Strategic Behavior in Unbalanced Matching Markets*

Peter Coles  Yannai Gonczarowski
*ebay Research Labs Hebrew University

Ran Shorrer
Harvard University and Harvard Business School

February 2014

Abstract

In this paper we explore how the balance of agents on the two sides of a matching market impacts their potential for strategic manipulation. Coles and Shorrer [5] previously showed that in large, balanced, uniform markets using the Men-Proposing Deferred Acceptance Algorithm, each woman’s best response to truthful behavior by all other agents is to truncate her list substantially. In fact, the optimal degree of truncation for such a woman goes to 100% of her list as the market size grows large. Recent findings of Ashlagi et. al. [2] demonstrate that in unbalanced random markets, the change in expected payoffs is small when one reverses which side of the market “proposes,” suggesting there is little potential gain from manipulation. Inspired by these findings, we study the implications of imbalance on strategic behavior in the incomplete information setting. We show that the “long” side has significantly reduced incentives for manipulation in this setting, but that the same doesn’t always apply to the “short” side. We also show that risk aversion and correlation in preferences affect the extent of optimal manipulation.

1 Introduction

A great success story in economic theory is the application of the Deferred Acceptance Algorithm (DAA), proposed by Gale and Shapley [10], to real world two-sided matching markets. The DAA and its variants have been used extensively in school choice settings [1], and most famously in the National Resident Matching Program [20]. The advantages of mechanisms using DAA over other mechanisms have been discussed extensively (See for example [19]). Importantly, it was shown that while no stable matching mechanism is strategy proof, mechanisms applying the DAA have truthful reporting as a dominant strategy for the proposing side [6, 18]. The choice of the proposing side has received some attention in the public domain and in the literature [20], but the general message that has emerged from this body of literature is that the choice of the proposing side has a small effect over agents’ utilities [20, 11, 13, 14, 2, 4].

*We thank Itai Ashlagi, Nicole Immorlica, Jacob Leshno, Yashodhan Kanoria, Assaf Romm and Al Roth for helpful discussions. We are especially grateful to Jenő Pál for his contributions to this paper. Shorrer thanks Microsoft Research for their hospitality while most of the paper was written. The authors can be contacted at rshorrer@hbs.edu.
This paper takes a different perspective on this issue. We look for the (exact) best responses of agents, and consider the degree of manipulation expected in the market. To do this, we restrict attention to truncation strategies, which are endowed with a natural metric for measuring the extent of manipulation (how many acceptable partners were declared unacceptable). This class of strategies was shown to be optimal in symmetric low-information settings [21, 7]. We derive comparative statics on the extent of manipulation as a function of risk aversion and correlation, and show that more risk averse agents submit longer lists (so they are “more truthful”) and that correlation in preferences also reduces the incentives to manipulate. These results are similar to the findings of Coles and Shorrer [5], but they are more general as we do not assume that the markets are balanced.

The main innovation in this paper is inspired by the results of Ashlagi, Kanoria and Leshno [2]. In contrast to the findings of Roth and Peranson [20] regarding the “large core” of markets when agents have long preference lists, and the related findings of Pittel [16], Ashlagi et. al. [2] show that if the number of agents on each side of the market is not balanced, the core becomes small in the typical case. So while the gap between men (women) expected partner ranks under the men and women proposing versions of the DAA is high in a balanced marked, even a slight imbalance “shrinks” this gap significantly. In light of this finding we ask: What are the effects of imbalance on the incentives to misrepresent one’s preferences? The answer is: it depends! Under the men-proposing version of the DAA, if there are more women than men, women optimally submit “long” lists. When the sides of the market are balanced, a woman facing truthful opponents should submit a short list; asymptotically she truncates 100% of her list. When women are over demanded (on the short side), we provide simulation evidence that extreme truncation is still optimal. We also show that truncation is “safe” when women are on the short side, but not when they are on the long side of the market. To summarize, the extent of optimal truncation may crucially depend on whether the strategic agents (the ones not on the proposing side) are on the long side or the short side of the market.

A market designer may prefer that agents submit either long or short lists. She may be concerned about the incentives for truthfulness for several reasons. For one, she may wish to advise participants that being truthful will not harm them, so as to “level the playing field” between savvy and naive agents [15, 9]. The number of matched agents and the (ex-post) stability of the match may also be affected [8]. An additional reason why designers may want to induce truthful reporting is that the submitted profiles may provide a signal as to the desirability of agents on the two sides of the market. In school choice settings, for example, truthful reporting allows school districts to learn about the actual desirability of different schools.

But market designers may also have reasons to favor shorter lists. From a computational perspective, running the DAA on shorter lists is faster. More importantly, the designer may think that there is a cost (actual or mental) to the system or to participants on one side of the market for generating a long preference list. For example, a school may be required to give each applicant a tour, paperwork may be required for each school that appears on an applicant’s list, and a student may simply find it hard to compare his 100th and 101st choices. We take no stand on whether ensuring truthfulness or promoting short lists is more desirable, but merely wish to provide advice to the market designer given objectives regarding list length.
2 Preliminaries

We begin by setting out the basic model of matching. Following Coles and Shorrer \[5\], and in contrast to some of the well-known papers in the field of matching, we endow agents with cardinal rather than ordinal preferences.

2.1 Marriage Markets and Stability

In this paper, only one-to-one two sided matching markets will be considered. We call these markets marriage markets for short, and label one side on the market as men $\mathcal{M}$, and the other as women $\mathcal{W}$. Both men and women are referred to as agents.

The preferences of man $m \in \mathcal{M}$ are given by a von Neumann-Morgenstern utility function $u_m : \mathcal{W} \cup \{m\} \rightarrow \mathbb{R}$. $u_m(w)$ is the utility that man $m$ derives from being matched with woman $w$ with certainty, and $u_m(m)$ is his utility from being unmatched. For simplicity, we assume that $u_m$ is one-to-one, so that there are no indifferences. Preferences for women are defined similarly. We denote by $u = \prod_{i \in \mathcal{M} \cup \mathcal{W}} u_i$ the profile of agents preferences.

