## Cyclotomic Factors of Coxeter Polynomials

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# Cyclotomic factors of Coxeter polynomials 

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#### Abstract

In this paper we show that the cyclotomic factors of the $E_{n}$ Coxeter polynomials depend only on the value of $n \bmod 360$, and come exclusively from spherical subdiagrams.


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## 1 Introduction

In this paper we determine which roots of unity are zeros of the $E_{n}$ Coxeter polynomial. We show these roots come exclusively from splittings of $E_{n}$ into spherical subdiagrams; in particular they always have order $2,3,5,8,12,18$, or 30 , and they only depend on the value of $n \bmod 360$ (provided we exclude the special case $n=9$ ).

The proof uses Mann's theorem on linear relations between roots of unity, and generalizes to other sequences of Coxeter diagrams where nodes are added to a separating edge.


Figure 1. The $E_{n}$ diagram.

The $\boldsymbol{E}_{\boldsymbol{n}}$ diagram. Coxeter systems are a useful source of Salem numbers, Pisot numbers and other interesting algebraic integers. For example, the smallest known Salem number arises from the Coxeter system $E_{10}$.

[^0]The $E_{n}$ Coxeter diagram, defined for $n \geq 3$, is shown in Figure 1. Note that $E_{3} \cong A_{2} \oplus A_{1}$. The $E_{n}$ diagram determines a quadratic form $B_{n}$ on $\mathbb{Z}^{n}$, and a reflection group $W_{n} \subset O\left(\mathbb{Z}^{n}, B_{n}\right)$ (see $\S 3$ ). The product of the generating reflections is a Coxeter element $w_{n} \in W_{n}$; it is well-defined up to conjugacy, since $E_{n}$ is a tree [Hum, $\left.\S 8.4\right]$.

The Coxeter number $h_{n}$ is the order of the Coxeter element $w_{n} \in W_{n}$, and its characteristic polynomial

$$
E_{n}(x)=\operatorname{det}\left(x I-w_{n}\right)
$$

is the Coxeter polynomial. Explicitly, for $n \geq 3$ we have:

$$
E_{n}(x)=\frac{x^{n-2} Q(x)+R(x)}{(x-1)}
$$

where $Q(x)=x^{3}-x-1$ and $R(x)=x^{3}+x^{2}-1$. (See e.g. [MRS, Lemma 5], [Hir2, §4.2] or Corollary 4.3 below.)

We can write $E_{n}(x)$ uniquely as a product of monic integral polynomials

$$
E_{n}(x)=C_{n}(x) S_{n}(x),
$$

where the zeros of the cyclotomic factor $C_{n}(x)$ are roots of unity, and those of the Salem factor $S_{n}(x)$ are not. Table 2 lists $E_{n}(x)$ for $n \leq 10$, along with its factorization into irreducibles and the Coxeter number $h_{n}$. Here $\Phi_{k}(x)$ is the cyclotomic polynomial for the primitive $k$ th roots of unity.
The spherical and affine cases. Since $E_{i}$ is a spherical diagram ( $B_{i}$ is positive definite) when $3 \leq i \leq 8$, we have $E_{i}(x)=C_{i}(x)$ (and $\left.S_{i}(x)=1\right)$ in this range.

The diagram $E_{9}$ is the affine version of $E_{8}$; its Coxeter element has infinite order, but still $E_{9}(x)=C_{9}(x)$. This is the only case where $E_{n}(x)$ has a multiple root (see Lemma 2.4 below).
The hyperbolic case. For $n \geq 10$, the diagram $E_{n}$ is hyperbolic; that is, the signature of $B_{n}$ is $(n-1,1)$. By [A'C] this implies that the factor $S_{n}(x)$ is a Salem polynomial: it is an irreducible, reciprocal polynomial, with a unique root $\lambda>1$ outside the unit disk. For $n=10, E_{n}(x)$ coincides with Lehmer's polynomial, and its root $\lambda \approx 1.1762808>1$ is the smallest known Salem number.

