# Foliations of Hilbert Modular Surfaces

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Foliations of Hilbert modular surfaces

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21 February, 2005

Abstract

The Hilbert modular surface $X_D$ is the moduli space of Abelian varieties $A$ with real multiplication by a quadratic order of discriminant $D > 1$. The locus where $A$ is a product of elliptic curves determines a finite union of algebraic curves $X_D(1) \subset X_D$.

In this paper we show the lamination $X_D(1)$ extends to an essentially unique foliation $F_D$ of $X_D$ by complex geodesics. The geometry of $F_D$ is related to Teichmüller theory, holomorphic motions, polygonal billiards and Lattès rational maps. We show every leaf of $F_D$ is either closed or dense, and compute its holonomy. We also introduce refinements $T_N(\nu)$ of the classical modular curves on $X_D$, leading to an explicit description of $X_D(1)$.

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1 Introduction

Let $D > 1$ be an integer congruent to 0 or 1 mod 4, and let $\mathcal{O}_D$ be the real quadratic order of discriminant $D$. The Hilbert modular surface

$$X_D = (\mathbb{H} \times \mathbb{H}) / \text{SL}(\mathcal{O}_D \oplus \mathcal{O}_D^\vee)$$

is the moduli space for principally polarized Abelian varieties $A_\tau = \mathbb{C}^2 / (\mathcal{O}_D \oplus \mathcal{O}_D^\vee \tau)$ with real multiplication by $\mathcal{O}_D$.

Let $X_D(1) \subset X_D$ denote the locus where $A_\tau$ is isomorphic to a polarized product of elliptic curves $E_1 \times E_2$. The set $X_D(1)$ is a finite union of disjoint, irreducible algebraic curves ($\S 4$), forming a lamination of $X_D$. Note that $X_D(1)$ is preserved by the twofold symmetry $\iota(\tau_1, \tau_2) = (\tau_2, \tau_1)$ of $X_D$.

In this paper we will show:

**Theorem 1.1** Up to the action of $\iota$, the lamination $X_D(1)$ extends to a unique foliation $\mathcal{F}_D$ of $X_D$ by complex geodesics.

(Here a Riemann surface in $X_D$ is a complex geodesic if it is isometrically immersed for the Kobayashi metric.)

**Holomorphic graphs.** The preimage $\tilde{X}_D(1)$ of $X_D(1)$ in the universal cover of $X_D$ gives a lamination of $\mathbb{H} \times \mathbb{H}$ by the graphs of countably many Möbius transformations. To foliate $X_D$ itself, in $\S 6$ we will show:

**Theorem 1.2** For any $(\tau_1, \tau_2) \notin \tilde{X}_D(1)$, there is a unique holomorphic function

$$f : \mathbb{H} \rightarrow \mathbb{H}$$

such that $f(\tau_1) = \tau_2$ and the graph of $f$ is disjoint from $\tilde{X}_D(1)$.

The graphs of such functions descend to $X_D$, and form the leaves of the foliation $\mathcal{F}_D$ ($\S 7$). The case $D = 4$ is illustrated in Figure 1.

**Modular curves.** To describe the lamination $X_D(1)$ explicitly, recall that the Hilbert modular surface $X_D$ is populated by infinitely many modular curves $F_N$ [Hir], [vG]. The endomorphism ring of a generic Abelian variety in $F_N$ is a quaternionic order $R$ of discriminant $N^2$.

In general $F_N$ can be reducible, and $R$ is not determined up to isomorphism by $N$. In $\S 3$ we introduce a refinement $F_N(\nu)$ of the traditional modular curves, such that the isomorphism class of $R$ is constant along
$F_N(\nu)$ and $F_N = \bigcup F_N(\nu)$. The additional finite invariant $\nu$ ranges in the ring $\mathcal{O}_D / (\sqrt{D})$ and its norm satisfies $N(\nu) = -N \mod D$. The curves $T_N = \bigcup F_{N/\ell^2}$ can be refined similarly, and we obtain:

**Theorem 1.3** The locus $X_D(1) \subset X_D$ is given by

$$X_D(1) = \bigcup T_N((e + \sqrt{D})/2),$$

where the union is over all integral solutions to $e^2 + 4N = D$, $N > 0$.

**Remark.** Although $X_D(1) = \bigcup T_{(D-e^2)/4}$ when $D$ is prime, in general (e.g. for $D = 12, 16, 20, 21, \ldots$) the locus $X_D(1)$ cannot be expressed as a union of the traditional modular curves $T_N$ (§3).

Here is a corresponding description of the lamination $\tilde{X}_D(1)$. Given $N > 0$ such that $D = e^2 + 4N$, let

$$\Lambda_D^N = \left\{ U = \begin{pmatrix} \mu & bD \\ -a & -\mu' \end{pmatrix} : a, b \in \mathbb{Z}, \mu \in \mathcal{O}_D, \det(U) = N \text{ and } \mu \equiv \pm(e + \sqrt{D})/2 \text{ in } \mathcal{O}_D / (\sqrt{D}) \right\}.$$ 

Let $\Lambda_D$ be the union of all such $\Lambda_D^N$. Choosing a real place $\nu_1 : \mathcal{O}_D \to \mathbb{R}$, we can regard $\Lambda_D$ as a set of matrices in $\text{GL}_2^+(\mathbb{R})$, acting by Möbius transformations on $\mathbb{H}$.

**Theorem 1.4** The lamination $\tilde{X}_D(1)$ of $\mathbb{H} \times \mathbb{H}$ is the union of the loci $\tau_2 = U(\tau_1)$ over all $U \in \Lambda_D$. 

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We also obtain a description of the locus $X_D(E) \subset X_D$ where $A_\tau$ admits an action of both $O_D$ and $O_E$ (§3).

**Quasiconformal dynamics.** Although its leaves are Riemann surfaces, $\mathcal{F}_D$ is not a holomorphic foliation. Its transverse dynamics is given instead by quasiconformal maps, which can be described as follows.

Let $q = q(z) \, dz^2$ be a meromorphic quadratic differential on $\mathbb{H}$. We say a homeomorphism $f : \mathbb{H} \to \mathbb{H}$ is a Teichmüller mapping relative to $q$ if it satisfies $\frac{\partial f}{\partial f} = \frac{\alpha q}{|q|}$ for some complex number $|\alpha| < 1$; equivalently, if $f$ has the form of an orientation-preserving real-linear mapping

$$f(x + iy) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = D_q(f) \begin{pmatrix} x \\ y \end{pmatrix}$$

in local charts where $q = dz^2 = (dx + idy)^2$.

Fix a transversal $\mathbb{H}_s = \{s\} \times \mathbb{H}$ to $\tilde{\mathcal{F}}_D$. Any $g \in \text{SL}(O_D \oplus O_D^\vee)$ acts on $\mathbb{H} \times \mathbb{H}$, permuting the leaves of $\tilde{\mathcal{F}}_D$. The permutation of leaves is recorded by the holonomy map

$$\phi_g : \mathbb{H}_s \to \mathbb{H}_s,$$

characterized by the property that $g(s, z)$ and $(s, \phi_g(z))$ lie on the same leaf of $\tilde{\mathcal{F}}_D$.

In §8 we will show:

**Theorem 1.5** The holonomy acts by Teichmüller mappings relative to a fixed meromorphic quadratic differential $q$ on $\mathbb{H}_s$. For $s = i$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have

$$D_q(\phi_g) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}_2(\mathbb{R}).$$

On the other hand, for $z \in \partial \mathbb{H}_s$ we have

$$\phi_g(z) = (a'z - b')/(-c'z + d');$$

in particular, the holonomy acts by Möbius transformations on $\partial \mathbb{H}_s$.

Here $(x + y\sqrt{D})' = (x - y\sqrt{D})$. Note that both Galois conjugate actions of $g$ on $\mathbb{R}^2$ appear, as different aspects of the holonomy map $\phi_g$.

**Quantum Teichmüller curves.** For comparison, consider an isometrically immersed Teichmüller curve

$$f : V \to \mathcal{M}_g,$$
generated by a holomorphic quadratic differential \((Y, q)\) of genus \(g\). For simplicity assume \(\text{Aut}(Y)\) is trivial. Then the pullback of the universal curve \(X = f^*(\mathcal{M}_{g,1})\) gives an algebraic surface

\[
p: X \to V
\]

with \(p^{-1}(v) = Y\) for a suitable basepoint \(v \in V\). The surface \(X\) carries a canonical foliation \(\mathcal{F}\), transverse to the fibers of \(p\), whose leaves map to Teichmüller geodesics in \(\mathcal{M}_{g,1}\). The holonomy of \(\mathcal{F}\) determines a map

\[
\pi_1(V, v) \to \text{Aff}^+(Y, q)
\]

giving an action of the fundamental group by Teichmüller mappings; and its linear part yields the isomorphism

\[
\pi_1(V, v) \cong \text{PSL}(Y, q) \subset \text{PSL}_2(\mathbb{R}),
\]

where \(\text{PSL}(Y, q)\) is the stabilizer of \((Y, q)\) in the bundle of quadratic differentials \(Q\mathcal{M}_g \to \mathcal{M}_g\). (See e.g. [V1], [Mc4, §2].)

The foliated Hilbert modular surface \((X_D, \mathcal{F}_D)\) presents a similar structure, with the fibration \(p : X \to V\) replaced by the holomorphic foliation \(\mathcal{A}_D\) coming from the level sets of \(\tau_1\) on \(\tilde{X}_D = \mathbb{H} \times \mathbb{H}\). This suggests that one should regard \((X_D, \mathcal{A}_D, \mathcal{F}_D)\) as a quantum Teichmüller curve, in the same sense that a 3-manifold with a measured foliation can be regarded as a quantum Teichmüller geodesic [Mc3].

**Question.** Does every fibered surface \(p : X \to C\) admit a foliation \(\mathcal{F}\) by Riemann surfaces transverse to the fibers of \(p\)?

**Complements.** We conclude in §9 by presenting the following related results.

1. Every leaf of \(\mathcal{F}_D\) is either closed or dense.

2. When \(D \neq d^2\), there are infinitely many eigenforms for real multiplication by \(\mathcal{O}_D\) that are isoperiodic but not isomorphic.

3. The Möbius transformations \(\Lambda_D\) give a maximal top-speed holomorphic motion of a discrete subset of \(\mathbb{H}\).

4. The foliation \(\mathcal{F}_4\) also arises as the motion of the Julia set in a Lattès family of iterated rational maps.
The link with complex dynamics was used to produce Figure 1.

**Notes and references.** The foliation $\mathcal{F}_D$ is constructed using the connection between polygonal billiards and Hilbert modular surfaces presented in [Mc4]. For more on the interplay of dynamics, holomorphic motions and quasiconformal mappings, see e.g. [MSS], [BR], [Sl], [Mc2], [Sul], [McS], [EKK] and [Dou]. A survey of the theory of holomorphic foliations of surfaces appears in [Br1]; see also [Br2] for the Hilbert modular case.

I would like to thank G. van der Geer, B. Gross and the referees for useful comments and suggestions.

## 2 Quaternion algebras

In this section we consider a real quadratic order $\mathcal{O}_D$ acting on a symplectic lattice $L$, and classify the quaternionic orders $R \subset \text{End}(L)$ extending $\mathcal{O}_D$.

