In Search of Ultimate-L the 19th Midrasha Mathematicae Lectures

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In search of Ultimate-L: [1]

The 19th Midrasha Mathematicae Lectures

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Abstract

We give a fairly complete account which first shows that the solution to the inner model problem for one supercompact cardinal will yield an ultimate version of L and then shows that the various current approaches to inner model theory must be fundamentally altered to provide that solution.

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1 Introduction

The Inner Model Program began with Gödel's discovery of L which in the modern view, is the first inner model. Of course it was Scott's Theorem, that if $V = L$ then there are no measurable cardinals, which set the stage for the necessity of the Inner Model Program.

By the early 1970's, the problem to extend the Inner Model Program to the level of supercompact cardinals had emerged as a key problem and the expectation was that in solving this problem the way would be open to extend the solution to much stronger large cardinals. The constructions of Kunen, solving the inner model problem for measurable cardinals, were generalized to solve the inner model problem at the level of Woodin cardinals in series of results driven primarily by seminal constructions of Mitchell and Steel and building on earlier work of Mitchell which had solved the inner model problem for strong cardinals[2].

The levels of Woodin cardinals represent key stages for the inner model program because the internal definability of the wellordering of the reals becomes progressively more complicated through the emergence of determinacy consequences.

By the year 2000, the Inner Model Program had been unconditionally extended by Neeman, [14], to the level of Woodin cardinals which are limits of strong cardinals and conditionally extended, [18], to the level of superstrong cardinals. The latter constructions require not only large cardinal hypotheses (an obvious necessity) but also *iteration hypotheses* which are abstract combinatorial hypotheses for *iterating* countable elementary substructures of rank initial segments of V. These basic hypotheses were first defined and analyzed by Martin and Steel.

The next advance was the extension of the Inner Model Program to the finite levels of supercompact cardinals, [24], again assuming the (same) Iteration Hypothesis that the earlier constructions were conditioned on.

About the same time in a decadal sense, there was a rather unexpected discovery. This was that if one could extend the Inner Model Program to the level of *one* supercompact cardinal then subject to a very general condition on the relationship of the supercompact cardinal of the inner model constructed and supercompact cardinals in V, the inner model constructed must be an *ultimate* version of L. In particular the Scott Effect would no longer apply.

This changed the entire framework for the Inner Model Program; from a program of the incremental understanding of large cardinals through the constructions of generalizations of L with V forever hopelessly out of reach because of Scott's Theorem and its descendents, into a program for perhaps understanding V itself.

The point here is that if there is an ultimate version of L which is compatible with *all* large cardinals and which must always exist in a version that is very close to V, then perhaps there is some version of an axiom that V is an ultimate version of L which is arguably true.

In fact a candidate for exactly such an axiom has been isolated, this is the axiom V = Ultimate-L, implicit in [20] and formally defined in [24].

This axiom strongly couples the *width* of the universe of sets to its *height* since in the context of the axiom V = Ultimate-L, one cannot change the width using Cohen's method of forcing without then changing the height. In particular, the axiom V = Ultimate-L renders Cohen's method of forcing completely useless as a method for establishing independence from the resulting conception of the universe of sets.

Coincident with these developments was another unexpected theorem. This is the HOD Dichotomy Theorem of [20] which is presented here in a more elegant form as Theorem 3.39. This theorem is arguably just an abstract generalization of Jensen's covering lemma. For this one simply recasts the covering lemma as the *Jensen Dichotomy Theorem* which shows that V must either be very *close* to L or

[2]see [6] for a far more thorough and elegant historical account.

very *far* from L.

The HOD Dichotomy Theorem generalizes this to HOD, showing that if there is an extendible cardinal then V must be either very close to HOD or very far from HOD. The existence of Ultimate-L would provide the explanation showing that in fact, unlike the Jensen Dichotomy Theorem, the HOD Dichotomy Theorem is not a dichotomy theorem since HOD must be close to V or equivalently that the "far" option is vacuous.

Of course HOD is not *canonical* in the way that L is since one can easily alter HOD by forcing. But that is not really relevant. The HOD Dichotomy Theorem, which is not a difficult theorem to prove, establishes an unexpected and deep connection between V and definability.

To illustrate, one curious corollary of the HOD Dichotomy Theorem is that if δ is an extendible cardinal then δ *must* be a measurable cardinal in HOD, see Theorem 3.40. Without the hypothesis that δ is an extendible cardinal, this conclusion need not hold even if δ is assumed to be a supercompact cardinal.

But maybe this is all just evidence that the inner model program *cannot* be extended to supercompact cardinals and moreover that there is an *anti-inner model theorem*.

Reinforcing this latter speculation are two points. First, the Jensen Dichotomy Theorem is a true dichotomy theorem since the existence of Silver's $0^\#$, which is implied by the existence of a measurable cardinal, *implies* V is very far from L. So perhaps the HOD Dichotomy Theorem is also a true dichotomy theorem and we simply have not yet discovered what plays the role of $0^\#$.

Now *if* the HOD Dichotomy Theorem is not a dichotomy theorem then one obtains a new generation of *inconsistency results* for the large cardinal hierarchy in the setting where the Axiom of Choice fails. This includes a mild strengthening of Reinhardt cardinals and it includes Berkeley cardinals.

Further one also obtains, but now in the context of the Axiom of Choice, that what seem like natural generalizations of axioms of *definable determinacy* are also false if sufficient large cardinals are assumed to exist.

Thus, and this is the second point, one could argue that it is quite reasonable to expect that there *are* axioms which play the role of $0^\#$ but in the context of the HOD Dichotomy.

In the next section we give a more detailed overview of this presentation and this brings me to a rather important underlying point. This point concerns the status of the Ultimate-L Project which is the program to prove the Ultimate-L Conjecture.

The Ultimate-L Conjecture, as defined in a slightly weaker form on page 102 in comparison to the original version implicit in [20] and defined in [24], is in essence three interrelated conjectures: first that there is no anti inner model theorem, second that the HOD Dichotomy Theorem is not a genuine dichotomy theorem, and third that (assuming sufficient large cardinals) Ultimate-L exists in close proximity to V.

The reference [25] is a manuscript in preparation with the goal of showing that if κ is a huge cardinal then the Ultimate-L Conjecture holds in $V_{\kappa+1}$ in a very slightly weakened form[3].

The issue of course is that until the manuscript is in final form, it is just a work in progress, no matter how confident one is of the eventual outcome.

Given the series of unexpected events to date on this subject, an abundance of caution seems prudent here. The approach in [25] is discussed in a bit more detail at the end of the next section and then again on page 89, in the context of the obstructions identified in the account.

Why then write this account now, before these issues are resolved? At the very least, something noteworthy has happened. The collective impact of all the obstructions which are the focus of this account, is that there are really very few mathematical options now for the form that any proof of the Ultimate-L Conjecture must take. This was not the case before and with hindsight that was a part of the whole problem.

[3] where the condition of weak Σ_2-definability is dropped.

Thus even if there are more surprises to come, this account presents a current snapshot of what is surely a critical and interesting point in the final story.

2 Overview

This is an expanded (and revised) version of the material presented first in a tutorial series of four lectures at the 19th Midrasha Mathematicae Meeting held at the Hebrew University and hosted by the Institute for Advanced Studies. I would like to thank the organizers and the IAS for their efforts in arranging the meeting and providing me the opportunity to give this lecture series. Also I wish to thank the participants for their close attention during the lectures.

Part of this material was given a second time in a week long tutorial series in the Summer School in Mathematical Logic held in Singapore in June, 2016, and hosted by the Institute for Mathematical Sciences (IMS) of the National University of Singapore. Here again I owe a considerable debt to the participants.

The purpose of this article, which was also the goal of the lectures, is to provide a fairly direct and complete account which first shows that the solution to the inner model problem for one supercompact cardinal will yield an ultimate version of L and then shows that the various current approaches to inner model theory must be fundamentally altered to provide that solution.

We examine the current approaches in a progression starting with the natural generalizations of $L[U]$ and ending with the modern framework based on partial extender models. This involves introducing many of the central notions of inner model theory.

The material from Section 3 and Section 4 is essentially all from [20] though the presentation is simplified quite a bit and some of the theorems have been strengthened. The material from Section 5 and Section 6 is new and combined with the material of Section 4 sets the stage for [25].

In fact, there are several changes here from the material given in the Midrasha Mathematicae lectures, particularly in Section 5. This was primarily driven by the goal to produce a version of Theorem 5.35 which could be used in [25].

A substantial portion of the final section is also new and deals with various possible formulations of the axiom, V = Ultimate-L. This revision of the material from the Midrasha Mathematicae lectures reflects more recent results from [25] and highlights how the AD^+-theory of determinacy enters the story by making possible a formulation of the axiom V = Ultimate-L which does *not* involve the detailed level-by-level construction of the actual model, or even the definitions of those levels.

It is interesting to note that for many of the standard generalizations of L which have been identified and studied, for example the partial extender models of Mitchell-Steel, the internal axiomatic characterization is not in general known once the models pass the level of having Woodin cardinals.

There are many reasons for this and not the least of these is the surprising fact that for the Mitchell-Steel models, most of the models *are* nontrivially the generic extension of another such model (if these models are simply assumed to be iterable), one just needs that within the models there is at least one Woodin cardinal, [24].

The three sections, Section 4, Section 5, and Section 6, indicate critical constraints which must be met and this turns out to provide sufficient information to convincingly predict what must happen and how. Part of this is what was expected but a significant part was completely unexpected and this concerns the issue of whether a construction of a fine-structural hierarchy based only on a general iteration hypothesis for V, could ever be vacuous. In fact I predict that what happens is much more extreme.

First, assuming the existence of a huge cardinal, the Weak (ω_1+1)-Iteration Hypothesis is consistently false and moreover the Weak Unique Branch Hypothesis outright false.

These iteration hypotheses are defined in Section 4.1 and are weak versions of what have become the standard iteration hypotheses used when outright constructions (based on just large cardinal hypotheses) are not known.

More surprising is the reason. This happens because otherwise one can prove the existence of fine-structural models and contradict the fundamental obstruction identified in Section 5.

The models constructed for this purpose are extender models and they are in the hierarchy of *non-strategic extender* models since no additional predicate for iterability is added. In particular even though the models are iterable, the iteration strategy is not added to the model. This is the traditional form of the fine-structural generalizations of L.

Thus I predict that a backgrounded construction of fine-structural models which succeeds based on what seems to be a natural iteration hypothesis can be *vacuous*.

The second prediction is that the essential core of the Ultimate-L Conjecture *holds* in $V_{\kappa+1}$ if κ is a huge cardinal. More precisely, if κ is a huge cardinal then there exists a transitive set M such that

(1) $M \vDash$ "$V =$ Ultimate-L",

(2) $\mathrm{Ord}^M = \kappa$ and $M \subset \mathrm{HOD}^{V_\kappa}$,

(3) For some $\delta < \kappa$, $(V_\kappa, M) \vDash$ "M is a weak extender model for δ is supercompact".

