Foliations of Hilbert Modular Surfaces

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Foliations of Hilbert modular surfaces

Curtis T. McMullen

21 February, 2005

Abstract

The Hilbert modular surface $X_D$ is the moduli space of Abelian varieties $A$ with real multiplication by a quadratic order of discriminant $D > 1$. The locus where $A$ is a product of elliptic curves determines a finite union of algebraic curves $X_D(1) \subset X_D$.

In this paper we show the lamination $X_D(1)$ extends to an essentially unique foliation $\mathcal{F}_D$ of $X_D$ by complex geodesics. The geometry of $\mathcal{F}_D$ is related to Teichmüller theory, holomorphic motions, polygonal billiards and Lattès rational maps. We show every leaf of $\mathcal{F}_D$ is either closed or dense, and compute its holonomy. We also introduce refinements $T_N(\nu)$ of the classical modular curves on $X_D$, leading to an explicit description of $X_D(1)$.

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1 Introduction

Let $D > 1$ be an integer congruent to 0 or 1 mod 4, and let $\mathcal{O}_D$ be the real quadratic order of discriminant $D$. The Hilbert modular surface

$$X_D = (\mathbb{H} \times \mathbb{H}) / \text{SL}(\mathcal{O}_D \oplus \mathcal{O}_D^\vee)$$

is the moduli space for principally polarized Abelian varieties $A_\tau = \mathbb{C}^2 / (\mathcal{O}_D \oplus \mathcal{O}_D^\vee \tau)$ with real multiplication by $\mathcal{O}_D$.

Let $X_D(1) \subset X_D$ denote the locus where $A_\tau$ is isomorphic to a polarized product of elliptic curves $E_1 \times E_2$. The set $X_D(1)$ is a finite union of disjoint, irreducible algebraic curves ($\S 4$), forming a lamination of $X_D$. Note that $X_D(1)$ is preserved by the twofold symmetry $\iota(\tau_1, \tau_2) = (\tau_2, \tau_1)$ of $X_D$.

In this paper we will show:

**Theorem 1.1** Up to the action of $\iota$, the lamination $X_D(1)$ extends to a unique foliation $\mathcal{F}_D$ of $X_D$ by complex geodesics.

(Here a Riemann surface in $X_D$ is a complex geodesic if it is isometrically immersed for the Kobayashi metric.)

**Holomorphic graphs.** The preimage $\tilde{X}_D(1)$ of $X_D(1)$ in the universal cover of $X_D$ gives a lamination of $\mathbb{H} \times \mathbb{H}$ by the graphs of countably many Möbius transformations. To foliate $X_D$ itself, in $\S 6$ we will show:

**Theorem 1.2** For any $(\tau_1, \tau_2) \notin \tilde{X}_D(1)$, there is a unique holomorphic function

$$f : \mathbb{H} \to \mathbb{H}$$

such that $f(\tau_1) = \tau_2$ and the graph of $f$ is disjoint from $\tilde{X}_D(1)$.

The graphs of such functions descend to $X_D$, and form the leaves of the foliation $\mathcal{F}_D$ ($\S 7$). The case $D = 4$ is illustrated in Figure 1.

**Modular curves.** To describe the lamination $X_D(1)$ explicitly, recall that the Hilbert modular surface $X_D$ is populated by infinitely many modular curves $F_N$ [Hir], [vG]. The endomorphism ring of a generic Abelian variety in $F_N$ is a quaternionic order $R$ of discriminant $N^2$.

In general $F_N$ can be reducible, and $R$ is not determined up to isomorphism by $N$. In $\S 3$ we introduce a refinement $F_N(\nu)$ of the traditional modular curves, such that the isomorphism class of $R$ is constant along
F_{N}(\nu) and F_{N} = \bigcup F_{N}(\nu). The additional finite invariant \nu ranges in the ring \mathcal{O}_{D}/(\sqrt{D}) and its norm satisfies N(\nu) = -N \mod D. The curves \mathcal{T}_{N} = \bigcup F_{N/\ell^2} can be refined similarly, and we obtain:

**Theorem 1.3** The locus \( X_{D}(1) \subset X_{D} \) is given by

\[
X_{D}(1) = \bigcup \mathcal{T}_{N}((e + \sqrt{D})/2),
\]

where the union is over all integral solutions to \( e^2 + 4N = D, N > 0 \).

**Remark.** Although \( X_{D}(1) = \bigcup T_{(D-e^2)/4} \) when \( D \) is prime, in general (e.g. for \( D = 12, 16, 20, 21, \ldots \)) the locus \( X_{D}(1) \) cannot be expressed as a union of the traditional modular curves \( T_{N} (\S 3) \).

Here is a corresponding description of the lamination \( \tilde{X}_{D}(1) \). Given \( N > 0 \) such that \( D = e^2 + 4N \), let

\[
\Lambda_{D}^{N} = \left\{ U = \begin{pmatrix} \mu & bD \\ -a & -\mu' \end{pmatrix} : a, b \in \mathbb{Z}, \mu \in \mathcal{O}_{D}, \det(U) = N \right\},
\]

and \( \mu \equiv \pm(e + \sqrt{D})/2 \) in \( \mathcal{O}_{D}/(\sqrt{D}) \).

Let \( \Lambda_{D} \) be the union of all such \( \Lambda_{D}^{N} \). Choosing a real place \( \iota_{1} : \mathcal{O}_{D} \rightarrow \mathbb{R} \), we can regard \( \Lambda_{D} \) as a set of matrices in \( \text{GL}_{2}^{+}(\mathbb{R}) \), acting by Möbius transformations on \( \mathbb{H} \).

**Theorem 1.4** The lamination \( \tilde{X}_{D}(1) \) of \( \mathbb{H} \times \mathbb{H} \) is the union of the loci \( \tau_{2} = U(\tau_{1}) \) over all \( U \in \Lambda_{D} \).
We also obtain a description of the locus $X_D(E) \subset X_D$ where $A_\tau$ admits an action of both $O_D$ and $O_E$ (§3).

**Quasiconformal dynamics.** Although its leaves are Riemann surfaces, $F_D$ is not a holomorphic foliation. Its transverse dynamics is given instead by quasiconformal maps, which can be described as follows.

Let $q = q(z) dz^2$ be a meromorphic quadratic differential on $\mathbb{H}$. We say a homeomorphism $f : \mathbb{H} \to \mathbb{H}$ is a Teichmüller mapping relative to $q$ if it satisfies $\frac{\partial f}{\partial \bar{f}} = \frac{\alpha q}{|q|}$ for some complex number $|\alpha| < 1$; equivalently, if $f$ has the form of an orientation-preserving real-linear mapping

$$f(x + iy) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = D_q(f) \begin{pmatrix} x \\ y \end{pmatrix}$$

in local charts where $q = dz^2 = (dx + idy)^2$.

Fix a transversal $\mathbb{H}_s = \{s\} \times \mathbb{H}$ to $\tilde{F}_D$. Any $g \in \text{SL}(O_D \oplus O_D^\vee)$ acts on $\mathbb{H} \times \mathbb{H}$, permuting the leaves of $\tilde{F}_D$. The permutation of leaves is recorded by the holonomy map

$$\phi_g : \mathbb{H}_s \to \mathbb{H}_s,$$

characterized by the property that $g(s, z)$ and $(s, \phi_g(z))$ lie on the same leaf of $\tilde{F}_D$.

In §8 we will show:

**Theorem 1.5** The holonomy acts by Teichmüller mappings relative to a fixed meromorphic quadratic differential $q$ on $\mathbb{H}_s$. For $s = i$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have

$$D_q(\phi_g) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}_2(\mathbb{R}).$$

On the other hand, for $z \in \partial \mathbb{H}_s$ we have

$$\phi_g(z) = (a'z - b')/(-c'z + d');$$

in particular, the holonomy acts by Möbius transformations on $\partial \mathbb{H}_s$.

Here $(x + y\sqrt{D})' = (x - y\sqrt{D})$. Note that both Galois conjugate actions of $g$ on $\mathbb{R}^2$ appear, as different aspects of the holonomy map $\phi_g$.

**Quantum Teichmüller curves.** For comparison, consider an isometrically immersed Teichmüller curve

$$f : V \to \mathcal{M}_g,$$
generated by a holomorphic quadratic differential \((Y, q)\) of genus \(g\). For simplicity assume \(\text{Aut}(Y)\) is trivial. Then the pullback of the universal curve \(X = f^*(\mathcal{M}_{g,1})\) gives an algebraic surface

\[
p: X \to V
\]

with \(p^{-1}(v) = Y\) for a suitable basepoint \(v \in V\). The surface \(X\) carries a canonical foliation \(\mathcal{F}\), transverse to the fibers of \(p\), whose leaves map to Teichmüller geodesics in \(\mathcal{M}_{g,1}\). The holonomy of \(\mathcal{F}\) determines a map

\[
\pi_1(V, v) \to \text{Aff}^+(Y, q)
\]

giving an action of the fundamental group by Teichmüller mappings; and its linear part yields the isomorphism

\[
\pi_1(V, v) \cong \text{PSL}(Y, q) \subset \text{PSL}_2(\mathbb{R}),
\]

where \(\text{PSL}(Y, q)\) is the stabilizer of \((Y, q)\) in the bundle of quadratic differentials \(Q\mathcal{M}_g \to \mathcal{M}_g\). (See e.g. [V1], [Mc4, §2]).

The foliated Hilbert modular surface \((X_D, \mathcal{F}_D)\) presents a similar structure, with the fibration \(p: X \to V\) replaced by the holomorphic foliation \(\mathcal{A}_D\) coming from the level sets of \(\tau_1\) on \(\tilde{X}_D = \mathbb{H} \times \mathbb{H}\). This suggests that one should regard \((X_D, \mathcal{A}_D, \mathcal{F}_D)\) as a quantum Teichmüller curve, in the same sense that a 3-manifold with a measured foliation can be regarded as a quantum Teichmüller geodesic [Mc3].

**Question.** Does every fibered surface \(p: X \to C\) admit a foliation \(\mathcal{F}\) by Riemann surfaces transverse to the fibers of \(p\)\

**Complements.** We conclude in §9 by presenting the following related results.

1. Every leaf of \(\mathcal{F}_D\) is either closed or dense.

2. When \(D \neq d^2\), there are infinitely many eigenforms for real multiplication by \(O_D\) that are isoperiodic but not isomorphic.

3. The Möbius transformations \(\Lambda_D\) give a maximal top-speed holomorphic motion of a discrete subset of \(\mathbb{H}\).

4. The foliation \(\mathcal{F}_4\) also arises as the motion of the Julia set in a Lattès family of iterated rational maps.
The link with complex dynamics was used to produce Figure 1.

Notes and references. The foliation $\mathcal{F}_D$ is constructed using the connection between polygonal billiards and Hilbert modular surfaces presented in [Mc4]. For more on the interplay of dynamics, holomorphic motions and quasiconformal mappings, see e.g. [MSS], [BR], [Sl], [Mc2], [Sul], [McS], [EKK] and [Dou]. A survey of the theory of holomorphic foliations of surfaces appears in [Br1]; see also [Br2] for the Hilbert modular case.

I would like to thank G. van der Geer, B. Gross and the referees for useful comments and suggestions.

2 Quaternion algebras

In this section we consider a real quadratic order $\mathcal{O}_D$ acting on a symplectic lattice $L$, and classify the quaternionic orders $R \subset \text{End}(L)$ extending $\mathcal{O}_D$.