We can now state our main result on the Coxeter polynomials $E_{n}(x)$.
Theorem 1.1 For all $n \neq 9$ :

1. The cyclotomic factor $C_{n}(x)$ is the least common multiple of the polynomials $\Phi_{2}(x), \Phi_{3}(x)$ and $E_{i}(x), 3 \leq i \leq 8$, that divide $E_{n}(x)$;
2. $E_{n}(x)$ is divisible by $E_{i}(x), 3 \leq i \leq 8$, iff $n \equiv i \bmod h_{i}$; and
3. $E_{n}(x)$ is divisible by $\Phi_{2}(x)$ iff $n=1 \bmod 2$, and by $\Phi_{3}(x)$ iff $n=0 \bmod 3$.

Corollary 1.2 The cyclotomic factor $C_{n}(x)$ only depends on $n \bmod 360$.

Corollary 1.3 The Salem factor $S_{n}(x)$ satisfies $n-15 \leq \operatorname{deg}\left(S_{n}\right) \leq n$.
The value $n-15$ is first attained when $n=349$.
Corollary 1.4 For $n \geq 10$, the polynomial $E_{n}(x)$ is irreducible (and hence $\left.E_{n}(x)=S_{n}(x)\right)$ iff $n \equiv 2,10,16,20,22,26$ or $28 \bmod 30$.

| $n$ | $h_{n}$ | Coxeter polynomial $E_{n}$ | Factorization |
| :---: | :---: | :---: | :---: |
| 3 | 6 | $1+2 x+2 x^{2}+x^{3}$ | $\Phi_{2}(x) \Phi_{3}(x)$ |
| 4 | 5 | $1+x+x^{2}+x^{3}+x^{4}$ | $\Phi_{5}(x)$ |
| 5 | 8 | $1+x+x^{4}+x^{5}$ | $\Phi_{2}(x) \Phi_{8}(x)$ |
| 6 | 12 | $1+x-x^{3}+x^{5}+x^{6}$ | $\Phi_{3}(x) \Phi_{12}(x)$ |
| 7 | 18 | $1+x-x^{3}-x^{4}+x^{6}+x^{7}$ | $\Phi_{2}(x) \Phi_{18}(x)$ |
| 8 | 30 | $1+x-x^{3}-x^{4}-x^{5}+x^{7}+x^{8}$ | $\Phi_{30}(x)$ |
| 9 | $\infty$ | $1+x-x^{3}-x^{4}-x^{5}-x^{6}+x^{8}+x^{9}$ | $\Phi_{1}(x)^{2} \Phi_{2}(x) \Phi_{3}(x) \Phi_{5}(x)$ |
| 10 | $\infty$ | $1+x-x^{3}-x^{4}-x^{5}-x^{6}-x^{7}+x^{9}+x^{10}$ | $S_{10}(x)$ |

Table 2. Coxeter polynomials for small $n$.


Figure 3. The $A_{n}$ diagram.

Joins of diagrams and periodicity. This behavior of $E_{n}$ can be understood as a consequence of two general phenomena.

For the first, recall that the $A_{n}$ diagram (Figure 3) has Coxeter polynomial

$$
A_{n}(x)=\frac{x^{n+1}-1}{x-1}=1+x+\cdots+x^{n}
$$

In $\S 3$ we will show:
Theorem 1.5 Let $F$ be the Coxeter diagram obtained by joining together diagrams $F_{1}, \ldots, F_{n}$ at a single new vertex $t$. Then any zero of two or more of the Coxeter polynomials $F_{i}(x)$ is also a zero of $F(x)$.

Noting that $E_{n}$ is a join of $E_{i}$ and $A_{n-i-1}$, we obtain:
Corollary 1.6 $E_{n}(x)$ is divisible by $\operatorname{gcd}\left(E_{i}(x), A_{n-i-1}(x)\right)$ for $3 \leq i<n-1$.

This result explains why the spherical Coxeter polynomials $E_{i}(x), 3 \leq i \leq 8$, occur as factors of $E_{n}(x)$. For example, $E_{38}$ is the join of $E_{8}$ and $A_{29}$. The zeros of $A_{29}(x)$ are the 30 th roots of unity (save $\zeta=1$ ); thus they include the zeros of $E_{8}(x)$, and consequently $E_{8}(x)$ divides $E_{38}(x)$. It also explains the occurrence of the cyclotomic factors $\Phi_{2}, \Phi_{3}$ and their product; these can occur as $\operatorname{gcd}\left(E_{3}, A_{n-4}\right)$, depending on the value of $n \bmod 6$.

