Semiclassical Virasoro symmetry of the quantum gravity S-matrix

Semiclassical Virasoro Symmetry of the Quantum Gravity $\mathcal{S}$-Matrix

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Abstract

It is shown that the tree-level $\mathcal{S}$-matrix for quantum gravity in four-dimensional Minkowski space has a Virasoro symmetry which acts on the conformal sphere at null infinity.

Contents

1 Introduction

BMS$^+$ transformations [1] comprise a subset of diffeomorphisms which act nontrivially on future null infinity of asymptotically Minkowskian space times, or $\mathcal{I}^+$. BMS$^-$ transformations act isomorphically on past null infinity, or $\mathcal{I}^-$. A particular 'diagonal' subgroup of the product group BMS$^+\times$BMS$^-$ has recently been shown [2] to be a symmetry of gravitational scattering. Ward identities of this diagonal symmetry relate $\mathcal{S}$-matrix elements with and without soft gravitons. These $\mathcal{S}$-matrix relations are not new [3]: they comprise Weinberg's soft graviton theorem [4]. More generally, the connection to soft theorems provides a new perspective on asymptotic symmetries in Minkowski space [5].

Over the decades a number of extensions/modifications to the BMS group have been proposed: e.g. the Newman-Unti group [6], the Spi group [7] and the extended BMS group [8, 9]. A criterion is needed to decide whether or not such extensions are 'physical'. Here we adopt the pragmatic approach that a Minkowskian asymptotic symmetry is physical if and only if it provides nontrivial relations among $\mathcal{S}$-matrix elements. We will view these $\mathcal{S}$-matrix relations as a definition of the symmetry.

In this paper we will show that, at tree-level, quantum gravity in asymptotically Minkowskian spaces in this sense has a physical Virasoro symmetry. The symmetry is implied by a recently proven soft theorem [10] and acts (diagonally) on the conformal S^2 at $\mathcal{I}$.

Our story begins with a conjecture of Barnich, Troessaert and Banks (BTB) [8, 9]. BMS^+ has an $SL(2, C)$ Lorentz subgroup generated by the six global conformal Killing vectors (CKVs) on the S^2 at $\mathcal{I}^+$. Locally, BTB showed that all of the infinitely many CKVs preserve the same asymptotic structure at $\mathcal{I}^+$ and are hence also candidate asymptotic symmetry generators. This larger set of vector fields was a priori excluded in the original work of BMS, who demanded that they be nonsingular everywhere on S^2. This restriction cuts the Virasoro group down to a mere $SL(2, C)$. BTB conjectured that the true asymptotic symmetry group of $\mathcal{I}^+$ is the 'extended BMS^+ group' generated by all CKVs. However it has not been clear if or in what sense the singular CKVs truly generate physical asymptotic symmetries.

Herein we consider, in the spirit of [2], a certain diagonal subgroup of (extended BMS^+)$\times$(extended BMS^-), denoted $\mathcal{X}$. Ward identities are derived for a Virasoro subgroup of $\mathcal{X}$. They are found to involve a soft graviton insertion with the Weinberg pole projected out, leaving the finite subleading term in the soft expansion. These Ward identities are in turn shown to be implied by a conjectured [11] soft relation schematically of the form

$$\lim_{\omega \to 0} \mathcal{M}_{n+1} = S^{(1)} \mathcal{M}_n. \tag{1.1}$$

Here $\mathcal{M}_{n+1}$ is an $n+1$-particle amplitude with a certain (pole-projected) energy ω soft graviton insertion, and $S^{(1)}$ involves the soft graviton momentum as well as the energies and angular momenta of the incoming and outgoing particles. Details are given below. The proof [10] of (1.1) for tree-level gravity amplitudes then implies a semiclassical Virasoro symmetry for the case of pure gravity. This demonstrates that the singularities in the generic CKVs do not, at least in this context, prevent them from generating physical symmetries.[1]

One might also hope to run the argument backwards and see to what extent the Virasoro symmetry of the $\mathcal{S}$-matrix implies the soft relation (1.1). In the case of supertranslations, the argument can be run in both directions [3]. However here we encounter several obstacles, including the need for a prescription for handling the CKV singularities and some zero mode issues. We leave this to future investigations.[2] Hence at this point the existence of a Virasoro symmetry is potentially a *weaker* condition than the validity of the soft relation (1.1).

[1] It may alternatively be possible to reach this conclusion without appealing to direct computations such as in [10] by carefully regulating the singularities and analyzing their effects. We do not attempt such an analysis herein.

[2] The Virasoro charges constructed in [8] may be useful for this purpose.

The analysis of [2, 3] related two structures which have been well-established and thoroughly studied over the last half-century: BMS symmetry and Weinberg's soft graviton theorem. Here the situation is rather different. We are relating two unestablished and understudied structures: asymptotic Virasoro symmetries and subleading soft graviton theorems. We hope the relation will illuminate both. In any case it is a rather different enterprise!

An important issue which we will not address is the quantum fate of the semiclassical Virasoro symmetry. Here the situation is currently up in the air. In [12, 13] it was shown that, in a standard regulator scheme, (1.1) receives IR divergent quantum corrections (at one loop only), which also make the $\mathcal{S}$-matrix ill-defined in this scheme. However in [14], the factor $S^{(1)}$ in (1.1) relating the 5 and 4 point amplitude was found to remain uncorrected at one loop in a scheme with the soft limit taken prior to removing the IR cutoff.[3] In the recent work [17, 18] (see also [19]) it was shown that a properly defined $\mathcal{S}$-matrix utilizing the gravity version of the Kulish-Faddeev construction [20] is free of all IR divergences. This may be the proper context for the discussion, as it is hard to have a symmetry of an $\mathcal{S}$-matrix without an $\mathcal{S}$-matrix! Should it ultimately be found that (1.1) does receive scheme-independent corrections, one must then determine whether it implies a quantum anomaly in the asymptotic Virasoro symmetry (which is potentially weaker than (1.1)), or a quantum deformation in its action on the amplitudes. Clearly highly relevant, but not yet fully incorporated into this discussion, is the low-energy theorem of Gross and Jackiw [21] who use dispersion theory to show that there is no correction to the first three terms[4] of the Born approximation to soft graviton-scalar scattering. This generalized the classic low-energy theorem for QED by Low [22]. Progress on the gravity version was recently made by White [19]. Clearly, there is much to understand!

The existence of a Virasoro symmetry potentially has far-reaching implications for Minkowski quantum gravity in general. However, at this point there are many basic unresolved points and it is too soon to tell what or if they might be. For example we do not know if the symmetry has quantum anomalies, what kind of representations appear,[5] the role of IR divergences or the connection to stringy Virasoro symmetries [5, 23]. Very recent developments indicate that these ideas, including the realization of the subleading soft theorem as a Virasoro symmetry, have a natural home in the twistor string [24, 25]. Since the symmetry

[3] [14] claims a result only for this one special case by direct computation. However, it has been suggested [15] that, using [16], a proof can be constructed in the scheme of [12, 13] that all loop corrections to S^1 in (1.1) are linked to discontinuities arising from infrared singularities and hence in the scheme of [14] (with the soft limit taken first) all loop corrections would disappear along with the discontinuities.

[4] $S^{(0)}$, $S^{(1)}$ and $S^{(2)}$ in the notation of [14].

[5] They may not be the familiar ones from the study of unitary 2D CFT on the sphere.

acts at the boundary, it is likely relevant to any holographic duality as long ago envisioned in [26].

This paper is organized as follows. Section 2 establishes notation and reviews a few salient formulae for asymptotically flat geometries. Section 3 describes the conjectured extended BMS$^{\pm}$ symmetry following [8]. In section 4 we define the diagonal subgroup $\mathcal{X}$ of (extended BMS^{+})$\times$(extended BMS^{-}) transformations, review Christodoulou-Klainerman (CK) spaces and define extended CK spaces by acting with $\mathcal{X}$. A prescription is given to define classical gravitational scattering from $\mathcal{I}^-$ to $\mathcal{I}^+$ and shown to be symmetric under $\mathcal{X}$. In section 5 the discussion of the quantum theory begins with the the action of extended BMS$^{\pm}$ generators on in and out states. A Ward identity is then derived which is equivalent to infinitesimal $\mathcal{X}$-invariance of $\mathcal{S}$. It relates amplitudes with and without a particular soft graviton insertion. Finally in section 6 we give the detailed form of the soft relation (1.1) and show that it implies the $\mathcal{X}$ Ward identity.