Quadratic orders. Given an integer $D > 0$, $D \equiv 0$ or 1 mod 4, the real quadratic order of discriminant $D$ is given by

$$\mathcal{O}_D = \mathbb{Z}[T]/(T^2 + bT + c),$$

where $D = b^2 - 4c$.

Let $K_D = \mathcal{O}_D \otimes \mathbb{Q}$. Provided $D$ is not a square, $K_D$ is a real quadratic field. Fixing an embedding $\iota_1 : K_D \to \mathbb{R}$, we obtain a unique basis

$$K_D = \mathbb{Q} \cdot 1 \oplus \mathbb{Q} \cdot \sqrt{D}$$

such that $\iota_1(\sqrt{D}) > 0$. The conjugate real embedding $\iota_2 : K_D \to \mathbb{R}$ is given by $\iota_2(x) = \iota_1(x')$, where $(a + b\sqrt{D})' = (a - b\sqrt{D})$.

Square discriminants. The case $D = d^2$ can be treated similarly, so long as we regard $x = \sqrt{d^2}$ as an element of $K_D$ satisfying $x^2 = d^2$ but $x \not\in \mathbb{Q}$. In this case the algebra $K_D \cong \mathbb{Q} \oplus \mathbb{Q}$ is not a field, so we must take care to distinguish between elements of the algebra such as

$$x = d - \sqrt{d^2} \in K_D,$$

and the corresponding real numbers

$$\iota_1(x) = d - d = 0, \quad \text{and} \quad \iota_2(x) = d + d = 2d.$$

Trace, norm and different. For simplicity of notation, we fix $D$ and denote $\mathcal{O}_D$ and $K_D$ by $\mathcal{O}$ and $K$. 


The trace and norm on $K$ are the rational numbers $\text{Tr}(x) = x + x'$ and $N(x) = xx'$. The inverse different is the fractional ideal

$$\mathcal{O}^\vee = \{ x \in K : \text{Tr}(xy) \in \mathbb{Z} \forall y \in \mathcal{O} \}.$$

It is easy to see that $\mathcal{O}^\vee = D^{-1/2} \mathcal{O}$, and thus the different $\mathcal{D} = (\mathcal{O}^\vee)^{-1} \subset \mathcal{O}$ is the principal ideal $(\sqrt{D})$. The trace and norm descend to give maps

$$\text{Tr}, N : \mathcal{O}/\mathcal{D} \to \mathbb{Z}/D,$$

satisfying

$$\text{Tr}(x)^2 = 4 N(x) \mod D. \quad (2.1)$$

When $D$ is odd, $\text{Tr} : \mathcal{O}/\mathcal{D} \to \mathbb{Z}/D$ is an isomorphism, and thus $(2.1)$ determines the norm on $\mathcal{O}/\mathcal{D}$. On the other hand, when $D = 4E$ is even, we have an isomorphism

$$\mathcal{O}/\mathcal{D} \cong \mathbb{Z}/2E \oplus \mathbb{Z}/2$$

given by $a + b\sqrt{E} \mapsto (a, b)$, and the trace and norm on $\mathcal{O}/\mathcal{D}$ are given by

$$\text{Tr}(a, b) = 2a \mod D, \quad N(a, b) = a^2 - Eb^2 \mod D.$$

**Symplectic lattices.** Now let $L \cong (\mathbb{Z}^{2g}, \left( \begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix} \right))$ be a unimodular symplectic lattice of genus $g$. (This lattice is isomorphic to the first homology group $H_1(\Sigma_g, \mathbb{Z})$ of an oriented surface of genus $g$ with the symplectic form given by the intersection pairing.)

Let $\text{End}(L) \cong M_{2g}(\mathbb{Z})$ denote the endomorphism ring of $L$ as a $\mathbb{Z}$-module. The Rosati involution $T \mapsto T^*$ on $\text{End}(L)$ is defined by the condition $\langle Tx, y \rangle = \langle x, T^*y \rangle$; it satisfies $(ST)^* = T^*S^*$, and we say $T$ is self-adjoint if $T = T^*$.

Specializing to the case $g = 2$, let $L$ denote the lattice

$$L = \mathcal{O} \oplus \mathcal{O}^\vee$$

with the unimodular symplectic form

$$\langle x, y \rangle = \text{Tr}(x \wedge y) = \text{Tr}_K(x_1y_2 - x_2y_1).$$

A standard symplectic basis for $L$ (satisfying $\langle a_i \cdot b_j \rangle = \delta_{ij}$) is given by

$$(a_1, a_2, b_1, b_2) = ((1, 0), (\gamma, 0), (0, -\gamma/\sqrt{D}), (0, 1/\sqrt{D})), \quad (2.2)$$
where $\gamma = (D + \sqrt{D})/2$.

The lattice $L$ comes equipped with a proper, self-adjoint action of $O$, given by

$$k \cdot (x_1, x_2) = (kx_1, kx_2).$$

(2.3)

Conversely, any proper, self-adjoint action of $O$ on a symplectic lattice of genus two is isomorphic to this model (see e.g. [Ru], [Mc7, Thm 4.1]). (Here an action of $R$ on $L$ is proper if it is indivisible: if whenever $T \in \text{End}(L)$ and $mT \in R$ for some integer $m \neq 0$, then $T \in R$.)

**Matrices.** The natural embedding of $L = O \oplus O^\vee$ into $K \oplus K$ determines an embedding of matrices

$$M_2(K) \to \text{End}(L \otimes \mathbb{Q}),$$

and hence a diagonal inclusion

$$K \to \text{End}(L \otimes \mathbb{Q})$$

extending the natural action (2.3) of $O$ on $L$. Every $T \in \text{End}(L \otimes \mathbb{Q})$ can be uniquely expressed in the form

$$T(x) = Ax + Bx', \quad A, B \in M_2(K),$$

where $(x_1, x_2)' = (x_1', x_2')$; and we have

$$T^*(x) = A^\dagger x + (B^\dagger)'x',$$

(2.4)

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\dagger = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

The automorphisms of $L$ as a symplectic $O$-module are given, as a subgroup of $M_2(K)$, by

$$\text{SL}(O \oplus O^\vee) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \begin{pmatrix} O & D \\ O^\vee & O \end{pmatrix} : ad - bc = 1 \right\}.$$

Compare [vG, p.12].

**Integrality.** An endomorphism $T \in \text{End}(L \otimes \mathbb{Q})$ is integral if it satisfies $T(L) \subset L$.

**Lemma 2.1** The endomorphism $\phi(x) = ax + bx'$ of $K$ satisfies $\phi(O) \subset O$ iff $a, b \in O^\vee$ and $a + b \in O$.

**Proof.** Since $x - x' \in \sqrt{D}\mathbb{Z}$ for all $x \in O$, the conditions on $a, b$ imply $\phi(x) = a(x - x') + (a + b)x' \in O$ for all $x \in O$. Conversely, if $\phi$ is integral, then $\phi(1) = a + b \in O$, and thus $a(x - x') \in O$ for all $x \in O$, which implies $a \in D^{-1/2}O = O^\vee$. $\blacksquare$
Corollary 2.2  The endomorphism $T(x) = kx + (a\ b\ c\ d)x'$ is integral iff we have
\[ a, b, c, d, k \in O^\vee \quad \text{and} \quad k + a, k - d \in O. \]

Proof. This follows from the preceding Lemma, using the fact that $kx + dx'$ maps $O^\vee$ to $O^\vee$ iff $kx - dx'$ maps $O$ to $O$. ■

Quaternion algebras. A rational quaternion algebra is a central simple algebra of dimension 4 over $\mathbb{Q}$. Every such algebra has the form
\[ Q \cong \mathbb{Q}[i, j]/(i^2 = a, j^2 = b, ij = -ji) = (a, b)_{\mathbb{Q}} \]
for suitable $a, b \in \mathbb{Q}^*$. Any $q \in Q$ satisfies a quadratic equation
\[ q^2 - \text{Tr}(q)q + \text{N}(q) = 0, \]
where $\text{Tr}, \text{N} : Q \to \mathbb{Q}$ are the reduced trace and norm.

An order $R \subset Q$ is a subring such that, as an additive group, we have $R \cong \mathbb{Z}^4$ and $Q \cdot R = Q$. Its discriminant is the square integer
\[ N^2 = |\det(\text{Tr}(q_iq_j))| > 0, \]
where $(q_i)_{4\times 1}$ is an integral basis for $R$. The discriminants of a pair of orders $R_1 \subset R_2$ are related by $N_1/N_2 = |R_2/R_1|^2$.

Generators. We say $V \in \text{End}(L)$ is a quaternionic generator if:
1. $V^* = -V$,
2. $V^2 = -N \in \mathbb{Z}$, $N \neq 0$,
3. $Vk = k'V$ for all $k \in K$, and
4. $k + D^{-1/2}V \in \text{End}(L)$ for some $k \in K$.

These conditions imply that $Q = K \oplus KV$ is a quaternion algebra isomorphic to $(D, -N)_{\mathbb{Q}}$. Conversely, we have:

Theorem 2.3  Any Rosati-invariant quaternion algebra $Q$ with
\[ K \subset Q \subset \text{End}(L \otimes \mathbb{Q}) \]
contains a unique pair of primitive quaternionic generators $\pm V$. 8
(A generator is *primitive* unless \((1/m)V, m > 1\) is also a generator.)

**Proof.** By a standard application of the Skolem-Noether theorem, we can write \(Q = K \oplus KW\) with \(0 \neq W^2 \in \mathbb{Q}\) and \(Wk = kW\) for all \(k \in K\). Then \(KW\) coincides with the subalgebra of \(Q\) anticommuting with the self-adjoint element \(\sqrt{D}\), so it is Rosati-invariant. The eigenspaces of *\(KW\) are exchanged by multiplication by \(\sqrt{D}\), so up to a rational multiple there is a unique nonzero \(V \in KW\) with \(V^* = -V\). A suitable integral multiple of \(V\) is then a generator, and a rational multiple is primitive. \(\blacksquare\)

**Corollary 2.4** *Quaternionic extensions* \(K \subset Q \subset \text{End}(L)\) *correspond bijectively to pairs of primitive generators* \(\pm V \in \text{End}(L)\).

**Generator matrices.** We say \(U \in M_2(K)\) is a *quaternionic generator matrix* if it has the form
\[
U = \begin{pmatrix} \mu & bD \\ -a & -\mu' \end{pmatrix}
\]  
(2.5)

with \(a, b \in \mathbb{Z}, \mu \in \mathcal{O}\) and \(N = \det(U) \neq 0\).

**Theorem 2.5** The endomorphism \(V(x) = Ux'\) is a quaternionic generator iff \(U\) is a quaternionic generator matrix.

