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Citation
McMullen, Curtis T. 1998. Complex earthquakes and Teichmuller theory. Journal of the American Mathematical Society 11: 283–320. Revised 2005.

Published Version
doi:10.1090/S0894-0347-98-00259-8

Complex earthquakes and Teichmüller theory

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4 March, 1996

Abstract

It is known that any two points in Teichmüller space are joined by an earthquake path. In this paper we show any earthquake path $\mathbb{R} \to T(S)$ extends to a proper holomorphic mapping of a simply-connected domain D into Teichmüller space, where $\mathbb{R} \subset D \subset \mathbb{C}$. These complex earthquakes relate Weil-Petersson geometry, projective structures, pleated surfaces and quasifuchsian groups.

Using complex earthquakes, we prove grafting is a homeomorphism for all 1-dimensional Teichmüller spaces, and we construct bending coordinates on Bers slices and their generalizations.

In the appendix we use projective surfaces to show the closure of quasifuchsian space is not a topological manifold.

Contents

*Research partially supported by the NSF.

1 Introduction

A hyperbolic Riemann surface is both a space of constant curvature and a complex manifold. These two natures are reflected in Teichmüller theory.

The complex nature of surfaces gives rise to extremal length, holomorphic quadratic differentials, the complex structure on Teichmüller space, quasiconformal deformations and Teichmüller geodesics. The geometric nature of surfaces gives rise to hyperbolic length, geodesic laminations, the symplectic structure on Teichmüller space, twist deformations and earthquake paths. While each theory is internally rich, important features also arise from their interplay, such as the Kähler metric on Teichmüller space.

In this paper we study *complex earthquakes*, another interface between the complex and geometric aspects of Teichmüller space. Real earthquakes include the classical Fenchel-Nielsen twist deformations around simple closed hyperbolic geodesics; they define paths in Teichmüller space that are easily understood from a geometric point of view. Complex earthquakes are obtained from these paths by analytic continuation.

Our main result shows the maximal analytic continuation of an earthquake path gives a proper holomorphic map of a disk into Teichmüller space. These *earthquake disks* can be compared to complex Teichmüller geodesics, which are also proper holomorphic disks.

The theory of earthquake disks brings into focus relations between Weil-Petersson geometry, projective structures and their holonomy, pleated surfaces and quasifuchsian groups. It also allows complex methods such as the Schwarz lemma and positivity of intersections to be brought to bear.

We give two applications to demonstrate the utility of earthquake disks. First, we show *grafting* (the imaginary counterpart of twisting) defines a homeomorphism for all one-dimensional Teichmüller spaces. This theorem supports the conjectural rigidity of 3-dimensional cone-manifolds. Second, we use earthquake disks to construct bending coordinates on Bers' embedding for Teichmüller space. These are 'polar coordinates' centered at the Fuchsian basepoint, with angular part a projective lamination and radial part the length of the bending locus.

In the Appendix we use related methods to demonstrate the topological complexity of the boundary of quasifuchsian space.

Statement of results. Let S be a compact oriented smooth surface or orbifold of negative Euler characteristic. Attached to S one has:

1. The *Teichmüller space* $T(S)$, parameterizing hyperbolic Riemann surfaces X marked by S;

2. The bundle $P(S) \to T(S)$ of complex projective structures on S (Riemann surfaces whose transition functions are Möbius transformations);
3. The variety $V(S)$ of irreducible representations of $\pi_1(S)$ in $PSL_2(\mathbb{C})$ that send $\pi_1(\partial S)$ to parabolics; and
4. The space of *measured laminations* $\mathcal{ML}(S)$; this is the completion of the space of weighted simple closed curves on S.

Given a simple closed geodesic γ on a hyperbolic surface $X \in T(S)$, we can construct a new surface by performing a right twist or *earthquake* along γ. The surface $\mathrm{tw}_{t\gamma}(X) \in T(S)$ is obtained by cutting along γ and re-gluing after twisting distance t to the right.

The related operation of *grafting* gives a new surface $\mathrm{gr}_{s\gamma}(X)$ by cutting along γ and inserting a flat cylinder of height s and circumference equal to the length of γ on X. Both operations extend by continuity to general measured laminations.

Theorem 1.1 (Earthquake disks) *For any lamination $\lambda \neq 0$ in $\mathcal{ML}(S)$ and any $X \in T(S)$, the earthquake path $\mathbb{R} \to T(S)$ given by*

$$t \mapsto \mathrm{tw}_{t\lambda}(X)$$

extends to a proper holomorphic map $D \to T(S)$, where D is a simply-connected domain and $\overline{\mathbb{H}} \subset D \subset \mathbb{C}$.

For $s > 0$ the complex earthquake map is given by

$$t + is \mapsto \mathrm{gr}_{s\lambda}(\mathrm{tw}_{t\lambda}(X)).$$

Its analytic continuation to negative values of s is defined via quasifuchsian groups. The proof that D is a disk uses:

Theorem 1.2 (Convexity) *The space of quasifuchsian groups is disk-convex in the representation variety: if*

$$f : \overline{\Delta} \to V(S)$$

is a holomorphic disk with $f(\partial\Delta) \subset QF(S)$, then $f(\Delta) \subset QF(S)$.

One-dimensional Teichmüller spaces. For Theorems 1.3 to 1.6 we assume $\dim_{\mathbb{C}} T(S) = 1$. Equivalently, we assume S is a surface or orbifold with s singular points, b boundary components and genus g, where $s+b+3g = 4$. By checking the degree of an earthquake disk we find:

Theorem 1.3 (Isomorphism) *The complex earthquake map $D \to T(S)$ is biholomorphic, sending $\overline{\mathbb{H}}$ to $\{Z : \ell_\lambda(Z) \le \ell_\lambda(X)\}$.*

Here $\ell_\lambda(X)$ denotes the length of the lamination λ in the hyperbolic metric on X.

From Theorem 1.3 and a Schwarz lemma argument we deduce:

Theorem 1.4 (Grafting bijection) *For any $\lambda \in \mathcal{ML}(S)$, the grafting map*

$$\mathrm{gr}_\lambda : T(S) \to T(S)$$

sending X to $\mathrm{gr}_\lambda(X)$ is a homeomorphism.

Previously grafting was known to be a homeomorphism in the countably many cases where $\lambda = \sum 2\pi m_i \gamma_i$ for integral weights $m_i > 0$ on disjoint simple closed curves γ_i [Tan]. Bijectivity of grafting is related to rigidity of cone-manifolds in §8.

Polar coordinates on a Bers slice. The space of quasifuchsian groups forms an open cell $QF(S) \subset V(S)$, admitting a natural biholomorphic parameterization

$$Q : T(S) \times T(\overline{S}) \to V(S)$$

such that $Q(X, Y)$ corresponds to a hyperbolic 3-manifold with conformal boundary $X \sqcup Y$. A *Bers slice* $B_Y \subset QF(S)$ is a model for Teichmüller space obtained by holding one factor fixed:

$$B_Y = \{Q(X, Y) : X \in T(S)\}.$$

Using the relation between earthquake disks and quasifuchsian groups, we obtain:

Theorem 1.5 (Bending coordinates) *Let $B_Y \subset V(S)$ be a Bers slice with basepoint $N = Q(\overline{Y}, Y)$. Then there is a natural homeomorphism*

$$(B_Y - N) \to \mathbb{P}\mathcal{ML}(S) \times (0, 1)$$

given by

$$Q(X, Y) \mapsto \left([\beta], \frac{\ell_\beta(Q(X, Y))}{\ell_\beta(Y)}\right),$$

where $Q(X, Y)$ has bending laminations $(\beta, \overline{\beta})$.

The inequality

$$\ell_\beta(X) \leq \ell_\beta(Q(X,Y)) \leq \ell_\beta(Y)$$

holds for any quasifuchsian manifold with bending laminations $(\beta, \overline{\beta})$ (Corollary 3.5), showing the length ratio in the Theorem is indeed in $(0,1)$.

As Y varies, the Bers embeddings $T(S) \to B_Y \subset V(S)$ range in a compact family of holomorphic mappings, allowing one to construct a generalized Bers slice by taking a limit. These slices also admit limiting bending coordinates.

Theorem 1.6 (Limit Bers slice) *Suppose $Y_n \to [\lambda]$ in $T(S) \cup \mathbb{P}\mathcal{ML}(S)$ and $B_{Y_n} \to B$. Then there is a homeomorphism*

$$B \to (\mathbb{P}\mathcal{ML}(S) - [\lambda]) \times \mathbb{R}^+,$$

given by

$$M = \lim Q(X, Y_n) \in B \;\mapsto\; \left([\beta],\ \frac{\ell_\beta(M)}{i(\beta,\lambda)}\right),$$

where β is the bending lamination of M.

When $[\lambda]$ is supported on a simple closed curve γ, the limit slice $B \subset \partial QF(S)$ is the Maskit model for $T(S)$; its points are geometrically finite groups with γ pinched to a rank-one cusp. The parameterization of the Maskit model by bending data as above was obtained by Keen and Series [KS].

When $[\lambda]$ is an irrational lamination, the limit Bers slice $B \subset \partial QF(S)$ consists of geometrically infinite groups with ending lamination λ.

Comparison with dynamics. A one-dimensional Bers slice B_Y in some ways resembles the family of iterated quadratic polynomials $f_\alpha(z) = \alpha z + z^2$, $|\alpha| < 1$. The ray in B_Y corresponding to a fixed bending lamination $[\beta]$ is like the ray of polynomials with $\arg(\alpha)$ fixed; the Fuchsian basepoint $Q(\overline{Y}, Y)$ corresponds to the map $f_0(z) = z^2$ whose Julia set is a circle; and the rational rays in either picture land at dynamical systems with parabolic points.

The topology of quasifuchsian space. In the Appendix we use related methods to give examples of the subtlety of variations of projective structures and algebraic limits. We show:

- The closure of quasifuchsian space $QF(S) \subset V(S)$ is not a topological manifold with boundary;

- There is a point $[\rho]$ on the boundary of a Bers slice such that every small neighborhood of ρ meets $QF(S)$ in a disconnected set; and
- The set of projective surfaces with univalent developing maps is not open among those with discrete holonomy.

We also provide a computer illustration of a slice through $QF(S)$. These results begin to reveal the intricacies of quasifuchsian space, which has hitherto seemed quite tame compared to other families of conformal dynamical systems.

Historical remarks and references. The theory of projective structures and Kleinian groups emerged, in the era of Fuchs, Schwarz, Poincaré and Klein, from the study of ordinary differential equations in the complex domain. See [Gr] for an extensive bibliography. Expository presentations of the theory of projective structures can be found in [Gun2] and [Mat].

The twist deformation was introduced by Fenchel and Nielsen as part of their geometric coordinates for Teichmüller space, and connected with the Weil-Petersson symplectic form by Wolpert [Wol2], [Wol3]. Modern treatments of Fenchel-Nielsen coordinates are given in [IT] and [Th3, §4.6]; see also [Gd], [Nag] and [Le] for background on Teichmüller theory.

Twisting was generalized from curves to laminations by Thurston, who showed any two points in Teichmüller space are related by a unique right earthquake [Th2]. This result was used by Kerckhoff in his solution to the Nielsen realization problem [Ker1], which contains a detailed discussion of earthquakes.

The grafting construction appears in works of Goldman, Hejhal and Maskit on 'exotic' projective structures [Gol], [Hej], [Msk]. These works concern grafting a 2π-annulus along a simple closed curve.

The bending lamination for the convex hull of hyperbolic 3-manifold was introduced by Thurston [Th1]. A detailed development is given by Epstein and Marden in [EpM]. Thurston also showed the bending lamination can be defined for an arbitrary projective structure, and thus every projective structure is given by a canonical grafting along a lamination. This result, Theorem 2.4 below, is presented by Kamishima and Tan in [KaT].

A precursor to the complex earthquakes we study here are the 'quakebend cocycles' of [EpM]. These cocycles give the holonomy of a complex earthquake, so a complex earthquake can be viewed as a lifting of the bending deformation to the space of projective structures. The present paper obtains the analytic continuation of this lift to negative bending, the simple-connectedness of the domain of the analytic continuation, and the properness of the resulting holomorphic disk. The last two properties are essential to applications and are shared by Teichmüller geodesics.

Our original motivation for developing complex earthquakes was the application to grafting given in Theorem 1.4 above. This application, and the examples of the Appendix, were in turn inspired by work of Shiga and Tanigawa on projective structures with discrete holonomy [ST], [Tan].

A second motivation was to show that certain totally degenerate groups are determined by their ending laminations and the length of their bending laminations (Theorem 1.6). These groups were computed explicitly in [Mc2, §3.7] and the present paper shows their uniqueness (see §7).

It is natural to ask if Theorems 1.4 and 1.5 hold true when $\dim T(S) > 1$. In higher-dimensional Teichmüller spaces one can construct earthquake polydisks $\overline{\mathbb{H}}^n \to T(S)$ by twisting and grafting along laminations with many transverse invariant measures (such as systems of simple closed curves). It seems difficult, however, to describe the image of these polydisks in terms of hyperbolic length, as in Theorem 1.3.

Outline of the paper. Detailed definitions and basic properties of the spaces and maps with which we will be concerned are provided in §2. The geometry of grafting and pleating is developed in §3 and §4, and in §5 we prove Theorem 1.2. These results are assembled in §6 to prove the existence of proper earthquake disks (Theorem 1.1). This section also contains an explicit calculation of an earthquake disk for the punctured torus. The results on one-dimensional Teichmüller spaces stated as Theorems 1.3 – 1.6 are deduced in §7. We conclude in §8 by relating grafting to hyperbolic cone-manifolds.

The topology of quasifuchsian space is discussed in the Appendix.

Acknowledgements. I would like to thank H. Tanigawa and D. Canary for stimulating discussions related to this work.

2 Laminations and deformations

This section briefly summarizes definitions and known results about the spaces and mappings with which we will be concerned. For detailed accounts the reader is referred to the references of the Introduction, especially [Ker1], [EpM] and [KaT]. Our principal goal is to introduce the complex earthquake map

$$\mathrm{Eq} : D(S) \to P(S)$$

and its natural domain of definition, $D(S) \subset \mathcal{ML}_{\mathbb{C}}(S) \times T(S)$.

We begin by defining the spaces $T(S)$, $P(S)$, $V(S)$ and $\mathcal{ML}(S)$.

$\boldsymbol{T(S)}$**.** The Teichmüller space $T(S)$ consists of pairs (f, X), where X is a hyperbolic Riemann surface of finite area and $f : \mathrm{int}(S) \to X$ is a diffeomorphism. The canonical orientation of X is required to agree with the given orientation of S. Two pairs (f_1, X_1) and (f_2, X_2) represent the same point in $T(S)$ if there is a holomorphic isomorphism $\alpha : X_1 \to X_2$ such that $\alpha \circ f_1$ is isotopic to f_2.

The space $T(S)$ is a finite-dimensional complex manifold, diffeomorphic to a cell.

$\boldsymbol{P(S)}$**.** A *projective structure* on a Riemann surface $X \in T(S)$ is given by an atlas of analytic charts whose transition functions are Möbius transformations. A canonical projective structure is provided by the Fuchsian uniformization $X = \mathbb{H}/\Gamma_X$.