**Quadratic orders.** Given an integer $D > 0$, $D \equiv 0$ or $1 \mod 4$, the real quadratic order of discriminant $D$ is given by

$$\mathcal{O}_D = \mathbb{Z}[T]/(T^2 + bT + c), \quad \text{where } D = b^2 - 4c.$$ 

Let $K_D = \mathcal{O}_D \otimes \mathbb{Q}$. Provided $D$ is not a square, $K_D$ is a real quadratic field. Fixing an embedding $\iota_1 : K_D \to \mathbb{R}$, we obtain a unique basis

$$K_D = \mathbb{Q} \cdot 1 \oplus \mathbb{Q} \cdot \sqrt{D}$$

such that $\iota_1(\sqrt{D}) > 0$. The conjugate real embedding $\iota_2 : K_D \to \mathbb{R}$ is given by $\iota_2(x) = \iota_1(x')$, where $(a + b\sqrt{D})' = (a - b\sqrt{D})$.

**Square discriminants.** The case $D = d^2$ can be treated similarly, so long as we regard $x = \sqrt{d^2}$ as an element of $K_D$ satisfying $x^2 = d^2$ but $x \notin \mathbb{Q}$. In this case the algebra $K_D \cong \mathbb{Q} \oplus \mathbb{Q}$ is not a field, so we must take care to distinguish between elements of the algebra such as

$$x = d - \sqrt{d^2} \in K_D,$$

and the corresponding real numbers

$$\iota_1(x) = d - d = 0, \quad \text{and} \quad \iota_2(x) = d + d = 2d.$$ 

**Trace, norm and different.** For simplicity of notation, we fix $D$ and denote $\mathcal{O}_D$ and $K_D$ by $K$ and $\mathcal{O}$. 

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The trace and norm on $K$ are the rational numbers $\text{Tr}(x) = x + x'$ and $N(x) = xx'$. The *inverse different* is the fractional ideal

$$\mathcal{O}^\vee = \{ x \in K : \text{Tr}(xy) \in \mathbb{Z} \forall y \in \mathcal{O} \}.$$ 

It is easy to see that $\mathcal{O}^\vee = D^{-1/2} \mathcal{O}$, and thus the *different* $D = (\mathcal{O}^\vee)^{-1} \subset \mathcal{O}$ is the principal ideal $(\sqrt{D})$. The trace and norm descend to give maps

$$\text{Tr}, N : \mathcal{O}/D \to \mathbb{Z}/D,$$

satisfying

$$\text{Tr}(x)^2 = 4 N(x) \mod D.$$ \hspace{1cm} (2.1)

When $D$ is odd, $\text{Tr} : \mathcal{O}/D \to \mathbb{Z}/D$ is an isomorphism, and thus (2.1) determines the norm on $\mathcal{O}/D$. On the other hand, when $D = 4E$ is even, we have an isomorphism

$$\mathcal{O}/D \cong \mathbb{Z}/2E \oplus \mathbb{Z}/2$$

given by $a + b\sqrt{E} \mapsto (a, b)$, and the trace and norm on $\mathcal{O}/D$ are given by

$$\text{Tr}(a, b) = 2a \mod D, \hspace{0.5cm} N(a, b) = a^2 - E b^2 \mod D.$$ 

**Symplectic lattices.** Now let $L \cong (\mathbb{Z}^{2g}, \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix})$ be a unimodular symplectic lattice of genus $g$. (This lattice is isomorphic to the first homology group $H_1(\Sigma_g, \mathbb{Z})$ of an oriented surface of genus $g$ with the symplectic form given by the intersection pairing.)

Let $\text{End}(L) \cong M_{2g}(\mathbb{Z})$ denote the endomorphism ring of $L$ as a $\mathbb{Z}$-module. The *Rosati involution* $T \mapsto T^*$ on $\text{End}(L)$ is defined by the condition $\langle Tx, y \rangle = \langle x, T^*y \rangle$; it satisfies $(ST)^* = T^*S^*$, and we say $T$ is *self-adjoint* if $T = T^*$.

Specializing to the case $g = 2$, let $L$ denote the lattice

$$L = \mathcal{O} \oplus \mathcal{O}^\vee$$

with the unimodular symplectic form

$$\langle x, y \rangle = \text{Tr}(x \wedge y) = \text{Tr}_Q^K(x_1y_2 - x_2y_1).$$

A standard symplectic basis for $L$ (satisfying $\langle a_i \cdot b_j \rangle = \delta_{ij}$) is given by

$$(a_1, a_2, b_1, b_2) = ((1,0), (\gamma, 0), (0, -\gamma'/\sqrt{D}), (0, 1/\sqrt{D})), \hspace{1cm} (2.2)$$
where $\gamma = (D + \sqrt{D})/2$.

The lattice $L$ comes equipped with a proper, self-adjoint action of $\mathcal{O}$, given by

$$k \cdot (x_1, x_2) = (kx_1, kx_2).$$

Conversely, any proper, self-adjoint action of $\mathcal{O}$ on a symplectic lattice of genus two is isomorphic to this model (see e.g. [Ru], [Mc7, Thm 4.1]). (Here an action of $R$ on $L$ is proper if it is indivisible: if whenever $T \in \text{End}(L)$ and $mT \in R$ for some integer $m \neq 0$, then $T \in R$.)

**Matrices.** The natural embedding of $L = \mathcal{O} \oplus \mathcal{O}^\vee$ into $K \oplus K$ determines an embedding of matrices

$$M_2(K) \to \text{End}(L \otimes \mathbb{Q}),$$

and hence a diagonal inclusion

$$K \to \text{End}(L \otimes \mathbb{Q})$$

extending the natural action (2.3) of $\mathcal{O}$ on $L$. Every $T \in \text{End}(L \otimes \mathbb{Q})$ can be uniquely expressed in the form

$$T(x) = Ax + Bx', \quad A, B \in M_2(K),$$

where $(x_1, x_2)' = (x_1', x_2')$; and we have

$$T^*(x) = A^\dagger x + (B^\dagger)'x',$$

where $(a \ b \ c \ d)^\dagger = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$.

The automorphisms of $L$ as a symplectic $\mathcal{O}$-module are given, as a subgroup of $M_2(K)$, by

$$\text{SL}(\mathcal{O} \oplus \mathcal{O}^\vee) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \begin{pmatrix} \mathcal{O} & \mathcal{D} \\ \mathcal{O}^\vee & \mathcal{O} \end{pmatrix} : ad - bc = 1 \right\}.$$  

Compare [vG, p.12].

**Integrality.** An endomorphism $T \in \text{End}(L \otimes \mathbb{Q})$ is integral if it satisfies $T(L) \subseteq L$.

**Lemma 2.1** The endomorphism $\phi(x) = ax + bx'$ of $K$ satisfies $\phi(\mathcal{O}) \subseteq \mathcal{O}$ iff $a, b \in \mathcal{O}^\vee$ and $a + b \in \mathcal{O}$.

**Proof.** Since $x - x' \in \sqrt{D}\mathbb{Z}$ for all $x \in \mathcal{O}$, the conditions on $a, b$ imply $\phi(x) = a(x - x') + (a + b)x' \in \mathcal{O}$ for all $x \in \mathcal{O}$. Conversely, if $\phi$ is integral, then $\phi(1) = a + b \in \mathcal{O}$, and thus $a(x - x') \in \mathcal{O}$ for all $x \in \mathcal{O}$, which implies $a \in D^{-1/2} \mathcal{O} = \mathcal{O}^\vee$. 

\[ \square \]
Corollary 2.2 The endomorphism \( T(x) = kx + (\begin{array}{cc} a & b \\ c & d \end{array}) x' \) is integral iff we have
\[ a, b, c, d, k \in \mathcal{O}^\vee \quad \text{and} \quad k + a, k - d \in \mathcal{O}. \]

**Proof.** This follows from the preceding Lemma, using the fact that \( kx + dx' \) maps \( \mathcal{O}^\vee \) to \( \mathcal{O}^\vee \) iff \( kx - dx' \) maps \( \mathcal{O} \) to \( \mathcal{O} \).

**Quaternions.** A rational quaternion algebra is a central simple algebra of dimension 4 over \( \mathbb{Q} \). Every such algebra has the form
\[ Q \cong \mathbb{Q}[i, j]/(i^2 = a, j^2 = b, ij = -ji) = \left( \begin{array}{c} a \\ b \end{array} \right) \]
for suitable \( a, b \in \mathbb{Q}^* \). Any \( q \in Q \) satisfies a quadratic equation
\[ q^2 - \text{Tr}(q)q + N(q) = 0, \]
where \( \text{Tr}, N : Q \to \mathbb{Q} \) are the reduced trace and norm.

An order \( R \subset Q \) is a subring such that, as an additive group, we have \( R \cong \mathbb{Z}^4 \) and \( Q \cdot R = Q \). Its discriminant is the square integer
\[ N^2 = |\det(\text{Tr}(q_iq_j))| > 0, \]
where \( (q_i)_4^1 \) is an integral basis for \( R \). The discriminants of a pair of orders \( R_1 \subset R_2 \) are related by \( N_1/N_2 = |R_2/R_1|^2 \).

**Generators.** We say \( V \in \text{End}(L) \) is a quaternionic generator if:
1. \( V^* = -V \),
2. \( V^2 = -N \in \mathbb{Z}, \ N \neq 0 \),
3. \(Vk = k'V \) for all \( k \in K \), and
4. \( k + D^{-1/2}V \in \text{End}(L) \) for some \( k \in K \).

These conditions imply that \( Q = K \oplus KV \) is a quaternion algebra isomorphic to \( \left( \frac{D, -N}{\mathbb{Q}} \right) \). Conversely, we have:

**Theorem 2.3** Any Rosati-invariant quaternion algebra \( Q \) with
\[ K \subset Q \subset \text{End}(L \otimes \mathbb{Q}) \]
contains a unique pair of primitive quaternionic generators \( \pm V \).
(A generator is primitive unless \((1/m)V, m > 1\) is also a generator.)

**Proof.** By a standard application of the Skolem-Noether theorem, we can write \(Q = K \oplus KW\) with \(0 \neq W^2 \in \mathbb{Q}\) and \(Wk = kW\) for all \(k \in K\). Then \(KW\) coincides with the subalgebra of \(Q\) anticommuting with the self-adjoint element \(\sqrt{D}\), so it is Rosati-invariant. The eigenspaces of \(*KW\) are exchanged by multiplication by \(\sqrt{D}\), so up to a rational multiple there is a unique nonzero \(V \in KW\) with \(V^* = -V\). A suitable integral multiple of \(V\) is then a generator, and a rational multiple is primitive.

**Corollary 2.4** Quaternionic extensions \(K \subset Q \subset \text{End}(L)\) correspond bijectively to pairs of primitive generators \(\pm V \in \text{End}(L)\).

**Generator matrices.** We say \(U \in M_2(K)\) is a quaternionic generator matrix if it has the form

\[
U = \begin{pmatrix} \mu & bD \\ -a & -\mu' \end{pmatrix}
\]  

(2.5)

with \(a, b \in \mathbb{Z}\), \(\mu \in \mathcal{O}\) and \(N = \det(U) \neq 0\).

**Theorem 2.5** The endomorphism \(V(x) = Ux'\) is a quaternionic generator iff \(U\) is a quaternionic generator matrix.

**Proof.** By (2.4) the condition \(V = -V^*\) is equivalent to \(U^\dagger = -U'\), and thus \(U\) can be written in the form (2.5) with \(a, b \in \mathbb{Q}\) and \(\mu \in K\). Assuming \(U^\dagger = -U'\), we have

\[
N = \det(U) = UU^\dagger = -UU' = -V^2,
\]

so \(V^2 \neq 0 \iff \det(U) \neq 0\). The condition that \(D^{-1/2}(k + V)\) is integral for some \(k\) implies, by Corollary 2.2, that the coefficients of \(U\) satisfy \(a, b \in \mathbb{Z}\) and \(\mu \in \mathcal{O}\); and given such coefficients for \(U\), the endomorphism \(D^{-1/2}(k + V)\) is integral when \(k = -\mu\).

