The Mandelbrot Set is Universal

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The Mandelbrot set is universal

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24 February, 1997

Abstract
We show small Mandelbrot sets are dense in the bifurcation locus for any holomorphic family of rational maps.

1 Introduction
Fix an integer $d \geq 2$, and let $p_c(z) = z^d + c$. The generalized Mandelbrot set $M_d \subset \mathbb{C}$ is defined as the set of $c$ such that the Julia set $J(p_c)$ is connected. Equivalently, $c \in M_d$ iff $p^n_c(0)$ does not tend to infinity as $n \to \infty$. The traditional Mandelbrot set is the quadratic version $M_2$.

A holomorphic family of rational maps over $X$ is a holomorphic map $f : X \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ where $X$ is a complex manifold and $\hat{\mathbb{C}}$ is the Riemann sphere. For each $t \in X$ the family $f$ specializes to a rational map $f_t : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, denoted $f_t(z)$. For convenience we will require that $X$ is connected and that $\deg(f_t) \geq 2$ for all $t$.

The bifurcation locus $B(f) \subset X$ is defined equivalently as the set of $t$ such that:

1. The number of attracting cycles of $f_t$ is not locally constant;
2. The period of the attracting cycles of $f_t$ is locally unbounded; or
3. The Julia set $J(f_t)$ does not move continuously (in the Hausdorff topology) over any neighborhood of $t$.

It is known that $B(f)$ is a closed, nowhere dense subset of $X$; its complement $X - B(f)$ is also called the J-stable set [MSS], [Mc2, §4.1].

As a prime example, $p_c(z) = z^d + c$ is a holomorphic family parameterized by $c \in \mathbb{C}$, and its bifurcation locus is $\partial M_d$. See Figure 1.

In this paper we show that every bifurcation set contains a copy of the boundary of the Mandelbrot set or its degree $d$ generalization. The Mandelbrot sets $M_d$ are thus universal; they are initial objects in the category

∗Research partially supported by the NSF. 1991 Mathematics Subject Classification: Primary 58F23, Secondary 30D05.
of bifurcations, providing a lower bound on the complexity of $B(f)$ for all families $f_t$.

For simplicity we first treat the case $X = \Delta = \{ t : |t| < 1 \}$.

**Theorem 1.1** For any holomorphic family of rational maps over the unit disk, the bifurcation locus $B(f) \subset \Delta$ is either empty or contains the quasiconformal image of $\partial M_d$ for some $d$.

The proof (§4) shows that $B(f)$ contains copies of $\partial M_d$ with arbitrarily small quasiconformal distortion, and controls the degrees $d$ that arise. For example we can always find a copy of $\partial M_d$ with $d \leq 2^{\text{deg}(f_t)} - 2$, and generically $B(f)$ contains a copy of $\partial M_2$ (see Corollary 4.4). Since the Theorem is local we have:

**Corollary 1.2** Small Mandelbrot sets are dense in $B(f)$.

There is also a statement in the dynamical plane:

**Theorem 1.3** Let $f$ be a holomorphic family of rational maps with bifurcations. Then there is a $d \geq 2$ such that for any $c \in M_d$ and $m > 0$, the family contains a polynomial-like map $f^n_t : U \rightarrow V$ hybrid conjugate to $z^d + c$ with $\text{mod}(U - V) > m$.

**Corollary 1.4** If $f$ has bifurcations then for any $\epsilon > 0$ there exists a $t$ such that $f_t(z)$ has a superattracting basin which is a $(1 + \epsilon)$-quasidisk.

**Proof.** The family contains a polynomial-like map $f^n_t : U \rightarrow V$ hybrid conjugate to $p_0(z) = z^d$, a map whose superattracting basin is a round disk. Since $\text{mod}(V - U)$ can be made arbitrarily large, the conjugacy can be made nearly conformal, and thus $f_t$ has a superattracting basin which is a $(1 + \epsilon)$-quasidisk.
For applications to Hausdorff dimension we recall:

\textbf{Theorem 1.5 (Shishikura)} For any \( d \geq 2 \), the Hausdorff dimension of \( \partial M_d \) is two. Moreover \( \text{H.dim}(J(p_c)) = 2 \) for a dense \( G_\delta \) of \( c \in \partial M_d \).

This result is stated for \( d = 2 \) in [Shi2] and [Shi1] but the argument generalizes to \( d \geq 2 \). Quasiconformal maps preserve sets of full dimension [GV], so from Theorems 1.1 and 1.3 we obtain:

\textbf{Corollary 1.6} For any family of rational maps \( f \) over \( \Delta \), the bifurcation set \( B(f) \) is empty or has Hausdorff dimension two.

\textbf{Corollary 1.7} If \( f \) has bifurcations, then \( \text{H.dim}(J(f_t)) = 2 \) for a dense set of \( t \in B(f) \).

For higher-dimensional families one has (§5):

\textbf{Corollary 1.8} For any holomorphic family of rational maps over a complex manifold \( X \), either \( B(f) = \emptyset \) or \( \text{H.dim}(B(f)) = \text{H.dim}(X) = 2 \dim_{\mathbb{C}} X \).

Similar results on Hausdorff dimension were obtained by Tan Lei, under a technical hypothesis on the family \( f \) [Tan].