Since we have assumed that agents’ preferences are one-to-one, they induce strict preference orderings on all possible partners and the possibility of remaining unmatched. For a man $m \in \mathcal{M}$ we denote by $P_m$ the preference list over $\mathcal{W} \cup \{m\}$ that is induced by $u_m$. For example, $P_m$ ranks $w_3$ higher than $w_1$ if $u_m(w_3) > u_m(w_1)$. We say that $w \in \mathcal{W}$ is acceptable for $m$ if $u_m(w) > u_m(m)$, so $m$ prefers being matched with $w$ over remaining single. We sometimes omit unacceptable mates from $m$’s preference list for notational convenience. Preference lists for women are defined similarly, and we denote by $P$ the profile of all preference lists.

A matching $\mu$ is a mapping from $\mathcal{M} \cup \mathcal{W}$ to itself, such that for each $m \in \mathcal{M}$ we have that $\mu(m) \in \mathcal{W} \cup \{m\}$, for each $w \in \mathcal{W}$ we have $\mu(w) \in \mathcal{M} \cup \{w\}$ and for each $x \in \mathcal{M} \cup \mathcal{W}$ $\mu^2(x) = x$. When $\mu(x) = x$ we say that $x$ is single or unmatched under the matching $\mu$. Otherwise, we refer to $\mu(w)$ as $w$’s husband and $\mu(m)$ as $m$’s wife under the matching $\mu$. We also use the terms partner and mate. The preferences over partners induce natural preference order over matchings, where each agent ranks the matchings according to the partner that is assigned to him.

A matching is individually rational if for every $x \in \mathcal{M} \cup \mathcal{W}$, the agent $x$ weakly prefers $\mu(x)$ to remaining single. A matching is blocked by a pair $(w, m) \in \mathcal{W} \times \mathcal{M}$ if both $w$ prefers $m$ to $\mu(w)$ and $m$ prefers $w$ to $\mu(m)$. A matching is stable if it is individually rational and not blocked by any pair. There always exists a stable matching in a market, but in general there may be more than one \[10\]. For given preferences, we say that a woman $w$ is achievable for a man $m$ if there exists a stable matching $\mu$ such that $\mu(w) = m$. A symmetric definition applies to women’s achievable mates.

2.2 The Men-Proposing Deferred Acceptance Algorithm

To prove that every marriage market has a stable matching, Gale and Shapley \[10\] proposed the Men-Proposing Deferred Acceptance Algorithm (MP-DA). It takes as an input a profile of preferences $P$ of a set of agents $\mathcal{M} \cup \mathcal{W}$ and outputs a stable matching $\mu^M[P]$. When $P$ is clear form the contexts, we
sometimes omit it and write $\mu^M$ instead of $\mu^M[P]$. The following is a description of the algorithm.

- **Step 1.** Each man proposes to the first woman on his preference list. Each woman then considers her offers, rejects all men deemed unacceptable, and if any others remain, rejects all but her most preferred mate.

- **Step k.** Each man who was rejected in step $k-1$ makes an offer to the next woman on his preference list. If his preference list is exhausted, or if he prefers bachelorhood to the next woman on his list, he makes no offer. Each woman behaves as in step 1, considering offers in hand (including any man she has retained from the previous step) and rejects all but her most preferred acceptable suitor.

- **Termination.** If in any step $k$, no man makes an offer, the algorithm terminates. Each woman is paired with her current mate and this matching is final.

Gale and Shapley show that this algorithm must terminate in finite time, and they provide a remarkable characteristic of the resulting outcome.

**Theorem.** (Gale-Shapley) The matching $\mu^M$ resulting from MP-DA is stable. Furthermore, for any other stable matching $\mu$, every man weakly prefers $\mu^M$ to $\mu$ and every woman weakly prefers $\mu$ to $\mu^M$.

Since there is no actual content to gender (it is just a label), it is clear that the women-proposing version of the algorithm (WP-DA) has identical but reversed properties. We denote its output given an input $P$ by $\mu^W[P]$.

As discussed by Roth [19], stability is a desirable property for a matching mechanism. But the theorem illustrates a particular feature of the stable matching produced by the MP-DA (WP-DA); it is the most desirable stable matching for men (women), and the least desirable for women (men). This paper focuses on the strategic incentives that emerge from this property under incomplete information, and their effects on the realized matchings given strategic reporting.

### 2.3 The Preference List Submission Problem

We now turn to study the incentive properties of stable matching mechanism which use the MP-DA. In a setting where agents are asked to report preferences lists to the mechanism, we consider if they have an incentive to report truthfully, or to submit a different preference list.

Consider a set of agents $M \cup W$. Agent $i \in M \cup W$ with preferences $u_i$ must submit a preference list $\hat{P}_i$ to MP-DA, where $\hat{P}_i$ is chosen from the set of $i$’s possible preference lists $\mathcal{P}_i$. The agent’s beliefs about what preference lists others will report are represented by the random variable $\hat{P}_{-i}$, which takes as its range $\mathcal{P}_{-i}$, the set of all possible preference list profiles for others. Note that since $u_i$ is a von Neumann-Morgenstern utility function, agent $i$ may compare outcomes in this incomplete information setting.

Agent $i$ solves the Preference List Submission Problem:

$$\max_{\hat{P}_i \in \mathcal{P}_i} \mathbb{E}[u_i(\mu^M[\hat{P}_i, \hat{P}_{-i}](i))].$$
Dubins and Freedman [6] and Roth [18] have shown that for any man \( m \) with preferences \( u_m \) and beliefs \( \tilde{P}_{-m} \), it is optimal for \( m \) to submit his true preference list \( P_m \) (which corresponds to \( u_m \)).