We let $P(S) \to T(S)$ denote the bundle whose fiber over X consists of all projective structures on X compatible with the given conformal structure and standard in the cusps. The latter condition means a neighborhood of each puncture of X is projectively isomorphic to a neighborhood of the origin in Δ^*, with respect to the Fuchsian projective structure $\Delta^* = \mathbb{H}/\langle z \mapsto z+1 \rangle$.

The difference between two projective structures is measured by a holomorphic quadratic differential on X (via the Schwarzian derivative) with at worst simple poles at the punctures (by our condition on the cusps). Thus $P(S)$ is a holomorphic bundle of finite-dimensional affine spaces over $T(S)$.

Note: the section of $P(S) \to T(S)$ which assigns to $X \in T(S)$ its Fuchsian projective structure is *not* holomorphic.

$\boldsymbol{V(S)}$**.** The space $V(S)$ consists of equivalence classes of irreducible representations

$$\rho : \pi_1(S) \to PSL_2(\mathbb{C})$$

such that $\rho(g)$ is parabolic for every $g \in \pi_1(\partial S)$. Representations that are conjugate in $PSL_2(\mathbb{C})$ define the same point in $V(S)$; thus

$$V(S) = \mathrm{Hom}^{\mathrm{irr}}_{\mathrm{parab}}(\pi_1(S), PSL_2(\mathbb{C}))/PSL_2(\mathbb{C}).$$

This space is a complex manifold [Gun1], [Fal], and

$$\dim V(S) = \dim P(S) = 2 \dim T(S).$$

$\boldsymbol{\mathcal{ML}(S)}$**.** Let $\mathcal{S}$ be the set of isotopy classes of essential, nonperipheral simple closed curves on S. (A curve is *peripheral* if it is parallel to the boundary.) For $\alpha, \beta \in \mathcal{S}$ the *intersection number* $i(\alpha, \beta)$ is the minimum number of points in which representatives of α and β must intersect.

The space $\mathcal{ML}(S)$ of *measured laminations* on S is the closure of $\mathbb{R}^+ \times \mathcal{S}$ in $\mathbb{R}^{\mathcal{S}}$ with respect to the embedding

$$(t, \alpha) \mapsto \langle t \cdot i(\alpha, \beta) \ : \ \beta \in \mathcal{S} \rangle.$$

We have $\mathcal{S} \subset \mathcal{ML}(S)$ by the map $\alpha \mapsto (1, \alpha)$. The intersection pairing extends to a continuous map $i : \mathcal{ML}(S) \times \mathcal{ML}(S) \to \mathbb{R}$.

A *geodesic lamination* on a hyperbolic surface is a closed set given as a disjoint union of complete, simple geodesics. On any surface $X \in T(S)$, any $\lambda \in \mathcal{ML}(S)$ can be represented by a compact, transversally measured geodesic lamination. The space of such compactly supported laminations is also sometimes denoted $\mathcal{ML}_0(S)$. An extensive discussion of measured laminations can be found, for example, in [Th1], [EpM], [Bon2] and [Ot].

To allow complex multiples of laminations, we let

$$\mathcal{ML}_{\mathbb{C}}(S) \ = \ \{(\lambda, z) \in \mathcal{ML}(S) \times \mathbb{C}\} / \langle (\lambda, tz) \sim (t\lambda, z) \ : \ t \in \mathbb{R}^+ \rangle.$$

This "tensor product" can be formed with other sets of scalars such as $\mathbb{R}$ and $\overline{\mathbb{H}}$, and we have

$$\mathcal{ML}(S) \subset \mathcal{ML}_{\mathbb{R}}(S) \subset \mathcal{ML}_{\overline{\mathbb{H}}}(S) \subset \mathcal{ML}_{\mathbb{C}}(S).$$

For $\lambda \in \mathcal{ML}_{\mathbb{C}}(S)$ the real and imaginary parts $\operatorname{Re} \lambda$, $\operatorname{Im} \lambda$ are multiples of a single real lamination and satisfy $\lambda = \operatorname{Re} \lambda + i \operatorname{Im} \lambda$.

The developing map for a projective structure. A projective surface $X \in P(S)$ determines a holomorphic *developing map*

$$\delta : \widetilde{X} \to \widehat{\mathbb{C}}$$

such that the lifted projective structure on the universal cover $\widetilde{X}$ agrees with the standard structure on $\widehat{\mathbb{C}}$ pulled back via δ. The map δ is unique up to an automorphism of $\widehat{\mathbb{C}}$, so we have a *holonomy homomorphism*

$$\rho : \pi_1(X) \to PSL_2(\mathbb{C})$$

such that $\delta(g \cdot x) = \rho(g) \cdot \delta(x)$. The *projective holonomy map*

$$\mu : P(S) \to V(S)$$

is defined by $X \mapsto [\rho]$.

Theorem 2.1 (Hejhal) *The holonomy map is a complex analytic local homeomorphism.*

See [Hej]; other treatments appear in [Ea], [Hub] and [Fal].

Lengths. For $X \in T(S)$ and $\lambda \in \mathcal{ML}(S)$, we denote by $\ell_\lambda(X)$ the length of λ in the hyperbolic metric on X. By [Ker2] we have:

Theorem 2.2 *The function $\ell : \mathcal{ML}(S) \times T(S) \to \mathbb{R}$ is continuous, and $\ell_\lambda(X)$ is a real-analytic function of X.*

For a simple closed curve γ, $\ell_{t\gamma}(X)$ is just t times the length of the geodesic representative of γ.

Twisting. Next we discuss deformations defined using laminations. We begin with the twist deformation

$$\mathrm{tw} : \mathcal{ML}_{\mathbb{R}}(S) \times T(S) \to T(S).$$

For a simple closed geodesic γ on a hyperbolic surface $X \in T(S)$, $\mathrm{tw}_{t\gamma}(X) \in T(S)$ is constructed by cutting X along γ, twisting distance t to the right, and re-gluing.

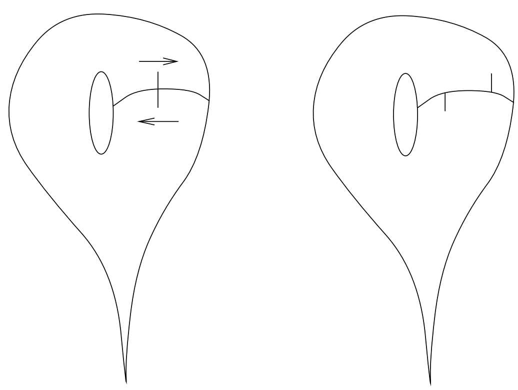

Figure 1. A right twist on a punctured torus.

By twisting to the *right* we mean that an observer standing on one side of γ will see the other side of the surface move to the right during the twist (see Figure 1). The notion of a right twist requires an orientation of X (provided by the complex structure) but not of γ.

A local model for twisting is the map on $\mathbb{H}$ given by

$$z \mapsto \begin{cases} e^t z & \text{if } z \in \mathbb{H}^-, \\ z & \text{if } z \in \mathbb{H}^+, \end{cases} \tag{2.1}$$

where $\mathbb{H}^\pm = \{z : \pm \operatorname{Re} z > 0\}$ and we have identified $\widetilde{X}$ with $\mathbb{H}$ so $\widetilde{\gamma} = i\mathbb{R}^+$.

The twist deformation extends by continuity to laminations; the more general deformations are called *earthquakes* (cf. [Ker1], [Ker2], [Th2], [EpM]). It extends to $\mathcal{ML}_\mathbb{R}(S)$ by twisting negative laminations to the left.

Twisting preserves the hyperbolic metric, and thus the Fuchsian projective structure, away from γ. Thus we obtain a mapping

$$\mathrm{Tw} : \mathcal{ML}_\mathbb{R}(S) \times T(S) \to P(S)$$

lifting $\mathrm{tw}(\cdot)$; the surface $\mathrm{Tw}_\lambda(X)$ is just $\mathrm{tw}_\lambda(X)$ with its Fuchsian projective structure. By [Ker1, Cor 2.6] we have:

Theorem 2.3 *The twist map* $\mathrm{Tw} : \mathcal{ML}_\mathbb{R}(S) \times T(S) \to P(S)$ *is continuous.*

Grafting. Next we describe the grafting maps:

$$\begin{tikzcd} \mathcal{ML}(S) \times T(S) \arrow[r, "\mathrm{Gr}"] \arrow[dr, "\mathrm{gr}"'] & P(S) \arrow[d] \\ & T(S). \end{tikzcd}$$

For a simple closed geodesic γ on a hyperbolic surface X, $\mathrm{gr}_{t\gamma}(X)$ is constructed by cutting along γ and inserting a Euclidean right cylinder $A(t)$ of height t and circumference $\ell_\gamma(X)$, with no twist (see Figure 2). The Euclidean and hyperbolic metrics piece together continuously to give a well-defined conformal structure.

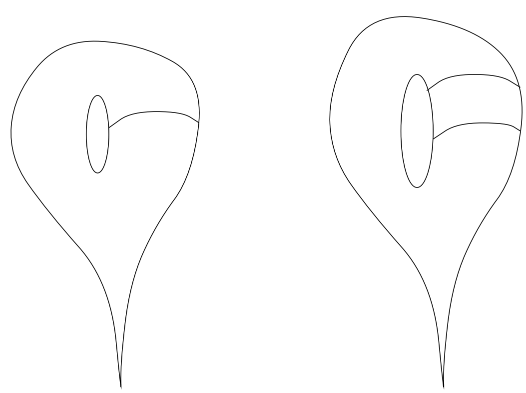

Figure 2. Grafting an annulus of height t along γ.

To define the *projective* surface $\mathrm{Gr}_{t\gamma}(X) \in P(S)$, we describe the inserted cylinder projectively as

$$A(t) = \widetilde{A}(t)/\langle z \mapsto e^{\ell_\gamma(X)} z \rangle,$$

where

$$\widetilde{A}(t) = \{z \in \mathbb{C}^* \ : \ \arg(z) \in [\pi, \pi + t]\}.$$

When $t \geq 2\pi$, one must interpret the projective surface $\widetilde{A}(t)$ as multi-sheeted. This projective structure on $A(t)$ fits together with the Fuchsian structure on $X - \gamma$ to define the projective structure on $\mathrm{Gr}_{t\gamma}(X)$. The metric $|dz|/|z|$ on $\widetilde{A}(t)$ makes $A(t)$ into a Euclidean cylinder of height t and circumference $\ell_\gamma(X)$, so $\mathrm{Gr}_{t\gamma(X)}$ is conformally identical to $\mathrm{gr}_{t\gamma}(X)$.

Identifying the universal cover of X with $\mathbb{H}$ so $\widetilde{\gamma} = i\mathbb{R}$, a local projective model for grafting is to cut along $\widetilde{\gamma}$, apply the map

$$z \mapsto \begin{cases} e^{it} z & \text{if } z \in \mathbb{H}^-, \\ z & \text{if } z \in \mathbb{H}^+, \end{cases} \tag{2.2}$$

and insert the strip $\widetilde{A}(t)$ to join the pieces together (see Figure 3). For large values of t this model should be thought of as a description of the developing map for $\mathrm{Gr}_{t\gamma}(X)$.

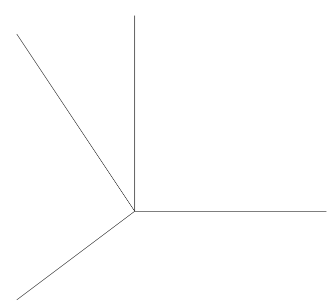

Figure 3. Local model for grafting.

Grafting extends by continuity from weighted simple curves to laminations; in fact we have [KaT]:

Theorem 2.4 (Thurston) *The grafting map* $\mathrm{Gr} : \mathcal{ML}(S) \times T(S) \to P(S)$ *is a homeomorphism.*

For the convenience of the reader we sketch the main idea in the proof of Theorem 2.4. Consider the universal cover $\widetilde{X}$ of a projective surface $X \in P(S)$. Because of the projective structure one has the notion of a maximal round disk $D \subset \widetilde{X}$. The ideal boundary of D is naturally a circle, an open subset of which lies in $\widetilde{X}$. Let $K(D) \subset D$ be the convex hull of the remainder of the ideal boundary with respect to the hyperbolic metric on D. As D varies, the hulls $K(D)$ cover $\widetilde{X}$ and have disjoint interiors. Those $K(D)$ which are single geodesics determine a lamination λ; those whose interior is nonempty piece together to form a hyperbolic surface Y such that $\mathrm{Gr}_\lambda(Y) = X$.

Complex earthquakes. We now combine twisting and grafting to form a single transformation

$$\mathrm{Eq} : \mathcal{ML}_{\overline{\mathbb{H}}}(S) \times T(S) \to P(S)$$

defined by

$$\mathrm{Eq}_\lambda(X) = \mathrm{Gr}_{\mathrm{Im}\,\lambda}(\mathrm{tw}_{\mathrm{Re}\,\lambda}(X)).$$

We call this deformation a *complex earthquake* along λ. By Theorems 2.3 and 2.4 we have:

Theorem 2.5 *The complex earthquake map* $\mathrm{Eq} : \mathcal{ML}_{\overline{\mathbb{H}}}(S) \times T(S) \to P(S)$ *is continuous.*

Later the domain of $\mathrm{Eq}_\lambda(X)$ will be enlarged to include certain laminations with negative imaginary part.

Most results about grafting, twisting and earthquakes are established by first assuming the lamination λ is a simple closed curve. To handle general laminations, one uses the density of weighted simple closed curves in $\mathcal{ML}(S)$ and the continuity asserted by Theorems 2.2 – 2.5 above.

For an arbitrary complex lamination, we define the *bending holonomy map*

$$\eta : \mathcal{ML}_{\mathbb{C}}(S) \times T(S) \to V(S)$$

by

$$\eta_\lambda(X) \quad = \quad \begin{cases} \mu \circ \mathrm{Eq}_\lambda(X) & \text{if } \mathrm{Im}\,\lambda \geq 0, \\ \overline{\mu} \circ \overline{\mathrm{Eq}}_{-\lambda}(\overline{X}) & \text{if } \mathrm{Im}\,\lambda \leq 0. \end{cases} \tag{2.3}$$

Recall that $\mu : P(S) \to V(S)$ sends a projective surface to its holonomy representation.

To explain the second formula above, let $\overline{S}$ denote S with its orientation reversed. Any $X \in T(S)$ has a complex conjugate $\overline{X} \in T(\overline{S})$ obtained by composing its charts with complex conjugation, and we have maps

$$\mathcal{ML}_{\overline{\mathbb{H}}}(\overline{S}) \times T(\overline{S}) \xrightarrow{\overline{\mathrm{Eq}}} P(\overline{S}) \xrightarrow{\overline{\mu}} V(\overline{S}).$$

But $\mathcal{ML}(S) = \mathcal{ML}(\overline{S})$ and $V(S) = V(\overline{S})$, since these spaces do not make reference to the orientation of S; with these identifications, $\overline{\mu} \circ \overline{\mathrm{Eq}}_{-\lambda}(\overline{X})$ lies in $V(S)$.

