The Mandelbrot Set is Universal

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The Mandelbrot set is universal

Curtis T. McMullen

24 February, 1997

Abstract

We show small Mandelbrot sets are dense in the bifurcation locus for any holomorphic family of rational maps.

1 Introduction

Fix an integer \(d \geq 2\), and let \(p_c(z) = z^d + c\). The generalized Mandelbrot set \(M_d \subset \mathbb{C}\) is defined as the set of \(c\) such that the Julia set \(J(p_c)\) is connected. Equivalently, \(c \in M_d\) iff \(p_c^n(0)\) does not tend to infinity as \(n \to \infty\). The traditional Mandelbrot set is the quadratic version \(M_2\).

A holomorphic family of rational maps over \(X\) is a holomorphic map

\[ f : X \times \hat{\mathbb{C}} \to \hat{\mathbb{C}} \]

where \(X\) is a complex manifold and \(\hat{\mathbb{C}}\) is the Riemann sphere. For each \(t \in X\) the family \(f\) specializes to a rational map \(f_t : \hat{\mathbb{C}} \to \hat{\mathbb{C}}\), denoted \(f_t(z)\). For convenience we will require that \(X\) is connected and that \(\deg(f_t) \geq 2\) for all \(t\).

The bifurcation locus \(B(f) \subset X\) is defined equivalently as the set of \(t\) such that:

1. The number of attracting cycles of \(f_t\) is not locally constant;
2. The period of the attracting cycles of \(f_t\) is locally unbounded; or
3. The Julia set \(J(f_t)\) does not move continuously (in the Hausdorff topology) over any neighborhood of \(t\).

It is known that \(B(f)\) is a closed, nowhere dense subset of \(X\); its complement \(X - B(f)\) is also called the \(J\)-stable set [MSS, [Mc2, §4.1].

As a prime example, \(p_c(z) = z^d + c\) is a holomorphic family parameterized by \(c \in \mathbb{C}\), and its bifurcation locus is \(\partial M_d\). See Figure 1.

In this paper we show that every bifurcation set contains a copy of the boundary of the Mandelbrot set or its degree \(d\) generalization. The Mandelbrot sets \(M_d\) are thus universal; they are initial objects in the category

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of bifurcations, providing a lower bound on the complexity of $B(f)$ for all families $f_t$.

For simplicity we first treat the case $X = \Delta = \{t : |t| < 1\}$.

**Theorem 1.1** For any holomorphic family of rational maps over the unit disk, the bifurcation locus $B(f) \subset \Delta$ is either empty or contains the quasiconformal image of $\partial M_d$ for some $d$.

The proof (§4) shows that $B(f)$ contains copies of $\partial M_d$ with arbitrarily small quasiconformal distortion, and controls the degrees $d$ that arise. For example we can always find a copy of $\partial M_d$ with $d \leq 2^{\deg(f_t) - 2}$, and generically $B(f)$ contains a copy of $\partial M_2$ (see Corollary 4.4). Since the Theorem is local we have:

**Corollary 1.2** Small Mandelbrot sets are dense in $B(f)$.

There is also a statement in the dynamical plane:

**Theorem 1.3** Let $f$ be a holomorphic family of rational maps with bifurcations. Then there is a $d \geq 2$ such that for any $c \in M_d$ and $m > 0$, the family contains a polynomial-like map $f^n_t : U \rightarrow V$ hybrid conjugate to $z^d + c$ with $\text{mod}(U - V) > m$.

**Corollary 1.4** If $f$ has bifurcations then for any $\epsilon > 0$ there exists a $t$ such that $f_t(z)$ has a superattracting basin which is a $(1 + \epsilon)$-quasidisk.

**Proof.** The family contains a polynomial-like map $f^n_t : U \rightarrow V$ hybrid conjugate to $p_0(z) = z^d$, a map whose superattracting basin is a round disk. Since $\text{mod}(V - U)$ can be made arbitrarily large, the conjugacy can be made nearly conformal, and thus $f_t$ has a superattracting basin which is a $(1 + \epsilon)$-quasidisk. \qed
For applications to Hausdorff dimension we recall:

**Theorem 1.5 (Shishikura)** For any \( d \geq 2 \), the Hausdorff dimension of \( \partial M_d \) is two. Moreover \( H.\dim(J(p_c)) = 2 \) for a dense \( G_\delta \) of \( c \in \partial M_d \).

This result is stated for \( d = 2 \) in [Shi2] and [Shi1] but the argument generalizes to \( d \geq 2 \). Quasiconformal maps preserve sets of full dimension [GV], so from Theorems 1.1 and 1.3 we obtain:

**Corollary 1.6** For any family of rational maps \( f \) over \( \Delta \), the bifurcation set \( B(f) \) is empty or has Hausdorff dimension two.

**Corollary 1.7** If \( f \) has bifurcations, then \( H.\dim(J(f_t)) = 2 \) for a dense set of \( t \in B(f) \).

For higher-dimensional families one has (§5):

**Corollary 1.8** For any holomorphic family of rational maps over a complex manifold \( X \), either \( B(f) = \emptyset \) or \( H.\dim(B(f)) = H.\dim(X) = 2 \dim \mathbb{C} X \).

Similar results on Hausdorff dimension were obtained by Tan Lei, under a technical hypothesis on the family \( f \) [Tan].