**Theorem.** (Dubins and Freedman; Roth) In the Preference List Submission Problem,

\[
P_m \in \arg \max_{P_m \in \mathcal{P}_m} \mathbb{E}[u_m(\mu^M[\tilde{P}_m, \tilde{P}_{-m}](m))].
\]

This is not the case for women, as they may misrepresent their preferences and get preferable outcomes in some settings [18]. A natural way to misrepresent one’s preferences is by submitting a truncated preferences list. A truncated preference list is identical to the original one, except that some acceptable partners are declared unacceptable. Denote by \( P^k_w \) the preference list which includes in order only \( w \)'s \( k \) most preferred men, and call this the \( k \)-truncation of her true preference list \( P_w \). If fewer than \( k \) men are acceptable to \( w \), then \( P^k_w \equiv P_w \). Truncation generates a simple tradeoff which is described by the following proposition:

**Proposition.** Let \( P \) be the preference list profile of all agents in \( \mathcal{M} \cup \mathcal{W} \). Then \( \mu^M[P^k_w, P_{-w}](w) \) is \( w \)'s least preferred achievable mate under \( P \) with rank \( \leq k \). Should no such mate exist, \( \mu^M[P^k_w, P_{-w}](w) = w \).

The proposition implies that when others’ submitted preferences lists are known with certainty it is easy to find a truncation strategy that would match the woman with her most preferred achievable partner, but also that when there exists uncertainty about others’ submitted lists truncation may yield each of the three possible results relative to truthful reporting:

1. No effect - when woman \( w \) has truncated below her least preferred achievable mate
2. Improvement - when woman \( w \) truncates above her least preferred achievable mate, and is matched with her least preferred achievable mate above the point of truncation
3. Turning unmatched - when woman \( w \) has an achievable mate, but has over-truncated by truncating above her most preferred achievable mate

Since the realized outcome depends on the realized profile that others submit, each truncation yields a lottery given the beliefs \( \tilde{P}_{-w} \), and the problem of choosing the optimal truncation corresponds to choosing the most preferable lottery.

### 2.3.1 Optimality of Truncation

Truncation is not the only possible misrepresentation of preferences. A woman could reverse two men in her preference list, list men as acceptable who are in fact unacceptable, drop men from the middle of her list, or use some combination of these. However, under some conditions, truncation is optimal.

The next proposition states that under certainty, women can do no better than to truncate [22].

**Proposition.** (Roth and Vande Vate) Suppose woman \( w \) has preferences \( u_w \) and knows others will report preference lists \( P_{-w} \) to MP-DA. Then truncating such that \( \mu^W(w) \) is the last acceptable partner on her list is an optimal strategy for \( w \).
Perhaps surprisingly, when a woman has very little information about the preference lists others might report, she again can do no better than to truncate. In order to gain from non-truncation misrepresentations, such as swapping the positions of two men in her reported preference list, a woman must have very specific information about the preference lists others report. Without such information, it is best to leave the men in their correct order. Roth and Rothblum \cite{RothRothblum2002} demonstrate this principle using the following framework.

Let woman \( w \)'s beliefs about reported preference lists of others be represented by \( \tilde{P}_{-w} \), a random variable taking on values in \( P_{-w} \). If \( P_{-w} \) is a preference list profile for agents \( -w \), define \( P_{m \leftrightarrow m'}_{-w} \) to be the preference list profile in which \( m \) and \( m' \) swap preference lists, and all women swap the positions of \( m \) and \( m' \) in their lists. We say that woman \( w \)'s beliefs are \((m,m')\)-symmetric if \( \Pr(\tilde{P}_{-w} = P_{-w}) = \Pr(\tilde{P}_{-w} = P_{m \leftrightarrow m'}_{-w}) \) for all \( P_{-w} \in P_{-w} \). For a subset \( M' \subseteq M \), beliefs \( \tilde{P}_{-w} \) are \( M' \)-symmetric if they are \((m,m')\)-symmetric for all \( m,m' \in M' \).

**Theorem.** (Roth and Rothblum) Suppose \( w \)'s beliefs about reported preference lists of others are \( M \)-symmetric. Then any preference list \( \hat{P}_w \) she might submit to MP-DA is weakly \( P_w \)-stochastically dominated by some truncation of her true preference list.\(^2\)

Hence, when \( w \) is certain about reported preference lists of her opponents, or when she has extreme, symmetric uncertainty, truncation is optimal.

### 2.3.2 The Truncation Problem

Even when truncation is not optimal, we may sometimes wish to restrict the choice set for women to truncations of her true preference list. We define the *Truncation Problem* for woman \( w \) with preferences \( u_w \) and beliefs \( \tilde{P}_{-w} \) on others’ submitted preference lists as

\[
\max_{k \in \{0,...,N\}} \mathbb{E}[u_w(\mu^M[\mu^M[P^k_{w}, \tilde{P}_{-w}](w)])].
\]

### 3 Optimal Truncation in Unbalanced Markets

Following Coles and Shorrer \cite{ColesShorrer2021} and Ashlagi et. al. \cite{Ashlagi2013} we consider a setting where each agent draws independently uniformly at random a complete preference list (so that all mates are acceptable).\(^3\) We assume further that for each agent \( i \), \( u_i(\cdot) \) is linear in the rank of \( i \)'s match, where being unmatched is treated as rank one below the lowest ranked mate. For a balanced uniform market with \( N \) men and \( N \) women, define

\[
k^*(N) \equiv \max_{k \in \{0,...,N\}} \left( \arg \max \mathbb{E}[u_w(\mu^M[P^k_{w}, \tilde{P}_{-w}](w))] \right).
\]

\(^1\)Ehlers \cite{Ehlers2022} provides weaker conditions, in the same spirit, under which truncation is still optimal.