2 Asymptotically flat geometry

2.1 Metrics

A general asymptotically Minkowskian metric can be expanded in $\frac{1}{r}$ around $\mathcal{I}^+$. In retarded Bondi coordinates it takes the form[6]

$$\begin{aligned} ds^2 =& - du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} \\ &+ \frac{2m_B}{r}du^2 + rC_{zz}dz^2 + rC_{\bar{z}\bar{z}}d\bar{z}^2 + 2g_{uz}dudz + 2g_{u\bar{z}}dud\bar{z} + ..., \end{aligned} \tag{2.1}$$

where the first line is the flat Minkowski metric, $\gamma_{z\bar{z}}$ (D_z) is the round metric (covariant derivative) on the unit S^2 and

$$g_{uz} = \frac{1}{2}D^zC_{zz} + \frac{1}{6r}C_{zz}D_zC^{zz} + \frac{2}{3r}N_z + \mathcal{O}(r^{-2}). \tag{2.2}$$

The Bondi mass aspect m_B, the angular momentum aspect N_z and C_{zz} depend only on $(u, z, \bar{z})$ and not r. The outgoing news tensor is defined by

$$N_{zz} \equiv \partial_u C_{zz}. \tag{2.3}$$

[6]We largely adopt the notation of [8] to which we refer the reader for further details.

$\mathcal{I}^+$ is the null surface $(r = \infty, u, z, \bar{z})$. We use the symbol $\mathcal{I}^+_+$ $(\mathcal{I}^+_-)$ to denote the future (past) boundary of $\mathcal{I}^+$ at $(r = \infty, u = \infty, z, \bar{z})$ $((r = \infty, u = -\infty, z, \bar{z}))$. This is depicted in figure 1.

There is an analogous construction on $\mathcal{I}^-$ with the metric given by

$$\begin{aligned} ds^2 &= -dv^2 + 2dvdr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} \\ &\quad + \frac{2m_B^-}{r}dv^2 + rD_{zz}dz^2 + rD_{\bar{z}\bar{z}}d\bar{z}^2 + 2g_{vz}dvdz + 2g_{v\bar{z}}dvd\bar{z} + ..., \end{aligned} \tag{2.4}$$

with

$$g_{vz} = -\frac{1}{2}D^zD_{zz} - \frac{1}{6r}D_{zz}D_zD^{zz} - \frac{2}{3r}N_z^- + \mathcal{O}(r^{-2}). \tag{2.5}$$

The $\mathcal{I}^-$ coordinate z in (2.4) is antipodally related to the $\mathcal{I}^+$ coordinate z in (2.1) in the sense that, for flat Minkowski space, a null geodesic begins and ends at the same value of z. Put another way, in the conformal compactifcation of asymptotically flat spaces, all of $\mathcal{I}$ is generated by null geodesics which run through spatial infinity i^0. These generators have the same constant z value on both $\mathcal{I}^+$ and $\mathcal{I}^-$. The incoming news tensor is defined by

$$M_{zz} \equiv \partial_v D_{zz}. \tag{2.6}$$

When expanding about flat Minkowski space we sometimes employ flat coordinates in which the flat metric takes the form

$$ds_F^2 = \eta_{\mu\nu}dx^\mu dx^\nu. \tag{2.7}$$

These are related to Bondi coordinates in flat space by

$$\begin{aligned} x^0 &= u + r = v - r, \\ x^1 + ix^2 &= \frac{2rz}{1+z\bar{z}}, \\ x^3 &= \frac{r(1-z\bar{z})}{1+z\bar{z}}. \end{aligned} \tag{2.8}$$

2.2 Constraints

The data in (2.1) are related by the constraint equations $G_{\mu\nu} = \mathcal{T}^M_{\mu\nu}$, where $\mathcal{T}^M_{\mu\nu}$ is the matter stress tensor and we adopt units in which $8\pi G = 1$. The leading term in the expansion of

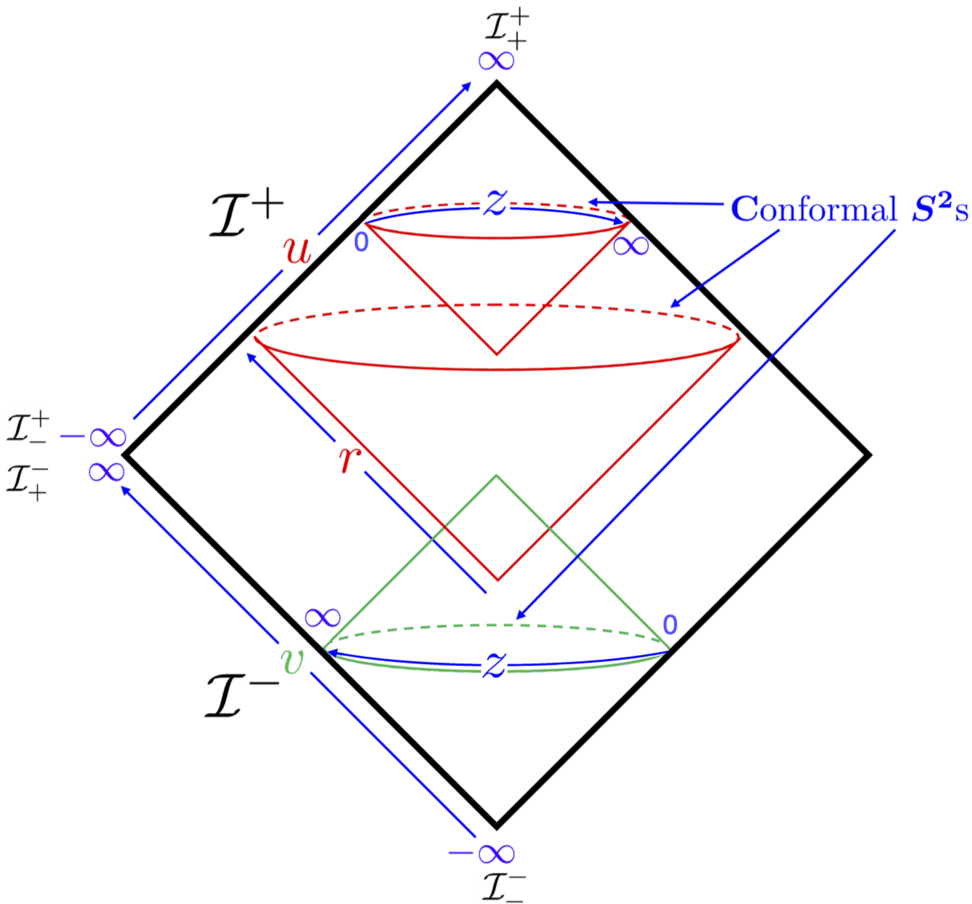

Figure 1: Penrose diagram for Minkowski space. Near $\mathcal{I}^+$ surfaces of constant retarded time u (red) are cone-like and intersect $\mathcal{I}^+$ in a conformal S^2 parametrized by $(z, \bar{z})$. Cone-like surfaces of constant advanced time v (green) intersect $\mathcal{I}^-$ in a conformal S^2 also parametrized by $(z, \bar{z})$. The future (past) S^2 boundary of $\mathcal{I}^+$ is labelled $\mathcal{I}^+_+$ ($\mathcal{I}^+_-$), while the future (past) boundary of $\mathcal{I}^-$ is labelled $\mathcal{I}^-_+$ ($\mathcal{I}^-_-$).