So in summary, I believe all the obstacles, along with their resolutions, have been finally identified and as a result it is now possible to prove that the core elements of Ultimate-L Conjecture, as specified above, hold in $V_{\kappa+1}$ if κ is a huge cardinal.

The methodology is to build the necessary witnesses for this through the construction of extender models in the hierarchy of *strategic-extender* models. This is the hierarchy of (iterable) extender models where each model is constructed from *two* predicates, one for the extender sequence and one for the iteration strategy.

The immediate question that this raises is how the construction of the strategic-extender models necessary to witness that the (strictly speaking, "weak") Ultimate-L Conjecture holds in $V_{\kappa+1}$ can possibly succeed when the construction of the *simpler* nonstrategic-extender models must fail since it is the construction of the latter which leads to the indicated contradiction.

The answer lies again in the problem of iterability. The construction of the strategic-extender models succeeds because one can prove that the models are iterable which one cannot do in the nonstrategic case. This is enabled by connecting with the general theory of AD^+-models and that connection does not exist in the nonstrategic case. This connection is through the HOD's as computed within the AD^+-models which in the relevant cases one verifies is a strategic-extender model as part of the induction.

The main obstacle to proving the Ultimate-L Conjecture in light of the obstructions identified here, is finding the technical reason why the hierarchy must transition from the nonstrategic-extender hierarchy to the strategic-extender hierarchy. But this is only a mystery if one accepts that no vacuous construction is possible *because* the iteration hypotheses one naturally uses must be *provable*. It is after surrendering on this point that the picture becomes what seems now so obvious: there is no obstacle here since the iteration hypotheses are *false* and this is because there *are* vacuous constructions.

Perhaps in the ideal world, this article would have been written a year from now after [25] was completely finished, thoroughly checked, and circulated. Of course then it would probably be a very different article and in any case, that is not this world and the making the predictions detailed above seems really the only option, short of saying nothing.

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Mathematicae, develop this material in suitable form for a tutorial, and so ultimately write this paper which at least for me has greatly clarified the situation.

Finally, I would like to also acknowledge support from the Newton Institute for Mathematical Sciences[5], the Institue for Mathematical Sciences (Singapore), and the National Science Foundation (US)[6,7].

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3 Weak extender models, universality, and the HOD Dichotomy

3.1 Supercompactness

We begin by reviewing the basic notions related to supercompact cardinals. Further details and the history of the development can be found in [6].

Definition 3.1. Suppose that κ is a regular cardinal and that $\kappa < \lambda$.

(1) $\mathcal{P}_\kappa(\lambda) = \{\sigma \subset \lambda \mid |\sigma| < \kappa\}$.

(2) Suppose that $U \subseteq \mathcal{P}(\mathcal{P}_\kappa(\lambda))$ is an ultrafilter.

a) U is *fine* if for each $\alpha < \lambda$,
$$\{\sigma \in \mathcal{P}_\kappa(\lambda) \mid \alpha \in \sigma\} \in U.$$

b) U is *normal* if for each function
$$f : \mathcal{P}_\kappa(\lambda) \to \lambda$$
such that
$$\{\sigma \in \mathcal{P}_\kappa(\lambda) \mid f(\sigma) \in \sigma\} \in U,$$
there exists $\alpha < \lambda$ such that
$$\{\sigma \in \mathcal{P}_\kappa(\lambda) \mid f(\sigma) = \alpha\} \in U.$$ □

Definition 3.2. Suppose that κ is an uncountable regular cardinal. Then κ is a *supercompact cardinal* if for each $\lambda > \kappa$ there exists an ultrafilter U on $\mathcal{P}_\kappa(\lambda)$ such that:

(1) U is κ-complete,

(2) U is a normal fine ultrafilter. □

The following basic lemma gives the connection between the two common formulations of supercompactness. One can require that the transitive class M and the embedding j each be Σ_2-definable in V from parameters.

Lemma 3.3. *Suppose κ is an uncountable regular cardinal. Then the following are equivalent.*

(1) *κ is a supercompact cardinal.*

(2) *For each $\lambda > \kappa$, there exists an elementary embedding*
$$j : V \to M$$
such that $\mathrm{CRT}(j) = \kappa$*, $j(\kappa) > \lambda$, and such that $M^\lambda \subset M$.*

Proof. Suppose κ is supercompact and $\lambda > \kappa$. Let U be a κ-complete normal fine ultrafilter on $\mathcal{P}_\kappa(\lambda)$. Let
$$j : V \to M \cong \mathrm{Ult}(V, U)$$
be the ultrapower embedding. Thus

(1.1) $j[\lambda] \in M$ and $j[\lambda] \in j(\mathcal{P}_\kappa(\lambda))$,

(1.2) $M = \{j(f)(j[\lambda]) \mid f \in V\}$.

Suppose $h : \lambda \to M$. For each $\alpha < \lambda$, let

$$f_\alpha : \mathcal{P}_\kappa(\lambda) \to V$$

be a function such that $h(\alpha) = j(f_\alpha)(j[\lambda])$. The function f_α exists by (1.1) and (1.2).

For each $\sigma \in \mathcal{P}_\kappa(\lambda)$ let

$$g_\sigma : \sigma \to V$$

be the function defined by $g_\sigma(\alpha) = f_\alpha(\sigma)$. Finally define

$$f : \mathcal{P}_\kappa(\lambda) \to V$$

by $f(\sigma) = g_\sigma$. Thus

$$j(f)(j[\lambda]) : j[\lambda] \to M$$

and

$$j(f)(j[\lambda]) \circ j|\lambda = h.$$

Therefore h is definable in M from $j(f)(j[\lambda])$ and so $h \in M$.

This proves that (1) implies (2). Now suppose that $\lambda > \kappa$ and that

$$j : V \to M$$

is an elementary embedding such that $\mathrm{CRT}(j) = \kappa$, $j(\kappa) > \lambda$, and such that $M^\lambda \subset M$. Thus

$$j[\lambda] \in j(\mathcal{P}_\kappa(\lambda)).$$

Let U be the set of all $A \subset \mathcal{P}_\kappa(\lambda)$ such that

$$j[\lambda] \in j(A).$$

Then U is a κ-complete normal fine ultrafilter on $\mathcal{P}_\kappa(\lambda)$. □

We shall need a specific variation of Solovay's Lemma on sets of measure one for normal fine κ-complete ultrafilters on $\mathcal{P}_\kappa(\lambda)$ where $\lambda > \kappa$ is a regular cardinal.

Lemma 3.4 (Solovay's Lemma). *Suppose that $\kappa < \lambda$ are regular cardinals and $<$ is a wellordering of $H(\lambda^+)$. Then there exists a set $X \subset \mathcal{P}_\kappa(\lambda)$ such that the following hold.*

(1) *Suppose U is a κ-complete, normal, fine, ultrafilter on $\mathcal{P}_\kappa(\lambda)$. Then $X \in U$.*

(2) *Suppose $\sigma, \tau \in X$ and $\sup(\sigma) = \sup(\tau)$. Then $\sigma = \tau$.*

(3) *X is uniformly definable in $(H(\lambda^+), <)$ from κ.*

Proof. Let $S = \{\alpha < \lambda \mid \mathrm{cof}(\alpha) = \omega\}$ and let

$$\langle S_\alpha : \alpha < \lambda \rangle$$

be the $<$-least partition of S into λ many stationary sets. Finally let X be the set of all $\sigma \in \mathcal{P}_\kappa(\lambda)$ such that

(1.1) $\omega < \mathrm{cof}(\sup(\sigma)) < \kappa$,

(1.2) σ is the set of $\alpha < \sup(\sigma)$ such that $S_\alpha \cap C \neq \emptyset$ for all closed cofinal subsets of $\sup(\sigma)$.

Then using the ultrapower embedding

$$j : V \to M \cong \mathrm{Ult}(V, U)$$

given by U, it follows that $j[\lambda] \in j(X)$ and so X witnesses the lemma. □

3.2 Weak Extender Models

The Inner Model Problem for supercompact cardinals has been a fundamental open problem for 40 years. Given the first solution of the inner model problem for measurable cardinals, this is the inner model $L[U]$ defined and analyzed in seminal work of Kunen, [7], and then Silver, a natural requirement for the solution at the level of a supercompact cardinal is that it should yield, or at least be compatible with, a *weak extender model* as defined below.

The original motivation here was to develop the theory of such weak extender models in order to either discover the relevant clues as to how to construct the fine-structural versions of such inner models, or conversely to conclude that the program cannot in general succeed. The latter would be an *anti-inner model theorem*.

Definition 3.5. A transitive class $N \vDash \text{ZFC}$ is a *weak extender model for δ is supercompact* if for every $\gamma > \delta$ there exists a δ-complete normal fine measure U on $\mathcal{P}_\delta(\gamma)$ such that

(1) $N \cap \mathcal{P}_\delta(\gamma) \in U$,

(2) $U \cap N \in N$. □

Analyzing *covering properties* between transitive models of ZFC has long been a fruitful subject of study. Such notions arise naturally between V and its generic extensions, and between V and canonical inner models of V, such as L.

Definition 3.6 (Hamkins [3]). Suppose N is a transitive class and that δ is a regular cardinal. Then N has the *δ-covering property* if for each $\sigma \subset N$ such that $|\sigma| < \delta$, there exists $\tau \in N$ such that

(1) $\sigma \subset \tau$,

(2) $|\tau| < \delta$. □

Remark 3.7. *V has the δ-covering property in $V[G]$ whenever G is V-generic for a partial order $\mathbb{P}$ which is $(<\delta)$-cc in V.* □

Lemma 3.8. *Suppose that N is a weak extender model for δ is supercompact. Then N has the δ-covering property.*

Proof. Let $\sigma \subset N$ be a set with $|\sigma| < \delta$. Since

$$N \vDash \text{ZFC}$$

we can reduce to the case that $\sigma \subset \text{Ord}$. Let $\lambda > \delta$ be such that $\sigma \subset \lambda$. Let U be a δ-complete normal fine ultrafilter on $\mathcal{P}_\delta(\lambda)$ such that

$$N \cap \mathcal{P}_\delta(\lambda) \in U.$$

Thus since U is fine and δ-complete,

$$\{\tau \in \mathcal{P}_\delta(\lambda) \mid \sigma \subset \tau\} \in U$$

and so there must exist

$$\tau \in \mathcal{P}_\delta(\lambda) \cap N$$

such that $\sigma \subset \tau$. □

Lemma 3.9. *Suppose that N is a weak extender model for δ is supercompact and that $\gamma > \delta$ is a regular cardinal in N. Then* $(\text{cof}(\gamma))^V = |\gamma|^V$.

Proof. Let U be a δ-complete normal fine ultrafilter on $\mathcal{P}_\delta(\gamma)$ such that

(1.1) $N \cap \mathcal{P}_\delta(\gamma) \in U$,

(1.2) $U \cap N \in N$.

By Solovay's Lemma applied within N, there exists a set

$$X \in N \cap U$$

such that π is 1-to-1 on X where $\pi(\sigma) = \sup(\sigma)$.

Let $C \subset \gamma$ be a closed cofinal set of ordertype $(\mathrm{cof}(\gamma))^V$.