\(^2\)\( \hat{P}_w \) is \( P_w \)-stochastically dominated by \( P'_w \) iff for any vNM utility function that corresponds to \( P_w \), the expected utility from submitting \( \hat{P}_w \) is at least as great as the expected utility from submitting \( P'_w \).

\(^3\)While this assumption is not very realistic for real markets, it may serve as an approximation for the behavior of the top tiers in a tiered market. For example, it may be the case that everyone agrees about the composition of the top tier of schools and students, but personal tastes causes the orderings to vary.\(^2\)
$k^*(N)$ describes woman $w$’s optimal point of truncation, given that the other women submit their true preference lists. If there are multiple optima, we conservatively select that which involves the least truncation. Coles and Shorrer [5] prove the following theorem.

**Theorem 1.** Let woman $w$ have uniform beliefs and preferences linear in rank (or any strictly increasing, convex transformation of such preferences). Then $\lim_{N \to \infty} \frac{k^*(N)}{N} = 0$.

Theorem 1 states that for balanced markets, as the market size grows large, the fraction of the list that an individual woman optimally truncates goes to 100%. The intuition behind this theorem can be gleaned from statistical facts about the most and least preferred achievable mates for women. In large balanced markets where preferences are uniform, the expected rank of the most preferred achievable mate of a woman (which is the same as the expected rank of her mate under WP-DA) is very low relative to the length of her list; it asymptotes to $\log N$ [16]. This suggests that a woman may safely truncate a large fraction of her list with little risk of becoming unmatched. Furthermore, the expected rank of a woman’s match under MP-DA is significantly worse, asymptoting to $\frac{N}{\log N}$ [16]. In fact, for large markets, Pittel [17] proved that the worst-off wife will be matched with a husband at the bottom of her list with probability approaching 1. This large gap in a woman’s expected most and least preferred achievable mates suggests that not only is it safe to truncate a large fraction of one’s list in large markets, but that a woman will also generate gains from such a truncation.

Figure 1 presents simulation results for balanced markets of size 10, 100, 1,000 and 10,000. It is clear from the figures that, when all other agents are truthful, the best response of a strategic woman is to submit a (very) short list. It also appears that the gains from truncation may be significant. In a market of size 10,000, the partner rank could potentially be reduced by about 1,000 in expectation (10% of the market size).

The recent paper by Ashlagi et. al. [2] implies that the large gap between the best and worst stable partners is a knife-edge case. When markets are even slightly unbalanced, under any stable matching the rank of the mates that an agent on the over demanded side of the market gets is approximately $\log N$ in expectation, while the other side can expect approximately $\frac{N}{\log N}$ [16]. Intuitively, these results imply that unbalanced matching markets typically have a “small core.” In turn this suggests that submitting a long list may constitute an optimal strategy in unbalanced uniform markets. In light of their findings, we inspect strategic behavior in the unbalanced setting.

### 3.1 The Case of More Women Than Men

Given $L \geq 0$, for a market with $N$ men and $N + L$ women define

$$k_s(N, L) = \min \left\{ \arg \max \left\{ \sum_{i=0}^{N} E\left[ u_w(\mu^M[P_{w_i}, \tilde{P}_{-w_i}](w)) \right] \right\} \right\}.$$  

---

4The approximations calculated by [2] involve multiplicative constants and describe expected payoffs conditional on an agent being assigned a partner.
Figure 1: Simulation Results for Truncation Payoffs. The graphs display \((N + 1)\) – an individual woman’s expected partner rank from truncating her list at each point \(k \in \{0, \ldots, N\}\) and submitting these preferences to MP-DA. Preference lists of the other agents are uniformly random, selected from the set of all possible full length preference list profiles, and payoffs are averaged over 100,000 draws. Markets are of size 10, 100, 1,000 and 10,000. 

\(k_s(N, L)\) describes woman \(w\)’s optimal point of truncation, given that the other women submit their true preference lists. If there are multiple optima, we conservatively select that which involves the most truncation. Note that for \(L = 0\), \(k_s(N, 0) \leq k^*(N)\) so the results of Theorem 1 apply to \(k_s(N, 0)\). We now have the following theorem, which constitutes a partial converse of Theorem 1. The theorem shows that the intuition from Ashlagi et. al. \([3]\) extends to the incomplete information setting when women are over demanded.

**Theorem 2.** Given \(L > 0\), consider a market with \(N\) men and \(N + L\) women. Let woman \(w\) have uniform beliefs and preferences linear in rank (or any strictly increasing, concave transformation of such preferences). Then \(\frac{k_s(N, L)}{N} \geq \frac{L}{L+1}\) so \(\lim_{N \to \infty} \frac{k_s(N, L)}{N} \geq \frac{L}{L+1}\). In particular \(\frac{k_s(N, L)}{N} \geq \frac{1}{2}\) and \(\lim_{N \to \infty} \frac{k_s(N, L)}{N} \geq \frac{1}{2}\).

**Proof.** Recall that a truncation by woman \(w\) could lead to one of three results:

1. No effect: woman \(w\) has truncated below her least preferred achievable mate
Figure 2: Simulation Results for Truncation Payoffs. The graphs display 101–an individual woman’s expected partner rank from truncating her list at each point $k \in \{0, \ldots, 100\}$ and submitting these preferences to MP-DA. Preference lists of the other agents are uniformly random, selected from the set of all possible full length preference list profiles, and payoffs are averaged over 100,000 draws. All markets have 100 men, and the number of women varies between 90, 95, 99, 101, 105 and 110.
2. Improvement: woman \( w \) truncates above her least preferred mate, and is matched with her least preferred achievable mate above the point of truncation.

3. Becoming unmatched: woman \( w \) has an achievable mate, but has over-truncated, truncating above her most preferred achievable mate.

