Further Pathologies in Algebraic Geometry

The Harvard community has made this article openly available. Please share how this access benefits you. Your story matters

<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Published Version</td>
<td>doi:10.2307/2372870</td>
</tr>
<tr>
<td>Citable link</td>
<td><a href="http://nrs.harvard.edu/urn-3:HUL.InstRepos:3446006">http://nrs.harvard.edu/urn-3:HUL.InstRepos:3446006</a></td>
</tr>
<tr>
<td>Terms of Use</td>
<td>This article was downloaded from Harvard University’s DASH repository, and is made available under the terms and conditions applicable to Other Posted Material, as set forth at <a href="http://nrs.harvard.edu/urn-3:HUL.InstRepos:dash.current.terms-of-use#LAA">http://nrs.harvard.edu/urn-3:HUL.InstRepos:dash.current.terms-of-use#LAA</a></td>
</tr>
</tbody>
</table>
FURTHER PATHOLOGIES IN ALGEBRAIC GEOMETRY.* 1

By David Mumford.

The following note is not strictly a continuation of our previous note [1]. However, we wish to present two more examples of algebro-geometric phenomena which seem to us rather startling. The first relates to characteristic \( p \) behaviour, and the second relates to the hypothesis of the completeness of the characteristic linear system of a maximal algebraic family. We will use the same notations as in [1].

I.

The first example is an illustration of a general principle that might be said to be indicated by many of the pathologies of characteristic \( p \):

A non-singular characteristic \( p \) variety is analogous to a general non-Kähler complex manifold; in particular, a projective embedding of such a variety is not as "strong" as a Kähler metric on a complex manifold; and the Hodge-Lefschetz-Dolbeault theorems on sheaf cohomology break down in every possible way.

In this case we wish to look at the two dimensional cohomology of an algebraic surface \( F \), non-singular, and of any characteristic but 0. The surface we shall choose will (a) be specialization of a characteristic 0 surface \( F' \), and (b) will satisfy \( q = h^{0,1} = h^{1,0} \). Consequently the second Betti number \( B_2 \) is the same, whether defined (i) as that of \( F' \) in the topological sense, (ii) as \( h^{2,0} + h^{1,1} + h^{0,2} \), or (iii) following Igusa [2], as \( \text{Deg}(c_2) + 4q - 2 \). Let \( \rho \) be the base number of \( F \). Igusa showed that, in fact, \( B_2 \geq \rho \). However, in characteristic 0, one has the stronger result, \( B_2 = h^{2,0} + h^{1,1} + h^{0,2} \geq 2\rho + \rho \) (where \( \rho = h^{2,0} = h^{0,2} \) is the geometric genus of \( F \)) as a result of the Hodge-Dolbeault theorems. Therefore the question arises whether this stronger inequality is valid in characteristic \( p \). The answer is no.

A rather complicated example was discovered in 1961 by J. Tate and A. Ogus. Here is a very simple example: let \( E \) be a super-singular elliptic

---

* Received July 30, 1962.
1 This work was supported by the Army Research Office (Durham).
curve of characteristic $p$ (i.e. such that the rank of $End(E)$ is 4). Let $F = E \times E$. In this case, in fact:

$$\rho = B_2 = 6; \quad p_g = 1.$$ 

Here $p_g = 1$ since the sheaf $\Omega_F^2 \cong \mathcal{O}_F$; and $B_2 = 6$ by Igusa's definition, for example, since $\text{Deg}(c_2) = 0$, and $q = 2$. Finally $\rho = 6$ since in general, for any two elliptic curves $E_1$ and $E_2$, one knows that the base number $\rho$ for $E_1 \times E_2$ equals 2 plus the rank of $\text{Hom}(E_1, E_2)$.

There remains one outstanding conjecture still neither proven nor disproven in characteristic $p$, which according to the general principle mentioned above ought to be false. This is the Regularity of the Adjoint, which may be stated as follows: if $V$ is a non-singular projective surface and if $H$ is a non-singular hyperplane section, then

$$H^1(\mathcal{O}_V) \to H^1(\mathcal{O}_H)$$

is injective.

II.

The second example concerns space curves in characteristic 0. Let $A$ be any family of non-singular space curves, and let $a \in A$ represent the curve $\gamma \subset P_3$. Let $T_a$ denote the Zariski tangent space to $A$ at $a$, and let $N$ denote the sheaf of sections of the normal bundle to $\gamma$ in $P_3$. Then it is well-known [3] that there is a "characteristic" map:

$$T_a \to H^0(N).$$

The problem of completeness consists in asking when, for given $\gamma$, there is a family $A$ containing $\gamma$ such that the characteristic map is surjective. Kodaira [3] has shown that such a family exists if $H^1(N) = (0)$. Our example shows that if $H^1(N) \neq (0)$, then there need not be such a family.