A family of rational maps \( f \) is algebraic if its parameter space \( X \) is a quasi-projective variety (such as \( \mathbb{C}^n \)) and the coefficients of \( f_t(z) \) are rational functions of \( t \). For example, \( p_c(z) = z^d + c \) is an algebraic family over \( X = \mathbb{C} \). Such families almost always contain bifurcations [Mc1]:

**Theorem 1.9** For any algebraic family of rational maps, either

1. The family is trivial (\( f_t \) and \( f_s \) are conformally conjugate for all \( t, s \in X \)); or

2. The family is affine (every \( f_t \) is critically finite and double covered by a torus endomorphism); or

3. The family has bifurcations (\( B(f) \neq \emptyset \)).

**Corollary 1.10** With rare exceptions, any algebraic family of rational maps exhibits small Mandelbrot sets in its parameter space.

---

1 This set of \( t \) can be improved to a dense \( G_\delta \) using Shishikura’s idea of hyperbolic dimension.
This Corollary was our original motivation for proving Theorem 1.1. As another application, for \( t \in \mathbb{C}^{d-1} \) let
\[
f_t(z) = z^d + t_1 z^{d-2} + \cdots + t_{d-1}
\]
and let
\[
\mathcal{C}_d = \{ t : J(f_t) \text{ is connected} \}
\]
denote the connectedness locus. Then we have:

**Corollary 1.11 (Tan Lei)** The boundary of the connectedness locus has full dimension; that is, \( \text{H. dim}(\partial \mathcal{C}_d) = \text{H. dim}(\mathcal{C}_d) = 2d - 2 \).

**Proof.** Consider the algebraic family \( g_a(z) = z^d + az^{d-1} \), which for \( a \neq 0 \) has all but one critical point fixed under \( g_a \). By Theorem 1.9, this family has bifurcations at some \( a \in \mathbb{C} \). Then there is a neighborhood \( U \) of \( (a, 0, \ldots, 0) \in \mathbb{C}^{d-2} \) such that for \( t \in U \) all critical points of \( f_t \) save one lie in an attracting or superattracting basin. If \( t \in B(f) \cap U \), then the remaining critical point has a bounded forward orbit under \( f_t \), but under a small perturbation tends to infinity. It follows that \( B(f) \cap U = \partial \mathcal{C}_d \cap U \neq \emptyset \), and thus \( \text{dim}(\partial \mathcal{C}_d) \geq \text{dim} B(f) = 2d - 2 \).

**Remark.** Rees has shown that the bifurcation locus has positive measure in the space of all rational maps of degree \( d \) [Rs]; it would be interesting to known general conditions on a family \( f \) such that \( B(f) \) has positive measure in the parameter space \( X \).

**Acknowledgements.** I would like to thank Tan Lei for sharing her results which prompted the writing of this note. Special cases of Theorem 1.1 were developed independently by Douady and Hubbard [DH, pp.332-336] and Eckmann and Epstein [EE].

## 2 Families of rational maps

In this section we begin a more formal study of maps with bifurcations.

**Definitions.** A **local bifurcation** is a holomorphic family of rational maps \( f_t(z) \) over the unit disk \( \Delta \), such that \( 0 \in B(f) \).

The following natural operations can be performed on \( f \) to construct new local bifurcations:

1. **Coordinate change:** replace \( f_t \) by \( m_t \circ f_t \circ m_t^{-1} \), where \( m : \Delta \times \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) is a holomorphic family of Möbius transformations.

2. **Iteration:** replace \( f_t(z) \) by \( f_t^n(z) \) for a fixed \( n \geq 1 \).
3. **Base change:** replace \( f_t(z) \) by \( f_{\phi(t)}(z) \), where \( \phi: \Delta \to \Delta \) is a nonconstant holomorphic map with \( \phi(0) \in B(f) \).

The first two operations leave the bifurcation locus unchanged, while the last transforms \( B(f) \) to \( \phi^{-1}(B(f)) \).

**Marked critical points.** We will also consider pairs \((f, c)\) consisting of a local bifurcation and a marked critical point; this means \( c: \Delta \to \hat{C} \) is holomorphic and \( f'_t(c_t) = 0 \). The operations above also apply to \((f, c)\); a coordinate change replaces \( c_t \) with \( m_t(c_t) \) and a base change replaces \( c_t \) with \( c_{\phi(t)} \).

**Misiurewicz points.** A marked critical point \( c \) of \( f \) is active if its forward orbit \( \langle f^n_t(c_t) : n = 1, 2, 3, \ldots \rangle \) fails to form a normal family of functions of \( t \) on any neighborhood of \( t = 0 \) in \( \Delta \). A parameter \( t \) is a Misiurewicz point for \((f, c)\) if the forward orbit of \( c_t \) under \( f_t \) lands on a repelling periodic cycle. If \( t = 0 \) is a Misiurewicz point, then either \( c \) is active or \( c_t \) is preperiodic for all \( t \).

**Proposition 2.1** If \( c \) is an active critical point, then \((f, c)\) has a sequence of distinct Misiurewicz points \( t_n \to 0 \).

