Cyclotomic factors of Coxeter polynomials

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Abstract

In this paper we show that the cyclotomic factors of the $E_n$ Coxeter polynomials depend only on the value of $n \mod 360$, and come exclusively from spherical subdiagrams.

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1 Introduction

In this paper we determine which roots of unity are zeros of the $E_n$ Coxeter polynomial. We show these roots come exclusively from splittings of $E_n$ into spherical subdiagrams; in particular they always have order 2, 3, 5, 8, 12, 18, or 30, and they only depend on the value of $n \mod 360$ (provided we exclude the special case $n = 9$).

The proof uses Mann’s theorem on linear relations between roots of unity, and generalizes to other sequences of Coxeter diagrams where nodes are added to a separating edge.

\[ \begin{array}{cccccccc}
1 & 2 & 4 & 5 & 6 & 7 & n-1 & n \\
\end{array} \]

Figure 1. The $E_n$ diagram.

The $E_n$ diagram. Coxeter systems are a useful source of Salem numbers, Pisot numbers and other interesting algebraic integers. For example, the smallest known Salem number arises from the Coxeter system $E_{10}$.

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The $E_n$ Coxeter diagram, defined for $n \geq 3$, is shown in Figure 1. Note that $E_3 \cong A_2 \oplus A_1$. The $E_n$ diagram determines a quadratic form $B_n$ on $\mathbb{Z}^n$, and a reflection group $W_n \subset O(\mathbb{Z}^n, B_n)$ (see §3). The product of the generating reflections is a Coxeter element $w_n \in W_n$; it is well-defined up to conjugacy, since $E_n$ is a tree [Hum, §8.4].

The Coxeter number $h_n$ is the order of the Coxeter element $w_n \in W_n$, and its characteristic polynomial

$$E_n(x) = \det(xI - w_n)$$

is the Coxeter polynomial. Explicitly, for $n \geq 3$ we have:

$$E_n(x) = \frac{x^{n-2}Q(x) + R(x)}{(x-1)},$$

where $Q(x) = x^3 - x - 1$ and $R(x) = x^3 + x^2 - 1$. (See e.g. [MRS, Lemma 5], [Hir2, §4.2] or Corollary 4.3 below.)

We can write $E_n(x)$ uniquely as a product of monic integral polynomials

$$E_n(x) = C_n(x)S_n(x),$$

where the zeros of the cyclotomic factor $C_n(x)$ are roots of unity, and those of the Salem factor $S_n(x)$ are not. Table 2 lists $E_n(x)$ for $n \leq 10$, along with its factorization into irreducibles and the Coxeter number $h_n$. Here $\Phi_k(x)$ is the cyclotomic polynomial for the primitive $k$th roots of unity.

The spherical and affine cases. Since $E_i$ is a spherical diagram ($B_i$ is positive definite) when $3 \leq i \leq 8$, we have $E_i(x) = C_i(x)$ (and $S_i(x) = 1$) in this range.

The diagram $E_9$ is the affine version of $E_8$; its Coxeter element has infinite order, but still $E_9(x) = C_9(x)$. This is the only case where $E_n(x)$ has a multiple root (see Lemma 2.4 below).

The hyperbolic case. For $n \geq 10$, the diagram $E_n$ is hyperbolic; that is, the signature of $B_n$ is $(n-1, 1)$. By [A'C] this implies that the factor $S_n(x)$ is a Salem polynomial: it is an irreducible, reciprocal polynomial, with a unique root $\lambda > 1$ outside the unit disk. For $n = 10$, $E_{10}(x)$ coincides with Lehmer’s polynomial, and its root $\lambda \approx 1.1762808 > 1$ is the smallest known Salem number.

We can now state our main result on the Coxeter polynomials $E_n(x)$.

**Theorem 1.1** For all $n \neq 9$:

1. The cyclotomic factor $C_n(x)$ is the least common multiple of the polynomials $\Phi_2(x), \Phi_3(x)$ and $E_i(x)$, $3 \leq i \leq 8$, that divide $E_n(x)$;
2. $E_n(x)$ is divisible by $E_i(x)$, $3 \leq i \leq 8$, iff $n \equiv i \mod h_i$; and
3. $E_n(x)$ is divisible by $\Phi_2(x)$ iff $n = 1 \mod 2$, and by $\Phi_3(x)$ iff $n = 0 \mod 3$.