In a balanced market, truncation worsened a woman’s outcome only when MP-DA terminated with one man exhausting his list. But when women outnumber men, truncation may cause MP-DA to terminate when a previously unmatched woman receives an offer. Note that when agents are truthful, at least one woman will have received no proposals prior to the truncation. Of course improvement upon truncation is possible only if \( w \) does not end up single.

Using the principle of deferred decisions, it is easy to see that conditional on a truncation making a difference, the probability of improvement is less than \( \frac{1}{L+1} \leq \frac{1}{2} \). To show this, recall that unmatched women have not received any proposals. Hence, it follows from the principle of deferred decisions and symmetry that, following truncation, any future proposal is at least as likely to be directed at these women as to \( w \). The algorithm terminates only when such a proposal happens.

Now consider the marginal benefit to \( w \) from omitting the lowest-ranked man from a list of length \( m+1 \). The most \( w \) can hope for is an improvement in her match of \( m \) ranks (from \( m+1 \) to \( m \)). If this omission instead leaves her unmatched, she drops \( N-m \) ranks (from \( m+1 \) to \( N+1 \)). Since the probability of becoming unmatched conditional on truncation having any effect is at least \( \frac{L}{1+L} \geq \frac{1}{2} \), the expected gain cannot be positive if \( m < \frac{L}{L+1} N \). Hence, the optimal list length for \( w \) is at least \( \frac{1}{L+1} N \geq \frac{1}{2} N \).

The left panels of Figure 2 illustrate our findings. We simulated markets with 100 men and 99, 95 and 90 women. In each market we generate independently and at random a full preference list for each agent. We then calculate an individual woman’s payoff, given that all other agents submit their true lists, for each possible level of truncation. Payoffs are depicted by 101 – partner rank, and we report the average result over 100,000 iterations.

The simulations support the findings of Theorem 2, as the optimal list length in all three markets is greater than 50. Indeed, the optimal list lengths are higher than 50, 84 and 91, the respective lower bounds the theorem indicates. In contrast to Figure 1, the balanced market case, it is almost impossible to detect the peak of the graphs. That is, not only should women submit long lists, but there is little to gain by truncating optimally. Note that as one would expect, women do worse as the competition on their side increases.

A few points related to Theorem 2 deserve attention. First, under uniform beliefs, Roth and Rothblum’s optimality theorem applies whether the market is balanced or not. This implies that the truncation strategies described in Theorems 1 and 2 are the best overall strategies, not just the optimal truncation strategies. Hence, we have a natural metric for the “distance” between the optimal strategy and truthfulness. The importance of Theorem 1 is in showing that the best response to straightforward behavior of others could be “far” from truthful, and so provides an important qualification to the literature which finds truthful reporting to be close to optimal [20, 11, 13, 14, 2]. Theorem 2 qualifies this previous finding.

We show that our example relies heavily on the fact that the number of women is not larger than the

\[ \frac{m+1}{2} \] in expectation - the expected rank of the remaining partners.
number of men.

A second point worth noting is that our choice of $k^*(N,L)$ was a conservative one. We could have
instead chosen to state the theorem using $k^*(N,L) \equiv \max \left\{ \arg \max_{k \in \{0,\ldots,N\}} E\left[u_w(\mu^M[P^k_w,\hat{P}_w](w))\right] \right\}$, break-
ing ties in favor of less truncation as in the definition of $k^*(N)$, and the theorem would of course still
hold (since $k^*(N,L) \geq k^*(N,L)$ by definition).

Finally, in contrast to most of the results in this strand of the literature, our theorem is not a “large
market result”; our result holds for any $N$ and $L$. That is, manipulation opportunities are minimal
(in the sense of distance from truthful submission) whenever women outnumber men, for unbalanced
markets of any size. However, a simple corollary of our result is that as imbalance in a market increases,
manipulation opportunities vanish altogether.

**Corollary 1.** Given a sequence $\{L_N\}$ with $\lim_{N} L_N = \infty$ and a sequence of uniform markets with $N$ men
and $N + L_N$ women, if woman $w$ has uniform beliefs and preferences linear in rank (or any strictly
increasing, concave transformation of such preferences), then $\lim_{N \to \infty} k^*_v(N, L_N) = 1$.

The simulation results presented in Figure 2 are consistent with the results of Ashlagi et. al. [2]. In
contrast to the relatively large gain that a woman may be able to realize by truncating in a balanced
market, when there are more women than men the graph of the expected payoff is much flatter between
the optimum and truthful reporting. It is also true that in this case truncation is relatively risky. One can
see from Figure 2 that, for example, in a market with 101 women and 100 men, submitting a list of length
30 exposes a woman to a significant risk of remaining unmatched. The following theorem formalizes this
observation by providing a lower bound on the probability of becoming unmatched following a relatively
conservative truncation.

**Theorem 3.** Fix $L > 0$ and $\delta \in (0,1)$. For $N$ large enough, in a uniform market with $N$ men and $N + L$
women, if all other agents report truthfully and woman $w$ submits a truncation list of length less than
$\delta N$, she will be unmatched with probability at least

$$\frac{49}{N + L}.$$  

Proof. From Pittel [17, Theorem 6.2] we know that in a balanced uniform market of size $N$, the probability
that the worst-off woman gets a mate ranked worse than $\delta N$ approaches 1. This probability only increases
when there are more women than men [12]. Now consider a woman truncating her list shorter than $\delta N$
while all other agents are truthful. From Pittel’s theorem, for large $N$ the probability that this truncation
makes a difference is at least $\frac{1}{N + L}$. For large enough $N$, this expression is greater than $\frac{499}{N}$. Conditional
on the truncation making a difference, using the principle of deferred decisions, the resulting chain of
rejections is at least as likely to terminate with a proposal to a woman that did not receive any proposals
until $w$ divorced her partner, as it is to return to $w$. This implies that even in the event that $w$ is matched
when she (and all others) report truthfully, by truncating her list to a size smaller than $\delta N$ she raises
her probability of being unmatched by at least $\frac{L}{N + L} \times \frac{499}{N} > \frac{49}{N}$. Multiplying both sides of the inequality
by the probability of $w$ being matched if she reports truthfully, and adding the probability of $w$ being
unmatched if she reports truthfully, produces the lower bound:

$$\frac{L}{N + L} + \left(1 - \frac{L}{N + L}\right) \frac{49}{N} = \frac{L}{N + L} + \frac{49}{N}.$$
Symmetry implies that \( w \) will remain unmatched with probability \( \frac{L}{N+L} \) no matter what (full) list she reports. We are interested in the increase in probability of being left unmatched relative to truthful reporting \( \frac{N^9}{N+L} \). While the increase does not appear to be large at first glance, several facts must be taken into account. First, this is a lower bound, and there is no good reason to suspect that it is tight. Moreover, the lower bound for the increase in probability is of the same order of magnitude as the probability of remaining unmatched under truthful reporting. Second, the degree of truncation of \( w \) may be minimal. The theorem allows \( w \) to submit 99% of her list and the results will still hold. A third point is that these results should be compared with the opposite case, where men outnumber women. This is exactly what we do in the next section.

But before we move on, we state a conjecture. Let a uniform market be the setting in which each agent is equally likely to have any full preference list. Additionally, agent utility depends on partner rank, agents on each side of the market identically value a match with their \( r \)th ranked choice and have identical value to being unmatched. Coles and Shorrer \cite{5} showed that:

**Theorem 4.** In uniform markets, there exists a symmetric equilibrium \( ((\sigma_m)^N, (\sigma_w)^N) \) where men each use the strategy \( \sigma_m \) of truthful reporting and women each use the strategy \( \sigma_w \), which is a mixture over truncation strategies.

In an imbalanced market were individual women have an incentive to submit long lists, we conjecture that submission of long lists is also optimal in equilibrium.

**Conjecture.** Fix \( L > 2 \). In a large uniform markets with \( N \) men and \( N + L \) women, where MP-DA is used, there exists a symmetric equilibrium where all women submit truncated lists longer than \( \frac{N}{2} \).

### 3.2 The Case of Fewer Women Than Men

The results of Ashlagi et. al. \cite{2} regarding the small core apply regardless of the direction of the imbalance. That is, no matter the size or direction of the imbalance, the expected potential improvement is small. One might therefore suspect that when men outnumber women, an analog to Theorem 2 would apply and room for manipulation would again be small. This, however, does not appear to be the case. The simulation results presented in the right panel of Figure 2 indicate that when there are fewer women than men, the optimal level of truncation may still be significant.

The figure depicts truncation payoffs for men in simulated markets with 100 men and 101, 105 and 110 women. In contrast to the case where women outnumber men, in this case the peaks of all three graphs involve lists of length shorter than 31.

Comparing the right and left panels of Figure 2 three additional facts stand out. The first is that women do much better when the balance tips slightly in their favor: payoffs with 99 women and 100 men are much higher than when there are 101 women. This difference becomes starker as the imbalance increase. This corroborates the findings of Ashlagi et. al. \cite{2} in the case when \( w \) reports truthfully.\(^6\)

\(^6\)In fact, their proof was for the balanced case, but the same proof would work for general uniform markets.
Also salient is that even though optimal truncation may be far from the truth, such a manipulation increases payoffs only minimally. This too could be deduced from their paper. The third salient feature is not a direct consequence of their findings (though is related to techniques used in their proofs). The simulations suggest that truncation is “safer” for women when they are over-demanded. That is, when there are more men than women, women may submit relatively short lists without facing a large risk of becoming unmatched, even if there is little gain from doing so.

The next theorem shows that this third fact holds more generally.

**Theorem 5.** Fix $L \geq 0$. For a uniform market with $N + L$ men and $N$ women, a woman that submits a truncation containing more than $L + (2 + a) \log^2 N$ men will be matched with probability at least $1 - O\left(N^{-c(a)}\right)$, where $c(a) = 2a \left[3 + (4a + 9)^{1/2}\right]^{-1}$. In particular a woman that submits a truncated list of more than $L + 10 \log^2 N$ men will be unmatched with probability at most $O\left(N^{-2}\right)$.

**Proof.** For notational simplicity we provide the proof for the case of $L+10 \log^2 N$, but analogous arguments would apply for all other cases. The proof has two steps. First, recall that in a market that uses $MP-DA$, adding men to the market makes the other men weakly worse-off and women weakly better-off [12]. Second, from Pittel [17, Theorem 6.1] we know that in a balanced market of size $N$, submitting a truncated list with $10 \log^2 N$ men ensures being matched with probability $1 - O\left(\frac{1}{N^2}\right)$. In a market with all the women and an arbitrary subset of the men containing $N$ agents, by submitting a truncated list with $L + 10 \log^2 N$ men, a woman submits a list containing, at least, her most preferred $10 \log^2 N$ men in the subset in order. By Pittel’s theorem, this ensures that the woman is matched with probability at least $1 - O\left(\frac{1}{N^2}\right)$. But the first point ensures that by adding the other $L$ men to the market all women are weakly better-off. In particular, no woman that would have been matched in the smaller market can become unmatched.

**Remark.** The statement of Theorem 5 is intentionally silent on the strategies of women $-w$. The proof shows that the statement holds when all other women are truthful. But the proof also holds whenever other women use truncation strategies, or any anonymous strategies. The logic is simple: truncation by other women only increases $w$’s probability of being matched given any list she submits.