**Proof.** This is a traditional normal families argument. Choose any 3 distinct repelling periodic points \( \{a_0, b_0, c_0\} \) for \( f_0 \), and let \( \{a_t, b_t, c_t\} \) be holomorphic functions parameterizing the corresponding periodic points of \( f_t \) for \( t \) near zero. Since \( \langle f^n_t(c_t) \rangle \) is not a normal family, it cannot avoid these three points, and any parameter \( t \) where \( f^n_t(c_t) \) meets \( a_t, b_t \) or \( c_t \) is a Misiurewicz point. \( \blacksquare \)

**Ramification.** Next we discuss the existence of univalent inverse branches for a single rational map \( F(z) \). Let \( d = \text{deg}(F, z) \) denote the local degree of \( F \) at \( z \in \hat{C} \); we have \( d > 1 \) iff \( z \) is a critical point of multiplicity \((d - 1)\). We say \( y \) is an unramified preimage of \( z \) if for some \( n \geq 0 \), \( F^n(y) = z \) and \( \text{deg}(F^n, y) = 1 \). We say \( z \) is unramified if it has infinitely many unramified preimages. In this case its unramified preimages accumulate on the full Julia set \( J(F) \).

**Proposition 2.2** If \( z \) has 5 distinct unramified preimages then it has infinitely many.

**Proof.** Let \( E \) be the set of all unramified preimages of \( z \), and let \( C \) be the critical points of \( F \). Then \( F^{-1}(E) \subset E \cup C \), so if \( |E| \) is finite then

\[
d|E| = \sum_{z \in F^{-1}(E)} 1 + \text{mult}(f', z) \leq |F^{-1}(E)| + 2d - 2 \leq |E| + 4d - 4
\]

and therefore \( |E| \leq 4 \). \( \blacksquare \)
Corollary 2.3 Let \((f, c)\) be a local bifurcation with marked critical point. Then the set of \(t\) such that \(c_t\) is ramified for \(f_t\) is either discrete or the whole disk.

**Proof.** By the previous Proposition, the ramified parameters are defined by a finite number of analytic equations in \(t\). ■

Proposition 2.4 After a suitable base change, any local bifurcation \(f\) can be provided with an active marked critical point \(c\) such that \(c_0\) is unramified for \(f_0\).

**Remark.** It is possible that all the active critical points are ramified at \(t = 0\). The base change in the Proposition will generally not preserve the central fiber \(f_0\).

**Proof.** The set \(C = \{(t, z) \in \Delta \times \hat{C} : f_t'(z) = 0\}\) is an analytic variety with a proper finite projection to \(\Delta\). By Puiseux series, after a base change of the form \(\phi(t) = e^{t^n}\) all the critical points of \(f\) can be marked by holomorphic functions \(\{c_1^t, \ldots, c_m^t\}\). Since \(t = 0\) is in the bifurcation set, by [Mc2, Thm. 4.2], there is an \(i\) such that \(\langle f^n_t(c_i^t)\rangle\) is not a normal family at \(t = 0\). That is, \(c_i^t\) is an active critical point.

Next we show \(c_i^t\) can be chosen so that for generic \(t\) it is disjoint from the forward orbits of all other critical points. If not, there is a \(c_j^t\) and \(n \geq 1\) such that \(f^n_t(c_j^t) = c_i^t\) for all \(t\). Then \(c_j^t\) is also active and we may replace \(c_i^t\) with \(c_j^t\). If the replacement process were to cycle, then \(c^t\) would be a periodic critical point, which is impossible because it is active. Thus we eventually achieve a \(c_i^t\) which is generically disjoint from the forward orbits of the other critical points.

In particular, there is a \(t\) such that \(c_i^t\) is unramified for \(f_t\). By Corollary 2.3, the set \(R \subset \Delta\) of parameters where \(c_i^t\) is ramified is discrete. By Proposition 2.1, there are Misiurewicz points \(t_n\) for \((f, c_i^t)\) with \(t_n \to 0\). Choose \(n\) such that \(t_n \not\in R\), and make a base change moving \(t_n\) to zero; then \(c_i^t\) is active, and \(c_i^0\) is unramified for \(f_0\). ■

Misiurewicz bifurcations. Let \((f, c)\) be a local bifurcation with a marked critical point. We say \((f, c)\) is a Misiurewicz bifurcation of degree \(d\) if

- **M1.** \(f_0(c_0)\) is a repelling fixed-point of \(f_0\);
- **M2.** \(c_0\) is unramified for \(f_0\);
- **M3.** \(f_t(c_t)\) is not a fixed-point of \(f_t\), for some \(t\); and
- **M4.** \(\deg(f_t, c_t) = d\) for all \(t\) sufficiently small.
Proposition 2.5 For any local bifurcation $(f, c)$ with $c$ active and $c_0$ unramified, there is a base change and an $n > 0$ such that $(f^n, c)$ is a Misiurewicz bifurcation.

Remark. The delicate point is condition (M4). The danger is that for every Misiurewicz parameter $t$, the forward orbit of $c_t$ might accidentally collide with another critical point before reaching the periodic cycle. We must avoid these collisions to make the degree of $f^n_t$ at $c_t$ locally constant.

Proof. There are Misiurewicz points $t_n \to 0$ for $(f, c)$, and $c_t$ is unramified for all $t$ near 0, so after a base change and replacing $f$ with $f^n$ we can arrange that $(f, c)$ satisfies conditions (M1), (M2) and (M3).

We can also arrange that $\deg(f_t, c_t) = d$ for all $t \neq 0$. However (M4) may fail because $\deg(f_t, c_t)$ may jump up at $t = 0$. This jump would occur if another critical of $f_t$ coincides with $c_t$ at $t = 0$.

To rule this out, we make a further perturbation of $f_0$. Let $a_t$ locally parameterize the repelling fixed-point of $f_t$ such that $f_0(c_0) = a_0$. Choose a neighborhood $U$ of $a_0$ such that for $t$ small, $f_t$ is linearizable on $U$ and $U$ is disjoint from the critical points of $f_t$. (This is possible since $f'_0(a_0) \neq 0$.)