Note that a right earthquake on X becomes a left earthquake on $\overline{X}$, due to reversal of orientation, so the two formulas in (2.3) agree when $\operatorname{Im} \lambda = 0$.

Pleated planes. Here is a more geometric description of the bending holonomy map η. A *pleated plane* is a continuous map

$$f : \mathbb{H}^2 \to \mathbb{H}^3$$

such that each $x \in \mathbb{H}^2$ lies on a geodesic γ mapped isometrically to a geodesic in $\mathbb{H}^3$. If only one such γ exists for a given x, then γ is a leaf in the *pleating lamination* λ of f.

Now suppose $\mathbb{H}^2$ is identified with the universal cover of a surface $X \in T(S)$. Then f is an *equivariant pleated plane* if there is a $[\rho] \in V(S)$ such that $f \circ g = \rho(g) \circ f$ for every deck transformation $g \in \pi_1(X) \cong \pi_1(S)$. This *holonomy representation* ρ is uniquely determined by f. If $\Gamma = \rho(\pi_1(X))$ happens to be a Kleinian group, then f descends to give a *pleated surface* $X \to \mathbb{H}^3/\Gamma$.

Any measured lamination $\lambda \in \mathcal{ML}_{\mathbb{R}}(S)$ determines an equivariant pleated plane by using the transverse measure on $\widetilde{\lambda} \subset \widetilde{X} \cong \mathbb{H}^2$ to prescribe *bending* of $\mathbb{H}^2$ inside $\mathbb{H}^3$. Bending is distinguish from grafting by the fact that t can be positive *or* negative, by bending in opposite directions. See [EpM, Chapter 3] for an extended discussion.

For a simple geodesic with transverse measure t, the local model of bending is compatible with the local model of grafting (2.2): $\mathbb{H}^+$ is mapped to the convex hull of $\mathbb{R}^+$ in $\mathbb{H}^3$, and $\mathbb{H}^-$ is mapped to the convex hull of $e^{it}\mathbb{R}^-$. By continuity, the models are also compatible for general laminations (see [EpM, Thm. 3.11.5]). Thus in terms of pleated planes, one can alternatively define the bending holonomy map by

$$\eta_{i\lambda}(X) = [\rho]$$

for $\lambda \in \mathcal{ML}_{\mathbb{R}}(S)$, where ρ is the holonomy representation of the pleated plane corresponding to $(\widetilde{X}, \widetilde{\lambda})$; and by

$$\eta_\lambda(X) = \eta_{i \operatorname{Im} \lambda}(\mathrm{tw}_{\operatorname{Re} \lambda}(X)) \tag{2.4}$$

for $\lambda \in \mathcal{ML}_{\mathbb{C}}(S)$.

Proposition 2.6 $\mathrm{Eq}_{t\lambda}(X)$ *and* $\eta_{t\lambda}(X)$ *vary holomorphically with respect to* $t \in \mathbb{H}$ *and* $t \in \mathbb{C}$ *respectively.*

Proof 1. For a simple geodesic γ and $t = u + iv$, the local model for the deformation

$$\mathrm{Eq}_{t\gamma}(X) = \mathrm{Gr}_{v\gamma}(\mathrm{tw}_{u\gamma}(X))$$

is a combination of equations (2.1) and (2.2). The combined effect of twisting and grafting is to reglue charts with the new transition function $\phi_t : z \mapsto e^{u+iv} z$. Since $\phi_t(z)$ is holomorphic in t, $\mathrm{Eq}_{t\gamma}(X)$ varies holomorphically in $P(S)$. To extend the result to general laminations, use the density of weighted simple closed curves in $\mathcal{ML}(S)$, the continuity of $Eq_\lambda(X)$ (Theorem 2.5) and the fact that a uniform limit of holomorphic maps is holomorphic.

The projective holonomy map $\mu : P(S) \to V(S)$ is holomorphic, so (2.3) exhibits $\eta_{t\lambda}(X)$ as a function continuous on $\mathbb{C}$ and holomorphic on $\mathbb{C} - \mathbb{R}$; therefore it is holomorphic on $\mathbb{C}$.

Proof 2. The holonomy map $\eta_{t\lambda}(X)$, thought of in terms of bendings, is analytic in t by [EpM, 3.8.1]. Since $\eta_{t\lambda}(X) = \mu \circ \mathrm{Eq}_{t\lambda}(X)$ and μ is an analytic local homeomorphism, $\mathrm{Eq}_{t\lambda}(X)$ is also analytic. ■

Quasifuchsian groups. Next we describe the relation between bending and grafting in a more geometric setting.

A *quasifuchsian group* $\Gamma \subset PSL_2(\mathbb{C})$ is a discrete subgroup stabilizing a quasidisk on the sphere. Any such Γ is quasiconformally conjugate to a Fuchsian group.

Let $QF(S) \subset V(S)$ denote the set of faithful representations such that $\Gamma = \rho(\pi_1(S))$ is quasifuchsian. The dynamics of Γ determines an invariant partition $\widehat{\mathbb{C}} = \Lambda \sqcup \Omega$, where the *limit set* Λ is a quasicircle, and the *domain of discontinuity* Ω is pair of disks. The *Kleinian 3-manifold*

$$M = (\mathbb{H}^3 \cup \Omega)/\Gamma$$

is diffeomorphic to $\mathrm{int}(S) \times [0, 1]$; it carries a hyperbolic structure on its interior and a projective structure on its boundary. Thus $[\rho] \in QF(S)$ determines a pair of projective surfaces

$$(\partial_p M, \overline{\partial}_p M) \in P(S) \times P(\overline{S})$$

such that

$$\partial M = \partial_p M \sqcup \overline{\partial}_p M.$$

We denote the underlying conformal structures on these surfaces by

$$(\partial_c M, \overline{\partial}_c M) \in T(S) \times T(\overline{S}).$$

Similarly, the convex core of M (the quotient of the convex hull of Λ by Γ) is bounded by a pair of hyperbolic surfaces

$$(\partial_h M, \overline{\partial}_h M) \in T(S) \times T(\overline{S}).$$

The faces of the convex hull are pleated surfaces with bending laminations

$$(\beta, \overline{\beta}) \in \mathcal{ML}(S) \times \mathcal{ML}(\overline{S}).$$

We have

$$[\rho] = \eta_\beta(\partial_h M) = \mu(\partial_p M).$$

We can regard $QF(S)$ as the space of *marked* quasifuchsian manifolds, where a *marking* of M is a choice of isomorphism between $\pi_1(M)$ and $\pi_1(S)$. The deformation theory of Kleinian groups yields [Bers1], [Kra]:

Theorem 2.7 (Bers) *There is a holomorphic bijection*

$$Q : T(S) \times T(\overline{S}) \to QF(S)$$

such that $M = Q(X, Y)$ *is the unique quasifuchsian manifold with* $(\partial_c M, \overline{\partial}_c M) = (X, Y)$.

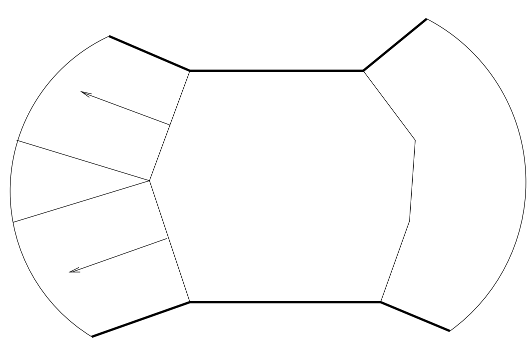

Figure 4. Normal projection from the faces of the convex core to ∂M.

Grafting gives a simple relation between the faces of the convex core and the boundary of M. For example, if $X = \partial_h M$ is bent along a simple closed curve γ with angle t, then orthogonal projection from $X - \gamma$ gives a projective map to $\partial_p M$ omitting an annulus isomorphic to $A(t)$ (see Figure 4). Therefore $\partial_p M = \mathrm{Gr}_{t\gamma}(X)$; more generally we have:

Theorem 2.8 *Let M be a quasifuchsian manifold with convex core bounded by hyperbolic surfaces (X, Y) with bending laminations $(\beta, \overline{\beta})$. Then the pair of projective surfaces bounding M satisfy*

$$(\partial_p M, \overline{\partial}_p M) \;\; = \;\; (\mathrm{Gr}_\beta(X), \mathrm{Gr}_{\overline{\beta}}(Y)).$$

Proof. A maximal round disk relative to the projective structure on $\partial_p M$ corresponds to a supporting hyperplane for the convex hull. Thus the definition of the transverse measure for a projective structure, $\phi(t)$ in [KaT, p.273], specializes to the definition of the bending measure for convex hull, $\beta(A)$ in [EpM, p.137]. Similarly, collapsing of the lamination on a projective surface corresponds to nearest-point projection to the boundary of the convex hull. ■

Analytic continuation of earthquakes. We now use quasifuchsian groups to give an explicit analytic continuation of $\mathrm{Eq}_\lambda(X)$.

Let $D(S)$ be the union of $\mathcal{ML}_{\overline{\mathbb{H}}}(S) \times T(S)$ and the component of $\eta^{-1}(QF(S))$ containing $\mathcal{ML}_{\mathbb{R}}(S) \times T(S)$. (Note that $\eta_\lambda(X) \in QF(S)$ for any real lamination λ, since the holonomy is Fuchsian.) We extend the complex earthquake deformation to a map

$$\mathrm{Eq} : D(S) \to P(S)$$

by setting

$$\mathrm{Eq}_\lambda(X) = \begin{cases} \mathrm{Gr}_{\mathrm{Im}\,\lambda}(\mathrm{tw}_{\mathrm{Re}\,\lambda}(X)) & \text{if } \mathrm{Im}\,\lambda \geq 0, \\ \partial_p M & \text{if } \mathrm{Im}\,\lambda \leq 0, \end{cases} \tag{2.5}$$

where M is the marked quasifuchsian 3-manifold corresponding to $[\rho] = \eta_\lambda(X)$.

This extension of $\mathrm{Eq}(\cdot)$ uses the fact that the quasifuchsian manifold $M = \eta_\lambda(X)$ exists for small positive *or* negative bending along a given lamination $\lambda \in \mathcal{ML}(S)$. For positive bending, $\partial_p M$ is described by grafting, so grafting extends to (certain) negative laminations by (2.5).

Theorem 2.9 *For $(\lambda, X) \in D(S)$ with $\operatorname{Im}\lambda \leq 0$, let M be the marked quasifuchsian manifold corresponding to the representation $\eta_\lambda(X)$. Then*

$$(\partial_p M, \overline{\partial}_p M) \quad = \quad (\mathrm{Eq}_\lambda(X), \mathrm{Eq}_{-\lambda}(\overline{X})) \tag{2.6}$$

and

$$\overline{\partial}_h M \quad = \quad \mathrm{tw}_{-\operatorname{Re}\lambda}(\overline{X}) \tag{2.7}$$

with bending lamination $\overline{\beta} = -\operatorname{Im}\lambda \in \mathcal{ML}(\overline{S})$.

Proof. The equation $\partial_p M = \mathrm{Eq}_\lambda(X)$ holds by definition.

To prove $\overline{\partial}_p M = \mathrm{Eq}_{-\lambda}(\overline{X})$, consider the subset U of parameters (λ, X) where this equality holds. Evidently U is closed and all real laminations are in U. By (2.3), the two sides of this equation are projective surfaces with the same holonomy; since $\mu : P(S) \to V(S)$ is a local homeomorphism, U is also open. Thus $U = \{(\lambda, X) \in D(S) : \operatorname{Im}\lambda \leq 0\}$ since the latter set is connected.

The characterization of $\overline{\partial}_h M$ is established similarly, using Theorem 2.8.

■

Theorem 2.10 *The earthquake deformation $\mathrm{Eq}_{t\lambda}(X)$ is holomorphic on the open set of t with $(t\lambda, X) \in D(S)$.*

Proof. We may assume $\lambda \in \mathcal{ML}(S)$. For $\operatorname{Im} t < 0$, $\partial_p M = \mu^{-1} \circ \eta_{t\lambda}(X)$, so analyticity of $\mathrm{Eq}(\cdot)$ follows from that of η and μ. We have already seen analyticity of $\mathrm{Eq}_{t\lambda}(X)$ for $\operatorname{Im} t > 0$, and the formulas in (2.5) agree and are continuous for $\operatorname{Im}\lambda = 0$. ■

In summary, the complex earthquake map arises from the Fenchel-Nielsen twist by analytic continuation to the open domain $D(S)$. We also define the map

$$\mathrm{eq} : D(S) \to T(S)$$

by recording the conformal structure underlying the projective surface $\mathrm{Eq}_\lambda(X)$. The relations between twists, complex earthquakes and holonomy are recorded in Figure 5.

To better understand the geometry of complex earthquakes, the next three sections develop general results on grafting, properness and convexity. The existence of properly embedded complex earthquake disks in Teichmüller space will follow.

$$
\begin{tikzcd}
\mathcal{ML}_{\mathbb{C}}(S) \times T(S) \arrow[r, "\eta"] & V(S) \\
D(S) \arrow[u, hook] \arrow[r, "Eq"] & P(S) \arrow[u, "\mu"'] \arrow[d] \\
\mathcal{ML}_{\mathbb{R}}(S) \times T(S) \arrow[u, hook] \arrow[ur, "Tw"] \arrow[r, "tw"] & T(S)
\end{tikzcd}
$$

Figure 5. Laminations and deformations.

Earthquake flows on $P(S)$. To give one more point of view, we briefly sketch how twisting and grafting along a lamination $\lambda \in \mathcal{ML}(S)$ can be thought of as the time-t maps for a holomorphic vector field on an open subset of $P(S)$.

To begin with, recall that the twists

$$\mathrm{tw}_{t\lambda} : T(S) \to T(S)$$

form a flow; that is, they obey the composition law

$$\mathrm{tw}_{(t+s)\lambda}(X) \quad = \quad \mathrm{tw}_{t\lambda}(\mathrm{tw}_{s\lambda}(X)).$$

There is a natural vector field τ_λ on $T(S)$ generating this flow; that is, such that

$$\left.\frac{d}{dt}\mathrm{tw}_{t\lambda}(X)\right|_{t=0} \quad = \quad \tau_\lambda.$$

Under the Fuchsian embedding $T(S) \subset V(S)$, the vector field τ_λ extends to a holomorphic vector field on the open set $R_\lambda \subset V(S)$ where λ is realizable. For example, suppose λ is a simple closed curve γ with transverse measure one. Then R_λ consists of those $[\rho]$ such that $\rho(\gamma)$ is hyperbolic. The vector $\tau_\lambda(\rho)$ can be represented by an explicit cocycle $\xi : \pi_1(S) \to sl_2(\mathbb{C})$ in group cohomology prescribing the effect of shearing along γ. The cocycle ξ varies holomorphically on $V(S)$ because the element of $sl_2(\mathbb{C})$ translating at unit speed along the axis of $\rho(\gamma)$ depends holomorphically on ρ. For general laminations, R_λ contains all ρ for which λ can be realized as an equivariant geodesic lamination in $\mathbb{H}^3$; in particular $QF(S) \subset R_\lambda$.