The second phenomenon underlying the behavior of $E_{n}$ is the following periodicity result, proved in $\S 4$.

Theorem 1.7 Let $F_{n}$ be a sequence of Coxeter diagrams obtained by adjoining two fixed diagrams to the ends of $A_{n}$. Assume $F_{n}(x) \in \mathbb{Z}[x]$. Then either
(i) The cyclotomic factor of $F_{n}(x)$ is periodic for all $n \gg 0$, or
(ii) The diagram $F_{n}$ is spherical or affine for all $n$.

In case (ii), $F_{n}$ (if connected) must be a re-indexing of one of the well-known spherical or affine series $A_{n}, B_{n}, D_{n}, \widetilde{B_{n}}, \widetilde{C_{n}}$ or $\widetilde{D_{n}}$.

This result, made effective, reduces Theorem 1.1 to a finite computation.
It would be interesting to find a general condition to insure that the cyclotomic factors of $F_{n}(x)$ come exclusively from its spherical subdiagrams, as is the case for $E_{n}(x)$.
Notes and references. For background on Coxeter systems, see e.g. [Bou] and [Hum]. More on the relationship between Coxeter systems, Salem numbers and Pisot numbers can be found in [Mc], [MRS], [Hir1] and [MS]. A version of Theorem 1.1 was proved independently, and by different arguments, by Bedford and $\operatorname{Kim}$ [BK, Thm. 2.4].

## 2 Roots of unity

Let $\zeta_{k}$ denote the primitive $k$ th root of unity $\exp (2 \pi i / k)$. In this section we formulate Mann's theorem, and use it to prove:

Theorem 2.1 Let $Q, R \in \mathbb{Z}[x]$ be polynomials, not both zero, such that

$$
\zeta_{k}^{n} Q\left(\zeta_{k}\right)+R\left(\zeta_{k}\right)=0
$$

for some $k \geq 1$ and $n \in \mathbb{Z}$. Then either $Q(x)= \pm x^{i} R(x)$ for some $i \in \mathbb{Z}$, or we have

$$
k \leq 2 s \max (\operatorname{deg} Q, \operatorname{deg} R)
$$

where $s$ is the product of the primes $p \leq \ell(Q)+\ell(R)$.
Here $\ell(P)$ denotes the number of terms in the polynomial $P$ (see below).
We then deduce Theorem 1.1 on the cyclotomic factor of $E_{n}(x)$.
Polar rational polygons. Let $\operatorname{Div}(\mathbb{C})$ denote the group of finite divisors on the complex plane. Any $D \in \operatorname{Div}(\mathbb{C})$ can be expressed as $D=\sum_{I} a_{i} \cdot z_{i}$ where each coefficient $a_{i} \in \mathbb{Z}$ is nonzero and $\operatorname{supp} D=\left\{z_{i}: i \in I\right\}$ is a set of distinct
points forming the support of $D$. There is a natural evaluation map $\operatorname{Div}(\mathbb{C}) \rightarrow \mathbb{C}$ defined by

$$
D \mapsto \sigma(D)=\sum a_{i} z_{i}
$$

We say $D$ is effective if its coefficients are positive.
A polar rational polygon (prp) is an effective divisor $D=\sum a_{i} \cdot z_{i}$ such that each $z_{i}$ is a root of unity and $\sigma(D)=0$. For each ordering of $I, D$ determines an (immersed) polygon in the plane with vertices $v_{i}=\sum_{j<i} a_{j} z_{j}$; its angles are rational multiples of $\pi$, and its sides are of integral length.

The length of a prp is given by $\ell(D)=|\operatorname{supp} D|$. Its order is the cardinality $o(D)$ of the subgroup of $\mathbb{C}^{*}$ generated by the roots of unity $\left\{z_{i} / z_{j}: i, j \in I\right\}$.