the G_{uu} constraint equation about $\mathcal{I}^+$ is

$$\partial_u m_B = \frac{1}{4}D_z^2 N^{zz} + \frac{1}{4}D_{\bar{z}}^2 N^{\bar{z}\bar{z}} - \frac{1}{2}T_{uu}^M - \frac{1}{4}N_{zz}N^{zz}, \tag{2.9}$$

where

$$T_{\mu\nu}^M(u,z,\bar{z}) = \lim_{r\to\infty} r^2 \mathcal{T}_{\mu\nu}^M(u,r,z,\bar{z}) \tag{2.10}$$

is the rescaled matter stress tensor which we have assumed falls of like $\frac{1}{r^2}$ near $\mathcal{I}^+$. The G_{uz} constraint gives

$$\begin{aligned}\partial_u N_z = &-\frac{1}{4}(D_z D_{\bar{z}}^2 C^{\bar{z}\bar{z}} - D_z^3 C^{zz}) - T_{uz}^M + \partial_z m_B + \frac{1}{16}D_z\partial_u(C_{zz}C^{zz}) \\ &- \frac{1}{4}N^{zz}D_z C_{zz} - \frac{1}{4}N_{zz}D_z C^{zz} - \frac{1}{4}D_z(C^{zz}N_{zz} - N^{zz}C_{zz}).\end{aligned} \tag{2.11}$$

Given the Bondi news, m_B, N_z and C_{zz} are all determined up to u-independent integration constants which are discussed below. The $\mathcal{I}^-$ constraints are

$$\partial_v m_B^- = \frac{1}{4}D_z^2 M^{zz} + \frac{1}{4}D_{\bar{z}}^2 M^{\bar{z}\bar{z}} + \frac{1}{2}T_{vv}^M + \frac{1}{4}M^{zz}M_{zz}, \tag{2.12}$$

$$\begin{aligned}\partial_v N_z^- = &\frac{1}{4}(D_z D_{\bar{z}}^2 D^{\bar{z}\bar{z}} - D_z^3 D^{zz}) - T_{vz}^M - \partial_z m_B^- + \frac{1}{16}D_z\partial_v(D_{zz}D^{zz}) \\ &- \frac{1}{4}M^{zz}D_z D_{zz} - \frac{1}{4}M_{zz}D_z D^{zz} - \frac{1}{4}D_z(D^{zz}M_{zz} - M^{zz}D_{zz}).\end{aligned} \tag{2.13}$$

3 Extended BMS$^\pm$ transformations

The extended BMS$^+$ group has been proposed [8, 9] as the asymptotic symmetry group at $\mathcal{I}^+$ of gravity on asymptotically flat spacetimes. It is generated by vector fields ξ^+ that locally preserve the asymptotic form (2.1) of the metric at $\mathcal{I}^+$

$$\mathcal{L}_{\xi^+} g_{ur} = \mathcal{O}(r^{-2}), \quad \mathcal{L}_{\xi^+} g_{uz} = \mathcal{O}(1), \quad \mathcal{L}_{\xi^+} g_{zz} = \mathcal{O}(r), \quad \mathcal{L}_{\xi^+} g_{uu} = \mathcal{O}(r^{-1}). \tag{3.1}$$

All such vector fields near $\mathcal{I}^+$ are of the form

$$\begin{aligned}\xi^+ =& (1+\frac{u}{2r})Y^{+z}\partial_z - \frac{u}{2r}D^{\bar{z}}D_z Y^{+z}\partial_{\bar{z}} - \frac{1}{2}(u+r)D_z Y^{+z}\partial_r + \frac{u}{2}D_z Y^{+z}\partial_u + c.c. \\ &+ f^+\partial_u - \frac{1}{r}(D^z f^+\partial_z + D^{\bar{z}} f^+\partial_{\bar{z}}) + D^z D_z f^+\partial_r,\end{aligned} \tag{3.2}$$

where f^+ is an arbitrary function on S^2 and here and elsewhere we suppress (in some cases metric-dependent) terms which are further subleading in $\frac{1}{r}$ and irrelevant to our analysis: see [8] for a recent treatment specifying these terms. Y^+ must be a conformal Killing vector on S^2 which obeys the equation

$$\partial_{\bar{z}} Y^{+z} = 0. \tag{3.3}$$

Globally there are six real vectors fields in an antisymmetric matrix $Y^z_{\mu\nu}$ obeying (3.3):

$$\begin{aligned} Y^z_{12} = iz, \;\; Y^z_{13} = -\tfrac{1}{2}(1+z^2), \;\; Y^z_{23} = -\tfrac{i}{2}(1-z^2), \\ Y^z_{03} = z, \;\; Y^z_{01} = -\tfrac{1}{2}(1-z^2), \;\; Y^z_{02} = -\tfrac{i}{2}(1+z^2). \end{aligned} \tag{3.4}$$

These generate the six Lorentz boosts and rotations on $\mathcal{I}^+$. Locally there are infinitely many solutions of the form $Y^z \sim z^n$ with poles somewhere on the sphere. In their original work [1], BMS excluded these singular vector fields. However in this paper we shall explore the conjecture of [8, 9] that all of these 'superrotations' should be included as part of the asymptotic symmetry group.

The extended BMS$^+$ group is a semi-direct product of superrotations with supertranslations. The supertranslations were recently analyzed in [2, 3]. For notational brevity we henceforth consider only the superrotation subgroup which has $f^+ = 0$ in (3.2) and reduces to

$$\xi^+ = (1 + \frac{u}{2r}) Y^{+z}\partial_z - \frac{u}{2r} D^{\bar{z}} D_z Y^{+z} \partial_{\bar{z}} - \frac{1}{2}(u+r) D_z Y^{+z} \partial_r + \frac{u}{2} D_z Y^{+z} \partial_u + c.c. \tag{3.5}$$

This maps $\mathcal{I}^+$ to itself via

$$\xi^+|_{\mathcal{I}^+} = Y^{+z}\partial_z + \frac{u}{2} D_z Y^{+z}\partial_u + c.c. \tag{3.6}$$

Similarly on $\mathcal{I}^-$ we have BMS$^-$ vector fields parametrized by Y^-

$$\xi^- = (1 - \frac{v}{2r}) Y^{-z}\partial_z + \frac{v}{2r} D^{\bar{z}} D_z Y^{-z} \partial_{\bar{z}} - \frac{1}{2}(r-v) D_z Y^{-z} \partial_r + \frac{v}{2} D_z Y^{-z} \partial_v + c.c. \tag{3.7}$$

Infinitesimal BMS$^+$ transformations act on the Bondi-gauge metric components as

$$\begin{aligned} \delta_{Y^+} C_{zz} &= \frac{u}{2}(D_z Y^{+z} + D_{\bar{z}} Y^{+\bar{z}})\partial_u C_{zz} + \mathcal{L}_{Y^+} C_{zz} - \frac{1}{2}(D_z Y^{+z} + D_{\bar{z}} Y^{+\bar{z}}) C_{zz} - u D_z^3 Y^{+z}, \\ \delta_{Y^+} N_{zz} &\equiv \partial_u \delta C_{zz} = \frac{u}{2}(D_z Y^{+z} + D_{\bar{z}} Y^{+\bar{z}})\partial_u N_{zz} + \mathcal{L}_{Y^+} N_{zz} - D_z^3 Y^{+z}. \end{aligned} \tag{3.8}$$

Similarly at $\mathcal{I}^-$

$$\begin{aligned}
\delta_{Y^-} D_{zz} &= \frac{v}{2}(D_z Y^{-z} + D_{\bar z} Y^{-\bar z})\partial_v D_{zz} + \mathcal{L}_{Y^-} D_{zz} - \frac{1}{2}(D_z Y^{-z} + D_{\bar z} Y^{-\bar z}) D_{zz} + v D_z^3 Y^{-z}, \\
\delta_{Y^-} M_{zz} &= \frac{v}{2}(D_z Y^{-z} + D_{\bar z} Y^{-\bar z})\partial_v M_{zz} + \mathcal{L}_{Y^-} M_{zz} + D_z^3 Y^{-z}.
\end{aligned} \tag{3.9}$$

4 $\mathcal{X}$ transformations

BMS$^\pm$ symmetries act on the physical data at $\mathcal{I}^\pm$ while preserving certain asymptotic structures such as the symplectic form [26]. They are not themselves symmetries of gravitational scattering: that is given some solution (D_{zz}, C_{zz}) of the gravitational scattering problem we cannot get a new one by acting with an element of BMS$^+$ or BMS$^-$. However for the case of supertranslations, it was argued in [2] that a certain diagonal subgroup of BMS$^+\times$BMS$^-$ is a symmetry of gravitational scattering in a suitable neighborhood [27] of flat space. This subgroup is generated by pairs of $SL(2, C)$ Killing vector fields and supertranslations $(Y^+, f^+; Y^-, f^-)$ obeying

$$Y^{+z}(z) = Y^{-z}(z) \equiv Y^z(z), \quad f^+(z,\bar z) = f^-(z, \bar z) \equiv f(z, \bar z), \tag{4.1}$$

with the understanding that the coordinate z is constant along null generators of $\mathcal{I}$ as they pass from $\mathcal{I}^-$ to $\mathcal{I}^+$ through spatial infinity i^0 in the conformal compactification of the spacetime. This means that points labelled by the same value of z on $\mathcal{I}^-$ and $\mathcal{I}^+$ lie at antipodal angles form the origin. This antipodal identification may sound a little odd at first, but in fact is required in order for the subgroup (4.1) to contain the usual global Poincare transformations.