Pulling back τ_λ via the holonomy map μ, we obtain a holomorphic vector field $\widetilde{\tau}_\lambda$ on an open subset $\widetilde{R_\lambda}$ of $P(S)$. The flow generated by this vector field is not complete: solutions to the equation $\dot{X}_t = \widetilde{\tau}_\lambda$ can reach infinity in finite time. However, the flow line through a point $X_0 \in T(S) \subset P(S)$ (under the Fuchsian embedding) is defined for all time and satisfies $X_t = \mathrm{Tw}_{t\lambda(X)}$. Similarly, this flow line is defined for all positive imaginary time and $X_{it} = \mathrm{Gr}_{t\lambda}(X)$.

Since the flow is holomorphic on $P(S)$, its restriction to $T(S)$ is real-analytic, and we have (cf. [Ker2], [Tan]):

Corollary 2.11 *The twisting and grafting maps are real-analytic functions of $X \in T(S)$.*

The flow generated by τ_λ can also be used to extend the deformation theory to more general projective surfaces; e.g. see [Gol] for grafting in the quasifuchsian case.

3 Geometry of grafting

In this section we study the shape of the Riemann surface obtained by grafting, and its relation to Kleinian groups, harmonic maps and the Weil-Petersson metric.

The projective metric. On any projective surface $Y \in P(S)$ we can consider two metrics: the hyperbolic (or Kobayashi) metric ρ_h, and the projective (or Thurston) metric ρ_p. For a tangent vector $v \in \mathrm{T}Y$, the hyperbolic length of $\rho_p(v)$ is the infimum of the hyperbolic lengths of vectors v' in $\mathrm{T}\mathbb{H}$ such that there exists a holomorphic map $f : \mathbb{H} \to Y$ sending v' to v. The projective metric is defined in the same way, with the added requirement that f is projective. (See [Tan] for additional remarks.)

For $y \in Y$, we have $\rho_h(y)/\rho_p(y) \leq 1$ with strict inequality unless the projective structure on Y is the standard one coming from its Fuchsian uniformization.

Now suppose $Y = \mathrm{Gr}_{t\gamma}(X)$ for some simple closed curve γ. Then Y is obtained from X by inserting a cylinder $A(t)$ along γ. It is not hard to see that the projective metric on Y is just the combination of the hyperbolic metric on X and the flat Euclidean metric on the cylinder $A(t)$. This concrete picture for ρ_p leads to:

Theorem 3.1 *For any $\alpha, \beta \in \mathcal{ML}(S)$ and any $X \in T(S)$, we have*

$$\ell_\alpha(\mathrm{gr}_\beta(X)) \;\; \leq \;\; \ell_\alpha(X) + i(\alpha, \beta).$$

The inequality is strict if both α and β are nonzero.

Proof. If $\beta = t\gamma$ is a multiple of a simple closed curve, then the ρ_p-length of any lamination α is bounded by $\ell_\alpha(X) + i(\alpha, \beta)$, since the intersection number $i(\alpha, \beta) = t\, i(\alpha, \gamma)$ gives the length needed to cross the inserted cylinder $A(t)$. Since weighted simple closed curves are dense in $\mathcal{ML}(S)$, the same bound holds for general β by continuity (using Theorems 2.2 and 2.4). When $\beta \neq 0$ we have $\rho_h < \rho_p$ so the hyperbolic length of any $\alpha \neq 0$ is strictly below its projective length. ∎

Since $i(\lambda, \lambda) = 0$ we see grafting along any lamination makes it shorter:

Corollary 3.2 *For any nonzero $\lambda \in \mathcal{ML}(S)$ and $X \in T(S)$, we have $\ell_\lambda(\mathrm{gr}_\lambda(X)) < \ell_\lambda(X)$.*

Bers slices. Next we would like to discuss negative grafting. We define

$$\mathrm{gr}_{-\lambda}(X) \;\; = \;\; \mathrm{eq}_{-i\lambda}(X) \;\; = \;\; \partial_p M$$

for any $\lambda \in \mathcal{ML}(S)$ with $(-i\lambda, X) \in D(S)$. Here M is the quasifuchsian manifold with holonomy $\eta_{-i\lambda}(X)$.

To begin we recall some results of Bers on quasifuchsian groups. For $Y \in T(\overline{S})$, the *Bers slice* $B_Y \subset QF(S)$ is the set of marked quasifuchsian 3-manifolds such that the conformal structure on one boundary component is fixed by the condition $\overline{\partial}_c M = Y$. The *Bers embedding*

$$b_Y : T(S) \to B_Y \subset V(S)$$

sends X to $Q(X, Y)$, the unique quasifuchsian manifold M with $(\partial_c M, \overline{\partial}_c M) = (X, Y)$.

Theorem 3.3 (Bers) *Any Bers slice B_Y has compact closure in $V(S)$.*

The proof in [Bers2] is based on the inequality[1]

$$\frac{1}{\ell_\gamma(M)} \;\; \geq \;\; \frac{1}{2}\left(\frac{1}{\ell_\gamma(\partial_c M)} + \frac{1}{\ell_\gamma(\overline{\partial}_c M)}\right) \tag{3.1}$$

[1]See [Mc1, Prop 6.4] for the corrected form of Bers' inequality we give here.

where γ is a closed curve on S and $\ell_\gamma(M)$ is the length of its geodesic representative in M (or 0 if none exists). This inequality implies

$$\ell_\gamma(M) \;\; \leq \;\; 2\ell_\gamma(Y)$$

for any $M \in B_Y$ and all γ; by this bound B_Y is confined to a compact subset of $V(S)$.

Bers' inequality extends by continuity to laminations, using Theorem 2.2. From it we can deduce the following complement to Corollary 3.2:

Theorem 3.4 *For any nonzero $\lambda \in \mathcal{ML}(S)$ we have*

$$\ell_\lambda(\mathrm{gr}_{-\lambda}(X)) > \ell_\lambda(X).$$

Proof. Let M be the quasifuchsian manifold with holonomy $\eta_{-i\lambda}(X)$. By Theorem 2.9, the face $\overline{\partial}_h M$ of the convex hull of M is isometric to $\overline{X}$ and pleated along λ, so $\ell_\lambda(M) = \ell_\lambda(\overline{X}) = \ell_\lambda(X)$. We also have $\overline{\partial}_c M = \mathrm{gr}_\lambda(\overline{X})$, so $\ell_\lambda(\overline{\partial}_c M) < \ell_\lambda(X)$ by Corollary 3.2. Finally $\ell_\lambda(\partial_c M) = \ell_\lambda(\mathrm{gr}_{-\lambda}(X))$, so Bers' inequality (3.1) yields

$$\frac{1}{\ell_\lambda(X)} > \frac{1}{2}\left(\frac{1}{\ell_\lambda(X)} + \frac{1}{\ell_\lambda(\mathrm{gr}_{-\lambda}(X))}\right)$$

and the estimate follows. ∎

Putting these results together we have:

Corollary 3.5 *For any quasifuchsian manifold M with bending lamination β on $\partial_h M$, we have*

$$\ell_\beta(\partial_c M) \leq \ell_\beta(M) = \ell_\beta(\partial_h M) \leq \ell_\beta(\overline{\partial}_c M).$$

The inequalities are strict unless M is Fuchsian.

Remark. For a simple closed curve γ, one can form the covering space Y_γ of $Y = \mathrm{gr}_{t\gamma}(X)$ corresponding to the subgroup $\langle\gamma\rangle \subset \pi_1(S)$. The surface Y_γ is a cylinder whose modulus is inversely proportional to $\ell_\gamma(Y)$. By studying how Y_γ is built from $A(t)$ and the universal cover of X, one can show

$$\ell_\gamma(\mathrm{gr}_{t\gamma}(X)) \;\; \leq \;\; \frac{\pi}{\pi + t}\ell_\gamma(X)$$

for any $t \geq 0$. A similar argument applied to the $\langle\gamma\rangle$-covering space of M gives

$$\ell_\gamma(\mathrm{gr}_{-t\gamma}(X)) \quad \geq \quad \frac{\pi}{\pi - t}\ell_\gamma(X)$$

for those $t > 0$ where negative grafting is defined. However a typical lamination λ is a limit of $t_n\gamma_n$ with $t_n \to 0$, so these inequalities yield no more than the preceding results in the limit.

Extremal length, harmonic maps and properness. Next we state a fundamental property of grafting.

Theorem 3.6 (Tanigawa) *(i) The grafting map* $\mathrm{gr}_\lambda : T(S) \to T(S)$ *is proper for any* $\lambda \in \mathcal{ML}(S)$. *(ii) If* $\lambda = \sum 2\pi m_i\gamma_i$ *for a system of disjoint simple closed curves* γ_i *with positive integral weights, then* gr_λ *is a real-analytic homeomorphism.*

See [Tan]. The condition in (ii) implies the holonomy of $\mathrm{Gr}_\lambda(X)$ is Fuchsian, in which case a result of Faltings shows gr_λ is a local homeomorphism [Fal, Thm. 12]. By properness, gr_λ is actually a global homeomorphism.

The proof of properness in [Tan] is based on the inequality

$$\ell_\lambda(X) \leq \frac{\ell_\lambda(X)^2}{L_\lambda(\mathrm{gr}_\lambda(X))} \leq 2E(h) \leq \ell_\lambda(X) + 2\,\mathrm{area}(X), \tag{3.2}$$

where

- $L_\lambda(\mathrm{gr}_\lambda(X))$ denotes extremal length,
- $\mathrm{area}(X)$ is the hyperbolic area, and
- $E(h)$ is the energy of the harmonic map $h : \mathrm{gr}_\lambda(X) \to X$ in the homotopy class compatible with markings.

Recall that the *extremal length* of a lamination is defined by

$$L_\lambda(Y) \quad = \quad \sup_\rho \frac{\ell_\lambda(Y,\rho)^2}{\mathrm{area}(Y,\rho)}$$

where the supremum is over Riemannian metrics ρ compatible with the given conformal structure on $Y \in T(S)$. From (3.2) we can also deduce:

Corollary 3.7 *For any nonzero* $\lambda \in \mathcal{ML}(S)$, *the map* $\overline{\mathbb{H}} \to T(S)$ *given by* $t \mapsto \mathrm{eq}_{t\lambda}(X)$ *is proper.*

Proof. Let $t_n \to \infty$ in $\overline{\mathbb{H}}$; we must show $Y_n = \mathrm{gr}_{\mathrm{Im}\, t_n}(X_n) \to \infty$ in $T(S)$, where $X_n = \mathrm{tw}_{\mathrm{Re}\, t_n}(X)$.

First suppose $\mathrm{Im}\, t_n$ is bounded. Then $\ell_{\mathrm{Im}\, t_n \lambda}(X_n) = (\mathrm{Im}\, t_n)\ell_\lambda(X)$ is bounded, so by (3.2) the energies of the harmonic maps $h_n : Y_n \to X_n$ are also bounded. On the other hand, $|\mathrm{Re}\, t_n| \to \infty$, so $X_n \to \infty$ in $T(S)$ (by properness of earthquake paths; cf. Corollary 4.3 below). For any fixed surface Y, the harmonic energy $E(h : Y \to X_n)$ tends to infinity [Wolf], so Y_n must follow X_n to infinity.

Now suppose $\mathrm{Im}\, t_n \to \infty$. Since extremal length satisfies $L_{t\lambda}(Y) = t^2 L_\lambda(Y)$, (3.2) implies $L_\lambda(Y_n) \le \ell_\lambda(X)/\mathrm{Im}\, t_n \to 0$, and so $Y_n \to \infty$ in this case too. ∎

Weil-Petersson geometry. To conclude this section we describe how grafting connects with the Weil-Petersson metric on Teichmüller space. Consider the length of a simple geodesic $\ell_\alpha(X)$ as function on $T(S)$. The main fact is that infinitesimally, grafting shortens the length of α as fast as possible.

Theorem 3.8 *For any simple closed curve $\alpha \in \mathcal{ML}(S)$ we have*

$$\frac{d}{dt}\,\mathrm{gr}_{t\alpha}(X) \;=\; -\nabla \ell_\alpha(X),$$

where the gradient is taken with respect to the Weil-Petersson metric.

Proof. On a Kähler manifold the gradient of a function is i times the Hamiltonian vector field it generates. For the Weil-Petersson symplectic form, the Hamiltonian flow generated by $-\ell_\alpha$ is the Fenchel-Nielsen rightward twist flow for α [Wol2, Thm 2.10]. Since the complex earthquake map $\mathrm{eq}_{t\alpha}(X)$ is holomorhic in t, we have

$$\frac{d}{dt}\,\mathrm{gr}_{t\alpha}(X) \;=\; i\frac{d}{dt}\,\mathrm{tw}_{t\alpha}(X)$$

and the Theorem follows. ∎

Corollary 3.9 *For any pair of simple closed curves α and β we have*

$$\frac{d}{dt}\ell_\alpha(\mathrm{gr}_{t\beta}(X)) \;=\; -\langle \nabla \ell_\alpha, \nabla \ell_\beta \rangle$$

with respect to the Weil-Petersson inner product.

Since $\langle \nabla \ell_\alpha, \nabla \ell_\beta \rangle$ is symmetric in α and β, we obtain the reciprocity law

$$\frac{d}{dt}\ell_\alpha(\mathrm{gr}_{t\beta}(X)) = \frac{d}{dt}\ell_\beta(\mathrm{gr}_{t\alpha}(X))$$

for the rate of change of lengths under grafting.

For $p \in \alpha \cap \beta \subset X$ let $\theta_p \in [0, \pi)$ denote the clockwise angle from α to β. Inserting a thin flat annulus $A(t)$ along β increases the length of α by $t \sin\theta_p + O(t^2)$ at each crossing. As in the proof of Theorem 3.1, the increase in projective length dominates $(d/dt)\ell_\alpha(\mathrm{gr}_{t\beta}(X))$, so we deduce:

Corollary 3.10 *For simple closed geodesics α and β on X, we have*

$$i(\alpha, \beta) \geq \sum_{p \in \alpha \cap \beta} \sin\theta_p \geq -\langle \nabla \ell_\alpha, \nabla \ell_\beta \rangle \tag{3.3}$$

in the Weil-Petersson inner product on the tangent space to $T(S)$ at X.

For example, this Corollary shows the vectors $\nabla \ell_{\alpha_i}$ for a system α_i of disjoint simple closed curves on X all lie in a half-space. This fact can also be seen geometrically: grafting along $\lambda = \sum \alpha_i$ decreases the lengths of all α_i simultaneously.

The sine estimate above recalls the cosine law [Wol1]

$$\frac{d}{dt}\ell_\alpha(\mathrm{tw}_{t\beta}(X)) = \sum_{p \in \alpha \cap \beta} \cos\theta_p; \tag{3.4}$$

however the second inequality in (3.3) cannot be replaced by equality (consider the case $\alpha = \beta$).