**Corollary 1.2** The cyclotomic factor $C_n(x)$ only depends on $n \mod 360$. 

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Corollary 1.3 The Salem factor $S_n(x)$ satisfies $n - 15 \leq \deg(S_n) \leq n$.

The value $n - 15$ is first attained when $n = 349$.

Corollary 1.4 For $n \geq 10$, the polynomial $E_n(x)$ is irreducible (and hence $E_n(x) = S_n(x)$) iff $n \equiv 2, 10, 16, 20, 22, 26$ or $28 \mod 30$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$h_n$</th>
<th>$E_n(x)$</th>
<th>Factorization</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>6</td>
<td>$1 + 2x + 2x^2 + x^3$</td>
<td>$\Phi_2(x)\Phi_3(x)$</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>$1 + x + x^2 + x^3 + x^4$</td>
<td>$\Phi_5(x)$</td>
</tr>
<tr>
<td>5</td>
<td>8</td>
<td>$1 + x + x^4 + x^5$</td>
<td>$\Phi_2(x)\Phi_8(x)$</td>
</tr>
<tr>
<td>6</td>
<td>12</td>
<td>$1 + x - x^3 + x^5 + x^6$</td>
<td>$\Phi_3(x)\Phi_{12}(x)$</td>
</tr>
<tr>
<td>7</td>
<td>18</td>
<td>$1 + x - x^3 - x^4 + x^6 + x^7$</td>
<td>$\Phi_2(x)\Phi_{18}(x)$</td>
</tr>
<tr>
<td>8</td>
<td>30</td>
<td>$1 + x - x^3 - x^4 - x^5 + x^7 + x^8$</td>
<td>$\Phi_{30}(x)$</td>
</tr>
<tr>
<td>9</td>
<td>$\infty$</td>
<td>$1 + x - x^3 - x^4 - x^5 - x^6 + x^8 + x^9$</td>
<td>$\Phi_1(x)^2\Phi_2(x)\Phi_3(x)\Phi_5(x)$</td>
</tr>
<tr>
<td>10</td>
<td>$\infty$</td>
<td>$1 + x - x^3 - x^4 - x^5 - x^6 - x^7 + x^9 + x^{10}$</td>
<td>$S_{10}(x)$</td>
</tr>
</tbody>
</table>

Table 2. Coxeter polynomials for small $n$.

![Figure 3. The $A_n$ diagram.](image)

Joins of diagrams and periodicity. This behavior of $E_n$ can be understood as a consequence of two general phenomena.

For the first, recall that the $A_n$ diagram (Figure 3) has Coxeter polynomial

$$A_n(x) = \frac{x^{n+1} - 1}{x - 1} = 1 + x + \cdots + x^n.$$  

In §3 we will show:

**Theorem 1.5** Let $F$ be the Coxeter diagram obtained by joining together diagrams $F_1, \ldots, F_n$ at a single new vertex $t$. Then any zero of two or more of the Coxeter polynomials $F_i(x)$ is also a zero of $F(x)$.

Noting that $E_n$ is a join of $E_i$ and $A_{n-i-1}$, we obtain:

**Corollary 1.6** $E_n(x)$ is divisible by $\gcd(E_i(x), A_{n-i-1}(x))$ for $3 \leq i < n - 1$.  

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This result explains why the spherical Coxeter polynomials $E_i(x)$, $3 \leq i \leq 8$, occur as factors of $E_n(x)$. For example, $E_{38}$ is the join of $E_8$ and $A_{29}$. The zeros of $A_{29}(x)$ are the 30th roots of unity (save $z = 1$); thus they include the zeros of $E_8(x)$, and consequently $E_8(x)$ divides $E_{38}(x)$. It also explains the occurrence of the cyclotomic factors $\Phi_2$, $\Phi_3$ and their product; these can occur as $\gcd(E_3, A_{n-4})$, depending on the value of $n \mod 6$.

The second phenomenon underlying the behavior of $E_n$ is the following periodicity result, proved in §4.

**Theorem 1.7** Let $F_n$ be a sequence of Coxeter diagrams obtained by adjoining two fixed diagrams to the ends of $A_n$. Assume $F_n(x) \in \mathbb{Z}[x]$. Then either

(i) The cyclotomic factor of $F_n(x)$ is periodic for all $n \gg 0$, or

(ii) The diagram $F_n$ is spherical or affine for all $n$.