In this paper we are interested in *extended* BMS$^+\times$BMS$^-$ transformations. We denote by $\mathcal{X}$ the subgroup of these transformations generated by vector fields asymptotic to (ξ^+, ξ^-) on $(\mathcal{I}^+, \mathcal{I}^-)$ subject to (4.1), where now Y^z is any of the infinitely many conformal Killing vectors on the sphere. Elements of $\mathcal{X}$ transform a solution (D_{zz}, C_{zz}) of the gravitational scattering problem to a new one (D'_{zz}, C'_{zz}) with different final and initial data. We will argue below that the new data is a new solution of the scattering problem.

4.1 Christodoulou-Klainerman spaces

We are interested in asymptotically flat solutions of the Einstein equation which revert to the vacuum in the far past and future. In particular we want to remain below the threshold for black hole formation. We will adopt the rigorous definition of such spaces given by

Christodoulou and Klainerman (CK) [27] who also proved their global existence and analyzed their asymptotic behavior.

CK studied asymptotically flat initial data in the center-of-mass frame on a maximal spacelike slice for which the Bach tensor $\varepsilon^{ijk}D^{(3)}_j G^{(3)}_{kl}$ of the induced three-metric decays like $r^{-7/2}$ (or faster) at spatial infinity and the extrinsic curvature like $r^{-5/2}$. This implies that in normal coordinates about infinity the leading part of the three-metric has the (conformally flat) Schwarzschild form, with corrections which decay like $r^{-3/2}$. CK showed that all such initial data which moreover satisfy a global smallness condition give rise to a global, i.e. geodesically complete, solution. We will refer to these solutions as CK spaces.

The smallness condition is satisfied in a finite neighborhood of Minkowski space, so this result established the stability of Minkowski space. Moreover many asymptotic properties of CK spaces at null infinity were derived in detail, see [28] for a summary. Here we note that the Bondi news N_{zz} vanishes on the boundaries of $\mathcal{I}^+$ as

$$N_{zz}(u) \sim |u|^{-3/2}, \tag{4.2}$$

or faster. Similarly on $\mathcal{I}^-$

$$M_{zz}(v) \sim |v|^{-3/2} \tag{4.3}$$

or faster. The Weyl curvature component Ψ_2^0 which in coordinates (2.1) is given by

$$\begin{aligned}\Psi_2^0(u,z,\bar z) &\equiv -\lim_{r\to\infty}(rC_{u\bar z r z}\gamma^{z\bar z}) \\ &= -m_B - \frac{1}{4}C_{zz}N^{zz} + \frac{1}{4}(D^zD^zC_{zz} - D^{\bar z}D^{\bar z}C_{\bar z\bar z})\end{aligned} \tag{4.4}$$

obeys

$$\Psi_2^0|_{\mathcal{I}^+_+} = 0, \tag{4.5}$$

while at $u=-\infty$

$$\Psi_2^0|_{\mathcal{I}^+_-} = -GM, \tag{4.6}$$

where G is Newton's constant and M is the ADM mass. Similar results pertain to $\mathcal{I}^-$.

In this paper we consider generalizations of pure gravity which include coupling massless matter which dissipates at late (early) times on $\mathcal{I}^+$ ($\mathcal{I}^-$) so that the system begins and ends in the vacuum. The CK analysis has not been fully generalized to this case, although there is no obvious reason analogs of (4.2)-(4.6) might not still pertain to a suitably defined neighborhood of the gravity+matter vacuum. In the absence of such a derivation (4.2)-(4.6)

will simply be imposed, in the matter-coupled case, as restrictions on the solutions under consideration.

4.2 Classical gravitational scattering

The classical problem of gravitational scattering is to find the outgoing data at $\mathcal{I}^+$ resulting from the evolution of given data on $\mathcal{I}^-$. We take the incoming data to be $D_{zz}(v, z, \bar{z})$ and the outgoing data to be $C_{zz}(u, z, \bar{z})$. The remaining metric components on $\mathcal{I}$ are then determined by constraints. We consider the geometries in the neighborhood of flat space defined by CK, which have $m_B = 0$ ($m_B^- = 0$) at $\mathcal{I}^+_+$ ($\mathcal{I}^-_-$). In particular we remain below the threshold for black hole formation.

A CK geometry, as described in (t, r, θ, ϕ) coordinates, does not quite provide a solution to this scattering problem. To find the in (out) data, one must perform a coordinate transformation to ingoing (outgoing) Bondi coordinates and determine D_{zz} (C_{zz}). This procedure is not unique: the coordinate transformations are ambiguous up to extended BMS$^\pm$ transformations on $\mathcal{I}^+$ or $\mathcal{I}^-$. D_{zz} and C_{zz} are not invariant under these transformations. Hence a solution of the scattering problem requires a prescription for fixing this ambiguity. A prescription to fix this ambiguity is to demand that

$$D_{zz}|_{\mathcal{I}^-_+} = C_{zz}|_{\mathcal{I}^+_-} = 0. \tag{4.7}$$

It was shown in [2] that the falloffs (4.2)-(4.6) imply this is always possible. One may then integrate the constraint equations to determine D_{zz} and C_{zz}, which will not in general vanish at $\mathcal{I}^+_+$ and $\mathcal{I}^-_-$.

This prescription does not give all near-flat solutions of the scattering problem. Indeed, all such solutions are in the center-of mass frame and have vanishing ADM three-momentum. However, given any solution of the scattering problem obeying (4.7), a new one with nonzero three-momentum may be obtained simply by acting with the boost element of $\mathcal{X}$. More generally, our prescription to define gravitational scattering is to take all solutions obtained by doing arbitrary $\mathcal{X}$ transformations on the solutions obeying (4.7). We shall refer to such scattering geometries, complete with $\mathcal{I}^\pm$ data, as *extended* CK spaces. Acting with an arbitrary finite conformal transformation $w(z)$ followed by an arbitrary finite supertranslation

f on (4.7) leads to the asymptotic behaviors for large negative u and positive v[7]

$$\begin{aligned} C_{ww}(u,w,\bar{w}) &\sim -2u(\partial_w z)^{1/2}\partial_w^2(\partial_w z)^{-1/2} - 2D_w^2 f + \mathcal{O}(u^{-3/2}),\\ D_{ww}(v,w,\bar{w}) &\sim 2v(\partial_w z)^{1/2}\partial_w^2(\partial_w z)^{-1/2} + 2D_w^2 f + \mathcal{O}(v^{-3/2}). \end{aligned} \tag{4.8}$$

We also have the relations at all the boundaries of $\mathcal{I}^{\pm}$

$$\begin{aligned} \partial_{\bar{z}} N_{zz}|_{\mathcal{I}^+_\pm} &= 0,\\ \partial_{\bar{z}} M_{zz}|_{\mathcal{I}^-_\pm} &= 0,\\ [D_{\bar{z}}^2 C_{zz} - D_z^2 C_{\bar{z}\bar{z}}]_{\mathcal{I}^+_\pm} &= 0,\\ [D_{\bar{z}}^2 D_{zz} - D_z^2 D_{\bar{z}\bar{z}}]_{\mathcal{I}^-_\pm} &= 0. \end{aligned} \tag{4.9}$$