A prp is primitive if it cannot be expressed as a $\operatorname{sum} D=D^{\prime}+D^{\prime \prime}$ of two other nonzero prp's. Every prp is a sum of primitive prp's.


Figure 4. Three primitive polar rational polygons.

We can now state the main result of [Man]:
Theorem 2.2 (Mann) Let $D$ be a primitive prp. Then the order $o(D)$ divides the product of the primes $p$ less than or equal to the length $\ell(D)$.

Examples. The regular $p$-gons are primitive prp's whenever $p$ is prime. The smallest primitive prp other than these has length 6 and order 15 ; it is given by

$$
D=\zeta_{5}+\zeta_{5}^{2}+\zeta_{5}^{3}+\zeta_{5}^{4}+\zeta_{6}+\zeta_{6}^{-1}
$$

The corresponding hexagon (for a suitable ordering of the terms in the prp), with sides of length one, is shown at the left in Figure 4. Two other primitive prp's, of length 7 and order 30, are shown in the center and at the right. Together with the regular $p$-gons for $p=3,5,7$, these are (up to rotation) all the primitive prp's of length $<8$ [Man].
Polynomials. Any polynomial $P(x) \in \mathbb{Z}[x]$ can be uniquely expressed in the form

$$
P(x)=\sum_{i \in I} \epsilon_{i} a_{i} x^{i}
$$

where $a_{i}>0$ and $\epsilon_{i}= \pm 1$. The length $\ell(P)=|I|$ is the number of terms in $P$.
Given $\zeta \in \mathbb{C}$, let $D P(\zeta)$ denote the effective divisor

$$
D P(\zeta)=\sum_{i \in I} a_{i} \cdot\left(\epsilon_{i} \zeta^{i}\right)
$$

If $\zeta$ is a root of unity and $P(\zeta)=0$, then $D P(\zeta)$ is a prp.
Proof of Theorem 2.1. Let $P(x)=x^{n} Q(x)+R(x)$. Then there are finite sums $Q(x)=\sum Q_{j}(x)$ and $R(x)=\sum R_{j}(x)$ such that

$$
D P\left(\zeta_{k}\right)=\sum_{j} D P_{j}\left(\zeta_{k}\right)=\sum_{j} \zeta_{k}^{n} D Q_{j}\left(\zeta_{k}\right)+D R_{j}\left(\zeta_{k}\right)
$$

gives a decomposition of $D P\left(\zeta_{k}\right)$ into primitive prps.
If $\ell\left(Q_{j}\right)>1$ for some $j$, then we have $o\left(D P_{j}\left(\zeta_{k}\right)\right) \geq k /(2 \operatorname{deg}(Q))$, since the ratio of any two roots of unity occurring in $D Q_{j}\left(\zeta_{k}\right)$ has the form $\pm \zeta_{k}^{e}$ with $1 \leq e \leq \operatorname{deg}(Q)$. By Mann's theorem, $o\left(D P_{j}\left(\zeta_{k}\right)\right)$ is bounded above by the product of the primes less than or equal to $\ell\left(P_{j}\right) \leq \ell(Q)+\ell(R)$, and so the desired upper bound for $k$ follows. The same argument applies if $\ell\left(R_{j}\right)>1$ for some $j$.

Now assume $\ell\left(Q_{j}\right)=\ell\left(R_{j}\right)=1$ for all $j$, but the desired bound on $k$ fails. Then $k>4 m$, where $m=\max (\operatorname{deg}(Q), \operatorname{deg}(R))$. Writing $Q_{j}(x)=a_{j} x^{e_{j}}$ and $R_{j}(x)=b_{j} x^{f_{j}}$, we have

$$
\zeta^{n} Q_{j}\left(\zeta_{k}\right)+R_{j}\left(\zeta_{k}\right)=a_{j} \zeta_{k}^{n+e_{j}}+b_{j} \zeta_{k}^{f_{j}}=0
$$