A family of rational maps \( f \) is \textit{algebraic} if its parameter space \( X \) is a quasi-projective variety (such as \( \mathbb{C}^n \)) and the coefficients of \( f_t(z) \) are rational functions of \( t \). For example, \( p_c(z) = z^d + c \) is an algebraic family over \( X = \mathbb{C} \). Such families almost always contain bifurcations [Mc1]:

\textbf{Theorem 1.9} For any algebraic family of rational maps, either

1. The family is trivial (\( f_t \) and \( f_s \) are conformally conjugate for all \( t, s \in X \)); or

2. The family is affine (every \( f_t \) is critically finite and double covered by a torus endomorphism); or

3. The family has bifurcations (\( B(f) \neq \emptyset \)).

\textbf{Corollary 1.10} With rare exceptions, any algebraic family of rational maps exhibits small Mandelbrot sets in its parameter space.

\footnote{This set of \( t \) can be improved to a dense \( G_\delta \) using Shishikura’s idea of hyperbolic dimension.}
This Corollary was our original motivation for proving Theorem 1.1.

As another application, for \( t \in \mathbb{C}^{d-1} \) let

\[
f_t(z) = z^d + t_1 z^{d-2} + \cdots + t_{d-1}
\]

and let

\[
C_d = \{ t : J(f_t) \text{ is connected} \}
\]
denote the connectedness locus. Then we have:

**Corollary 1.11 (Tan Lei)** The boundary of the connectedness locus has full dimension; that is, \( H. \dim(\partial C_d) = H. \dim(C_d) = 2d - 2 \).

**Proof.** Consider the algebraic family \( g_a(z) = z^d + az^{d-1} \), which for \( a \neq 0 \) has all but one critical point fixed under \( g_a \). By Theorem 1.9, this family has bifurcations at some \( a \in \mathbb{C} \). Then there is a neighborhood \( U \) of \( (a, 0, \ldots, 0) \in \mathbb{C}^{d-2} \) such that for \( t \in U \) all critical points of \( f_t \) save one lie in an attracting or superattracting basin. If \( t \in B(f) \cap U \), then the remaining critical point has a bounded forward orbit under \( f_t \), but under a small perturbation tends to infinity. It follows that \( B(f) \cap U = \partial C_d \cap U \neq \emptyset \), and thus \( \dim(\partial C_d) \geq \dim B(f) = 2d - 2 \).

**Remark.** Rees has shown that the bifurcation locus has positive measure in the space of all rational maps of degree \( d \) \([Rs]\); it would be interesting to known general conditions on a family \( f \) such that \( B(f) \) has positive measure in the parameter space \( X \).

**Acknowledgements.** I would like to thank Tan Lei for sharing her results which prompted the writing of this note. Special cases of Theorem 1.1 were developed independently by Douady and Hubbard \([DH, pp.332-336]\) and Eckmann and Epstein \([EE]\).

## 2 Families of rational maps

In this section we begin a more formal study of maps with bifurcations.

**Definitions.** A local bifurcation is a holomorphic family of rational maps \( f_t(z) \) over the unit disk \( \Delta \), such that \( 0 \in B(f) \).

The following natural operations can be performed on \( f \) to construct new local bifurcations:

1. **Coordinate change:** replace \( f_t \) by \( m_t \circ f_t \circ m_t^{-1} \), where \( m : \Delta \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}} \) is a holomorphic family of Möbius transformations.

2. **Iteration:** replace \( f_t(z) \) by \( f_t^n(z) \) for a fixed \( n \geq 1 \).
3. **Base change:** replace $f_t(z)$ by $f_{\phi(t)}(z)$, where $\phi : \Delta \to \Delta$ is a nonconstant holomorphic map with $\phi(0) \in B(f)$.

The first two operations leave the bifurcation locus unchanged, while the last transforms $B(f)$ to $\phi^{-1}(B(f))$.

**Marked critical points.** We will also consider pairs $(f, c)$ consisting of a local bifurcation and a marked critical point; this means $c : \Delta \to \hat{\mathbb{C}}$ is holomorphic and $f'_t(c_t) = 0$. The operations above also apply to $(f, c)$; a coordinate change replaces $c_t$ with $m_t(c_t)$ and a base change replaces $c_t$ with $c_{\phi(t)}$.

**Misiurewicz points.** A marked critical point $c$ of $f$ is active if its forward orbit $\langle f^n_t(c_t) : n = 1, 2, 3, \ldots \rangle$ fails to form a normal family of functions of $t$ on any neighborhood of $t = 0$ in $\Delta$. A parameter $t$ is a Misiurewicz point for $(f, c)$ if the forward orbit of $c_t$ under $f_t$ lands on a repelling periodic cycle. If $t = 0$ is a Misiurewicz point, then either $c$ is active or $c_t$ is preperiodic for all $t$.

**Proposition 2.1** If $c$ is an active critical point, then $(f, c)$ has a sequence of distinct Misiurewicz points $t_n \to 0$.

**Proof.** This is a traditional normal families argument. Choose any 3 distinct repelling periodic points \{a_0, b_0, c_0\} for $f_0$, and let \{a_t, b_t, c_t\} be holomorphic functions parameterizing the corresponding periodic points of $f_t$ for $t$ near zero. Since $\langle f^n_t(c_t) \rangle$ is not a normal family, it cannot avoid these three points, and any parameter $t$ where $f^n_t(c_t)$ meets $a_t, b_t$ or $c_t$ is a Misiurewicz point.

