Teichmüller Curves in Genus Two: Torsion Divisors and Ratios of Sines

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Teichmüller curves in genus two: Torsion divisors and ratios of sines

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1 Introduction

Let $\mathcal{M}_g$ denote the moduli space of Riemann surfaces of genus g, and $\Omega\mathcal{M}_g \to \mathcal{M}_g$ the bundle of pairs (X, ω), where $\omega \neq 0$ is a holomorphic 1-form on $X \in \mathcal{M}_g$.

There is a natural action of $\mathrm{GL}_2^+(\mathbb{R})$ on $\Omega\mathcal{M}_g$, whose orbits project to complex geodesics in $\mathcal{M}_g$. The projection of an orbit is almost always dense. If the stabilizer $\mathrm{SL}(X, \omega) \subset \mathrm{SL}_2(\mathbb{R})$ of a given form is a lattice, however, then the projection of its orbit gives a closed, algebraic *Teichmüller curve*

$$f : V = \mathbb{H}/\,\mathrm{SL}(X, \omega) \to \mathcal{M}_g.$$

A Teichmüller curve is *primitive* if it does not arise from a curve in $\mathcal{M}_h$, $h < g$, via a covering construction. Our main result completes the classification of primitive Teichmüller curves in genus two.

Theorem 1.1 *The decagon form $\omega = dx/y$ on the curve $y^2 = x(x^5 - 1)$ generates the only primitive Teichmüller curve $f : V \to \mathcal{M}_2$ coming from a form with simple zeros.*

In contrast to this uniqueness, in genus two there are countably many primitive Teichmüller curves coming from forms with double zeros. Each such curve is uniquely determined by its discriminant and its spin invariant [Mc3].

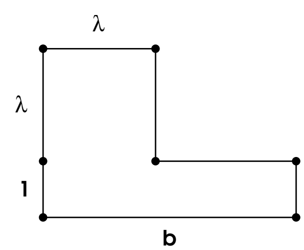

Figure 1. The billiard table $L(b, e)$, with $\lambda = (e + \sqrt{e^2 + 4b})/2$.

Billiards. As a corollary of Theorem 1.1, we obtain a related classification of billiards in polygons.

Let $P \subset \mathbb{C}$ be a polygon with angles in $\pi\mathbb{Q}$. Via an unfolding construction, P determines a holomorphic 1-form $(X, \omega) \in \Omega\mathcal{M}_g$ such that billiard trajectories in P correspond to geodesics on $(X, |\omega|)$. We refer to g as the *genus* of P.

For example, let P denote the right triangle with smallest internal angle $\pi/10$. Then P unfolds, under repeated reflection through its long sides, to give a tiling of the regular decagon Q by 20 triangles. The associated 1-form (X, ω) is obtained from (Q, dz) by identifying opposite sides; it has genus two, and agrees with the decagon form cited above.

We say P and P' are *equivalent* if $A \cdot (X, \omega) = (X', \omega')$ for some $A \in \mathrm{GL}_2^+(\mathbb{R})$. Equivalence is a form of scissors congruence that respects the billiard dynamics.

Veech showed that if $\mathrm{SL}(X, \omega)$ is a lattice, then the billiard flow in P is *dynamically optimal*: every trajectory is either periodic or uniformly distributed, and the number of types of periodic trajectories of length $\leq L$ is asymptotic to $c(P) \cdot L^2$; see [V].

Theorem 1.2 *Let $P \subset \mathbb{C}$ be a polygon of genus two. Then the billiard flow in P is dynamically optimal if and only if P is equivalent to:*

- *A polygon tiled by congruent 30-60-90 triangles or isoceles right triangles;*
- *The L-shaped polygon $L(b, e)$, for some $b, e \in \mathbb{Z}$ (Figure 1); or*
- *The 18-72-90 triangle that tiles the regular decagon.*

Otherwise, there is a billiard trajectory in P that is neither periodic nor dense.

Proof. By [Mc4, Thm. 7.1], either $\mathrm{SL}(X, \omega)$ is a lattice, or P has a trajectory that is neither periodic nor dense.

Assume we are in the former case. Then P is dynamically optimal, and (X, ω) generates a Teichmüller curve $f : V \to \mathcal{M}_2$.

If ω has a double zero, then P is equivalent to an L-shaped polygon $L(b, e)$ [Mc3, Cor. 1.3], and all such polygons are optimal

If ω has a pair of simple zeros and the Teichmüller curve it generates is primitive, then P is equivalent to the 18-72-90 triangle by Theorem 1.1.

Otherwise, ω is the pullback of a form of genus one via a covering $\pi : X \to \mathbb{C}/\Lambda$ branched over just one point. We can assume Λ is the square or hexagonal lattice, and thus P is equivalent to a polygon tiled as above. (Examples of this type include rectangles with rational barriers [EMS].) Conversely $\mathrm{SL}(X, \omega)$ is commensurable to $\mathrm{SL}_2(\mathbb{Z})$ whenever P can be tiled in this way, and thus P is optimal. ∎

Infinitely generated groups. Theorem 1.1 also yields a large family of examples where $\mathrm{SL}(X, \omega)$ is infinitely generated.

Theorem 1.3 *Let (X, ω) be a form of genus two with simple zeros, such that the trace field of $\mathrm{SL}(X, \omega)$ is irrational. Then either:*

- *(X, ω) is equivalent to the decagon form, or*
- *$\mathrm{SL}(X, \omega)$ is an infinitely generated group.*

This follows from [Mc2, §10].

Torsion divisors. The proof of our main result relies on:

Theorem 1.4 (Möller) *Suppose $\mathrm{SL}(X, \omega)$ is a lattice with trace field K, and $\deg(K/\mathbb{Q})$ agrees with the genus of X. Then the divisor $[P - Q]$ has finite order in $\mathrm{Jac}(X)$, whenever P and Q are two zeros of ω.*

See [Mo, Cor 3.4]. The proof also uses an elementary fact about cyclotomic fields (§2):

Theorem 1.5 *There are only 15 pairs of rational numbers* $0 < \alpha < \beta < 1/2$ *such that* $\sin(\pi\alpha)/\sin(\pi\beta)$ *is a quadratic irrational.*

To see how these results are related to Theorem 1.1, let $f : V \to \mathcal{M}_2$ be a primitive Teichmüller curve generated by a form $(X, \omega) \in \Omega\mathcal{M}_2$ with simple zeros. Normalizing by the action of $\mathrm{GL}_2^+(\mathbb{R})$, we can assume

$$(X, \omega) = (E_1, \omega_1) \underset{I}{\#} (E_2, \omega_2)$$

is the connected sum of two forms of genus one, with period lattices given by

$$\begin{aligned} \mathrm{Per}(\omega_1) &= \mathbb{Z}(\lambda, 0) \oplus \mathbb{Z}(0, \lambda), \quad \lambda = (e + \sqrt{e^2 + 4bc})/2, \quad \text{and} \\ \mathrm{Per}(\omega_2) &= \mathbb{Z}(b, 0) \oplus \mathbb{Z}(a, c) \end{aligned}$$

for suitable integers (a, b, c, e) (see §4). We can then use the diagonal matrices $A_s = \left(\begin{smallmatrix} 1 & 0 \\ 0 & s \end{smallmatrix}\right)$ to push (X, ω) to the boundary of moduli space, and obtain a limiting stable Abelian differential $(Y, dx/y)$. The limit Y is a stable curve of genus zero with two nodes, that can be expressed in the form

$$y^2 = (1 - x^2)(x - A)^2(x - B)^2$$

with $-1 < A < B < 1$. (The nodal curve that arises from the decagon form, determined in §5, is shown in Figure 2.)