for all $j$. Consequently $\zeta_{k}^{f_{j}-e_{j}}= \pm \zeta_{k}^{n}$ for all $j$. This implies $f_{j}-e_{j}$ is constant $\bmod k$ or $\bmod (k / 2)$ (depending on the parity of $k$ ). But $k>4 m$ and $\left(f_{j}-e_{j}\right) \in$ $[-m, m]$, so the difference of exponents $i=f_{j}-e_{j}$ is also constant in $\mathbb{Z}$. We then have

$$
a_{j} \zeta_{k}^{n-i+f_{j}}+b_{j} \zeta_{k}^{f_{j}}=0
$$

for all $j$; thus $\epsilon=\zeta_{k}^{n-i}= \pm 1$ and $\epsilon a_{j}+b_{j}=0$, which gives $\epsilon x^{i} Q_{j}(x)+R_{j}(x)=0$ and hence $Q(x)= \pm x^{-i} R(x)$.

Application to $\boldsymbol{E}_{\boldsymbol{n}}$. Now recall that for $n \geq 3$ we have

$$
E_{n}(x)(x-1)=x^{n-2}\left(x^{3}-x-1\right)+\left(x^{3}+x^{2}-1\right)=x^{n-2} Q(x)+R(x)
$$

Since $\operatorname{deg}(Q)=\operatorname{deg}(R)=3$ and $\ell(Q)+\ell(R)=6$, the Theorem above implies:
Corollary 2.3 If $E_{n}\left(\zeta_{k}\right)=0$, then $k \leq 180$.
Lemma 2.4 The polynomial $E_{n}(x)$ is separable for all $n \neq 9$.
Proof. The only possible multiple roots of $E_{n}(x)$ are in its cyclotomic factor $C_{n}(x)$. But for $|x|=1$ we have

$$
\left|\left(E_{n}(x)(x-1)\right)^{\prime}\right|>(n-2)|Q(x)|-\left|Q^{\prime}(x)\right|-\left|R^{\prime}(x)\right|>0.3(n-2)-9
$$

so $E_{n}(x)$ is separable once $n \geq 32$. The remaining cases are easily checked individually.

Proof of Theorem 1.1. It is straightforward to verify that the Theorem is correct for $3 \leq n \leq 182$. Thus $E_{n}\left(\zeta_{k}\right)=0$ for some $n$ in this range, $n \neq 9$, iff $k \in\{2,3,5,8,12,18,30\}=K$.

By separability, the cyclotomic factor only depends on the roots of unity where $E_{n}\left(\zeta_{k}\right)=0$. But the vanishing of $E_{n}\left(\zeta_{k}\right)$ only depends on the value of $n \bmod k$, so by Corollary 2.3 no new roots of unity can occur as zeros of $E_{n}(x)$ for $n>182$. So once the Theorem is checked for all $n \leq 182$ it also holds for all larger values of $n$.

## 3 Joins

In this section we define the join of a collection of Coxeter systems, and establish the following more precise version of Theorem 1.5.

Theorem 3.1 Let $(W, S)$ be the join of Coxeter systems $\left(W_{i}, S_{i}\right)_{i=1}^{m}$, with bicolored Coxeter elements $w_{i}$. Suppose $\lambda$ is an eigenvalue of $w_{i}$ with multiplicity $m_{i} \geq 0$. Then $\lambda$ occurs as an eigenvalue of the bicolored Coxeter element $w \in W$ with multiplicity at least $\left(\sum m_{i}\right)-1$.

Coxeter systems. Recall that a Coxeter system $(W, S)$ is an abstract group $W$ with a distinguished set of generators $S$, such that the product st $\in W$ of two generators has finite order $m_{s t} \geq 2$, the generators themselves have order 2 , and these relations give a presentation for $W$.

The pair $(W, S)$ determines a quadratic form $B$ on $\mathbb{R}^{S}$ with matrix $B_{s t}=$ $-2 \cos \left(\pi / m_{s t}\right)$, and a geometric representation $W \hookrightarrow O\left(\mathbb{R}^{S}, B\right)$ where the generators act by the reflections

$$
\begin{equation*}
s \cdot v=v-B\left(e_{s}, v\right) e_{s} \tag{3.1}
\end{equation*}
$$

The Coxeter diagram $F$ of $(W, S)$ is the (undirected) graph with vertex set $S$ and an edge of weight $m_{s t}-2$ joining $s$ to $t$ whenever $m_{s t}>2$. By convention an unlabeled edge has weight one, and $i$ parallel unlabeled edges indicate a single edge of weight $i$.