**Ramification.** Next we discuss the existence of univalent inverse branches for a single rational map $F(z)$. Let $d = \text{deg}(F, z)$ denote the local degree of $F$ at $z \in \hat{\mathbb{C}}$; we have $d > 1$ iff $z$ is a critical point of multiplicity $(d - 1)$. We say $y$ is an unramified preimage of $z$ if for some $n \geq 0$, $F^n(y) = z$ and $\text{deg}(F^n, y) = 1$. We say $z$ is unramified if it has infinitely many unramified preimages. In this case its unramified preimages accumulate on the full Julia set $J(F)$.

**Proposition 2.2** If $z$ has 5 distinct unramified preimages then it has infinitely many.

**Proof.** Let $E$ be the set of all unramified preimages of $z$, and let $C$ be the critical points of $F$. Then $F^{-1}(E) \subset E \cup C$, so if $|E|$ is finite then

$$d|E| = \sum_{z \in F^{-1}(E)} 1 + \text{mult}(f', z) \leq |F^{-1}(E)| + 2d - 2 \leq |E| + 4d - 4$$

and therefore $|E| \leq 4$. 


Corollary 2.3 Let \((f, c)\) be a local bifurcation with marked critical point. Then the set of \(t\) such that \(c_t\) is ramified for \(f_t\) is either discrete or the whole disk.

**Proof.** By the previous Proposition, the ramified parameters are defined by a finite number of analytic equations in \(t\). \(\blacksquare\)

Proposition 2.4 After a suitable base change, any local bifurcation \(f\) can be provided with an active marked critical point \(c\) such that \(c_0\) is unramified for \(f_0\).

**Remark.** It is possible that all the active critical points are ramified at \(t = 0\). The base change in the Proposition will generally not preserve the central fiber \(f_0\).

**Proof.** The set \(C = \{(t, z) \in \Delta \times \hat{\mathbb{C}} : f'_t(z) = 0\}\) is an analytic variety with a proper finite projection to \(\Delta\). By Puiseux series, after a base change of the form \(\phi(t) = \epsilon t^n\) all the critical points of \(f\) can be marked by holomorphic functions \(\{c^1_t, \ldots, c^m_t\}\). Since \(t = 0\) is in the bifurcation set, by [Mc2, Thm. 4.2], there is an \(i\) such that \(\langle f^n_t(c^i_t) \rangle\) is not a normal family at \(t = 0\). That is, \(c^i\) is an active critical point.

Next we show \(c^i\) can be chosen so that for generic \(t\) it is disjoint from the forward orbits of all other critical points. If not, there is a \(c^j\) and \(n \geq 1\) such that \(f^n_t(c^j_t) = c^i_t\) for all \(t\). Then \(c^j\) is also active and we may replace \(c^i\) with \(c^j\). If the replacement process were to cycle, then \(c^i\) would be a periodic critical point, which is impossible because it is active. Thus we eventually achieve a \(c^i\) which is generically disjoint from the forward orbits of the other critical points.

In particular, there is a \(t\) such that \(c^i_t\) is unramified for \(f_t\). By Corollary 2.3, the set \(R \subset \Delta\) of parameters where \(c^i_t\) is ramified is discrete. By Proposition 2.1, there are Misiurewicz points \(t_n\) for \((f, c^i)\) with \(t_n \to 0\). Choose \(n\) such that \(t_n \not\in R\), and make a base change moving \(t_n\) to zero; then \(c^i\) is active, and \(c^i_0\) is unramified for \(f_0\). \(\blacksquare\)

Misiurewicz bifurcations. Let \((f, c)\) be a local bifurcation with a marked critical point. We say \((f, c)\) is a Misiurewicz bifurcation of degree \(d\) if

- M1. \(f_0(c_0)\) is a repelling fixed-point of \(f_0\);
- M2. \(c_0\) is unramified for \(f_0\);
- M3. \(f_t(c_t)\) is not a fixed-point of \(f_t\), for some \(t\); and
- M4. \(\text{deg}(f_t, c_t) = d\) for all \(t\) sufficiently small.
Proposition 2.5  For any local bifurcation \((f, c)\) with \(c\) active and \(c_0\) unramified, there is a base change and an \(n > 0\) such that \((f^n, c)\) is a Misiurewicz bifurcation.

Remark.  The delicate point is condition (M4). The danger is that for every Misiurewicz parameter \(t\), the forward orbit of \(c_t\) might accidentally collide with another critical point before reaching the periodic cycle. We must avoid these collisions to make the degree of \(f^n\) at \(c_t\) locally constant.

Proof.  There are Misiurewicz points \(t_n \to 0\) for \((f, c)\), and \(c_t\) is unramified for all \(t\) near 0, so after a base change and replacing \(f\) with \(f^n\) we can arrange that \((f, c)\) satisfies conditions (M1), (M2) and (M3).

We can also arrange that \(\text{deg}(f_t, c_t) = d\) for all \(t \neq 0\). However (M4) may fail because \(\text{deg}(f_t, c_t)\) may jump up at \(t = 0\). This jump would occur if another critical of \(f_t\) coincides with \(c_t\) at \(t = 0\).

To rule this out, we make a further perturbation of \(f_0\). Let \(a_t\) locally parameterize the repelling fixed-point of \(f_t\) such that \(f_0(a_0) = a_0\). Choose a neighborhood \(U\) of \(a_0\) such that for \(t\) small, \(f_t\) is linearizable on \(U\) and \(U\) is disjoint from the critical points of \(f_t\). (This is possible since \(f'_0(a_0) \neq 0\).)