**The invariant \(\nu(U)\).** Given generator matrix \(U = \begin{pmatrix} \mu & bD \\ -a & -\mu' \end{pmatrix}\), let \(\nu(U)\) denote the image of \(\mu\) in the finite ring \(\mathcal{O}/D\). It is easy to check that

\[
\nu(U) = \pm \nu(g'Ug^{-1})
\]

for all \(g \in \text{SL}(\mathcal{O} \oplus \mathcal{O}^\vee)\), and that its norm satisfies

\[
N(\nu(U)) \equiv -N \bmod D.
\]  

(2.6)
Quaternionic orders. Let \( V(x) = Ux' \), and let

\[
R_U = (K \oplus KV) \cap \text{End}(L)
\]

Then \( R_U \) is a Rosati-invariant order in the quaternion algebra generated by \( V \). Clearly \( O \subset R_U \), so we can also regard \((R_U, \ast)\) as an involutive algebra over \( O \). We will show that \( N = \det(U) \) and \( \nu(U) \) determine \((R_U, \ast)\) up to isomorphism.

Models. We begin by constructing a model algebra \((R_N(\nu), \ast)\) over \( O_D \) for every \( \nu \in O \setminus D \) with \( N(\nu) = -N \neq 0 \) mod \( D \).

Let \( Q_N = K \oplus KV \) be the abstract quaternion algebra with the relations \( V^2 = -N \) and \( V k = k'V \). Define an involution on \( Q_N \) by \((k_1 + k_2 V)^* = (k_1 - k_2 V)\), and let \( R_N(\nu) \) be the order in \( Q_N \) defined by

\[
R_N(\nu) = \{ \alpha + \beta V : \alpha, \beta \in O, \alpha + \beta \nu \in O \}.
\]  

(2.7)

Note that \( O' \cdot D \subset O \), so the definition of \( R_N(\nu) \) depends only on the class of \( \nu \) in \( O \setminus D \). To check that \( R_N(\nu) \) is an order, note that

\[
(\alpha + \beta V)(\gamma + \delta V) = (\kappa + \lambda V) = (\alpha \gamma - N \beta \delta') + (\alpha \delta + \beta \gamma')V;
\]

since \(-N \equiv N(\nu) = \nu \nu' \mod D\), we have

\[
\kappa + \nu \lambda \equiv (\alpha \gamma + \nu \nu' \beta \delta') + \nu (\alpha \delta + \beta \gamma')
= (\alpha + \beta \nu)(\gamma' + \delta' \nu') + \alpha (\gamma - \gamma' + \nu \delta - \nu' \delta')
\equiv 0 + 0 \mod O,
\]

and thus \( R_U \) is closed under multiplication.

**Theorem 2.6** The quaternionic order \( R_N(\nu) \) has discriminant \( N^2 \).

**Proof.** Note that the inclusions

\[
O \oplus O V \subset R_N(\nu) \subset O' \oplus O' V
\]

each have index \( D \). The quaternionic order \( O \oplus O V \) has discriminant \( D^2 N^2 \), since \( V^2 = -N \) and \( \text{Tr} | O V = 0 \), and thus \( R_N(\nu) \) has discriminant \( N^2 \).
Theorem 2.7 We have \((R_N(\nu), \ast) \cong (R_M(\mu), \ast)\) iff \(N = M\) and \(\nu = \pm \mu\).

Proof. The element \(V \in R_N(\nu)\) is, up to sign, the order’s unique primitive generator, in the sense that \(V^* = -V, Vk = k'V\) for all \(k \in \mathcal{O}_D, V^2 \neq 0, k + D^{-1/2}V \in R_N(\nu)\) for some \(k \in K\), and \(V\) is not a proper multiple of another element in \(R_N(\nu)\) with the same properties. Thus the structure of \((R_N(\nu), \ast)\) as an \(\mathcal{O}_D\)-algebra determines \(V \in R_N(\nu)\) up to sign, and \(V\) determines \(N = -V^2\) and the constant \(\nu \in \mathcal{O}/\mathcal{D}\) in the relation \(\alpha + \beta \nu \in \mathcal{O}\) defining \(R_N(\nu) \subset K \oplus KV\). \(\blacksquare\)

Theorem 2.8 If \(U\) is a primitive generator matrix, then we have

\[(R_U, \ast) \cong (R_N(\nu), \ast)\]

where \(N = \text{det}(U)\) and \(\nu = \nu(U)\).

Proof. Setting \(V(x) = Ux'\), we need only verify that \((K \oplus KV) \cap \text{End}(L)\) coincides with the order \(R_N(\nu)\) defined by (2.7). To see this, let

\[T(x) = \alpha x + \beta V(x) = \alpha x + \beta \begin{pmatrix} \mu & bD \\ -a & -\mu' \end{pmatrix} x'\]

in \(K \oplus KV\). By Corollary 2.2, \(T\) is integral iff

(i) \(a\beta, b\beta, \mu\beta, \mu'\beta \in \mathcal{O}'\),

(ii) \(\alpha \in \mathcal{O}'\),

(iii) \(\alpha + \beta \mu \in \mathcal{O}\) and

(iv) \(\alpha + \beta \mu' \in \mathcal{O}\).

Using (iii), condition (iv) can be replaced by

(iv') \(\beta(\mu - \mu')/\sqrt{D} \in \mathcal{O}'\).

Since \(U\) is primitive, the ideal \((a, b, \mu, (\mu - \mu')/\sqrt{D})\) is equal to \(\mathcal{O}\). Thus (i) and (iv') together are equivalent to the condition \(\beta \in \mathcal{O}'\), and we are left with the definition of \(R_N(\nu)\). \(\blacksquare\)
Remark. In general, the invariants det($U$) and $\nu(U)$ do not determine the embedding $R_U \subset \text{End}(L)$ up to conjugacy. For example, when $D$ is odd, the generator matrices $U_1 = \begin{pmatrix} 0 & D^2 \\ -D & 0 \end{pmatrix}$ and $U_2 = \begin{pmatrix} 0 & D^3 \\ -1 & 0 \end{pmatrix}$ have the same invariants, but the corresponding endomorphisms are not conjugate in $\text{End}(L)$ because

$$L/V_1(L) \cong (\mathbb{Z}/D \times \mathbb{Z}/D^2)^2$$

while

$$L/V_2(L) \cong \mathbb{Z}/D \times \mathbb{Z}/D^2 \times \mathbb{Z}/D^3.$$  

Extra quadratic orders. Finally we determine when the algebra $R_N(\nu)$ contains a second, independent quadratic order $O_E$.

**Theorem 2.9** The algebra $(R_N(\nu), \ast)$ contains a self-adjoint element $T \notin O_D$ generating a copy of $O_E$ iff there exist $e, \ell \in \mathbb{Z}$ such that

$$ED = e^2 + 4N\ell^2, \quad \ell \neq 0$$

and $(e + E\sqrt{D})/2 + \ell\nu \equiv 0 \mod D$.

**Proof.** Given $e, \ell$ as above, let

$$T = \alpha + \beta V = D^{-1/2} \left( \frac{e + E\sqrt{D}}{2} + \ell V \right).$$

Then we have $T = T^\ast$, $T \in R_N(\nu)$ and $T^2 - eT + (E - E^2)/4 = 0$; therefore $\mathbb{Z}[T] \cong O_E$. A straightforward computation shows that, conversely, any independent copy of $O_E$ in $R_N(\nu)$ arises as above.  

For additional background on quaternion algebras, see e.g. [Vi], [MR] and [Mn].

\section{Modular curves and surfaces}

In this section we describe modular curves on Hilbert modular surfaces from the perspective of the Abelian varieties they determine.

**Abelian varieties.** A *principally polarized Abelian variety* is a complex torus $A \cong \mathbb{C}^g/L$ equipped with a unimodular symplectic form $\langle x, y \rangle$ on $L \cong \mathbb{Z}^g$, whose extension to $L \otimes \mathbb{R} \cong \mathbb{C}^g$ satisfies

$$\langle x, y \rangle = \langle ix, iy \rangle \quad \text{and} \quad \langle x, ix \rangle \geq 0.$$
The ring $\text{End}(A) = \text{End}(L) \cap \text{End}(C^g)$ is Rosati invariant, and coincides with the endomorphism ring of $A$ as a complex Lie group. We have $\text{Tr}(TT^*) \geq 0$ for all $T \in \text{End}(A)$.

Every Abelian variety can be presented in the form

$$A = C^g/(\mathbb{Z}^g \oplus \Pi\mathbb{Z}^g),$$

where $\Pi$ is an element of the Siegel upper halfplane

$$\mathcal{H}_g = \{ \Pi \in M_g(\mathbb{C}) : \Pi^t = \Pi \text{ and } \text{Im}(\Pi) \text{ is positive-definite} \}.$$

The symplectic form on $L = \mathbb{Z}^g \oplus \Pi\mathbb{Z}^g$ is given by $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. Any two such presentations of $A$ differ by an automorphism of $L$, so the moduli space of abelian varieties of genus $g$ is given by the quotient space

$$A_g = \mathcal{H}_g / \text{Sp}_{2g}(\mathbb{Z}).$$

**Real multiplication.** As in §2, let $D > 0$ be the discriminant of a real quadratic order $O_D$, and let $K = O \otimes \mathbb{Q}$. Fix a real place $\iota_1 : K \to \mathbb{R}$, and set $\iota_2(k) = \iota_1(k')$.

We will regard $K$ as a subfield of the reals, using the fixed embedding $\iota_1 : K \subset \mathbb{R}$. The case $D = d^2$ is treated with the understanding that the real numbers $(k, k')$ implicitly denote $(\iota_1(k), \iota_2(k))$, $k \in K$.

An Abelian variety $A \in A_2$ admits real multiplication by $O_D$ if there is a self-adjoint endomorphism $T \in \text{End}(A)$ generating a proper action of $\mathbb{Z}[T] \cong O_D$ on $A$. Any such variety can be presented in the form

$$A_T = C^2/(O_D \oplus O_D^\vee \tau) = C^2/\phi_T(L),$$

where $\tau = (\tau_1, \tau_2) \in \mathbb{H} \times \mathbb{H}$ and where $L = O \oplus O^\vee$ is embedded in $C^2$ by the map

$$\phi_T(x_1, x_2) = (x_1 + x_2 \tau_1, x_1' + x_2' \tau_2).$$

As in §2, the symplectic form on $L$ is given by $\langle x, y \rangle = \text{Tr}_{O_D^\vee}^L(x \wedge y)$, and the action of $O_D$ on $C^2 \supset L$ is given simply by $k \cdot (z_1, z_2) = (kz_1, k'z_2)$.