Let $b_t \in U - \{a_t\}$ be a parameterized repelling periodic point close to $a_t$. Then $b_t$ has preimages in $U$ accumulating on $a_t$. Choose $s$ near 0 such that $f_s(c_s)$ hits one of these preimages (such an $s$ exists by the argument principle and (M3)). For this special parameter, $c_s$ first maps close to $a_s$, then remains in $U$ until it finally lands on $b_s$. Since there are no critical points in $U$, we have $\deg(f^i_s, c) = d$ for all $i > 0$.

Making a base change moving $s$ to $t = 0$, we find that $(f^n, c)$ satisfies (M1-M4) for $n$ a suitable multiple of the period of $b_s$. 

3 The Misiurewicz cascade

In this section we show that when a Misiurewicz point bifurcates, it produces a cascade of polynomial-like maps.

Definitions. A polynomial-like map $g : U \to V$ is a proper, holomorphic map between simply-connected regions with $\overline{U}$ compact and $\overline{U} \subset V \subset \mathbb{C}$ [DH]. Its filled Julia set is defined by

$$K(g) = \bigcap_{1}^{\infty} g^{-n}(V);$$

it is the set of points that never escape from $U$ under forward iteration.

Any polynomial such as $p_c(z) = z^d + c$ can be restricted to a polynomial-like map $p_c : U \to V$ of degree $d$ with the same filled Julia set. Moreover small analytic perturbations of $p_c : U \to V$ are also polynomial-like.
A degree $d$ Misiurewicz bifurcation $(f, c)$ gives rise to polynomial-like maps $f^n_t : B_0 \to B_n$, by the following mechanism. For small $t$, a small ball $B_0$ about the critical point $c_t$ maps to a small ball $B_1$ close to, but not containing, the fixed-point of $f_t$. The iterates $B_i = f_i^t(B)$ then remain near the fixed-point for a long time, ultimately expanding by a large factor. Finally for suitable $t$, as $B_i$ escapes from the fixed-point it maps back over $B_0$, resulting in a degree $d$ map $f^n_t : B_0 \to B_n \supset B_0$. Since most of the images $\langle B_i \rangle$ lie in the region where $f_t$ behaves linearly, the first-return map $f^n_t : B_0 \to B_n$ behaves like a polynomial of degree $d$.

This scenario leads to a cascade of families of polynomial-like maps, indexed by the return time $n$. Here is a precise statement.

**Theorem 3.1** Let $(f, c)$ be a degree $d$ Misiurewicz bifurcation, and fix $R > 0$. Then for all $n \gg 0$, there is a coordinate change depending on $n$ such that $c_t = 0$ and

$$f^n_t(z) = z^d + \xi + O(\epsilon_n)$$

whenever $|z|, |\xi| \leq R$. Here $t = t_n(1 + \gamma_n \xi)$, $t_n$ and $\gamma_n$ are nonzero, and $\gamma_n$, $t_n$ and $\epsilon_n$ tend to zero as $n \to \infty$.

The constants in $O(\cdot)$ above depend on $f$ and $R$ but not on $n$.

The proof yields more explicit information. Let $\lambda_0 = f'_0(f_0(c_0))$ be the multiplier of the fixed-point on which $c_0$ lands, and let $r$ be the multiplicity of intersection of the graph of $c_t$ and the graph of this fixed-point at $t = 0$. Then for $t = t_n$, the critical point $c_t$ is periodic with period $n$, and we have:

$$t_n \sim C\lambda_0^{-n/r},$$

$$\gamma_n = C'\lambda_0^{-n/(d-1)},$$

$$\epsilon_n = n(|\lambda_0|^{-n/r} + |\lambda_0|^{-n/(d-1)}),$$

for certain constants $C, C'$ depending on $f$. Due to the choice of roots, there are $r$ possibilities for $t_n$ and $(d-1)$ for $\gamma_n$; the Theorem is valid for all choices. Finally for $\xi$ fixed and $t = t_n(1 + \gamma_n \xi)$, the map $f^n_t$ is polynomial-like near $c_t$ for all $n \gg 0$, and in the original $z$-coordinate its filled Julia set satisfies

$$\text{diam } K(f^n_t) \sim |\lambda_0|^{-n/(d-1)}.$$

**Notation.** We adopt the usual conventions: $a_n = O(b_n)$, $a_n \asymp b_n$, $a_n \sim b_n$ and $n \gg 0$ mean $|a_n| < C|b_n|$, $(1/C)|b_n| < |a_n| < C|b_n|$, $a_n/b_n \to 1$ and $n \geq N$, where $C$ and $N$ are implicit constants.

**Proof.** We will make several constructions that work on a small neighborhood of $t = 0$. First, let $a_t$ parameterize the repelling fixed-point of $f_t$ such
that \( a_0 = f_0(c_0) \). Let \( \lambda_t = f'_t(a_t) \) be its multiplier. There is a holomorphically varying coordinate chart \( u = \phi_t(z) \) defined near \( z = a_t \) such that

\[
\phi_t \circ f_t \circ \phi_t^{-1}(u) = \lambda_t u
\]

for \( u \) near 0. We call \( u = \phi_t(z) \) the linearizing coordinate; note that \( u = 0 \) at \( a_t \).