Let \(b_t \in U - \{a_t\}\) be a parameterized repelling periodic point close to \(a_t\). Then \(b_t\) has preimages in \(U\) accumulating on \(a_t\). Choose \(s\) near 0 such that \(f_s(c_s)\) hits one of these preimages (such an \(s\) exists by the argument principle and (M3)). For this special parameter, \(c_s\) first maps close to \(a_s\), then remains in \(U\) until it finally lands on \(b_s\). Since there are no critical points in \(U\), we have \(\text{deg}(f^i_s, c) = d\) for all \(i > 0\).

Making a base change moving \(s\) to \(t = 0\), we find that \((f^n, c)\) satisfies (M1-M4) for \(n\) a suitable multiple of the period of \(b_s\).

3  The Misiurewicz cascade

In this section we show that when a Misiurewicz point bifurcates, it produces a cascade of polynomial-like maps.

Definitions.  A polynomial-like map \(g : U \to V\) is a proper, holomorphic map between simply-connected regions with \(\overline{U}\) compact and \(\overline{U} \subset V \subset \mathbb{C}\) [DH]. Its filled Julia set is defined by

\[
K(g) = \bigcap_{n=1}^{\infty} g^{-n}(V);
\]

it is the set of points that never escape from \(U\) under forward iteration.

Any polynomial such as \(p_c(z) = z^d + c\) can be restricted to a polynomial-like map \(p_c : U \to V\) of degree \(d\) with the same filled Julia set. Moreover small analytic perturbations of \(p_c : U \to V\) are also polynomial-like.
A degree $d$ Misiurewicz bifurcation $(f, c)$ gives rise to polynomial-like maps $f_t^n : B_0 \to B_n$, by the following mechanism. For small $t$, a small ball $B_0$ about the critical point $c_t$ maps to a small ball $B_1$ close to, but not containing, the fixed-point of $f_t$. The iterates $B_i = f_t^i(B)$ then remain near the fixed-point for a long time, ultimately expanding by a large factor. Finally for suitable $t$, as $B_i$ escapes from the fixed-point it maps back over $B_0$, resulting in a degree $d$ map $f_t^n : B_0 \to B_n \supset B_0$. Since most of the images $\langle B_i \rangle$ lie in the region where $f_t$ behaves linearly, the first-return map $f_t^n : B_0 \to B_n$ behaves like a polynomial of degree $d$.

This scenario leads to a cascade of families of polynomial-like maps, indexed by the return time $n$. Here is a precise statement.

**Theorem 3.1** Let $(f, c)$ be a degree $d$ Misiurewicz bifurcation, and fix $R > 0$. Then for all $n \gg 0$, there is a coordinate change depending on $n$ such that $c_t = 0$ and

$$f_t^n(z) = z^d + \xi + O(\epsilon_n)$$

whenever $|z|, |\xi| \leq R$. Here $t = t_n(1 + \gamma_n\xi)$, $t_n$ and $\gamma_n$ are nonzero, and $\gamma_n, t_n$ and $\epsilon_n$ tend to zero as $n \to \infty$.

The constants in $O(\cdot)$ above depend on $f$ and $R$ but not on $n$.

The proof yields more explicit information. Let $\lambda_0 = f_0'(f_0(c_0))$ be the multiplier of the fixed-point on which $c_0$ lands, and let $r$ be the multiplicity of intersection of the graph of $c_t$ and the graph of this fixed-point at $t = 0$. Then for $t = t_n$, the critical point $c_t$ is periodic with period $n$, and we have:

$$t_n \sim C\lambda_0^{-n/r}, \quad (3.1)$$

$$\gamma_n = C'\lambda_0^{-n/(d-1)}, \quad \text{and} \quad (3.2)$$

$$\epsilon_n = n(|\lambda_0|^{-n/r} + |\lambda_0|^{-n/(d-1)}), \quad (3.3)$$

for certain constants $C, C'$ depending on $f$. Due to the choice of roots, there are $r$ possibilities for $t_n$ and $(d-1)$ for $\gamma_n$; the Theorem is valid for all choices. Finally for $\xi$ fixed and $t = t_n(1 + \gamma_n\xi)$, the map $f_t^n$ is polynomial-like near $c_t$ for all $n \gg 0$, and in the original $z$-coordinate its filled Julia set satisfies

$$\text{diam } K(f_t^n) \leq |\lambda_0|^{-n/(d-1)}.$$

**Notation.** We adopt the usual conventions: $a_n = O(b_n), a_n \asymp b_n, a_n \sim b_n$ and $n \gg 0$ mean $|a_n| < C|b_n|$, $(1/C)|b_n| < |a_n| < C|b_n|$, $a_n/b_n \to 1$ and $n \geq N$, where $C$ and $N$ are implicit constants.

**Proof.** We will make several constructions that work on a small neighborhood of $t = 0$. First, let $a_t$ parameterize the repelling fixed-point of $f_t$ such
that $a_0 = f_0(c_0)$. Let $\lambda_t = f'_t(a_t)$ be its multiplier. There is a holomorphically varying coordinate chart $u = \phi_t(z)$ defined near $z = a_t$ such that

\[ \phi_t \circ f_t \circ \phi_t^{-1}(u) = \lambda_t u \]  

(3.4)

for $u$ near 0. We call $u = \phi_t(z)$ the linearizing coordinate; note that $u = 0$ at $a_t$.

We next arrange that $u = 1$ is an unramified preimage of $c_t$. Since $c_0$ is unramified by (M2), its unramified preimages accumulate on $a_0$. Let $b_0$ be one such preimage, with $f_0^p(b_0) = c_0$ and $b_0$ in the domain of $\phi_0$. Then $b_0$ prolongs to a holomorphic function $b_t$ with $f_t^p(b_t) = c_t$. Replacing $\phi_t$ by $\phi_t(z)/\phi_t(b_t)$, we can assume $u = \phi_t(b_t) = 1$.