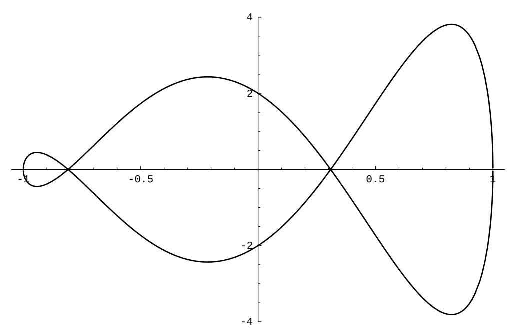

Figure 2. A nodal curve in the boundary of $\mathcal{M}_2$.

Writing $(A, B) = (-\cos \pi\alpha, \cos \pi\beta)$ with $0 < \alpha, \beta < 1$, we find

$$\frac{\sin(\pi\alpha)}{\sin(\pi\beta)} = \frac{\lambda}{b} \tag{1.1}$$

by a residue calculation (§4). On the other hand, by Theorem 1.4 the divisor $D_\infty = P - Q$ is torsion in $\mathrm{Jac}(Y)$, where $(dx/y) = P + Q$ consists of the two points in Y lying over $x = \infty$. But it is straightforward to show if D_∞ is torsion in $\mathrm{Jac}(Y)$, then α and β are rational (§2). Thus (1.1) must agree with one of the 15 quadratic irrational sine ratios occurring in Theorem 2.1.

This fact radically constrains the possibilities for (X, ω), leading to 22 cases listed in §6. Only the decagon form (which comes from $\sin(\pi/5)/\sin(2\pi/5) = (-1+\sqrt{5})/2$) passes a further rationality test, completing the proof.

Curves on Hilbert modular surfaces. We conclude by formulating a purely algebraic corollary of Theorem 1.1.

Let $D > 0$ be the discriminant of an order $\mathcal{O}_D$ in a real quadratic field. For each $n > 0$, let $\Omega E_D[n] \subset \mathcal{M}_2$ denote the set of forms (X, ω) such that

- $\mathrm{Jac}(X)$ admits real multiplication by $\mathcal{O}_D$, with ω as an eigenform, and
- The divisor $[P - Q]$ has order n in $\mathrm{Jac}(X)$, where P and Q are the zeros of ω.

We denote the projection of $\Omega E_D[n]$ to $\mathcal{M}_2$ by $W_D[n]$. The locus $W_D[1]$ is called the *Weierstrass curve* in [Mc3].

Theorem 1.6 *The variety $W_D[1]$ is 1-dimensional for all D, as is the locus $W_5[5]$. Otherwise $W_D[n]$ is zero-dimensional.*

This result follows from:

Theorem 1.7 *The primitive Teichmüller curves generated by forms of genus two coincide with the 1-dimensional irreducible components of $\bigcup W_D[n]$.*

Proof. Let $\Omega R_D \subset \Omega\mathcal{M}_2$ denote the set of eigenforms (X, ω) for real multiplication by $\mathcal{O}_D$ such that the relative periods of ω lie in the rational span of its absolute periods, i.e.

$$\int_P^Q \omega \in \mathrm{Per}(\omega) \otimes \mathbb{Q}$$

where P and Q are the zeros of ω. The locus ΩR_D is a countable union of $\mathrm{GL}_2^+(\mathbb{R})$-orbits, so its projection $R_D \subset \mathcal{M}_2$ is a countable union of complex geodesics.

By the Abel-Jacobi Theorem, if $n(P - Q)$ is a principal divisor then $n \int_P^Q \xi$ is a period of ξ for every $\xi \in \Omega(X)$. Thus $W_D[n] \subset R_D$, so $\dim W_D[n] \le 1$.

Now let $f : V \to \mathcal{M}_2$ be a primitive Teichmüller curve. Then $f(V)$ is contained in the locus of real multiplication by [Mc1], and $f(V) \subset W_D[n]$ by Theorem 1.4. Conversely, if $V \subset W_D[n]$ is an irreducible curve, then V is also an algebraic component of R. Hence V is a Teichmüller curve, which is primitive because D is not a square. ∎

Proof of Theorem 1.6. As we will see in §5, the curve generated by the decagon form lies in $W_5[5]$. Moreover, the forms in $\Omega W_D[n]$ have simple zeros iff $n > 1$. Thus the theorem is immediate from [Mc3] when $n = 1$, and from Theorem 1.1 when $n > 1$. ∎

It would be interesting to have a similar algebraic description of the primitive Teichmüller curves in higher genus.

Notes and references. Theorem 1.1 was conjectured in [Mc3], and established for all Teichmüller curves with discriminant $D \leq 400$ in [Mc4]. For additional results related to Teichmüller curves in genus two, see [V], [GJ], [EMS], [HS], [HL], and [Ca]. I would like to thank J. Harris and M. Möller for useful discussions.

2 Ratios of sines

In this section we will show:

Theorem 2.1 *For each $d > 0$, there are only finitely many pairs of rational numbers $0 < \alpha < \beta \leq 1/2$ such that*

$$\sin(\pi\alpha)/\sin(\pi\beta)$$

is an algebraic number of degree d over $\mathbb{Q}$.

Lemma 2.2 *Let $f_t(x) = \sin(tx)/(t\sin(x))$ with $t > 1$. If x_1, tx_1 and x_2 all lie in the interval $[0, \pi/2]$, and $f_t(x_1) = f_t(x_2)$, then $x_1 = x_2$.*

Proof. Since $2/\pi \leq \sin(x)/x \leq 1$ for all $x \in [0, \pi/2]$, we have $f_t(x_1) \geq 2/\pi \geq 1/2$, which implies

$$(2/\pi)x_2 \leq \sin(x_2) \leq \frac{\sin(x_2)}{|\sin(tx_2)|} = \frac{1}{|tf_t(x_2)|} = \frac{1}{tf_t(x_1)} \leq 2/t.$$

Thus x_1 and x_2 both belong to the interval $[0, \pi/t]$. But it is easy to see that $f_t(x)$ is strictly decreasing on this interval (with $f_t(0) = 1$ and $f_t(\pi/t) = 0$), so the condition $f_t(x_1) = f_t(x_2)$ implies $x_1 = x_2$. ■

Proof of Theorem 2.1. Fix a degree $d > 0$, and let $A = \pi(a/p)$ and $B = \pi(b/q)$ be rational multiples of π in lowest terms, such that $0 < A < B < \pi/2$ and $\sin(A)/\sin(B)$ has degree d over $\mathbb{Q}$. To show there are only finitely many possibilities for the pair $(a/p, b/q)$, we will show that $n = \operatorname{lcm}(p, q)$ is bounded by a constant N_d depending only on d.

