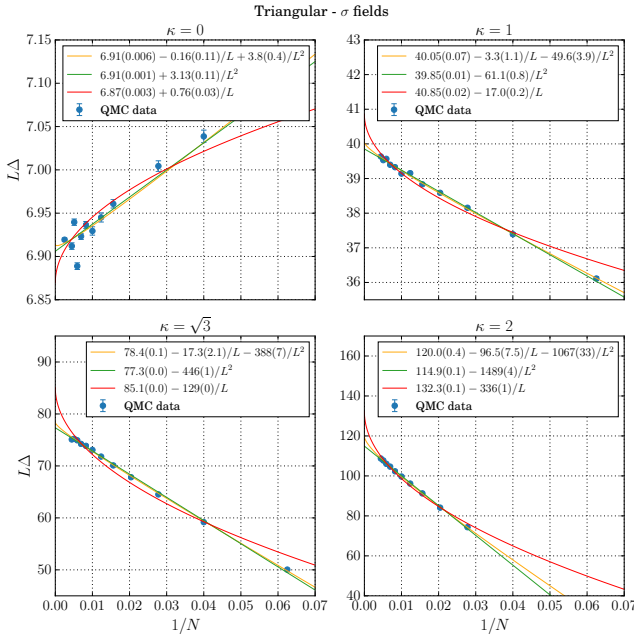


FIG. 9. Comparison of different fitting approaches for the extrapolation of the finite-size σ_T levels in different κ sectors. The values in parantheses give the pure fitting error for the given values.

Speed of light from QMC

For each lattice, in order to extract the speed of light c , we proceed as follows. We first extract with QMC the energies $E_L^{\sigma_T}(\kappa)$ of the lowest \mathbb{Z}_2 odd levels at a given system size L . We then extrapolate those to the thermodynamic limit $E^{\sigma_T}(\kappa)$. We finally fit a line $E^{\sigma_T}(\kappa) = \delta E + c \cdot \kappa$ to this extrapolated data, in the interval $[\kappa_{\min}, \kappa_{\max}]$ [44]. Since we expect the levels at small momenta to be affected by the periodic boundary conditions, we take $\kappa_{\min} > 0$. For large momenta,

the finite-size curvature effects render the thermodynamic-limit extrapolation ill-defined, and therefore one has to introduce an ultraviolet cutoff κ_{\max} . There is ambiguity in the choice of κ_{\min} and κ_{\max} , but for all *reasonable* choices (i.e. so that enough points lie within the linear regime in this interval), one gets a value for c with a fitting asymptotic standard error of less than 0.5%. However, the value of c thus obtained varies by about 1% (2%) for the square and triangular lattices (square-octagon, honeycomb and kagome lattices) across the various choices of fitting intervals. This leads to the speed of light estimates given in Tab. I.

Lattice	c/J	(prev. works)
Square	3.323 ± 0.033	3.01 ± 0.09 [26]
Square-Octagon	5.126 ± 0.103	
Triangular	2.047 ± 0.020	
Honeycomb	2.923 ± 0.058	
Kagome	2.013 ± 0.040	

TABLE I. Speed of light for each lattice geometry, from QMC.

Spectrum of the Wilson-Fisher CFT on a torus: ϵ -expansion

In this appendix we elaborate on the calculation of the spectrum from the ϵ expansion. A more detailed exposition which generalizes to the $O(N)$ model and includes deviations from the critical point will be presented in a future publication. The Hamiltonian is

$$H = \int d^d x \left[\frac{1}{2} \Pi^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{s}{2} \phi^2 + \frac{u}{4!} \phi^4 \right] \quad (17)$$

where we will always tune to the critical point, $s = s_c$, $u = u^*$, in final expressions. The system is taken to be on $d/2$ copies of a 2-torus with modular parameter $\tau = \tau_1 + i\tau_2$ and area $\mathcal{A} = \text{Im}(\omega_2 \omega_1^*) = \tau_2 L^2$. We will use complex coordinates, $x = x_1 + ix_2$, for each copy of the torus (see Fig. 1 in the main text).