In case (ii), $F_n$ (if connected) must be a re-indexing of one of the well-known spherical or affine series $A_n, B_n, D_n, \tilde{B}_n, \tilde{C}_n$ or $\tilde{D}_n$.

This result, made effective, reduces Theorem 1.1 to a finite computation.

It would be interesting to find a general condition to insure that the cyclotomic factors of $F_n(x)$ come exclusively from its spherical subdiagrams, as is the case for $E_n(x)$.

**Notes and references.** For background on Coxeter systems, see e.g. [Bou] and [Hum]. More on the relationship between Coxeter systems, Salem numbers and Pisot numbers can be found in [Mc], [MRS], [Hir1] and [MS]. A version of Theorem 1.1 was proved independently, and by different arguments, by Bedford and Kim [BK, Thm. 2.4].

## 2 Roots of unity

Let $\zeta_k$ denote the primitive $k$th root of unity $\exp(2\pi i/k)$. In this section we formulate Mann’s theorem, and use it to prove:

**Theorem 2.1** Let $Q, R \in \mathbb{Z}[x]$ be polynomials, not both zero, such that

$$\zeta_k^n Q(\zeta_k) + R(\zeta_k) = 0$$

for some $k \geq 1$ and $n \in \mathbb{Z}$. Then either $Q(x) = \pm x^i R(x)$ for some $i \in \mathbb{Z}$, or we have

$$k \leq 2s \max(\deg Q, \deg R),$$

where $s$ is the product of the primes $p \leq \ell(Q) + \ell(R)$.

Here $\ell(P)$ denotes the number of terms in the polynomial $P$ (see below).

We then deduce Theorem 1.1 on the cyclotomic factor of $E_n(x)$.

**Polar rational polygons.** Let $\text{Div}(\mathbb{C})$ denote the group of finite divisors on the complex plane. Any $D \in \text{Div}(\mathbb{C})$ can be expressed as $D = \sum_I a_i \cdot z_i$ where each coefficient $a_i \in \mathbb{Z}$ is nonzero and $\supp D = \{ z_i : i \in I \}$ is a set of distinct
points forming the support of $D$. There is a natural evaluation map $\text{Div}(\mathbb{C}) \rightarrow \mathbb{C}$ defined by
$$D \mapsto \sigma(D) = \sum a_i z_i.$$ 
We say $D$ is effective if its coefficients are positive.

A polar rational polygon (prp) is an effective divisor $D = \sum a_i \cdot z_i$ such that each $z_i$ is a root of unity and $\sigma(D) = 0$. For each ordering of $I$, $D$ determines an (immersed) polygon in the plane with vertices $v_i = \sum_{j < i} a_j z_j$; its angles are rational multiples of $\pi$, and its sides are of integral length.

The length of a prp is given by $\ell(D) = |\text{supp} D|$. Its order is the cardinality $o(D)$ of the subgroup of $\mathbb{C}^*$ generated by the roots of unity $\{z_i/z_j : i, j \in I\}$.

A prp is primitive if it cannot be expressed as a sum $D = D' + D''$ of two other nonzero prp’s. Every prp is a sum of primitive prp’s.

![Figure 4. Three primitive polar rational polygons.](image)

We can now state the main result of [Man]:

**Theorem 2.2 (Mann)** Let $D$ be a primitive prp. Then the order $o(D)$ divides the product of the primes $p$ less than or equal to the length $\ell(D)$.

**Examples.** The regular $p$-gons are primitive prp’s whenever $p$ is prime. The smallest primitive prp other than these has length 6 and order 15; it is given by
$$D = \zeta_5 + \zeta_5^2 + \zeta_5^3 + \zeta_5^4 + \zeta_6 + \zeta_6^{-1}.$$ 
The corresponding hexagon (for a suitable ordering of the terms in the prp), with sides of length one, is shown at the left in Figure 4. Two other primitive prp’s, of length 7 and order 30, are shown in the center and at the right. Together with the regular $p$-gons for $p = 3, 5, 7$, these are (up to rotation) all the primitive prp’s of length $< 8$ [Man].

