Amenable Coverings of Complex Manifolds and Holomorphic Probability Measures

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Amenable coverings of complex manifolds and holomorphic probability measures

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1 Introduction.

Let $\pi : Y \to X$ be a covering map between complex manifolds X and Y. For many holomorphic objects (such as functions or forms) one can define a pushforward operator π_* carrying objects on Y to objects on X by summing over the fiber. The pushforward is *efficient* if there is little cancellation in the sum. To achieve efficiency, there must be coherence in phase on different sheets of the covering.

In this paper we will demonstrate a direct connection between *efficiency* of the push-forward π_* and *amenability* of the covering π. The latter is a purely combinatorial property of the pair of groups $\pi_1(Y) \subset \pi_1(X)$; the covering is amenable if there exists a $\pi_1(X)$-invariant finitely additive probability measure on the coset space $\pi_1(X)/\pi_1(Y)$. Amenability is discussed in more detail in §3.

The connection will be established by studying complex-valued measures on the fibers of the covering, varying holomorphically with respect to the base.

Motivating Example. To any Riemann surface X we can associate the Banach space $Q(X)$ of holomorphic quadratic differentials $\phi(z)dz^2$ such that

$$||\phi|| \;=\; \int_X |\phi| \;<\; \infty.$$

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Then associated to a covering $Y \to X$ there is an operator

$$\Theta_{Y/X} : Q(Y) \to Q(X)$$

given by $\phi \mapsto \pi_*(\phi)$. It is easy to see that $||\Theta_{Y/X}|| \leq 1$.

Let B_X and B_Y denote the open unit balls in $Q(X)$ and $Q(Y)$ respectively.

Theorem 1.1 ([Mc1]) *Let $Y \to X$ be a covering of a hyperbolic Riemann surface X. Either:*

1. *The covering is amenable, and $\Theta(B_Y) = B_X$, or*
2. *The covering is nonamenable, and $\overline{\Theta(B_Y)}$ is contained in the interior of B_X.*

Corollary 1.2 (Kra's Theta conjecture) *For the universal cover $\pi : \Delta \to X$ of a hyperbolic surface of finite area,*

$$||\Theta_{\Delta/X}|| < 1.$$

In other words, there is always a definite inefficiency in the representation of ϕ as a pushforward $\pi_*(\psi)$.

Remark. Let Γ be the Fuchsian group of deck transformations for the universal covering $\Delta \to X$. Then given $\psi \in Q(\Delta)$, its pushforward can be expressed by the classical Poincaré series [Poin]

$$\Theta(\psi) = \sum_{\gamma \in \Gamma} \gamma^*(\psi).$$

Thus sum is clearly Γ-invariant, so it defines an element of $Q(X)$.

Proof of the corollary. For a surface of finite hyperbolic area, it is easy to see that the universal cover is nonamenable (cf. [Mc1]), and it is well-known that $Q(X)$ is finite dimensional. Thus $\overline{\Theta(B_\Delta)}$ is a compact subset of B_X, so $||\Theta|| < 1$. ∎

These results have implications for Teichmüller theory and for the construction of hyperbolic structures on 3-manifolds; see [Mc4], [Mc3] for an expository account and [Mc1], [Mc2] for details.

Recently Barrett and Diller gave an elegant and surprising new proof of Theorem 1.1 in the case of the universal covering (and thus a new proof of the Theta conjecture), as part of their work on affine bundles and holomorphic averaging [BD]. Here we present an understanding of their argument which applies to all coverings, not just the universal covering, and which leads to a short proof of the full Theorem 1.1.

Outline of the paper. §2 states the main result of this paper (Theorem 2.1, Efficiency implies amenability) and applies it to give a new proof of Theorem 1.1. Theorem 2.1 is proved in §3. We conclude in §4 with a geometric sketch of the Barrett-Diller proof of the Theta conjecture, to help indicate the connection between their ideas and ours.

Remark. Theorem 1.1 is not quite sufficient for the applications of [Mc2]. For example, we need a uniform bound on $||\Theta_{Y_n/X}||$ for a certainly countable collection of covering spaces Y_n. The proof in [Mc1] has the advantage of yielding an effective bound on $||\Theta||$.