As discussed in the main text, a gapless field theory on a finite volume leads to incurable infrared divergences due to the zero-momentum component of the fields. The solution to this problem is to split the fields into a zero-mode part and a finite momentum part,

$$\begin{aligned} \phi(x) &= \mathcal{A}^{-\frac{1-d}{4}} \varphi + \psi(x) \\ \Pi(x) &= \mathcal{A}^{-\frac{d+1}{4}} \pi + p(x) \end{aligned} \quad (18)$$

where the zero mode terms have been normalized such that they are dimensionless and satisfy the commutation relation $[\varphi, \pi] = i$. The fields $\chi(x)$ and $p(x)$ only have finite-momentum modes in their Fourier series:

$$\begin{aligned} \psi(x) &= \frac{1}{\mathcal{A}^{d/4}} \sum_{k \neq 0} \frac{e^{ik \cdot x}}{\sqrt{2|k|}} (b(k) + b^\dagger(-k)) \\ p(x) &= -\frac{i}{\mathcal{A}^{d/4}} \sum_{k \neq 0} \sqrt{\frac{|k|}{2}} e^{ik \cdot x} (b(k) - b^\dagger(-k)) \end{aligned} \quad (19)$$

The momentum sums are over $d/2$ copies of the complex dual lattice to the torus, and the dot product is given by $k \cdot x = \text{Re}(kx^*)$. With this decomposition, the Hamiltonian can be split up as

$$H = H_0 + V \quad (20)$$

with

$$\begin{aligned} H_0 &= \sum_{k \neq 0} |k| b^\dagger(k) b(k) \\ V &= \frac{1}{\sqrt{\mathcal{A}}} \left[\frac{1}{2} \pi^2 + \frac{1}{2} \mathcal{A} s \varphi^2 + \frac{u \mathcal{A}^{\epsilon/2}}{4!} \varphi^4 \right] \\ &+ \frac{u \mathcal{A}^{\epsilon/2}}{\sqrt{\mathcal{A}}} \frac{\varphi^2}{8} \sum_{\mathbf{k} \neq 0} \frac{\chi(-k) \chi(k)}{\mathcal{A}^{1/2} |k|} \\ &+ \frac{u \mathcal{A}^{\epsilon/2}}{\sqrt{\mathcal{A}}} \frac{\varphi}{6} \sum_{k, k' \neq 0} \frac{\chi(k) \chi(k') \chi(-k - k')}{(8 \mathcal{A}^{3/2} \omega_k \omega_{k'} \omega_{k+k'})^{1/2}} \\ &+ \frac{u \mathcal{A}^{\epsilon/2}}{\sqrt{\mathcal{A}}} \frac{1}{4!} \sum_{\substack{k_i \neq 0 \\ k_1 + k_2 = k_3 + k_4}} \frac{\chi(k_1) \chi(k_2) \chi(k_3) \chi(k_4)}{4 (\mathcal{A}^2 |k_1| |k_2| |k_3| |k_4|)^{1/2}} \quad (21) \end{aligned}$$

where we define $\chi(k) \equiv b(k) + b^\dagger(-k)$. Here, \mathcal{H}_0 describes a Fock spectrum of finite momentum states, and the interaction Hamiltonian V contains all terms involving the zero mode and nonlinearities. In this paper, we will always set the ground state energy to zero; a future publication will discuss the universal dependence of the ground state energy on L and on relevant perturbations $s - s_c$.

At zeroth order, our states are given by finite momentum Fock states multiplied by arbitrary functionals of the zero mode,

$$H_0 \Psi[\varphi] |k, k', \dots\rangle = (|k| + |k'| + \dots) \Psi[\varphi] |k, k', \dots\rangle \quad (22)$$

Since we can multiply by any normalizable functional $\Psi[\varphi]$, these states are infinitely degenerate. This degeneracy is broken in perturbation theory. We use a perturbation method due to C. Bloch which is well-suited to degenerate problems [40]. For a review of this method and its relation to other effective Hamiltonian methods, see Ref. [59]. The main idea is to consider each degenerate subspace separately, but construct an effective Hamiltonian within each subspace whose eigenvalues give the exact energy. So if we consider a degenerate subspace of H_0 ,

$$H_0 |\alpha_0\rangle = \epsilon_0 |\alpha_0\rangle \quad (23)$$

this perturbation method constructs a new operator H_{eff} which acts on this subspace but gives the exact energy levels,

$$H_{eff} |\alpha_0\rangle = E_\alpha |\alpha_0\rangle \quad (24)$$

where $E_\alpha = \epsilon_0 + \mathcal{O}(V)$.