For small $t$, the composition $f_t^p \circ \phi_t^{-1}$ is univalent near $u = 1$. By applying a coordinate change $z \mapsto m_t(z)$, where $m_t$ is a Möbius transformation depending on $t$, we can arrange that $c_t = 0$ and that

\[ f_t^p \circ \phi_t^{-1}(u) = (u - 1) + O((u - 1)^2) \]  

(3.5)

on $B(1, \epsilon)$.

Since $\deg(f_t, 0) = d$ for $t$ near 0 by (M4), we have

\[ \phi_t \circ f_t(z) = \sum A_i(t)z^i \]  

(3.6)

\[ = A_0(t) + A_d(0)z^d(1 + O(|z| + |t|)) \]  

(3.7)

with $A_d(0) \neq 0$. Here $A_0(t) = f_t(0)$ is the $u$-coordinate of the critical value. By (M3), $c_t$ is not pre-fixed for all $t$, so there is an $r > 0$ such that

\[ A_0(t) = t^r B(t) \]  

(3.8)

where $B(0) \neq 0$.

\[ \begin{array}{c}
\text{Figure 2. Visiting the repelling fixed-point}
\end{array} \]
More precisely, for $t = t_n$ we will arrange that $c_t$ maps first close to $a_t$, then lands after $n$ iterates on $b_t$, and thus returns in $p$ further iterates to $c_t$; see Figure 2. In the $u$-coordinate system, $f_t$ is linear and $b_t = 1$, so the equation $f_t^{n+1}(c_t) = b_t$ becomes

$$\lambda^n_t A_0(t) = 1 \quad \text{when } t = t_n. \quad (3.10)$$

By the argument principle, for $n \gg 0$ this equation has a solution $t_n$ close to any root of the approximation $\lambda^n_0 t^p B(0) = 1$ obtained from (3.8). Moreover

$$t_n \sim B(0)^{-1} \lambda_0^{-n/r}$$

(verifying (3.1)), and $t_n$ satisfies (3.9) because $f_t^n(b_t) = c_t$. (There are actually be $r$ solutions for $t_n$ for a given $n$; any one of the $r$ solutions will do.)

We now turn to the estimate of $f_t^{1+n+p}(z)$ for $(t, z)$ near $(t_n, 0)$. We will assume throughout that $t = t_n + s$ and that:

$$|z| \text{ and } |s/t_n| \text{ are } O(\Lambda^{-n/(d-1)}) \quad (3.11)$$

where $\Lambda = |\lambda_0| > 1$. (To see this is the correct scale at which to work, suppose $\text{diam}(B) \approx \text{diam} f_t^{1+n+p}(B)$, where $B$ is a ball centered at $z = 0$. Then $\text{diam} f_t(B) \approx (\text{diam} B)^d$, and $f_t^n$ is expanding by a factor of about $\Lambda^n$, while $f_t^p$ is univalent, so we get $\text{diam} B \approx \Lambda^n (\text{diam} B)^d$, or $\text{diam} B \approx \Lambda^{-n/(d-1)}$. Similarly $|f_t^{1+n+p}(0)| \approx \Lambda^n (s/t_n) t_n^p = (s/t_n) = O(\text{diam} B)$ when $s$ is as above.)

It is also convenient to set

$$\tilde{\Lambda} = \min(\Lambda^{1/(d-1)}, \Lambda^{1/r}) > 1,$$

so that we may assert:

$$z \text{ and } t \text{ are } O(\tilde{\Lambda}^{-n}). \quad (3.12)$$

By (3.11) the $n$ iterates of $f_t(z)$ lie within the domain of linearization, so by (3.7) we have

$$\phi_t \circ f_t^{1+n}(z) = \lambda^n_t A_0(t) + \lambda^n_t A_0(d) z^{d} (1 + O(|z| + |t|)).$$

The first term is approximately 1. Indeed, $\lambda^n_t = \lambda^n_{t_n} (1 + O(ns))$, so by (3.8) we have

$$\lambda^n_t A_0(t) = \lambda^n_{t_n} (t_n + s)^r B(t_n + s)$$

$$= \lambda^n_{t_n} (1 + O(ns)) \cdot t_n^r \left(1 + \frac{s}{t_n} \right)^r \cdot B(t_n)(1 + O(s))$$

$$= \lambda^n_{t_n} A_0(t_n) \left(1 + r \frac{s}{t_n} + O((s/t_n)^2) + O(ns) \right)$$

$$= 1 + r \frac{s}{t_n} + O((s/t_n)^2) + O(ns)$$
by (3.10). Similarly, $\lambda^n_t = \lambda^n_0(1 + O(t))$, so
\[
\phi_t \circ f_t^{1+n}(z) - 1 = \\
\lambda^n_0 A_0(d) z^d(1 + O(|z| + |nt|)) + r \frac{s}{t_n} + O((s/t_n)^2) + O(ns) = \\
\lambda^n_0 A_0(d) z^d + r \frac{s}{t_n} + O(n\Lambda^{-n/(d-1)}\tilde{\Lambda}^{-n}),
\]

using (3.11) and (3.12). The expression above as a whole is $O(\Lambda^{-n/(d-1)})$, so composing with the univalent map $f_t^p \circ \phi_t^{-1}$ introduces (by (3.5)) an additional error of size $O(\Lambda^{-2n/(d-1)})$, which is already accounted for in the $O(\cdot)$ above. Thus the expression above also represents $f_t^{1+n+p}(z)$.