4 Pleated surfaces

Large earthquakes along a fixed $\lambda \in \mathcal{ML}(S)$ move a given Riemann surface X off to infinity in Teichmüller space. After a large earthquake, can grafting along λ move the surface back into a compact subset of $T(S)$? The answer is *no* for positive grafting by Corollary 3.7 above. To establish properness of earthquake disks (Theorem 6.2), we also need to analyze the case of negative grafting. It is handled by the general result below.

Theorem 4.1 *Fix a nonzero lamination $\lambda \in \mathcal{ML}(S)$ and a surface $X \in T(S)$. Let $X_n = \mathrm{tw}_{t_n\lambda}(X)$, where $|t_n| \to \infty$. Let $f_n : \widetilde{X_n} \to \mathbb{H}^3$ be any* *sequence of equivariant pleated planes, each pleated along the support of* $\widetilde{\lambda}$. *Then*

$$[\rho_n] \to \infty \quad \textit{in } V(S),$$

where

$$\rho_n : \pi_1(S) \cong \pi_1(X_n) \to PSL_2(\mathbb{C})$$

are the corresponding holonomy representations.

Since $\eta_{t\lambda}(X)$ can be described as the holonomy of a pleated plane, we have:

Corollary 4.2 *The holonomy* $\eta_{t\lambda}(X) \to \infty$ *in* $V(S)$ *as* $|\operatorname{Re} t| \to \infty$.

The special case where t is real yields:

Corollary 4.3 *The earthquake path* $t \mapsto \mathrm{tw}_{t\lambda}(X)$ *gives a proper embedding* $\mathbb{R} \to T(S)$.

Remark. The holonomy $\eta_{t\lambda}(X)$ need not tend to infinity as $|\operatorname{Im} t| \to \infty$; for example, if λ is a simple closed curve with transverse measure one, then $\eta_{t\lambda}(X)$ is periodic under $t \mapsto t + 2\pi i$.

To give the proof of Theorem 4.1 we first need some facts about earthquakes and pleating. Let α and β be a pair of measured laminations on a hyperbolic surface. For each $p \in \alpha \cap \beta$ recall $\theta_p \in [0, \pi)$ denotes the clockwise angle from α to β at p. (This convention is consistent with right earthquakes.) Then a generalization of (3.4) gives

$$\frac{d}{dt}\ell_\alpha(\mathrm{tw}_{t\beta}(X)) \quad = \quad \int_{\alpha\cap\beta} \cos\theta_p \, d\alpha \times d\beta, \tag{4.1}$$

where the integral over $\alpha\cap\beta$ is with respect to the product of their transverse measures [Ker2, Prop. 2.5]. From [Ker1, Prop. 3.5] we also have:

Theorem 4.4 *For each* $p \in \alpha \cap \beta$, *the angle* $\theta_p(t)$ *between* α *and* β *on* $\mathrm{tw}_{t\beta}(X)$ *is a decreasing function of* t.

We remark that a point $p \in \alpha\cap\beta$ is labeled by the pair of leaves on which it lies, and this labeling allows one to canonically identify intersections of laminations on different surfaces in Teichmüller space.

The proof of Theorem 4.4 is suggested in Figure 6. A right earthquake transports the geodesic representative of α on X to a set of geodesic segments

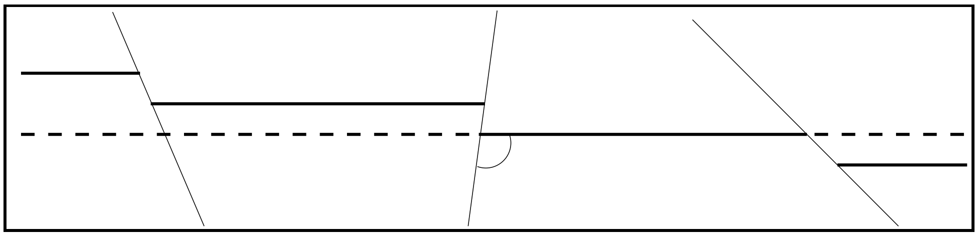

Figure 6. An earthquake decreases θ_p.

on $\mathrm{tw}_{t\beta}(X)$. The prolongations of these segments to complete geodesics (one of which is shown by a dotted line) are disjoint. The broken segments converge to points on the circle at infinity which are clockwise from the endpoints of any of their prolongations. (In the Figure, this means the geodesic representative of α runs from above the dotted line on the left to below it on the right.) Thus the geodesic representative of α on $\mathrm{tw}_{t\beta}(X)$ is rotated towards β, and θ_p decreases.

The argument above can easily be made quantitative, yielding the formula

$$\frac{d\theta_{p(0)}}{dt} = -\int_{-\infty}^{\infty} e^{-|s|} \sin\theta_{p(s)}\, d\beta(s), \tag{4.2}$$

where $p(s)$ parameterizes the leaf of α through p by arclength, and $d\beta(s)$ is the measure on the leaf coming from the transverse measure of β. Indeed, translating a geodesic through $p(s)$ an infinitesimal distance ϵ in the normal direction moves its endpoints through visual angle ϵ as seen from $p(s)$, and hence through angle $e^{-|s|}\epsilon$ as seen from $p(0)$. The integral above totals these effects. We only need (4.2) in the case where α and β are simple curves.

We also need to relate lengths on X to the holonomy of a pleated plane. Let $f : \widetilde{X} \to \mathbb{H}^3$ be an equivariant pleated plane with holonomy ρ, and let α be a simple closed curve on X. Let

$$\ell_\alpha(\rho) = \inf_{x\in\mathbb{H}^3} d(x, \rho(\alpha)x)$$

denote the translation length of α under ρ. We now replace β by the pleating locus λ and prove:

Lemma 4.5 *If f is pleated along $\widetilde{\lambda}$, and $0 \le \theta_p < \epsilon$ for all $p \in \alpha\cap\lambda$, then*

$$\ell_\alpha(\rho) > (1-\delta)\ell_\alpha(X),$$

where $\delta \to 0$ as $\epsilon \to 0$.

Proof. Let $\widetilde{\alpha} \subset \widetilde{X}$ be a lift of the geodesic $\alpha \subset X$ with stabilizer $g : \widetilde{X} \to \widetilde{X}$ in the group of deck transformations.

Consider any geodesic segment $[a, b] \subset \widetilde{\alpha}$ of length one. We claim

$$d(f(a), f(b)) > 1 - 4\epsilon. \tag{4.3}$$

If $[a, b]$ is disjoint from $\widetilde{\lambda}$, then it is mapped isometrically to $\mathbb{H}^3$ by f so this inequality is immediate. Otherwise $[a, b]$ meets a leaf L of $\widetilde{\lambda}$. The condition $\theta_p < \epsilon$ implies $[a, b]$ is nearly parallel to L, and that the nearest points a', b' to a, b on L satisfy $d(a, a'), d(b, b') < \epsilon$. Since L is in the pleating locus, we have $d(f(a'), f(b')) = d(a', b') > 1 - 2\epsilon$; since $f : \widetilde{X} \to \mathbb{H}^3$ is distance-decreasing, we have

$$d(f(a), f(b)) > d(f(a'), f(b')) - 2\epsilon > 1 - 4\epsilon.$$

From (4.3) it follows easily that $\rho(g)$ is hyperbolic, that $f(\widetilde{\alpha})$ lies within a small tube about the geodesic stabilized by $\rho(g)$, and that there is a small δ such that $d(f(x), f(y)) > (1 - \delta)d(x, y)$ for any well-separated points x and y on $\widetilde{\alpha}$. Letting $y = g^n(x)$ for $n \gg 0$ we obtain the Lemma. ∎

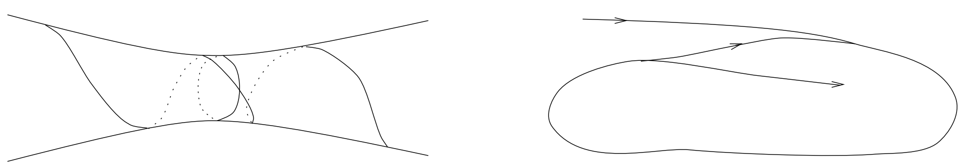

Figure 7. Constructing a simple closed curve α nearly parallel to λ.

Proof of Theorem 4.1. We will prove divergence of ρ_n when $\operatorname{Re} t_n \to \infty$; the case where $\operatorname{Re} t_n \to -\infty$ is completely analogous.

To begin we construct a simple closed geodesic α on X such that $i(\alpha, \lambda) > 0$ and for all $p \in \alpha \cap \lambda$ the angle θ_p between α and λ satisfies $\theta_p < \epsilon$. Here ϵ should be chosen in $(0, \pi/2)$, and small enough that $(1 - \delta) > 1/2$ in Lemma 4.5.

If all leaves of λ are closed, then α can be constructed by starting with any curve that meets λ and applying high powers of right Dehn twists around the components of λ (see Figure 7(a)). By (4.2), the geodesic representatives of the resulting curves have $\theta_p \to 0$ at the finite set of points p meeting λ.

If some leaf λ_0 of λ is not closed, then α can be constructed by closing λ_0 with a leftward bridge of small geodesic curvature near one of its accumulation points (see Figure 7(b)). By basic hyperbolic geometry, a loop with small geodesic curvature is C^1-close to its geodesic representative. Since nearby leaves of λ are nearly parallel, the intersections of α with λ all have small angle (even though they may be uncountable in number). Indeed the angle can be made arbitrarily small by taking a very close return of λ_0 and an almost geodesic bridge.

By Theorem 4.4, the angle $\theta_p(t)$ of a crossing between α and λ on $\mathrm{tw}_{t\lambda}(X)$ is a decreasing function of t. Thus $\theta_p(t) < \epsilon$ for all $t > 0$. By (4.1), we have

$$\frac{d}{dt}\ell_\alpha(\mathrm{tw}_{t\lambda}(X)) \quad = \quad \int_{\alpha\cap\lambda} \cos\theta_p \, d\alpha \times d\beta \quad > \quad \cos(\epsilon)\, i(\alpha, \lambda) \quad > \quad 0$$

because $\epsilon < \pi/2$. Therefore the length of α on $\mathrm{tw}_{t\lambda}(X)$ tends to infinity as $t \to \infty$. By Lemma 4.5,

$$\ell_{\rho_n}(\alpha) \quad > \quad (1-\delta)\ell_\alpha(\mathrm{tw}_{t_n\lambda}(X));$$

thus the translation length of α tends to infinity and so $\rho_n \to \infty$ in $V(S)$.
■

Question. Is it true more generally that $\eta_{\lambda_n}(X) \to \infty$ in $V(S)$ for any $\lambda_n \in \mathcal{ML}_{\mathbb{C}}(S)$ with $\operatorname{Re}\lambda_n \to \infty$?

5 Convexity of representations

Let us say a subset K of a complex manifold X is *disk-convex* if, for every continuous map $f : \overline{\Delta} \to K$, holomorphic on Δ, the condition $f(S^1) \subset K$ implies $f(\Delta) \subset K$. In this section we will prove:

Theorem 5.1 *The space of quasifuchsian groups $QF(S)$ is disk-convex in $V(S)$.*

The proof is based on general results about representations into $PSL_2(\mathbb{C})$. Let G be a nonelementary group (one containing no abelian subgroup of finite index). Let

$$R(G) \quad = \quad \mathrm{Hom}(G, PSL_2(\mathbb{C}))/\,\mathrm{PSL}_2(\mathbb{C})$$

be the space of representations $\rho : G \to PSL_2(\mathbb{C})$, modulo conjugacy. A map $f : Z \to R(G)$ from a complex manifold into $R(G)$ is *holomorphic* if it is locally of the form $f(z) = [\rho_z]$, where $\rho_z(g)$ is holomorphic in z for every $g \in G$. Using the theory of holomorphic motions one can establish [Sul, Thm. 1], [Bers3]:

Theorem 5.2 (Sullivan) *If $f : Z \to R(G)$ is holomorphic and the representations $f(Z)$ are all faithful, then they are also quasiconformally conjugate to one another.*

Lemma 5.3 *Let $f : \overline{\Delta} \to R(G)$ be a holomorphic disk. Suppose $f(t)$ is faithful for all t near S^1, and $f|S^1$ is homotopic through faithful representations to a constant map. Then $f(\overline{\Delta})$ consists entirely of faithful representations.*

Proof. Let $\rho_t = f(t) \in R(G)$. Suppose there is an $s \in \Delta$ such that $K = \mathrm{Ker}(\rho_s)$ is nontrivial. We will show $f|S^1$ is homotopically nontrivial.

Since G is nonelementary, its normal subgroup K and the faithful image $\rho_1(K)$ are also nonelementary. Therefore we can choose $g \in K$ of infinite order with $\rho_1(g)$ not parabolic. Setting $T(t) = \mathrm{tr}^2(\rho_t(g))$, we obtain a nonconstant holomorphic function on $\overline{\Delta}$ (since $T(s) = 4$ and $T(1) \neq 4$).

Now $T(t) = 4\cos^2(\pi\theta)$ with $\theta \in \mathbb{Q}$ iff $\rho_t(g)$ is a finite order elliptic element in $PSL_2(\mathbb{C})$. Since ρ_t is faithful near S^1, these values must be avoided, and since T is an open map, it sends S^1 into $\mathbb{C} - [0, 4]$. But $T(s) = 4$, so the winding number of $T : S^1 \to (\mathbb{C} - [0, 4])$ is positive by the argument principle. Thus any extension of T from S^1 to a continuous function on Δ must have the interval $[0, 4]$ in its image, so any homotopy of $f|S^1$ to a constant map must pass through a representation with a power of g in its kernel. Therefore $f|S^1$ gives an essential loop in the space of faithful representations. ∎

Proof of Theorem 5.1. Let $f : \overline{\Delta} \to V(S)$ be a holomorphic disk with $f(S^1) \subset QF(S)$. Since $QF(S)$ is isomorphic to $T(S) \times T(\overline{S})$, it is contractible, so $f|S^1$ is homotopically trivial as a loop in the space of faithful representations of $\pi_1(S)$. By the preceding Lemma, $f(\Delta)$ consists of faithful representations, and they are all quasiconformally conjugate by Sullivan's theorem. Therefore $f(\Delta) \subset QF(S)$. ∎

Remarks. Bers and Ehrenpreis showed Teichmüller space is holomorphically convex [BE]; another proof, using length functions, appears in [Wol4]. By a remarkable application of Grunsky's inequality, Shiga proved that $T(S)$

is convex with respect to holomorphic functions on $Q(X)$ under Bers' embedding into the vector space of holomorphic quadratic differentials on a fixed $X \in T(S)$ [Sh]. This implies $T(S)$ is disk-convex in $Q(X)$.

Question. Is $QF(S)$ convex with respect to holomorphic functions on $V(S)$?

6 Earthquake disks

In this section we will prove Theorem 1.1 on the existence of properly embedded complex earthquake disks in Teichmüller space. The proof relies on the results of the preceding three sections.

Let $\lambda \in \mathcal{ML}(S)$ be a nonzero lamination and let $X \in T(S)$ be a Riemann surface. Define

$$h : \mathbb{C} \to V(S)$$

by $h(t) = \eta_{t\lambda}(X)$, and let $D(X, \lambda)$ be the union of $\overline{\mathbb{H}}$ and the component of $h^{-1}(QF(S))$ containing 0. Then $D(X, \lambda)$ is the largest connected set of t including 0 such that $(t\lambda, X) \in D(S)$ for all $t \in D(X, \lambda)$.