The expression for H_{eff} can be obtained perturbatively in V , which was the main result of Bloch's work [40]. To leading

order, the effective Hamiltonian for a given degenerate subspace is given by

$$H_{eff} = \epsilon_0 P_0 + P_0 V P_0 + P_0 V \frac{1 - P_0}{\epsilon_0 - \mathcal{H}_0} V P_0 + \dots \quad (25)$$

where P_0 is the projection operator onto the degenerate subspace of interest. At this order in perturbation theory the effective Hamiltonian is hermitian, although to higher orders one needs to make a unitary transformation insure hermiticity [40, 59].

As a definite example, we give the effective Hamiltonian acting on the Fock vacuum, $\Psi[\varphi] |0\rangle$. From Eq. (25), the effective Hamiltonian takes the form

$$H_{eff, k=0} = h_{k=0} |0\rangle \langle 0|$$

$$h_{k=0} = \langle 0 | V | 0 \rangle - \langle 0 | V \left(\frac{1 - |0\rangle \langle 0|}{H_0} \right) V | 0 \rangle + \dots \quad (26)$$

When this acts on the ground state manifold, it generates a Schrödinger equation for the zero-mode functional,

$$h_{k=0} \Psi[\varphi] = E \Psi[\varphi] \quad (27)$$

We now obtain $h_{k=0}$ from evaluating the expectation values in Eq. (26). This involves UV divergent sums which requires renormalization, but the renormalization constants and RG equations will be identical to the infinite volume case as a consequence of finite-size scaling. We renormalize the theory using dimensional regularization with minimal subtraction as detailed in Ref. [60], and then set the couplings to their fixed point values. The analytic continuation of divergent loop sums to arbitrary dimension can be found, for example, in Refs. [38, 60, 61].

The one-loop result for $h_{k=0}$ can be written

$$h(\varphi) = \frac{1}{\sqrt{\mathcal{A}}} \left(\frac{1}{2} \pi^2 + \frac{R}{2} \varphi^2 + \frac{U}{4!} \varphi^4 \right), \quad (28)$$

where R and U are dimensionless universal quantities which form a power series in ϵ . These constants will also depend on τ , and our expression for them is in terms of integrals over Riemann theta functions which need to be evaluated numerically. As we will justify below, we need to calculate R to order ϵ and U to order ϵ^2 to obtain the spectrum to one-loop. Given the commutation relation $[\varphi, \pi] = i$, the momentum acts on the zero-mode functional as

$$\pi^2 \Psi[\varphi] = -\frac{d^2}{d\varphi^2} \Psi[\varphi] \quad (29)$$

Therefore, at leading order in the ϵ -expansion, the low-energy spectrum of the Ising model on the torus maps onto the spectrum of a one-dimensional quantum anharmonic oscillator with universal coefficients.

In spite of the effective Hamiltonian being an ordinary series in ϵ , the oscillator is strongly coupled for small ϵ . This

can be seen by performing the canonical transformation $\varphi \rightarrow U^{-1/6}\varphi$ and $\pi \rightarrow U^{1/6}\pi$, which takes

$$\begin{aligned} & \left(\frac{\pi^2}{2} + \frac{R}{2}\varphi^2 + \frac{U}{4!}\varphi^4 \right) \\ & \rightarrow U^{1/3} \left(\frac{\pi^2}{2} + \frac{RU^{-2/3}}{2}\varphi^2 + \frac{1}{4!}\varphi^4 \right) \end{aligned} \quad (30)$$

Since both R and U are $\mathcal{O}(\epsilon)$, this latter form implies that the energy eigenvalues are an expansion in $\epsilon^{1/3}$, and the coefficients of the expansion are given by a pure quartic oscillator perturbed by a quadratic term. The latter problem has been widely studied in the literature, and the coefficients of this expansion to high order can be found in Ref. [41].