We next arrange that \( u = 1 \) is an unramified preimage of \( c_t \). Since \( c_0 \) is unramified by (M2), its unramified preimages accumulate on \( a_0 \). Let \( b_0 \) be one such preimage, with \( f_0^p(b_0) = c_0 \) and \( b_0 \) in the domain of \( \phi_0 \). Then \( b_0 \) prolongs to a holomorphic function \( b_t \) with \( f_t^p(b_t) = c_t \). Replacing \( \phi_t \) by \( \phi_t(z)/\phi_t(b_t) \), we can assume \( u = \phi_t(b_t) = 1 \).

For small \( t \), the composition \( f_t^p \circ \phi_t^{-1} \) is univalent near \( u = 1 \). By applying a coordinate change \( z \mapsto m_t(z) \), where \( m_t \) is a Möbius transformation depending on \( t \), we can arrange that \( c_t = 0 \) and that

\[
f_t^p \circ \phi_t^{-1}(u) = (u - 1) + O((u - 1)^2)
\]

on \( B(1, \epsilon) \).

Since \( \deg(f_t, 0) = d \) for \( t \) near 0 by (M4), we have

\[
\phi_t \circ f_t(z) = \sum A_i(t)z^i = A_0(t) + A_d(0)z^d(1 + O(|z| + |t|))
\]

with \( A_d(0) \neq 0 \). Here \( A_0(t) = f_t(0) \) is the \( u \)-coordinate of the critical value. By (M3), \( c_t \) is not pre-fixed for all \( t \), so there is an \( r > 0 \) such that

\[
A_0(t) = t^r B(t)
\]

where \( B(0) \neq 0 \).

![Figure 2. Visiting the repelling fixed-point](image)

Next for \( n \gg 0 \) we choose \( t_n \) such that

\[
f_t^{1+n+p}(c_t) = c_t \quad \text{when } t = t_n.
\]
More precisely, for \( t = t_n \) we will arrange that \( c_t \) maps first close to \( a_t \), then lands after \( n \) iterates on \( b_t \), and thus returns in \( p \) further iterates to \( c_t \); see Figure 2. In the \( u \)-coordinate system, \( f_t \) is linear and \( b_t = 1 \), so the equation \( f_t^{n+1}(c_t) = b_t \) becomes

\[
\lambda^n A_0(t) = 1 \quad \text{when } t = t_n. \tag{3.10}
\]

By the argument principle, for \( n \gg 0 \) this equation has a solution \( t_n \) close to any root of the approximation \( \lambda_0^n t^n B(0) = 1 \) obtained from (3.8). Moreover

\[
t_n \sim B(0)^{-1/\lambda_0^n}
\]

(verifying (3.1)), and \( t_n \) satisfies (3.9) because \( f^n_t(b_t) = c_t \). (There are actually \( r \) solutions for \( t_n \) for a given \( n \); any one of the \( r \) solutions will do.)

We now turn to the estimate of \( f_t^{1+n+p}(z) \) for \( (t, z) \) near \( (t_n, 0) \). We will assume throughout that \( t = t_n + s \) and that:

\[
|z| \text{ and } |s/t_n| \text{ are } O(\Lambda^{-n/(d-1)}) \tag{3.11}
\]

where \( \Lambda = |\lambda_0| > 1 \). (To see this is the correct scale at which to work, suppose \( \text{diam}(B) \sim \text{diam} f_t^{1+n+p}(B) \), where \( B \) is a ball centered at \( z = 0 \). Then \( \text{diam} f_t(B) \sim \text{diam} B)^d \), and \( f^n_t \) is expanding by a factor of about \( \Lambda^n \), while \( f^n_t \) is univalent, so we get \( \text{diam} B \sim \Lambda^n \text{diam}(B)^d \), or \( \text{diam} B \sim \Lambda^{-n/(d-1)} \). Similarly \( |f_t^{1+n+p}(0)| \sim \Lambda^n (s/t_n) t_n^r \approx (s/t_n) = O(\text{diam} B) \) when \( s \) is as above.)

It is also convenient to set

\[
\hat{\Lambda} = \min(\Lambda^{1/(d-1)}, \Lambda^{1/r}) > 1,
\]

so that we may assert:

\[
z \text{ and } t \text{ are } O(\hat{\Lambda}^{-n}). \tag{3.12}
\]

By (3.11) the \( n \) iterates of \( f_t(z) \) lie within the domain of linearization, so by (3.7) we have

\[
\phi_t \circ f_t^{1+n}(z) = \lambda_t^n A_0(t) + \lambda_t^n A_0(d) z^d(1 + O(|z| + |t|)).
\]

The first term is approximately 1. Indeed, \( \lambda^n_t = \lambda^n_{t_n}(1 + O(ns)) \), so by (3.8) we have

\[
\lambda^n_t A_0(t) = \lambda^n_{t_n}(t_n + s)^r B(t_n + s)
\]

\[
= \lambda^n_{t_n}(1 + O(ns)) \cdot t_n^r \left(1 + \frac{s}{t_n}\right)^r \cdot B(t_n)(1 + O(s))
\]

\[
= \lambda^n_{t_n} A_0(t_n) \left(1 + r \frac{s}{t_n} + O((s/t_n)^2) + O(ns)\right)
\]

\[
= 1 + r \frac{s}{t_n} + O((s/t_n)^2) + O(ns)
\]
by (3.10). Similarly, $\lambda^n_t = \lambda^n_0 (1 + O(t))$, so
\[
\phi_t \circ f_t^{1+n}(z) - 1 = \lambda^n_0 A_0(d) z^d (1 + O(|z| + |nt|)) + r \frac{s}{t_n} + O((s/t_n)^2) + O(ns) = \\
\lambda^n_0 A_0(d) z^d + r \frac{s}{t_n} + O(n \Lambda^{-n/(d-1)} \Lambda^{-n}),
\]
using (3.11) and (3.12). The expression above as a whole is $O(\Lambda^{-n/(d-1)})$, so composing with the univalent map $f_t^p \circ \phi_t^{-1}$ introduces (by (3.5)) an additional error of size $O(n \Lambda^{-n/(d-1)} \Lambda^{-n})$, which is already accounted for in the $O(\cdot)$ above. Thus the expression above also represents $f_t^{1+n+p}(z)$.