**Eigenforms.** The Abelian variety $A_T$ comes equipped with a distinguished pair of normalized eigenforms $\eta_1, \eta_2 \in \Omega(A_T)$. Using the isomorphism $H_1(A_T, \mathbb{Z}) \cong L$, these forms are characterized by the property that

$$\phi_T(C) = \left( \int_C \eta_1, \int_C \eta_2 \right).$$
Modular surfaces. If we change the identification \( L \cong H_1(A_{\tau}, \mathbb{Z}) \) by an automorphism \( g \) of \( L \), we obtain an isomorphic Abelian variety \( A_{g, \tau} \). Thus the moduli space of Abelian varieties with real multiplication by \( \mathcal{O}_D \) is given by the Hilbert modular surface

\[
X_D = (\mathbb{H} \times \mathbb{H}) / \text{SL}(\mathcal{O}_D \oplus \mathcal{O}_D^\vee).
\]

The point \( g(\tau) \) is characterized by the property that

\[
\phi_{g, \tau} = \chi(g, \tau) \phi_{\tau} \circ g^{-1}
\]

for some matrix \( \chi(g, \tau) \in \text{GL}_2(\mathbb{C}) \); explicitly, we have

\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\cdot
\begin{pmatrix}
\tau_1 \\
\tau_2
\end{pmatrix}
= \begin{pmatrix}
ar_1 - b & a' \tau_2 - b' \\
-c \tau_1 + d' & -c' \tau_2 + d'
\end{pmatrix}
\]

(3.3)

and

\[
\chi(g, \tau) = \begin{pmatrix}
(d - c \tau_1)^{-1} & 0 \\
0 & (d' - c' \tau_2)^{-1}
\end{pmatrix}.
\]

(3.4)

A point \([\tau] \in X_D\) gives an Abelian variety \([A_{\tau}] \in \mathcal{A}_2\) with a chosen embedding \( \mathcal{O}_D \rightarrow \text{End}(A_{\tau}) \). Similarly, a point \( \tau \in X_D = \mathbb{H} \times \mathbb{H} \) gives an Abelian variety with a distinguished isomorphism or marking, \( L \cong H_1(A_{\tau}, \mathbb{Z}) \), sending \( \mathcal{O}_D \) into \( \text{End}(A_{\tau}) \).

Modular embedding. The modular embedding

\[
p_D : X_D \rightarrow \mathcal{A}_2
\]

is given by \([\tau] \mapsto [A_{\tau}]\). To write \( p_D \) explicitly, note that the embedding \( \phi_{\tau} : L \rightarrow \mathbb{C}^2 \) can be expressed with respect to the basis \((a_1, a_2, b_1, b_2)\) for \( L \) given in (2.2) by the matrix

\[
\phi_{\tau} = \begin{pmatrix}
1 & \gamma & -\tau_1 \gamma'/\sqrt{D} & \tau_1/\sqrt{D} \\
1 & \gamma' & \tau_2 \gamma'/\sqrt{D} & -\tau_2/\sqrt{D}
\end{pmatrix} = (A, B).
\]

Consequently we have \( A_{\tau} \cong \mathbb{C}^2/(\mathbb{Z}^2 \oplus \Pi \mathbb{Z}^2) \), where

\[
\Pi = \overline{p_D}(\tau) = A^{-1} B = \frac{1}{D} \begin{pmatrix}
\tau_1 (\gamma')^2 + \tau_2 \gamma^2 & -\tau_1 \gamma' - \tau_2 \gamma \\
-\tau_1 \gamma' - \tau_2 \gamma & \tau_1 + \tau_2
\end{pmatrix}.
\]

The map \( X_D \rightarrow p_D(X_D) \) has degree two.
Modular curves. Given a matrix \( U(x) = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in M_2(K) \cap \text{End}(L) \) such that \( U' = -U^* \), let \( V(x) = Ux' \) and define

\[
\mathbb{H}_U = \{ \tau \in \mathbb{H} \times \mathbb{H} : V \in \text{End}(A_{\tau}) \}.
\]

It is straightforward to check that

\[
\mathbb{H}_U = \left\{ (\tau_1, \tau_2) : \tau_2 = \frac{d\tau_1 + b}{c\tau_1 + a} \right\};
\]

indeed, when \( \tau_1 \) and \( \tau_2 \) are related as above, the map \( \phi_{\tau} : L \to \mathbb{C}^2 \) satisfies

\[
\phi_{\tau}(V(x)) = \begin{pmatrix} 0 & a + c\tau_1 \\ a' + c'\tau_2 & 0 \end{pmatrix} \phi_{\tau}(x),
\]

exhibiting the complex-linearity of \( V \). Note that \( \mathbb{H}_U = \emptyset \) if \( \det(U) < 0 \).

We now restrict attention to the case where \( U \) is a generator matrix. Then by the results of §2, we have:

**Theorem 3.1** The ring \( \text{End}(A_{\tau}) \) contains a quaternionic order extending \( \mathcal{O}_D \) if and only if \( \tau \in \mathbb{H}_U \) for some generator matrix \( U \).

Let \( F_U \subset X_D \) denote the projection of \( \mathbb{H}_U \) to the quotient \( (\mathbb{H} \times \mathbb{H})/\text{SL}(\mathcal{O}_D \oplus \mathcal{O}_D^*) \).

Following [Hir, §5.3], we define the modular curve \( F_N \) by

\[
F_N = \bigcup \{ F_U : U \text{ is a primitive generator matrix with } \det(U) = N \}.
\]

It can be shown that \( F_N \) is an algebraic curve on \( X_D \).

To describe this curve more precisely, let

\[
F_N(\nu) = \{ F_U : U \text{ is primitive, } \det(U) = N \text{ and } \nu(U) = \pm \nu \},
\]

where \( \nu \in \mathcal{O}_D / \mathcal{D}_D \). Note that we have

\[
F_N(\nu) \neq \emptyset \iff N(\nu) = -N \mod D
\]

by equation (2.6), \( F_N(\nu) = F_N(-\nu) \), and \( F_N = \bigcup F_N(\nu) \).

The results of §2 give the structure of the quaternion ring generated by \( V(x) = Ux' \).

**Theorem 3.2** The curve \( F_N(\nu) \subset X_D \) coincides with the locus of Abelian varieties such that

\[
\mathcal{O}_D \subset R \subset \text{End}(A_{\tau}),
\]

for some properly embedded quaternionic order \((R, *)\) isomorphic to \((R_N(\nu), *)\).
Corollary 3.3 The curve $F_N$ is the locus where $O_D \subset \text{End}(A_{\tau})$ extends to a properly embedded, Rosati-invariant quaternionic order of discriminant $N^2$.

Two quadratic orders. We can now describe the locus $X_D(E)$ of Abelian varieties with an independent, self-adjoint action of $O_E$. (We do not require the action of $O_E$ to be proper.)

To state this description, it is useful to define:

$$T_N = \bigcup \{ F_U : \det(U) = N \} = \bigcup F_{N/2},$$

and

$$T_N(\nu) = \bigcup \{ F_U : \det(U) = N, \nu(U) = \pm \nu \}.$$

Then Theorem 2.9 implies:

Theorem 3.4 The locus $X_D(E)$ is given by

$$X_D(E) = \bigcup T_N((e + E\sqrt{D})/2),$$

where the union is over all $N > 0$ and $e \in \mathbb{Z}$ such that $ED = e^2 + 4N$.

Corollary 3.5 We have $X_D(1) = \bigcup \{ T_N((e + \sqrt{D})/2) : e^2 + 4N = D \}$.

Refined modular curves. To conclude we show that in general the expression $F_N = \bigcup F_N(\nu)$ gives a proper refinement of $F_N$. First note:

Theorem 3.6 We have $F_N(\nu) = F_N$ iff $\pm \nu$ are the only solutions to

$$N(\xi) = -N \mod D, \quad \xi \in O_D/D_D.$$

Corollary 3.7 If $D = p$ is prime, then $F_N = F_N(\nu)$ whenever $F_N(\nu) \neq \emptyset$.

Proof. In this case, according to (2.1), the norm map

$$N : O_D/D_D \overset{\text{Tr}}{\rightarrow} \mathbb{Z}/p \rightarrow \mathbb{Z}/p$$

is given by $N(\xi) = \xi^2/4$. Since $F_N(\nu) \neq \emptyset$, we have $N(\nu) = -N$; and since $\mathbb{Z}/p$ is a field, $\pm \nu$ are the only solutions to this equation. \qed
Corollary 3.8 When $D$ is prime, we have $X_D(E) = \bigcup T_{(E + D - e^2)/4}$.

Now consider the case $D = 21$, the first odd discriminant which is not a prime. Then the norm map is still given by $N(\xi) = \xi^2/4$ on $\mathcal{O}_D/\mathcal{D} \cong \mathbb{Z}/D$, but now $\mathbb{Z}/D$ is not a field. For example, the equation $\xi^2 = 1 \mod D$ has four solutions, namely $\xi = 1, 8, 13$ or $20$. These give four solutions to the equation $N(\xi) = -5$, and hence contribute two distinct terms to the expression

$$F_5 = \bigcup F_5(\nu) = F_5((1 + \sqrt{21})/2) \cup F_5((8 + \sqrt{21})/2).$$

Only one of these terms appears in the expression for $X_{21}(1)$. In fact, since $21 = 1^2 + 4 \cdot 5 = 3^2 + 4 \cdot 3$, by Corollary 3.5 we have

$$X_{21}(1) = F_3 \cup F_5((1 + \sqrt{21})/2) \neq F_3 \cup F_5.$$

(The full curve $F_3$ appears because the only solutions to $N(\xi) = \xi^2/4 = -3 \mod 21$ are $\xi = \pm 3$.)

Using Theorem 3.6, it is similarly straightforward to check other small discriminants; for example:

Theorem 3.9 For $D \leq 30$ we have $X_D(1) = \bigcup_{\nu^2 + 4N = D} T_N$ when $D = 4, 5, 8, 9, 13, 17, 25$ and $29$, but not when $D = 12, 16, 20, 21, 24$ or $28$.

Notes. For more background on modular curves and surfaces, see [Hir], [HZ2], [HZ1], [BL], [Mc7, §4] and [vG]. Our $U = \left( \begin{smallmatrix} \mu & bD \\ -a & -\mu' \end{smallmatrix} \right)$ corresponds to the skew-Hermitian matrix $B = \sqrt{D} \left( \begin{smallmatrix} \mu \\ \mu' \end{smallmatrix} bD \left( \begin{smallmatrix} a \\ c \end{smallmatrix} \right) \right)$ in [vG, Ch. V]. Note that (3.3) agrees with the standard action $(a\tau + b)/(c\tau + d)$ up to the automorphism $(a \ b \ c \ d) \mapsto (a \ b \ -c \ d)$ of $\text{SL}_2(K)$. We remark that $X_D$ can also be presented as the quotient $(\mathbb{H} \times -\mathbb{H})/\text{SL}_2(\mathcal{O}_D)$, using the fact that $\sqrt{D} = -\sqrt{D'}$; on the other hand, the surfaces $(\mathbb{H} \times \mathbb{H})/\text{SL}_2(\mathcal{O}_D)$ and $X_D$ are generally not isomorphic (see e.g. [HH].)

It is known that the intersection numbers $\langle T_N, T_M \rangle$ form the coefficients of a modular form [HZ1], [vG, Ch. VI]. The results of [GKZ] suggest that the intersection numbers of the refined modular curves $T_N(\nu)$ may similarly yield a Jacobi form.

4 Laminations

In this section we show algebraically that $\tilde{X}_D(1)$ gives a lamination of $\mathbb{H} \times \mathbb{H}$ by countably many disjoint hyperbolic planes. We also describe these
laminations explicitly for small values of $D$. Another proof of laminarity appears in §7.

**Jacobian varieties.** Let $\Omega(X)$ denote the space of holomorphic 1-forms on a compact Riemann surface $X$. The *Jacobian* of $X$ is the Abelian variety $\text{Jac}(X) = \Omega(X)^*/H_1(X, \mathbb{Z})$, polarized by the intersection pairing on 1-cycles.