**Proof.** By (2.4) the condition \(V = -V^*\) is equivalent to \(U^\dagger = -U'\), and thus \(U\) can be written in the form (2.5) with \(a, b \in \mathbb{Q}\) and \(\mu \in K\). Assuming \(U^\dagger = -U'\), we have
\[
N = \det(U) = UU^\dagger = -UU' = -V^2,
\]
so \(V^2 \neq 0 \iff \det(U) \neq 0\). The condition that \(D^{-1/2}(k+V)\) is integral for some \(k\) implies, by Corollary 2.2, that the coefficients of \(U\) satisfy \(a, b \in \mathbb{Z}\) and \(\mu \in \mathcal{O}\); and given such coefficients for \(U\), the endomorphism \(D^{-1/2}(k+V)\) is integral when \(k = -\mu\). \(\blacksquare\)

**The invariant \(\nu(U)\).** Given generator matrix \(U = \begin{pmatrix} \mu & bD \\ -a & -\mu' \end{pmatrix}\), let \(\nu(U)\) denote the image of \(\mu\) in the finite ring \(\mathcal{O}/\mathcal{D}\). It is easy to check that
\[
\nu(U) = \pm \nu(g'Ug^{-1})
\]
for all \(g \in \text{SL}(\mathcal{O} \oplus \mathcal{O}^\vee)\), and that its norm satisfies
\[
N(\nu(U)) \equiv -N \mod D. \quad (2.6)
\]
Quaternionic orders. Let $V(x) = Ux'$, and let

$$R_U = (K \oplus KV) \cap \text{End}(L).$$

Then $R_U$ is a Rosati-invariant order in the quaternion algebra generated by $V$. Clearly $O \subset R_U$, so we can also regard $(R_U, *)$ as an involutive algebra over $O$. We will show that $N = \det(U)$ and $\nu(U)$ determine $(R_U, *)$ up to isomorphism.

Models. We begin by constructing a model algebra $(R_N(\nu), *)$ over $O_D$ for every $\nu \in O / D$ with $N(\nu) = -N \neq 0 \mod D$.

Let $Q_N = K \oplus KV$ be the abstract quaternion algebra with the relations $V^2 = -N$ and $Vk = k'V$. Define an involution on $Q_N$ by $(k_1 + k_2 V)^* = (k_1 - k_2 V)$, and let $R_N(\nu)$ be the order in $Q_N$ defined by

$$R_N(\nu) = \{ \alpha + \beta V : \alpha, \beta \in O^\vee, \alpha + \beta \nu \in O \}.$$  \hfill (2.7)

Note that $O^\vee \cdot D \subset O$, so the definition of $R_N(\nu)$ depends only on the class of $\nu$ in $O / D$. To check that $R_N(\nu)$ is an order, note that

$$(\alpha + \beta V)(\gamma + \delta V) = (\kappa + \lambda V) = (\alpha \gamma - N \beta \delta') + (\alpha \delta + \beta \gamma')V;$$

since $-N \equiv N(\nu) = \nu \nu' \mod D$, we have

$$\kappa + \nu \lambda \equiv (\alpha \gamma + \nu \nu' \beta \delta') + \nu (\alpha \delta + \beta \gamma')$$

$$= (\alpha + \beta \nu)(\gamma' + \delta' \nu') + \alpha (\gamma - \gamma' + \nu \delta - \nu' \delta')$$

$$\equiv 0 + 0 \mod O,$$

and thus $R_U$ is closed under multiplication.

Theorem 2.6 The quaternionic order $R_N(\nu)$ has discriminant $N^2$.

Proof. Note that the inclusions

$$O \oplus O V \subset R_N(\nu) \subset O^\vee \oplus O^\vee V$$

each have index $D$. The quaternionic order $O \oplus O V$ has discriminant $D^2 N^2$, since $V^2 = -N$ and $\text{Tr} | O V = 0$, and thus $R_N(\nu)$ has discriminant $N^2$. 

10
Theorem 2.7 We have \((R_N(\nu), \ast) \cong (R_M(\mu), \ast)\) iff \(N = M\) and \(\nu = \pm \mu\).

Proof. The element \(V \in R_N(\nu)\) is, up to sign, the order’s unique primitive generator, in the sense that \(V^* = -V, V k = k' V\) for all \(k \in \mathcal{O}_D, V^2 \neq 0, k + D^{-1/2} V \in R_N(\nu)\) for some \(k \in K\), and \(V\) is not a proper multiple of another element in \(R_N(\nu)\) with the same properties. Thus the structure of \((R_N(\nu), \ast)\) as an \(\mathcal{O}_D\)-algebra determines \(V \in R_N(\nu)\) up to sign, and \(V\) determines \(N = -V^2\) and the constant \(\nu \in \mathcal{O} / \mathcal{D}\) in the relation \(\alpha + \beta\nu \in \mathcal{O}\) defining \(R_N(\nu) \subset K \oplus KV\). 

Theorem 2.8 If \(U\) is a primitive generator matrix, then we have
\[(R_U, \ast) \cong (R_N(\nu), \ast)\]
where \(N = \det(U)\) and \(\nu = \nu(U)\).

Proof. Setting \(V(x) = Ux'\), we need only verify that \((K \oplus KV) \cap \text{End}(L)\) coincides with the order \(R_N(\nu)\) defined by (2.7). To see this, let
\[T(x) = \alpha x + \beta V(x) = \alpha x + \beta \begin{pmatrix} \mu & b D \\ -a & -\mu' \end{pmatrix} x'\]
in \(K \oplus KV\). By Corollary 2.2, \(T\) is integral iff
\[(i) \ a\beta, b\beta, \mu\beta, \mu'\beta \in \mathcal{O}^\vee,\]
\[(ii) \ \alpha \in \mathcal{O}^\vee,\]
\[(iii) \ \alpha + \beta\mu \in \mathcal{O} \text{ and }\]
\[(iv) \ \alpha + \beta\mu' \in \mathcal{O}.\]
Using (iii), condition (iv) can be replaced by
\[(iv') \ \beta(\mu - \mu')/\sqrt{D} \in \mathcal{O}^\vee.\]
Since \(U\) is primitive, the ideal \((a, b, \mu, (\mu - \mu')/\sqrt{D})\) is equal to \(\mathcal{O}\). Thus (i) and (iv') together are equivalent to the condition \(\beta \in \mathcal{O}^\vee\), and we are left with the definition of \(R_N(\nu)\). 

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Remark. In general, the invariants \( \det(U) \) and \( \nu(U) \) do not determine the embedding \( R_U \subset \text{End}(L) \) up to conjugacy. For example, when \( D \) is odd, the generator matrices \( U_1 = \begin{pmatrix} 0 & D^2 \\ -D & 0 \end{pmatrix} \) and \( U_2 = \begin{pmatrix} 0 & D^3 \\ -1 & 0 \end{pmatrix} \) have the same invariants, but the corresponding endomorphisms are not conjugate in \( \text{End}(L) \) because
\[
L/V_1(L) \cong (\mathbb{Z}/D \times \mathbb{Z}/D^2)^2
\]
while
\[
L/V_2(L) \cong \mathbb{Z}/D \times \mathbb{Z}/D^2 \times \mathbb{Z}/D^3.
\]

Extra quadratic orders. Finally we determine when the algebra \( R_N(\nu) \) contains a second, independent quadratic order \( \mathcal{O}_E \).

**Theorem 2.9** The algebra \( (R_N(\nu),\ast) \) contains a self-adjoint element \( T \notin \mathcal{O}_D \) generating a copy of \( \mathcal{O}_E \) iff there exist \( e, \ell \in \mathbb{Z} \) such that
\[
ED = e^2 + 4N\ell^2, \quad \ell \neq 0
\]
and \( (e + E\sqrt{D})/2 + \ell\nu = 0 \mod D \).

**Proof.** Given \( e, \ell \) as above, let
\[
T = \alpha + \beta V = D^{-1/2} \left( \frac{e + E\sqrt{D}}{2} + \ell V \right).
\]
Then we have \( T = T^* \), \( T \in R_N(\nu) \) and \( T^2 - eT + (E - E^2)/4 = 0 \); therefore \( \mathbb{Z}[T] \cong \mathcal{O}_E \). A straightforward computation shows that, conversely, any independent copy of \( \mathcal{O}_E \) in \( R_N(\nu) \) arises as above.

For additional background on quaternion algebras, see e.g. [Vi], [MR] and [Mn].

### 3 Modular curves and surfaces

In this section we describe modular curves on Hilbert modular surfaces from the perspective of the Abelian varieties they determine.

**Abelian varieties.** A *principally polarized Abelian variety* is a complex torus \( A \cong \mathbb{C}^g/L \) equipped with a unimodular symplectic form \( \langle x, y \rangle \) on \( L \cong \mathbb{Z}^{2g} \), whose extension to \( L \otimes \mathbb{R} \cong \mathbb{C}^g \) satisfies
\[
\langle x, y \rangle = \langle ix, iy \rangle \quad \text{and} \quad \langle x, ix \rangle \geq 0.
\]
The ring $\text{End}(A) = \text{End}(L) \cap \text{End}(C^g)$ is Rosati invariant, and coincides with the endomorphism ring of $A$ as a complex Lie group. We have $\text{Tr}(TT^*) \geq 0$ for all $T \in \text{End}(A)$.

Every Abelian variety can be presented in the form

$$A = C^g / (\mathbb{Z}^g \oplus \Pi \mathbb{Z}^g),$$

where $\Pi$ is an element of the Siegel upper halfplane

$$\mathcal{H}_g = \{ \Pi \in M_g(\mathbb{C}) : \Pi^t = \Pi \text{ and } \text{Im}(\Pi) \text{ is positive-definite} \}.$$

The symplectic form on $L = \mathbb{Z}^g \oplus \Pi \mathbb{Z}^g$ is given by $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. Any two such presentations of $A$ differ by an automorphism of $L$, so the moduli space of abelian varieties of genus $g$ is given by the quotient space

$$A_g = \mathcal{H}_g / \text{Sp}_{2g}(\mathbb{Z}).$$

**Real multiplication.** As in §2, let $D > 0$ be the discriminant of a real quadratic order $\mathcal{O}_D$, and let $K = \mathcal{O} \otimes \mathbb{Q}$. Fix a real place $\iota_1 : K \to \mathbb{R}$, and set $\iota_2(k) = \iota_1(k')$.

We will regard $K$ as a subfield of the reals, using the fixed embedding $\iota_1 : K \subset \mathbb{R}$. The case $D = d^2$ is treated with the understanding that the real numbers $(k, k')$ implicitly denote $(\iota_1(k), \iota_2(k))$, $k \in K$.

An Abelian variety $A \in A_2$ admits real multiplication by $\mathcal{O}_D$ if there is a self-adjoint endomorphism $T \in \text{End}(A)$ generating a proper action of $\mathbb{Z}[T] \cong \mathcal{O}_D$ on $A$. Any such variety can be presented in the form

$$A_\tau = C^2 / (\mathcal{O}_D \oplus \mathcal{O}_D^\vee \tau) = C^2 / \phi_\tau(L),$$

where $\tau = (\tau_1, \tau_2) \in \mathbb{H} \times \mathbb{H}$ and where $L = \mathcal{O} \oplus \mathcal{O}^\vee$ is embedded in $\mathbb{C}^2$ by the map

$$\phi_\tau(x_1, x_2) = (x_1 + x_2 \tau_1, x_1' + x_2' \tau_2).$$

As in §2, the symplectic form on $L$ is given by $\langle x, y \rangle = \text{Tr}_D(x \wedge y)$, and the action of $\mathcal{O}_D$ on $\mathbb{C}^2 \supset L$ is given simply by $k \cdot (z_1, z_2) = (kz_1, k'z_2)$.