**Polynomials.** Any polynomial $P(x) \in \mathbb{Z}[x]$ can be uniquely expressed in the form
$$P(x) = \sum_{i \in I} \epsilon_i a_i x^i,$$
where $a_i > 0$ and $\epsilon_i = \pm 1$. The length $\ell(P) = |I|$ is the number of terms in $P$. Given $\zeta \in \mathbb{C}$, let $DP(\zeta)$ denote the effective divisor
$$DP(\zeta) = \sum_{i \in I} a_i \cdot (\epsilon_i \zeta^i).$$
If \( \zeta \) is a root of unity and \( P(\zeta) = 0 \), then \( DP(\zeta) \) is a prp.

**Proof of Theorem 2.1.** Let \( P(x) = x^nQ(x) + R(x) \). Then there are finite sums \( Q(x) = \sum Q_j(x) \) and \( R(x) = \sum R_j(x) \) such that

\[
DP(\zeta_k) = \sum_j DP_j(\zeta_k) = \sum_j \zeta_k^eDP_j(\zeta_k) + DR_j(\zeta_k)
\]
gives a decomposition of \( DP(\zeta_k) \) into primitive prps.

If \( \ell(Q_j) > 1 \) for some \( j \), then we have \( o(DP_j(\zeta_k)) \geq k/(2 \deg(Q)) \), since the ratio of any two roots of unity occurring in \( DQ_j(\zeta_k) \) has the form \( \pm \zeta_k^e \) with \( 1 \leq e \leq \deg(Q) \). By Mann’s theorem, \( o(DP_j(\zeta_k)) \) is bounded above by the product of the primes less than or equal to \( \ell(P_j) \leq \ell(Q) + \ell(R) \), and so the desired upper bound follows. The same argument applies if \( \ell(R_j) > 1 \) for some \( j \).

Now assume \( \ell(Q_j) = \ell(R_j) = 1 \) for all \( j \), but the desired bound on \( k \) fails. Then \( k > 4m \), where \( m = \max(\deg(Q), \deg(R)) \). Writing \( Q_j(x) = a_jx^{e_j} \) and \( R_j(x) = b_jx^{f_j} \), we have

\[
\zeta^Q_j(\zeta_k) + R_j(\zeta_k) = a_j\zeta_k^{e_j} + b_j\zeta_k^{f_j} = 0
\]
for all \( j \). Consequently \( \zeta_k^{f_j-e_j} = \pm \zeta_k^e \) for all \( j \). This implies \( f_j - e_j \) is constant mod \( k \) or mod \( k/2 \) (depending on the parity of \( k \)). But \( k > 4m \) and \( (f_j - e_j) \in [-m, m] \), so the difference of exponents \( i = f_j - e_j \) is also constant in \( \mathbb{Z} \). We then have

\[
a_j\zeta_k^{n-i+f_j} + b_j\zeta_k^{f_j} = 0
\]
for all \( j \); thus \( \epsilon = \zeta_k^{n-i} = \pm 1 \) and \( a_jb_j = 0 \), which gives \( \epsilon x^iQ_j(x) + R_j(x) = 0 \) and hence \( Q(x) = \pm x^{-i}R(x) \).

**Application to \( E_n \).** Now recall that for \( n \geq 3 \) we have

\[
E_n(x)(x-1) = x^{n-2}(x^3 - x - 1) + (x^3 + x^2 - 1) = x^{n-2}Q(x) + R(x).
\]
Since \( \deg(Q) = \deg(R) = 3 \) and \( \ell(Q) + \ell(R) = 6 \), the Theorem above implies:

**Corollary 2.3** If \( E_n(\zeta_k) = 0 \), then \( k \leq 180 \).

**Lemma 2.4** The polynomial \( E_n(x) \) is separable for all \( n \neq 9 \).

**Proof.** The only possible multiple roots of \( E_n(x) \) are in its cyclotomic factor \( C_n(x) \). But for \( |x| = 1 \) we have

\[
|E_n(x)(x-1)| > (n-2)|Q(x)| - |Q'(x)| - |R'(x)| > 0.3(n-2) - 9,
\]
so \( E_n(x) \) is separable once \( n \geq 32 \). The remaining cases are easily checked individually.
Proof of Theorem 1.1. It is straightforward to verify that the Theorem is correct for $3 \leq n \leq 182$. Thus $E_n(\zeta_k) = 0$ for some $n$ in this range, $n \neq 9$, if $k \in \{2, 3, 5, 8, 12, 18, 30\} = K$.