Definition. For any $X \in T(S)$ and nonzero lamination $\lambda \in \mathcal{ML}(S)$, the *complex earthquake map*

$$f : D(X, \lambda) \to T(S)$$

is given by $f(t) = \mathrm{eq}_{t\lambda}(X)$.

Figure 8. The domain $D(X, \alpha)$ of the earthquake map for a simple closed curve α.

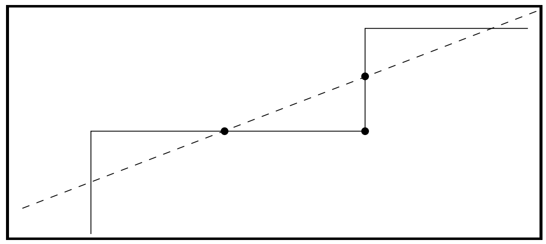

Figure 9. Length after twisting.

Example: the punctured torus. Let S be a surface of genus one with one boundary component; then $\pi_1(S) = \langle a, b \rangle$ is a free group. Choose a hyperbolic structure $X \in T(S)$ so the geodesics (α, β) representing a and b cross at right angles (i.e., so X is 'rectangular'). Then one can explicitly compute the representation $\rho_t = \eta_{t\alpha}(X)$ resulting from bending along α, and from this information draw a picture of $D(X, \alpha)$ (see Figure 8).

To explain the computation of bending, first note that ρ_t is essentially determined by the traces of the generators a and b.[2] We claim these traces (up to sign) are given explicitly by:

$$(A_t, B_t) \quad = \quad (\operatorname{tr} \rho_t(a), \operatorname{tr} \rho_t(b)) \quad = \quad (A_0, B_0 \cosh(t/2)). \tag{6.1}$$

To check this formula, suppose t is real, and set $X_t = \operatorname{tw}_{t\alpha}(X)$ and $L_t = \ell_{X_t}(\beta)$. Then the homotopy class b is represented on X_t by a right-angled broken geodesic, with pieces of length L_0 and t; by considering the straight representative of this broken geodesic (shown as a dotted line in Figure 9), we obtain a right triangle with sides $t/2$, $L_0/2$ and $L_t/2$. By hyperbolic trigonometry we have

$$\cosh(L_t/2) = \cosh(L_0/2)\cosh(t/2),$$

and since $B_t = 2\cosh(L_t/2)$ we obtain (6.1) for t real. The case where t is complex follows by analytic continuation.[3]

Of course $A_t = A_0$ because twisting along α does not change the holonomy around α.

Using this information, one can compute a picture of $D(X, \alpha) \subset \mathbb{C}$. Figure 8 depicts the case $A_0 = 3$; the region $D(X, \alpha)$ is shown in white,

[2] Given $\operatorname{tr} \rho(a)$ and $\operatorname{tr} \rho(b)$, there are two choices for ρ, differing by the automorphism $(a, b) \mapsto (a, b^{-1})$.

[3] Some related explicit bending calculations appear in [PS].

with the real axis running through it. The condition that X_0 is rectangular implies $\operatorname{tr}\rho_0(ab) = \operatorname{tr}\rho_0(ab^{-1})$, which gives

$$4(A_0^2 + B_0^2) = A_0^2 B_0^2$$

and thereby determines $B_0 = 6/\sqrt{5}$. From (6.1) we then know ρ_t for each $t \in \mathbb{C}$. Coloring black the points t in the lower half-plane such that the traces (A_t, B_t) do not give a quasifuchsian group, we obtain Figure 8. The portion of $D(X, \alpha)$ shown runs from $\operatorname{Re} t = -8$ to 8; it is periodic under translation by $\ell_\alpha(X) = 1.92485\ldots$, since twisting by $\ell_\alpha(X)$ results in a full Dehn twist around α. It is also easy to see that $D(X, \alpha) \subset \{t : \operatorname{Im}(t) > -\pi\}$, because the convex hull cannot be bent by more that π. This bound holds quite generally for complex earthquakes along simple closed curves.

Next we prove a general result evident in Figure 8:

Lemma 6.1 *The domain $D(X, \lambda)$ is simply-connected.*

Proof. Consider any smoothly bounded disk $U \subset \mathbb{C} - \mathbb{H}$ with $\partial U \subset D(X, \lambda)$. Then $h(\partial U) \subset QF(S)$, where $h(t) = \eta_{t\lambda}(X)$. Since $QF(S)$ is disk-convex in $V(S)$ (Theorem 5.1), we have $h(U) \subset QF(S)$ and therefore $U \subset D(X, \lambda)$. It follows that $D(X, \lambda)$ is simply-connected. ■

Thus the proof of Theorem 1.1 is completed by:

Theorem 6.2 (Properness) *The complex earthquake map $f : D(X, \lambda) \to T(S)$ is proper.*

Proof. Let $t_n \to \infty$ in $D(X, \lambda)$; we must show

$$Y_n = \operatorname{eq}_{t_n\lambda}(X) \to \infty$$

in $T(S)$. It suffices to establish $Y_n \to \infty$ along a subsequence.

Since Corollary 3.7 handles the upper halfplane, we may assume $\operatorname{Im} t_n < 0$. Then there are quasifuchsian manifolds M_n with holonomy $\eta_{t_n\lambda}(X)$ such that

$$(\partial_c M_n, \overline{\partial}_h M_n) = (Y_n, Z_n).$$

By (2.7) we have

$$Z_n = \operatorname{tw}_{-\operatorname{Re} t_n \lambda}(\overline{X})$$

and the pleating map $f_n : Z_n \to M_n$ sending Z_n to one face of the convex core of M_n has bending lamination $\beta_n = -\operatorname{Im} t_n \lambda$.

Now if $|\operatorname{Re} t_n|$ is unbounded, we can pass to a subsequence such that $|\operatorname{Re} t_n| \to \infty$. Then $M_n \to \infty$ in $V(S)$, as can be seen by applying Theorem 4.1 to the pleated surfaces f_n. By Bers' inequality (3.1), M_n must eventually exit any compact family of Bers slices; since $M_n \in B_{Y_n}$, we have $Y_n \to \infty$.

Finally assume $|\operatorname{Re} t_n|$ is bounded. Then Z_n ranges in a compact subset of $T(\overline{S})$. On any fixed Z_n, the measure of a short transversal to the bending lamination $\beta_n = -\operatorname{Im} t_n \lambda$ cannot exceed 2π, since the pleating map f_n is injective. We conclude that $|\operatorname{Im} t_n|$ is bounded as well, so we may assume $t_n \to t_\infty \in \mathbb{C}$.

To reach a contradiction, suppose Y_n does not tend to infinity. Then we can pass to a subsequence such that

$$\partial_c M_n = Y_n \to Y_\infty \in T(S).$$

By (2.6), the other face of the conformal boundary satisfies

$$\overline{\partial}_c M_n = \operatorname{eq}_{-t_n\lambda}(\overline{X}) \to \operatorname{eq}_{-t_\infty\lambda}(\overline{X}).$$

Since both components of the conformal boundary of M_n converge in Teichmüller space, $M_\infty = \lim M_n$ is quasifuchsian. Therefore $t_\infty \in D(X, \lambda)$, contrary to the assumption that $t_n \to \infty$.

We conclude that Y_n also tends to infinity when $|\operatorname{Re} t_n|$ is bounded. ∎

Remark. By Corollary 3.2 and Theorem 3.4, for $t \in D(X, \lambda)$ we have the estimates

$$\ell_\lambda(\operatorname{eq}_{t\lambda}(X)) \begin{cases} < \ell_\lambda(X) & \text{if } \operatorname{Im} t > 0, \\ = \ell_\lambda(X) & \text{if } \operatorname{Im} t = 0, \text{ and} \\ > \ell_\lambda(X) & \text{if } \operatorname{Im} t < 0. \end{cases} \tag{6.2}$$

In particular, the length of X is unchanged only along the twist path, where t is real.

7 One-dimensional Teichmüller spaces

In this section we prove the results originally stated as Theorems 1.3, 1.4, 1.5 and 1.6. Throughout this section we make the standing assumption:

The Teichmüller space $T(S)$ is one-dimensional.

The first Theorem is also the central tool.

Theorem 7.1 *The complex earthquake map $f : D(X, \lambda) \to T(S)$ is a holomorphic bijection, sending $\overline{\mathbb{H}}$ to $\{Y \,:\, \ell_\lambda(Y) \leq \ell_\lambda(X)\}$.*

Proof. Let $L = \{Z \in T(S) \,:\, \ell_\lambda(Z) = \ell_\lambda(X)\}$. Then we have $f^{-1}(L) \subset \mathbb{R}$ by (6.2). Any two points in Teichmüller space are related by a *unique* right earthquake [Th2], so $f : \mathbb{R} \to L$ is bijective. Thus f is a proper map of degree one, hence an isomorphism. The characterization of $f(\overline{\mathbb{H}})$ follows from (6.2). ∎

Theorem 7.2 *The grafting map*

$$\mathrm{gr}_\lambda : T(S) \to T(S)$$

is a homeomorphism for any $\lambda \in \mathcal{ML}(S)$.

Lemma 7.3 *Let $g : \mathbb{H} \to \mathbb{H}$ be a holomorphic map satisfying $g(z + a) = g(z) + b$, where $b > a > 0$. Then g has no fixed point.*

Proof. The map $h(z) = g(az)/b$ commutes with $T(z) = z+1$, so it descends to a map of the punctured disk $\Delta^* \cong \mathbb{H}/\langle T\rangle$ to itself. By the Schwarz Lemma, $\operatorname{Im} h(z) \geq \operatorname{Im} z$ and thus $\operatorname{Im} g(z) \geq (b/a) \operatorname{Im} z > \operatorname{Im} z$. ∎

Proof of Theorem 7.2. Assume $\mathrm{gr}_\lambda(X) = \mathrm{gr}_\lambda(Y)$; we must show $X = Y$. Let

$$f_{\lambda,X} : \mathbb{H} \to U_X = \{Z \in T(S) \,:\, \ell_\lambda(Z) < \ell_\lambda(X)\}$$

be the holomorphic homeomorphism given by $f_{\lambda,X}(t) = \mathrm{eq}_{t\lambda}(X)$, and similarly for Y. Then

$$f_{\lambda,X}(i) = \mathrm{gr}_\lambda(X) = \mathrm{gr}_\lambda(Y) = f_{\lambda,Y}(i).$$

Since the graphs of holomorphic maps between Riemann surfaces intersect positively, the graphs of $f_{\gamma,X}$ and $f_{\gamma,Y}$ also meet for some simple closed curve γ. Indeed, $\lambda = \lim s_n\gamma_n$, $f_{s_n\gamma_n,X}(t) \to f_{\lambda,X}(t)$ uniformly on compact sets by Theorem 2.5, and $f_{s_n\gamma_n,X}(t) = f_{\gamma_n,X}(s_n t)$. Similar reasoning applies to $f_{\lambda,Y}$. Thus the graphs of $f_{\gamma,X}$ and $f_{\gamma,Y}$ meet for any $\gamma = \gamma_n$ with n sufficiently large.

We may assume (by interchanging the roles of X and Y if necessary) that $\ell_\gamma(X) \leq \ell_\gamma(Y)$, so $U_X \subset U_Y$. Define a holomorphic map $g : \mathbb{H} \to \mathbb{H}$ by the composition

$$\mathbb{H} \xrightarrow{f_{\gamma,X}} U_X \subset U_Y \xrightarrow{f_{\gamma,Y}^{-1}} \mathbb{H};$$

then

$$\mathrm{eq}_{t\gamma}(X) = \mathrm{eq}_{g(t)\gamma}(Y).$$

Since the graphs of $f_{\gamma,X}$ and $f_{\gamma,Y}$ meet, the map g has a fixed-point in $\mathbb{H}$.

Let $\tau : T(S) \to T(S)$ be the automorphism of Teichmüller space determined by a right Dehn twist on γ. Since a full twist around γ is the same as τ, we have

$$f_{\gamma,X}(t + \ell_\gamma(X)) \quad = \quad \tau \circ f_{\gamma,X}(t)$$

and similarly for Y; therefore

$$g(t + \ell_\gamma(X)) = t + \ell_\gamma(Y).$$

Since g has a fixed point in $\mathbb{H}$, the Lemma above implies $\ell_\gamma(X) = \ell_\gamma(Y)$. Then $X = \mathrm{tw}_{s\gamma}(Y)$ for some $x \in \mathbb{R}$, so $\mathrm{eq}_{t\gamma}(X) = \mathrm{eq}_{(t+s)\gamma}(Y)$ and $g(t) = t + s$. Since g has a fixed point, $s = 0$, $X = Y$ and grafting is injective. ■

Bending coordinates for a Bers slice. Let $B_Y \subset V(S)$ be the Bers slice for $Y \in T(\overline{S})$, and let $N = Q(\overline{Y}, Y)$ be the unique Fuchsian group it contains. For any $M \in B_Y - N$, the bending lamination $\beta \in \mathcal{ML}(S)$ of $\partial_h M$ is nonzero, and $\ell_\beta(M) = \ell_\beta(\partial_h M) < \ell_\beta(Y)$ by Corollary 3.5. Thus we have a continuous map

$$p : (B_Y - N) \to \mathbb{P}\mathcal{ML}(S) \times (0, 1)$$

given by

$$p(M) \quad = \quad \left([\beta], \; \frac{\ell_\beta(M)}{\ell_\beta(Y)} \right).$$

We now complete the proof of Theorem 1.5 by showing:

Theorem 7.4 *The map p is a bijection.*

Proof. Consider any $(\beta, L) \in \mathcal{ML}(S) \times \mathbb{R}$ with $\beta \neq 0$ and $0 < L < \ell_\beta(Y)$. Suppose $M \in QF(S)$ has bending lamination $s\beta$ (for some $s > 0$) and $\ell_\beta(M) = L$. To show p is injective, we will show M is uniquely determined by the data (β, L).

The first observation is that L determines the geometry of $\partial_h M$ up to a real earthquake along β. That is, fixing any $Z \in T(S)$ with $\ell_\beta(Z) = L$, the earthquake homeomorphism $f : D(Z, \beta) \to T(S)$ sends $\mathbb{R}$ to the set of Riemann surfaces where β has length L. Thus $\partial_h M = \mathrm{eq}_{r\beta}(Z)$ for some $r \in \mathbb{R}$.

Next, note that the base surface Y (with its orientation reversed) is obtained from $\partial_h M$ by negative grafting; that is, $\overline{Y} = \mathrm{gr}_{-s\beta}(\partial_h M)$. Therefore

$$\overline{Y} = \mathrm{eq}_{(r-is)\beta}(Z). \tag{7.1}$$

Since the complex earthquake map

$$f : D(Z, \beta) \to T(S)$$

is a homeomorphism, there is a unique $t = r - is$ satisfying (7.1). Since M has holonomy $\eta_{t\beta}(Z)$, it is uniquely determined by the data (β, L).