The product of the generators $w=s_{1} \cdots s_{n}$ of $W$, taken in any order, is a Coxeter element of $(W, S)$. If the diagram $F$ is a tree, then the conjugacy class of $w$ is independent of the choice of ordering. If $F$ is bipartite (meaning we can write $S=S_{0} \sqcup S_{1}$ with all edges connecting $S_{0}$ to $S_{1}$ ), then the bicolored Coxeter element

$$
w=\prod S_{0} \prod S_{1}
$$

is well-defined up to conjugacy (cf.[Mc, §5]). Thus in Theorem 3.1 we implicitly assume the Coxeter systems $\left(W_{i}, s_{i}\right)$ are bipartite.

The Coxeter polynomial of a bipartite Coxeter system $(W, S)$ is the characteristic polynomial

$$
F(x)=\operatorname{det}(x I-w)
$$

of its bicolored Coxeter elements. We generally denote it using the same symbol as the diagram. Note that if the diagram $F$ has no multiple edges, then $W$ preserves the lattice $\mathbb{Z}^{S}$ and thus $F(x) \in \mathbb{Z}[x]$.
Pointed Coxeter systems. A pointed Coxeter system is a triple ( $W, S, s$ ) with $s \in S$. It is determined up to isomorphism by a pointed diagram $(F, s)$. By deleting $s$, we obtain a Coxeter subsystem ( $\widehat{W}, \widehat{S}$ ) with Coxeter polynomial $\widehat{F}(x)$.

We let $\left(A_{n}, i\right)$ and $\left(E_{n}, i\right)$ denote the $A_{n}$ and $E_{n}$ diagrams with the $i$ th vertex distinguished, using the numbering in Figures 1 and 3.
Joins. The join $(W, S)$ of pointed Coxeter systems $\left(W_{i}, S_{i}, s_{i}\right)_{i=1}^{m}$ is defined by taking an independent generator $t$, setting $S=\{t\} \bigcup S_{i}$, and setting

$$
W=\left(W_{1} * \cdots * W_{m} *\langle t\rangle\right) /\left\langle t^{2}=\left(s_{1} t\right)^{3}=\cdots=\left(s_{m} t\right)^{3}=\mathrm{id}\right\rangle
$$

The corresponding diagram $F$ is obtained from $\sqcup F_{i}$ by adding a new vertex $t$ and connecting it to each $s_{i}$ with a single edge (see Figure 5). If all the diagrams $F_{i}$ are bipartite, so is $F$.

In Theorem 3.1, basepoints $s_{i} \in S_{i}$ must be chosen to make the join $(W, S)$ well-defined, but the conclusion holds independent of the choice of basepoints.


Figure 5. The join of $A_{3}, B_{2}$ and $D_{4}$.

Proof of Theorem 3.1. Let $(W, S)$ be the join of $\left(W_{i}, S_{i}\right)_{1}^{m}$. By equation (3.1), a given reflection $s(v)$ only changes the coordinate $v_{s}$ of a vector $v \in \mathbb{R}^{S}$. Thus we have natural inclusions $W_{i} \subset W$ compatible with the inclusions $\mathbb{R}^{S_{i}} \subset \mathbb{R}^{S}$.

Since $s, t \in S$ commute whenever they are not joined by an edge in the Coxeter diagram, we can write the bicolored Coxeter element $w \in W$ in the form

$$
w=t w_{1} \cdots w_{m}
$$

Let $E_{i} \subset \mathbb{C}^{S_{i}} \subset \mathbb{C}^{S}$ be the $\lambda$-eigenspaces for $w_{i}$, extended by zero in the remaining coordinates. By (3.1) we have $w_{i} \mid E_{j}=\mathrm{id}$ for $i \neq j$. Thus $\oplus E_{i}$ is a $\lambda$-eigenspace for $w_{1} \cdots w_{m}$. Since $t(v)$ only changes $v_{t}$, there is a codimensionone subspace $E \subset \oplus E_{i}$ such that $t \mid E=\mathrm{id}$. Consequently the multiplicity of $\lambda$ as an eigenvalue for $w$ is bounded below by

$$
\operatorname{dim}(E)=\left(\sum \operatorname{dim}\left(E_{i}\right)\right)-1=\left(\sum m_{i}\right)-1
$$