In the case of genus two, any principally polarized Abelian variety $A$ is either a Jacobian or a product of polarized elliptic curves. The latter case occurs if $A$ admits real multiplication by $\mathcal{O}_1$, generated by projection to one of the factors of $A \cong B_1 \times B_2$. In particular, we have:

**Theorem 4.1** For any $D \geq 4$, the locus of Jacobian varieties in $X_D$ is given by $X_D - X_D(1)$.

**Laminations.** To describe $X_D(1)$ in more detail, given $N > 0$ such that $D = e^2 + 4N$ let

$$\Lambda_D^N = \{ U \in M_2(K) : U \text{ is a generator matrix, } \det(U) = N \text{ and } \nu(U) \equiv \pm(e + \sqrt{D})/2 \mod D_D \},$$

and let $\Lambda_D$ be the union of all such $\Lambda_D^N$. Note that if $U$ is in $\Lambda_D$, then $-U, U'$ and $U^*$ are also in $\Lambda_D$.

By Corollary 3.5, the preimage of $X_D(1)$ in $\tilde{X}_D = \mathbb{H} \times \mathbb{H}$ is given by:

$$\tilde{X}_D(1) = \bigcup \{ \mathbb{H}_U : U \in \Lambda_D \}.$$

Note that each $\mathbb{H}_U$ is the graph of a Möbius transformation.

**Theorem 4.2** The locus $\tilde{X}_D(1)$ gives a lamination of $\mathbb{H} \times \mathbb{H}$ by countably many hyperbolic planes.

(This means any two planes in $\tilde{X}_D(1)$ are either identical or disjoint.)

For the proof, it suffices to show that the difference $g \circ h^{-1}$ of two Möbius transformations in $\Lambda_D$ is never elliptic. Since $\Lambda_D$ is invariant under $U \mapsto U^* = (\det(U))U^{-1}$, this in turn follows from:

**Theorem 4.3** For any $U_1, U_2 \in \Lambda_D$, we have $\text{Tr}(U_1U_2)^2 \geq 4 \det(U_1U_2)$.

**Proof.** By the definition of $\Lambda_D$, we can write $D = e_i^2 + 4\det(U_i) = e_i^2 + 4N_i$, where $e_i \geq 0$. We can also assume that

$$U_i = \begin{pmatrix} \mu_i & b_iD \\ -a_i & -\mu_i' \end{pmatrix}$$

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satisfies
\[ \mu_i \equiv (x_i + y_i \sqrt{D})/2 \equiv (e_i + \sqrt{D})/2 \mod D \]
(replacing \( U_i \) with \(-U_i \) if necessary). It follows that \( y_i \) is odd and \( x_i = e_i \mod D \), which implies
\[ \text{Tr}(U_1 U_2) \equiv \text{Tr}(\mu_1 \mu_2) = (x_1 x_2 + D y_1 y_2)/2 \equiv (e_1 e_2 - D)/2 \mod D. \quad (4.1) \]
(The factor of 1/2 presents no difficulties, because \( x_i \) is even when \( D \) is even.)

Now suppose
\[ \text{Tr}(U_1 U_2)^2 < 4 \det(U_1 U_2) = 4N_1 N_2. \quad (4.2) \]
Then we have \( |\text{Tr}(U_1 U_2)| < 2\sqrt{N_1 N_2} \leq D/2 \), and thus (4.1) implies
\[ \text{Tr}(U_1 U_2) = (e_1 e_2 - D)/2. \]
But this implies
\[ 4 \text{Tr}(U_1 U_2)^2 = (D - e_1 e_2)^2 \geq (D - e_1^2)(D - e_2^2) = (4N_1)(4N_2) = 16 \det(U_1 U_2), \]
contradicting (4.2).

**Small discriminants.** To conclude we record a few cases where \( \Lambda_D \) admits a particularly economical description.

For concreteness, we will present \( \Lambda_D \) as a set matrices in \( \text{GL}_2^+(\mathbb{R}) \) using the chosen real place \( \iota_1 : K \to \mathbb{R} \). This works even when \( D = d^2 \), since both \( \mu \) and \( \mu' \) appear on the diagonal of \( U \in \Lambda_D \) (no information is lost). Under the standard action \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = (az + b)/(cz + d) \) of \( \text{GL}_2^+(\mathbb{R}) \) on \( \mathbb{H} \), we can then write
\[ \tilde{X}_D(1) = \bigcup_{\Lambda_D} \{(\tau_1, \tau_2) : \tau_2 = U(\tau_1)\}. \]
This holds despite the twist in the definition (3.5) of \( \mathbb{H}_U \), because \( \Lambda_D \) is invariant under \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & b \\ c & a \end{pmatrix} \).

**Theorem 4.4** For \( D = 4, 5, 8, 9 \) and 13 respectively, we have:
\[ \Lambda_4 = \{ U \in M_2(\mathbb{Z}) : \text{det}(U) = 1 \text{ and } U \equiv (\cdot \cdot 0) \mod 4 \}, \]
\[ \Lambda_5 = \{ U = \begin{pmatrix} \mu & bD \\ -a & -\mu' \end{pmatrix} : \text{det}(U) = 1 \}, \]
\[ \Lambda_8 = \Lambda_8^1 \cup \Lambda_8^2 = \{ U = \begin{pmatrix} \mu & bD \\ -a & -\mu' \end{pmatrix} : \text{det}(U) = 1 \text{ or } 2 \}, \]
\[ \Lambda_9 = \{ U \in M_2(\mathbb{Z}) : \text{det}(U) = 2 \text{ and } U \equiv (\cdot \cdot 0) \mod 9 \}, \text{ and} \]
\[ \Lambda_{13} = \Lambda_{13}^1 \cup \Lambda_{13}^3 = \{ U = \begin{pmatrix} \mu & bD \\ -a & -\mu' \end{pmatrix} : \text{det}(U) = 1 \text{ or } 3 \}, \]

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where it is understood that $a, b \in \mathbb{Z}$ and $\mu \in \mathcal{O}_D$.

**Proof.** Recall from Theorem 3.9 that $X_D(1) = \bigcup_{e^2 + 4N = D} T_N$ when $D = 4, 5, 8, 9$ and 13. When this equality holds, we can ignore the condition on $\nu(U)$ in the definition of $\Lambda_D$. The cases $D = 5, 8$ and 13 then follow directly from the definition of $\Lambda_D^N$. For $D = 9$, we note that any integral matrix satisfying $\det \begin{pmatrix} x & 9b \\ -a & y \end{pmatrix} = 2$ also satisfies $x + y = 0 \mod 3$, and thus it can be written in the form $\begin{pmatrix} \mu & bD \\ -a & -\mu' \end{pmatrix}$ with $\mu = \frac{(x - y) + (x + y)\sqrt{3}/3}{2}$. Similar considerations apply when $D = 4$.

5 Foliations of Teichmüller space

In this section we introduce a family of foliations $\mathcal{F}_i$ of Teichmüller space, related to normalized Abelian differentials and their periods $\tau_{ij} = \int_{b_i} \omega_j$. We then show:

**Theorem 5.1** There is a unique holomorphic section of the period map $\tau_{ii} : T_g \to \mathbb{H}$ through any $Y \in T_g$. Its image is the leaf of $\mathcal{F}_i$ containing $Y$.

The case $g = 2$ will furnish the desired foliations of Hilbert modular surfaces.

**Abelian differentials.** Let $Z_g$ be a smooth oriented surface of genus $g$. Let $T_g$ be the Teichmüller space of Riemann surfaces $Y$, each equipped with an isotopy class of homeomorphism or marking $Z_g \to Y$. The marking determines a natural identification between $H_1(Z_g)$ and $H_1(Y)$ used frequently below.

Let $\Omega T_g \to T_g$ denote the bundle of nonzero Abelian differentials $(Y, \omega)$, $\omega \in \Omega(Y)$. For each such form we have a period map $I(\omega) : H_1(Z_g, \mathbb{Z}) \to \mathbb{C}$ given by $I(\omega) : C \to \int_C \omega$. There is a natural action of $\text{GL}_2^+(\mathbb{R})$ on $\Omega T_g$, satisfying

$$I(A \cdot \omega) = A \circ I(\omega) \quad (5.1)$$
under the identification $\mathbb{C} = \mathbb{R}^2$ given by $x + iy = (x, y)$.

Each orbit $GL_2^+(\mathbb{R}) \cdot (Y, \omega)$ projects to a complex geodesic

$$f : \mathbb{H} \to T_g,$$

which can be normalized so that $f(i) = Y$ and

$$\nu = \frac{df}{dt} \bigg|_{t=i} = \frac{i}{2}\overline{\omega}.$$

The subspace of $H^1(Z_g, \mathbb{R})$ spanned by $(\text{Re}\, \omega, \text{Im}\, \omega)$ is constant along each orbit (cf. [Mc7, §3]).

**Symplectic framings.** Now let $(a_1, \ldots, a_g, b_1, \ldots, b_g)$ be a real symplectic basis for $H_1(Z_g, \mathbb{R})$ (with $\langle a_i, b_i \rangle = -\langle b_i, a_i \rangle = 1$ and all other products zero). Then for each $Y \in T_g$, there exists a unique basis $(\omega_1, \ldots, \omega_g)$ of $\Omega(Y)$ such that $\int_{a_i} \omega_j = \delta_{ij}$. The period matrix

$$\tau_{ij}(Y) = \int_{b_i} \omega_j$$

then determines an embedding

$$\tau : T_g \to \mathfrak{H}_g.$$

This agrees with the usual Torelli embedding, up to composition with an element of $\text{Sp}_{2g}(\mathbb{R})$. Note that $\text{Im}(\tau_{ii}(Y)) > 0$ since $\text{Im}\, \tau$ is positive definite.

The normalized 1-forms $(\omega_i)$ give a splitting

$$\Omega(Y) = \oplus_i^g \mathbb{C}\omega_i = \oplus_i^g F_i(Y),$$

and corresponding subbundles $F_i T_g \subset \Omega T_g$.

**Complex subspaces.** Let $(a_i^*, b_i^*)$ denote the dual basis for $H^1(Z_g, \mathbb{R})$, and let $S_i$ be the span of $(a_i^*, b_i^*)$. It easy to check that the following conditions are equivalent:

1. $S_i$ is a complex subspace of $H^1(Y, \mathbb{R}) \cong \Omega(Y)$.
2. $S_i$ is spanned by $(\text{Re}\, \omega_i, \text{Im}\, \omega_i)$.
3. The period matrix $\tau(Y)$ satisfies $\tau_{ij} = 0$ for all $j \neq i$. 

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Let\( T_g(S_i) \subset T_g \) denote the locus where these condition hold. Note that condition (3) defines a totally geodesic subset
\[
H_i \cong \mathbb{H} \times \mathfrak{H}_{g-1} \subset \mathfrak{H}_g
\]
such that\( T_g(S_i) = \tau^{-1}(H_i) \).

**Foliations.** Next we show that the complex geodesics generated by the forms \((Y, \omega_i)\) give a foliation of Teichmüller space.

**Theorem 5.2** The sub-bundle \( F_i T_g \subset \Omega T_g \) is invariant under the action of \( \text{GL}_2^+(\mathbb{R}) \), as is its restriction to \( T_g(S_i) \).

**Proof.** The invariance of \( F_i T_g \) is immediate from (5.1). To handle the restriction to \( T_g(S_i) \), recall that the span \( W \) of \((\text{Re} \omega_i, \text{Im} \omega_i)\) is constant along orbits; thus the condition \( W = S_i \) characterizing \( T_g(S_i) \) is preserved by the action of \( \text{GL}_2^+(\mathbb{R}) \).