5 $\mathcal{X}$ Ward identity

5.1 Quantum states

In the quantum theory, incoming (outgoing) states on $\mathcal{I}^-$ ($\mathcal{I}^+$) are presumed to form representations of extended BMS$^-$ (BMS$^+$). In this subsection we will describe the action of an infinitesimal Virasoro transformation δ_Y parameterized by Y^z on a generic Fock-basis in-state. For $\mathcal{I}^-$ we define

$$Q^-(Y^-)|in\rangle = -i\delta_{Y^-}|in\rangle, \tag{5.1}$$

and similarly we define $Q^+(Y^+)$ on $\mathcal{I}^+$.[8] Q^- may be decomposed into a hard and soft part as

$$Q^- = Q_H^- + Q_S^-, \tag{5.2}$$

where Q_H^- generates the diffeomorphism $\xi^-(Y^-)$ on the incoming hard particles, and Q_S^- creates a soft graviton. Let us denote an in-state comprised of n particles with energies E_k incoming at points z_k for $k = 1,\ldots,n$ on the conformal S^2 by

$$|z_1, z_2, \ldots\rangle. \tag{5.3}$$

[7] Interestingly the news tensor at the boundary $\mathcal{I}^+_-$ obeys $N_{ww}|_{\mathcal{I}^+_-} = -2(\partial_w z)^{1/2}\partial_w^2(\partial_w z)^{-1/2}$, which is the transformation law for a 2D CFT stress tensor.

[8] Explicit expressions for the proper BMS$^{\pm}$ charges as integrals of fields on $\mathcal{I}$ were worked out in detail in [3] and shown to generate the proper BMS$^{\pm}$ symmetries. Expressions for the Virasoro charges $Q^{\pm}$ are given in [8], but were not shown to generate the symmetries. In this paper such explicit expressions will not be needed: transformation laws for the states suffice.

Then the hard action is simply to act with $\xi^{-\mu}\partial_{k\mu}$ on each scalar particle

$$Q_H^-|z_1, z_2, ...\rangle = -i \sum_k \left(Y^{-z}(z_k)\partial_{z_k} - \frac{E_k}{2} D_z Y^{-z}(z_k)\partial_{E_k} \right) |z_1, z_2, ...\rangle, \tag{5.4}$$

Here $-(1 + E_k\partial_{E_k})$ arises from the Fourier transform of $v\partial_v$, and the coefficient of $D_z Y^{-z}$ is shifted by one half as in [5] due to the $r\partial_r$ term in (3.7). For spinning particles we must replace $Y^{-z}(z_k)\partial_{z_k}$ with the Lie derivative $\mathcal{L}_{Y^-(z_k)}$.[9]

To determine Q_S^-, note that the inhomogeneous transformation of the incoming Bondi news M_{zz} is

$$\delta_{Y^-} M_{zz}(v, z, \bar{z}) = D_z^3 Y^{-z}. \tag{5.5}$$

The action of Q_S^- on a state must implement this shift. It follows that

$$[Q_S^-, M_{zz}] = -i D_z^3 Y^{-z}. \tag{5.6}$$

Using the commutator [26]

$$[M_{\bar{z}\bar{z}}(v, z, \bar{z}), M_{ww}(v', w, \bar{w})] = 2i\gamma_{z\bar{z}}\delta^2(z - w)\partial_v \delta(v - v'), \tag{5.7}$$

one concludes that, up to a total derivative commuting with M_{zz},

$$Q_S^- = \frac{1}{2} \int_{\mathcal{I}^-} dv d^2z D_z^3 Y^{-z} v M^z_{\ \bar{z}}. \tag{5.8}$$

This reproduces the linear term in the full expression for the charge given in [8].[10] Q_S^- is a zero-frequency operator (because of the v integral) linear in the metric fluctuation. Acting on the in - vacuum, it creates a soft graviton with polarization tensor proportional to $D_z^3 Y^{-z}$. The explicit form of the momentum space creation operator will be constructed below in subsection 5.3. Altogether then Q^- maps the n-particle states into themselves plus an n-hard+1-soft state:

$$Q^-|z_1, z_2, ...\rangle = -i \sum_{k=1}^{n} \left(Y^{-z}(z_k)\partial_{z_k} - \frac{E_k}{2} D_z Y^{-z}(z_k)\partial_{E_k} \right) |z_1, z_2, ...\rangle \ + Q_S^-|z_1, z_2, ...\rangle. \tag{5.9}$$

[9] More explicitly if we have a particle of helicity h, and Rindler energy $-iv\partial_v = E_R$, the parentheses in (5.4) are of the form $Y^z\partial_z + Y^{\bar{z}}\partial_{\bar{z}} + h_R D_z Y^z + h_L D_{\bar{z}} Y^{\bar{z}}$ where for helicity h, the 'conformal weights' (see e.g. [5]) are $h_R = \frac{h}{2} - \frac{1}{2}E\partial_E = \frac{1}{2}(h + 1 + iE_R)$, $h_L = -\frac{h}{2} - \frac{1}{2}E\partial_E = \frac{1}{2}(-h + 1 + iE_R)$.

[10] The formula in [8] differs by a total derivative which improves the large $|v|$ behavior and may be essential in a more general context. The slightly simpler expression here is sufficient for the present purpose.

Similarly Virasoro transformations on $\mathcal{I}^+$ are decomposed as

$$Q^+ = Q_H^+ + Q_S^+ \tag{5.10}$$

and we denote out-states comprised of m particles with energies E_k outgoing at points z_k for $k = n+1, ...n+m$ by

$$\langle z_{n+1}, z_{n+2}, ...|. \tag{5.11}$$

One finds

$$\begin{aligned}\langle z_{n+1}, z_{n+2}, ...|Q^+ = i \sum_{k=n+1}^{n+m} \left(Y^{+z}(z_k)\partial_{z_k} - \tfrac{E_k}{2} D_z Y^{+z}(z_k)\partial_{E_k}\right) \langle z_{n+1}, z_{n+2}, ...| \\ + \langle z_{n+1}, z_{n+2}, ...|Q_S^+,\end{aligned} \tag{5.12}$$

where

$$Q_S^+ = -\frac{1}{2}\int_{\mathcal{I}^+} dud^2z D_z^3 Y^{+z} u N_{\bar{z}}^{z}. \tag{5.13}$$

5.2 $\mathcal{X}$-invariance of $\mathcal{S}$

In this section we derive a quantum Ward identity from the assumption that $\mathcal{X}$-invariance survives quantization. The quantum version of infinitesimal $\mathcal{X}$ invariance of classical gravitational scattering is, using (4.1)

$$\langle out|Q^+(Y)\mathcal{S} - \mathcal{S}Q^-(Y)|in\rangle = 0, \tag{5.14}$$

for any pair of in and out states $(|in\rangle, |out\rangle)$. Let us define the normal-ordered soft graviton insertion

$$: Q_S(Y)\mathcal{S} := Q_S^+(Y)\mathcal{S} - \mathcal{S}Q_S^-(Y). \tag{5.15}$$

(5.14) is then the Ward identity

$$\begin{aligned}\langle z_{n+1}, z_{n+2}, ...| : Q_S\mathcal{S} : |z_1, z_2, ...\rangle = \\ -i\sum_{k=1}^{n+m} \left(Y^z(z_k)\partial_{z_k} - \tfrac{E_k}{2} D_z Y^z(z_k)\partial_{E_k}\right) \langle z_{n+1}, z_{n+2}, ...|\mathcal{S}|z_1, z_2, ...\rangle,\end{aligned} \tag{5.16}$$

where the k sum now runs over both in and out particles and again for spinning particles the Lie derivative replaces the ordinary one on the right hand side. This relates the derivatives of any $\mathcal{S}$-matrix element to the same $\mathcal{S}$-matrix element with a particular soft graviton insertion.