2 Statement of results

To set up the main theorem we require:

- $\pi : Y \to X$: a covering map between (connected) complex manifolds;
- $\mathcal{L} \to X$: a holomorphic line-bundle over X;
- $\mathcal{L}' \to Y$: the pull-back of $\mathcal{L}$ to Y;
- $\phi : X \to \mathcal{L}$: a holomorphic section of $\mathcal{L}$, *not identically zero*; and
- $\psi_n : Y \to \mathcal{L}'$: a sequence of holomorphic sections such that $\pi_*(\psi_n) = \phi$.

Pushforward. Here is a more precise explanation of what we mean by $\pi_*(\psi) = \phi$. Let $U \subset X$ be any open ball. Then $\pi^{-1}(U)$ is a collection V_i of disjoint balls in Y, and π admits a holomorphic inverse $\rho_i : U \to V_i$ for each i. If $\sum \rho_i^*(\psi)$ converges to ϕ uniformly on compact subsets of U, for every such U, then we say $\pi_*(\psi) = \phi$.

Efficiency. The sequence ψ_n is *efficient* if $\pi_*|\psi_n|$ converges to $|\phi|$ uniformly on compact subsets of X.

Here $|\phi|$ is a section of the oriented real line-bundle $|\mathcal{L}|$ naturally attached to $\mathcal{L}$. In down-to-earth terms, one can trivialize $\mathcal{L}$ over a ball $U \subset X$, so the ψ and ϕ become functions; then efficiency means $\sum |\rho_i^*(\psi_n)|$ is close to $|\phi|$ when n is large.

Theorem 2.1 (Efficiency implies amenability) *If $\pi : Y \to X$ admits an efficient sequence $\psi_n : Y \to \mathcal{L}'$ with $\pi_*(\psi_n) = \phi$, then the covering π is amenable.*

We record the following complementary result:

Theorem 2.2 *If $\pi : Y \to X$ is amenable and $\phi = \pi_*(\psi)$ for some $\psi : Y \to \mathcal{L}'$, then there is an efficient sequence ψ_n with $\pi_*(\psi_n) = \phi$ and $\pi_*|\psi_n| \leq \pi_*|\psi|$.*

When the covering is regular with deck group $\mathbb{Z}$ generated by g, the ψ_n can be given by

$$\psi_n = \frac{1}{n}\sum_{k=1}^{n}(g^k)^*\psi.$$

For a general amenable covering, the proof of Theorem 2.2 is a straightforward generalization of [Mc1, Theorem 9.1]; we omit the details.

Remark. Theorems 2.1 and 2.2 also hold for finite dimensional affine bundles; for simplicity we stick with the versions above. The versions for line bundles have applications to bounded symmetric domains (see the Appendix to [Mc1]).

Proof of Theorem 1.1. Assuming the results above, we present a new proof of the main facts concerning $||\Theta||$.

Let $\pi : Y \to X$ be a covering of Riemann surfaces. Quadratic differentials on X and Y are sections of bundles $\mathcal{L}$ and $\mathcal{L}'$ obtained by squaring the canonical bundles. Note that a section of $|\mathcal{L}|$ is naturally an area form on X (since $|\phi| = |\phi(z)||dz|^2$) and that for $\psi \in Q(Y)$,

$$||\psi|| \;\; = \;\; \int_X \pi_*|\psi_n|.$$

Now suppose the covering is amenable. Let ϕ be a member of B_X, so $||\phi|| = 1 - \epsilon < 1$. By a result of Ahlfors and Bers, $\phi = \pi_*(\psi)$ for some ψ in $Q(Y)$ (see [Kra]). By Theorem 2.2, there is an efficient sequence ψ_n with $\pi_*(\psi_n) = \phi$ and $\pi_*|\psi_n| \leq \pi_*|\psi|$. Since the latter is integrable, there is compact set $K \subset X$ such that $\int_{X-K} \pi_*|\psi_n| < \epsilon/2$ for every n. By efficiency, $\pi_*|\psi_n|$ tends uniformly to $|\phi|$ on K. Since $\int_K |\phi| < 1 - \epsilon$, for n large enough $||\psi_n|| = \int_X \pi_*|\psi_n| < 1$ and thus $\Theta(B_Y) = B_X$. This is the first part of Theorem 1.1.