**Corollary 5.3** The foliation of \( F_i T_g \) by \( \text{GL}_2^+(\mathbb{R}) \) orbits projects to a foliation \( \mathcal{F}_i \) of \( T_g \) by complex geodesics.

**Corollary 5.4** The locus \( T_g(S_i) \) is also foliated by \( \mathcal{F}_i \); any leaf meeting \( T_g(S_i) \) is entirely contained therein.

**Proof of Theorem 5.1.** The proof uses Ahlfors’ variational formula [Ah] and follows the same lines as the proof of [Mc4, Thm. 4.2]; it is based on the fact that the leaves of \( \mathcal{F}_i \) are the geodesics along which the periods of \( \omega_i \) change most rapidly.

Let \( s : \mathbb{H} \to T_g \) be a holomorphic section of \( \tau_{ii} \). Let \( v \in \mathbb{T} \) be a unit tangent vector with respect to the hyperbolic metric \( \rho = |dz|/(2 \text{Im} z) \) of constant curvature \(-4\), mapping to \( Ds(v) \in T_Y T_g \). By the equality of the Teichmüller and Kobayashi metrics [Gd, Ch. 7], \( Ds(v) \) is represented by a Beltrami differential \( \nu = \nu(z) d\overline{z}/dz \) on \( Y \) with \( \|\nu\|_\infty \leq 1 \). But \( s \) is a section, so the composition
\[
\tau_{ii} \circ s : \mathbb{H} \to \mathbb{H}
\]
is the identity; thus the norm of its derivative, given by Ahlfors’ formula as
\[
\|D(\tau_{ii} \circ s)(\nu)\| = \left| \int_Y \omega^2_i \nu \right| \left| \int_Y |\omega_i|^2 \right|,
\]
is one. It follows that \( \nu = \overline{\omega_i}/\omega_i \) up to a complex scalar of modulus one, and thus \( Ds(v) \) is tangent to the complex geodesic generated by \((Y, \omega_i)\). Equivalently, \( s(\mathbb{H}) \) is everywhere tangent to the foliation \( \mathcal{F}_i \); therefore its image is the unique leaf through \( Y \).
6 Genus two

We can now obtain results on Hilbert modular surfaces by specializing to the case of genus two. In this section we will show:

**Theorem 6.1** There is a unique holomorphic section of $\tau_1$ passing through any given point of $\mathbb{H} \times \mathbb{H} - \tilde{X}_D(1)$.

Here $\tau_1 : \mathbb{H} \times \mathbb{H} \to \mathbb{H}$ is simply projection onto the first factor. This result is a restatement of Theorem 1.2; as in §1, we assume $D \geq 4$.

**Framings for real multiplication.** Let $g = 2$, and choose a symplectic isomorphism

$$L = H_1(Z_g, \mathbb{Z}) \cong \mathcal{O}_D \oplus \mathcal{O}_D^\vee.$$

We then have an action of $\mathcal{O}_D$ on $H_1(Z_g, \mathbb{Z})$, and the elements $\{a, b\} = \{(1, 0), (0, 1)\}$ in $L$ give a distinguished basis for

$$H_1(Z_g, \mathbb{Q}) = L \otimes \mathbb{Q} \cong K^2$$

as a vector space over $K = \mathcal{O}_D \otimes \mathbb{Q}$. Using the two Galois conjugate embeddings $K \to \mathbb{R}$, we obtain an orthogonal splitting

$$H_1(Z_g, \mathbb{R}) = L \otimes \mathbb{R} = V_1 \oplus V_2$$

such that $k \cdot (C_1, C_2) = (kC_1, k'C_2)$. The projections $(a_i, b_i)$ of $a, b \in L$ to each summand yield bases for $V_i$, which taken together give a standard symplectic basis for $H_1(Z_g, \mathbb{R})$. (Note that $(a_i, b_i)$ is generally not an integral symplectic basis; indeed, when $K$ is a field, the elements $(a_i, b_i)$ do not even lie in $H_1(Z_g, \mathbb{Q})$.)

Let $S^D_i \subset H^1(Z_g, \mathbb{R})$ be the span of the dual basis $a_i^*, b_i^*$.

**Theorem 6.2** The ring $\mathcal{O}_D \subset \text{End}(L)$ acts by real multiplication on $\text{Jac}(Y)$ if and only if $Y \in T_g(S^D_1)$.

**Proof.** Since $g = 2$ we have $S^D_2 = (S^D_1)^\perp$, and thus $T_g(S^D_1) = T_g(S^D_2)$. But $\text{Jac}(Y)$ has real multiplication iff $S^D_1$ and $S^D_2$ are complex subspaces of $H^1(Y, \mathbb{R}) \cong \Omega(Y)$ so the result follows. (Cf. [Mc4, Lemma 7.4].)
Sections. Let $E_D = X_D - X_D(1)$ denote the space of Jacobians in $X_D$, and $\bar{E}_D = \mathbb{H} \times \mathbb{H} - \bar{X}_D(1)$ its preimage in the universal cover. (The notation comes from [Mc7, §4], where we consider the space of eigenforms $\Omega E_D$ as a closed, $\text{GL}_2^+(\mathbb{R})$-invariant subset of $\Omega \mathcal{M}_g$.)

By the preceding result, the Jacobian of any $Y \in T_g(S_D^1)$ is an Abelian variety with real multiplication. Moreover, the marking of $Y$ determines a marking

$$L \cong H_1(Y, \mathbb{Z}) \cong H_1(\text{Jac}(Y), \mathbb{Z})$$

of its Jacobian, and thus a map

$$\text{Jac} : T_g(S_D^1) \to \bar{E}_D = \bar{X}_D - \bar{X}_D(1).$$

The basis $(a_i, b_i)$ yields a pair of normalized forms $\omega_1, \omega_2 \in \Omega(Y)$. Similarly, we have a pair of normalized eigenforms $\eta_1, \eta_2 \in \Omega(A_\tau)$ for each $\tau \in \bar{X}_D$, characterized by (3.2). Under the identification $\Omega(Y) = \Omega(Jac(Y))$, we find:

**Theorem 6.3** The forms $\omega_i$ and $\eta_i$ are equal for any $Y \in T_g(S_D^1)$. Thus $\text{Jac}(Y) = A_{(\tau_1, \tau_2)}$, where

$$\begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} = \tau_{ij}(Y) = \left( \int_{b_i} \omega_j \right).$$  \hspace{1cm} (6.1)

**Proof.** The period map $\phi_\tau : L \to \mathbb{C}^2$ for $A_\tau = \text{Jac}(Y)$ is given by

$$\phi_\tau(C) = \left( \int_C \eta_1, \int_C \eta_2 \right) = (x_1 + x_2 \tau_1, x'_1 + x'_2 \tau_2),$$

where $C = (x_1, x_2) \in \mathcal{O}_D \oplus \mathcal{O}_D'$; in particular, we have

$$\phi_\tau(a) = \phi_\tau(1, 0) = (1, 1).$$

Since $\phi_\tau$ diagonalizes the action of $K$, we also have

$$\phi_\tau(C) = \left( \int_{C_1} \eta_1, \int_{C_2} \eta_2 \right)$$

for any $C = C_1 + C_2 \in L \otimes \mathbb{R} = V_1 \oplus V_2$. Setting $C = a$, this implies $\phi_\tau(a_1) = (1, 0)$ and $\phi_\tau(a_2) = (0, 1)$; thus $\int_a \eta_j = \delta_{ij}$, and therefore $\eta_i = \omega_i$ for $i = 1, 2$. Similarly, we have

$$\phi_\tau(b) = (\tau_1, \tau_2) = (\tau_{11}, \tau_{22}),$$

which implies $Y$ and $A_\tau$ are related by (6.1).
Corollary 6.4 We have a commutative diagram

\[
\begin{array}{ccc}
\mathcal{T}_g(S_1^D) & \xrightarrow{\text{Jac}} & \tilde{E}_D \\
\tau_{11} \downarrow & & \tau_1 \downarrow \\
\mathbb{H} & & 
\end{array}
\]

Proof of Theorem 6.1. Using the Torelli theorem, it follows easily that \( \text{Jac} : \mathcal{T}_g(S_1^D) \to \tilde{E}_D \) is a holomorphic covering map. Since \( \mathbb{H} \) is simply-connected, any section \( s \) of \( \tau_1 \) lifts to a section \( \text{Jac}^{-1} \circ s \) of \( \tau_{11} \). Thus Theorem 5.1 immediately implies Theorem 6.1. \( \blacksquare \)

7 Holomorphic motions

In this section we use the theory of holomorphic motions to define and characterize the foliation \( \mathcal{F}_D \).

Holomorphic motions. Given a set \( E \subset \hat{\mathbb{C}} \) and a basepoint \( s \in \mathbb{H} \), a holomorphic motion of \( E \) over \( (\mathbb{H}, s) \) is a family of injective maps

\[
F_t : E \to \hat{\mathbb{C}}, \quad t \in \mathbb{H},
\]

such that \( F_s(z) = z \) and \( F_t(z) \) is a holomorphic function of \( t \).

A holomorphic motion of \( E \) has a unique extension to a holomorphic motion of its closure \( \overline{E} \); and each map \( F_t : E \to \hat{\mathbb{C}} \) extends to a quasiconformal homeomorphism of the sphere. In particular, \( F_t|\text{int}(E) \) is quasiconformal (see e.g. [Dou]).

These properties imply:

Theorem 7.1 Let \( P \) be a partition of \( \mathbb{H} \times \mathbb{H} \) into disjoint graphs of holomorphic functions. Then:

1. \( P \) is the set of leaves of a transversally quasiconformal foliation \( \mathcal{F} \) of \( \mathbb{H} \times \mathbb{H} \); and
2. If we adjoin the graphs of the constant functions \( f : \mathbb{H} \to \partial \mathbb{H} \) to \( P \), we obtain a continuous foliation of \( \mathbb{H} \times \mathbb{H} \).

The foliation \( \mathcal{F}_D \). Recall that every component of \( \tilde{X}_D(1) \subset \mathbb{H} \times \mathbb{H} \) is the graph of a Möbius transformation. By Theorem 6.1, there is a unique partition of \( \mathbb{H} \times \mathbb{H} - \tilde{X}_D(1) \) into the graphs of holomorphic maps as well.
Taken together, these graphs form the leaves of a foliation \( \tilde{\mathcal{F}}_D \) of \( \mathbb{H} \times \mathbb{H} \) by the preceding result. Since \( \tilde{X}_D(1) \) is invariant under \( SL(O_D \oplus O_D^\vee) \), the foliation \( \tilde{\mathcal{F}}_D \) descends to a foliation \( \mathcal{F}_D \) of \( X_D \).

To characterize \( \mathcal{F}_D \), recall that the surface \( X_D \) admits a holomorphic involution \( \iota(\tau_1, \tau_2) = (\tau_2, \tau_1) \) which preserves \( X_D(1) \).

**Theorem 7.2** The only leaves shared by \( \mathcal{F}_D \) and \( \iota(\mathcal{F}_D) \) are the curves in \( X_D(1) \).