Consider the field extension L/K, where $K = \mathbb{Q}(\sin(A))$ and $L = K(\sin(B))$. Then $\deg(K/\mathbb{Q})$ and $\deg(L/\mathbb{Q})$ are comparable to $\phi(p)$ and $\phi(n)$ respectively, where ϕ is Euler's phi-function. We also have $\deg(L/K) \leq d$, so $\phi(n)/\phi(p) = O(d)$, which implies $(n/p) \leq C_d$ for some constant C_d.

Next, recall that the Galois conjugates of $\sin(A)/\sin(B)$ are given by

$$\mu(k) = \sin(kA)/\sin(kB),$$

where k ranges in

$$(\mathbb{Z}/2n)^* \cong \operatorname{Gal}(\mathbb{Q}(e^{\pi i/n})/\mathbb{Q}).$$

α	β	$\sin(\pi\alpha)/\sin(\pi\beta)$	Trace	Norm
$1/12$	$5/12$	$2-\sqrt{3}$	4	1
$1/10$	$1/2$	$(-1+\sqrt{5})/4$	$-1/2$	$-1/4$
$1/12$	$1/4$	$(-1+\sqrt{3})/2$	-1	$-1/2$
$1/10$	$3/10$	$(3-\sqrt{5})/2$	3	1
$1/8$	$3/8$	$-1+\sqrt{2}$	-2	-1
$1/6$	$1/3$	$\sqrt{3}/3$	0	$-1/3$
$1/6$	$3/10$	$(-1+\sqrt{5})/2$	-1	-1
$1/10$	$1/6$	$(-1+\sqrt{5})/2$	-1	-1
$1/5$	$2/5$	$(-1+\sqrt{5})/2$	-1	-1
$1/6$	$1/4$	$\sqrt{2}/2$	0	$-1/2$
$1/4$	$1/2$	$\sqrt{2}/2$	0	$-1/2$
$1/4$	$5/12$	$-1+\sqrt{3}$	-2	-2
$3/10$	$1/2$	$(1+\sqrt{5})/4$	$1/2$	$-1/4$
$1/4$	$1/3$	$\sqrt{6}/3$	0	$-2/3$
$1/3$	$1/2$	$\sqrt{3}/2$	0	$-3/4$

Table 3. The fifteen quadratic irrational sine ratios.

Thus by replacing $\mu(1)$ with $\mu(k)$ with $ka = 1 \bmod p$, we can assume $a = 1$.

Let t be the least integer greater than 1 such that $\gcd(t, 2n) = 1$ and $\mu(t) = \mu(1)$. We then have

$$\frac{\sin(tA)}{\sin(A)} = \frac{\sin(tB)}{\sin(B)}. \tag{2.1}$$

We also have the bound $t \leq (5 \log(2n))^d$, since there is an integer $1 < k < 5\log(2n)$ relatively prime to n [Mc3, Thm. 9.1], and $\mu(k^i) = \mu(1)$ for some $i \leq d$. This shows

$$0 < tA = \pi ta/p \leq \pi C_d (5\log(2n))^d/n,$$

and therefore $tA < \pi/2$ whenever n exceeds a suitable constant N_d depending only on d.

But if $tA < \pi/2$, then we can apply the preceding Lemma to (2.1) to conclude that $A = B$, a contradiction. Consequently $n \leq N_d$, and therefore the number of sine ratios of degree d over $\mathbb{Q}$ is finite. ∎

Quadratic sine ratios. The only rational sine ratio occurring as above is $\sin(\pi/6)/\sin(\pi/2) = 1/2$. Similarly, a straightforward computation using the bounds of the preceding proof establishes:

Theorem 2.3 *The ratio* $\sin(\pi\alpha)/\sin(\pi\beta)$ *is a quadratic irrational for exactly 15 pairs of rational numbers* $0 < \alpha < \beta \leq 1/2$.

These pairs are listed in Table 3.

For related results, see [Man].

3 Torsion divisors

In this section we review the classical relationship between Pell's equation over $\mathbb{C}[x]$ and torsion divisors on hyperelliptic curves.

Hyperelliptic curves. Let C_0 be the hyperelliptic curve defined by the equation

$$y^2 = p(x),$$

where $p(x) \in \mathbb{C}[x]$ has degree $2g - 2$ and $g \geq 0$. The curve $C_0 \subset \mathbb{C}^2$ extends to a unique complete curve C of arithmetic genus g, with two points $P, Q \in C$ lying over $x = \infty$. The completion is smooth if all the zeros of $p(x)$ are simple; otherwise, it has singularities lying over the multiple zeros of p.

Note that the hyperelliptic involution $(x, y) \mapsto (x, -y)$ interchanges P and Q. We refer to $D_\infty = P - Q \in \operatorname{Div}(C)$ as the *divisor at infinity*; it is well-defined up to sign.

The divisor nD_∞ is *principal* if the following equivalent conditions hold:

1. There is a regular function $f : C_0 \to \mathbb{C}^*$ with a pole of order n at P, and a zero of order n at Q.

2. The class $n[D] \in \mathrm{Pic}(C)$ is trivial.

3. The curve C is irreducible, and $h^0(C, \mathcal{O}_C(nD)) > 0$.

In case (1) we write $nD_\infty = (f)$. The divisor D_∞ is *torsion* if nD_∞ is principal for some $n > 0$.

Theorem 3.1 *The divisor nD, $n > 0$ is principal if and only if there are polynomials $a, b \in \mathbb{C}[x]$, with $\deg(a) = n$, satisfying Pell's equation*

$$a(x)^2 - p(x)b(x)^2 = 1. \tag{3.1}$$

Proof. Assume $a, b \in \mathbb{C}[x]$ satisfy (3.1). Then the polynomial $f(x,y) = a(x) + yb(x) \in \mathbb{C}[x,y]/(y^2 - p(x))$ satisfies

$$f(x,y) \cdot f(x,-y) = (a + yb)(a - yb) = 1,$$

so it defines a regular map $f : C_0 \to \mathbb{C}^*$. Since $a(x)$ and $yb(x)$ both have poles of order n at P and Q, $f(x,y)$ has a pole of order n at one of these points, and hence a zero of order n at the other; therefore nD is a principal divisor.

Conversely, suppose nD is the divisor of a regular map $f : C \to \mathbb{P}^1$. Then the product $f(x,y)f(x,-y) = c$ is a nonzero constant, since it has no zeros or poles. Replacing f with $f/\sqrt{c}$, we can assume $c = 1$. Then the rational functions

$$a(x,y) = \frac{f(x,y) + f(x,-y)}{2}, \quad b(x,y) = \frac{f(x,y) - f(x,-y)}{2y} \tag{3.2}$$

are invariant under the hyperelliptic involution, and have no poles on C_0; so they are actually polynomials in x, satisfying (3.1). ∎

Remark. The solution to $(f) = nD_\infty$ is a *unit* in the coordinate ring

$$\mathcal{O}(C_0) = \mathbb{C}[x,y]/(y^2 - p(x)),$$

and $a^2 - pb^2$ is its norm over $\mathbb{C}[x]$.