Let

$$j : V \to M$$

be the ultrapower embedding given by U. Thus $j[\gamma]$ is the unique element σ of $j(X)$ such that

$$\sup(\sigma) = \sup(j[\gamma]).$$

But C is closed cofinal in γ and so

$$\sup(j[\gamma]) \in j(C).$$

Therefore

$$\{\sigma \in X \mid \sup(\sigma) \in C\} \in U.$$

Further, since U is fine,

$$\cup\{\sigma \in X \mid \sup(\sigma) \in C\} = \gamma.$$

Therefore $|\gamma|^V = |C|^V \cdot \delta = (\mathrm{cof}(\gamma))^V \cdot \delta$.

Finally γ is a regular cardinal in N and N has the δ-covering property and so

$$(\mathrm{cof}(\gamma))^V \geq \delta.$$

Thus $|\gamma|^V = |C|^V \cdot \delta = (\mathrm{cof}(\gamma))^V \cdot \delta = (\mathrm{cof}(\gamma))^V$. □

Theorem 3.10. *Suppose that N is a weak extender model for δ is supercompact and that $\gamma > \delta$ is a singular cardinal. Then γ is a singular cardinal in N and*

$$(\gamma^+)^N = \gamma^+.$$

Proof. If γ is a regular cardinal in N then by Lemma 3.9, $\mathrm{cof}(\gamma) = |\gamma|$ which contradicts that γ is singular.

Let $\lambda = (\gamma^+)^N$. Then λ is a regular cardinal in N and so again by Lemma 3.9, $\mathrm{cof}(\lambda) = |\lambda| \geq \gamma$. But $\mathrm{cof}(\lambda)$ is a regular cardinal and so $\mathrm{cof}(\lambda) > \gamma$. This implies that $\lambda = \gamma^+$. □

3.3 Extendible cardinals and Magidor's Lemma

A natural strengthening of the notion of a supercompact cardinal is given by the notion of an extendible cardinal. Again [6] is an excellent reference for further details, both historical and mathematical.

Definition 3.11. Suppose that δ is a cardinal. Then δ is an *extendible cardinal* if for each $\lambda > \delta$ there exists an elementary embedding

$$\pi : V_{\lambda+1} \to V_{\pi(\lambda)+1}$$

such that $\mathrm{CRT}(\pi) = \delta$ and $\pi(\delta) > \lambda$. □

Lemma 3.12 (Magidor, [9]). *Suppose that δ is a regular cardinal. Then the following are equivalent.*

(1) *δ is supercompact.*

(2) *For each $\lambda > \delta$ there exist $\bar{\delta} < \bar{\lambda} < \delta$ and an elementary embedding*

$$\pi : V_{\bar{\lambda}+1} \to V_{\lambda+1}$$

such that $\mathrm{CRT}(\pi) = \bar{\delta}$ and such that $\pi(\bar{\delta}) = \delta$. □

Lemma 3.13. *Suppose that N is a weak extender model for δ is supercompact. Then for each $\lambda > \delta$ and for each $a \in V_\lambda$, there exist $\bar{\delta} < \bar{\lambda} < \delta$, $\bar{a} \in V_{\bar{\lambda}}$, and an elementary embedding*

$$\pi : V_{\bar{\lambda}+1} \to V_{\lambda+1}$$

such that the following hold.

(1) $\text{CRT}(\pi) = \bar{\delta}$, $\pi(\bar{\delta}) = \delta$, *and* $\pi(\bar{a}) = a$.

(2) $\pi(N \cap V_{\bar{\lambda}}) = N \cap V_\lambda$.

(3) $\pi|(N \cap V_{\bar{\lambda}}) \in N$.

Proof. By increasing λ and replacing a by the pair (a, λ) if necessary, we can reduce to the case that

$$\lambda = |V_\lambda|$$

and that $\text{cof}(\lambda) = \omega$. Thus $|N \cap V_\lambda|^N = \lambda$. Fix a bijection

$$\rho : \lambda \to N \cap V_\lambda$$

such that $\rho \in N$.

Let U be a δ-complete normal fine ultrafilter on $\mathcal{P}_\delta(\lambda)$ such that

(1.1) $N \cap \mathcal{P}_\delta(\lambda) \in U$,

(1.2) $U \cap N \in N$.

For each $\sigma \in \mathcal{P}_\delta(\lambda)$, let

$$X_\sigma = \{\rho(\alpha) \mid \alpha \in \sigma\}.$$

Let Z be the set of all $\sigma \in \mathcal{P}_\delta(\lambda)$ such that

$$X_\sigma \prec N \cap V_\sigma.$$

Thus $Z \in U$. For each $\sigma \in Z$, let M_σ be the transitive collapse of X_σ. The key claim is:

(2.1) $\{\sigma \in Z \mid M_\sigma = N \cap V_\alpha$ where α is the ordertype of $\sigma\} \in U$.

This follows easily by working in N and considering the ultrapower embedding,

$$j_W : N \to M_W \cong \text{Ult}(N, W)$$

where $W = U \cap N$. The relevant points are:

(3.1) $j_W[\lambda] \in M_W$,

(3.2) $W = \{A \subset \mathcal{P}_\kappa(\lambda) \cap N \mid A \in N$ and $j_W[\lambda] \in j_W(A)\}$.

Now let

$$j_U : V \to M_U \cong \text{Ult}(V, U)$$

be the ultrapower embedding (now computed in V). Thus since $|V_\lambda| = \lambda$ and since $\text{cof}(\lambda) = \omega$,

$$(M_U)^{V_{\lambda+1}} \subset M_U$$

and so $j_U|V_{\lambda+1} \in M_U$. Further by (2.1),

$$j_U(N \cap V_\lambda) \cap V_\lambda = N \cap V_\lambda.$$

Thus the following hold where as usual $j_U(N)$ denotes that class

$$j_U = \cup\{j_U(N \cap V_\alpha) \mid \alpha \in \text{Ord}\}.$$

(4.1) $j_U|(N \cap V_\lambda) \in j_U(N)$.

(4.2) $(\text{cof}(\lambda))^N < \delta$.

(4.3) $j_U|(N \cap V_{\lambda+1}) \in j_U(N)$ (since $j_U[\lambda] \in j_U(N)$).

(4.4) $j_U(N) \cap V_\lambda = N \cap V_\lambda$.

Note that (4.1)–(4.4) imply that the conclusion of the lemma holds for $(j_U(a), j_U(\lambda))$ in M_U for $j_U(N)$. Therefore the conclusion of the lemma holds in V for (a, λ) relative to N. ◻

3.4 Elementary embeddings of weak extender models

We now prove that if δ is an extendible cardinal and N is a weak extender model for δ is supercompact, then N has a remarkable closure property relative to elementary embeddings of N with critical point at least δ.

This theorem is in a natural sense a strong generalization of the following corollary of a theorem of Dodd and Jensen. By a recent result of Jensen and Steel, the analogous theorem holds for essentially all large cardinal notions below the level of a Woodin cardinal. Here we focus on singular cardinals in V and N simply because of the conclusion of Theorem 3.10.

Theorem 3.14 (after Dodd-Jensen). *Suppose that $N \vDash$ ZFC is an inner model such that*

$$\gamma^+ = (\gamma^+)^N$$

for a proper class of singular cardinals which are singular in N. Suppose in V there is a measurable cardinal. Then in N, there is an inner model with a measurable cardinal. □

Theorem 3.14 is just one of a series of theorems which show that if $N \vDash$ ZFC is an inner model such that

$$\gamma^+ = (\gamma^+)^N$$

for a proper class of singular cardinals which are singular in N, then N has inner models for various large cardinal hypotheses that hold in V. For such inner models N which are constructed as *enlargements* of L, the large cardinal hypotheses which can hold in V cannot exceed the level of large cardinal hypotheses which *hold* in N. At levels beyond that of a Woodin cardinal, the precise generalizations involve some version of *correctness* or *iterability*.

By Theorem 3.10, if N is a weak extender model for the supercompactness of δ, then

$$\gamma^+ = (\gamma^+)^N$$

and γ is singular in N, for *all* singular cardinals $\gamma > \delta$.

Therefore, Theorem 3.14 and its generalizations suggest that N should contain inner models of any large cardinal hypothesis which holds in V and moreover if N is actually an enlargement of L then these large cardinal hypotheses should hold in N.

In fact we obtain much more and we shall prove two versions of this, Theorem 3.15 and the more general Theorem 3.26 which is formulated in terms of *extenders*.

Theorem 3.15. *Suppose that N is a weak extender model for δ is supercompact and $\gamma > \delta$ is a cardinal in N. Suppose that*

$$j : H(\gamma^+)^N \to H(j(\gamma)^+)^N$$

is an elementary embedding such that $\delta \leq \text{CRT}(j)$. Then $j \in N$.

Proof. Fix $\lambda > j(\gamma)$ such that $\lambda = |V_\lambda|$. Letting $a = j$, by Lemma 3.13, there exist $\bar{\delta} < \bar{\lambda} < \delta$, $\bar{a} \in V_{\bar{\lambda}}$, and an elementary embedding

$$\pi : V_{\bar{\lambda}+1} \to V_{\lambda+1}$$

such that the following hold.

(1.1) $\text{CRT}(\pi) = \bar{\delta}$, $\pi(\bar{\delta}) = \delta$, and $\pi(\bar{a}) = a$.

(1.2) $\pi(N \cap V_{\bar{\lambda}}) = N \cap V_\lambda$.

(1.3) $\pi|(N \cap V_{\bar{\lambda}}) \in N$.

Thus $\bar{a} = \bar{j}$ where

$$\bar{j} : H(\bar{\gamma}^+)^N \to H(\bar{j}(\bar{\gamma})^+)^N.$$

It suffices to prove:

(2.1) $\bar{j} \in N$;

since $\pi(\bar{j}) = j$ and since $\pi|(N \cap V_{\bar{\lambda}}) \in N$.

Let

$$E = \left\{(A, \xi) \mid A \in \mathcal{P}(\bar{\gamma}) \cap N, \xi < \bar{j}(\bar{\gamma}), \text{ and } \xi \in \bar{j}(A)\right\}$$

We prove that $E \in N$. This implies that

$$\bar{j}| \left(\mathcal{P}(\bar{\gamma}) \cap N\right) \in N$$

which implies that $\bar{j} \in N$.

The key point is:

(3.1) $\pi| \left(H(\bar{\gamma}^+)\right)^N \in \left(H(\gamma^+)\right)^N$.

This is because $\pi|(N \cap V_{\bar{\lambda}}) \in N$ noting that $(H(\gamma^+))^N$ is closed under γ-sequences in N.

Let

$$\pi^* = \pi| \left(H(\bar{\gamma}^+)\right)^N \in \left(H(\gamma^+)\right)^N.$$

Thus $\pi^* \in (H(\gamma^+))^N$ and so $\pi^* \in \text{dom}(j)$.