The above form for the effective Hamiltonian shows that the ϵ expansion on the torus results in a reordering of the perturbation expansion, since powers of φ effectively carry a factor of $\epsilon^{-1/6}$. This reordering is what justifies our calculating R to order ϵ and U to order ϵ^2 above. A detailed analysis shows that the one-loop expansion of the energy levels is accurate to order $\epsilon^{4/3}$, since the leading two-loop correction to the energy is of order $\epsilon^{5/3}$. This leading two-loop correction is to the coefficient R , and we also need to add terms of the form $p^2\varphi^4 + c.c.$ and φ^6 to the effective Hamiltonian at the same order.

For the finite momentum states, the effective Hamiltonian also takes the form of a strongly-coupled oscillator, but the coefficients will depend on the momentum. In addition, the effective Hamiltonian will couple different Fock states with the same energy and momentum whenever possible, which can lead to a multi-dimensional Hamiltonian which mixes Fock states; for these Hamiltonians the effect of these off-diagonal terms were computed numerically.

Finally, we note that since the anti-periodic sectors in the Ising* transition do not have a zero mode, the calculation is straight-forward. The energy levels are a normal expansion in ϵ , and we simply need to compute the $\mathcal{O}(\epsilon)$ correction to the energy using first-order perturbation theory.

Complete low-energy spectrum for Ising CFT with modular parameter $\tau = i$

$\tau = i$	$\kappa = 0$	$\kappa = 1 (\times 4)$	$\kappa = \sqrt{2} (\times 4)$	$\kappa = 2 (\times 4)$	$\kappa = \sqrt{5} (\times 8)$	$\kappa = 2\sqrt{2} (\times 4)$	Denomination
\tilde{E}/c	0						I
	1.28						σ_T
	4.71						ε_T
		6.79					$\sigma_T + \Delta\kappa$
	8.77						σ'_T
		9.52	9.44				$\varepsilon_T + \Delta\kappa$
	12.9			11.6			
					13.15		
					13.5		
		13.6					
	14.1						
	14.4			14.5			
					14.9	14.67	
						15.4	
			15.6				
		16.0					
	17.3			17.6			
	17.7						
	17.9						
			18.4				
						18.46	

TABLE II. Low-energy spectrum $\tilde{E}/c = (E - E_0)\sqrt{N}/c$ for the Ising QFT with $\tau = i$ obtained from ED/QMC on the square lattice. c denotes the speed of light (see Tab. I). Unshaded (shaded) cells are even (odd) under spin-inversion. Blue colored values are obtained from QMC+ED, the other values from ED alone. The degeneracy of the finite-momentum levels is given in brackets, all levels for $\kappa = 0$ are not degenerate, some very close levels may, however, be actually degenerate in the thermodynamic limit. The given values are obtained by linear fits of the finite-size levels \tilde{E}_N/c as a function of $1/N$ and should be accurate up to variations of the last given digit. Obtaining more precise values is a non-trivial task as the values from QMC are the result of a series of fits and ED data shows larger finite-size effects for higher levels in the spectrum and for larger momentum κ , where the available finite-size momenta already lie within the non-linear regime of the dispersion relation close to the Brillouin zone boundary. The last column shows our denomination of the levels as it was used in the main text. See Tab. III for a comparison with ϵ -expansion results.

$\tau = i$	$\kappa = 0$	$\kappa_2 = 0$	$\kappa = 1$	$\kappa = \sqrt{2}$	$\kappa_2 = \sqrt{2}$	$\kappa = 2$	$\kappa = \sqrt{5}$	$\kappa = 2\sqrt{2}$	Denomination
\tilde{E}/c	0								1
	1.825								σ_T
	5.16								ε_T
				6.40					$\sigma_T + \Delta\kappa$
				8.88					$\varepsilon_T + \Delta\kappa$
		9.02			9.01				σ'_T
				12.41	11.35				
							12.60		
							12.72		
		12.95				12.95			
		13.39							
					14.867				14.15
				15.51			14.873		15.51
			16.00				15.83		16.32
				16.50					17.78

TABLE III. Low-energy spectrum for the Ising QFT with modular parameter $\tau = i$ from ε -expansion. The notation κ_2 indicates "two-particle" states (but this distinction loses meaning for higher κ).