By separability, the cyclotomic factor only depends on the roots of unity where $E_n(\zeta_k) = 0$. But the vanishing of $E_n(\zeta_k)$ only depends on the value of $n \mod k$, so by Corollary 2.3 no new roots of unity can occur as zeros of $E_n(x)$ for $n > 182$. So once the Theorem is checked for all $n \leq 182$ it also holds for all larger values of $n$.

3 Joins

In this section we define the join of a collection of Coxeter systems, and establish the following more precise version of Theorem 1.5.

**Theorem 3.1** Let $(W, S)$ be the join of Coxeter systems $(W_i, S_i)_{i=1}^m$, with bi-colored Coxeter elements $w_i$. Suppose $\lambda$ is an eigenvalue of $w_i$ with multiplicity $m_i \geq 0$. Then $\lambda$ occurs as an eigenvalue of the bi-colored Coxeter element $w \in W$ with multiplicity at least $(\sum m_i) - 1$.

Coxeter systems. Recall that a Coxeter system $(W, S)$ is an abstract group $W$ with a distinguished set of generators $S$, such that the product $st \in W$ of two generators has finite order $m_{st} \geq 2$, the generators themselves have order 2, and these relations give a presentation for $W$.

The pair $(W, S)$ determines a quadratic form $B$ on $\mathbb{R}^S$ with matrix $B_{st} = -2 \cos(\pi/m_{st})$, and a geometric representation $W \hookrightarrow O(\mathbb{R}^S, B)$ where the generators act by the reflections

$$s \cdot v = v - B(e_s, v)e_s. \quad (3.1)$$

The Coxeter diagram $F$ of $(W, S)$ is the (undirected) graph with vertex set $S$ and an edge of weight $m_{st} - 2$ joining $s$ to $t$ whenever $m_{st} > 2$. By convention an unlabeled edge has weight one, and $i$ parallel unlabeled edges indicate a single edge of weight $i$.

The product of the generators $w = s_1 \cdots s_n$ of $W$, taken in any order, is a Coxeter element of $(W, S)$. If the diagram $F$ is a tree, then the conjugacy class of $w$ is independent of the choice of ordering. If $F$ is bipartite (meaning we can write $S = S_0 \sqcup S_1$ with all edges connecting $S_0$ to $S_1$), then the bicolored Coxeter element

$$w = \prod S_0 \prod S_1$$

is well-defined up to conjugacy (cf.[Mc, §5]). Thus in Theorem 3.1 we implicitly assume the Coxeter systems $(W_i, s_i)$ are bipartite.

The Coxeter polynomial of a bipartite Coxeter system $(W, S)$ is the characteristic polynomial

$$F(x) = \det(xI - w)$$

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of its bicolored Coxeter elements. We generally denote it using the same symbol as the diagram. Note that if the diagram $F$ has no multiple edges, then $W$ preserves the lattice $\mathbb{Z}^S$ and thus $F(x) \in \mathbb{Z}[x]$.

**Pointed Coxeter systems.** A pointed Coxeter system is a triple $(W, S, s)$ with $s \in S$. It is determined up to isomorphism by a pointed diagram $(F, s)$. By deleting $s$, we obtain a Coxeter subsystem $(\hat{W}, \hat{S})$ with Coxeter polynomial $\hat{F}(x)$.

We let $(A_n, i)$ and $(E_n, i)$ denote the $A_n$ and $E_n$ diagrams with the $i$th vertex distinguished, using the numbering in Figures 1 and 3.

**Joins.** The join $(W, S)$ of pointed Coxeter systems $(W_i, S_i, s_i)_{i=1}^m$ is defined by taking an independent generator $t$, setting $S = \{t\} \cup S_i$, and setting

$$W = (W_1 * \cdots * W_m * \{t\}) / (t^2 = (s_1t)^3 = \cdots = (s_m t)^3 = \text{id}).$$

The corresponding diagram $F$ is obtained from $\sqcup F_i$ by adding a new vertex $t$ and connecting it to each $s_i$ with a single edge (see Figure 5). If all the diagrams $F_i$ are bipartite, so is $F$.

In Theorem 3.1, basepoints $s_i \in S_i$ must be chosen to make the join $(W, S)$ well-defined, but the conclusion holds independent of the choice of basepoints.