In fact, in our example, this incompleteness holds for every curve in an open set of the corresponding Chow variety. Consequently, it is also an example where the Hilbert scheme [4] has a multiple component, i.e. is not reduced at one of its generic points. Another corollary of this example is obtained by blowing up such a space curve $\gamma \subset P_3$ to a surface $E$ in a new three-dimensional variety $V_3$. Then Kodaira [5] has shown essentially that the local moduli scheme of the variety $V_3$ is isomorphic to the germ of the
Hilbert scheme of $P_3$ at the point corresponding to $\gamma$. Therefore we have constructed a non-singular projective three-dimensional variety whose local moduli scheme is nowhere reduced; in other words, any small deformation of $V_3$ is a variety the number of whose moduli is less than the dimension of $H^1(\Theta)$ (where $\Theta$ is the sheaf of vector fields).

The curves $\gamma$ that we have in mind have degree 14 and genus 24. In the following, $h$ will stand for the divisor class on $\gamma$ induced by plane sections, and $K_{\gamma}$ will stand for the canonical divisor on $\gamma$; also $F$ will stand for a cubic or quartic surface in $P_3$, and $H$ will stand for the (Cartier) divisor class on $F$ induced by plane sections. The first step is partial classification of all space curves of this degree and genus, which confirms the results of M. Noether's well-known table [6].

(A) Any non-singular space curve $\gamma$ of degree 14 and genus 24 is contained in a pencil $P$ of quartic surfaces.

Proof. Since $\text{Deg}(4h) = 56$, and $\text{Deg}(K_{\gamma}) = 46$, the linear system $|4h|_{\gamma}$ is non-special, and has dimension $56 - 24 = 32$. Since there is a 34-dimensional family of quartics in $P_3$, (A) follows.

There are 2 cases: (a) the pencil has no fixed components, and (b) the pencil has fixed components. In case (a), note first that if $F'$ and $F''$ span $P$, then $F' \cdot F'' = \gamma + c$, where $c$ is a conic. Now $c$ has at most double points, hence $\gamma + c$ has at most triple points. Therefore no point $x$ is a double point for both $F'$ and $F''$. Noting that both $F'$ and $F''$ are non-singular and transversal along $\gamma - c$, hence at all but a finite number of points of $\gamma$, it follows that almost every $F \in P$ is non-singular everywhere along $\gamma$.

(B) Every algebraic family of space curves of type (a) has dimension less than or equal 56.

Proof. It is enough to show that every family of pairs $(\gamma, F)$ consisting of such curves $\gamma$, and quartics $F \supset \gamma$, $F$ being non-singular along $\gamma$, has dimension at most 57. Now since all such quartics contains conics, they are not generic [?], and there is at most a $34 - 1 = 33$ dimensional family of quartics $F$ involved in such a family of pairs. Moreover, the dimension of the set of all $\gamma$ on one such $F$ can be computed from the Riemann-Roch theorem on $F$:

\footnote{Here and below, $|D|_V$ always means the linear system on $V$ in which the Cartier divisor $D$ varies. Also, $(D^2)_V$ always denotes the self-intersection of $D$, as a divisor class on $D$ (assuming $D$ effective).}
\[ \dim |\gamma|_F = \frac{\text{Deg}(\gamma^2)_F}{2} + 1 + \{\dim H^i(\mathcal{O}_F(\gamma)) - \dim H^2(\mathcal{O}_F(\gamma))\}. \]

But \((\gamma^2)_F \equiv K_\gamma\) on \(\gamma\), hence \(\text{Deg}(\gamma^2)_F = 46\). Moreover, \(H^4(\mathcal{O}_F(\gamma))\) is dual to \(H^{2-i}(\mathcal{O}_F(-\gamma))\) by Serre duality. This cohomology group can be computed from the exact sequence:

\[ 0 \rightarrow \mathcal{O}_F(-\gamma) \rightarrow \mathcal{O}_F \rightarrow \mathcal{O}_\gamma \rightarrow 0. \]

It follows that both are zero, hence \(\dim |\gamma|_F = 24\). Therefore, indeed, the set of pairs \((\gamma, F)\) has dimension at most \(33 + 24 = 57\).\(^3\)

Now consider case (b). Such a \(\gamma\) must be contained in a reducible quartic, hence in a plane, a quadric, or a cubic surface. The first two possibilities are readily checked and it happens that they contain no curves of the required degree and genus. Moreover, such a curve is contained in a unique cubic surface \(F\), because \(\text{Deg}(\gamma) = 14 > 9 = \text{Deg}(F' \cdot F'')\), for two distinct cubic surfaces \(F'\) and \(F''\). We will say that \(\gamma\) is of type \((b_0)\) if the cubic \(F\) is non-singular; otherwise, we will say that \(\gamma\) is of type \((b_1)\).