5.3 Mode expansions

We wish to express $Q_S^{\pm}$ in terms of standard momentum space soft graviton creation and annihilation operators. The flat space graviton mode expansion is[11]

$$h_{\mu\nu}^{\rm out}(x) = \sum_{\alpha=\pm} \int \frac{d^3q}{(2\pi)^3} \frac{1}{2\omega_q} \left[\varepsilon_{\mu\nu}^{\alpha*}(\vec{q}) a_\alpha^{\rm out}(\vec{q}) e^{iq\cdot x} + \varepsilon_{\mu\nu}^{\alpha}(\vec{q}) a_\alpha^{\rm out}(\vec{q})^\dagger e^{-iq\cdot x}\right], \tag{5.17}$$

where $q^0 = \omega_q = |\vec{q}|$, $\alpha = \pm$ are the two helicities and

$$[a_\alpha^{\rm out}(\vec{q}), a_\beta^{\rm out}(\vec{q}')^\dagger] = 2\omega_q \delta_{\alpha\beta}(2\pi)^3 \delta^3\left(\vec{q} - \vec{q}'\right). \tag{5.18}$$

The outgoing gravitons with momentum q correspond to final-state insertions of $a_\alpha^{\rm out}(\vec{q})$. It is convenient to parametrize the graviton four-momentum by $(\omega_q, w, \bar{w})$

$$q^\mu = \frac{\omega_q}{1+w\bar{w}}\left(1+w\bar{w}, w+\bar{w}, i\left(\bar{w}-w\right), 1-w\bar{w}\right), \tag{5.19}$$

with polarization tensors

$$\begin{aligned} \varepsilon^{\pm\mu\nu} &= \varepsilon^{\pm\mu}\varepsilon^{\pm\nu}, \\ \varepsilon^{+\mu}(\vec{q}) &= \frac{1}{\sqrt{2}}\left(\bar{w}, 1, -i, -\bar{w}\right), \\ \varepsilon^{-\mu}(\vec{q}) &= \frac{1}{\sqrt{2}}\left(w, 1, i, -w\right). \end{aligned} \tag{5.20}$$

These obey $\varepsilon^{\pm\mu} q_\mu = \varepsilon^{\pm\mu}{}_\mu = 0$ and

$$\varepsilon_{\bar{z}}^{+}\left(\vec{q}\right) = \partial_{\bar{z}} x^\mu \varepsilon_\mu^{+}\left(\vec{q}\right) = \frac{\sqrt{2}r\left(1+z\bar{w}\right)}{\left(1+z\bar{z}\right)^2}, \qquad \varepsilon_{\bar{z}}^{-}\left(\vec{q}\right) = \partial_{\bar{z}} x^\mu \varepsilon_\mu^{-}\left(\vec{q}\right) = \frac{\sqrt{2}rz\left(w-z\right)}{\left(1+z\bar{z}\right)^2}. \tag{5.21}$$

In retarded Bondi coordinates

$$C_{\bar{z}\bar{z}}(u, z, \bar{z}) = 2 \lim_{r\to\infty} \frac{1}{r} h_{\bar{z}\bar{z}}^{\rm out}(r, u, z, \bar{z}). \tag{5.22}$$

Using $h_{\bar{z}\bar{z}}^{\rm out} = \partial_{\bar{z}} x^\mu \partial_{\bar{z}} x^\nu h_{\mu\nu}^{\rm out}$ and the mode expansion

$$C_{\bar{z}\bar{z}} = 2 \lim_{r\to\infty} \frac{1}{r} \partial_{\bar{z}} x^\mu \partial_{\bar{z}} x^\nu \sum_{\alpha=\pm} \int \frac{d^3q}{(2\pi)^3} \frac{1}{2\omega_q} \left[\varepsilon_{\mu\nu}^{\alpha*}(\vec{q}) a_\alpha^{\rm out}(\vec{q}) e^{-i\omega_q u - i\omega_q r(1-\cos\theta)} + h.c.\right], \tag{5.23}$$

[11]Here we take $g_{\mu\nu} = \eta_{\mu\nu} + \sqrt{32\pi G} h_{\mu\nu} = \eta_{\mu\nu} + 2h_{\mu\nu}$.

where θ is the angle between $\vec{x}$ and $\vec{q}$. This integral is dominated for large r by the contribution near θ=0:

$$C_{\bar{z}\bar{z}} = -\frac{i}{4\pi^2}\hat{\varepsilon}^+_{\bar{z}\bar{z}}\int_0^\infty d\omega_q[a^{\rm out}_-(\omega_q\hat{x})e^{-i\omega_q u} - a^{\rm out}_+(\omega_q\hat{x})^\dagger e^{i\omega_q u}]. \tag{5.24}$$

Here, $\hat{x}$ is parameterized by $(z,\bar{z})$

$$\hat{x} \equiv \frac{\vec{x}}{r} = \frac{1}{1+z\bar{z}}(z+\bar{z}, i(\bar{z}-z), 1-z\bar{z}) \tag{5.25}$$

and

$$\hat{\varepsilon}^+_{\bar{z}\bar{z}} = \frac{\partial_{\bar{z}}x^\mu\partial_{\bar{z}}x^\nu}{r^2}\varepsilon^+_{\mu\nu} = \frac{2}{(1+z\bar{z})^2}. \tag{5.26}$$

Define:

$$N^\omega_{\bar{z}\bar{z}} \equiv \int e^{i\omega u}\partial_u C_{\bar{z}\bar{z}}du. \tag{5.27}$$

Then from the large r saddle point expansion of (5.23), we have:

$$\begin{aligned} N^\omega_{\bar{z}\bar{z}} &= -\tfrac{1}{2\pi}\hat{\varepsilon}^+_{\bar{z}\bar{z}}\omega a^{\rm out}_-(\omega\hat{x}), \\ N^{-\omega}_{\bar{z}\bar{z}} &= -\tfrac{1}{2\pi}\hat{\varepsilon}^+_{\bar{z}\bar{z}}\omega a^{\rm out}_+(\omega\hat{x})^\dagger, \end{aligned} \tag{5.28}$$

with $\omega > 0$ in both cases. We define $N^{(1)}_{\bar{z}\bar{z}}$ as:

$$\begin{aligned} N^{(1)}_{\bar{z}\bar{z}} &\equiv \textstyle\int du u N_{\bar{z}\bar{z}} \\ &= -\lim_{\omega\to 0}\tfrac{i}{2}(\partial_\omega N^\omega_{\bar{z}\bar{z}} + \partial_{-\omega}N^{-\omega}_{\bar{z}\bar{z}}) \\ &= \tfrac{i}{4\pi}\hat{\varepsilon}^+_{\bar{z}\bar{z}}\lim_{\omega\to 0}(1+\omega\partial_\omega)[a^{\rm out}_-(\omega\hat{x}) - a^{\rm out}_+(\omega\hat{x})^\dagger]. \end{aligned} \tag{5.29}$$

A mode expansion analogous to (5.24) can be defined for $D_{\bar{z}\bar{z}}$ on $\mathcal{I}^-$

$$D_{\bar{z}\bar{z}} = -\frac{i}{4\pi^2}\hat{\varepsilon}^+_{\bar{z}\bar{z}}\int_0^\infty d\omega_q[a^{\rm in}_-(\omega_q\hat{x})e^{-i\omega_q v} - a^{\rm in}_+(\omega_q\hat{x})^\dagger e^{i\omega_q v}], \tag{5.30}$$

from which we find

$$\begin{aligned} M^\omega_{\bar{z}\bar{z}} &= -\tfrac{1}{2\pi}\hat{\varepsilon}^+_{\bar{z}\bar{z}}\omega a^{\rm in}_-(\omega\hat{x}), \\ M^{-\omega}_{\bar{z}\bar{z}} &= -\tfrac{1}{2\pi}\hat{\varepsilon}^+_{\bar{z}\bar{z}}\omega a^{\rm in}_+(\omega\hat{x})^\dagger, \end{aligned} \tag{5.31}$$

and

$$\begin{aligned} M^{(1)}_{\bar{z}\bar{z}} &= -\lim_{\omega\to 0}\tfrac{i}{2}(\partial_\omega M^\omega_{\bar{z}\bar{z}} + \partial_{-\omega}M^{-\omega}_{\bar{z}\bar{z}}) \\ &= \tfrac{i}{4\pi}\hat{\varepsilon}^+_{\bar{z}\bar{z}}\lim_{\omega\to 0}(1+\omega\partial_\omega)[a^{\rm in}_-(\omega\hat{x}) - a^{\rm in}_+(\omega\hat{x})^\dagger]. \end{aligned} \tag{5.32}$$