Now fix $A \in \mathcal{P}(\bar{\gamma}) \cap N$ and $\xi < \bar{j}(\bar{\gamma})$. Thus

$$\begin{aligned} \xi \in \bar{j}(A) &\iff \pi(\xi) \in \pi(\bar{j})(\pi(A)) \\ &\iff \pi(\xi) \in j(\pi(A)) \\ &\iff \pi(\xi) \in j(\pi^*(A)) \\ &\iff \pi(\xi) \in j(\pi^*)(j(A)) = j(\pi^*)(A). \end{aligned}$$

Thus E can be computed from $\pi|\bar{j}(\bar{\gamma})$ and $j(\pi^*)$. Both these functions are in N and so $E \in N$. □

We recall the large cardinal hypothesis that κ is n-huge.

Definition 3.16. Suppose $n < \omega$. Then κ is *n-huge* if there exists an elementary embedding

$$j : V \to M$$

such that $\text{CRT}(j) = \kappa$ and such that $M^{\kappa_n} \subset M$ where

$$\langle \kappa_i : i < \omega \rangle$$

is the sequence where $\kappa_0 = \kappa = \text{CRT}(j)$ and for all $i < \omega$, $\kappa_{i+1} = j(\kappa_i)$. □

Note that κ is 0-huge if and only if κ is a measurable cardinal. However if κ is 1-huge then in V_κ there are extendible cardinals and much more.

The following typical corollary of Theorem 3.15 illustrates the universality, for large cardinal hypotheses, of weak extender models for supercompactness.

Theorem 3.17. *Suppose that N is a weak extender model for δ is supercompact. Suppose that for each $n < \omega$, there is a proper class of n-huge cardinals. Then in N, for each $n < \omega$, there is a proper class of n-huge cardinals.* □

Theorem 3.18 (Kunen,[8]). *Suppose that λ is a cardinal. Then there is no non-trivial elementary embedding*

$$j : V_{\lambda+2} \to V_{\lambda+2}.$$

Proof. Let j be given. Note that $V_{\lambda+2}$ is logically equivalent to $H(|V_{\lambda+1}|^+)$ and so j yields an elementary embedding

$$\pi : H(\lambda^{++}) \to H(\lambda^{++}).$$

Note that $\pi(\lambda) = \lambda$ and $\pi(\lambda^+) = \lambda^+$.

Let $S = \{\alpha < \lambda^+ \mid \mathrm{cof}(\alpha) = \omega\}$ and let $\langle S_\alpha : \alpha < \lambda^+\rangle$ be a partition of S into stationary sets. Let

$$\langle T_\alpha : \alpha < \lambda^+\rangle = \pi(\langle S_\alpha : \alpha < \lambda^+\rangle).$$

Let $C = \{\alpha \in S \mid \pi(\alpha) = \alpha\}$. Thus C is ω-closed and cofinal in λ^+. By the elementarity of π, for each $\alpha < \lambda^+$, T_α is a stationary subset of S and so for each $\alpha < \lambda^+$,

$$C \cap T_\alpha \neq \emptyset.$$

Let $\kappa = \mathrm{CRT}(\pi)$ and choose

$$\xi \in C \cap T_\kappa.$$

Finally choose $\beta < \lambda^+$ such that $\xi \in S_\beta$. Then

$$\xi = \pi(\xi) \in \pi(S_\beta) = T_{\pi(\beta)}.$$

This implies $\pi(\beta) = \kappa$ which contradicts that $\kappa = \mathrm{CRT}(\pi)$. □

Theorem 3.19. *Let N be a weak extender model for δ is supercompact. Then there is no nontrivial elementary embedding $j : N \to N$ such that $\delta \leq \mathrm{CRT}(j)$.*

Proof. By Theorem 3.15, for each $\kappa > \delta$, $j|(N \cap V_{\kappa+1}) \in N$. Thus j is amenable to N and in particular there must exist a cardinal λ of N such that $\mathrm{CRT}(j) < \lambda$, $j(\lambda) = \lambda$, and such that

$$j|(V_{\lambda+2} \cap N) \in N.$$

This contradicts Kunen's Theorem. □

3.5 Extenders

For our purposes, the theory ZF\Powerset is formulated with the Collection Axiom in place of the Replacement Axiom. Over this base theory, the various formulations of the Axiom of Choice are *not* all equivalent, and the Wellordering Principle is the strongest among the usual variations. Thus we define ZFC\Powerset to be the theory ZF\Powerset (with the Collection Axiom) together with the Wellordering Principle.

The issue which arises from which formulation of the Axiom of Choice to use is the following. Suppose that M and N are transitive models of ZFC\Powerset and that

$$\pi : M \to N$$

is an elementary embedding which is cofinal in the sense that $N = \cup\{\pi(a) \mid a \in M\}$. Suppose π is the identity on the ordinals. Must π be the identity?

If one uses the Wellordering Principle, then the answer is yes, π must be the identity. If however one uses the usual formulation of the Axiom of Choice then the answer is no, π need not be the identity. We give an example.

Let $L[G]$ be a generic extension of L for adding ω_2^L many Cohen reals and let $L[G][g]$ be a generic extension of $L[G]$ for adding ω_2^L-many Cohen reals.

Now define

$$M = L_{\omega_2^L}(\mathbb{R}^{L[G]})[G]$$

and define

$$N = L_{\omega_2^L}(\mathbb{R}^{L[G][g]})[G][g],$$

where each is viewed as a transitive set. Thus in each case we are constructing over the reals from an additional predicate. Note that $\mathcal{P}(\omega)$ exists in both M and N but for example, $\mathcal{P}(\omega_1)$ does not exist in either M or N.

It follows by the homogeneity of Cohen forcing that both M and N are models ZFC\Powerset with the usual formulation of the Axiom of Choice and that the natural map

$$\pi : M \to N$$

where $\pi(\mathbb{R}^M) = \mathbb{R}^N$ is an elementary embedding.

Finally for purposes of constructing inner models, one is really only interested transitive models of ZFC\Powerset which are of the form $L_\alpha[P]$, and in this situation the various possible formulations of ZFC\Powerset discussed above, are all equivalent.

Definition 3.20. Suppose that M and N are transitive models of ZFC\Powerset and that

$$\pi : M \to N$$

is an elementary embedding. Then π is *cofinal* if

$$N = \cup \{\pi(a) \mid a \in M\}. \qquad \square$$

Definition 3.21. Suppose that M and N are transitive models of ZFC\Powerset and that

$$\pi : M \to N$$

is a cofinal elementary embedding which is not the identity.

Let $\kappa = \mathrm{CRT}(\pi)$ and suppose that $\eta \in \mathrm{Ord}^N$. Let $\hat{\eta}$ be least such that

$$\eta \leq \pi(\hat{\eta}).$$

For each $a \in [\eta]^{<\omega}$, let

$$E_a = \left\{A \in N \cap \mathcal{P}\left([\hat{\eta}]^{|a|}\right) \mid a \in \pi(A)\right\}.$$

Let $E = \langle E_a : a \in [\eta]^{<\omega}\rangle$. Then:

(1) E is an *M-extender*.

(2) η is the *length* of E.

(3) κ is the *critical point* of E. $\square$

Definition 3.22. Suppose that M is a transitive model of ZFC\Powerset and that

$$E = \langle E_a : a \in [\eta]^{<\omega}\rangle$$

is an M-extender. Then

$$\mathrm{Ult}_0(M, E) = \lim_{a \in [\eta]^{<\omega}} \mathrm{Ult}_0(M, E_a). \qquad \square$$

Remark 3.23. Following the conventions in inner model theory we use the notation $\mathrm{Ult}_0(M, E)$ instead of $\mathrm{Ult}(M, E)$. The reason is that in the general case where M is not assumed to be a model of ZFC\Powerset there can exist more complicated ultrapowers which one can define, these include the *fine-structural* ultrapowers $\mathrm{Ult}_n(M, E)$. $\square$

Lemma 3.24. *Suppose that M is a transitive model of* ZFC\Powerset *and that*

$$E = \langle E_a : a \in [\eta]^{<\omega}\rangle$$

is an M-extender. Then:

(1) $\mathrm{Ult}_0(M, E)$ *is wellfounded.*

(2) *Let M_E be the transitive collapse of* $\mathrm{Ult}_0(M, E)$ *and let*

$$\pi_E : M \to M_E$$

be the ultrapower embedding. Then:

(a) *π_E is a cofinal elementary embedding.*

(b) $\mathrm{CRT}(\pi_E) < \eta < \mathrm{Ord}^{M_E}$.

(c) *Let F be the M–extender of length η given by π_E. Then $F = E$.* □

The following theorem which is the Universality Theorem for weak extender models, is the general version of Theorem 3.15 and this is formulated simply in terms of N-extenders with no assumptions whatsoever on the *strength* of the extenders.

This version of universality is optimal in that it characterizes when an N-extender (which has large enough critical point) must belong to N in the simplest possible terms.

Remark 3.25. We note that Theorem 3.26 implies Theorem 3.15. The only issue is that given

$$j : H(\gamma^+)^N \to H(j(\gamma)^+)^N$$

as in the hypothesis of Theorem 3.15 and letting E be the $H(\gamma^+)^N$-extender of length $j(\gamma)$ given by j, one must verify $\mathrm{Ult}_0(N, E)$ is wellfounded so that E is also an N-extender.

The point here is that if

$$\pi_E : N \to M_E \cong \mathrm{Ult}_0(N, E)$$

is the ultrapower embedding then

$$\pi_E | H(\gamma^+)^N = j$$

and so for each $A \in \mathcal{P}(\gamma) \cap N$, $\pi_E(A) \in N$.

The wellfoundedness of $\mathrm{Ult}_0(N, E)$ follows by using j and appealing to the δ-covering property of N. □

Theorem 3.26 (The Universality Theorem). *Suppose that N is a weak extender model for δ is supercompact and that E is an N-extender of length η with critical point $\kappa_E \geq \delta$. Let*

$$\pi_E : N \to M_E \cong \mathrm{Ult}_0(N, E)$$

be the ultrapower embedding. Then the following are equivalent.

(1) *For each $A \subset \eta$, $\pi_E(A) \cap \eta \in N$.*

(2) *$E \in N$.*

Proof. Trivially (2) implies (1) and so it suffices to prove (1) implies (2). The proof that (1) implies (2) is just a reworking of the proof of Theorem 3.15.

Let ι be least such that $\pi_E(\iota) \geq \eta$. Thus ι is a cardinal of N. Fix $\lambda > \eta$ such that $\lambda = |V_\lambda|$ and such that $\mathrm{cof}(\lambda) > \iota$. Thus

$$\pi_E(\lambda) = \lambda.$$

By Lemma 3.13, there exist $\bar{\delta} < \bar{\lambda} < \delta$, $\bar{E} \in V_{\bar{\lambda}}$, and an elementary embedding

$$\pi : V_{\bar{\lambda}+1} \to V_{\lambda+1}$$

such that the following hold.

(1.1) $\mathrm{CRT}(\pi) = \bar{\delta}$, $\pi(\bar{\delta}) = \delta$, and $\pi(\bar{E}) = E$.

(1.2) $\pi(N \cap V_{\bar{\lambda}}) = N \cap V_\lambda$.