(C) Every maximal algebraic family of curves \(\gamma\) of type \((b_0)\) has dimension 56.

Proof. Let \(\gamma\) be a curve of type \((b_0)\), and let \(F\) be the corresponding cubic surface. Since \(K_F = -H\), by the Riemann-Roch theorem on \(F\):

\[ \dim |\gamma|_F = \frac{\text{Deg}(\gamma \cdot \gamma + H)_F}{2} + \{\dim H^i(\mathcal{O}_F(\gamma)) - \dim H^2(\mathcal{O}_F(\gamma))\}. \]

But \(K_F = \gamma \cdot (\gamma + K_F)\), hence \(46 = \text{Deg}(\gamma^2)_F - \text{Deg}(\gamma \cdot H)_F = \text{Deg}(\gamma^2)_F - 14\), hence \(\text{Deg}(\gamma^2)_F = 60\). Also, \(H^i(\mathcal{O}_F(\gamma))\) is dual to \(H^{2-i}(\mathcal{O}_F(-H - \gamma))\), and this group can be computed from the exact sequence:

\[ 0 \rightarrow \mathcal{O}_F(-H - \gamma) \rightarrow \mathcal{O}_F \rightarrow \mathcal{O}_{(H, \gamma)} \rightarrow 0. \]

Since \(H + \gamma\) is a reduced and connected curve, \(H^0(\mathcal{O}_{H, \gamma}) = k\) (constants), and this implies \(H^1(\mathcal{O}_F(-H - \gamma)) = (0)\) for \(i = 1\) and \(2\). Putting all this together, we see that \(\dim |\gamma|_F = 37\). Since there is a 19-dimensional family

\(^3\)It may be objected that we have used the Riemann-Roch theorem, and Serre duality as though \(F\) were non-singular. But since \(F\) is non-singular along \(\gamma\), the former can be proved by means of the exact sequence:

\[ 0 \rightarrow \mathcal{O}_F \rightarrow \mathcal{O}_F(\gamma) \rightarrow \mathcal{O}_F(\gamma^2)_F \rightarrow 0. \]

And the latter can be proven either (a) directly by resolving the singularities of \(F\) and comparing the cohomology on \(F\) and on its resolution, or (b) as a consequence of Grothendieck's general theory [8]. In the second case, one merely has to note that \(F\) is always a Cohen-Macaulay variety; and since it is a quartic surface the canonical sheaf is simply \(\mathcal{O}_F\) itself.
of cubic surfaces, (C) follows if we show that a generic $\gamma$ in a maximal algebraic family is contained in a generic cubic surface. But let $\gamma \subset F$ be any curve of the family. Then recalling that the divisor class group of any non-singular cubic surface is the same as that of any other, it follows that if the set of all non-singular cubic surfaces are suitably parametrized the invertible sheaf $\mathcal{O}_F(\gamma)$ will be a specialization of an invertible sheaf $L$ defined on the generic cubic surface $F^*$. And since $H^i(\mathcal{O}_F(\gamma)) = (0)$ for $i = 1$ and 2, by the upper semi-continuity of cohomology [9], we conclude that $H^i(L) = (0)$ for $i = 1$ and 2, and that all sections of $\mathcal{O}_F(\gamma)$ are specializations of sections of $L$. Therefore $\dim H^0(L) = 38$; and since almost all sections of $\mathcal{O}_F(\gamma)$ are non-singular, so are almost sections of $L$. Hence there is a non-singular $\gamma^* \subset F^*$ specializing to $\gamma \subset F$. QED.

Now suppose $C$ is the Chow variety of non-singular curves of degree 14, and genus 24. Let $C_b \subset C$ be the locus of curves of type $(b)$, and let $C_{b_1} \subset C_b$ be the locus of curves of type $(b_1)$. Then it is clear that $C_b$ and $C_{b_1}$ are closed (possibly reducible) subvarieties of $C$. By (B) and (C), every component of $C - C_b$ has dimension $\leq 56$, and every component of $C_b - C_{b_1}$ has dimension $= 56$. Therefore if $C_0$ equals $C$ minus $C_{b_1}$ and minus the closure of $C - C_b$, $C_0$ is an open set in the Chow variety, of dimension 56, and parametrizing almost all curves of type $(b_0)$.