**Eigenforms.** The Abelian variety $A_\tau$ comes equipped with a distinguished pair of normalized eigenforms $\eta_1, \eta_2 \in \Omega(A_\tau)$. Using the isomorphism $H_1(A_\tau, \mathbb{Z}) \cong L$, these forms are characterized by the property that

$$\phi_\tau(C) = \left( \int_C \eta_1, \int_C \eta_2 \right).$$

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Modular surfaces. If we change the identification \( L \cong H_1(A, \mathbb{Z}) \) by an automorphism \( g \) of \( L \), we obtain an isomorphic Abelian variety \( A_g \). Thus the moduli space of Abelian varieties with real multiplication by \( \mathcal{O}_D \) is given by the Hilbert modular surface

\[
X_D = ((\mathbb{H} \times \mathbb{H})/ \text{SL}(\mathcal{O}_D \oplus \mathcal{O}_D^\vee)).
\]

The point \( g(\tau) \) is characterized by the property that

\[
\phi_{g, \tau} = \chi(g, \tau) \phi_\tau \circ g^{-1}
\]

for some matrix \( \chi(g, \tau) \in \text{GL}_2(\mathbb{C}) \); explicitly, we have

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (\tau_1, \tau_2) = \left( \begin{array}{cc} a\tau_1 - b & a'\tau_2 - b' \\ -c\tau_1 + d' & -c'\tau_2 + d' \end{array} \right)
\]

and

\[
\chi(g, \tau) = \begin{pmatrix} (d - c\tau_1)^{-1} & 0 \\ 0 & (d' - c'\tau_2)^{-1} \end{pmatrix}.
\]

A point \([\tau] \in X_D\) gives an Abelian variety \([A] \in \mathcal{A}_2\) with a chosen embedding \( \mathcal{O}_D \to \text{End}(A) \). Similarly, a point \( \tau \in X_D = \mathbb{H} \times \mathbb{H} \) gives an Abelian variety with a distinguished isomorphism or marking, \( L \cong H_1(A, \mathbb{Z}) \), sending \( \mathcal{O}_D \) into \( \text{End}(A) \).

Modular embedding. The modular embedding

\[
p_D : X_D \to \mathcal{A}_2
\]

is given by \([\tau] \mapsto [A] \). To write \( p_D \) explicitly, note that the embedding \( \phi_\tau : L \to \mathbb{C}^2 \) can be expressed with respect to the basis \((a_1, a_2, b_1, b_2)\) for \( L \) given in (2.2) by the matrix

\[
\phi_\tau = \begin{pmatrix} 1 & \gamma & -\tau_1\gamma' / \sqrt{D} & \tau_1 / \sqrt{D} \\ 1 & \gamma' & \tau_2\gamma / \sqrt{D} & -\tau_2 / \sqrt{D} \end{pmatrix} = (A, B).
\]

Consequently we have \( A \cong \mathbb{C}^2 / (\mathbb{Z}^2 \oplus \Pi \mathbb{Z}^2) \), where

\[
\Pi = \tilde{p}_D(\tau) = A^{-1}B = \frac{1}{D} \begin{pmatrix} \tau_1(\gamma')^2 + \tau_2\gamma^2 & -\tau_1\gamma' - \tau_2\gamma \\ -\tau_1\gamma' - \tau_2\gamma & \tau_1 + \tau_2 \end{pmatrix}.
\]

The map \( X_D \to p_D(X_D) \) has degree two.
Modular curves. Given a matrix \( U(x) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(K) \cap \text{End}(L) \) such that \( U' = -U^* \), let \( V(x) = Ux' \) and define
\[
\mathbb{H}_U = \{ \tau \in \mathbb{H} \times \mathbb{H} : V \in \text{End}(A_{\tau}) \}.
\]
It is straightforward to check that \( \mathbb{H}_U = \{ (\tau_1, \tau_2) : \tau_2 = \frac{d\tau_1 + b}{c\tau_1 + a} \} \); (3.5)
indeed, when \( \tau_1 \) and \( \tau_2 \) are related as above, the map \( \phi_{\tau} : L \to \mathbb{C}^2 \) satisfies
\[
\phi_{\tau}(V(x)) = \begin{pmatrix} 0 & a + c\tau_1 \\ a' + c'\tau_2 & 0 \end{pmatrix} \phi_{\tau}(x),
\]
exhibiting the complex-linearity of \( V \). Note that \( \mathbb{H}_U = \emptyset \) if \( \det(U) < 0 \).

We now restrict attention to the case where \( U \) is a generator matrix. Then by the results of §2, we have:

**Theorem 3.1** The ring \( \text{End}(A_{\tau}) \) contains a quaternionic order extending \( \mathcal{O}_D \) if and only if \( \tau \in \mathbb{H}_U \) for some generator matrix \( U \).

Let \( F_U \subset X_D \) denote the projection of \( \mathbb{H}_U \) to the quotient \( (\mathbb{H} \times \mathbb{H})/SL(\mathcal{O}_D \oplus \mathcal{O}_D') \).
Following [Hir, §5.3], we define the modular curve \( F_N \) by
\[
F_N = \bigcup \{ F_U : U \text{ is a primitive generator matrix with } \det(U) = N \}.
\]
It can be shown that \( F_N \) is an algebraic curve on \( X_D \).

To describe this curve more precisely, let
\[
F_N(\nu) = \{ F_U : U \text{ is primitive, } \det(U) = N \text{ and } \nu(U) = \pm \nu \},
\]
where \( \nu \in \mathcal{O}_D/\mathcal{D}_D \). Note that we have
\[
F_N(\nu) \neq \emptyset \iff N(\nu) = -N \mod D
\]
by equation (2.6), \( F_N(\nu) = F_N(-\nu) \), and \( F_N = \bigcup F_N(\nu) \).

The results of §2 give the structure of the quaternion ring generated by \( V(x) = Ux' \).

**Theorem 3.2** The curve \( F_N(\nu) \subset X_D \) coincides with the locus of Abelian varieties such that
\[
\mathcal{O}_D \subset R \subset \text{End}(A_{\tau}),
\]
for some properly embedded quaternionic order \((R, \ast) \) isomorphic to \( (R_N(\nu), \ast) \).
Corollary 3.3  The curve $F_N$ is the locus where $O_D \subset \text{End}(A_\tau)$ extends to a properly embedded, Rosati-invariant quaternionic order of discriminant $N^2$.

Two quadratic orders. We can now describe the locus $X_D(E)$ of Abelian varieties with an independent, self-adjoint action of $O_E$. (We do not require the action of $O_E$ to be proper.)

To state this description, it is useful to define:

$$T_N = \bigcup \{ F_U : \det(U) = N \} = \bigcup F_{N/\ell^2},$$

and

$$T_N(\nu) = \bigcup \{ F_U : \det(U) = N, \nu(U) = \pm \nu \}.$$

Then Theorem 2.9 implies:

Theorem 3.4  The locus $X_D(E)$ is given by

$$X_D(E) = \bigcup T_N((e + E\sqrt{D})/2),$$

where the union is over all $N > 0$ and $e \in \mathbb{Z}$ such that $ED = e^2 + 4N$.

Corollary 3.5  We have $X_D(1) = \bigcup \{ T_N((e + \sqrt{D})/2) : e^2 + 4N = D \}$.

Refined modular curves. To conclude we show that in general the expression $F_N = \bigcup F_N(\nu)$ gives a proper refinement of $F_N$. First note:

Theorem 3.6  We have $F_N(\nu) = F_N$ iff $\pm \nu$ are the only solutions to

$$N(\xi) = -N \mod D, \quad \xi \in O_D/\mathcal{D}_D.$$

Corollary 3.7  If $D = p$ is prime, then $F_N = F_N(\nu)$ whenever $F_N(\nu) \neq \emptyset$.

Proof.  In this case, according to (2.1), the norm map

$$N : O_D/\mathcal{D}_D \cong \mathbb{Z}/p \to \mathbb{Z}/p$$

is given by $N(\xi) = \xi^2/4$. Since $F_N(\nu) \neq \emptyset$, we have $N(\nu) = -N$; and since $\mathbb{Z}/p$ is a field, $\pm \nu$ are the only solutions to this equation.

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Corollary 3.8 When $D$ is prime, we have $X_D(E) = \bigcup T_{(E_D - \epsilon^2)/4}$.

Now consider the case $D = 21$, the first odd discriminant which is not a prime. Then the norm map is still given by $N(\xi) = \xi^2/4$ on $O_D / D \cong \mathbb{Z}/D$, but now $\mathbb{Z}/D$ is not a field. For example, the equation $\xi^2 = 1 \mod D$ has four solutions, namely $\xi = 1, 8, 13, 20$. These give four solutions to the equation $N(\xi) = -5$, and hence contribute two distinct terms to the expression

$$F_5 = \bigcup F_5(\nu) = F_5((1 + \sqrt{21})/2) \cup F_5((8 + \sqrt{21})/2).$$

Only one of these terms appears in the expression for $X_D(1)$. In fact, since $21 = 1^2 + 4 \cdot 5 = 3^2 + 4 \cdot 3$, by Corollary 3.5 we have

$$X_{21}(1) = F_3 \cup F_5((1 + \sqrt{21})/2) \neq F_3 \cup F_5.$$

(The full curve $F_3$ appears because the only solutions to $N(\xi) = \xi^2/4 = -3 \mod 21$ are $\xi = \pm 3$.)

Using Theorem 3.6, it is similarly straightforward to check other small discriminants; for example:

Theorem 3.9 For $D \leq 30$ we have $X_D(1) = \bigcup_{\nu^2 + 4N=D} T_N$ when $D = 4, 5, 8, 9, 13, 17, 25$ and 29, but not when $D = 12, 16, 20, 21, 24$ or 28.

Notes. For more background on modular curves and surfaces, see [Hir], [HZ2], [HZ1], [BL], [Mc7, §4] and [vG]. Our $U = \left( \begin{smallmatrix} \mu & bD \\ -a & -\mu' \end{smallmatrix} \right)$ corresponds to the skew-Hermitian matrix $B = \sqrt{D} \left( \begin{smallmatrix} a & \mu' \\ \mu & bD \end{smallmatrix} \right)$ in [vG, Ch. V]. Note that (3.3) agrees with the standard action $(a\tau + b)/(c\tau + d)$ up to the automorphism $(a \ b \ c \ d) \mapsto (a \ b \ c \ d)$ of $SL_2(K)$. We remark that $X_D$ can also be presented as the quotient $(\mathbb{H} \times -\mathbb{H})/SL_2(O_D)$, using the fact that $\sqrt{D} = -\sqrt{D}$; on the other hand, the surfaces $(\mathbb{H} \times \mathbb{H})/SL_2(O_D)$ and $X_D$ are generally not isomorphic (see e.g. [HH]).

It is known that the intersection numbers $\langle T_N, T_M \rangle$ form the coefficients of a modular form [HZ1], [vG, Ch. VI]. The results of [GKZ] suggest that the intersection numbers of the refined modular curves $T_N(\nu)$ may similarly yield a Jacobi form.

4 Laminations

In this section we show algebraically that $\tilde{X}_D(1)$ gives a lamination of $\mathbb{H} \times \mathbb{H}$ by countably many disjoint hyperbolic planes. We also describe these
laminations explicitly for small values of $D$. Another proof of laminarity appears in §7.

**Jacobian varieties.** Let $\Omega(X)$ denote the space of holomorphic 1-forms on a compact Riemann surface $X$. The Jacobian of $X$ is the Abelian variety $\text{Jac}(X) = \Omega(X)^*/H_1(X, \mathbb{Z})$, polarized by the intersection pairing on 1-cycles.