Genus zero. Recall (e.g. from [Ri]) that the Chebyshev polynomials $T_n(x)$ and $U_n(x)$ of the first and second kinds are characterized by:

$$T_n(\cos\theta) = \cos(n\theta), \quad U_{n-1}(\cos\theta) = \sin(n\theta)/\sin(\theta).$$

Theorem 3.2 *Let C be the conic $y^2 = x^2 - 1$. Then nD_∞ is principal for every $n > 0$, and the corresponding solutions to Pell's equation are given by*

$$T_n(x)^2 - (x^2 - 1)U_{n-1}(x)^2 = 1.$$

Proof. Under the isomorphism $\mathbb{P}^1 \cong C$ given by

$$t \mapsto (x(t), y(t)) = \left(\frac{t+t^{-1}}{2}, \frac{t-t^{-1}}{2}\right),$$

the points $t=0$ and $t=\infty$ lie over $x=\infty$; thus $f(t)=t^n$ satisfies $(f)=nD_\infty$. Setting $t=e^{i\theta}$, we have $(x,y)=(\cos\theta, i\sin\theta)$, and thus equation (3.2) implies that $(a,b)=(\pm T_n, \pm U_{n-1})$. Indeed, Pell's equation amounts simply to the identity

$$T_n(x)^2 - (x^2-1)U_{n-1}(x)^2 = \cos^2(n\theta) + \sin^2(n\theta) = 1.$$

■

Corollary 3.3 *The divisor D_∞ is torsion on the curve*

$$y^2 = (x^2-1)q(x)^2$$

if and only if $q(x)$ has simple zeros, all contained in $(-1,1)\cap\cos\pi\mathbb{Q}$.

Proof. By the preceding theorem, we can solve Pell's equation

$$a(x)^2 - (x^2-1)q(x)^2 b(x)^2 = 1$$

if and only if $q(x)$ divides $U_{n-1}(x)$ for some n. Since the zeros of $U_{n-1}(x)$ coincide with the set $(-1,1)\cap\cos(\pi/n)\mathbb{Z}$, and each has multiplicity one, the result follows. ■

Limits. We conclude by noting that torsion divisors behave well under limits, so long as the limiting curve is irreducible.

Theorem 3.4 *Let C_k and C be irreducible hyperelliptic curves of arithmetic genus g, defined by a convergent sequence of polynomials*

$$y^2 - p_k(x) \;\to\; y^2 - p(x) \;\in \mathbb{C}[x,y].$$

Suppose the divisor at infinity is torsion of order $\le n$ for each C_k. Then the same is true for C.

Proof. By assumption there exist polynomial solutions to Pell's equation $a_k(x)^2 - p_k(x)b_k(x)^2 = 1$ of uniformly bounded degree. If the coefficients of (a_k, b_k) are bounded, a convergent subsequence yields a solution to the equation $a(x)^2 - p(x)b(x)^2 = 1$, so the divisor at infinity is torsion for C. Otherwise, a limit of suitable rescalings of the polynomials (a_k, b_k) yields a nontrivial solution to the equation $a(x)^2 - p(x)b(x)^2 = 0$; this implies $p(x)$ is a square, contrary to irreducibility of C. ■

Remark. The sequence $y^2 = x^2 - 1/k^2 \to y^2 - x^2$ gives a simple example showing Theorem 3.4 fails without irreducibility.

Notes and references. The theory of torsion divisors on hyperelliptic curves was studied by Abel in [Ab]; for more recent developments, see [HMP] [BC], [BGL], and [Yu].

4 Teichmüller curves in genus two

In this section we recall the theory of Teichmüller curves in genus two, and the prototypical forms that generate them. We then establish:

Theorem 4.1 *If the prototypical form* (X, ω) *of type* (a, b, c, e) *and width* $0 < t < 1$ *generates a primitive Teichmüller curve*

$$f : V \to \mathcal{M}_2,$$

then there are rational numbers $0 < \alpha < \beta < 1/2$ *satisfying*

$$\frac{\sin(\pi\alpha)}{\sin(\pi\beta)} = \frac{\lambda}{b} \quad \text{and} \quad t = \beta + \alpha\frac{b}{\lambda}, \tag{4.1}$$

where $\lambda = (e + \sqrt{e^2 + 4bc})/2$.

In the course of the proof, we will obtain an explicit formula for certain nodal curves $Y \in \partial\mathcal{M}_2$ that arise from the cusps of V.

Teichmüller curves. Let $\mathcal{M}_g$ denote the moduli space of Riemann surfaces of genus g, and let

$$\Omega\mathcal{M}_g \to \mathcal{M}_g$$

denote the bundle of pairs (X, ω) such that ω is a nonzero holomorphic 1-form on $X \in \mathcal{M}_g$. The space $\Omega\mathcal{M}_g$ carries a natural action of $\mathrm{GL}_2^+(\mathbb{R})$, whose orbits project to complex geodesics in $\mathcal{M}_g$ (see e.g. [Mc1]). The stabilizer of a given form will be denoted by $\mathrm{SL}(X, \omega)$; it is a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$.

If $\mathrm{SL}(X, \omega)$ is a lattice in $\mathrm{SL}_2(\mathbb{R})$, then the projection of its orbit gives a holomorphic *Teichmüller curve*

$$f : V = \mathbb{H}/\,\mathrm{SL}(X, \omega) \to \mathcal{M}_g.$$

The image of $f(V) \subset \mathcal{M}_g$ is an algebraic curve, isometrically immersed for the Teichmüller metric. We say $f : V \to \mathcal{M}_g$ is *primitive* if the form (X, ω) is not the pullback of a holomorphic form on a curve of lower genus.

Stable curves. Let $\overline{\mathcal{M}_g}$ denote the compactification of moduli space by stable curves. By passing to the normalization

$$\pi : \widetilde{Y} \to Y,$$

any stable curve $Y \in \overline{\mathcal{M}_g}$ can be regarded as a smooth curve $\widetilde{Y}$ with certain pairs of points (y_i, y_i') identified to form nodes.

Let $\Omega(Y)$ denote the space of holomorphic 1-forms on $\widetilde{Y} - \bigcup\{y_i, y_i'\}$ with at worst simple poles, of opposite residues, at each pair of points (y_i, y_i'). Then $\Omega\mathcal{M}_g \to \mathcal{M}_g$ extends to the bundle

$$\Omega\overline{\mathcal{M}_g} \to \overline{\mathcal{M}_g}$$

of *stable Abelian differentials*, consisting of pairs (Y, ξ) with $\xi \in \Omega(Y)$. This bundle can be regarded as the relative dualizing sheaf for the universal curve $\overline{\mathcal{C}_g} \to \overline{\mathcal{M}_g}$ [HM, Ch. 3].