We will now prove the contrapositive of the second part. If $\overline{\Theta(B_Y)}$ meets the unit sphere in $Q(X)$, then there is a $\phi \in Q(X)$ with $||\phi|| = 1$, and a sequence $\psi_n \in Q(Y)$ such that

$$\Theta(\psi_n) \;\; = \;\; \pi_*(\psi_n) \;\; = \;\; \phi$$

and $||\psi_n|| \to 1$. Since $\pi_*|\psi_n| \geq |\phi|$ and

$$||\psi_n|| - ||\phi|| = \int_X \pi_*|\psi_n| - |\phi| \to 0,$$

we have that $\pi_*|\psi_n|$ tends to $|\phi|$ in the L^1-sense. By Cauchy's integral formula, the L^1 norm of an analytic function controls its modulus of continuity, so $\pi_*|\psi_n| \to |\phi|$ uniformly on compact subsets of X. Therefore ψ_n is an efficient sequence. By Theorem 2.1, the covering is amenable. ■

3 Holomorphic families of measures

This section provides the proof of Theorem 2.1.

Complex probability measures. Let A be a set and let $L^\infty(A)$ denote the Banach space of bounded, complex-valued functions $f(a)$ on A with the norm $||f|| = \sup |f(a)|$.

A *complex probability measure* on A is a map $\mu : A \to \mathbb{C}$ such that $||\mu|| = \sum_A |\mu(a)| < \infty$ and $\sum \mu(a) = 1$. The measure $\mu(B)$ of a set $B \subset A$ is $\sum_B \mu(a)$. The measure μ determines an element of the dual space $L^\infty(A)^*$ by

$$\mu(f) \;=\; \sum_A \mu(a) f(a).$$

Amenability. A *mean* on A is a complex-linear map

$$m : L^\infty(A) \to \mathbb{C}$$

such that $m(f)$ is real if f is real-valued, $m(f) \geq 0$ if $f \geq 0$ and the mean of the constant function $f = 1$ is one. It follows that m is continuous, so m belongs to $L^\infty(A)^*$.

Remark. Elements of $L^\infty(A)^*$ can be interpreted as finitely additive measures on A. From this point of view, a mean is simply a positive finitely additive probability measure.

Let G be a group acting transitively on the set A. The action of G is *amenable* if there exists a G-invariant mean on A. Note that amenability is a property of the pair (G, H) where H is the stabilizer of a point on A, since we may identify A and G/H.

Example. The action of $G = \mathbb{Z}$ on itself by translation is amenable. However it seems impossible to constructively exhibit a translation invariant mean that is defined for *all* bounded functions. Thus it is useful to have a more concrete criterion for amenability, and this is provided by the Følner-Rosenblatt condition [Ros]:

> An action is amenable if and only if there is a net A_α of finite subsets of A such that for every g in G,
> $$|gA_\alpha \Delta A_\alpha|/|A_\alpha| \to 0$$
> as $\alpha \to \infty$.

Here $B\Delta C = (B-C)\cup(C-B)$ denotes the symmetric difference. Frequently the net can be replaced by a sequence. When $G = A = \mathbb{Z}$ we can take $A_n = \{0, 1, \ldots, n\}$.

Remark. In [Mc1], the proof of Theorem 1.1 employs the Følner-Rosenblatt criterion; the present proof works directly with linear functionals on $L^\infty(A)$.

A more complete discussion of amenability can be found in [Mc1], [Gre] and [Pier].

Amenable coverings. We now return to the setting of the introduction, and define a *covering* $\pi : Y \to X$ to be amenable if the action of $\pi_1(X)$ on the coset space $\pi_1(X)/\pi_1(Y)$ is amenable.

Proof of Theorem 2.1. Starting with an efficient sequence $\pi_*(\psi_n) = \phi$, we will show that the covering $\pi : Y \to X$ is amenable.

Let $E \subset X$ be the analytic subvariety on which ϕ vanishes. To simplify notation, let
$$A(x) \quad = \quad \pi^{-1}(x)$$
denote the fiber of the covering over x.