In the case of genus two, any principally polarized Abelian variety $A$ is either a Jacobian or a product of polarized elliptic curves. The latter case occurs iff $A$ admits real multiplication by $O_1$, generated by projection to one of the factors of $A \cong B_1 \times B_2$. In particular, we have:

**Theorem 4.1** For any $D \geq 4$, the locus of Jacobian varieties in $X_D$ is given by $X_D - X_D(1)$.

**Laminations.** To describe $X_D(1)$ in more detail, given $N > 0$ such that $D = e^2 + 4N$ let

$$\Lambda^N_D = \{ U \in M_2(K) : U \text{ is a generator matrix, } \det(U) = N \text{ and } \nu(U) \equiv \pm(e + \sqrt{D})/2 \text{ mod } D \},$$

and let $\Lambda_D$ be the union of all such $\Lambda^N_D$. Note that if $U$ is in $\Lambda_D$, then $-U, U'$ and $U^*$ are also in $\Lambda_D$.

By Corollary 3.5, the preimage of $X_D(1)$ in $\tilde{X}_D = \mathbb{H} \times \mathbb{H}$ is given by:

$$\tilde{X}_D(1) = \bigcup \{ \mathbb{H}_U : U \in \Lambda_D \}.$$

Note that each $\mathbb{H}_U$ is the graph of a Möbius transformation.

**Theorem 4.2** The locus $\tilde{X}_D(1)$ gives a lamination of $\mathbb{H} \times \mathbb{H}$ by countably many hyperbolic planes.

(This means any two planes in $\tilde{X}_D(1)$ are either identical or disjoint.)

For the proof, it suffices to show that the difference $g \circ h^{-1}$ of two Möbius transformations in $\Lambda_D$ is never elliptic. Since $\Lambda_D$ is invariant under $U \mapsto U^* = (\det(U))^{-1}$, this in turn follows from:

**Theorem 4.3** For any $U_1, U_2 \in \Lambda_D$, we have $\text{Tr}(U_1U_2)^2 \geq 4\det(U_1U_2)$. 

**Proof.** By the definition of $\Lambda_D$, we can write $D = e^2_i + 4\det(U_i) = e^2_i + 4N_i$, where $e_i \geq 0$. We can also assume that

$$U_i = \begin{pmatrix} \mu_i & b_iD \\ -a_i & -\mu'_i \end{pmatrix}.$$
satisfies

\[ \mu_i \equiv (x_i + y_i \sqrt{D})/2 \equiv (e_i + \sqrt{D})/2 \mod D \]

(replacing \( U_i \) with \(-U_i \) if necessary). It follows that \( y_i \) is odd and \( x_i = e_i \mod D \), which implies

\[ \text{Tr}(U_1 U_2) \equiv \text{Tr}(\mu_1 \mu_2) = (x_1 x_2 + D y_1 y_2)/2 \equiv (e_1 e_2 - D)/2 \mod D. \]  

(4.1)

(The factor of 1/2 presents no difficulties, because \( x_i \) is even when \( D \) is even.)

Now suppose \( \text{Tr}(U_1 U_2)^2 < 4 \det(U_1 U_2) = 4 N_1 N_2. \) (4.2)

Then we have \( |\text{Tr}(U_1 U_2)| < 2 \sqrt{N_1 N_2} \leq D/2 \), and thus (4.1) implies

\[ \text{Tr}(U_1 U_2) = (e_1 e_2 - D)/2. \]

But this implies

\[ 4 \text{Tr}(U_1 U_2)^2 = (D - e_1 e_2)^2 \geq (D - e_1^2)(D - e_2^2) = (4 N_1)(4 N_2) = 16 \det(U_1 U_2), \]

contradicting (4.2).

**Small discriminants.** To conclude we record a few cases where \( \Lambda_D \) admits a particularly economical description.

For concreteness, we will present \( \Lambda_D \) as a set matrices in \( \text{GL}_2^+(\mathbb{R}) \) using the chosen real place \( \iota_1 : K \to \mathbb{R} \). This works even when \( D = d^2 \), since both \( \mu \) and \( \mu' \) appear on the diagonal of \( U \in \Lambda_D \) (no information is lost). Under the standard action \((a \ b \ c \ d) \cdot z = (az + b)/(cz + d)\) of \( \text{GL}_2^+(\mathbb{R}) \) on \( \mathbb{H} \), we can then write

\[ \tilde{X}_D(1) = \bigcup_{\Lambda_D} \{ (\tau_1, \tau_2) : \tau_2 = U(\tau_1) \}. \]

This holds despite the twist in the definition (3.5) of \( \mathbb{H}_U \), because \( \Lambda_D \) is invariant under \((a \ b \ c \ d) \mapsto (d \ b \ c \ a)\).

**Theorem 4.4** For \( D = 4, 5, 8, 9 \) and 13 respectively, we have:

\[ \Lambda_4 = \{ U \in M_2(\mathbb{Z}) : \det(U) = 1 \text{ and } U \equiv (\begin{smallmatrix} * & 0 \\ * & 2 \end{smallmatrix}) \mod 4 \}, \]

\[ \Lambda_5 = \{ U = \begin{pmatrix} \mu & bD \\ -a & -\mu' \end{pmatrix} : \det(U) = 1 \}, \]

\[ \Lambda_8 = \Lambda_8^1 \cup \Lambda_8^2 = \{ U = \begin{pmatrix} \mu & bD \\ -a & -\mu' \end{pmatrix} : \det(U) = 1 \text{ or } 2 \}, \]

\[ \Lambda_9 = \{ U \in M_2(\mathbb{Z}) : \det(U) = 2 \text{ and } U \equiv (\begin{smallmatrix} * & 0 \\ * & 9 \end{smallmatrix}) \mod 9 \}, \]  

and

\[ \Lambda_{13} = \Lambda_{13}^1 \cup \Lambda_{13}^3 = \{ U = \begin{pmatrix} \mu & bD \\ -a & -\mu' \end{pmatrix} : \det(U) = 1 \text{ or } 3 \}, \]  

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where it is understood that \( a, b \in \mathbb{Z} \) and \( \mu \in \mathcal{O}_D \).

**Proof.** Recall from Theorem 3.9 that \( X_D(1) = \bigcup_{c^2 + 4N = D} T_N \) when \( D = 4, 5, 8, 9 \) and 13. When this equality holds, we can ignore the condition on \( \nu(U) \) in the definition of \( \Lambda_D \). The cases \( D = 5, 8 \) and 13 then follow directly from the definition of \( \Lambda_D^N \). For \( D = 9 \), we note that any integral matrix satisfying \( \det \left( \begin{array}{cc} x & 9b \\ -a & y \end{array} \right) = 2 \) also satisfies \( x + y = 0 \mod 3 \), and thus it can be written in the form \( \left( \begin{array}{cc} \mu & bD \\ -a & -\mu' \end{array} \right) \) with
\[
\mu = \frac{(x - y) + (x + y)\sqrt{3}/3}{2}.
\]
Similar considerations apply when \( D = 4 \). \( \blacksquare \)

## 5 Foliations of Teichmüller space

In this section we introduce a family of foliations \( \mathcal{F}_i \) of Teichmüller space, related to normalized Abelian differentials and their periods \( \tau_{ij} = \int b_i \omega_j \). We then show:

**Theorem 5.1** There is a unique holomorphic section of the period map
\[
\tau_{ii} : T_g \to \mathbb{H}
\]
through any \( Y \in T_g \). Its image is the leaf of \( \mathcal{F}_i \) containing \( Y \).

The case \( g = 2 \) will furnish the desired foliations of Hilbert modular surfaces.

**Abelian differentials.** Let \( Z_g \) be a smooth oriented surface of genus \( g \). Let \( T_g \) be the Teichmüller space of Riemann surfaces \( Y \), each equipped with an isotopy class of homeomorphism or marking \( Z_g \to Y \). The marking determines a natural identification between \( H_1(Z_g) \) and \( H_1(Y) \) used frequently below.

Let \( \Omega T_g \to T_g \) denote the bundle of nonzero Abelian differentials \((Y, \omega)\), \( \omega \in \Omega(Y) \). For each such form we have a period map
\[
I(\omega) : H_1(Z_g, \mathbb{Z}) \to \mathbb{C}
\]
given by \( I(\omega) : C \to \int_C \omega \). There is a natural action of \( \text{GL}_2^+(\mathbb{R}) \) on \( \Omega T_g \), satisfying
\[
I(A \cdot \omega) = A \circ I(\omega) \quad (5.1)
\]
under the identification $C = \mathbb{R}^2$ given by $x + iy = (x, y)$.

Each orbit $\text{GL}_2^+(\mathbb{R}) \cdot (Y, \omega)$ projects to a complex geodesic

$$f : \mathbb{H} \to T_g,$$

which can be normalized so that $f(i) = Y$ and

$$\nu = \frac{df}{dt} \bigg|_{t=i} = \frac{i \overline{\omega}}{2 \omega}.$$ 

The subspace of $H^1(Z_g, \mathbb{R})$ spanned by $(\text{Re} \omega, \text{Im} \omega)$ is constant along each orbit (cf. [Mc7, §3]).

**Symplectic framings.** Now let $(a_1, \ldots, a_g, b_1, \ldots, b_g)$ be a real symplectic basis for $H_1(Z_g, \mathbb{R})$ (with $\langle a_i, b_i \rangle = -\langle b_i, a_i \rangle = 1$ and all other products zero). Then for each $Y \in T_g$, there exists a unique basis $(\omega_1, \ldots, \omega_g)$ of $\Omega(Y)$ such that $\int a_i \omega_j = \delta_{ij}$. The period matrix

$$\tau_{ij}(Y) = \int_{b_i} \omega_j$$

then determines an embedding

$$\tau : T_g \to \delta_g.$$

This agrees with the usual Torelli embedding, up to composition with an element of $\text{Sp}_{2g}(\mathbb{R})$. Note that $\text{Im}(\tau_{ii}(Y)) > 0$ since $\text{Im} \tau$ is positive definite.

The normalized 1-forms $(\omega_i)$ give a splitting

$$\Omega(Y) = \oplus_i \mathbb{C} \omega_i = \oplus_i F_i(Y),$$

and corresponding subbundles $F_i T_g \subset \Omega T_g$.

**Complex subspaces.** Let $(a_i^*, b_i^*)$ denote the dual basis for $H^1(Z_g, \mathbb{R})$, and let $S_i$ be the span of $(a_i^*, b_i^*)$. It easy to check that the following conditions are equivalent:

1. $S_i$ is a complex subspace of $H^1(Y, \mathbb{R}) \cong \Omega(Y)$.
2. $S_i$ is spanned by (Re $\omega_i$, Im $\omega_i$).
3. The period matrix $\tau(Y)$ satisfies $\tau_{ij} = 0$ for all $j \neq i$. 

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Let $T_g(S_i) \subset T_g$ denote the locus where these conditions hold. Note that condition (3) defines a totally geodesic subset

$$H_i \cong \mathbb{H} \times S_{g-1} \subset T_g$$

such that $T_g(S_i) = \tau^{-1}(H_i)$.

**Foliations.** Next we show that the complex geodesics generated by the forms $(Y, \omega_i)$ give a foliation of Teichmüller space.

**Theorem 5.2** The sub-bundle $F_i T_g \subset \Omega T_g$ is invariant under the action of $GL_2^+(\mathbb{R})$, as is its restriction to $T_g(S_i)$.

**Proof.** The invariance of $F_i T_g$ is immediate from (5.1). To handle the restriction to $T_g(S_i)$, recall that the span $W$ of $(\text{Re} \omega_i, \text{Im} \omega_i)$ is constant along orbits; thus the condition $W = S_i$ characterizing $T_g(S_i)$ is preserved by the action of $GL_2^+(\mathbb{R})$. \[\square\]

**Corollary 5.3** The foliation of $F_i T_g$ by $GL_2^+(\mathbb{R})$ orbits projects to a foliation $F_i$ of $T_g$ by complex geodesics.