(1.3) $\pi|(N \cap V_{\bar{\lambda}}) \in N$.

Thus $\bar{E}$ is an N-extender. Let $\bar{\eta}$ be the length of $\bar{E}$, let

$$\pi_{\bar{E}} : N \to \mathrm{Ult}_0(N, \bar{E})$$

be the ultrapower embedding, and let $\bar{\iota}$ be least such that $\pi_{\bar{E}}(\bar{\iota}) \geq \bar{\eta}$. Thus

(2.1) $\pi(\bar{\iota}) = \iota$,

(2.2) $\pi(\pi_{\bar{E}} \cap V_{\bar{\lambda}}) = \pi_E \cap V_\lambda$.

Let

$$P_{\bar{E}} = \{(A, \xi) \mid A \in \mathcal{P}(\bar{\iota}) \cap N, \xi < \bar{\eta} \text{ and } \xi \in \pi_{\bar{E}}(A)\}$$

We prove that $P_{\bar{E}} \in N$. This implies $\bar{E} \in N$ and so $E \in N$ since $\pi(\bar{E}) = E$.

Now fix $A \in \mathcal{P}(\bar{\iota}) \cap N$ and $\xi < \bar{\eta}$. We have that $\pi|(N \cap V_\lambda) \in N$ and so letting

$$\pi^* = \pi|(N \cap V_\lambda)$$

we have:

$$\begin{aligned} \xi \in \pi_{\bar{E}}(A) \quad &\Longleftrightarrow \quad \pi(\xi) \in \pi(\pi_{\bar{E}} \cap V_{\bar{\lambda}})(\pi(A)) \\ &\Longleftrightarrow \quad \pi(\xi) \in \pi_E(\pi(A)) \\ &\Longleftrightarrow \quad \pi(\xi) \in \pi_E(\pi^*(A)) \\ &\Longleftrightarrow \quad \pi(\xi) \in \pi_E(\pi^*)(\pi_E(A)) = \pi_E(\pi^*)(A). \end{aligned}$$

Thus $P_{\bar{E}}$ can be computed from $\pi|\bar{\eta}$ and $\pi_E(\pi^*)$. We have $\pi|\bar{\eta} \in N$ but only that

$$\pi_E(\pi^*) \in M_E \cong \mathrm{Ult}_0(N, E).$$

However we only need

$$\{(A, \pi_E(\pi^*)(A) \cap \eta) \mid A \in \mathcal{P}(\bar{\iota}) \cap N\} \in N$$

in order to show that $P_{\bar{E}} \in N$.

Working in N and since we have both that $\pi^* \in N$ and that $\bar{\iota} < \delta$, we can choose $Z \subset \iota$ such that

(3.1) $Z \in N$,

(3.2) For all $\delta \leq \theta \leq \iota$,

$$\{(A, \pi^*(A) \cap \theta) \mid A \in \mathcal{P}(\bar{\iota}) \cap N\} \in L[Z \cap \theta].$$

But then $\pi_E(Z) \cap \eta \in N$ and so by the elementarity of π_E,

$$\{(A, \pi_E(\pi^*)(A) \cap \eta) \mid A \in \mathcal{P}(\bar{\iota}) \cap N\} \in L[\pi_E(Z) \cap \eta] \subset N.$$

This proves $\bar{E} \in N$ and so $E \in N$. □

Remark 3.27. Suppose that E is an L-extender of length η. Then

$$L \cong \mathrm{Ult}_0(L, E)$$

and so $\pi_E(A) \in L$ for all $A \in L$. □

As a corollary of Theorem 3.26, we obtain the direct transference of Woodin cardinals to weak extender models for supercompactness. This easily generalizes to the appropriate versions of essentially any current large cardinal hypothesis.

Theorem 3.28. *Suppose that N is a weak extender model for δ is supercompact and that $\theta > \delta$ is a Woodin cardinal. Then θ is a Woodin cardinal in N.*

Proof. By the definition of a Woodin cardinal, θ is a Woodin cardinal if for all $A \subset V_\theta$, there exists $\delta < \kappa < \theta$ such that for all $\kappa < \lambda < \theta$ with $|V_\lambda| = \lambda$, there is a V-extender E such that

(1.1) $\mathrm{CRT}(j_E) = \kappa$, $\mathrm{LTH}(E) = \lambda$, and $j_E(\kappa) > \lambda$, (1.2) $V_\lambda \subset M_E$ and $j_E(A \cap V_\kappa) \cap V_\lambda = A \cap V_\lambda$.

where

$$j_E : V \to M_E \cong \text{Ult}_0(V, E)$$

is the ultrapower embedding.

But then for all $A \in \mathcal{P}(V_\theta) \cap N$, there exists $\delta < \kappa < \theta$ such that for all $\kappa < \lambda < \theta$ with $|V_\lambda| = \lambda$, there is an N-extender E such that

(2.1) $\text{CRT}(j_E) = \kappa$, $\text{LTH}(E) = \lambda$, and $j_E(\kappa) > \lambda$,

(2.2) $j_E(N \cap V_\kappa) \cap V_\lambda = N \cap V_\lambda$,

(2.3) $V_\lambda \cap N \subset M_E$ and $j_E(A \cap V_\kappa) \cap V_\lambda = A \cap V_\lambda$.

where

$$j_E : N \to M_E \cong \text{Ult}_0(N, E)$$

is the ultrapower embedding.

By Lemma 3.26 and with E as above, $(E|\eta)|N \in N$ for all $\eta < \lambda$ (since $\kappa_E = \kappa > \delta$) and so since λ can be chosen cofinally large in θ, θ is a Woodin cardinal in N. □

Definition 3.29. (1) E is an *extender* if E is a V-extender.

(2) An extender, E, of length η is *λ-complete* if

$$\eta^\lambda \subseteq M$$

where $M = \text{Ult}_0(V, E)$. □

Suppose that E is an extender with critical point κ, $\mathbb{P} \in V_\kappa$, and $G \subseteq \mathbb{P}$ is V-generic. Then E naturally defines an extender in $V[G]$ and

$$(j_E)^{V[G]}|V = (j_E)^V.$$

Lemma 3.30. *Suppose that $\delta < \kappa$, E is an extender which is δ-complete with critical point κ, and that*

$$j : V \to M \subseteq V[G]$$

is a generic elementary embedding such that

(i) *$M = \{j(f)(\alpha) \mid \alpha < \delta$ and $f \in V\}$,*

(ii) *G is V-generic for some partial order $\mathbb{P} \in V$ such that $|\mathbb{P}| \leq \delta$ in V.*

Then $(j_E)^{V[G]}|M = (j_F)^M$ where $F = j(E)$.

Proof. By (i), $M = \text{Ult}_0(V, H)$ where H is a V-extender of length δ.

Let $\eta = \text{LTH}(E)$ and for each $a \in [\eta]^\delta$ let E_a be the ultrafilter,

$$E_a = \left\{A \subseteq [\hat{\eta}]^\delta \mid a \in j_E(A)\right\},$$

where $\hat{\eta} = \min\{\gamma \mid \eta \leq j_E(\gamma)\}$.

Since E is δ-complete for each $a \in [\eta]^\delta$, $a \in \text{Ult}_0(V, E)$ and so E_a is defined.

Suppose that $a \subseteq b$ and $b \in [\eta]^\delta$. Then there is a natural elementary embedding,

$$j_{a,b} : \text{Ult}_0(V, E_a) \to \text{Ult}_0(V, E_b).$$

This defines a directed system indexed by the directed set, $([\eta]^\delta, \subseteq)$ with limit, $\text{Ult}_0(V, E)$.

This is just the usual analysis of $\text{Ult}_0(V, E)$ as the limit of a directed system of ultrapowers except here the underlying directed set is $([\eta]^\delta, \subseteq)$ instead of the directed set, $([\eta]^{<\omega}, \subseteq)$.

Let $X = [\hat{\eta}]^\delta$. For each $a \in [\eta]^\delta$, $E_a \subseteq \mathcal{P}(X)$ and E_a is an ultrafilter on X. Fix $a \in [\eta]^\delta$.

We first show the following. Suppose that

$$f : X \to M$$

is a function in $V[G]$. Then there exists a function

$$f^* : j(X) \to M$$

such that $f^* \in M$ and such that

$$\{y \in X \mid f(y) = f^*(j(y))\} \in (E_a)_G$$

where $(E_a)_G$ is the ultrafilter in $V[G]$ generated by E_a.

Fix f and work in $V[G]$. For each $y \in X$ there exists a pair (g_y, α_y) such that

(1.1) $\alpha_y < \delta$,

(1.2) $g_y \in V$,

(1.3) $f(y) = j(g_y)(\alpha_y)$.

This defines a function

$$F : X \to V$$

where for all $y \in X$, $F(y) = (g_y, \alpha_y)$.

Since E_a is κ-complete and since $|\mathbb{P}|^V \leq \delta < \kappa$, it follows that there exists $Z \in E_a$ and there exists $\alpha < \delta$ such that

(2.1) $F|Z \in V$,

(2.2) $\alpha_y = \alpha$ for all $y \in Z$.

Define

$$f^* : j(X) \to M$$

by $f^*(t) = 0$ if $t \notin j(Z)$ and if $t \in j(Z)$ then

$$f^*(t) = j(F)_t(\alpha)$$

where for each $y \in X$, $F_y = g_y$.

Thus for each $y \in Z$,

$$f^*(j(y)) = j(F)_{j(y)}(\alpha) = (j(F_y))(\alpha) = (j(g_y))(\alpha) = (j(g_y))(\alpha_y) = f(y),$$

and so f^* is as required.

What we have done is show that for each $a \in [\eta]^\delta$ the lemma holds with E replaced by E_a. This special case is due to Steel.

Now we use the hypothesis that E is δ-complete. Suppose $b \in j([\eta]^\delta)$. Then there exists $\alpha < \delta$ and a function

$$g : \delta \to [\eta]^\delta$$

such that $j(g)(\alpha) = b$ noting that $\delta \leq j(\delta)$. Let $a = \cup\{g(\beta) \mid \beta < \delta\}$. Thus $a \in [\eta]^\delta$, $a \in V$ and $b \subseteq j(a)$.

Thus

$$\left\{j(a) \mid a \in [\eta]^\delta\right\}$$

is cofinal in the directed set,

$$\left\{a \mid a \in j([\eta]^\delta)\right\},$$

and so $\mathrm{Ult}_0(M, j(E))$ is the limit of $\mathrm{Ult}_0(M, j(E_a))$ over the directed set $([\eta]^\delta, \subseteq)^V$ and the lemma follows by the correspondence of functions established above. □

There is a useful corollary of Lemma 3.30 which allows one to generate a variety of weak extender models for the supercompactness of some cardinal δ, and which have various other properties.

The main motivation for this is to show that weak extender models for supercompactness need not be so close to V as to render the notion useless as a requirement for inner model theory at the level of supercompactness. The latter is a natural speculation given for example the Universality Theorem, Theorem 3.26.