We begin by defining, for each x in $X - E$, a sequence of complex probability measures $\mu_n(x)$ on $A(x)$. Let
$$\mu_n(x)(y) = \frac{\psi_n(y)}{\pi^*\phi(x)}.$$
Since $\mathcal{L}' = \pi^*\mathcal{L}$, ψ_n and $\pi^*\phi$ are sections of the same line bundle and their ratio is a complex number (finite because x is not in E). Note that
$$\frac{1}{\phi(x)} \sum_{y\in A(x)} \psi_n(y) \quad = \quad \frac{\phi(x)}{\phi(x)} = 1$$
by the definition of pushforward, so these are indeed probability measures. The measures $\mu_n(x)$ are sections of the flat bundle of Banach spaces $L^\infty(A(x))^*$.

Test functions. A *test function* (f, U) is a bounded locally constant function on $\pi^{-1}(U)$, where U is an open subset of $X - E$. The function f can be thought of as a constant section of $L^\infty(A(x))$. A family of functionals $\mu(x) \in L^\infty(A(x))$ determines a map $< \mu, f >: U \to \mathbb{C}$ by $x \mapsto \mu(x)(f)$.

Since $\pi : Y \to X$ is a covering map, any bounded function on a fiber of π can be can be prolonged to a test function. Thus $\mu(x)$ is determined by its values on test functions.

Proposition 3.1 1. *The measures $\mu_n(x)$ depend holomorphically on x in $X - E$. That is, $< \mu_n, f >$ is holomorphic on U for any test function (f, U).*

2. *For any compact $K \subset X - E$, $||\mu_n(z)|| \to 1$ uniformly on K as n tends to infinity.*

Proof. Let (f, U) be a test function. Then

$$\mu_n(x)(f) \quad = \quad \frac{1}{\phi(x)} \sum_{\pi(y)=x} \psi_n(y) f(y).$$

The sum converges uniformly on compact sets by the definition of pushforward, so $\mu_n(x)$ is holomorphic.

As for the second statement,

$$||\mu_n(x)|| \quad = \quad \sum_{y \in A(x)} |\mu_n(x)(y)| \quad = \quad \frac{\pi_* |\psi_n|(x)}{|\phi(x)|},$$

and by the definition of efficiency this quantity tends to one uniformly on K as $n \to \infty$. ∎

Proposition 3.2 *There exists a subnet μ_α of μ_n which converges weak*-uniformly on compact sets to a holomorphic functional $\nu(x) \in L^\infty(A(x))^*$.*

This means that $< \mu_\alpha, f >$ converges to $< \nu, f >$ uniformly on compact sets for any test function (f, U).

Proof. Let $L^\infty(A)^*_1$ denote the unit ball endowed with the weak* topology. By theorems of Tychonoff and Alaoglu [Roy], the space

$$\prod_X L^\infty(A(x))^*_1$$

is compact. Since $||\mu_n|| \to 1$, the μ_n are locally bounded, so there is a subnet $\mu_\alpha(x)$ which converges pointwise to $\nu(x)$. Now for each test function (f, U), and each compact $K \subset U$, $< \mu_\alpha, f >$ is a pointwise convergent net of uniformly bounded holomorphic functions. Since bounded holomorphic functions are equicontinuous, the convergence is uniform and the limit is holomorphic. ■

Proposition 3.3 *The functional ν is a locally constant mean on $A(x)$.*

Here locally constant means that $< \nu, f >$ is constant on U for any test function (f, U) defined over a connected set U.

Proof. Since μ_α is a net of complex probability measures, $< \mu_\alpha, 1 >= 1$, so the same is true for ν.

We now appeal to the following easy fact: if a_i is a sequence of complex numbers, with $\sum a_i = 1$ and $\sum |a_i| = 1 + \epsilon$, then $\sum_{\operatorname{Re} a_i < 0} |a_i| \leq \epsilon$ and $\sum |\operatorname{Im} a_i| < \delta(\epsilon)$ where $\delta \to 0$ as $\epsilon \to 0$. Since μ_α is a subnet of μ_n, we have

$$\pi_* |\mu_\alpha(x)| = \sum_{y \in A(x)} |\mu_\alpha(y)| \to 1,$$

and we have just noted that $\sum_{A(x)} \mu_\alpha(y) = 1$. It follows that the limiting functional $\nu(x)$ is real and $\nu(x)(f) \geq 0$ if $f \geq 0$. Thus ν is a mean.