**Corollary 5.4** The locus $T_g(S_i)$ is also foliated by $F_i$: any leaf meeting $T_g(S_i)$ is entirely contained therein.

**Proof of Theorem 5.1.** The proof uses Ahlfors’ variational formula [Ah] and follows the same lines as the proof of [Mc4, Thm. 4.2]; it is based on the fact that the leaves of $\mathcal{F}_i$ are the geodesics along which the periods of $\omega_i$ change most rapidly.

Let $s : \mathbb{H} \to T_g$ be a holomorphic section of $\tau_{ii}$. Let $v \in T \mathbb{H}$ be a unit tangent vector with respect to the hyperbolic metric $\rho = |dz|/(2\text{Im} z)$ of constant curvature $-4$, mapping to $Ds(v) \in T_Y T_g$. By the equality of the Teichmüller and Kobayashi metrics [Gd, Ch. 7], $Ds(v)$ is represented by a Beltrami differential $\nu = \nu(z)dz/dz$ on $Y$ with $\|\nu\|_\infty \leq 1$. But $s$ is a section, so the composition

$$\tau_{ii} \circ s : \mathbb{H} \to \mathbb{H}$$

is the identity; thus the norm of its derivative, given by Ahlfors’ formula as

$$\|D(\tau_{ii} \circ s)(\nu)\| = \left| \int_Y \omega_i^2 \nu \right| / \left| \int_Y |\omega_i|^2 \right|,$$

is one. It follows that $\nu = \overline{\nu}/\omega_i$ up to a complex scalar of modulus one, and thus $Ds(v)$ is tangent to the complex geodesic generated by $(Y, \omega_i)$. Equivalently, $s(\mathbb{H})$ is everywhere tangent to the foliation $\mathcal{F}_i$; therefore its image is the unique leaf through $Y$. \[\square\]
6 Genus two

We can now obtain results on Hilbert modular surfaces by specializing to the case of genus two. In this section we will show:

**Theorem 6.1** There is a unique holomorphic section of $\tau_1$ passing through any given point of $\mathbb{H} \times \mathbb{H} - \tilde{X}_D(1)$.

Here $\tau_1 : \mathbb{H} \times \mathbb{H} \to \mathbb{H}$ is simply projection onto the first factor. This result is a restatement of Theorem 1.2; as in §1, we assume $D \geq 4$.

**Framings for real multiplication.** Let $g = 2$, and choose a symplectic isomorphism

$$L = H_1(Z_g, \mathbb{Z}) \cong \mathcal{O}_D \oplus \mathcal{O}_D^\vee.$$

We then have an action of $\mathcal{O}_D$ on $H_1(Z_g, \mathbb{Z})$, and the elements $\{a, b\} = \{(1, 0), (0, 1)\}$ in $L$ give a distinguished basis for

$$H_1(Z_g, \mathbb{Q}) = L \otimes \mathbb{Q} \cong K^2$$

as a vector space over $K = \mathcal{O}_D \otimes \mathbb{Q}$. Using the two Galois conjugate embeddings $K \to \mathbb{R}$, we obtain an orthogonal splitting

$$H_1(Z_g, \mathbb{R}) = L \otimes \mathbb{R} = V_1 \oplus V_2$$

such that $k \cdot (C_1, C_2) = (kC_1, k'C_2)$. The projections $(a_i, b_i)$ of $a, b \in L$ to each summand yield bases for $V_i$, which taken together give a standard symplectic basis for $H_1(Z_g, \mathbb{R})$. (Note that $(a_i, b_i)$ is generally not an integral symplectic basis; indeed, when $K$ is a field, the elements $(a_i, b_i)$ do not even lie in $H_1(Z_g, \mathbb{Q})$.)

Let $S_i^D \subset H^1(Z_g, \mathbb{R})$ be the span of the dual basis $a_i^*, b_i^*$.

**Theorem 6.2** The ring $\mathcal{O}_D \subset \text{End}(L)$ acts by real multiplication on $\text{Jac}(Y)$ if and only if $Y \in T_g(S_1^D)$.

**Proof.** Since $g = 2$ we have $S_2^D = (S_1^D)^\perp$, and thus $T_g(S_1^D) = T_g(S_2^D)$. But $\text{Jac}(Y)$ has real multiplication iff $S_1^D$ and $S_2^D$ are complex subspaces of $H^1(Y, \mathbb{R}) \cong \Omega(Y)$ so the result follows. (Cf. [Mc4, Lemma 7.4].)
**Sections.** Let $E_D = X_D - X_D(1)$ denote the space of Jacobians in $X_D$, and $\tilde{E}_D = \mathbb{H} \times \mathbb{H} - \tilde{X}_D(1)$ its preimage in the universal cover. (The notation comes from [Mc7, §4], where we consider the space of eigenforms $\Omega E_D$ as a closed, $\text{GL}_2^+(\mathbb{R})$-invariant subset of $\Omega M_g$.)

By the preceding result, the Jacobian of any $Y \in T_g(S^D_1)$ is an Abelian variety with real multiplication. Moreover, the marking of $Y$ determines a marking

$$L \cong H_1(Y, \mathbb{Z}) \cong H_1(\text{Jac}(Y), \mathbb{Z})$$

of its Jacobian, and thus a map

$$\text{Jac} : T_g(S^D_1) \to \tilde{E}_D = \tilde{X}_D - \tilde{X}_D(1).$$

The basis $(a_i, b_i)$ yields a pair of normalized forms $\omega_1, \omega_2 \in \Omega(Y)$. Similarly, we have a pair of normalized eigenforms $\eta_1, \eta_2 \in \Omega(A_\tau)$ for each $\tau \in \tilde{X}_D$, characterized by (3.2). Under the identification $\Omega(Y) = \Omega(\text{Jac}(Y))$, we find:

**Theorem 6.3** The forms $\omega_i$ and $\eta_i$ are equal for any $Y \in T_g(S^D_1)$. Thus $\text{Jac}(Y) = A_{(\tau_1, \tau_2)}$, where

$$\left( \begin{array}{cc} \tau_1 & 0 \\ 0 & \tau_2 \end{array} \right) = \tau_{ij}(Y) = \left( \int_{b_i} \omega_j \right). \quad (6.1)$$

**Proof.** The period map $\phi_\tau : L \to \mathbb{C}^2$ for $A_\tau = \text{Jac}(Y)$ is given by

$$\phi_\tau(C) = \left( \int_C \eta_1, \int_C \eta_2 \right) = (x_1 + x_2\tau_1, x'_1 + x'_2\tau_2),$$

where $C = (x_1, x_2) \in \mathcal{O}_D \oplus \mathcal{O}_D'$; in particular, we have

$$\phi_\tau(a) = \phi_\tau(1, 0) = (1, 1).$$

Since $\phi_\tau$ diagonalizes the action of $K$, we also have

$$\phi_\tau(C) = \left( \int_{C_1} \eta_1, \int_{C_2} \eta_2 \right)$$

for any $C = C_1 + C_2 \in L \otimes \mathbb{R} = V_1 \oplus V_2$. Setting $C = a$, this implies $\phi_\tau(a_1) = (1, 0)$ and $\phi_\tau(a_2) = (0, 1)$; thus $\int_a \eta_j = \delta_{ij}$, and therefore $\eta_i = \omega_i$ for $i = 1, 2$. Similarly, we have

$$\phi_\tau(b) = (\tau_1, \tau_2) = (\tau_{11}, \tau_{22}),$$

which implies $Y$ and $A_\tau$ are related by (6.1).
Corollary 6.4 We have a commutative diagram

\[ \begin{array}{ccc}
T_g(S^D_1) & \xrightarrow{\text{Jac}} & \tilde{E}_D \\
\downarrow{\tau_{11}} & & \downarrow{\tau_1} \\
\mathbb{H} & & \mathbb{H}.
\end{array} \]

Proof of Theorem 6.1. Using the Torelli theorem, it follows easily that \( \text{Jac} : T_g(S^D_1) \to \tilde{E}_D \) is a holomorphic covering map. Since \( \mathbb{H} \) is simply-connected, any section \( s \) of \( \tau_1 \) lifts to a section \( \text{Jac}^{-1} \circ s \) of \( \tau_{11} \). Thus Theorem 5.1 immediately implies Theorem 6.1.

7 Holomorphic motions

In this section we use the theory of holomorphic motions to define and characterize the foliation \( \mathcal{F}_D \).

Holomorphic motions. Given a set \( E \subset \hat{\mathbb{C}} \) and a basepoint \( s \in \mathbb{H} \), a holomorphic motion of \( E \) over \( (\mathbb{H}, s) \) is a family of injective maps

\[ F_t : E \to \hat{\mathbb{C}}, \quad t \in \mathbb{H}, \]

such that \( F_s(z) = z \) and \( F_t(z) \) is a holomorphic function of \( t \).

A holomorphic motion of \( E \) has a unique extension to a holomorphic motion of its closure \( \overline{E} \); and each map \( F_t : E \to \hat{\mathbb{C}} \) extends to a quasiconformal homeomorphism of the sphere. In particular, \( F_t|\text{int}(E) \) is quasiconformal (see e.g. [Dou]).

These properties imply:

Theorem 7.1 Let \( P \) be a partition of \( \mathbb{H} \times \mathbb{H} \) into disjoint graphs of holomorphic functions. Then:

1. \( P \) is the set of leaves of a transversally quasiconformal foliation \( \mathcal{F} \) of \( \mathbb{H} \times \mathbb{H} \); and

2. If we adjoin the graphs of the constant functions \( f : \mathbb{H} \to \partial\mathbb{H} \) to \( P \), we obtain a continuous foliation of \( \mathbb{H} \times \bar{\mathbb{H}} \).

The foliation \( \mathcal{F}_D \). Recall that every component of \( \bar{X}_D(1) \subset \mathbb{H} \times \mathbb{H} \) is the graph of a Möbius transformation. By Theorem 6.1, there is a unique partition of \( \mathbb{H} \times \mathbb{H} - \bar{X}_D(1) \) into the graphs of holomorphic maps as well.
Taken together, these graphs form the leaves of a foliation $\tilde{\mathcal{F}}_D$ of $\mathbb{H} \times \mathbb{H}$ by the preceding result. Since $\tilde{X}_D(1)$ is invariant under $\text{SL}(O_D \oplus O^\vee_D)$, the foliation $\tilde{\mathcal{F}}_D$ descends to a foliation $\mathcal{F}_D$ of $X_D$.

To characterize $\mathcal{F}_D$, recall that the surface $X_D$ admits a holomorphic involution $\iota(\tau_1, \tau_2) = (\tau_2, \tau_1)$ which preserves $X_D(1)$.

**Theorem 7.2** The only leaves shared by $\mathcal{F}_D$ and $\iota(\mathcal{F}_D)$ are the curves in $X_D(1)$.

**Proof.** Let $f : \mathbb{H} \to \mathbb{H}$ be a holomorphic function whose graph $F$ is both a leaf of $\tilde{\mathcal{F}}_D$ and $\iota(\tilde{\mathcal{F}}_D)$. Then $\iota(F)$ is also a graph, so $f$ is an isometry. But if $F \cap \tilde{X}_D(1) = \emptyset$, then $F$ lifts to a leaf of the foliation $\mathcal{F}_1$ of Teichmüller space, and hence $f$ is a contraction by [Mc4, Thm. 4.2].