The generic elementary embeddings given by the *Stationary Tower* at Woodin cardinals $\delta < \kappa$ give many examples of j which satisfy the conditions of Lemma 3.31 and with any given uncountable regular cardinal below κ as the critical point.

However, we shall only use Lemma 3.31 (with the partial order $\mathbb{P}$ trivial so that $V = V[G]$) to obtain Lemma 3.32 which shows that Lemma 3.19 is optimal.

Lemma 3.31. *Suppose that $\delta < \kappa$, κ is supercompact, and that*

$$j : V \to M \subseteq V[G]$$

is a generic elementary embedding such that

(i) $M = \{j(f)(\alpha) \mid \alpha < \delta \text{ and } f \in V\}$,

(ii) *G is V-generic for some partial order $\mathbb{P} \in V$ such that $|\mathbb{P}| \leq \delta$ in V.*

Then in $V[G]$, M is a weak extender model for κ is supercompact.

Proof. By Lemma 3.30, for each extender $E \in V$, if (in V),

(1.1) $\mathbb{P} \in V_{\text{CRT}(E)}$,

(1.2) $\rho(E) = \text{LTH}(E)$,

(1.3) $\text{cof}(\text{LTH}(E)) > \delta$,

then in $V[G]$, $E_G \cap M \in M$ where E_G is the extender in $V[G]$ generated by E. The point is that E is δ-complete and so by Lemma 3.30,

$$j(E) = E_G \cap M.$$

Since κ is supercompact in V, the class of all such extenders, E_G, witnesses that κ is supercompact in $V[G]$. The corollary follows. □

Lemma 3.32 shows that the restriction on critical points in Theorem 3.19 is necessary and in addition, combined with Theorem 3.45 shows that the case where $N = \text{HOD}$ is quite different.

Lemma 3.32. *Suppose that δ is a supercompact cardinal. Then there is a weak extender model, N, for δ is supercompact such that for each λ there is a nontrivial elementary embedding*

$$j : N \to N$$

with $\text{CRT}(j) < \delta$ such that $j(\lambda) = \lambda$.

Proof. Let $\kappa < \delta$ be a measurable cardinal and let U be a normal κ-complete uniform ultrafilter on κ. Let $\langle (M_n, U_n, j_{n,n+1}) : n < \omega \rangle$ be the iteration of (V, U) of length ω. Thus

(1.1) $(M_0, U_0) = (V, U)$,

(1.2) $M_{n+1} = \text{Ult}_0(M_n, U_n)$ and $j_{n,n+1} : M_n \to M_{n+1}$ is the ultrapower embedding.

(1.3) $U_{n+1} = j_{n,n+1}(U_n)$.

Let

$$M_\omega = \lim_{n<\omega} M_n$$

be the direct limit under the composition of the elementary embeddings,

$$\langle j_{n,n+1} : n < \omega\rangle.$$

Thus M_ω is wellfounded and so for each $\lambda \in \text{Ord}$,

$$j_{n,n+1}(\lambda) = \lambda$$

for all sufficiently large $n < \omega$.

Define $N = M_\omega$ and let

$$j_{0,\omega} : V \to N$$

be the associated elementary embedding.

Let $\eta = j_{0,\omega}(\kappa)$. Then $\eta < (2^\kappa)^+ < \delta$ and

$$N = \text{Ult}_0(V, E)$$

where E is the extender of length η given by $j_{0,\omega}$.

Thus N is a weak extender model for δ is supercompact by Lemma 3.31 (with $\mathbb{P}$ trivial so that $V = V[G]$).

Finally for all $n < \omega$, $j_{n,n+1}(N) = N$ and so for all $\lambda \in \text{Ord}$, for all sufficiently large $n < \omega$, $j_{n,n+1}|N$ yields an elementary embedding

$$j : N \to N$$

such that $j(\lambda) = \lambda$, and this proves the lemma. □

3.6 The HOD Dichotomy Theorem

Jensen's Covering Lemma is naturally formulated as a dichotomy theorem:

Theorem 3.33 (Jensen). *One of the following holds.*

(1) *Suppose γ is a singular cardinal. Then γ is singular in L and $\gamma^+ = (\gamma^+)^L$.*

(2) *Every uncountable cardinal is inaccessible in L.* □

The following theorem is arguably an abstract generalization of the Jensen Covering Lemma when stated as above in the form of a dichotomy theorem. We shall prove a strong version of Theorem 3.34 as the HOD Dichotomy Theorem, Theorem 3.39 below.

Theorem 3.34. *Assume that δ is an extendible cardinal. Then one of the following holds.*

(1) *For every singular cardinal $\gamma > \delta$, γ is singular in* HOD *and $(\gamma^+)^{\text{HOD}} = \gamma^+$.*

(2) *Every regular cardinal $\gamma \geq \delta$ is a measurable cardinal in* HOD. □

Definition 3.35. Let λ be an uncountable regular cardinal. Then λ is *ω-strongly measurable in* HOD if there exists $\kappa < \lambda$ such that:

(1) $(2^\kappa)^{\text{HOD}} < \lambda$.

(2) There is no partition $\langle S_\alpha \mid \alpha < \kappa\rangle$ of $\text{cof}(\omega) \cap \lambda$ into stationary sets such that $\langle S_\alpha \mid \alpha < \kappa\rangle \in \text{HOD}$. □

Lemma 3.36. *Suppose that λ is an uncountable regular cardinal and that $\mathcal{F}$ is a λ-complete uniform filter on λ. Let*

$$\mathbb{B} = \mathcal{P}(\lambda)/I$$

where I is the ideal dual to $\mathcal{F}$. Suppose that $\mathbb{B}$ is γ-cc for some γ such that $2^\gamma < \lambda$. Then $|\mathbb{B}| \leq 2^\gamma$ and $\mathbb{B}$ is atomic.

Proof. It suffices to prove that $\mathbb{B}$ is atomic. Equivalently, it suffices to show that if $A \subseteq \lambda$ and $A \notin I$ then there exists $B \subseteq A$ such that $B \notin I$ and such that B cannot be split into 2 sets each of which is I-positive.

This in turn reduces to simply proving that $\mathbb{B}$ has an atom since if $\mathbb{B}$ is not atomic then we can replace I by the ideal generated by $I \cup \{A\}$ where A/I is the join in $\mathbb{B}$ of all the atoms of $\mathbb{B}$.

Therefore we assume toward a contradiction that $\mathbb{B}$ has no atoms. Let

$$\langle (P_\alpha, Z_\alpha) : \alpha < \Theta \rangle$$

be a maximal sequence such that $\Theta \leq \gamma + 1$ and such that for all $\alpha < \beta$,

(1.1) $2^{|\beta|} < \lambda$,

(1.2) $Z_\beta \in \mathcal{F}$ and $Z_\beta \subseteq Z_\alpha$,

(1.3) P_α is a partition of Z_β into I-positive sets,

(1.4) P_β refines P_α,

(1.5) for each $B \in P_\alpha$, there exist distinct $X, Y \in P_\beta$ such that $X \cup Y \subset B$.

For each $\alpha < \Theta$, $|P_\alpha| < \gamma$ since $\mathbb{B}$ is γ-cc. We prove

(2.1) $2^{|\Theta|} \geq \lambda$.

Assume toward a contradiction that $2^{|\Theta|} < \lambda$. Thus $\Theta < \lambda$ and so $Z \in \mathcal{F}$ where

$$Z = \cap \{Z_\alpha \mid \alpha < \Theta\}.$$

Define an equivalence relation $\sim$ on Z by $\xi_1 \sim \xi_2$ if for all $\alpha < \Theta$, for all $A \in P_\alpha$, $\xi_1 \in A$ if and only if $x_2 \in A$.

We have:

(3.1) $2^{|\Theta|} < \lambda$ and $2^\gamma < \lambda$,

(3.2) for each $\alpha < \lambda$, $|P_\alpha| < \gamma$.

Therefore $|Z/{\sim}| < \lambda$. But then $Z_\Theta \in \mathcal{F}$ where

$$Z_\Theta = \cup \{[\xi]_\sim \mid \xi \in Z \text{ and } [\xi]_\sim \notin I\}$$

and where for each $\xi \in Z$, $[\xi]_\sim$ is the $\sim$-equivalence class of ξ.

Define $P_\Theta = \{[\xi]_\sim \mid \xi \in Z_\Theta\}$. This contradicts the maximality of the sequence

$$\langle (P_\alpha, Z_\alpha) : \alpha < \Theta \rangle.$$

This proves that $2^{|\Theta|} \geq \lambda$. But this implies that $\Theta > \gamma$. Fix $\xi \in Z_\gamma$. For each $\alpha < \gamma$, let $X_\alpha \in P_\alpha$ be such that $\xi \in X_\alpha$. Thus

$$\langle X_\alpha : \alpha < \gamma \rangle$$

is a decreasing sequence of I-positive sets and for each $\alpha < \gamma$, $X_{\alpha+1} \backslash X_\alpha$ is I-positive. This yields an antichain in $\mathbb{B}$ is cardinality γ which contradicts that $\mathbb{B}$ is γ-cc. □

Lemma 3.37. *Assume λ is ω-strongly measurable in* HOD. *Then*

$$\text{HOD} \vDash \lambda \textit{ is a measurable cardinal.}$$

Proof. Let $S = \left\{\alpha < \lambda \mid (\mathrm{cof}(\alpha))^V = \omega\right\}$ and let

$$\mathcal{F} = \{A \in \mathcal{P}(\kappa) \cap \mathrm{HOD} \mid S \backslash A \text{ is not a stationary subset of } \lambda \text{ in } V\}.$$

Thus $\mathcal{F} \in \mathrm{HOD}$ and in HOD, $\mathcal{F}$ is a λ-complete uniform filter on λ. Since λ is ω-strongly measurable in HOD, there exists $\gamma < \lambda$ such that in HOD:

(1.1) $2^\gamma < \lambda$,

(1.2) $\mathcal{P}(\lambda)/I$ is γ-cc where I is the ideal dual to $\mathcal{F}$.

Therefore by Lemma 3.36, the Boolean algebra

$$(\mathcal{P}(\lambda) \cap \mathrm{HOD})\,/I$$

is atomic. □

Theorem 3.38. *Suppose that δ is an extendible cardinal. Then the following are equivalent.*

(1) HOD *is a weak extender model for δ is supercompact.*

(2) *There exists a regular cardinal $\lambda \geq \delta$ which is not ω-strongly measurable in* HOD.

Proof. By Theorem 3.10, (1) implies that for every singular cardinal $\gamma > \delta$,

$$\gamma^+ = (\gamma^+)^{\mathrm{HOD}}$$

and by Lemma 3.37, this implies (2).

Thus it suffices to show that (2) implies (1). We first prove:

(1.1) For each $\alpha > \delta$ there exists a regular cardinal $\lambda > \alpha$ such that λ is not ω-strongly measurable in HOD.