A real-valued holomorphic function is constant. Since $< \nu, f >$ is holomorphic for any test function f, and real-valued if f is real-valued, it follows that ν is locally constant. ■

Completion of the proof of Theorem 2.1. Let x_0 be a basepoint in $X - E$, y_0 a point in Y lying over x_0. The choice of y_0 determines an action of $G = \pi_1(X, x_0)$ on $A(x_0)$, with stabilizer H of y_0 equal to $\pi_1(Y, y_0)$. Thus $\nu(x_0)$ determines a mean on $L^\infty(G/H)$.

We claim this mean is G-invariant. Since E has real codimension two, any loop in $\pi_1(X, x_0)$ has a representative γ which avoids E. The monodromy of the locally constant family of sets $A(x)$ around γ is exactly the action of γ on the fiber $A(x_0) \cong G/H$. But $\nu(x)$ is a locally constant functional on $L^\infty(A(x))$, so it is invariant under the action of γ.

Thus ν is an invariant mean on $L^\infty(G/H)$ and the covering $Y \to X$ is amenable. ■

4 A bridge to affine bundles

To conclude, we give an interpretation of the Barrett-Diller proof of the Theta conjecture [BD], which may illuminate the connection of their work with the argument above.

Definitions. An *affine automorphism* of $\mathbb{C}$ is simply a transformation of the form $z \mapsto az + b$, where $a, b \in \mathbb{C}$ and $a \neq 0$. A *complex affine line* L is a space equipped with a bijection $L \to \mathbb{C}$ well-defined up to composition with affine automorphisms.

Given a countable bounded set $A \subset L$ and a complex probability measure μ on A, define the *barycenter* β of μ by

$$\beta \;=\; \sum_{a \in A} \mu(a) a.$$

Since $\sum_A \mu(a) = 1$, the point $\beta \in L$ is well-defined: it is independant of the particular identification of L with $\mathbb{C}$ needed to form the sum.

Unlike the barycenter of a positive measure, the barycenter of a complex measure can lie outside the convex hull of its support. But it is easy to see:

Proposition 4.1 *If $\sum_A |\mu(a)| = 1 + \epsilon$, and A is contained in a round disk $D(p, r)$ in L, then the barycenter of μ lies in $D(p, r(1 + \epsilon))$.*

Now let $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ denote the Riemann sphere. For any point $z \in \widehat{\mathbb{C}}$, there is an automorphism (Möbius transformation) $\delta : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that $\delta(z) = \infty$. This δ is well-defined up to composition with an affine automorphism. Consequently:

> The complement $L = \widehat{\mathbb{C}} - \{z\}$ of any point $z \in \widehat{\mathbb{C}}$ carries the structure of an affine line.

For μ a complex probability measure on a countable set $A \subset \widehat{\mathbb{C}}$, let $\mathrm{bary}(\mu, z)$ denote the barycenter of μ with respect to the natural affine structure on $\widehat{\mathbb{C}} - \{z\}$. This point is defined so long as $\overline{A}$ does not meet z.

The barycenter is *natural*, in the sense that $\mathrm{bary}(\delta_* \mu, \delta z) = \delta\, \mathrm{bary}(\mu, z)$ for any automorphism δ of $\widehat{\mathbb{C}}$.

Barycenter approach to the Theta conjecture. With these preliminaries in place, we now sketch the Barrett-Diller proof of the Theta conjecture.

Let $\Delta = \{z \,:\, |z| < 1\}$ denote the unit disk, and let $\Sigma = \widehat{\mathbb{C}} - \overline{\Delta}$ denote the complementary disk about infinity.

Let X be a hyperbolic Riemann surface of finite volume. Then we can find a covering map $\pi : \Sigma \to X$ which presents X as the quotient of the disk Σ by the action of a Fuchsian group Γ.