Finally we make a linear change of coordinates of the form $z \mapsto \alpha_n z$, conjugating the expression above to
\[
f_t^{1+n+p}(z) = \alpha_n^{1-d} \lambda^n_0 A_0(d) z^d + \alpha_n r \frac{s}{t_n} + O(n\alpha_n \Lambda^{-n/(d-1)}\tilde{\Lambda}^{-n}).
\]
Setting $\alpha_n = (\lambda^n_0 A_0(d))^{1/(d-1)}$ to normalize the coefficient of $z^d$, we have $|\alpha_n| \asymp \Lambda^{n/(d-1)}$ and thus:
\[
f_t^{1+n+p}(z) = z^d + \alpha_n r \frac{s}{t_n} + O(n\Lambda^{-n})
\]
\[
= z^d + \xi + O(\epsilon_n)
\]
with $t = t_n(1 + \gamma_n \xi)$, $\gamma_n$ and $\epsilon_n$ as in (3.2) and (3.3). Notice that if $|z|$ and $|\xi|$ are bounded by $R$ in the expression above, then (3.11) is satisfied in our original coordinates. Reindexing $n$, we obtain the Theorem. ■

4 Small Mandelbrot sets

We now show the Misiurewicz cascade leads to small Mandelbrot sets in parameter space. From this we deduce Theorems 1.1 and 1.3 of the Introduction.

**Hybrid conjugacy.** Let $g_1, g_2$ be polynomial-like maps of the same degree. A hybrid conjugacy is a quasiconformal map $\phi$ between neighborhoods of $K(g_1)$ and $K(g_2)$ such that $\phi \circ g_1 = g_2 \circ \phi$ and $\overline{\partial} \phi|K(g_1) = 0$. We say $g_1$ and $g_2$ are hybrid equivalent if such a conjugacy exists. By a basic result of Douady and Hubbard, every polynomial-like map $g$ of degree $d$ is hybrid equivalent to a polynomial of degree $d$, unique up to affine conjugacy if $K(g)$ is connected [DH, Theorem 1].
Theorem 4.1 Let \((f, c)\) be a degree \(d\) Misiurewicz bifurcation. Then the parameter space \(\Delta\) contains quasiconformal copies \(\mathcal{M}_d^n\) of the degree \(d\) Mandelbrot set \(M_d\), converging to the origin, with \(\partial \mathcal{M}_d^n\) contained in the bifurcation locus \(B(f)\).

More precisely, for all \(n \gg 0\) there are homeomorphisms
\[
\phi_n : M_d \to \mathcal{M}_d^n \subset \Delta
\]
such that:

1. \(f_t^n\) is hybrid equivalent to \(z^d + \xi\) whenever \(t = \phi_n(\xi)\);
2. \(d(0, \mathcal{M}_d^n) \asymp |\lambda_0|^{-n/r}\);
3. \(\text{diam}(\mathcal{M}_d^n)/d(0, \mathcal{M}_d^n) \asymp |\lambda_0|^{-n/(d-1)}\);
4. \(\phi_n\) extends to a quasiconformal map of the plane with dilatation bounded by \(1 + O(\epsilon_n)\); and
5. \(\psi_n^{-1} \circ \phi_n(\xi) = \xi + O(\epsilon_n)\), where \(\psi_n(\xi) = t_n(1 + \gamma_n \xi)\).

The notation is from (3.1) – (3.3).

We begin by recapitulating some ideas from [DH]. Let \(\Delta(R) = \{z : |z| < R\}\), and let
\[
g_{\xi}(z) = z^d + \xi + h(\xi, z)
\]
be a holomorphic family of mappings defined for \((\xi, z) \in \Delta(R) \times \Delta(R)\), where \(R > 10\) and \(g_{\xi}(0) = 0\). Let \(\mathcal{M} \subset \Delta(R)\) be the set of \(\xi\) such that the forward orbit \(g_{\xi}^n(0)\) remains in \(\Delta(R)\) for all \(n > 0\).

Lemma 4.2 There is a \(\delta > 0\) such that if \(\sup |h(\xi, z)| = \epsilon < \delta\) then there is a homeomorphism
\[
\phi : M_d \to \mathcal{M}
\]
such that for all \(\xi \in M_d\), \(g_{\phi(\xi)}\) is hybrid equivalent to \(z^d + \xi, |\phi(\xi) - \xi| < O(\epsilon)\), and \(\phi\) extends to a \(1 + O(\epsilon)\)-quasiconformal map of the plane.

Proof. Let \(p_{\xi}(z) = z^d + \xi\). Since \(R > 10\) we have \(M_d \subset \Delta(R)\) and \(K(p_{\xi}) \subset \Delta(R)\) for all \(\xi \in M_d\); indeed these sets have capacity one, so their diameters are bounded by \(4\) [Ah]. In addition, for \(\xi \in M_d\) the map \(p_{\xi} : U \to \Delta(R)\) is polynomial-like, where \(U = p_{\xi}^{-1}(\Delta(R))\). By continuity, when \(\sup |h|\) is sufficiently small, \(\mathcal{M}\) is compact and \(g_{\xi}\) is polynomial-like for all \(\xi \in \mathcal{M}\).