Prototypical forms. We now specialize to the case $g = 2$. Following [Mc4, §3], we say a set of integers $p = (a, b, c, e)$ is a *splitting prototype* of discriminant $D > 0$ if:

$$\begin{array}{lll} D = e^2 + 4bc, & 0 \le a < \gcd(b,c), & c + e < b, \\ 0 < b, & 0 < c, \quad \text{and} & \gcd(a,b,c,e) = 1. \end{array}$$

Let $\lambda = (e + \sqrt{D})/2$ and let $I = [0, t\lambda]$, where $t \in (0, 1]$. Let $(E_i, \omega_i) = (\mathbb{C}/\Lambda_i, dz), i = 1, 2$ be the forms of genus one with period lattices $\Lambda_1 = \mathbb{Z}(\lambda, 0) \oplus \mathbb{Z}(0, \lambda)$ and $\Lambda_2 = \mathbb{Z}(b, 0) \oplus \mathbb{Z}(a, c)$.

The *prototypical form* of *type* p and *width* t is given by the connected sum

$$(X, \omega) = (E_1, \omega_1) \underset{I}{\#} (E_2, \omega_2).$$

The connected sum is obtained by slitting each torus open along the projection of the arc I, and then gluing corresponding edges. The form (X, ω) can also be expressed geometrically as the quotient $(P, dz)/\sim$ of the polygon $P \subset \mathbb{C}$ shown in Figure 4.

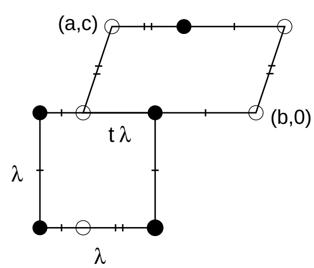

Figure 4. The prototypical form of type (a, b, c, e) and width t.

By [Mc4, §1 and Cor. 3.7] we have:

Theorem 4.2 *Every Teichmüller curve generated by a form in $\Omega\mathcal{M}_2$ is also generated by a prototypical form. The resulting curve is primitive if and only if the discriminant $D = e^2 + 4bc$ is not a square.*

Stable sums. The operation of connected sum extends continuously way to the case where the tori $E_i \cong \mathbb{C}/\Lambda_i$ degenerate to cylinders $C_i \cong \mathbb{C}/\mathbb{Z}a_i \cong \mathbb{C}^*$. In this case the connected sum gives a stable curve in the stratum $\mathcal{M}_{0,4} \subset \partial\mathcal{M}_2$, with the ends (y_i, y_i') of each cylinder C_i identified to form nodes.

Theorem 4.3 *Let $I = [0, v]$, and assume $0 < v < r < s$. Then the connected sum*

$$(Y, \omega) = (\mathbb{C}/\mathbb{Z}r, dz) \underset{I}{\#} (\mathbb{C}/\mathbb{Z}s, dz) \tag{4.2}$$

is proportional to the stable Abelian differential dx/y on a nodal curve of the form

$$y^2 = (x^2 - 1)(x + \cos(\pi\alpha))^2 (x - \cos(\pi\beta))^2, \tag{4.3}$$

where $\alpha \in (0, 1/2)$, $\beta \in (0, 1)$, $\sin(\pi\alpha)/\sin(\pi\beta) = r/s$ and $v = \alpha s + \beta r$.

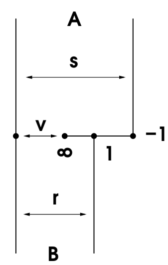

Figure 5. Connected sum of cylinders.

Proof. Start by expressing Y as a degree two branched covering $\pi : Y \to \mathbb{P}^1$ of the projective line, with hyperelliptic involution η. Then the form $(Y, \omega^2)/\eta$ gives a meromorphic quadratic differential $(\mathbb{P}^1, q)$ on the quotient $\mathbb{P}^1$. Since the periods r, s and v of ω are real, $(\mathbb{P}^1, q)$ can be constructed from a rectangular polygon $(P, (dz/2)^2)$ in $\mathbb{C}$. More precisely, if we define P by gluing the infinite rectangles $[0, s] \times [0, \infty)$ and $[0, r] \times [0, -\infty)$ together along the segment $[0, v]$, then the double of $(P, dz^2/4)$ along its edges can be identified with $(\mathbb{P}^1, q)$.

We can normalize so that the interior of P corresponds to the upper half plane $\mathbb{H} \subset \mathbb{P}^1$, its boundary to the extended real axis, and its vertices to the ordered set $(\infty, -1, A, B, 1) \subset \mathbb{R}$. The image of I on $\mathbb{P}^1$ is then a loop based at ∞, crossing the real axis between A and B.

Since $-1 < A < B < 1$, there are unique constants $0 < \alpha, \beta < 1$ such that $(-\cos \pi\alpha, \cos \pi\beta) = (A, B)$. Since $\pi : Y \to \mathbb{P}^1$ is branched over $x = \pm 1$ and has nodes over $x = A, B$, Y is isomorphic to the hyperelliptic curve (4.3), with $\pi(x, y) = x$. Therefore $\omega = C dx/y$, for some $C \neq 0$, since these two forms have the same zeros and poles. Consequently we have

$$q = \omega^2 = \frac{C^2\, dx^2}{(x^2 - 1)(x - A)^2 (x - B)^2},$$

which implies

$$\frac{\mathrm{Res}_B(q)}{\mathrm{Res}_A(q)} = \left(\frac{r}{s}\right)^2 = \frac{1 - A^2}{1 - B^2} = \left(\frac{\sin(\pi\alpha)}{\sin(\pi\beta)}\right)^2.$$

Thus $\sin(\pi\alpha)/\sin(\pi\beta) = r/s$, because both sides are positive.

Using the fact that $\mathrm{Res}_A(q) = s^2/(2\pi i)^2$ and $\mathrm{Res}_B(q) = r^2/(2\pi i)^2$, it is then straightforward to compute

$$v = \int_I \omega = \int_{-i\infty}^{+i\infty} \sqrt{q} = \alpha s + \beta r,$$

where the integral is taken along a path that crosses $\mathbb{R}$ between A and B. The choice of square-root is dictated by the condition $v > 0$.

Finally note that the inequalities $r < s$ and $A < B$ imply

$$0 \leq \sin(\pi\alpha) < \sin(\pi\beta) \quad \text{and} \quad -\cos(\pi\alpha) < \cos(\pi\beta);$$

since $\sin^2(\theta) + \cos^2(\theta) = 1$, we must have $-\cos(\pi\alpha) < 0$ and therefore $\alpha < 1/2$. ∎

Theorem 4.4 *Let (X, ω) be the prototypical form of type (a, b, c, e) and width t, and let $A_s = \left(\begin{smallmatrix} 1 & 0 \\ 0 & s \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{R})$. Then as $s \to \infty$ we have*

$$A_s \cdot (X, \omega) \ \to \ (\mathbb{C}/\mathbb{Z}\lambda, dz) \mathop{\#}_{I} (\mathbb{C}/\mathbb{Z}b, dz) \tag{4.4}$$

in $\Omega\overline{\mathcal{M}_2}$, where $I = [0, t\lambda]$.