The Coxeter polynomial of a join. Here is an alternative approach to the result above. When $F$ is the join of $\left(F_{i}, s_{i}\right)_{1}^{m}$, a straightforward matrix computation yields the following useful formula for its Coxeter polynomial:

$$
\begin{equation*}
F(x)=F_{1}(x) \cdots F_{m}(x)\left((x+1)-x \sum_{1}^{m} \frac{\widehat{F}_{i}(x)}{F_{i}(x)}\right) \tag{3.2}
\end{equation*}
$$

Cf. [CDS, Prob 9, p.78], [MRS, Cor. 4].
By writing the Coxeter element of $\left(W_{i}, S_{i}\right)$ with $s_{i}$ at the end, one can verify that the order of vanishing of its Coxeter polynomial satisfies ord $\left(P_{i}, \lambda\right)-1 \leq$ $\operatorname{ord}\left(\widehat{P}_{i}, \lambda\right)$. Thus equation (3.2) implies

$$
\operatorname{ord}(F, \lambda) \geq-1+\sum \operatorname{ord}\left(F_{i}, \lambda\right)
$$

This inequality is equivalent to Theorem 3.1 when the quadratic form $B$ of $(W, S)$ is non-degenerate, as it is for $E_{n}, n \neq 9$.

## 4 Decorating $A_{n}$

In this section we generalize our results on $E_{n}$ to more general diagrams $F_{n}$ of the form shown in Figure 6. Our main result is:

Theorem 4.1 Let $F_{n}$ be the sequence of Coxeter diagrams obtained by attaching pointed diagrams $(B, s)$ and $(C, t)$ to the ends of $A_{n}$. Assume $F_{n}(x) \in \mathbb{Z}[x]$ for all $n$. Then either

1. The diagram $F_{n}$ is spherical or affine for all $n$, or
2. The cyclotomic factor of $F_{n}(x)$ is periodic for $n \gg 0$.


Figure 6 . The diagram $F_{n}$ obtained by attaching $(B, s)$ and $(C, t)$ to the ends of $A_{n}$.

Coxeter polynomials. We begin by determining the Coxeter polynomial $F_{n}(x)$. First, by repeatedly applying equation (3.2) with $m=1$, we obtain:

Proposition 4.2 The Coxeter polynomial of the diagram $B_{n}$ obtained by attaching $(B, s)$ to one end of $A_{n}$ satisfies:

$$
B_{n}(x)(x-1)=x^{n+1}(B(x)-\widehat{B}(x))+(x \widehat{B}(x)-B(x)) .
$$

Here is an example:
Corollary 4.3 For $n \geq 4$, we have

$$
E_{n}(x)(x-1)=x^{n-2}\left(x^{3}-x-1\right)+\left(x^{3}+x^{2}-1\right)
$$

Proof. Take $(B, s)=\left(A_{4}, 2\right)$; then $B(x)=A_{4}(x), \widehat{B}(x)=A_{1}(x) A_{2}(x)$, and $B_{n}(x)=E_{n+4}(x)$. Thus $B(x)-\widehat{B}(x)=x\left(x^{3}-x-1\right)$ and $x \widehat{B}(x)-B(x)=$ $x^{3}+x^{2}-1$, which gives

$$
E_{n+4}(x)(x-1)=x^{n+2}\left(x^{3}-x-1\right)+\left(x^{3}+x^{2}-1\right)
$$

Since $F_{n}$ is the join of $B_{n-1}$ and $C$, by applying equation (3.2) once more we find:

Proposition 4.4 The Coxeter polynomials of $F_{n},(B, s)$ and $(C, t)$ are related by $F_{n}(x)(x-1)=x^{n+1} Q(x)-R(x)$, where

$$
\begin{aligned}
Q(x) & =(B(x)-\widehat{B}(x))(C(x)-\widehat{C}(x)) \quad \text { and } \\
R(x) & =(x \widehat{B}(x)-B(x))(x \widehat{C}(x)-C(x))
\end{aligned}
$$

We will also need the following result. Let $\beta\left(F_{n}\right) \geq 1$ denote the largest real zero of $F_{n}(x)$; equivalently, the spectral radius of the bicolored Coxeter element for $F_{n}$.