By results of Douady and Hubbard, we can also choose \(\delta\) small enough that \(|h| < \delta\) implies there is a homeomorphism
\[
\phi : M_d \to \mathcal{M}
\]
such that \( g_{\phi(\xi)} \) is hybrid equivalent to \( z^d + \xi \) [DH, Prop. 21].

Now assume \(|h| < \epsilon < \delta\). For \( t \in \Delta \) let \( M_t \) denote the parameters where the critical point remains bounded for the family

\[
g_{\xi,t} = z^d + \xi + t\frac{\delta}{\epsilon} h(\xi, z),
\]

and define \( \phi_t : M_d \rightarrow M_t \) as above. Then \( \phi_t \) is a family of injections, with \( \phi_0(z) = z \), and \( \phi_t(\xi) \) is a holomorphic function of \( t \) for every \( \xi \). (For example this is clear at \( \xi \in \partial M_d \) because Misiurewicz points are dense in \( \partial M_d \); for the general case see [DH, Prop. 22].)

By a theorem of Słodkowski [Sl] (cf. [Dou], [BR]), \( \phi_t(z) \) prolongs to a holomorphic motion of the entire plane, and its complex dilatation \( \mu_t = \overline{\partial} \phi_t / \partial \phi_t \) gives a holomorphic map of the unit disk into the unit ball in \( L^\infty(\hat{\mathbb{C}}) \). By the Schwarz lemma, \( \|\mu_t\|_\infty \leq |t| \); since \( \phi_t = \phi_t / \epsilon \), we obtain a quasiconformal extension of \( \phi \) with dilatation \( 1 + O(\epsilon) \). The bound on \( |\phi(\xi) - \xi| \) similarly results by applying the Schwarz Lemma to the map \( \Delta \rightarrow \hat{\mathbb{C}} - \{0,1,\infty\} \) given by \( t \mapsto \phi_t(\xi) \), once three points have been normalized to remain fixed during the motion.

**Proof of Theorem 4.1.** Fix \( R > 10 \). For all \( n \gg 0 \), Theorem 3.1 provides a family of rational maps of the form

\[
g_{\xi}(z) = f^n_t(z) = z^d + \xi + O(\epsilon_n)
\]

defined for \( (\xi, z) \in \Delta(R) \times \Delta(R) \), where \( t = \psi_n(\xi) = t_n(1 + \gamma_n \xi) \). The preceding Lemma gives homeomorphisms \( \tilde{\phi}_n : M_d \rightarrow \tilde{M}_{\xi}^0 \subset \Delta(R) \) for all \( n \gg 0 \). Setting \( \phi_n = \tilde{\phi}_n \circ \tilde{\phi}_n \), the Theorem results from the Lemma and the bounds (3.1) – (3.3).

**Example.** The quadratic family \((f, c) = (z^2 + t - 2, 0)\) is a Misiurewicz bifurcation of degree \( d = 2 \), with \( \lambda_0 = 2 \) and \( r = 1 \). Thus \( M_2 \) contains small copies \( \mathcal{M}_2 \) of itself near \( c = -2 \), with \( d(M_2) \times -2) \sim 4^{-n} \) and \( \text{diam} \mathcal{M}_2 \sim 16^{-n} \).

**Consequences.** Assembling the preceding results, we may now prove the Theorems stated in the Introduction. Here is a more precise form of Theorem 1.1:

**Theorem 4.3** Let \( f \) be a holomorphic family of rational maps over the unit disk with bifurcations. Then there is a nonempty list of degrees

\[
D \subset \{2, 3, \ldots, 2^{\deg(f_t) - 2}\}
\]

such that for any \( \epsilon > 0 \) and \( d \in D \), \( B(f) \) contains the image of \( \partial M_d \) under a \((1 + \epsilon)\)-quasiconformal map.

If the critical points of \( f \) are marked \( \{c_1^t, \ldots, c^n_t\} \) such that
\( (i \leq N) \iff c^i \text{ is active and } c^i_t \text{ is unramified for some } t, \)

then we may take

\[
D = \{ \inf \sup_k \deg(f^k_t, c^i_t) : i \leq N \}.
\]

Proof. Let \( B_0 = B(f) \). After a base change we can assume that \( f \) is a local bifurcation with critical points marked as above. By Proposition 2.4, there is at least one active, unramified critical point, so \( N \geq 1 \). For any \( i \leq N \), we can make a base change so \( c^i_t \) is active and unramified; then by Proposition 2.5, a further base change makes \((f^n, c^i_t)\) into a degree \( d \) Misiurewicz bifurcation.

Let \( d_i = \inf \sup_k \deg(f^k_t, c^i_t) \). We claim \( d = d_i \leq 2^{2\deg(f_t) - 2} \). Indeed, \( \deg(f^n_t, c^i_t) \) assumes its minimum outside a discrete set, and it is equal to \( d \) near \( t = 0 \), so \( d_i \geq d \). On the other hand, \( c^i_0 \) lands on a repelling periodic cycle, so \( \deg(f^k_0, c^i_0) = d \) for all \( k > n \), and therefore \( d_i \leq d \). Finally \( d \) is largest if \( c^i \) hits all the other critical points of \( f \) before reaching the repelling cycle; in this case \( d = (p_1 + 1)(p_2 + 1) \cdots (p_m + 1) \) for some partition \( p_1 + p_2 + \cdots + p_m = 2d - 2 \). The product is maximized by the partition \( 1 + 1 + \cdots + 1 \), so \( d \leq 2^{2\deg(f_t) - 2} \).