**Proof.** Let \( f : \mathbb{H} \to \mathbb{H} \) be a holomorphic function whose graph \( F \) is both a leaf of \( \tilde{\mathcal{F}}_D \) and \( \iota(\tilde{\mathcal{F}}_D) \). Then \( \iota(F) \) is also a graph, so \( f \) is an isometry. But if \( F \cap \tilde{X}_D(1) = \emptyset \), then \( F \) lifts to a leaf of the foliation \( F_1 \) of Teichmüller space, and hence \( f \) is a contraction by [Mc4, Thm. 4.2].

**Corollary 7.3** The only leaves of \( \tilde{\mathcal{F}}_D \) that are graphs of Möbius transformations are those belonging to \( \tilde{X}_D(1) \).

**Complex geodesics.** Let us say \( \mathcal{F} \) is a foliation by complex geodesics if each leaf is a hyperbolic Riemann surface, isometrically immersed for the Kobayashi metric. We can then characterize \( \mathcal{F}_D \) as follows.

**Theorem 7.4** Up to the action of \( \iota \), \( \mathcal{F}_D \) is the unique extension of the lamination \( X_D(1) \) to a foliation of \( X_D \) by complex geodesics.

**Proof.** Let \( \mathcal{F} \) be a foliation by complex geodesics extending \( X_D(1) \). Then every leaf of its lift \( \tilde{\mathcal{F}} \) to \( \tilde{X}_D \) is a Kobayashi geodesic for \( \mathbb{H} \times \mathbb{H} \). But a complex geodesic in \( \mathbb{H} \times \mathbb{H} \) is either the graph of a holomorphic function or its inverse, so every leaf belongs to either \( \tilde{\mathcal{F}}_D \) or \( \iota(\tilde{\mathcal{F}}_D) \). Consequently every leaf of \( \mathcal{F} \) is a leaf of \( \mathcal{F}_D \) or \( \iota(\mathcal{F}_D) \). Since these foliations have no leaves in common on the open set \( U = X_D - X_D(1) \), \( \mathcal{F} \) coincides with one or the other.

**Stable curves.** The Abelian varieties \( E \times F \) in \( X_D(1) \) are the Jacobians of certain stable curves with real multiplication, namely the nodal curves \( Y = E \vee F \) obtained by gluing \( E \) to \( F \) at a single point. If we adjoin these stable curves to \( M_2 \), we obtain a partial compactification \( M_2^* \) which maps isomorphically to \( A_2 \). The locus \( X_D(1) \) can then be regarded as the projection to \( X_D \) of a finite set of \( GL_2^+(\mathbb{R}) \) orbits in \( \Omega M_2^* \), giving another proof that it is a lamination.
8 Quasiconformal dynamics

In this section we use the relative period map $\rho = \int_{y_1}^{y_2} \eta_1$ to define a meromorphic quadratic differential $q = (d\rho)^2$ transverse to $\mathcal{F}_D$. We then show the transverse dynamics of $\mathcal{F}_D$ is given by Teichmüller mappings relative to $q$.

**Absolute periods.** The level sets of $\tau_1$ form the leaves of a holomorphic foliation $\tilde{A}_D$ on $\mathbb{H} \times \mathbb{H}$ which covers foliation $A_D$ of $X_D$. By (3.2), every $\tau = (\tau_1, \tau_2)$ determines a pair of eigenforms $\eta_1, \eta_2 \in \Omega(A_\tau)$ such that the absolute periods

$$\int_C \eta_1, \quad C \in H_1(A_\tau, \mathbb{Z})$$

are constant along the leaves of $\tilde{A}_D$. Since every leaf of $\tilde{F}_D$ is the graph of a function $f : \mathbb{H} \rightarrow \mathbb{H}$, we have:

**Theorem 8.1** The foliation $A_D$ is transverse to $\mathcal{F}_D$.

**The Weierstrass curve.** Recall that $E_D \subset X_D$ denotes the locus of Jacobians with real multiplication by $O_D$. For $[A_\tau] = \text{Jac}(Y) \in E_D$ we can regard the eigenforms $\eta_1, \eta_2$ as holomorphic 1-forms in $\Omega(Y) \cong \Omega(A_\tau)$.

Let $W_D \subset E_D$ denote the locus where $\eta_1$ has a double zero on $Y$. By [Mc5] we have:

**Theorem 8.2** The locus $W_D$ is an algebraic curve with one or two irreducible components, each of which is a leaf of $\mathcal{F}_D$.

We refer to $W_D$ as the Weierstrass curve, since $\eta_1$ vanishes at a Weierstrass point of $Y$.

**Relative periods.** Let $E_D(1, 1) = X_D - (W_D \cup X_D(1))$ denote the Zariski open set where $\eta_1$ has a pair of simple zeros, and let $\tilde{E}_D(1, 1)$ be its preimage in the universal cover $\tilde{X}_D$. Let

$$\mathbb{H}_s = \{s\} \times \mathbb{H} \subset \mathbb{H} \times \mathbb{H},$$

and let $\mathbb{H}_s^* = \mathbb{H}_s \cap \tilde{E}_D(1, 1)$.

For each $\tau \in \mathbb{H}_s^*$, let $y_1, y_2$ denote the zeros of the associated form $\eta_1 \in \Omega(Y)$. We can then define the (multivalued) relative period map $\rho_s : \mathbb{H}_s^* \rightarrow \mathbb{C}$ by

$$\rho_s(\tau) = \int_{y_1}^{y_2} \eta_1.$$
To make $\rho_s(\tau)$ single-valued, we must (locally) choose (i) an ordering of the zeros $y_1$ and $y_2$, and (ii) a path on $Y$ connecting them.

**Quadratic differentials.** Let $z$ be a local coordinate on $\mathbb{H}_s$, and recall that the absolute periods of $\eta_1$ are constant along $\mathbb{H}_s$. Thus if we change the choice of path from $y_1$ to $y_2$, the derivative $d\rho/dz$ remains the same; and if we interchange $y_1$ and $y_2$, it changes only by sign. Thus the quadratic differential

$$q = (d\rho/dz)^2 \, dz^2$$

is globally well-defined on $\mathbb{H}_s^*$.

**Theorem 8.3** The form $q$ extends to a meromorphic quadratic differential on $\mathbb{H}_s$, with simple zeros where $\mathbb{H}_s$ meets $\bar{W}_D$, and simple poles where it meets $\bar{X}_D(1)$.

**Proof.** It is a general result that the period map provides holomorphic local coordinates on any stratum of $\Omega \mathcal{M}_g$ (see [V2], [MS, Lemma 1.1], [KZ]). Thus $\rho_s|_{\mathbb{H}_s^*}$ is holomorphic with $d\rho_s \neq 0$, and hence $q|_{\mathbb{H}_s^*}$ is a nowhere vanishing holomorphic quadratic differential.

To see $q$ acquires a simple zero when $\eta_1$ acquires a double zero, note that the relative period map $\rho(t) = \int_{\sqrt{t}}^{\sqrt{t}} (z^2 - t) \, dz = (-4/3)t^{3/2}$ of the local model $\eta_t = (z^2 - t) \, dz$ satisfies $(d\rho/dt)^2 = 4t$. Similarly, a point of $\mathbb{H}_s \cap \bar{X}_D(1)$ is locally modeled by the family of connected sums

$$(Y_t, \eta_t) = (E_1, \omega_1) \# (E_2, \omega_2),$$

with $I = [0, \rho(t)] = [0, \pm \sqrt{t}]$. Since $(d\rho/dt)^2 = 1/(4t)$, at these points $q$ has simple poles.

See [Mc7, §6] for more on connected sums.

**Teichmüller maps.** Now let $f : \mathbb{H}_s \to \mathbb{H}_t$ be a quasiconformal map. We say $f$ is a Teichmüller map, relative to a holomorphic quadratic differential $q$, if its complex dilatation satisfies

$$\mu(f) = \left( \frac{\partial f / \partial \overline{z}}{\partial f / \partial z} \right) \cdot \frac{dz}{d\overline{z}} = \alpha \frac{\overline{q}}{|q|}$$
for some $\alpha \in \mathbb{C}^*$. This is equivalent to the condition that $w = f(z)$ is real-linear in local coordinates where $q = dz^2$ and $dw^2$ respectively. In such charts we can write

$$w = w_0 + D_q(f) \cdot z,$$

with $D_q(f) \in \text{SL}_2(\mathbb{R})$. We refer to $D_q(f)$ as the linear part of $f$; it is only well-defined up to sign, since $z \mapsto -z$ preserves $dz^2$.

**Theorem 8.4** Given $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(\mathcal{O}_D \oplus \mathcal{O}_D^\vee)$ and $s \in \mathbb{H}$, let $\mathbb{H}_t = g(\mathbb{H}_s)$. Then the linear part of $g : \mathbb{H}_s \to \mathbb{H}_t$ is given by $D_q(g) : z = (d - cs)^{-1}z$.

**Proof.** Since the Riemann surfaces $Y$ at corresponding points of $\mathbb{H}_s$ and $\mathbb{H}_t$ differ only by marking, the relative period maps $\rho_s$ and $\rho_t$ differ only by the normalization of $\eta_1$. This discrepancy is accounted for by equation (3.4), which gives $\rho_t/\rho_s = \chi(g, s) = (d - cs)^{-1}$. Since the coordinates $\rho_s$ and $\rho_t$ linearize $q$, the map $D_q(g)$ is given by multiplication by $(d - cs)^{-1}$.

Now let $C_{st} : \mathbb{H}_s \to \mathbb{H}_t$ be the unique map such that $z$ and $C_{st}(z)$ lie on the same leaf of $\tilde{F}_D$.

**Theorem 8.5** The linear part of $C_{st}$ is given by $D_q(C_{st}) = A_tA_s^{-1}$, where $A_u = \begin{pmatrix} 1 & \text{Re}(u) \\ 0 & \text{Im}(u) \end{pmatrix} \in \text{PSL}_2(\mathbb{R})$.

**Proof.** By the definition of $\mathcal{F}_D$, the forms $\eta_1$ at corresponding points of $\mathbb{H}_s$ and $\mathbb{H}_t$ are related by some element $B \in \text{GL}_2^+(\mathbb{R})$ acting on $\Omega T_R$. Thus $\rho_t = B \circ \rho_s$ and therefore $D_q(C_{st}) = B$. Since the action of $B$ on the absolute periods of $\eta_1$ satisfies

$$B(\mathcal{O}_D \oplus \mathcal{O}_D^\vee s) = \mathcal{O}_D \oplus \mathcal{O}_D^\vee t$$

(in the sense of equation (3.1)), we have $B(1) = 1$ and $B(s) = t$, and thus $B = A_tA_s^{-1}$ as above.

**Dynamics.** Every leaf of $\tilde{F}_D$ meets the transversal $\mathbb{H}_s$ in a single point. Thus the action of $g \in \text{SL}(\mathcal{O}_D \oplus \mathcal{O}_D^\vee)$ on the space of leaves determines a holonomy map

$$\phi_g : \mathbb{H}_s \to \mathbb{H}_s,$$

characterized by the property that $(s, \phi_g(z))$ lies on the same leaf as $g(s, z)$.

**Theorem 8.6** The group $\text{SL}(\mathcal{O}_D \oplus \mathcal{O}_D^\vee)$ acts on $\mathbb{H}_s$ by Teichmüller mappings, satisfying $D_q(\phi_g) = g$ in the case $s = i$. 

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As usual we regard $g$ as a real matrix using $\iota_1 : K \to \mathbb{R}$.

**Proof.** Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and $t = (as - b)/(cs + d)$; then $\mathbb{H}_t = g(\mathbb{H}_s)$.