![Figure 5. The join of $A_3$, $B_2$ and $D_4$.](image)

**Proof of Theorem 3.1.** Let $(W, S)$ be the join of $(W_i, S_i)_{i=1}^m$. By equation (3.1), a given reflection $s(v)$ only changes the coordinate $v_s$ of a vector $v \in \mathbb{R}^S$. Thus we have natural inclusions $W_i \subset W$ compatible with the inclusions $\mathbb{R}^S_i \subset \mathbb{R}^S$.

Since $s,t \in S$ commute whenever they are not joined by an edge in the Coxeter diagram, we can write the bicolored Coxeter element $w \in W$ in the form

$$w = tw_1 \cdots w_m.$$

Let $E_i \subset \mathbb{C}^S_i \subset \mathbb{C}^S$ be the $\lambda$-eigenspaces for $w_i$, extended by zero in the remaining coordinates. By (3.1) we have $w_i|E_j = \text{id}$ for $i \neq j$. Thus $\oplus E_i$ is a $\lambda$-eigenspace for $w_1 \cdots w_m$. Since $t(v)$ only changes $v_t$, there is a codimension-one subspace $E \subset \oplus E_i$ such that $t|E = \text{id}$. Consequently the multiplicity of $\lambda$ as an eigenvalue for $w$ is bounded below by

$$\dim(E) = \left( \sum \dim(E_i) \right) - 1 = \left( \sum m_i \right) - 1.$$
The Coxeter polynomial of a join. Here is an alternative approach to the result above. When \( F \) is the join of \((F_i, s_i)_{i=1}^{m}\), a straightforward matrix computation yields the following useful formula for its Coxeter polynomial:

\[
F(x) = F_1(x) \cdots F_m(x) \left( x + 1 - x \sum_{i=1}^{m} \frac{\tilde{F}_i(x)}{F_i(x)} \right).
\]

(3.2)

Cf. [CDS, Prob 9, p.78], [MRS, Cor. 4].

By writing the Coxeter element of \((W_i, S_i)\) with \(s_i\) at the end, one can verify that the order of vanishing of its Coxeter polynomial satisfies \(\text{ord}(P_i, \lambda) - 1 \leq \text{ord}(\tilde{P}_i, \lambda)\). Thus equation (3.2) implies

\[
\text{ord}(F, \lambda) \geq -1 + \sum \text{ord}(F_i, \lambda).
\]

This inequality is equivalent to Theorem 3.1 when the quadratic form \(B\) of \((W, S)\) is non-degenerate, as it is for \(E_n, n \neq 9\).

4 Decorating \(A_n\)

In this section we generalize our results on \(E_n\) to more general diagrams \(F_n\) of the form shown in Figure 6. Our main result is:

**Theorem 4.1** Let \(F_n\) be the sequence of Coxeter diagrams obtained by attaching pointed diagrams \((B, s)\) and \((C, t)\) to the ends of \(A_n\). Assume \(F_n(x) \in \mathbb{Z}[x]\) for all \(n\). Then either

1. The diagram \(F_n\) is spherical or affine for all \(n\), or
2. The cyclotomic factor of \(F_n(x)\) is periodic for \(n \gg 0\).

\[
\begin{array}{c}
\bullet & 1 & 2 & \cdots & n-1 & n & t \\
B & & & & & & \\
\bullet & 1 & 2 & \cdots & n-1 & n & t \\
C & & & & & & \\
\end{array}
\]

Figure 6. The diagram \(F_n\) obtained by attaching \((B, s)\) and \((C, t)\) to the ends of \(A_n\).

Coxeter polynomials. We begin by determining the Coxeter polynomial \(F_n(x)\). First, by repeatedly applying equation (3.2) with \(m = 1\), we obtain:
Proposition 4.2 The Coxeter polynomial of the diagram $B_n$ obtained by attaching $(B, s)$ to one end of $A_n$ satisfies:

$$B_n(x)(x - 1) = x^{n+1}(B(x) - \tilde{B}(x)) + (x\tilde{B}(x) - B(x)).$$

Here is an example:

Corollary 4.3 For $n \geq 4$, we have

$$E_n(x)(x - 1) = x^{n-2}(x^3 - x - 1) + (x^3 + x^2 - 1).$$

Proof. Take $(B, s) = (A_4, 2)$; then $B(x) = A_4(x)$, $\tilde{B}(x) = A_1(x)A_2(x)$, and $B_n(x) = E_{n+4}(x)$. Thus $B(x) - \tilde{B}(x) = x(x^3 - x - 1)$ and $x\tilde{B}(x) - B(x) = x^3 + x^2 - 1$, which gives

$$E_{n+4}(x)(x - 1) = x^{n+2}(x^3 - x - 1) + (x^3 + x^2 - 1).$$

Since $F_n$ is the join of $B_{n-1}$ and $C$, by applying equation (3.2) once more we find:

Proposition 4.4 The Coxeter polynomials of $F_n$, $(B, s)$ and $(C, t)$ are related by $F_n(x)(x - 1) = x^{n+1}Q(x) - R(x)$, where

$$Q(x) = (B(x) - \tilde{B}(x))(C(x) - \tilde{C}(x))$$

and

$$R(x) = (x\tilde{B}(x) - B(x))(x\tilde{C}(x) - C(x)).$$

We will also need the following result. Let $\beta(F_n) \geq 1$ denote the largest real zero of $F_n(x)$; equivalently, the spectral radius of the bicolored Coxeter element for $F_n$.

Proposition 4.5 (Hoffman–Smith) If $\beta(F_n) > 1$, then $\beta(F_n) \neq \beta(F_{n+1})$.

Proof. Let $A_{st} = 2I - B_{st}$ denote the symmetric ‘adjacency matrix’ for the $F_n$ diagram, and $\alpha(F_n)$ its spectral radius. Then since $\beta(F_n) > 1$, we have

$$\alpha(F_n) = (2 + \beta(F_n) + \beta(F_n)^{-1})^{1/2} > 2,$$

(see e.g. [Mc, Thm. 5.1]).

By [HS, Lem. 2.3 and Prop. 2.4], the condition $\alpha(F_n) > 2$ implies that $\alpha(F_{n+1}) < \alpha(F_n)$ if we are adding nodes to an internal path, and that $\alpha(F_{n+1}) > \alpha(F_n)$ if we are adding nodes to an external path (i.e. if $(B, s)$ or $(C, t)$ is equal to $(A_1, 1)$.) (The proof in [HS] is given for graphs, but it applies without change to Coxeter diagrams, using the following key fact: if $s$ is an endpoint of a maximal $A_k$ embedded in $F_n$, either $s$ is an endpoint of $F_n$, or $\sum_{t \neq s} A_{st} \geq 1 + \sqrt{2}$.)

In particular, we have $\alpha(F_{n+1}) \neq \alpha(F_n)$, and hence $\beta(F_n) \neq \beta(F_{n+1})$. □
Proof of Theorem 4.1. Since \( \text{deg } B > \text{deg } \hat{B} \) and \( \text{deg } C > \text{deg } \hat{C} \), we have \( Q(x) \neq 0 \). By Theorem 2.1, either:

(i) \( F_n(x)(x - 1) = (x^n \pm x^i)Q(x) \), or

(ii) Only finitely many \( k \) satisfy \( F_n(\zeta_k) = 0 \) for some \( n \).

In case (i), the zeros of \( F_n(x) \) outside the unit circle must be constant as \( n \) varies. This implies the spectral radius \( \beta(F_n) \) of the bicolored Coxeter element is constant; hence \( \beta(F_n) = 1 \) by Proposition 4.5, which means \( F_n \) is spherical or affine by [A'C].

For case (ii), fix \( k \) such that \( F_n(\zeta_k) = 0 \). Clearly the values of \( F_n(\zeta_k) \) are periodic in \( n \). To complete the proof, we must show the order of vanishing of \( F_n \) at \( \zeta_k \) is also periodic. For this we may assume \( Q(x) \) and \( R(x) \) are relatively prime. Then \( Q(\zeta_k) \neq 0 \), and hence for all \( n \gg 0 \), \( F'_n(\zeta_k) \neq 0 \), since the dominant term in the derivative is \( (n + 1)\zeta_k^nQ(\zeta_k) \). Consequently the cyclotomic zeros of \( F_n \) are simple for all \( n \gg 0 \), and the proof is complete. \( \blacksquare \)

Notes. For a survey of results on the largest eigenvalues of graphs, including the inequality of Hoffman and Smith used above, see [CR].
References


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