**Corollary 7.3** The only leaves of $\tilde{\mathcal{F}}_D$ that are graphs of Möbius transformations are those belonging to $\tilde{X}_D(1)$.

**Complex geodesics.** Let us say $\mathcal{F}$ is a foliation by complex geodesics if each leaf is a hyperbolic Riemann surface, isometrically immersed for the Kobayashi metric. We can then characterize $\mathcal{F}_D$ as follows.

**Theorem 7.4** Up to the action of $\iota$, $\mathcal{F}_D$ is the unique extension of the lamination $X_D(1)$ to a foliation of $X_D$ by complex geodesics.

**Proof.** Let $\mathcal{F}$ be a foliation by complex geodesics extending $X_D(1)$. Then every leaf of its lift $\tilde{\mathcal{F}}$ to $\tilde{X}_D$ is a Kobayashi geodesic for $\mathbb{H} \times \mathbb{H}$. But a complex geodesic in $\mathbb{H} \times \mathbb{H}$ is either the graph of a holomorphic function or its inverse, so every leaf belongs to either $\tilde{\mathcal{F}}_D$ or $\iota(\tilde{\mathcal{F}}_D)$. Consequently every leaf of $\mathcal{F}$ is a leaf of $\mathcal{F}_D$ or $\iota(\mathcal{F}_D)$. Since these foliations have no leaves in common on the open set $U = X_D - X_D(1)$, $\mathcal{F}$ coincides with one or the other.

**Stable curves.** The Abelian varieties $E \times F$ in $X_D(1)$ are the Jacobians of certain stable curves with real multiplication, namely the nodal curves $Y = E \vee F$ obtained by gluing $E$ to $F$ at a single point. If we adjoin these stable curves to $\mathcal{M}_2$, we obtain a partial compactification $\mathcal{M}_2^*$ which maps isomorphically to $A_2$. The locus $X_D(1)$ can then be regarded as the projection to $X_D$ of a finite set of $\text{GL}_2^+(\mathbb{R})$ orbits in $\Omega\mathcal{M}_2^*$, giving another proof that it is a lamination.
8 Quasiconformal dynamics

In this section we use the relative period map $\rho = \int_{y_1}^{y_2} \eta_1$ to define a meromorphic quadratic differential $q = (d\rho)^2$ transverse to $\mathcal{F}_D$. We then show the transverse dynamics of $\mathcal{F}_D$ is given by Teichmüller mappings relative to $q$.

**Absolute periods.** The level sets of $\tau_1$ form the leaves of a holomorphic foliation $\tilde{A}_D$ on $\mathbb{H} \times \mathbb{H}$ which covers foliation $A_D$ of $X_D$. By (3.2), every $\tau = (\tau_1, \tau_2)$ determines a pair of eigenforms $\eta_1, \eta_2 \in \Omega(A_\tau)$ such that the absolute periods

$$\int_C \eta_1, \quad C \in H_1(A_\tau, \mathbb{Z})$$

are constant along the leaves of $\tilde{A}_D$. Since every leaf of $\tilde{F}_D$ is the graph of a function $f : \mathbb{H} \to \mathbb{H}$, we have:

**Theorem 8.1** The foliation $A_D$ is transverse to $\mathcal{F}_D$.

**The Weierstrass curve.** Recall that $E_D \subset X_D$ denotes the locus of Jacobians with real multiplication by $\mathcal{O}_D$. For $[A_\tau] = \text{Jac}(Y) \in E_D$ we can regard the eigenforms $\eta_1, \eta_2$ as holomorphic 1-forms in $\Omega(Y) \cong \Omega(A_\tau)$.

Let $W_D \subset E_D$ denote the locus where $\eta_1$ has a double zero on $Y$. By [Mc5] we have:

**Theorem 8.2** The locus $W_D$ is an algebraic curve with one or two irreducible components, each of which is a leaf of $\mathcal{F}_D$.

We refer to $W_D$ as the *Weierstrass curve*, since $\eta_1$ vanishes at a Weierstrass point of $Y$.

**Relative periods.** Let $E_D(1, 1) = X_D - (W_D \cup X_D(1))$ denote the Zariski open set where $\eta_1$ has a pair of simple zeros, and let $\tilde{E}_D(1, 1)$ be its preimage in the universal cover $\tilde{X}_D$. Let

$$\mathbb{H}_s = \{s\} \times \mathbb{H} \subset \mathbb{H} \times \mathbb{H},$$

and let $\mathbb{H}_s^* = \mathbb{H}_s \cap \tilde{E}_D(1, 1)$.

For each $\tau \in \mathbb{H}_s^*$, let $y_1, y_2$ denote the zeros of the associated form $\eta_1 \in \Omega(Y)$. We can then define the (multivalued) relative period map $\rho_s : \mathbb{H}_s^* \to \mathbb{C}$ by

$$\rho_s(\tau) = \int_{y_1}^{y_2} \eta_1.$$
To make $\rho_s(\tau)$ single-valued, we must (locally) choose (i) an ordering of the zeros $y_1$ and $y_2$, and (ii) a path on $Y$ connecting them.

**Quadratic differentials.** Let $z$ be a local coordinate on $H_s$, and recall that the absolute periods of $\eta_1$ are constant along $H_s$. Thus if we change the choice of path from $y_1$ to $y_2$, the derivative $d\rho/dz$ remains the same; and if we interchange $y_1$ and $y_2$, it changes only by sign. Thus the quadratic differential

$$q = (d\rho/dz)^2 \, dz^2$$

is globally well-defined on $H^*_s$.

**Theorem 8.3** The form $q$ extends to a meromorphic quadratic differential on $H_s$, with simple zeros where $H_s$ meets $\tilde{W}_D$, and simple poles where it meets $\tilde{X}_D(1)$.

**Proof.** It is a general result that the period map provides holomorphic local coordinates on any stratum of $\Omega M_g$ (see [V2], [MS, Lemma 1.1], [KZ]). Thus $\rho_s|H^*_s$ is holomorphic with $d\rho_s \neq 0$, and hence $q|H^*_s$ is a nowhere vanishing holomorphic quadratic differential.

To see $q$ acquires a simple zero when $\eta_1$ acquires a double zero, note that the relative period map $\rho(t) = \int_{-\sqrt{t}}^{\sqrt{t}} (z^2 - t) \, dz = (-4/3)t^{3/2}$ of the local model $\eta_t = (z^2 - t) \, dz$ satisfies $(d\rho/dt)^2 = 4t$. Similarly, a point of $H_s \cap \tilde{X}_D(1)$ is locally modeled by the family of connected sums

$$(Y_t, \eta_t) = (E_1, \omega_1) \# (E_2, \omega_2),$$

with $I = [0, |\rho(t)|] = [0, \pm \sqrt{t}]$. Since $(d\rho/dt)^2 = 1/(4t)$, at these points $q$ has simple poles. ■

See [Mc7, §6] for more on connected sums.

**Teichmüller maps.** Now let $f : H_s \to H_t$ be a quasiconformal map. We say $f$ is a Teichmüller map, relative to a holomorphic quadratic differential $q$, if its complex dilatation satisfies

$$\mu(f) = \left( \frac{\partial f/\partial \bar{z}}{\partial f/\partial z} \right) \frac{dz}{d\bar{z}} = \frac{\bar{q}}{|q|}$$

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for some \( \alpha \in \mathbb{C}^* \). This is equivalent to the condition that \( w = f(z) \) is real-linear in local coordinates where \( q = dz^2 \) and \( dw^2 \) respectively. In such charts we can write
\[
    w = w_0 + D_q(f) \cdot z,
\]
with \( D_q(f) \in \text{SL}_2(\mathbb{R}) \). We refer to \( D_q(f) \) as the \textit{linear part of} \( f \); it is only well-defined up to sign, since \( z \mapsto -z \) preserves \( dz^2 \).

**Theorem 8.4** Given \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(\mathcal{O}_D \oplus \mathcal{O}_D^\vee) \) and \( s \in \mathbb{H} \), let \( \mathbb{H}_t = g(\mathbb{H}_s) \). Then the linear part of \( g : \mathbb{H}_s \rightarrow \mathbb{H}_t \) is given by \( D_q(g) \cdot z = (d - cs)^{-1}z \).

**Proof.** Since the Riemann surfaces \( Y \) at corresponding points of \( \mathbb{H}_s \) and \( \mathbb{H}_t \) differ only by marking, the relative period maps \( \rho_s \) and \( \rho_t \) differ only by the normalization of \( \eta_1 \). This discrepancy is accounted for by equation (3.4), which gives \( \rho_t/\rho_s = \chi(g, s) = (d - cs)^{-1} \). Since the coordinates \( \rho_s \) and \( \rho_t \) linearize \( q \), the map \( D_q(g) \) is given by multiplication by \( (d - cs)^{-1} \).

Now let \( C_{st} : \mathbb{H}_s \rightarrow \mathbb{H}_t \) be the unique map such that \( z \) and \( C_{st}(z) \) lie on the same leaf of \( \tilde{F}_D \).

**Theorem 8.5** The linear part of \( C_{st} \) is given by \( D_q(C_{st}) = A_tA_s^{-1} \), where \( A_u = \begin{pmatrix} 1 & \text{Re}(u) \\ 0 & \text{Im}(u) \end{pmatrix} \in \text{PSL}_2(\mathbb{R}) \).

**Proof.** By the definition of \( F_D \), the forms \( \eta_1 \) at corresponding points of \( \mathbb{H}_s \) and \( \mathbb{H}_t \) are related by some element \( B \in \text{GL}_2^+(\mathbb{R}) \) acting on \( \Omega T_g \). Thus \( \rho_t = B \circ \rho_s \) and therefore \( D_q(C_{st}) = B \). Since the action of \( B \) on the absolute periods of \( \eta_1 \) satisfies
\[
    B(\mathcal{O}_D \oplus \mathcal{O}_D^\vee s) = \mathcal{O}_D \oplus \mathcal{O}_D^\vee t
\]
(in the sense of equation (3.1)), we have \( B(1) = 1 \) and \( B(s) = t \), and thus \( B = A_tA_s^{-1} \) as above.

**Dynamics.** Every leaf of \( \tilde{F}_D \) meets the transversal \( \mathbb{H}_s \) in a single point. Thus the action of \( g \in \text{SL}(\mathcal{O}_D \oplus \mathcal{O}_D^\vee) \) on the space of leaves determines a \textit{holonomy map}
\[
    \phi_g : \mathbb{H}_s \rightarrow \mathbb{H}_s,
\]
characterized by the property that \( (s, \phi_g(z)) \) lies on the same leaf as \( g(s, z) \).

**Theorem 8.6** The group \( \text{SL}(\mathcal{O}_D \oplus \mathcal{O}_D^\vee) \) acts on \( \mathbb{H}_s \) by \textit{Teichmüller mappings}, satisfying \( D_q(\phi_g) = g \) in the case \( s = i \).
(As usual we regard $g$ as a real matrix using $\iota_1 : K \to \mathbb{R}$.)

**Proof.** Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and $t = (as - b)/(-cs + d)$; then $\mathbb{H}_t = g(\mathbb{H}_s)$.