Proof. The rank two lattices

$$\Lambda_1^s = \mathbb{Z}(\lambda, 0) \oplus \mathbb{Z}(0, s\lambda), \quad \Lambda_2^s = \mathbb{Z}(b, 0) \oplus \mathbb{Z}(sa, sc)$$

giving the summands of

$$A_s \cdot (X, \omega) = (\mathbb{C}/\Lambda_1^s, dz) \mathop{\#}_{I} (\mathbb{C}/\Lambda_2^s, dz)$$

converge geometrically to the rank one discrete groups $\mathbb{Z}(\lambda, 0)$ and $\mathbb{Z}(b, 0)$. ∎

Proof of Theorem 4.1. Assume $\mathrm{SL}(X, \omega)$ is a lattice for the prototypical form of type (a, b, c, e) and width $0 < t < 1$. Then by the Theorems 4.3 and 4.4 the closure of $\mathrm{GL}_2^+(\mathbb{R}) \cdot (X, \omega)$ in $\Omega\overline{\mathcal{M}_2}$ contains the stable Abelian differential $(Y, dx/y)$, where Y is the hyperelliptic curve

$$y^2 = (x^2 - 1)(x + \cos(\pi\alpha))^2 (x - \cos(\pi\beta))^2 \tag{4.5}$$

and $0 < \alpha < \beta < 1/2$ satisfy (4.1).

The zeros P, Q of dx/y lie over $x = \infty$, so their difference $P - Q$ is the divisor at infinity for (4.5). Moreover, by Theorem 1.4, D_∞ is a limit of torsion divisors of constant order on nearby curves in $f(V)$. By Theorem 3.4, D_∞ is itself torsion, and hence α and β are rational by Corollary 3.3. ∎

5 The regular decagon

In this section we illustrate the results of the preceding section using the Teichmüller curve generated by the decagon form.

Algebraic description. Let (X, ω) denote the *decagon form*, given by $\omega = dx/y$ on the hyperelliptic curve

$$y^2 = x(x^5 - 1).$$

By symmetry considerations, this form can be also obtained from a regular decagon (Q, dz) in $\mathbb{C}$ by identifying opposite sides.

Theorem 5.1 *The decagon form generates the Teichmüller*

$$f : V \to \mathcal{M}_2$$

given by $f(t) = [X_t]$, where X_t is the hyperelliptic curve $y^2 = q_t(x)$,

$$q_t(x) = x\left(-1 - 2\,t^5 - t^{10} + 25\,t^4\,x - 50\,t^3\,x^2 + 35\,t^2\,x^3 - 10\,t\,x^4 + x^5\right),$$

and t ranges in the space $V = (\mathbb{P}^1 - \{t\, :\, t^{10} = 1\})/D_{10}$.

Here D_{10} is the dihedral group generated by $t \mapsto 1/t$ and $t \mapsto \exp(2\pi i/5)t$. The curves X_t and X_{gt} are isomorphic for all $g \in D_{10}$, so f is well-defined on V. We will also see that the 1-forms lying over X_t in the orbit $\mathrm{GL}_2^+(\mathbb{R}) \cdot (X, \omega)$ are proportional to dx/y.

The proof will be deferred to the end of the section.

Stable limits. The Teichmüller curve $f : V \to \mathcal{M}_2$ is also generated by the prototypical form of type $(a, b, c, e) = (0, 1, 1, -1)$ and $t = (2 + \gamma)/5$, where $\gamma = (1 + \sqrt{5})/2$ is the golden mean [Mc4, Lemma 8.2]. For this prototype, we have $\lambda = (e + \sqrt{e^2 + 4bc})/2 = \gamma^{-1}$.

To verify Theorem 4.1, note that as $t \to 1$, the 1-form $(X_t, dx/y)$ converges to the stable Abelian differential $(X_1, dx/y)$ on the nodal curve

$$y^2 = q_1(x) = x(x - 4)(1 - 3x + x^2)^2.$$

This curve is isomorphic to

$$y^2 = (x^2 - 1)(4x^2 + 2x - 1)^2 = (x^2 - 1)(x - A)^2(x - B)^2,$$

where $\{A, B\} = \{-\cos(\pi/5), \cos(2\pi/5)\} = \{(-1 \pm \sqrt{5})/4\}$. Thus the corresponding ratio of sines satisfies

$$\frac{\sin(\pi/5)}{\sin(2\pi/5)} = \frac{-1 + \sqrt{5}}{2} = \frac{\lambda}{b},$$

and the width is given by

$$t = \frac{2 + \gamma}{5} = \frac{2}{5} + \frac{1}{5}\frac{b}{\lambda},$$

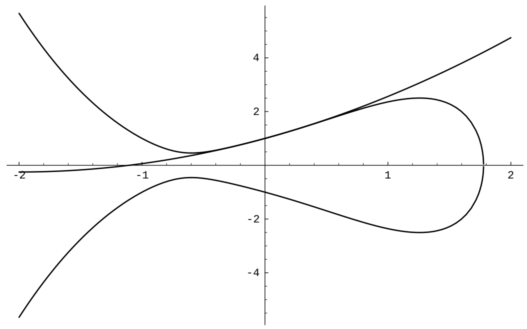

Figure 6. The divisor D_0 has order 5.

both as predicted by (4.1). The graph of $y^2 = (1 - x^2)(4x^2 + 2x - 1)$ is shown in Figure 2 of the Introduction.

Torsion divisors. Note that the zeros $P, Q \in X_t$ of $\omega_t = dx/y$ lie over $x = \infty$, so their difference $P - Q$ is the divisor at infinity. According to Theorem 1.4, this divisor is torsion whenever X_t is irreducible. In fact, one can see explicitly that:

Theorem 5.2 *For any t such that $t^5 + 1 \neq 0$, $5D_\infty$ is a principal divisor on the curve $y^2 = q_t(x)$.*

Proof. It is convenient to replace q_t with its reciprocal polynomial

$$\begin{aligned} r_t(x) &= x^6 q_t(x^{-1}) \\ &= 1 - 10\,t\,x + 35\,t^2\,x^2 - 50\,t^3\,x^3 + 25\,t^4\,x^4 - x^5 - 2\,t^5\,x^5 - t^{10}\,x^5, \end{aligned}$$

so that D_∞ becomes the divisor D_0 lying over $x = 0$. We then find the remarkable factorization

$$r_t(x) = (5t^2x^2 - 5tx + 1)^2 - (t^5 + 1)^2 x^5.$$

This means that the conic

$$y = f_t(x) = 5t^2x^2 - 5tx + 1$$

intersects the curve $y^2 = r_t(x)$ to order 5 over $x = 0$ (see Figure 6 for the case $t = -1/4$). Consequently, $F_t(x, y) = (y - f_t(x))/(y + f_t(x))$ gives a rational function with divisor $(F_t) = 5D_0$. ∎

Remark. The corresponding solution to Pell's equation $a_t(x)^2 - b_t(x)^2 q_t(x) = 1$ satisfies

$$b_t(x) = \frac{2(5\,t^2 - 5\,t\,x + x^2)}{(1+t^5)^2}.$$

Regular polygons. We now proceed to the proof of Theorem 5.1.