We are interested in the matrix element

$$\begin{aligned}
&\langle out|N^{(1)}_{\bar{z}\bar{z}}\mathcal{S}+\mathcal{S}M^{(1)}_{\bar{z}\bar{z}}|in\rangle \\
&=\tfrac{i}{4\pi}\hat{\varepsilon}^{+}_{\bar{z}\bar{z}}\lim_{\omega\to 0}(1+\omega\partial_\omega)\langle out|(a^{\rm out}_{-}(\omega\hat{x})-a^{\rm out}_{+}(\omega\hat{x})^{\dagger})\mathcal{S}+\mathcal{S}(a^{\rm in}_{-}(\omega\hat{x})-a^{\rm in}_{+}(\omega\hat{x})^{\dagger})|in\rangle \\
&=\tfrac{i}{4\pi}\hat{\varepsilon}^{+}_{\bar{z}\bar{z}}\lim_{\omega\to 0}(1+\omega\partial_\omega)\langle out|a^{\rm out}_{-}(\omega\hat{x})\mathcal{S}-\mathcal{S}a^{\rm in}_{+}(\omega\hat{x})^{\dagger}|in\rangle,
\end{aligned} \tag{5.33}$$

which is $\langle out|\mathcal{S}|in\rangle$ with soft graviton insertions.[12] Such insertions generically have Weinberg poles behaving as $\frac{1}{\omega}$. However the prefactor $1+\omega\partial_\omega$ projects out this pole, leaving the subleading $\mathcal{O}(\omega^0)$ soft factor. Equation (5.33) and its hermitian conjugate are related to the Q_S matrix element by

$$\begin{aligned}
&\langle out|:Q_S\mathcal{S}:|in\rangle \\
&=-\tfrac{1}{2}\int d^2z\gamma^{z\bar{z}}D^3_zY^z\langle out|N^{(1)}_{\bar{z}\bar{z}}\mathcal{S}+\mathcal{S}M^{(1)}_{\bar{z}\bar{z}}|in\rangle \\
&=-\tfrac{i}{8\pi}\lim_{\omega\to 0}(1+\omega\partial_\omega)\int d^2z\gamma^{z\bar{z}}D^3_zY^z\hat{\varepsilon}^{+}_{\bar{z}\bar{z}}\langle out|a^{\rm out}_{-}(\omega\hat{x})\mathcal{S}-\mathcal{S}a^{\rm in}_{+}(\omega\hat{x})^{\dagger}|in\rangle.
\end{aligned} \tag{5.34}$$

Given the asymptotic behavior (4.8) near i^0, the boundary relation $N_{\bar{z}\bar{z}}|_{\mathcal{I}^+_-}=-M_{\bar{z}\bar{z}}|_{\mathcal{I}^-_+}$ establishes a correspondence between the in and out modes, such that the contributions to the matrix element (5.34) from the $a^{\rm out}_{-}(\omega\hat{x})$ and $-a^{\rm in}_{+}(\omega\hat{x})^{\dagger}$ insertions are equal.

6 From soft theorem to Virasoro symmetry

In this section we begin by assuming the subleading-soft relation[13]

$$\lim_{\omega\to 0}(1+\omega\partial_\omega)\langle z_{n+1},z_{n+2},...|a_{-}(q)\mathcal{S}|z_1,z_2,...\rangle=S^{(1)-}\langle z_{n+1},z_{n+2},...|\mathcal{S}|z_1,z_2,...\rangle, \tag{6.1}$$

with

$$S^{(1)-}\;=\;-i\sum_k\frac{p_{k\mu}\varepsilon^{-\mu\nu}q^\lambda J_{k\lambda\nu}}{p_k\cdot q}. \tag{6.2}$$

Here $J_{k\lambda\nu}\equiv L_{k\lambda\nu}+S_{k\lambda\nu}$ is the total ingoing orbital+spin angular momentum of the kth particle which obeys the global conservation law $\sum J_{k\lambda\nu}=0$. We note the $(1{+}\omega\partial_\omega)$ prefactor on the left hand side projects out the would-be Weinberg pole accompanying a soft insertion.

[12] Here we assume that $|in\rangle$ and $\langle out|$ states contain no soft gravitons.
[13] A single soft graviton insertion has the ω expansion

$$\langle z_{n+1},z_{n+2},...|a_{-}(q)\mathcal{S}|z_1,z_2,...\rangle=\left(S^{(0)-}+S^{(1)-}\right)\langle z_{n+1},z_{n+2},...|\mathcal{S}|z_1,z_2,...\rangle+\mathcal{O}(\omega).$$

For notational brevity we consider the contribution for negative polarization: the general formula has an $S^{(1)}$ with a general polarization tensor replacing (6.2). We will show that (6.1) implies the Ward identity (5.16), which in turn is equivalent to infinitesimal $\mathcal{X}$-invariance of the $\mathcal{S}$-matrix. Although the relation (6.1) potentially has wider validity, the only case in which it is known to be a theorem is tree-level gravitons [10]. Hence only for this case do we claim the results of this section imply a Virasoro symmetry.

Gauge invariance provides an important check on this formula. Amplitudes must vanish for pure gauge gravitons with polarizations

$$\varepsilon_\Lambda^{\mu\nu} = q^\mu \Lambda^\nu + q^\nu \Lambda^\mu \tag{6.3}$$

for any Λ. Inserting this into (6.2) we find

$$iS^{(1)}(\varepsilon_\Lambda) = q^\mu \Lambda^\nu \sum_k J_{k\mu\nu} + \sum_k \frac{p_k \cdot \Lambda q^\mu q^\nu J_{k\mu\nu}}{p_k \cdot q}. \tag{6.4}$$

The first terms vanishes by global angular momentum conservation, while the second vanishes by antisymmetry of $J_{k\mu\nu}$. This is very similar to the gauge invariance of the Weinberg pole, which vanishes due to global energy-momentum conservation or equivalently translational symmetry. The Weinberg soft theorem implies that this global translational symmetry is promoted to a local supertranslational symmetry on the sphere [3], because there is one symmetry for every angle $\vec{q}$. In this section we will see a parallel story for rotational invariance: the soft relation (6.1) implies that rotations are promoted to a local superrotational - equivalently Virasoro - symmetry on the sphere.

The first step is to write the hard particle momenta p_k, the soft graviton momentum q and chosen polarization $\varepsilon^{-\mu\nu} = \varepsilon^{-\mu}\varepsilon^{-\nu}$ in terms of the points z_k and z at which they arrive on the on the asymptotic S^2 and their energies E_k, ω

$$\begin{aligned}
p_k^\mu &= \tfrac{E_k}{1+z_k\bar{z}_k}\left(1+z_k\bar{z}_k, \bar{z}_k+z_k, i(\bar{z}_k-z_k), 1-z_k\bar{z}_k\right), \\
q^\mu &= \tfrac{\omega}{1+z\bar{z}}\left(1+z\bar{z}, \bar{z}+z, i(\bar{z}-z), 1-z\bar{z}\right), \\
\varepsilon^{-\mu} &= \tfrac{1}{\sqrt{2}}(z, 1, i, -z).
\end{aligned} \tag{6.5}$$

One then finds for the orbital terms

$$S^{(1)-} = \sum_k \left(\frac{E_k(z-z_k)(1+z\bar{z}_k)}{(\bar{z}_k-\bar{z})(1+z_k\bar{z}_k)} \partial_{E_k} + \frac{(z-z_k)^2}{(\bar{z}_k-\bar{z})} \partial_{z_k} \right). \tag{6.6}$$

The spin term will be added in below. This expression obeys

$$\gamma^{z\bar{z}} D_z^3 (\hat{\varepsilon}^+_{\bar{z}\bar{z}} S^{(1)-}) = -2\pi \sum_k \big(D_z \delta^{(2)}(z - z_k) E_k \partial_{E_k} + 2\delta^{(2)}(z - z_k)\partial_{z_k} \big). \tag{6.7}$$

Multiplying both sides of (6.1) by $D_z^3 Y^z \hat{\varepsilon}^{+z}_{\bar{z}}$ and integrating over the soft graviton angle z gives

$$\begin{aligned} &\langle z_{n+1}, z_{n+2}, ...| : Q_S \mathcal{S} : |z_1, z_2, ...\rangle = \\ &\qquad - i \sum_k \left(Y^z(z_k)\partial_{z_k} - \tfrac{E_k}{2} D_z Y^z(z_k) \partial_{E_k} \right) \langle z_{n+1}, z_{n+2}, ...|\mathcal{S}|z_1, z_2, ...\rangle, \end{aligned} \tag{6.8}$$

which is exactly the Ward identity (5.16) arising from an asymptotic Virasoro symmetry, minus the so-far-omitted spin terms.