The group Γ acts on the whole Riemann sphere, and $\overline{X} = \Delta/\Gamma$ is also a Riemann surface, the "complex conjugate" of X. More precisely, reflection through the unit circle commutes with the action of Γ, and so it determines an antiholomorphic homeomorphism $\rho : X \to \overline{X}$.

Now suppose $||\Theta_{\Delta/X}|| = 1$. Since the unit ball in $Q(X)$ is compact, we can find a $\phi \in Q(X)$ and a sequence $\psi_n \in Q(\Delta)$ such that $\Theta_{\Delta/X}(\psi_n) = \phi$ and $\lim ||\psi_n|| = ||\phi|| = 1$. From this we will deduce a contradiction.

Let $\psi = \pi^*\phi$ denote the pullback of ϕ to Σ, and let $E \subset \Sigma$ be the discrete set of zeros of ψ.

Pick a point p in Δ, and let A denote the orbit Γp. For $z \in \Sigma - E$, define a complex probability measure $\mu_n(z)$ on A by

$$\mu_n(z)(\gamma p) \;=\; \frac{\psi_n(\gamma^{-1}z)}{\psi(z)}.$$

We claim that for any $\delta \in \Gamma$, the pushforward $\delta_*\mu_n(z) = \mu_n(\delta z)$. Indeed,

$$(\delta_*\mu_n(z))(\gamma p) \;=\; \mu_n(z)(\delta^{-1}\gamma p) \;=\; \frac{\psi_n(\gamma^{-1}\delta z)}{\psi(z)} \;=\; \frac{\psi_n(\gamma^{-1}\delta z)}{\psi(\delta z)} \;=\; \mu_n(\delta z)(\gamma p),$$

where we have used the fact that ψ is Γ-invariant.

Define $f_n : \Sigma - E \to \widehat{\mathbb{C}}$ by

$$f_n(z) \;=\; \mathrm{bary}(\mu_n(z), z);$$

i.e. $f_n(z)$ is the barycenter of $\mu_n(z)$ with respect to z. It is not hard to verify that f_n is holomorphic. Moreover, for any $\delta \in \Gamma$,

$$f_n(\delta z) \;=\; \mathrm{bary}(\mu_n(\delta z), \delta z) \;=\; \mathrm{bary}(\delta_*\mu_n(z), \delta z) \;=\; \delta \mathrm{bary}(\mu_n(z), z) \;=\; \delta f_n(z)$$

by naturality of the barycenter.

To conclude the proof, use the fact that $||\psi_n|| \to ||\phi|| = 1$ to construct a subsequence f_{n_k} converging to $g(z)$ uniformly on compact subsets of $\Sigma - E$. By Proposition 4.1 above, one may show that in the limit, $g(z)$ lies within the convex hull of A with respect to the affine structure determined by z. In particular, g maps $\Sigma - E$ into the closure of the unit disk Δ. Since bounded analytic functions have no isolated singularities, g can be extended to a map $\Sigma \to \overline{\Delta}$.

The key property of this map is that it inherits equivariance from the f_n: that is, $g(\delta z) = \delta g(z)$ for all δ in Γ.

It is easy to see that no such g exists. If g assumes a value in $\partial\Delta$ it must be constant. The constant must be a common fixed point for all elements of Γ, which is absurd for a cofinite volume Fuchsian group.

Otherwise g maps Σ into Δ. Composing with the reflection ρ through the unit circle, we obtain a map $\rho \circ g : \Sigma \to \Sigma$ such that $\rho \circ g(\delta z) = \delta(\rho \circ g(z))$ for δ in Γ. This map descends to an antiholomorphic map $h : X \to X$ which induces the identity on $\pi_1(X)$.

By the Schwarz lemma, h must be an orientation reversing isometry, since it cannot shrink the length of a closed geodesic. Thus the original map g is a Möbius transformation exchanging Σ and Δ. But since g commutes with Γ, it preserves the attracting fixed points of hyperbolic elements of Γ; these are dense in the circle, so g is the identity. This contradicts the fact that g maps Σ to Δ. ■

Remark. It is hard to imagine an analogue of the orientation-reversing map h for a general covering $Y \to X$. The proof of Theorem 2.1 was obtained by studying the measures μ_n.

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