Finally we show every (β, L) as above arises for some M, so p is surjective. To see this, recall that by Theorem 1.3 f maps the part of $D(Z, \beta)$ in the lower half-plane to the set of Riemann surfaces where β is longer than L. Since $L < \ell_\beta(\overline{Y})$, there exists $t = r - is$, $s < 0$ satisfying (7.1). Then the quasifuchsian manifold M with holonomy $\eta_{t\beta}(Z)$ has bending lamination $s\beta$ of length L and $\overline{\partial}_c M = Y$, so M lies in B_Y and it realizes the given data $([\beta], L)$. ■

Limit Bers slices. By Bers' inequality (3.1), the holomorphic Bers embeddings

$$\langle b_Y : T(S) \to V(S) \ : \ Y \in T(\overline{S}) \rangle$$

form a normal family on $T(S)$. Consider any sequence $Y_n \to \infty$ in $T(\overline{S})$ such that the embeddings b_{Y_n} converge. We call the map

$$b : T(S) \to V(S)$$

given by

$$b(X) \ = \ \lim b_{Y_n}(X) = \lim Q(X, Y_n)$$

a *limit Bers embedding*, and its image $B \subset V(S)$ a *limit Bers slice.*

Let us also assume that $Y_n \to [\lambda] \in \mathbb{P}\mathcal{ML}(S)$ in Thurston's compactification of Teichmüller space. Since $\dim T(S) = 1$ this means there are $C_n \to \infty$ such that

$$\frac{\ell_\alpha(Y_n)}{C_n} \to i(\alpha, \lambda) \tag{7.2}$$

for any $\alpha \in \mathcal{ML}(S)$ that is not proportional to λ. (See [FLP], [Bon2] or [Ot] for a presentation of Thurston's compatification.)

All groups in B are quasiconformally conjugate. One can show that the limit Bers slices are of two possible types. If $[\lambda]$ is represented by a simple closed curve, then every $M \in B$ is geometrically finite, with an accidental parabolic on $[\lambda]$. Otherwise, every $M \in B$ is totally degenerate, and we will see below (Corollary 7.6) that every $M \in B$ has ending lamination $[\lambda]$.[4]

A Kleinian manifold $M \in B$ has a unique boundary component $\partial_p M$ carrying the full fundamental group of M. Equivalently, the domain of discontinuity of the corresponding Kleinian group Γ has a unique invariant component Ω', and $\partial_p M \in P(S)$ is the marked projective surface Ω'/Γ.

It is not hard to verify that most of the discussion for boundaries of quasifuchsian manifolds carries over to $\partial_p M$. For example,

$$\partial_p M \quad = \quad \lim \partial_p Q(X, Y_n)$$

in $P(S)$, and $\partial_p M$ is conformally isomorphic to X. Letting $\partial_h M$ denote the face of the convex hull of M correspond to $\partial_p M$, and β its bending lamination, we have

$$\partial_p M \quad = \quad \mathrm{Gr}_\beta(\partial_h M).$$

By Theorem 2.4, the bending lamination β for M is the limit of those for $\partial_h Q(X, Y_n)$.

Since no $M \in B$ is Fuchsian, $\beta \neq 0$. Thus we can define a map

$$p : B \to \mathbb{P}\mathcal{ML}(S) \times \mathbb{R}^+$$

by

$$p(M) \quad = \quad \left([\beta], \ \frac{\ell_\beta(M)}{i(\beta, \lambda)} \right).$$

We now complete the proof of Theorem 1.6 by showing:

Theorem 7.5 *The map p establishes a homeomorphism*

$$B \cong (\mathbb{P}\mathcal{ML}(S) - [\lambda]) \times \mathbb{R}^+.$$

Proof. Consider any $(\beta, L) \in \mathcal{ML}(S) \times \mathbb{R}$ with $L > 0$ and $i(\beta, \lambda) \neq 0$. We will show:

(i) There exists an $M \in B$ with bending lamination proportional to β and $\ell_\beta(M) = L$;

(ii) This M is unique; and

(iii) The bending lamination of $M \in B$ is never proportional to λ.

[4]By a recent result of Minsky, the lamination $[\lambda]$ uniquely determines the limit slice B [Min].

These three results will complete the proof.

To begin, note for all $n \gg 0$ there are unique quasifuchsian groups $Q(X_n, Y_n)$ realizing the data $([\beta], L)$ on the part of their convex hull boundary facing X_n. Indeed, $\ell_\beta(Y_n) \to \infty$ by (7.2), so $\ell_\beta(Y_n) > L$ for $n \gg 0$, and thus by Theorem 7.4 there is a unique X_n such that $Q(X_n, Y_n)$ is pleated along $[\beta]$ and $\ell_\beta Q(X_n, Y_n) = L$.

Proof of (i). Pick a surface $Z \in T(S)$ with $\ell_\beta(Z) = L$. Since $Q(X_n, Y_n)$ realizes the data $([\beta], L)$ on its convex hull boundary, we can write

$$X_n = \mathrm{eq}_{t_n\beta}(Z)$$

and

$$\partial_p Q(X_n, Y_n) = \mathrm{Eq}_{t_n\beta}(Z)$$

for a unique $t_n \in \mathbb{H}$.

We claim $Q(X_n, Y_n)$ ranges in a compact subset of $V(S)$. Indeed, if X_n is bounded in $T(S)$, then $Q(X_n, Y_n)$ is bounded in $V(S)$ by Bers' inequality (3.1), so we have compactness. On the other hand, if $X_n \to \infty$ in $T(S)$, then $X_n \to [\beta]$ in Thurston's compactification $T(S) \cup \mathbb{P}\mathcal{ML}(S)$, because $\ell_\beta(X_n) \le L$ by Corollary 3.5. Since $Y_n \to [\lambda]$ and $\beta \cup \lambda$ binds the surface S, in this case the sequence $\langle Q(X_n, Y_n)\rangle$ is contained in a compact subset of $V(S)$ by Thurston's Double Limit Theorem [Th4], [Ot, Ch. 5].

Since $Q(X_n, Y_n)$ is bounded in $V(S)$, the sequence $|\operatorname{Re} t_n|$ is bounded by Theorem 4.1. But then $|\operatorname{Im} t_n|$ is also bounded, since the total bending along a short transversal to β on $\partial_h Q(X_n, Y_n)$ is less than 2π. Thus we can pass to a subsequence $\langle n_i\rangle$ such that $Q(X_{n_i}, Y_{n_i}) \to M$ in $V(S)$ and $t_{n_i} \to s \in \mathbb{H}$. Then $X_{n_i} \to X = \mathrm{eq}_{s\beta}(Z)$ in $T(S)$.

We claim $M = b(X)$ and M realizes the data $([\beta], L)$. Indeed,

$$M = \lim Q(X_{n_i}, Y_{n_i}) = \lim Q(X, Y_{n_i}) = b(X)$$

since the Teichmüller distance from X_{n_i} to X_∞ tends to zero. Similarly

$$\partial_p M = \lim \partial_p Q(X_{n_i}, Y_{n_i}) = \lim \mathrm{Eq}_{t_{n_i}\beta}(Z) = \mathrm{Eq}_{s\beta}(Z),$$

so the bending lamination of M is $[\beta]$ and $\ell_\beta(M) = \ell_\beta(Z) = L$.

Proof of (ii). Now consider any $M = b(X) \in B$ realizing the data $([\beta], L)$ on its convex core boundary. We will show $M = \lim Q(X_n, Y_n)$ and thus M is unique.

The bending lamination of M is $y\beta$ for some $y > 0$. Let $Z = \partial_h M$ and define

$$f(t) = \mu \circ \mathrm{Eq}_{t\beta}(Z),$$

so

$$f(iy) = \mu(\mathrm{Eq}_{iy\beta}(Z)) = \mu(\mathrm{Gr}_{y\beta}(Z)) = \mu(\partial_p M) = M.$$

Then the holomorphic disk

$$f : \mathbb{H} \to V(S)$$

meets the limit Bers embedding

$$b : T(S) \to V(S);$$

in fact $b(X) = f(iy) = M$. It is easy to see the intersection is isolated, so it persists under perturbation of b. This means $b = \lim b_{Y_n}$ admits a sequence $(X_n', t_n) \to (X, iy)$ such that

$$b_{Y_n}(X_n') = Q(X_n', Y_n) = f(t_n) \tag{7.3}$$

and thus

$$Q(X_n', Y_n) \to M.$$

By (7.3), the projective surfaces $Z_n' = \partial_p Q(X_n', Y_n)$ and $Z_n = \mathrm{Eq}_{t_n\beta}(Z)$ have the same holonomy. Since the developing map for Z_n' is univalent, so is that of any limit, and we conclude $Z_n' \to \partial_p M$. By continuity of complex earthquakes, $Z_n \to \mathrm{Eq}_{iy\beta}(Z) = \partial_p M$ as well. But the holonomy map $\mu : P(S) \to V(S)$ is a local homeomorphism (Theorem 2.1), so $Z_n' = Z_n$ for all $n \gg 0$.

Thus

$$\partial_p Q(X_n', Y_n) = \mathrm{Eq}_{t_n\beta}(Z)$$

for all $n \gg 0$, and therefore $Q(X_n', Y_n)$ realizes the data $([\beta], L)$ on its convex hull boundary. Since the manifold in the slice B_{Y_n} realizing this data is unique, we have $X_n = X_n'$ and $M = \lim Q(X_n, Y_n)$ as desired.

Proof of (iii). According to Thurston [Th1], [Bon1], for any hyperbolic manifold on the boundary of a Bers slice such as $M \in B$, there is at least one $[\epsilon] \in \mathbb{P}\mathcal{ML}(S)$ with $\ell_\epsilon(M) = 0$. The union of the supports of such ϵ's is the *ending lamination* of M. Since the ending lamination is a quasi-isometry invariant, it is the same for all $M \in B$. But for each $[\beta] \neq [\lambda]$, we have in (i) constructed an $M \in B$ with $\ell_\beta(M) > 0$. By a process of elimination, we find $\ell_\lambda(M) = 0$ for all $M \in B$, and thus $[\lambda]$ never occurs as the bending lamination.

Thus p maps B bijectively to $(\mathbb{P}\mathcal{ML}(S) - [\lambda]) \times \mathbb{R}^+$. ■

Remark. The last part of the proof also shows:

Corollary 7.6 *If $Q(X, Y_n) \to M \in \partial QF(S)$ then $Y_n \to [\lambda]$, where $[\lambda]$ is the ending lamination for M.*

Example: totally degenerate groups. For any pseudo-Anosov mapping class $\psi \in \mathrm{Mod}(S)$ and any $Y \in T(\overline{S})$, it is shown in [Mc2, §3.5] that the Bers slices $B_{\psi^n(Y)}$ converge to a limit Bers slice B_ψ that is independent of Y. A numerical example of a representation $[\rho] \in B_\psi$ is also given, where S is a torus with an orbifold point of order 2,

$$\pi_1(S) \;=\; \langle a, b \,:\, [a,b]^2 = 1\rangle,$$

ψ is the mapping class $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, and the bending locus is the simple closed curve $\langle a\rangle$ with length determined by $\operatorname{tr}\rho(a) = 3$ [Mc2, §3.7].

The preceding Theorem shows rigorously that there exists a *unique* totally degenerate group with these properties.

8 Grafting and cone-manifolds

We conclude with a result that relates grafting to the conjectural rigidity of cone-manifolds.

A *hyperbolic cone-manifold* (M^3, L) is a smooth manifold with a complete path metric that is a Riemannian hyperbolic metric except along a geodesic link $L \subset M^3$. Along a component L_i of L the metric has a cone-like singularity; a disk orthogonal to L has total angle θ_i at its intersection with L_i. If there are integers n_i such that $\theta_i = 2\pi/n_i$, then (M^3, L) is an orbifold.

Grafting gives a construction of hyperbolic cone-manifolds, as follows. Let γ_i be a collection of simple closed geodesics on $X \in T(S)$, and let $\lambda = \sum \alpha_i \gamma_i$ be a measured lamination supported on $\bigcup \gamma_i$. Initially let us also suppose that bending X along λ results in a quasifuchsian manifold Q. Then X and $Y = \mathrm{gr}_\lambda(X)$ can be thought of as components of the convex core boundary and conformal boundary of Q respectively.

Now remove the convex core from Q, retain the component joining X to Y, and double along X. The result is a cone manifold $M(X, \lambda)$ homeomorphic to $\mathrm{int}(S) \times (0, 1)$, containing X as a totally geodesic submanifold. Under the embedding $X \hookrightarrow M(X, \lambda)$, the geodesics γ_i map to the components L_i of the cone locus L; along L_i we have cone angle $\theta_i = 2(\pi + \alpha_i)$. The cone-manifold has a natural conformal boundary, $\partial M(X, \lambda) = Y \sqcup \overline{Y}$.

This quasifuchsian construction is useful to visualize $M(X, \lambda)$ when the bending is small, but in fact the cone-manifold $M(X, \lambda)$ can be constructed for any $\lambda \in \mathcal{ML}(S)$.

To see this, first let

$$C(\alpha, \ell) = \{(z,t) \in \mathbb{H}^3 \cong \mathbb{C} \times \mathbb{R} \,:\, 0 \leq \arg(z) \leq \alpha\}/\langle (z,t) \mapsto (e^{\ell}z, e^{\ell}t)\rangle.$$

Then $C(\alpha, \ell)$ is a hyperbolic manifold bounded by totally geodesic half-cylinders meeting with angle α along a geodesic of length ℓ. For $\alpha \geq 2\pi$ the region $C(\alpha, \ell)$ above should be interpreted as having multiple sheets.

Next, take the Fuchsian manifold $M(X, 0)$ in which X is realized as a totally geodesic surface, and cut along the half-cylinders through γ_i orthogonal to X. Finally, glue in two copies of $C(\alpha_i, \ell_i)$ along γ_i, one on each side of X, where $\ell_i = \ell_X(\gamma_i)$. The result is a cone-manifold $M(X, \lambda)$, now defined for all laminations and agreeing with the quasifuchsian construction for small bending. We still have $\partial M(X, \lambda) = Y \sqcup \overline{Y}$.

A traditional geometrically finite manifold M is determined up to isometry by its topology, the parabolic locus and the conformal structure on ∂M. For cone-manifolds one has:

Conjecture 8.1 (Rigidity) *A geometrically finite cone-manifold (M, L) is determined up to isometry by the topology of the pair (M, L), the parabolic locus, the cone angles θ_i along L_i, and the conformal structure on ∂M.*

Local rigidity is known for closed cone-manifolds with cone angles $0 < \theta_i < 2\pi$, by work of Hodgson and Kerckhoff [HK]. The case of cone-manifolds with angles in excess of 2π, such as $M(X, \lambda)$, is currently open.

We may now state:

Theorem 8.2 *Local rigidity of geometrically finite cone-manifolds implies the grafting map*

$$gr_\lambda : T(S) \to T(S)$$

is a homeomorphism for all laminations $\lambda \in \mathcal{ML}(S)$ supported on simple closed curves.