Intuition for Theorem 5 may come from considering markets with large imbalances. Consider, for example, uniform markets with $N$ women and $(1 + \lambda) N$ men, for positive $\lambda$. In these markets, $MP-DA$ terminates only after $\lambda N$ men have proposed to all of the women. Since preferences are independent, this implies (using the “principle of deferred decisions”) that even in the men optimal stable matching, a woman is matched with a man high on her list with high probability (her expected partner rank is lower than $\frac{N}{\lambda N} = \frac{1}{\lambda}$).

Rather than being an analog, Theorem 5 stands in sharp contrast to Theorem 3. The ratio between the length of the lists described in Theorem 5 and the ones described in Theorem 3 approaches 0 as $N$ grows large (since the lists from Theorem 5 are much shorter). Yet the ratio of the increases in the probability of becoming unmatched approaches infinity if the (short) list in the setting of Theorem 5 is chosen to be sufficiently long (e.g. $11 \log^2 N$).
To illustrate Theorem 5, we present additional simulation evidence. We simulated a market with 1000 men and 999 women, and estimated the returns to truncation for a woman $w$ given that all other agents are truthful, reporting average results over 100,000 iterations. The results are summarized in the left panel of Figure 3. While difficult to observe with the naked eye, the maximum is attained at 89, so that in terms of list length, the best response is still “far” from truthful reporting.

Figure 3: Simulation Results for Truncation Payoffs. In markets with 1000 men and 999 or 1001 women, the graph displays 1001 – an individual woman’s expected partner rank from truncating her list at each point $k \in \{0, \ldots, 1000\}$ and submitting these preferences to MP-DA. Preference lists of the other agents are selected, uniformly at random, from the set of all possible full length preference list profiles, and payoffs are averaged over 100,000 draws.

4 Other Aspects Impacting the Optimal Level of Truncation

Coles and Shorrer [5] provide several comparative statics for the optimal level of truncation in the case of balanced markets. We demonstrate that these hold in the unbalanced case as well.

4.1 Truncation and Risk Aversion

As discussed previously, truncation is a risky strategy. Compared to truthful reporting, truncation may offer some benefit, but over-truncating can lead to large losses depending on the profile of preferences that is submitted by others. One might expect agents with more conservative attitudes toward risk to shy away from this proposition. In this section, we formalize this intuition.

We consider a general setting, with arbitrary preferences for woman $w$ and beliefs about reported preferences of others. Let $\psi(\cdot)$ be any strictly increasing, concave transformation. We claim that for any beliefs about others, woman $w$ with preferences $u_w(\cdot)$ will truncate more than a woman $w_\psi$ who has identical beliefs, but preferences given by $\psi(u_w(\cdot))$. 

14
We fix $w$’s preferences to be $u_w(\cdot)$, and define the shorthand
\[ v(k, P_{-w}) \equiv u_w(\mu^M[P^k_{w}, P_{-w}](w)), \]
w’s payoff from submitting truncated preference list $P^k_w$. Now define
\[ v_\psi(k, P_{-w}) \equiv \psi(u_w(\mu^M[P^k_{w}, P_{-w}](w))), \]
the payoff from submitting truncated preference list $P^k_w$ for a woman $w$ with preferences $\psi(u_w(\cdot))$.

The following theorem states that if $w$ prefers truncating less to more, then $w$ definitely prefers truncating less to more.

**Theorem 6.** Let $\tilde{\mathcal{P}}_{-w}$ be any random variable distributed over $\mathcal{P}_{-w}$. Then $\forall k \in \{1, \ldots, N - 1\}, \forall t \in \{1, \ldots, N - k\}$ we have
\[ \mathbb{E}[v(k, \tilde{\mathcal{P}}_{-w})] \leq \mathbb{E}[v(k + t, \tilde{\mathcal{P}}_{-w})] \Rightarrow \]
\[ \mathbb{E}[v_\psi(k, \tilde{\mathcal{P}}_{-w})] \leq \mathbb{E}[v_\psi(k + t, \tilde{\mathcal{P}}_{-w})]. \]
Furthermore, if i) $\psi(\cdot)$ is strictly concave, and ii) under $\tilde{\mathcal{P}}_{-w}$, each man is achievable for $w$ with positive probability, then the second inequality is strict.

We can now use Theorem 6 to sort optimal truncation points based on degree of concavity.

**Corollary 2.** Let $k^l_w$ be the minimum optimal truncation point (by rank) and let $k^h_w$ be the maximum optimal truncation point for woman $i \in \{w, w_\psi\}$. Then $k^l_w \leq k^l_{w_\psi}$ and $k^h_w \leq k^h_{w_\psi}$. Furthermore, if conditions i) and ii) from Theorem 6 hold, then $k^h_w \leq k^l_{w_\psi}$.

We omit the proofs, as they are straightforward analogs of the proofs of Theorem 5 and Corollary 1 in Coles and Shorrer [5]. The key insight in the analysis is the interpretation of truncation as a risky lottery, and then mapping the additional risk associated with incremental truncation to an extra lottery a woman must face. If a woman doesn’t like to face the extra lottery, then certainly a woman with more concave preferences will not want to face it. Note that despite pertaining to risk aversion, the results in this section do not restrict the structure of $u_w(\cdot)$ in any way. For example, we do not require $u_w(\cdot)$ to be “concave.” Rather, it is the relative concavity that is crucial.

This result can offer advice to a market designer. If she wishes to see long lists, for example since her objective is to maximize the number of matches, a market designer may wish to choose the less risk averse side to be the “proposers” in the Deferred Acceptance Algorithm. If the two sides of the market are identical in all regards except for their risk preferences, the more risk averse side will be less likely to truncate, even if manipulations increase their expected partner rank. Lower levels of truncation will increase the number of realized matches, and consequently, reduce the number of participants left unmatched. However, in making this choice, the market designer should take other market features into consideration as well, as we demonstrate in the next section.
4.2 Truncation and Correlated Preferences

Coles and Shorrer [5] provided theoretical and empirical evidence that, in the balanced setting, correlation in preferences of agents on one side of the market reduces their incentive to truncate. In this section, we show that their findings generalize to unbalanced markets.