Let Y be the hyperelliptic curve $y^2 = x^n - 1$, and let $\xi = dx/y$. The form (Y, ξ) can also be described geometrically as the quotient $(Q, dz)/\sim$, where $Q \subset \mathbb{C}$ is a pair of regular n-gons joined along an edge, and its other edges are identified in opposite pairs (Figure 7).

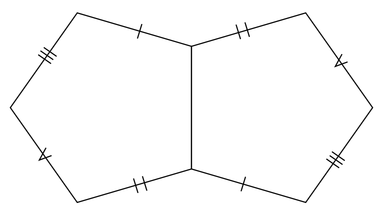

Figure 7. A polygonal model for the form dx/y on the curve $y^2 = x^5 - 1$.

It is known that $\mathrm{SL}(Y, \xi)$ is a lattice for all $n \geq 3$; in fact $\mathrm{SL}(Y, \xi)$ is a triangle group of signature $(2, n, \infty)$ when n is odd, and signature $(n/2, \infty, \infty)$ when n is even [V].

Moreover, by [Lo] the Teichmüller curve

$$F : V \cong \mathbb{H}/\,\mathrm{SL}(Y, \xi) \to \mathcal{M}_g \tag{5.1}$$

can be described algebraically as follows. For $t \in \mathbb{P}^1$, let Y_t be the hyperelliptic curve given by

$$y^2 = p_t(x) = \prod_{i=0}^{n} (x - \zeta^i - t\zeta^{-i}), \tag{5.2}$$

where $\zeta = e^{2\pi i/n}$. Let $G \subset \mathrm{Aut}(\mathbb{P}^1)$ denote the dihedral group generated by $t \mapsto 1/t$ and $t \mapsto \zeta^2 t$. Then the isomorphism class of Y_t depends only on the value of $t \in \mathbb{P}^1/G$, and the map (5.1) is given on the quotient space

$$V = (\mathbb{P}^1 - \{t \,:\, t^n = 1\})/G$$

simply by $F(t) = [Y_t] \in \mathcal{M}_g$. (See [Me] for a related construction.) The forms $(Y_t, \xi_t) = (Y_t, dx/y)$ all lie in the orbit $\mathrm{GL}_2^+(\mathbb{R}) \cdot (Y, \xi)$, so they also generated the Teichmüller curve $F : V \to \mathcal{M}_g$.

Now suppose $n = 2k$ is even. Then only even powers of x occur in $p_t(x)$, and therefore $\rho(x, y) = (-x, -y)$ is an automorphism of Y_t. Using the change of variables $(x, y) \leftarrow (x^2, xy)$, the quotient $X_t = Y_t/\langle \rho \rangle$ can be expressed as the hyperelliptic curve

$$y^2 = q_t(x), \quad \text{where } q_t(x^2) = x^2 p_t(x). \tag{5.3}$$

The form $2dx/y$ is also ρ-invariant, so it descends to the form $\omega_t = dx/y$ on X_t. Since $\deg q_t(u) = 1 + k$, ω_t has a single zero (lying over $x = \infty$) when k is even, and a pair of distinct zeros when k is odd.

Note that X_0 is the hyperelliptic curve $y^2 = q_0(x) = x(x^k - 1)$. Geometrically, the form (X_0, ω_0) is obtained from a *single* regular $2k$-gon (P, dz) by identifying opposite sides. Thus the regular $2k$-gon generates the Teichmüller curve

$$f : V \to \mathcal{M}_h$$

given by $f(t) = [X_t] = [y^2 - q_t(x)]$.

Proof of Theorem 5.1. For the decagon, we simply specialize to the case $n = 10$ in (5.2), and use (5.3) to obtain a formula for $q_t(x)$. ∎

6 Uniqueness

In this section we complete the proof of our main result by showing:

Theorem 6.1 *The only primitive Teichmüller curve $f : V \to \mathcal{M}_2$ generated by a form with simple zeros comes from the regular decagon.*

Sine ratios. We begin by using the list of quadratic irrational sine ratios to show:

Theorem 6.2 *Suppose the prototypical form (X, ω) of type (a, b, c, e) and width $0 < t < 1$ generates a primitive Teichmüller curve. Let $f = \gcd(b, c, e)$, and write $(a, b, c, e) = (a, fB, fC, fE)$. Then (B, C, E) and t are equal to one of the 22 possibilities listed in Table 8.*

Proof. Let $\lambda = (e+\sqrt{e^2 + 4bc})/2$ as usual, and let $\mu = (E+\sqrt{E^2 + 4BC})/(2B)$. Then $\mu = \lambda/b$. Since the Teichmüller curve generated by (X, ω) is primitive, μ is a quadratic irrational.

According to Theorem 4.1, there exist rationals $\alpha \in (0, 1/2)$ and $\beta \in (0, 1)$ such that

$$\sin(\pi\alpha)/\sin(\pi\beta) = \mu \quad \text{and} \quad t = \beta + \alpha\mu^{-1}.$$

Thus α and $\min(\beta, 1-\beta)$ must correspond to one of the 15 quadratic irrational sine ratios listed in Table 3 of §2. The values of α and β determine μ and t by the formulas above. They also determine the triple (B, C, E), since this is the smallest positive integral vector proportional to $(1, -\mathrm{N}(\mu), \mathrm{Tr}(\mu))$.

It is thus straightforward to convert Table 3 into Table 8, in which the values of (α, β) in the first two columns give rise to the values of (B, C, E) and t in the columns that follow. The cases $(\alpha, \beta) = (1/12, 5/12)$ and $(1/10, 3/10)$ in Table 3 are excluded, because they give $c/b = -N(\mu) < 0$, contrary to the condition $b, c > 0$ in the definition of a prototype. ∎

α	β	(B,C,E)	t
$1/10$	$1/2$	$(4,1,-2)$	$(6+\sqrt{5})/10$
$1/12$	$1/4$	$(2,1,-2)$	$(4+\sqrt{3})/12$
$1/12$	$3/4$	$(2,1,-2)$	$(10+\sqrt{3})/12$
$1/8$	$3/8$	$(1,1,-2)$	$(4+\sqrt{2})/8$
$1/8$	$5/8$	$(1,1,-2)$	$(6+\sqrt{2})/8$
$1/6$	$1/3$	$(3,1,0)$	$(2+\sqrt{3})/6$
$1/6$	$2/3$	$(3,1,0)$	$(4+\sqrt{3})/6$
$1/6$	$3/10$	$(1,1,-1)$	$(23+5\sqrt{5})/60$
$1/6$	$7/10$	$(1,1,-1)$	$(47+5\sqrt{5})/60$
$1/10$	$1/6$	$(1,1,-1)$	$(13+3\sqrt{5})/60$
$1/10$	$5/6$	$(1,1,-1)$	$(53+3\sqrt{5})/60$
$1/5$	$2/5$	$(1,1,-1)$	$(5+\sqrt{5})/10$
$1/5$	$3/5$	$(1,1,-1)$	$(7+\sqrt{5})/10$
$1/6$	$1/4$	$(2,1,0)$	$(3+2\sqrt{2})/12$
$1/6$	$3/4$	$(2,1,0)$	$(9+2\sqrt{2})/12$
$1/4$	$1/2$	$(2,1,0)$	$(2+\sqrt{2})/4$
$1/4$	$5/12$	$(1,2,-2)$	$(13+3\sqrt{3})/24$
$1/4$	$7/12$	$(1,2,-2)$	$(17+3\sqrt{3})/24$
$3/10$	$1/2$	$(4,1,2)$	$(2+3\sqrt{5})/10$
$1/4$	$1/3$	$(3,2,0)$	$(8+3\sqrt{6})/24$
$1/4$	$2/3$	$(3,2,0)$	$(16+3\sqrt{6})/24$
$1/3$	$1/2$	$(4,3,0)$	$(9+4\sqrt{3})/18$

Table 8. The 22 candidates.