Finally we make a linear change of coordinates of the form $z \mapsto \alpha_n z$, conjugating the expression above to
\[
f_t^{1+n+p}(z) = \alpha_n^{1-d} \lambda^n_0 A_0(d) z^d + \alpha_n r \frac{s}{t_n} + O(n \alpha_n \Lambda^{-n/(d-1)} \Lambda^{-n}).
\]
Setting $\alpha_n = (\lambda^n_0 A_0(d))^{1/(d-1)}$ to normalize the coefficient of $z^d$, we have $|\alpha_n| \asymp \Lambda^{n/(d-1)}$ and thus:
\[
f_t^{1+n+p}(z) = z^d + \alpha_n r \frac{s}{t_n} + O(n \Lambda^{-n})
\]
with $t = t_n(1 + \gamma_n \xi)$, $\gamma_n$ and $\epsilon_n$ as in (3.2) and (3.3). Notice that if $|z|$ and $|\xi|$ are bounded by $R$ in the expression above, then (3.11) is satisfied in our original coordinates. Reindexing $n$, we obtain the Theorem.  

\section{Small Mandelbrot sets}

We now show the Misiurewicz cascade leads to small Mandelbrot sets in parameter space. From this we deduce Theorems 1.1 and 1.3 of the Introduction.

\textbf{Hybrid conjugacy.} Let $g_1, g_2$ be polynomial-like maps of the same degree. A \textit{hybrid conjugacy} is a quasiconformal map $\phi$ between neighborhoods of $K(g_1)$ and $K(g_2)$ such that $\phi \circ g_1 = g_2 \circ \phi$ and $\overline{\partial} \phi | K(g_1) = 0$. We say $g_1$ and $g_2$ are \textit{hybrid equivalent} if such a conjugacy exists. By a basic result of Douady and Hubbard, every polynomial-like map $g$ of degree $d$ is hybrid equivalent to a polynomial of degree $d$, unique up to affine conjugacy if $K(g)$ is connected [DH, Theorem 1].
Theorem 4.1 Let \((f, c)\) be a degree \(d\) Misiurewicz bifurcation. Then the parameter space \(\Delta\) contains quasiconformal copies \(M^d_n\) of the degree \(d\) Mandelbrot set \(M_d\), converging to the origin, with \(\partial M^d_n\) contained in the bifurcation locus \(B(f)\).

More precisely, for all \(n \gg 0\) there are homeomorphisms

\[ \phi_n : M_d \to M^d_n \subset \Delta \]

such that:

1. \(f^n_t\) is hybrid equivalent to \(z^d + \xi\) whenever \(t = \phi_n(\xi)\);
2. \(d(0, M^d_n) \approx |\lambda_0|^{-n/r}\);
3. \(\operatorname{diam}(M^d_n)/d(0, M^d_n) \approx |\lambda_0|^{-n/(d-1)}\);
4. \(\phi_n\) extends to a quasiconformal map of the plane with dilatation bounded by \(1 + O(\epsilon_n)\); and
5. \(\psi_n^{-1} \circ \phi_n(\xi) = \xi + O(\epsilon_n), \) where \(\psi_n(\xi) = t_n(1 + \gamma_n \xi)\).

The notation is from (3.1) – (3.3).

We begin by recapitulating some ideas from [DH]. Let \(\Delta(R) = \{z : |z| < R\}\), and let

\[ g_\xi(z) = z^d + \xi + h(\xi, z) \]

be a holomorphic family of mappings defined for \((\xi, z) \in \Delta(R) \times \Delta(R)\), where \(R > 10\) and \(g'_\xi(0) = 0\). Let \(M \subset \Delta(R)\) be the set of \(\xi\) such that the forward orbit \(g_\xi^n(0)\) remains in \(\Delta(R)\) for all \(n > 0\).

Lemma 4.2 There is a \(\delta > 0\) such that if \(\sup |h(\xi, z)| = \epsilon < \delta\) then there is a homeomorphism

\[ \phi : M_d \to M \]

such that for all \(\xi \in M_d\), \(g_\phi(\xi)\) is hybrid equivalent to \(z^d + \xi\), \(|\phi(\xi) - \xi| < O(\epsilon)\), and \(\phi\) extends to a \(1 + O(\epsilon)\)-quasiconformal map of the plane.

Proof. Let \(p_\xi(z) = z^d + \xi\). Since \(R > 10\) we have \(M_d \subset \Delta(R)\) and \(K(p_\xi) \subset \Delta(R)\) for all \(\xi \in M_d\); indeed these sets have capacity one, so their diameters are bounded by 4 [Ah]. In addition, for \(\xi \in M_d\) the map \(p_\xi : U \to \Delta(R)\) is polynomial-like, where \(U = p_\xi^{-1}(\Delta(R))\). By continuity, when \(\sup |h|\) is sufficiently small, \(M\) is compact and \(g_\xi\) is polynomial-like for all \(\xi \in M\).