Since $\phi_g(z)$ is obtained from $g(s, z)$ by combing it along the leaves of $\mathcal{F}_D$ back into $\mathbb{H}_s$, we have $\phi_g(s, z) = C_{ts}(g(s, z))$. Thus the chain rule implies

$$D_q(\phi_g) \cdot z = B \cdot z = A_s \circ A_t^{-1}(z/(-cs + d)).$$

Now assume $s = i$. Then we have $B(ai - b) = A_t^{-1}(t) = i$ and $B(-ci + d) = A_t^{-1}(1) = 1$; therefore $B^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ and thus $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = g$. □

**Corollary 8.7** The foliation $\mathcal{F}_D$ carries a natural transverse invariant measure.

**Proof.** Since $\det D_q(\phi_g) = 1$ for all $g$, the form $|q|$ gives a holonomy-invariant measure on the transversal $\mathbb{H}_s$.

Finally we show that, although $\phi_g|\mathbb{H}_s$ is quasiconformal, its continuous extension to $\partial \mathbb{H}_s$ is a Möbius transformation.

**Theorem 8.8** For any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(\mathcal{O}_D \oplus \mathcal{O}_D^*)$ and $z \in \partial \mathbb{H}_s$, we have

$$\phi_g(z) = (a'z - b')/(-c'z + d').$$

**Proof.** By Theorem 7.1, the combing maps $C_{st}$ extend to the identity on $\partial \mathbb{H}_s$. Thus $(t, \phi_g(z)) = g(s, z)$, and the result follows from equation (3.3). □

Note: if we use the transversal $\mathbb{H}_t$ instead of $\mathbb{H}_s$, the holonomy simply changes by conjugation by $C_{st}$.

**9 Further results**

In this section we summarize related results on the density of leaves, isoperiodic forms, holomorphic motions and iterated rational maps.

**I. Density of leaves.** By [Mc7], the closure of the complex geodesic $f : \mathbb{H} \to \mathcal{M}_2$ generated by a holomorphic 1-form is either an algebraic curve, a Hilbert modular surface or the whole moduli space. Since the leaves of $\mathcal{F}_D$ are examples of such complex geodesics, we obtain:
Theorem 9.1 Every leaf of $\mathcal{F}_D$ is either a closed algebraic curve, or a dense subset of $X_D$.

It is easy to see that the union of the closed leaves is dense when $D = d^2$. On the other hand, the classification of Teichmüller curves in [Mc5] and [Mc6] implies:

Theorem 9.2 If $D$ is not a square, then $\mathcal{F}_D$ has only finitely many closed leaves. These consist of the components of $W_D \cup X_D(1)$ and, when $D = 5$, the Teichmüller curve generated by the regular decagon.

II. Isoperiodic forms. Next we discuss interactions between the foliations $\mathcal{F}_D$ and $\mathcal{A}_D$. When $D = d^2$ is a square, the surface $X_D$ is finitely covered by a product, and hence every leaf of $\mathcal{A}_D$ is closed.

Theorem 9.3 If $D$ is not a square, then every leaf $L$ of $\mathcal{A}_D$ is dense in $X_D$, and $L \cap F$ is dense in $F$ for every leaf $F$ of $\mathcal{F}_D$.

Proof. The first result follows from the fact that $SL(O_D \oplus O_D^\vee)$ is a dense subgroup of $SL_2(\mathbb{R})$, and the second follows from the first by transversality of $\mathcal{A}_D$ and $\mathcal{F}_D$. 

Let us say a pair of 1-forms $(Y_i, \omega_i) \in \Omega M_g$ are isoperiodic if there is a symplectic isomorphism

$$\phi : H_1(Y_1, \mathbb{Z}) \rightarrow H_1(Y_2, \mathbb{Z})$$

such that the period maps

$$I(\omega_i) : H_1(Y_i, \mathbb{Z}) \rightarrow \mathbb{C}$$

satisfy $I(\omega_1) = I(\omega_2) \circ \phi$. Since the absolute periods of $\eta_1$ are constant along the leaves of $\mathcal{A}_D$, from the preceding result we obtain:

Corollary 9.4 The $SL_2(\mathbb{R})$-orbit of any eigenform for real multiplication by $O_D$, $D \neq d^2$, contains infinitely many isoperiodic forms.

For a concrete example, let $Q \subset \mathbb{C}$ be a regular octagon containing $[0,1]$ as an edge. Identifying opposite sides of $Q$, we obtain the octagonal form

$$(Y, \omega) = (Q, dz)/\sim$$
of genus two.

Let $\mathbb{Z}[\zeta] \subset \mathbb{C}$ denote the ring generated by $\zeta = (1+i)/\sqrt{2} = \exp(2\pi i/8)$, equipped with the symplectic form

$$\langle z_1, z_2 \rangle = \text{Tr}_Q \left( (\zeta + \zeta^2 + \zeta^3)z_1\bar{z}_2/4 \right).$$

Then it is easy to check that:

1. The octagonal form $\omega$ has a single zero of order 2, and
2. Its period map $I(\omega)$ sends $H_1(Y, \mathbb{Z})$ to $\mathbb{Z}[\zeta]$ by a symplectic isomorphism.

However, these two properties do not determine $(Y, \omega)$ uniquely. Indeed, $\omega$ is an eigenform for real multiplication by $\mathcal{O}_8$, so the preceding Corollary ensures there are infinitely many isoperiodic forms $(Y_i, \omega_i)$ in its $\text{SL}_2(\mathbb{R})$ orbit. In other words we have:

**Corollary 9.5** There are infinite many fake octagonal forms in $\Omega \mathcal{M}_2$.

Note that the forms $(Y_i, \omega_i)$ cannot be distinguished by their relative periods either, since they all have double zeros.

A similar statement can be formulated for the pentagonal form on the curve $y^2 = x^5 - 1$.

**III. Top-speed motions.** Let $F_t : E \to \mathbb{H}$ be a holomorphic motion of $E \subset \mathbb{H}$ over $(\mathbb{H}, s)$. By the Schwarz lemma, we have $\|dF_t(z)/dt\| \leq 1$ with respect to the hyperbolic metric on $\mathbb{H}$. Let us say $F_t$ is a top-speed holomorphic motion if equality holds everywhere; equivalently, if $t \mapsto F_t(z)$ is an isometry of $\mathbb{H}$ for every $z \in E$.

A top-speed holomorphic motion is maximal if it cannot be extended to a top-speed motion of a larger set $E' \supset E$.

**Theorem 9.6** For any discriminant $D \geq 4$, the map

$$F_t(U(s)) = U(t), \quad U \in \Lambda_D$$

gives a maximal top-speed holomorphic motion of $E = \Lambda_D \cdot s$ over $(\mathbb{H}, s)$.

**Proof.** Let $t \mapsto f(t) = F_t(z)$ be an extension of the motion to a point $z \notin E$. Then the graph of $f$ is a leaf of $\tilde{\mathcal{F}}_D$, since it is disjoint from $\tilde{X}_D(1)$. But the only leaves that are graphs of Möbius transformations are those in $\tilde{X}_D(1)$, by Corollary 7.3.

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Corollary 9.7 The group $\Gamma(2) = \{A \in \text{SL}_2(\mathbb{Z}) : A \equiv I \mod 2\}$ gives a maximal top-speed holomorphic motion of $E = \Gamma(2) \cdot s$ over $(\mathbb{H}, s)$.

Proof. We have $\Gamma(2) = g \Lambda_4 g^{-1}$, where $g = \begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix}$ (Theorem 4.4).

IV. Iterated rational maps. Finally we explain how the foliation $\mathcal{F}_4$ of $X_4$ arises in complex dynamics.

First recall that the moduli space of elliptic curves can be described as the quotient orbifold $\mathcal{M}_1 = \tilde{\mathcal{M}}_1 / S_3$, where

$$\tilde{\mathcal{M}}_1 = \mathbb{H} / \Gamma(2) \cong \mathbb{C} - \{0, 1\}.$$  

The deck group $S_3$ also acts diagonally on $\tilde{\mathcal{M}}_1 \times \tilde{\mathcal{M}}_1$, preserving the diagonal $\Delta$.

Theorem 9.8 For $D = 4$, we have $(X_D, X_D(1)) \cong (\tilde{\mathcal{M}}_1 \times \tilde{\mathcal{M}}_1, \Delta) / S_3$.

Proof. Since $\mathcal{O}_4^\vee = (1/2) \mathcal{O}_4$, the surface $X_4$ is isomorphic to $(\mathbb{H} \times \mathbb{H}) / \text{SL}_2(\mathcal{O}_4)$. In these coordinates we have $\Lambda_4 = \Gamma(2)$. Since

$$\text{SL}_2(\mathcal{O}_4) \cong \{(A_1, A_2) \in \text{SL}_2(\mathbb{Z}) : A_1 \equiv A_2 \mod 2\}$$

contains $\Gamma(2) \times \Gamma(2)$ as a subgroup of index 6, the result follows.

Now consider, for each $t \in \tilde{\mathcal{M}}_1$, the elliptic curve $E_t$ defined by $y^2 = x(x-1)(x-t)$. There is a unique rational map $f_t : \mathbb{P}^1 \to \mathbb{P}^1$ such that

$$x(2P) = f_t(x(P))$$

with respect to the usual group law on $E_t$. Indeed, using the fact that $-2P$ lies on the tangent line to $E_t$ at $P$, we find

$$f_t(z) = \frac{(z^2 - t)^2}{4z(z - 1)(z - t)}.$$ 

Note that the postcritical set

$$P(f_t) = \bigcup \{f^n_t(z) : n > 0, f'_t(z) = 0\}$$

coincides with the branch locus $\{0, 1, t, \infty\}$ of the map $x : E_t \to \mathbb{P}^1$.

The rational maps $f_t(z)$ form a stable family of Lattès examples. It is well-known that the Julia set of any Lattès example is the whole Riemann sphere; and that in any stable family, the Julia set varies by a holomorphic motion respecting the dynamics (see e.g. [MSS], [Mc1, Ch. 4], [Mil].)
Theorem 9.9 As \( t \) varies in \( \tilde{\mathcal{M}}_1 \), the holomorphic motion of \( J(f_t) \) sweeps out the lift of the foliation \( \mathcal{F}_4 \) to the covering space \( \tilde{\mathcal{M}}_1 \times \tilde{\mathcal{M}}_1 \) of \( X_4 \).

Proof. Let \( \mathcal{G} \) be the foliation of \( \tilde{\mathcal{M}}_1 \times \mathbb{P}^1 \) swept out by \( J(f_t) \). Since the holomorphic motion respects the dynamics, it preserves the post-critical set, and thus the leaves of \( \mathcal{G} \) include the loci \( z = 0, 1, \infty \) as well as the diagonal \( t = z \). In particular, \( \mathcal{G} \) restricts to a foliation of the finite cover \( \tilde{\mathcal{M}}_1 \times \tilde{\mathcal{M}}_1 - \Delta \) of \( X_4 - X_4(1) \). Since each leaf of \( \mathcal{G} \) lifts to the graph of a holomorphic function in the universal cover \( \mathbb{H} \times \mathbb{H} \), it lies over a leaf of \( \mathcal{F}_D \) by the uniqueness part of Theorem 1.2. ■

Algebraic curves. The loci \( f^n_t(z) = \infty \) form a dense set of algebraic leaves of \( \mathcal{G} \) that can easily be computed inductively. The real points of these curves are graphed in Figure 1; thus the figure depicts the lift of \( \mathcal{F}_4 \) to the finite cover \( \tilde{\mathcal{M}}_1 \times \tilde{\mathcal{M}}_1 \) of \( X_4 \).

References