Proposition 4.5 (Hoffman-Smith) If $\beta\left(F_{n}\right)>1$, then $\beta\left(F_{n}\right) \neq \beta\left(F_{n+1}\right)$.
Proof. Let $A_{s t}=2 I-B_{s t}$ denote the symmetric 'adjacency matrix' for the $F_{n}$ diagram, and $\alpha\left(F_{n}\right)$ its spectral radius. Then since $\beta\left(F_{n}\right)>1$, we have

$$
\alpha\left(F_{n}\right)=\left(2+\beta\left(F_{n}\right)+\beta\left(F_{n}\right)^{-1}\right)^{1 / 2}>2,
$$

(see e.g. [Mc, Thm. 5.1]).
By [HS, Lem. 2.3 and Prop. 2.4], the condition $\alpha\left(F_{n}\right)>2$ implies that $\alpha\left(F_{n+1}\right)<\alpha\left(F_{n}\right)$ if we are adding nodes to an internal path, and that $\alpha\left(F_{n+1}\right)>$ $\alpha\left(F_{n}\right)$ if we are adding nodes to an external path (i.e. if $(B, s)$ or $(C, t)$ is equal to $\left(A_{i}, 1\right)$.) (The proof in [HS] is given for graphs, but it applies without change to Coxeter diagrams, using the following key fact: if $s$ is an endpoint of a maximal $A_{k}$ embedded in $F_{n}$, either $s$ is an endpoint of $F_{n}$, or $\sum_{t \neq s} A_{s t} \geq 1+\sqrt{2}$.)

In particular, we have $\alpha\left(F_{n+1}\right) \neq \alpha\left(F_{n}\right)$, and hence $\beta\left(F_{n}\right) \neq \beta\left(F_{n+1}\right)$.

Proof of Theorem 4.1. Since $\operatorname{deg} B>\operatorname{deg} \widehat{B}$ and $\operatorname{deg} C>\operatorname{deg} \widehat{C}$, we have $Q(x) \neq 0$. By Theorem 2.1, either:
(i) $F_{n}(x)(x-1)=\left(x^{n} \pm x^{i}\right) Q(x)$, or
(ii) Only finitely many $k$ satisfy $F_{n}\left(\zeta_{k}\right)=0$ for some $n$.

In case (i), the zeros of $F_{n}(x)$ outside the unit circle must be constant as $n$ varies. This implies the spectral radius $\beta\left(F_{n}\right)$ of the bicolored Coxeter element is constant; hence $\beta\left(F_{n}\right)=1$ by Proposition 4.5 , which means $F_{n}$ is spherical or affine by $\left[\mathrm{A}^{\prime} \mathrm{C}\right]$.

For case (ii), fix $k$ such that $F_{n}\left(\zeta_{k}\right)=0$. Clearly the values of $F_{n}\left(\zeta_{k}\right)$ are periodic in $n$. To complete the proof, we must show the order of vanishing of $F_{n}$ at $\zeta_{k}$ is also periodic. For this we may assume $Q(x)$ and $R(x)$ are relatively prime. Then $Q\left(\zeta_{k}\right) \neq 0$, and hence for all $n \gg 0, F_{n}^{\prime}\left(\zeta_{k}\right) \neq 0$, since the dominant term in the derivative is $(n+1) \zeta_{k}^{n} Q\left(\zeta_{k}\right)$. Consequently the cyclotomic zeros of $F_{n}$ are simple for all $n \gg 0$, and the proof is complete.

Notes. For a survey of results on the largest eigenvalues of graphs, including the inequality of Hoffman and Smith used above, see [CR].

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