The spin contribution comes from evaluating:

$$S_S^{(1)-} = -i \sum_k \frac{p_{k\lambda} \varepsilon^{-\lambda\nu} q^\mu S_{k\mu\nu}}{p_k \cdot q}. \tag{6.9}$$

In terms of the helicity h defined by

$$h p_\mu = -\frac{1}{2} \varepsilon_{\mu\nu\lambda\rho} S^{\nu\lambda} p^\rho, \tag{6.10}$$

one finds

$$S_S^{(1)-} = \sum_k \frac{(z - z_k)(1 + z\bar{z}_k)}{(\bar{z} - \bar{z}_k)(1 + z_k \bar{z}_k)} h_k, \tag{6.11}$$

while the third derivative obeys

$$\gamma^{z\bar{z}} D_z^3 (\hat{\varepsilon}^+_{\bar{z}\bar{z}} S_S^{(1)-}) = 2\pi \sum_k h_k D_z \delta^{(2)}(z - z_k). \tag{6.12}$$

Hence the spin contribution for the helicity states corrects (6.8) to:

$$\begin{aligned} \langle z_{n+1}, z_{n+2}, ...| : Q_S \mathcal{S} : |z_1, z_2, ...\rangle \;\; &= -i \sum_k \left(Y^z(z_k)\partial_{z_k} - \tfrac{E_k}{2} D_z Y^z(z_k) \partial_{E_k} + \tfrac{h_k}{2} D_z Y^z(z_k) \right) \\ &\qquad\qquad \langle z_{n+1}, z_{n+2}, ...|\mathcal{S}|z_1, z_2, ...\rangle, \end{aligned} \tag{6.13}$$

in agreement with the spin-corrected version of (5.16).

In conclusion the soft relation (6.1), whenever valid, implies a Virasoro symmetry of the quantum gravity $\mathcal{S}$-matrix.

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References

[1] H. Bondi, M. G. J. van der Burg, A. W. K. Metzner, "Gravitational waves in general relativity VII. Waves from isolated axisymmetric systems", Proc. Roy. Soc. Lond. A 269, 21 (1962); R. K. Sachs, "Gravitational waves in general relativity VIII. Waves in asymptotically flat space-time", Proc. Roy. Soc. Lond. A 270, 103 (1962).

[2] A. Strominger, "On BMS Invariance of Gravitational Scattering," arXiv:1312.2229 [hep-th].

[3] T. He, V. Lysov, P. Mitra and A. Strominger, "BMS supertranslations and Weinberg's soft graviton theorem," arXiv:1401.7026 [hep-th].

[4] S. Weinberg, "Infrared photons and gravitons," Phys. Rev. **140**, B516 (1965).

[5] A. Strominger, "Asymptotic Symmetries of Yang-Mills Theory," arXiv:1308.0589 [hep-th].

[6] E. J.Newman and T. W. J. Unti, "Behavior of asymptotically flat empty spaces" J. Math. Phys **3**, 891 (1962).

[7] A. Ashtekar and R. O. Hansen, "A unified treatment of null and spatial infinity in general relativity. I - Universal structure, asymptotic symmetries, and conserved quantities at spatial infinity," J. Math. Phys. **19**, 1542 (1978).

[8] G. Barnich and C. Troessaert, "Symmetries of asymptotically flat 4 dimensional spacetimes at null infinity revisited," Phys. Rev. Lett. **105**, 111103 (2010) [arXiv:0909.2617 [gr-qc]]; "Supertranslations call for superrotations," PoS CNCFG **2010**, 010 (2010), [arXiv:1102.4632 [gr-qc]] G. Barnich and C. Troessaert, "BMS charge algebra," JHEP **1112**, 105 (2011) [arXiv:1106.0213 [hep-th]].

[9] See footnote 18 in T. Banks, "A Critique of pure string theory: Heterodox opinions of diverse dimensions," hep-th/0306074.

[10] F. Cachazo and A. Strominger, "Evidence for a New Soft Graviton Theorem," arXiv:1404.4091 [hep-th].

[11] A. Strominger,"Virasoro Notes" unpublished (2013).

[12] Z. Bern, S. Davies and J. Nohle, "On Loop Corrections to Subleading Soft Behavior of Gluons and Gravitons," arXiv:1405.1015 [hep-th].

[13] S. He, Y. -t. Huang and C. Wen, "Loop Corrections to Soft Theorems in Gauge Theories and Gravity," arXiv:1405.1410 [hep-th].

[14] F. Cachazo and E. Y. Yuan, "Are Soft Theorems Renormalized?," arXiv:1405.3413 [hep-th].

[15] Z. Bern, private communication.

[16] Z. Bern and G. Chalmers, "Factorization in one loop gauge theory," Nucl. Phys. B **447**, 465 (1995) [hep-ph/9503236].

[17] R. Akhoury, R. Saotome and G. Sterman, "Collinear and Soft Divergences in Perturbative Quantum Gravity," Phys. Rev. D **84**, 104040 (2011) [arXiv:1109.0270 [hep-th]].

[18] J. Ware, R. Saotome and R. Akhoury, "Construction of an asymptotic S matrix for perturbative quantum gravity," arXiv:1308.6285 [hep-th].

[19] C. D. White, "Factorization Properties of Soft Graviton Amplitudes," JHEP **1105**, 060 (2011) [arXiv:1103.2981 [hep-th]].

[20] P. P. Kulish and L. D. Faddeev, "Asymptotic conditions and infrared divergences in quantum electrodynamics," Theor. Math. Phys. **4**, 745 (1970) [Teor. Mat. Fiz. **4**, 153 (1970)].

[21] D. J. Gross and R. Jackiw, "Low-Energy Theorem for Graviton Scattering," Phys. Rev. **166**, 1287 (1968).

[22] F. E. Low, "Scattering of light of very low frequency by systems of spin 1/2," Phys. Rev. **96**, 1428 (1954).

[23] T. Banks, "The Super BMS Algebra, Scattering and Holography," arXiv:1403.3420 [hep-th].

[24] T. Adamo, E. Casali and D. Skinner, "Perturbative gravity at null infinity," arXiv:1405.5122 [hep-th].

[25] Y. Geyer, A. E. Lipstein and L. Mason, "Ambitwistor strings at null infinity and subleading soft limits," arXiv:1406.1462 [hep-th].

[26] A. Ashtekar, "Asymptotic Quantization of the Gravitational Field," Phys. Rev. Lett. **46**, 573 (1981); A. Ashtekar and M. Streubel, "Symplectic Geometry of Radiative Modes and Conserved Quantities at Null Infinity," Proc. Roy. Soc. Lond. A **376**, 585 (1981); A. Ashtekar, "Asymptotic Quantization: Based On 1984 Naples Lectures," Naples, Italy: Bibliopolis, (1987).

[27] D. Christodoulou and S. Klainerman, "The Global nonlinear stability of the Minkowski space," Princeton University Press, Princeton, 1993.

[28] D. Christodoulou, "Nonlinear nature of gravitation and gravitational wave experiments," Phys. Rev. Lett. **67**, 1486 (1991).