By results of Douady and Hubbard, we can also choose \(\delta\) small enough that \(|h| < \delta\) implies there is a homeomorphism

\[ \phi : M_d \to M \]
such that $g_{\phi(\xi)}$ is hybrid equivalent to $z^d + \xi$ [DH, Prop. 21].

Now assume $|h| < \epsilon < \delta$. For $t \in \Delta$ let $\mathcal{M}_t$ denote the parameters where the critical point remains bounded for the family

$$g_{\xi, t} = z^d + \xi + t\frac{\delta}{\epsilon} h(\xi, z),$$

and define $\phi_t : M_d \to \mathcal{M}_t$ as above. Then $\phi_t$ is a family of injections, with $\phi_0(z) = z$, and $\phi_t(\xi)$ is a holomorphic function of $t$ for every $\xi$. (For example this is clear at $\xi \in \partial M_d$ because Misiurewicz points are dense in $\partial M_d$; for the general case see [DH, Prop. 22].)

By a theorem of Słodkowski [Sl] (cf. [Dou], [BR]), $\phi_t(z)$ prolongs to a holomorphic motion of the entire plane, and its complex dilatation $\mu_t = \partial \phi_t / \partial \phi_t$ gives a holomorphic map of the unit disk into the unit ball in $L^\infty(\hat{\mathbb{C}})$. By the Schwarz lemma, $\|\mu_t\|_{\infty} \leq |t|$; since $\phi = \phi_\epsilon / \delta$, we obtain a quasiconformal extension of $\phi$ with dilatation $1 + O(\epsilon)$. The bound on $|\phi(\xi) - \xi|$ similarly results by applying the Schwarz Lemma to the map $\Delta \to \hat{\mathbb{C}} - \{0, 1, \infty\}$ given by $t \mapsto \phi_t(\xi)$, once three points have been normalized to remain fixed during the motion.

**Proof of Theorem 4.1.** Fix $R > 10$. For all $n \gg 0$, Theorem 3.1 provides a family of rational maps of the form

$$g_{\xi}(z) = f^n_{\psi}(z) = z^d + \xi + O(\epsilon_n)$$

defined for $(\xi, z) \in \Delta(R) \times \Delta(R)$, where $t = \psi_n(\xi) = t_n(1 + \gamma_n \xi)$. The preceding Lemma gives homeomorphisms $\tilde{\phi}_n : M_d \to \tilde{\mathcal{M}}_d \subset \Delta(R)$ for all $n \gg 0$. Setting $\phi_n = \psi_n \circ \tilde{\phi}_n$, the Theorem results from the Lemma and the bounds (3.1) – (3.3).

**Example.** The quadratic family $(f, c) = (z^2 + t - 2, 0)$ is a Misiurewicz bifurcation of degree $d = 2$, with $\lambda_0 = 2$ and $r = 1$. Thus $M_2$ contains small copies $\mathcal{M}_2^n$ of itself near $c = -2$, with $d(\mathcal{M}_2^n, -2) \asymp 4^{-n}$ and $\operatorname{diam} \mathcal{M}_2^n \asymp 16^{-n}$.

**Consequences.** Assembling the preceding results, we may now prove the Theorems stated in the Introduction. Here is a more precise form of Theorem 1.1:

**Theorem 4.3** Let $f$ be a holomorphic family of rational maps over the unit disk with bifurcations. Then there is a nonempty list of degrees

$$D \subset \{2, 3, \ldots, 2^{\deg(f_t) - 2}\}$$

such that for any $\epsilon > 0$ and $d \in D$, $B(f)$ contains the image of $\partial M_d$ under a $(1 + \epsilon)$-quasiconformal map.

If the critical points of $f$ are marked $\{c^1_t, \ldots, c^m_t\}$ such that
\[ (i \leq N) \iff c^i \text{ is active and } c^i \text{ is unramified for some } t, \]
then we may take
\[ D = \{ \inf_{i} \sup_{k} \deg(f^k_i, c^i) : i \leq N \}. \]

**Proof.** Let \( B_0 = B(f) \). After a base change we can assume that \( f \) is a local bifurcation with critical points marked as above. By Proposition 2.4, there is at least one active, unramified critical point, so \( N \geq 1 \). For any \( i \leq N \), we can make a base change so \( c^i \) is active and unramified; then by Proposition 2.5, a further base change makes \((f^n, c^i)\) into a degree \( d \) Misiurewicz bifurcation.

Let \( d_i = \inf_{i} \sup_{k} \deg(f^k_i, c^i) \). We claim \( d = d_i \leq 2^{2\deg(f^t) - 2} \). Indeed, \( \deg(f^n_i, c^i) \) assumes its minimum outside a discrete set, and it is equal to 2 near \( t = 0 \), so \( d_i \geq d \). On the other hand, \( c^0 \) lands on a repelling periodic cycle, so \( \deg(f^n_k, c^0) = d \) for all \( k > n \), and therefore \( d_i \leq d \). Finally \( d \) is largest if \( c^i \) hits all the other critical points of \( f \) before reaching the repelling cycle; in this case \( d = (p_1 + 1)(p_2 + 1) \cdots (p_m + 1) \) for some partition \( p_1 + p_2 + \cdots + p_m = 2d - 2 \). The product is maximized by the partition 1 + 1 + \cdots + 1, so \( d \leq 2^{2\deg(f^t) - 2} \).