Fix a regular cardinal $\lambda_0 \geq \delta$ such that λ_0 is not ω-strongly measurable in HOD. Let $\kappa > \lambda_0$ be such that $\kappa > \alpha$ and

$$V_\kappa \prec_{\Sigma_2} V.$$

Thus

$$V_\kappa \vDash \text{“}\lambda_0 \text{ is not } \omega\text{-strongly measurable in HOD”}.$$

Since δ is extendible, there exists an elementary embedding

$$\pi : V_{\kappa+1} \to V_{\pi(\kappa)+1}$$

such that $\mathrm{CRT}(\pi) = \delta$ and $\pi(\delta) > \kappa > \alpha$. Thus

$$V_{\pi(\kappa)} \vDash \text{“}\pi(\lambda_0) \text{ is not } \omega\text{-strongly measurable in HOD”}.$$

But

$$(\mathrm{HOD})^{V_{\pi(\kappa)}} \subset \mathrm{HOD}$$

and so $\pi(\lambda_0)$ is not ω-strongly measurable in HOD. This proves (1.1).

Fix $\kappa_0 > \delta$ and let $\kappa > \kappa_0$ be such that $|V_\kappa| = \kappa$. Let $\lambda_0 > 2^\kappa$ be a regular cardinal which is not ω-strongly measurable in HOD and let $\lambda > \lambda_0$ be such that

$$V_\lambda \prec_{\Sigma_2} V.$$

Thus $\lambda = |V_\lambda|$ and $\mathrm{HOD} \cap V_\lambda = (\mathrm{HOD})^{V_\lambda}$.

Let $S = \{\alpha < \lambda_0 \mid \mathrm{cof}(\alpha) = \omega\}$. Thus there exists a partition

$$\langle S_\alpha : \alpha < \kappa\rangle \in \mathrm{HOD}$$

of S into stationary subsets of S.

Let

$$\pi : V_{\lambda+1} \to V_{\pi(\lambda)+1}$$

be an elementary embedding such that $\mathrm{CRT}(\pi) = \delta$ and $\pi(\delta) > \lambda$.

Let $T = \pi(S)$ and let

$$\langle T_\alpha : \alpha < \pi(\kappa)\rangle = \pi(\langle S_\alpha : \alpha < \kappa\rangle).$$

Thus:

(2.1) $\pi(\lambda_0)$ is a regular cardinal.

(2.2) $T = \{\alpha < \pi(\lambda_0) \mid \mathrm{cof}(\alpha) = \omega\}$.

(2.3) $\langle T_\alpha : \alpha < \pi(\kappa)\rangle$ is a partition of T into stationary sets.

(2.4) $\langle T_\alpha : \alpha < \pi(\kappa)\rangle \in (\mathrm{HOD})^{V_{\pi(\lambda)}}$.

Let

$$\Theta = \sup\{\pi(\xi) \mid \xi < \lambda_0\}.$$

Thus $\Theta < \pi(\lambda_0)$. Let σ be the set of all $\alpha < \pi(\kappa)$ such that

$$T_\alpha \cap C \neq \emptyset$$

for all closed cofinal subsets $C \subseteq \Theta$. Therefore

$$\sigma = \{\pi(\alpha) \mid \alpha < \kappa\}.$$

But $\sigma \in (\mathrm{HOD})^{V_{\pi(\lambda)}}$ since

$$\langle T_\alpha : \alpha < \pi(\kappa)\rangle \in (\mathrm{HOD})^{V_{\pi(\lambda)}}.$$

This proves that

$$\pi|\kappa \in (\mathrm{HOD})^{V_{\pi(\lambda)}}.$$

But there is a bijection

$$\rho : \kappa \to \mathrm{HOD} \cap V_\kappa$$

such that $\rho \in (\mathrm{HOD})^{V_\lambda}$ and so

$$\pi|(\mathrm{HOD} \cap V_\kappa) \in (\mathrm{HOD})^{V_{\pi(\lambda)}}.$$

Let U_0 be the normal fine ultrafilter on $\mathcal{P}_\delta(\kappa_0)$ given by π. Thus

(3.1) $\mathcal{P}_\delta(\kappa_0) \cap \mathrm{HOD} \in U_0$,

(3.2) $U_0 \cap \mathrm{HOD} \in (\mathrm{HOD})^{V_{\pi(\lambda)}} \subset \mathrm{HOD}$.

This proves that HOD is a weak extender model for δ is supercompact. □

We now come to the HOD Dichotomy Theorem. There are various equivalent versions but the following is sufficient for our purposes.

Theorem 3.39 (HOD Dichotomy Theorem). *Suppose that δ is an extendible cardinal. Then one of the following holds.*

(1) *Every regular cardinal $\kappa \geq \delta$ is ω-strongly measurable in* HOD. *Further:*

  (a) HOD *is not a weak extender for the supercompactness of any λ.*

  (b) *There is no weak extender model N for the supercompactness of some λ such that $N \subseteq$* HOD.

(2) *No regular cardinal $\kappa \geq \delta$ is ω-strongly measurable in* HOD. *Further:*

  (a) HOD *is a weak extender model for the supercompactness of δ.*

  (b) *Every singular cardinal $\gamma > \delta$ is singular in* HOD *and*

$$\gamma^+ = (\gamma^+)^{\mathrm{HOD}}.$$

Proof. Assume toward a contradiction that κ and γ are regular cardinals, each greater than or equal to δ, such that κ is not ω-strongly measurable in HOD and that γ is ω-strongly measurable in HOD.

Since γ is ω-strongly measurable in HOD, there exists a stationary set

$$S \subset \{\alpha < \gamma \mid \mathrm{cof}(\alpha) = \omega\}$$

such that

(1.1) $S \in \mathrm{HOD}$,

(1.2) $\mathcal{F} \cap (\mathrm{HOD} \cap \mathcal{P}(\gamma))$ is an ultrafilter;

where $\mathcal{F}$ is the club filter (of V) restricted to S.

Let

$$U = \mathcal{F} \cap (\mathrm{HOD} \cap \mathcal{P}(\gamma)).$$

Thus in HOD, U is a γ-complete, normal, uniform ultrafilter on γ.

By Theorem 3.38, HOD is a weak extender model for δ is supercompact. Therefore by Lemma 3.8, HOD has the δ-covering property and so for each $\xi \in S$,

$$(\mathrm{cof}(\xi))^{\mathrm{HOD}} < \delta.$$

Thus

$$\left\{\xi < \gamma \mid (\mathrm{cof}(\xi))^{\mathrm{HOD}} < \xi\right\} \in U.$$

This contradicts that in HOD, U is a γ-complete, normal, uniform ultrafilter on γ. □

The HOD Dichotomy Theorem has an interesting corollary and a much stronger version is given by Theorem 5.28.

Theorem 3.40. *Suppose that δ is an extendible cardinal. Then δ is a measurable cardinal in* HOD.

Proof. By Lemma 3.37, we can reduce to the case that δ is not ω-strongly measurable in HOD. But then by Theorem 3.39, HOD is a weak extender model for δ is supercompact and so δ is a supercompact cardinal in HOD. □

One can by a more careful argument generalize the previous theorem and obtain the following variation.

Theorem 3.41. *Suppose there exists an elementary embedding*

$$j : V_{\kappa+\omega} \to V_{j(\kappa)+\omega}$$

with $\mathrm{CRT}(j) = \kappa$*. Then there is a measurable cardinal in* HOD. □

3.7 The HOD Hypothesis

Definition 3.42 (The HOD Hypothesis). There exists a proper class of regular cardinals λ which are not ω-strongly measurable in HOD. □

Remark 3.43. (1) It is not known if there can exist 4 regular cardinals which *are* ω-strongly measurable in HOD.

(2) Suppose γ is a singular strong limit cardinal of uncountable cofinality. It is not known if γ^+ can ever be ω-strongly measurable in HOD. □

The following theorem is an immediate corollary of the HOD Dichotomy Theorem.

Theorem 3.44 (HOD Hypothesis). *Suppose that δ is an extendible cardinal. Then* HOD *is a weak extender model for δ is supercompact.* □

Comparing the next theorem with Lemma 3.32 shows that the case of HOD being a weak extender model for the supercompactness of some δ, is quite different than the case of an arbitrary transitive class N.

Theorem 3.45 (HOD Hypothesis). *Suppose that there is an extendible cardinal. Then there is an ordinal λ such that for all $\gamma > \lambda$, if*

$$j : \mathrm{HOD} \cap V_{\gamma+1} \to \mathrm{HOD} \cap V_{j(\gamma)+1}$$

is an elementary embedding with $j(\lambda) = \lambda$ then $j \in \mathrm{HOD}$.

Proof. Let δ be an extendible cardinal and let $\lambda_0 = \delta^{+\omega}$ be the ω-th cardinal above δ. Clearly $(\mathrm{cof}(\lambda_0))^{\mathrm{HOD}} = \omega$. Further by Theorem 3.10 and Theorem 3.44,

$$(\lambda_0^+)^{\mathrm{HOD}} = \lambda_0^+.$$

Therefore if $\eta < \lambda_0^+$ then $(\mathrm{cof}(\eta))^{\mathrm{HOD}} < \lambda_0$. Let κ_0 be least such that

$$\left\{\eta < \lambda_0^+ \mid \mathrm{cof}(\eta) = \omega \text{ and } (\mathrm{cof}(\eta))^{\mathrm{HOD}} = \kappa_0\right\}$$

is stationary in λ_0^+.

Define $\lambda = \lambda_0 + \kappa_0$. We show that λ is as required. Suppose $\gamma > \lambda$ and

$$j : \mathrm{HOD} \cap V_{\gamma+1} \to \mathrm{HOD} \cap V_{j(\gamma)+1}$$

is an elementary embedding such that $j(\lambda) = \lambda$. By Theorem 3.44, if $j|\delta$ is the identity then $j \in \mathrm{HOD}$. Therefore we have only to prove that $j|\delta$ is the identity. Since $\kappa_0 < \lambda_0$ and since $j(\lambda) = \lambda$, $j(\lambda_0) = \lambda_0$ and $j(\kappa_0) = \kappa_0$.

Clearly j induces canonically an elementary embedding

$$j^* : (H(\lambda_0^{++}))^{\mathrm{HOD}} \to (H(\lambda_0^{++}))^{\mathrm{HOD}}$$

with the property that $j|\lambda_0 = j^*|\lambda_0$.

Let

$$S = \left\{\eta < \lambda_0^+ \mid \mathrm{cof}(\eta) = \omega \text{ and } (\mathrm{cof}(\eta))^{\mathrm{HOD}} = \kappa_0\right\}.$$

Thus since S is stationary in λ_0^+ and since

$$(\lambda_0^+)^{\mathrm{HOD}} = \lambda_0^+,$$

there is a partition

$$\langle S_\alpha : \alpha < \lambda_0^+\rangle \in \mathrm{HOD}$$

of S into stationary sets. Let

$$\langle T_\beta : \beta < \lambda_0^+\rangle = j^*(\langle S_\alpha : \alpha < \lambda_0^+\rangle).$$

Note that if $\eta \in S$ and if η is closed under j^* then $j^*(\eta) = \eta$. This is because $(\mathrm{cof}(\eta))^{\mathrm{HOD}} = \kappa_0$ and because $j^*(\kappa_0) = \kappa_0$.