We consider first the case of perfectly correlated preferences on women’s side of the market. In this case, there exist a unique stable matching, and so women have no incentive to truncate their lists at all when all others report truthfully. If women are uncertain about men’s preferences, truncation may only lead them to a worse outcome, provided that others are truthful.

While perfect correlation and independence are easy to model, partial correlation may appear in many forms. In this paper, we focus on one simple such form. Consider the Preference List Submission Problem for woman w with preferences \( u_w \) and beliefs \( \tilde{P}_{-w} \) about reported preference lists of opponents. Let \( p(\cdot, \cdot) \) be the probability mass function for w’s beliefs. That is,

\[
p(P_M, P_{W \setminus \{w\}}) = \begin{cases} p^M(P_M) & \text{if } P_w = P_w \forall \hat{w} \in W \setminus \{w\} \\ 0 & \text{otherwise} \end{cases}
\]

\( p^C(\cdot, \cdot) \) is the distribution that preserves the marginal distribution over men’s preferences \( p^M(\cdot) \), but where the other women share the preferences of w.

Define beliefs \( p^\alpha(\cdot) \) by

\[
p^\alpha(P_{-w}) = (1 - \alpha)p(P_{-w}) + \alpha p^C(P_{-w}).
\]

Hence, as \( \alpha \) varies from 0 to 1, \( p^\alpha \) ranges from \( p \) to \( p^C \). The marginal distribution over men’s preferences remains fixed, while the correlation of women’s preferences steadily increases (the distribution remains constant if \( p = p^C \)).

The set of optimal truncation points for woman w with preferences \( u_w \) and beliefs indexed by \( \alpha \) is given by

\[
k^*(\alpha, p, u_w) = \arg \max_{k \in \{0, \ldots, N\}} \mathbb{E}_{p^\alpha}[v(k, \hat{P}_{-w})].
\]

Notice that since the choice set is finite, \( k^*(\cdot, \cdot) \) will be non-empty.

Let \( k^h(\alpha, p, u_w) = \max[k^*(\alpha, p, u_w)] \) and \( k^l(\alpha, p, u_w) = \min[k^*(\alpha, p, u_w)] \), the optimal choices involving the least and most truncation respectively.

The following proposition states that for any preferences \( u_w \) and beliefs p, as we increase the degree of correlation \( \alpha \), woman w should truncate less.

**Proposition 1.** Let \( \alpha, \alpha' \in [0, 1] \) with \( \alpha' > \alpha \). Then \( k^l(\alpha', p, u_w) \geq k^l(\alpha, p, u_w) \) and \( k^h(\alpha', p, u_w) \geq k^h(\alpha, p, u_w) \).
The proof of the proposition is analogous to the proof of Proposition 2 in Coles and Shorrer [5], and is therefore omitted.

The anticipated level of correlation in the environment might influence the advice a market designer can offer participants. If correlation is high, the designer can safely advise participants to report truthfully, and it is in their best interest to do so. With low correlation (sufficiently heterogeneous preferences), players may anticipate gains from truncation, which if acted on, could lead to unstable matching.

5 Conclusion

In this paper, we study optimal strategic behavior in unbalanced one-to-one matching markets, where matchings are determined by the Deferred Acceptance Algorithm and agents have incomplete information about the preferences of others. We focus on truncation strategies, which are attractive for agents as they are simple and always weakly increase the probability of being matched with more-preferred mates. From a computational perspective, this reduces significantly the dimensions of the strategy space, allowing us to use simulations to pinpoint optimal behavior. This restriction also produces a natural metric on the extent of manipulation: the shorter the lists submitted, the further they are from truthfulness. This allows us to make relative statements about optimal list lengths.

The main innovation of this paper is in studying the effect of imbalance in the number of agents on the two sides of a market on their potential for manipulation. We study a stylized setting which we term a uniform market, and find that the degree of manipulation observed in this setting critically depends on the direction of the imbalance. When women are on the long side of the market (there are more women than men), we find that the incentives for women to manipulate are significantly diminished compared to a balanced market. This finding is consistent with the intuition of Ashlagi et. al. [2], who find that the expected gap between an agent’s highest and lowest achievable mates is small in unbalanced uniform markets.

By contrast, when men outnumber women, we provide evidence that a woman’s best response to truthful behavior by others involves a significant degree of truncation. This finding qualifies results that suggest opportunities for manipulation in such settings are minimal (e.g. in terms of potential gain in utility [14]). We further show that truncation is safe when women are on the short side (more men than women) but not when they are on the long side.

We also provide comparative statics regarding the extent of manipulation, regardless of the direction of size of a market imbalance. When women are more risk averse, they should be less aggressive in their degree of truncation. Correlation in women’s preferences also reduces their incentive to truncate.

Matching mechanisms based on the Deferred Acceptance Algorithm are used extensively in a variety of entry level labor markets [19] and in school choice [15]. One advantage of DAA is that it induces truthful reporting as a dominant strategy for one side of the market [18, 6]. This alone is an argument market designers have used to decide which side will be the “proposing” one [19].

In addition to shedding light on strategic behavior in unbalanced markets generally, our paper introduces a new factor that might be considered when selecting the proposing side: direction of imbalance. By selecting the over-demanded side to propose, potential for strategic manipulation is minimized. Selecting
the over-demanded side to receive offers leaves room for significant, safe manipulation. While simplistic and stylized in many respects, our result is a first effort to extend the logic market designers rely on in choosing the proposing side. Future work should find more general environments in which the extent of manipulation may be compared, and explore the interaction between the different forces that determine the incentives to manipulate.

References