Since $\phi_g(z)$ is obtained from $g(s, z)$ by combing it along the leaves of $\tilde{F}_D$ back into $\mathbb{H}_s$, we have $\phi_g(s, z) = C_{ts}(g(s, z))$. Thus the chain rule implies

$$D_q(\phi_g) \cdot z = B \cdot z = A_s \circ A_t^{-1}(z/(cs + d)).$$

Now assume $s = i$. Then we have $B(ai - b) = A_t^{-1}(t) = i$ and $B(-ci + d) = A_t^{-1}(1) = 1$; therefore $B^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ and thus $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = g$. ■

**Corollary 8.7** The foliation $F_D$ carries a natural transverse invariant measure.

**Proof.** Since $\det D_q(\phi_g) = 1$ for all $g$, the form $|q|$ gives a holonomy-invariant measure on the transversal $\mathbb{H}_s$.

Finally we show that, although $\phi_g|\mathbb{H}_s$ is quasiconformal, its continuous extension to $\partial \mathbb{H}_s$ is a Möbius transformation.

**Theorem 8.8** For any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(\mathcal{O}_D \oplus \mathcal{O}_D^\vee)$ and $z \in \partial \mathbb{H}_s$, we have

$$\phi_g(z) = (a'z - b')/(-c'z + d').$$

**Proof.** By Theorem 7.1, the combing maps $C_{st}$ extend to the identity on $\partial \mathbb{H}_s$. Thus $(t, \phi_g(z)) = g(s, z)$, and the result follows from equation (3.3). ■

Note: if we use the transversal $\mathbb{H}_t$ instead of $\mathbb{H}_s$, the holonomy simply changes by conjugation by $C_{st}$.

**9 Further results**

In this section we summarize related results on the density of leaves, isoperiodic forms, holomorphic motions and iterated rational maps.

**I. Density of leaves.** By [Mc7], the closure of the complex geodesic $f : \mathbb{H} \to \mathcal{M}_2$ generated by a holomorphic 1-form is either an algebraic curve, a Hilbert modular surface or the whole moduli space. Since the leaves of $F_D$ are examples of such complex geodesics, we obtain:
Theorem 9.1 Every leaf of $\mathcal{F}_D$ is either a closed algebraic curve, or a dense subset of $X_D$.

It is easy to see that the union of the closed leaves is dense when $D = d^2$. On the other hand, the classification of Teichmüller curves in [Mc5] and [Mc6] implies:

Theorem 9.2 If $D$ is not a square, then $\mathcal{F}_D$ has only finitely many closed leaves. These consist of the components of $W_D \cup X_D(1)$ and, when $D = 5$, the Teichmüller curve generated by the regular decagon.

II. Isoperiodic forms. Next we discuss interactions between the foliations $\mathcal{F}_D$ and $\mathcal{A}_D$. When $D = d^2$ is a square, the surface $X_D$ is finitely covered by a product, and hence every leaf of $\mathcal{A}_D$ is closed.

Theorem 9.3 If $D$ is not a square, then every leaf $L$ of $\mathcal{A}_D$ is dense in $X_D$, and $L \cap F$ is dense in $F$ for every leaf $F$ of $\mathcal{F}_D$.

Proof. The first result follows from the fact that $\text{SL}(\mathcal{O}_D \oplus \mathcal{O}_D')$ is a dense subgroup of $\text{SL}_2(\mathbb{R})$, and the second follows from the first by transversality of $\mathcal{A}_D$ and $\mathcal{F}_D$.

Let us say a pair of 1-forms $(Y_i, \omega_i) \in \Omega M_g$ are isoperiodic if there is a symplectic isomorphism

$$\phi : H_1(Y_1, \mathbb{Z}) \to H_1(Y_2, \mathbb{Z})$$

such that the period maps

$$I(\omega_i) : H_1(Y_i, \mathbb{Z}) \to \mathbb{C}$$

satisfy $I(\omega_1) = I(\omega_2) \circ \phi$. Since the absolute periods of $\eta_1$ are constant along the leaves of $\mathcal{A}_D$, from the preceding result we obtain:

Corollary 9.4 The $\text{SL}_2(\mathbb{R})$-orbit of any eigenform for real multiplication by $\mathcal{O}_D$, $D \neq d^2$, contains infinitely many isoperiodic forms.

For a concrete example, let $Q \subset \mathbb{C}$ be a regular octagon containing $[0, 1]$ as an edge. Identifying opposite sides of $Q$, we obtain the octagonal form

$$(Y, \omega) = (Q, dz)/ \sim$$
of genus two.

Let \( \mathbb{Z}[\zeta] \subset \mathbb{C} \) denote the ring generated by \( \zeta = (1+i)/\sqrt{2} = \exp(2\pi i/8) \), equipped with the symplectic form

\[
\langle z_1, z_2 \rangle = \text{Tr}_Q^\mathbb{Q}((\zeta + \zeta^2 + \zeta^3)z_1z_2/4).
\]

Then it is easy to check that:

1. The octagonal form \( \omega \) has a single zero of order 2, and

2. Its period map \( I(\omega) \) sends \( H_1(Y, \mathbb{Z}) \) to \( \mathbb{Z}[\zeta] \) by a symplectic isomorphism.

However, these two properties do not determine \((Y, \omega)\) uniquely. Indeed, \( \omega \) is an eigenform for real multiplication by \( \mathcal{O}_8 \), so the preceding Corollary ensures there are infinitely many isoperiodic forms \((Y_i, \omega_i)\) in its \( \text{SL}_2(\mathbb{R}) \) orbit. In other words we have:

**Corollary 9.5** There are infinite many fake octagonal forms in \( \Omega \mathcal{M}_2 \).

Note that the forms \((Y_i, \omega_i)\) cannot be distinguished by their relative periods either, since they all have double zeros.

A similar statement can be formulated for the pentagonal form on the curve \( y^2 = x^5 - 1 \).

### III. Top-speed motions.

Let \( F_t : E \to \mathbb{H} \) be a holomorphic motion of \( E \subset \mathbb{H} \) over \((\mathbb{H}, s)\). By the Schwarz lemma, we have \( \|dF_t(z)/dt\| \leq 1 \) with respect to the hyperbolic metric on \( \mathbb{H} \). Let us say \( F_t \) is a top-speed holomorphic motion if equality holds everywhere; equivalently, if \( t \mapsto F_t(z) \) is an isometry of \( \mathbb{H} \) for every \( z \in E \).

A top-speed holomorphic motion is **maximal** if it cannot be extended to a top-speed motion of a larger set \( E' \supset E \).

**Theorem 9.6** For any discriminant \( D \geq 4 \), the map

\[
F_t(U(s)) = U(t), \quad U \in \Lambda_D
\]

gives a maximal top-speed holomorphic motion of \( E = \Lambda_D \cdot s \) over \((\mathbb{H}, s)\).

**Proof.** Let \( t \mapsto f(t) = F_t(z) \) be an extension of the motion to a point \( z \notin E \). Then the graph of \( f \) is a leaf of \( \tilde{\mathcal{F}}_D \), since it is disjoint from \( \tilde{X}_D(1) \). But the only leaves that are graphs of Möbius transformations are those in \( \tilde{X}_D(1) \), by Corollary 7.3. \( \blacksquare \)
Corollary 9.7 The group $\Gamma(2) = \{ A \in \text{SL}_2(\mathbb{Z}) : A \equiv I \mod 2 \}$ gives a maximal top-speed holomorphic motion of $E = \Gamma(2) \cdot s$ over $(\mathbb{H}, s)$.

Proof. We have $\Gamma(2) = g \Lambda_4 g^{-1}$, where $g = \begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix}$ (Theorem 4.4).

IV. Iterated rational maps. Finally we explain how the foliation $\mathcal{F}_4$ of $X_4$ arises in complex dynamics.

First recall that the moduli space of elliptic curves can be described as the quotient orbifold $\mathcal{M}_1 = \tilde{\mathcal{M}}_1 / S_3$, where

$$\tilde{\mathcal{M}}_1 = \mathbb{H} / \Gamma(2) \cong \mathbb{C} - \{0, 1\}.$$ 

The deck group $S_3$ also acts diagonally on $\tilde{\mathcal{M}}_1 \times \tilde{\mathcal{M}}_1$, preserving the diagonal $\Delta$.

Theorem 9.8 For $D = 4$, we have $(X_D, X_D(1)) \cong (\tilde{\mathcal{M}}_1 \times \tilde{\mathcal{M}}_1, \Delta) / S_3$.

Proof. Since $\mathcal{O}_4 = (1/2) \mathcal{O}_4$, the surface $X_4$ is isomorphic to $(\mathbb{H} \times \mathbb{H}) / \text{SL}_2(\mathcal{O}_4)$. In these coordinates we have $\Lambda_4 = \Gamma(2)$. Since

$$\text{SL}_2(\mathcal{O}_4) \cong \{ (A_1, A_2) \in \text{SL}_2(\mathbb{Z}) : A_1 \equiv A_2 \mod 2 \}$$

contains $\Gamma(2) \times \Gamma(2)$ as a subgroup of index 6, the result follows.

Now consider, for each $t \in \tilde{\mathcal{M}}_1$, the elliptic curve $E_t$ defined by $y^2 = x(x - 1)(x - t)$. There is a unique rational map $f_t : \mathbb{P}^1 \to \mathbb{P}^1$ such that

$$x(2P) = f_t(x(P))$$

with respect to the usual group law on $E_t$. Indeed, using the fact that $-2P$ lies on the tangent line to $E_t$ at $P$, we find

$$f_t(z) = \frac{(z^2 - t)^2}{4z(z - 1)(z - t)}.$$

Note that the postcritical set

$$P(f_t) = \bigcup \{ f_t^n(z) : n > 0, f_t'(z) = 0 \}$$

coincides with the branch locus $\{0, 1, t, \infty\}$ of the map $x : E_t \to \mathbb{P}^1$.

The rational maps $f_t(z)$ form a stable family of Lattès examples. It is well-known that the Julia set of any Lattès example is the whole Riemann sphere; and that in any stable family, the Julia set varies by a holomorphic motion respecting the dynamics (see e.g. [MSS], [Mc1, Ch. 4], [Mil].)
Theorem 9.9  As \( t \) varies in \( \widetilde{M}_1 \), the holomorphic motion of \( J(f_t) \) sweeps out the lift of the foliation \( F_4 \) to the covering space \( \widetilde{M}_1 \times \widetilde{M}_1 \) of \( X_4 \).

Proof. Let \( \mathcal{G} \) be the foliation of \( \widetilde{M}_1 \times \mathbb{P}^1 \) swept out by \( J(f_t) \). Since the holomorphic motion respects the dynamics, it preserves the post-critical set, and thus the leaves of \( \mathcal{G} \) include the loci \( z = 0, 1, \infty \) as well as the diagonal \( t = z \). In particular, \( \mathcal{G} \) restricts to a foliation of the finite cover \( \widetilde{M}_1 \times \widetilde{M}_1 - \Delta \) of \( X_4 - X_4(1) \). Since each leaf of \( \mathcal{G} \) lifts to the graph of a holomorphic function in the universal cover \( \mathbb{H} \times \mathbb{H} \), it lies over a leaf of \( \mathcal{F}_D \) by the uniqueness part of Theorem 1.2.

Algebraic curves. The loci \( f^n_t(z) = \infty \) form a dense set of algebraic leaves of \( \mathcal{G} \) that can easily be computed inductively. The real points of these curves are graphed in Figure 1; thus the figure depicts the lift of \( \mathcal{F}_4 \) to the finite cover \( \widetilde{M}_1 \times \widetilde{M}_1 \) of \( X_4 \).

References