By Theorem 4.1, the bifurcation locus \( B(f) \) contains almost conformal copies \( \partial M^n_d \) of \( \partial M_d \) accumulating at \( t = 0 \), with \( \text{diam}(\partial M^n_d) \ll d(0, \partial M^n_d) \). Letting \( \phi : \Delta \to \Delta \) denote the composition of all the base-changes occurring so far, we have \( B(f) = \phi^{-1}(B_0) \). Then \( \phi \) is univalent and nearly linear on \( \partial M^n_d \) for \( n \gg 0 \), so \( \phi(\partial M^n_d) \subset B_0 \) is a \((1 + \epsilon)\)-quasiconformal copy of \( \partial M_d \).

Let \( \text{Rat}_d \) be the space of all rational maps of degree \( d \); it is a Zariski-open subset of \( \mathbb{P}^{2d+1} \). We now make precise the statement that a generic family contains a copy of the standard Mandelbrot set.

**Corollary 4.4** There is a countably union of proper subvarieties \( R \subset \text{Rat}_d \) such that for any local bifurcation, either \( f_t \in R \) for all \( t \), or \( B(f) \) contains a copy of \( \partial M_2 \).

**Proof.** On a finite branched cover \( X \) of \( \text{Rat}_d \), the critical points of \( f \in \text{Rat}_d \) can be marked \( \{c^1(f), \ldots, c^{2d-2}(f)\} \). Clearly \( \deg(f^n, c_i(f)) = 2 \) outside a proper subvariety \( V_{n, i} \) of \( X \). Let \( R \) be the union of the images of these varieties in \( \text{Rat}_d \), and apply the preceding argument to see \( D = \{2\} \) if some \( f_t \notin R \).

**Proof of Theorem 1.3.** The proof follows the same lines as that of Theorem 4.3; to get \( \text{mod}(V - U) \) large one takes \( R \) large in Theorem 3.1. Thus Theorem 1.3 also holds for all \( d \in D \).
5 Hausdorff dimension

In this section we prove Corollary 1.8: for any holomorphic family $f$ of rational maps over a complex manifold $X$, we have $\text{H.dim}(B(f)) = \text{H.dim}(X)$ if $B(f) \neq \emptyset$.

Recall that the Hausdorff dimension of a metric space $X$ is the infimum of the set of $\delta \geq 0$ such that there exists coverings $X = \bigcup X_i$ with $\sum (\text{diam } X_i)^\delta$ arbitrarily small.

**Lemma 5.1** Let $Y$ be a metric space, $X$ a subset of $Y \times [0, 1]$. Then

$$\text{H.dim}(X) \geq 1 + \inf \text{H.dim}(X_t)$$

where $X_t = \{y : (y, t) \in X\}$.

Here $Y \times [0, 1]$ is given the product metric.

**Proof.** Fix $\delta$ with $\delta + 1 > \text{H.dim}(X)$. For any $n$ there is a covering $X \subset \bigcup B(y_i, r_i) \times I_i$ with $|I_i| = r_i$ and $\sum r_i^{\delta+1} < 4^{-n}$. Note that $X_t \subset \bigcup_{t \in I_i} B(y_i, r_i)$

and

$$\int_0^1 \sum_{t \in I_i} r_i^\delta \, dt = \sum r_i^{\delta+1} < 4^{-n}.$$ 

Let $E_n$ be the set of $t$ where the integrand exceeds $2^{-n}$; then $\sum m(E_n) < \sum 2^{-n} < \infty$. Thus almost every $t$ belongs to at most finitely many $E_n$, so almost every $X_t$ admits infinitely many coverings with $\sum r_i^\delta < 2^{-n} \to 0$. Therefore $\delta \geq \inf \text{H.dim}(X_t)$, and the Theorem follows.

The Lemma above is related to the product formula

$$\text{H.dim}(X \times Y) \geq \text{H.dim}(X) + \text{H.dim}(Y);$$

see [Fal, Ch. 5] and references therein.

**Proof of Corollary 1.8.** Suppose $B(f) \neq \emptyset$. Then there is a $t_0 \in B(f)$ and a locally parameterized periodic point $a(t)$ of period $n$ such that $a(t)$ changes from attracting to repelling near $t_0$ [MSS], [Mc2, §4.1]. More formally this means the multiplier $\lambda(t) = (f^n)'(a(t))$ is not locally constant and $|\lambda(t_0)| = 1$.

Choosing local coordinates we can reduce to the case $X = \Delta^n$ and $t_0 = 0$. Let $\Delta_s = \Delta \times \{s\}$ for $s \in \Delta^{n-1}$. For coordinates in general position, $\lambda(t)$ is nonconstant on $\Delta_0$. Shrinking the $\Delta^{n-1}$ factor, we can also assume $a(t)$ changes from attracting to repelling in the family $f|\Delta_s$ for all $s$. Then

$$B(f)_s = B(f) \cap \Delta_s \supset B(f|\Delta_s) \neq \emptyset$$
and $H.\dim B(f|\Delta_s) = 2$ by Corollary 1.6. Applying the Lemma above to $B(f) \subset \Delta \times \Delta^{n-1}$ we find

$$H.\dim(B(f)) \geq (2n - 2) + \inf_s H.\dim B(f)_s = 2n = H.\dim(X).$$

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\[\Box\]
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References


Mathematics Department, Harvard University, 1 Oxford St, Cambridge, MA 02138-2901