By Theorem 4.1, the bifurcation locus \( B(f) \) contains almost conformal copies \( \partial M^d_{n,i} \) of \( \partial M^d_t \) accumulating at \( t = 0 \), with \( \text{diam}(\partial M^d_{n,i}) \ll d(0, \partial M^d_t) \). Letting \( \phi : \Delta \to \Delta \) denote the composition of all the base-changes occurring so far, we have \( B(f) = \phi^{-1}(B_0) \). Then \( \phi \) is univalent and nearly linear on \( \partial M^d_t \) for \( n \gg 0 \), so \( \phi(\partial M^d_{n,i}) \subset B_0 \) is a \((1 + \epsilon)\)-quasiconformal copy of \( \partial M^d_t \).

Let \( \text{Rat}_d \) be the space of all rational maps of degree \( d \); it is a Zariski-open subset of \( \mathbb{P}^{2d+1} \). We now make precise the statement that a generic family contains a copy of the standard Mandelbrot set.

**Corollary 4.4** There is a countably union of proper subvarieties \( R \subset \text{Rat}_d \) such that for any local bifurcation, either \( f_t \in R \) for all \( t \), or \( B(f) \) contains a copy of \( \partial M^d_2 \).

Proof. On a finite branched cover \( X \) of \( \text{Rat}_d \), the critical points of \( f \in \text{Rat}_d \) can be marked \( \{c^1(f), \ldots, c^{2d-2}(f)\} \). Clearly \( \deg(f^n, c_i(f)) = 2 \) outside a proper subvariety \( V_{n,i} \) of \( X \). Let \( R \) be the union of the images of these varieties in \( \text{Rat}_d \), and apply the preceding argument to see \( D = \{2\} \) if some \( f_t \notin R \).

**Proof of Theorem 1.3.** The proof follows the same lines as that of Theorem 4.3; to get \( \text{mod}(V-U) \) large one takes \( R \) large in Theorem 3.1. Thus Theorem 1.3 also holds for all \( d \in D \).
5 Hausdorff dimension

In this section we prove Corollary 1.8: for any holomorphic family \( f \) of rational maps over a complex manifold \( X \), we have \( \text{H.dim}(B(f)) = \text{H.dim}(X) \) if \( B(f) \neq \emptyset \).

Recall that the Hausdorff dimension of a metric space \( X \) is the infimum of the set of \( \delta \geq 0 \) such that there exists coverings \( X = \bigcup X_i \) with \( \sum (\text{diam } X_i)^\delta \) arbitrarily small.

Lemma 5.1 Let \( Y \) be a metric space, \( X \) a subset of \( Y \times [0, 1] \). Then
\[
\text{H.dim}(X) \geq 1 + \inf \text{H.dim}(X_t)
\]
where \( X_t = \{ y : (y, t) \in X \} \).

Here \( Y \times [0, 1] \) is given the product metric.

**Proof.** Fix \( \delta \) with \( \delta + 1 > \text{H.dim}(X) \). For any \( n \) there is a covering \( X \subset \bigcup B(y_i, r_i) \times I_i \) with \( |I_i| = r_i \) and \( \sum r_i^{\delta + 1} < 4^{-n} \). Note that
\[
X_t \subset \bigcup_{t \in I_i} B(y_i, r_i)
\]
and
\[
\int_0^1 \sum_{t \in I_i} r_i^\delta \, dt = \sum r_i^{\delta + 1} < 4^{-n}.
\]
Let \( E_n \) be the set of \( t \) where the integrand exceeds \( 2^{-n} \); then \( \sum m(E_n) < \sum 2^{-n} < \infty \). Thus almost every \( t \) belongs to at most finitely many \( E_n \), so almost every \( X_t \) admits infinitely many coverings with \( \sum r_i^{\delta} < 2^{-n} \to 0 \). Therefore \( \delta \geq \inf \text{H.dim}(X_t) \), and the Theorem follows.

The Lemma above is related to the product formula
\[
\text{H.dim}(X \times Y) \geq \text{H.dim}(X) + \text{H.dim}(Y);
\]
see [Fal, Ch. 5] and references therein.

**Proof of Corollary 1.8.** Suppose \( B(f) \neq \emptyset \). Then there is a \( t_0 \in B(f) \) and a locally parameterized periodic point \( a(t) \) of period \( n \) such that \( a(t) \) changes from attracting to repelling near \( t_0 \) [MSS], [Mc2, §4.1]. More formally this means the multiplier \( \lambda(t) = (f^n)'(a(t)) \) is not locally constant and \( |\lambda(t_0)| = 1 \).

Choosing local coordinates we can reduce to the case \( X = \Delta^n \) and \( t_0 = 0 \). Let \( \Delta_s = \Delta \times \{ s \} \) for \( s \in \Delta^{n-1} \). For coordinates in general position, \( \lambda(t) \) is nonconstant on \( \Delta_0 \). Shrinking the \( \Delta^{n-1} \) factor, we can also assume \( a(t) \) changes from attracting to repelling in the family \( f|\Delta_s \) for all \( s \). Then
\[
B(f)_s = B(f) \cap \Delta_s \supset B(f|\Delta_s) \neq \emptyset
\]
and $\text{H. dim } B(f|\Delta_s) = 2$ by Corollary 1.6. Applying the Lemma above to $B(f) \subset \Delta \times \Delta^{n-1}$ we find

$$\text{H. dim}(B(f)) \geq (2n - 2) + \inf_s \text{H. dim } B(f)_s = 2n = \text{H. dim}(X).$$

\[ \square \]

References


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