Ratios of moduli. To complete the proof, we will use the following rationality constraint.

Theorem 6.3 *If the prototypical form (X,ω) of type (a,b,c,e) and width t generates a Teichmüller curve, then the ratio*

$$\frac{(1-t)(bc/g+(n+1)\lambda)}{t\lambda-ng} \tag{6.1}$$

is rational, where $\lambda=(e+\sqrt{e^2+4bc})/2$, $g=\gcd(a,b)$ and $n=\lfloor t\lambda/g\rfloor$.

(Here $\lfloor x\rfloor$ denotes the greatest integer less than or equal to x.)

Proof. Consider the foliation $\mathcal{F}$ of $(X,|\omega|)$ by vertical geodesics, corresponding to vertical lines in the polygon $P\subset\mathbb{C}$ in Figure 4. The first return map of $\mathcal{F}$ to the transversal $L=[0,b]$, forming the base of the upper parallelogram in P, is given by $f(x)=x+a \bmod b$. Since f is periodic, all the leaves of $\mathcal{F}$ are closed.

More precisely, the leaves of $\mathcal{F}$ disjoint from L sweep out a cylinder C_1 of height $h_1=(1-t)\lambda$ and circumference $c_1=\lambda$. The leaves that do meet L cross it b/g times, since this is the period of f. These leaves sweep out a pair of cylinders C_2 and C_3, distinguished by their intersection numbers with $I=[0,t\lambda]\subset L$: we have $C_2\cdot I=n$ and $C_3\cdot I=n+1$. Since $C_i\cdot I$ also counts the number of times that C_i passes through the $\lambda\times\lambda$ square inside of P, these cylinders have circumferences $c_2=bc/g+n\lambda$ and $c_3=bc/g+(n+1)\lambda$. Similarly one can check that their heights are given by $h_2=(n+1)g-t\lambda$ and $h_3=t\lambda-ng$.

Now if $\mathrm{SL}(X,\omega)$ is a lattice, then the moduli $m_i=h_i/c_i$ of the cylinders C_i have rational ratios (see e.g. [V, 2.4], [Mc1, Lemma 9.7]). In particular, the ratio m_1/m_3 given by (6.1) is rational. ■

Example. For the prototypical form (X,ω) of type $(a,b,c,e)=(0,1,1,-1)$ and width $t=(2+\gamma)/5$ (generating the regular decagon), the moduli of the 3 cylinders of the vertical foliation satisfy $[m_1:m_2:m_3]=[1:2:1]$. We have $n=0$, $g=1$, and the quantity (6.1) is simply 1.

Proof of Theorem 6.1. Let $f:V\to\mathcal{M}_2$ be a primitive Teichmüller curve generated by a holomorphic 1-form (X,ω) with simple zeros. By Theorem 4.2 we may assume (X,ω) is a prototypical form, of type (a,b,c,e) and width $0<t<1$. Let $f=\gcd(b,c,e)$ and write $(a,b,c,e)=(a,fB,fC,fE)$; then (B,C,E) and t agree with one of the 22 possibilities listed in Table 8.

By the definition of a prototype, we have $\gcd(a,b,c,e)=\gcd(a,f)=1$. Therefore

$$g=\gcd(a,b)=\gcd(a,fB)=\gcd(a,B)$$

is a divisor of B. So for each value of (B,C,E,t), there are only finitely many possibilities for g.

For concreteness, suppose the candidate prototype satisfies $(B,C,E)=(1,1,-1)$ and $t=(5+\sqrt{5})/10$. Then $\mu=\lambda/b=(-1+\sqrt{5})/2$, and $g=1$.

Substituting $(b, c) = (fB, fC)$ and $\lambda = \mu f B$ into (6.1), Theorem 6.3 implies the ratio

$$\frac{f(3f - f^2 - 5n + 8fn - 5n^2 + \sqrt{5}(f^2 - 2fn + 3n^2 - f + 3n))}{2(f^2 - 5n^2)}$$

is rational, where

$$n = \lfloor tfB\mu/g \rfloor = \lfloor f/\sqrt{5} \rfloor.$$

Equivalently, we have

$$p(f, n) = f^2 - 2fn + 3n^2 - f + 3n = 0. \tag{6.2}$$

It is straightforward to verify that the only positive integral solution to (6.2) (with $n = \lfloor f/\sqrt{5} \rfloor$) is $f = 1$. Indeed, for large values of f we have $n \approx f/\sqrt{5}$, and thus $p(f, n)$ grows like f^2, leaving only finitely many cases to check. (A plot of $p(f, \lfloor f/\sqrt{5} \rfloor)$ is shown in Figure 9).

The case $f = 1$ gives $(a, b, c, e) = (0, 1, 1, -1)$, so it corresponds to the regular decagon.

A similar calculation shows there are no Teichmüller curves at all coming from the remaining candidates in Table 8. For example, suppose $(B, C, E) = (2, 1, -2)$, $t = (4 + \sqrt{3})/12$ and $g = 2$. Then $\mu = (1 + \sqrt{3})/2$, and we find

$$p(f, n) = 23f^2 - 48fn + 216n^2 - 24f + 216n = 0,$$

where $n = \lfloor(-1 + 3\sqrt{3})f/24\rfloor$. For $f = 1, 2, 3, 4, 5, \ldots$ we have $p(f, n) = -1, 44, 135, 272, 455, \ldots$, and quadratic growth insures $p(f, n) > 0$ for all $f > 5$.

The remaining cases are similar. ∎

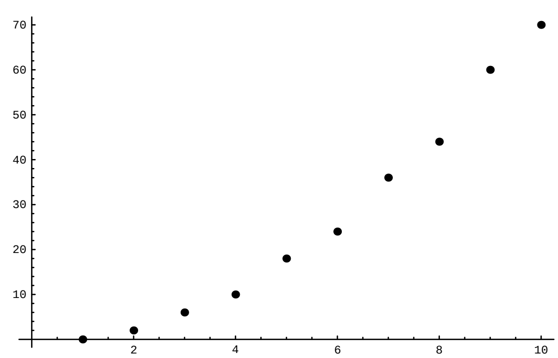

Figure 9. Plot of $p(f, \lfloor f/\sqrt{5} \rfloor)$.

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