Proof. We first show that local rigidity implies gr_λ is locally injective.

Suppose $X_n \to X$ in $T(S)$ and $Y = \mathrm{gr}_\lambda(X) = \mathrm{gr}_\lambda(X_n)$ for all n. Then the cone-manifolds $M(X_n, \lambda)$ have the same conformal boundary and bending angles for all n. By local rigidity, $M(X, \lambda) = M(X_n, \lambda)$ for all n sufficiently large. But X_n is determined by $M(X_n, \lambda)$, since it is isometric to a totally geodesic surface in $M(X_n, \lambda)$. Therefore $X_n = X$ for $n \gg 0$, and grafting is locally injective.

Since gr_λ is proper (Theorem 3.6), local injectivity implies global homeomorphism. ∎

Similarly, positive results on injectivity of grafting (such as Theorems 1.4 and 3.6) can be seen as evidence of rigidity in the presence of cone angles greater than 2π.

A Appendix: The topology of quasifuchsian space

This appendix provides some examples that illustrate the subtlety of projective structures and algebraic limits. We find that despite the disk-convexity of $QF(S)$ in $V(S)$ (Theorem 5.1), the geometry of $\overline{QF(S)}$ is quite complicated. We will show:

Theorem A.1 *The closure of $QF(S)$ in $V(S)$ is not a topological manifold with boundary.*

In fact there exists a $[\rho] \in \partial QF(S)$ such that $U \cap QF(S)$ is disconnected for any sufficiently small neighborhood U of $[\rho]$.

The case of a punctured torus. Consider the case where S is a surface of genus one with one boundary component. Using ending laminations, one can prolong Bers' isomorphism

$$QF(S) \cong T(S) \times T(\overline{S})$$

to natural map

$$\overline{QF(S)} \xrightarrow{\nu} (\widehat{T}(S) \times \widehat{T}(\overline{S})) - D,$$

where $\widehat{T}(S) = T(S) \cup \mathbb{P}\mathcal{ML}(S)$ is Thurston's compactification of Teichmüller space, $\overline{S}$ is S with its orientation reversed, and

$$D \subset \mathbb{P}\mathcal{ML}(S) \times \mathbb{P}\mathcal{ML}(\overline{S})$$

is the diagonal. The target of ν *is* a manifold with boundary (it is homeomorphic to a 4-ball with an unknotted circle removed from its boundary), so the Theorem shows ν is not a homeomorphism.

On the other hand, Minsky has shown that ν is a bijection and that the closure of any Bers slice in $QF(S)$ is homeomorphic to a disk [Min]. Thus each Bers slice is tame, but the family of all slices is nevertheless quite complicated. For further discussion of the continuity of ν see [Min, §12.3].

The quasifuchsian space of a punctured torus can also be studied experimentally. Figure 10 shows a computer-generated linear slice through $QF(S)$, revealing some of the topological complexity suggested by Theorem A.1. Since $\pi_1(S) = \langle a, b \rangle$ and $[\rho(a), \rho(b)]$ is parabolic, points in $V(S)$ are essentially determined by the data $(\operatorname{tr} \rho(a), \operatorname{tr} \rho(b)) = (\alpha, \beta) \in \mathbb{C}^2$, which can be specified arbitrarily. In this picture β is fixed at $2 + 6i$, and α ranges in the square of width 8 centered at $\alpha = 2 + 6i$ in $\mathbb{C}$; the white region shows where $(\alpha, \beta) \in QF(S)$. By the results of §5, each component of the white

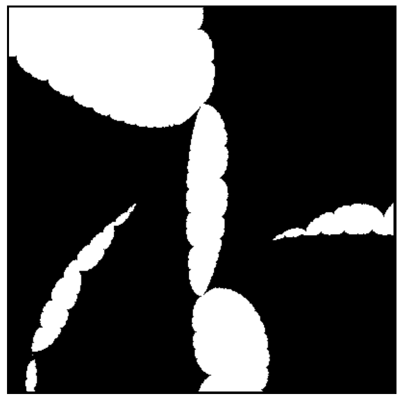

Figure 10. A slice of $QF(S)$.

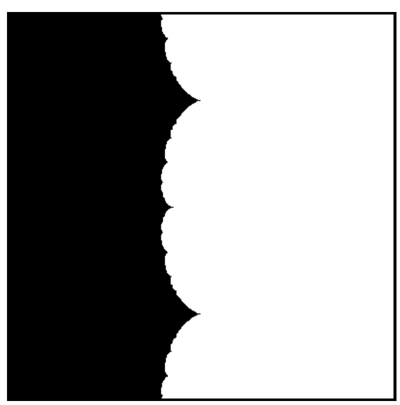

Figure 11. The Maskit embedding of $T(S)$.

is a disk; as shown in the Figure, in general the white region has many components, some of which touch.

The closely related Maskit embedding of Teichmüller space, lying in the slice $\beta = 2$, has been studied in detail by D. Wright [MMW], [Wr], [KS]. The Maskit embedding is disjoint from $QF(S)$ (since $\rho(b)$ is parabolic), but its boundary is a cusped curve locally of the same character as the boundary in Figure 10. However the Maskit embedding is topologically a single disk, and therefore simpler than a general slice of $QF(S)$ (see Figure 11).

Exotic projective structures. The proof of Theorem A.1 will use the holonomy map $\mu : P(S) \to T(S)$. Let us say a projective surface with quasifuchsian holonomy is *standard* if its developing map is injective; otherwise it is *exotic*.

Under μ the standard surfaces map bijectively $QF(S)$; the standard surface corresponding to a quasifuchsian manifold $M \in QF(S)$ is simply $\partial_p M$. The other components of $\mu^{-1}(QF(S))$ consist of exotic surfaces; examples are discussed in [Msk], [Hej] and [Gol]. In [Tan] grafting is used to produce exotic surfaces in every fiber of $P(S) \to T(S)$ Here we will show:

Theorem A.2 *There exists a projective surface $Z \in P(S)$ which is a limit of both standard and exotic surfaces.*

Corollary A.3 *The set $U(S)$ of projective surfaces with injective developing maps is not open in $K(S)$, the set of surfaces with discrete holonomy.*

Proof. The surface Z is in $U(S)$, since it is a limit of standard surfaces; but Z is also a limit of exotic surfaces, and the latter are contained in $K(S) - U(S)$. ■

This Corollary suggests that $K(S)$ may have a combinatorial structure similar to that of the Mandelbrot set, with $U(S)$ playing the role of the main cardioid and with limbs of $K(S)$ attached along the cusps in $\partial U(S)$. The space $K(S)$ is studied in [ST]. We also remark that by Minsky's work, for S a punctured torus the ending invariants give a *homeomorphism*

$$U(S) \xrightarrow{\nu} T(S) \times \widehat{T}(\overline{S});$$

working in $P(S)$ is similar to working in a Bers' slice, and the boundary behavior of quasifuchsian space seems to be tamer in $P(S)$ than in $V(S)$.

Our examples are based on those of Kerckhoff and Thurston [KT], with an added twist discovered by Anderson and Canary [AC]. The key ingredient is the following:

Lemma A.4 *There exists a sequence of quasifuchsian manifolds $M_n \in QF(S)$ such that:*

1. *The algebraic limit M_∞ of M_n lies in the boundary of a Bers' slice B_Z, but*

2. *The geometric limit N of M_n is distinct from M_∞, and $\pi_1(M_\infty) \subset \pi_1(N)$ does not correspond to any component of ∂N.*

Proof. Choose a nonempty system of disjoint simple closed curves $C \subset S$, none peripheral and no two parallel. The proof will show:

- We can take $M_n = Q(\tau^n(X), \tau^{2n}(Y))$, where τ is a product of Dehn twists around the curves in C;

- In the algebraic limit M_∞, the curves C become rank-one cusps; and

- In the geometric limit N, the curves C give rise to cusps of rank two.

To begin the construction, let N (the candidate geometric limit) be a geometrically finite Kleinian manifold homeomorphic to $\mathrm{int}(S) \times [0,1] - (C \times 1/2)$. There are many ways to construct such a manifold. For example, when C is a maximal system of disjoint simple curves, N can be constructed from a maximal cusp N_0 in the boundary of a Bers slice, chosen so all curves in C have been pinched to rank-one cusps. To build N, cut away the ends of N_0 bounded by totally geodesic triply-punctured spheres, and then double across the resulting boundary.

Given N, choose a basis $\langle \lambda_i, \mu_i \rangle$ for the fundamental group of each rank two cusp, with λ_i homotopic to $C_i \times 1$ and μ_i trivial in $S \times [0,1]$. Performing $(1,n)$ Dehn filling on all the cusps, we obtain a new manifold N_n together with an inclusion

$$F_n : N \to N_n.$$

Here $\pi_1(N_n)$ is obtained from $\pi_1(N)$ by adding the relations $\mu_i = \lambda_i^n$, and N_n is homeomorphic to $S \times [0,1]$.

By a result of Thurston, there are complete hyperbolic metrics on N_n converging to N geometrically; that is, so the metric distortion of the filling map F_n tends to zero on compact subsets of N as $n \to \infty$. In fact we may take $N_n = Q(X, \tau^n Y)$, where $X \sqcup Y$ is the conformal boundary of N and τ is a simultaneous Dehn twist around the components of C (see [KT]). Here the marking $S \to N_n$ is the composition of F_n with the inclusion $S \to S \times 1 \subset N$.

Now comes the twist. Start with the embedding $f_0 : \mathrm{int}(S) \to N$ given by $f_0(x) = (x, 3/4)$, and surger it to obtain immersion $f : \mathrm{int}(S) \to N$

wrapping once around each boundary component of a tubular neighborhood of $C \times 1/2$. The surgery is carried out as follows: along each component C_i of C, insert a band $C_i \times [0,1]$ into S, and send it to the tube $C_i \times S^1$ around $C_i \times 1/2$ using the map $[0,1] \to S^1$ that identifies the endpoints of the interval.

Let $M_n \in QF(S)$ be the hyperbolic manifold N_n marked by the composition

$$S \xrightarrow{f} N \xrightarrow{F_n} N_n.$$

Then $M_n = Q(\tau^n(X), \tau^{2n}(Y))$. (Note that $F_n \circ f$ is homotopic to an *embedding* of S into N_n, since the cusp has been filled.) Since F_n converges to an isometry, M_n converges geometrically to N and algebraically to the subgroup M_∞ of N represented by $f_*(\pi_1(S))$. Since $f : \mathrm{int}(S) \to N$ is not homotopic into ∂N, $\pi_1(M_\infty)$ does not represent the fundamental group of either component of ∂N. Finally $\pi_1(M_\infty)$ is a geometrically finite surface group, whose only accidental parabolics correspond to C. Therefore M_∞ lies on the boundary of some Bers slice B_Z. ∎

Example. The case where C is a separating curve on a surface of genus 2 is shown in Figure 12. The manifold N is homotopy equivalent to the union of S and a torus along C, so its fundamental group is given by

$$\pi_1(N) = \langle a, b, c, d, \mu, \lambda \ : \ \lambda = [a,b] = [c,d], [\mu, \lambda] = 1 \rangle.$$

The surface $S \times 1 \subset N$ corresponds to the subgroup generated by $\langle a, b, c, d \rangle$, while the immersed surface corresponds to $\langle a, b, \mu c \mu^{-1}, \mu d \mu^{-1} \rangle$.

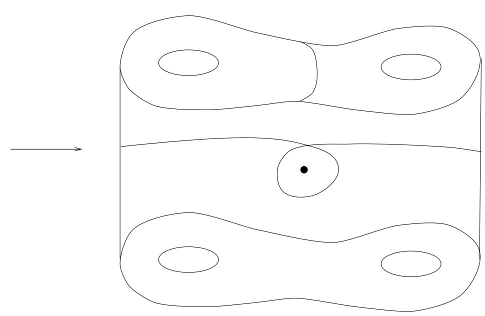

Figure 12. An immersed surface wrapping once around the cusp.

The covering $\pi : \mathrm{int}(M_\infty) \to \mathrm{int}(N)$ is depicted in Figure 13. Note that $\pi : \partial_h M \to N$ is a pleated surface modeling the immersed surface $f : S \to N$.

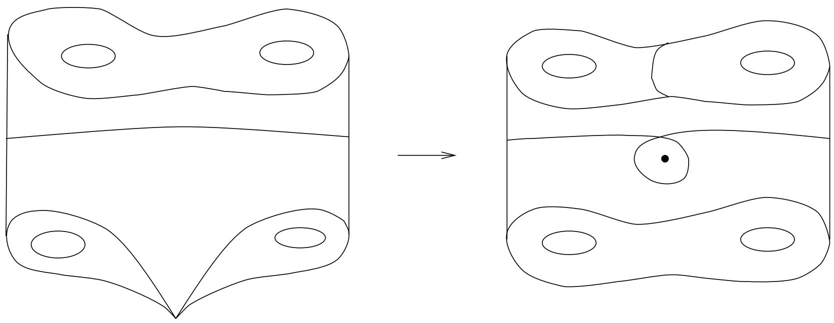

Figure 13. The covering space corresponding to the immersed surface.

Proof of Theorem A.2. Consider any M_n converging to M_∞ algebraically and to N geometrically as in the Lemma. Since M_∞ is in the boundary of a Bers slice, the projective surface $Z = \partial_p M_\infty$ is defined and Z is the limit of standard surfaces $Z'_n = \partial M'_n$, where $M'_n \in B_Z$ tends to M_∞.

The holonomy map is a local homeomorphism, so there also are projective surfaces $Z_n \to Z$ with $\mu(Z_n) = M_n$. Since Z does not represent a boundary component of N, the image $\delta(\widetilde{Z})$ of its universal cover under the developing map meets the limit set Λ of $\pi_1(N)$. But $M_n \to N$ geometrically, so $\Lambda \subset \liminf \Lambda_n$, where Λ_n is the limit set of $\pi_1(M_n)$. Thus the developing image of $\widetilde{Z_n}$ meets Λ_n for all $n \gg 0$. Since Λ_n is the limit set of $\mu(Z_n)$, we have shown Z_n is exotic for all $n \gg 0$. ∎

Proof of Theorem A.1. We have seen there is a projective surface such that $Z = \lim Z_n = \lim Z'_n$, where Z_n are exotic and Z'_n are standard. Let U be a neighborhood of $[\rho] = \mu(Z) \in V(S)$. Since μ is a local homeomorphism, a neighborhood V of Z maps homeomorphically to U when U is sufficiently small. Since V contains both standard and exotic surfaces, $V \cap \mu^{-1}(QF(S))$ is disconnected; therefore $U \cap QF(S)$ is also disconnected.

By the theory of holomorphic motions (Theorem 5.2), $QF(S) \subset V(S)$ is the interior of its closure (since all groups in $\mathrm{int}(\overline{QF(S)})$ are quasiconformally conjugate). If $\overline{QF(S)} \subset V(S)$ were a topological manifold, then there would be a small neighborhood U of $[\rho] \in \partial QF(S)$ meeting the manifold's interior $QF(S)$ in a connected set, contrary to what we have just seen. Thus $\overline{QF(S)}$ is not a manifold with boundary. ∎

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