We shall now single out a set of components of $C_0$ such that, if $N$ is the normal sheaf to a $\gamma$ in one of these components, then $\dim H^0(N) = 57$. In fact, we say that $\gamma \subset F$ is of type $(b'_0)$ if there is a line $E$ on $F$ such that $\gamma \equiv 4H + 2E$ on $F$. Then the corresponding $C'_0 \subset C_0$ which is the locus of such curves is clearly closed in $C_0$. But it is also open: if $\gamma^* \subset F^*$ specializes to $\gamma \subset F$, and if $\gamma \equiv 4H + 2E$ on $F$, then first of all, there is a line $E^* \subset F^*$ (possibly only rationally defined after a suitable base extension) which specializes to $E$; and secondly, since the divisor class group is discrete and constant for all non-singular cubics,

$$\gamma - 4H - 2E \equiv 0$$

implies $\gamma^* - 4H^* - 2E^* \equiv 0$.

Therefore $\gamma^*$ is of type $(b'_0)$.

(D) If $\gamma \subset F$ is of type $(b'_0)$, then $\dim H^0(N) = 57$.

Proof. Let $N_F$ be the sheaf of normal vector fields to $\gamma$ and in $F$, and let $N_P$ be the sheaf of normal vector fields to $F$, and in $P_3$, which are defined along $\gamma$. Then we have the sequence:

$$0 \rightarrow N_F \rightarrow N \rightarrow N_P \rightarrow 0.$$
But if \( D \) is a non-singular divisor on a non-singular variety \( V \), then its normal sheaf is isomorphic to \( \mathcal{O}_D((D^2)_V) \). Therefore \( N_F \cong \mathcal{O}_\gamma((\gamma^2)_F) \) and \( N_F \cong \mathcal{O}_\gamma(3h) \). But since \( K_\gamma \equiv (\gamma^2)_F + \gamma \cdot K_F \equiv (\gamma^2)_F - h \), it follows that \((\gamma^2)_F\) is a non-special divisor, of degree 60 in fact. Therefore \( H^1(N_F) = (0) \) and \( \dim H^0(N_F) = 60 - (24 - 1) = 37 \). On the other hand, by the Riemann-Roch theorem for curves,

\[
\dim H^0(\mathcal{O}_\gamma(3h)) = 42 - (24 - 1) + \dim H^0(\mathcal{O}_\gamma(K_\gamma - 3h)) = 19 + \dim H^0(\mathcal{O}_\gamma(6h - 4h)) = 19 + \dim H^0(\mathcal{O}_\gamma(2\gamma \cdot E))
\]

(using the hypothesis \( \gamma \equiv 4H + 2E \)). But now, use the exact sequence:

\[
0 \to \mathcal{O}_F(-4H) \to \mathcal{O}_F(2E) \to \mathcal{O}_\gamma(2\gamma \cdot E) \to 0.
\]

It is readily seen that \( H^i(\mathcal{O}_F(-4H)) = (0) \) for \( i = 0 \) and 1, and that \( \dim H^0(\mathcal{O}_F(2E)) = 1 \). Putting all this information together we conclude:\n\[
\dim H^0(N) = 3\gamma + 19 + 1 = 57. \quad \text{QED}
\]

It remains only to note:

(E) If \( F \) is any non-singular cubic surface, and \( E \subseteq F \) is any line, there exist non-singular curves \( \gamma \in |4H + 2E| \), and they have degree 14 and genus 24.

Proof. The degree and genus of such a \( \gamma \) are computed by the usual formulae, recalling that \( \text{Deg}(E^2)_F = -1 \). To see that such a \( \gamma \) exists, it suffices, by the characteristic 0 Bertini theorem, to prove that \( |4H + 2E| \) has no base points. But the only possible base points are the points of \( E \), and we use the exact sequence:

\[
0 \to \mathcal{O}_F(4H + E) \to \mathcal{O}_F(4H + 2E) \to \mathcal{O}_E(2) \to 0.
\]

Since the sections of \( \mathcal{O}_E(2) \) have no base points, it suffices to prove \( H^1(\mathcal{O}_F(4H + E)) = (0) \). But this follows from the sequence:

\[
0 \to \mathcal{O}_F(4H) \to \mathcal{O}_F(4H + E) \to \mathcal{O}_E(3) \to 0,
\]

since \( H^1(\mathcal{O}_F(4H)) = (0) \), and \( H^1(\mathcal{O}_E(3)) = (0) \). \quad \text{QED.}

Harvard University.
REFERENCES.