Complete low-energy spectrum for Ising CFT with modular parameter $\tau = \frac{1}{2} + \frac{\sqrt{3}}{2}i$

$\tau = \frac{1}{2} + \frac{\sqrt{3}}{2}i$	$\kappa = 0$	$\kappa = 1 (\times 6)$	$\kappa = \sqrt{3} (\times 6)$	$\kappa = 2 (\times 6)$	$\kappa = \sqrt{7} (\times 12)$	Denomination
\tilde{E}/c	0					1
	1.35					σ_T
	5.03					ε_T
			7.77			$\sigma_T + \Delta\kappa$
						σ'_T
		9.33	10.53			$\varepsilon_T + \Delta\kappa$
				13.14		
		14.5		14.6		
			14.8			
		15.1 ($\times 2$)			15.10	
			16.1		15.9	
					16.2	
		18.3		16.7		
				18.6		
		19.8 ($\times 2$)				19.82

TABLE IV. Low-energy spectrum for the Ising QFT with $\tau = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ obtained from ED/QMC on the triangular lattice. See Tab. II for further details and Tab. V for a comparison with ε -expansion results.

$\tau = \frac{1}{2} + \frac{\sqrt{3}}{2}i$	$\kappa = 0$	$\kappa = 1 (\times 6)$	$\kappa = \sqrt{3} (\times 6)$	$\kappa = 2 (\times 6)$	$\kappa = \sqrt{7} (\times 12)$	Denomination
\tilde{E}/c	0					1
	1.96					σ_T
	5.55					ε_T
		7.38				$\sigma_T + \Delta\kappa$
	9.72					σ'_T
		10.02				$\varepsilon_T + \Delta\kappa$
				12.68		
		13.83				
	14.31					
	14.44				14.54	
					14.67	
		14.90		14.90		
				15.08		
	15.23					
	16.66					
	17.59				16.96	
				17.97		
		18.13	18.13			
		18.24				
			18.84			
19.61					19.28	

TABLE V. Low-energy spectrum for the Ising QFT with modular parameter $\tau = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ from ϵ -expansion.

Complete low-energy spectrum for Ising* CFT with $\tau = i$

$\tau = i$	(P,P)			(P,A)/(A,P)			(A,A)			Denomination
	$\kappa = 0$	$\kappa = 1$	$\kappa = \sqrt{2}$	$\kappa = 0$	$\kappa = 1$	$\kappa = \sqrt{2}$	$\kappa = 0$	$\kappa = 1$	$\kappa = \sqrt{2}$	
\tilde{E}/c	0			0.45			0.62			1
	4.71				7.3					$1'_T$
				8.5			9.0			$1''_T$
		9.52								ε_T
					10.4			10.3	9.6	
				11.6		11.4				

TABLE VI. Low-energy spectrum for the Ising* CFT with $\tau = i$ obtained from ED on the square lattice. The four distinct topological sectors are indicated by the corresponding boundary conditions (P,A) etc. in the two directions around the torus, where A(P) denotes (anti-)periodic boundary conditions (See main text for further details). The four lowest-lying levels constitute the topological four-fold degenerate groundstate manifold in the Toric Code phase and are still remarkably low in energy at criticality. See Tab. VIII for a comparison with ϵ -expansion results and Tab. II for further details.

$\tau = i$	(P,P)			(P,A)/(A,P)			(A,A)			Denomination
	$\kappa = 0$	$\kappa = 1$	$\kappa = \sqrt{2}$	$\kappa = 0$	$\kappa = 1$	$\kappa = \sqrt{2}$	$\kappa = 0$	$\kappa = 1$	$\kappa = \sqrt{2}$	
\tilde{E}/c	0			0.19			0.35			1
	5.16			7.54	6.88					$1'_T$
		8.88								$1''_T$
			11.35				9.75	9.75	9.18	ε_T

TABLE VII. Low-energy spectrum for the Ising* CFT with $\tau = i$ obtained from ϵ -expansion.