Therefore for all $\beta < \lambda_0^+$, $T_\beta \cap S$ is stationary in λ_0^+ if and only if $\beta = j^*(\alpha)$ for some $\alpha < \lambda_0^+$. This implies that

$$\{j^*(\alpha) \mid \alpha < \lambda_0^+\} \in \mathrm{HOD}$$

since $\left\{\beta < \lambda_0^+ \mid T_\beta \cap S \text{ is stationary in } \lambda_0^+\right\} \in \mathrm{HOD}$. But by the elementarity of j^* and since $j^*(S) = S$, for all $\beta < \lambda_0^+$,

$$\mathrm{HOD} \vDash \text{“}T_\beta \cap S \text{ is stationary in } \lambda_0^+\text{”},$$

which implies (since $\left\{j^*(\alpha) \mid \alpha < \lambda_0^+\right\} \in \mathrm{HOD}$) that $j^*|\lambda_0^+$ is the identity. Thus

$$\mathrm{CRT}(j) > \delta$$

and so by Theorem 3.15 and Theorem 3.44, $j \in \mathrm{HOD}$. □

Theorem 3.46 (HOD Hypothesis). *Suppose that there exists an extendible cardinal. Then there is no sequence of non-trivial elementary embeddings,*

$$j_i : \mathrm{HOD} \to \mathrm{HOD}$$

such that the direct limit,

$$\lim_{i<\omega} j_i \circ \cdots \circ j_0(\mathrm{HOD}),$$

is wellfounded.

Proof. Assume toward a contradiction that the direct limit is wellfounded. Then for every ordinal λ,

$$j_i(\lambda) = \lambda$$

for all sufficiently large $i < \omega$. Therefore by Theorem 3.45, j_i must be the identity for all sufficiently large $i < \omega$. □

Theorem 3.47 (HOD Hypothesis). *Suppose that there exists an extendible cardinal. Let T be the Σ_2-theory of V with ordinal parameters. Then there is no non-trivial elementary embedding,*

$$j : (\mathrm{HOD}, T) \to (\mathrm{HOD}, T).$$

Proof. By Theorem 3.45, there exists $\lambda \in \mathrm{Ord}$ such that for all $\gamma > \lambda$, if

$$k : \mathrm{HOD} \cap V_{\gamma+1} \to \mathrm{HOD} \cap V_{k(\gamma)+1}$$

is an elementary embedding with $k(\lambda) = \lambda$, then $k \in \mathrm{HOD}$. Let λ_0 be the least such λ. Clearly λ_0 is definable in V and so λ_0 is definable in (HOD, T).

Suppose toward a contradiction that

$$j : (\mathrm{HOD}, T) \to (\mathrm{HOD}, T)$$

is a non-trivial elementary embedding. Therefore $j(\lambda_0) = \lambda_0$ and so for all $\gamma > \lambda_0$,

$$j| \ \mathrm{HOD} \cap V_{\gamma+1} \in \mathrm{HOD},$$

which is a contradiction. □

3.8 The HOD Conjecture

The HOD Dichotomy Theorem together with the speculation that there is an extension of inner model theory to the level of supercompact cardinals suggests the following conjecture. Of course one could modify the conjecture by replacing the theory

$$\text{ZFC} + \text{“There is a supercompact cardinal”}$$

with the theory

$$\text{ZFC} + \text{“There is an extendible cardinal”}$$

or even by a still stronger theory, but at this stage its seems rather unlikely that this is actually necessary. However, the weaker conjecture obtained from the stronger theory given by some (reasonable) large cardinal hypothesis might be easier to prove.

Definition 3.48 (HOD Conjecture). The theory

$$\text{ZFC} + \text{“There is a supercompact cardinal”}$$

proves the HOD Hypothesis. □

We end this section by listing several consequences of the HOD Conjecture. These are in the context of just ZF and suggest there may be rather surprising approximations to the Axiom of Choice which simply follow from the existence of large cardinals (such as extendible cardinals). Details can be found in [20]. There is a much stronger version of Theorem 3.49 in [25] but for the purposes of this account that version is not really relevant. The stronger version simply reduces the rank of the parameter a to nearly the least supercompact cardinal (where supercompactness is as defined in [20]).

Theorem 3.49 (ZF). *Assume the* HOD *Conjecture. Suppose δ is an extendible cardinal. Then there is a transitive class $M \subseteq V$ such that:*

(1) $M \models \text{ZFC}$.

(2) *M is $\Sigma_2(a)$-definable for some $a \in V_\delta$.*

(3) *Every set of ordinals is generic over M for some partial order $\mathbb{P} \in V_\delta$.*

(4) *$M \models$ "δ is an extendible cardinal".* □

Theorem 3.50 (ZF). *Assume the* HOD *Conjecture. Suppose δ is an extendible cardinal. Then for all $\lambda > \delta$ there is no non-trivial elementary embedding $j : V_{\lambda+2} \to V_{\lambda+2}$.* □

Theorem 3.49 suggests the following conjecture which if provable would show an extraordinary connection between the existence of extendible cardinals and the Axiom of Choice.

Definition 3.51 (Axiom of Choice Conjecture (ZF)). Suppose that δ is an extendible cardinal and that $G \subset \text{Coll}(\omega, V_\delta)$ is V-generic. Then the Axiom of Choice holds in $V[G]$. □

For the statement of the following theorem $L(\mathcal{P}(\text{Ord}))$ denotes the transitive class given by the union:

$$\cup \{L(\mathcal{P}(\alpha)) \mid \alpha \in \text{Ord}\}.$$

This is the smallest inner model of ZF which contains all sets of ordinals.

Theorem 3.52 (ZF). *Assume the* HOD *Conjecture. Suppose that δ is an extendible cardinal. Then the following hold in $L(\mathcal{P}(\text{Ord}))$.*

(1) *δ is an extendible cardinal.*

(2) *The Axiom of Choice Conjecture.* □

We make a final comment. Assuming ZF, the Axiom of Choice holds if and only if

$$L(\mathcal{P}(\text{Ord})) \models \text{Axiom of Choice}.$$

Thus while proving the Axiom of Choice Conjecture would argue for the Axiom of Choice just from the existence of an extendible cardinal, by Theorem 3.52, just proving the HOD Conjecture would also suffice for this purpose.

4 The coding obstruction

If one can prove the following conjecture then one verifies a minor weakening of the HOD Conjecture.

Conjecture 4.1. *Suppose δ is an extendible cardinal. Then there exists a weak extender model N for δ is supercompact such that*

$$N \subseteq \text{HOD}. \qquad \square$$

Defining a weak extender model for δ is a measurable cardinal in the natural fashion, Kunen's theory of $L[U]$ yields:

Theorem 4.2 (after Kunen). *Suppose that δ is a measurable cardinal. Then there exists a weak extender model N for δ is measurable such that*

$$N \subseteq \text{HOD}. \qquad \square$$

Thus one just needs to generalize Kunen's construction of $L[U]$ to the level of supercompact cardinals. The purpose of this section is to show that this cannot easily be done. Before giving the details we introduce the key notion of an *iteration tree* which is the basis on which *iterability hypotheses* are formulated. Iteration trees were first defined by Steel and the basic theory is given in [11]. The definition we give is from [20] and is more general in that a wider class of extenders is allowed.

We also prove a preliminary positive result, Theorem 4.31, which implies one of the results implicit in [11], that assuming a natural iteration hypothesis, Kunen's theorem *can* be (directly) generalized far beyond the level of measurable cardinals and up to the level of *superstrong cardinals*. Superstrong cardinals are defined at the beginning of Section 5.

4.1 Iteration trees and iteration hypotheses

We review some definitions from [20]. To be consistent with the terminology used in the fine-structure theory of extender models, the premice of [20] we shall call coarse premice. The definition of a coarse premouse is below.

Definition 4.3. A *coarse premouse* is a pair (M, δ) such that M is transitive, $\delta \in M$, and:

(1) $M \vDash \text{ZC} + \Sigma_2$-Replacement.

(2) Suppose that $F : M_\delta \to M \cap \text{Ord}$ is definable from parameters in M, then F is bounded in M.

(3) δ is strongly inaccessible in M. $\square$

We fix some notation.

Definition 4.4. If E is an extender then:

(1) $j_E : V \to M_E \cong \text{Ult}_0(V, E)$ is the ultrapower embedding.

(2) ξ is a generator of E if $\xi \neq j_E(f)(s)$ for all $s \in [\xi]^{<\omega}$ and $f \in V$.

(3) $\nu_E = \sup\{\xi + 1 \mid \xi \text{ is a generator of } E\}$.

(4) $\kappa_E = \text{CRT}(E) = \text{CRT}(j_E)$ and $\kappa^*_E = j_E(\kappa_E)$.

(5) $\rho(E) = \sup\{\alpha \mid V_\alpha \subset \text{Ult}_0(V, E)\}$.

(6) $\iota_E = \sup\{\alpha \mid j_E(\alpha) < \nu_E\}$.

(7) $\text{SP}(E)$ is the set of all cardinals $\gamma \leq \iota_E$ such that there is a generator ξ of E such that $\sup(j_E[\gamma]) \leq \xi < j_E(\gamma)$.

(8) E is *ω-huge* if $\rho(E) = \lambda$ where $\lambda > \kappa_E$ is least such that $j_E(\lambda) = \lambda$. □

Remark 4.5. (1) Note that $\rho(E) \geq \kappa_E + 1$, ι_E is a cardinal, and $\iota_E = \sup(\text{SP}(E))$. However assuming for example that there is a supercompact cardinal, it is not always that case that $\iota_E \in \text{SP}(E)$.

(2) $\text{SP}(E)$ is the set of cardinals ι for which E induces uniform ultrafilters on ι, these are the *spaces* associated to the (uniform) ultrafilters of E. Every cardinal $\iota \in \text{SP}(E)$ must have cofinality at least κ_E, however $\text{SP}(E)$ need not contain even all the regular cardinals ι such that $\kappa_E \leq \iota < \iota_E$. □

Definition 4.6. Suppose that (M, δ) is a coarse premouse. An *iteration tree*, $\mathcal{T}$, on (M, δ) of length η is a tree order $<_{\mathcal{T}}$ on η with minimum element 0 and which is a suborder of the standard order, together with a sequence

$$\langle M_\alpha, E_\beta, j_{\gamma,\alpha} : \alpha < \eta, \beta + 1 < \eta, \gamma <_{\mathcal{T}} \alpha \rangle$$

such that the following hold.

(1) $M_0 = M$,

(2) $j_{\gamma,\alpha} : M_\gamma \to M_\alpha$ for all $\gamma <_{\mathcal{T}} \alpha < \eta$,

(3) Suppose that $\alpha + 1 < \eta$. Then $\alpha + 1$ has an immediate predecessor, α^*, in the tree order $<_{\mathcal{T}}$ and:

a) $E_\alpha \in j_{0,\alpha}(M \cap V_\delta)$ and $M_\alpha \vDash$ "E_α is an extender which is not ω-huge" ;

b) If $\alpha^* < \alpha$ then $\iota_{E_\alpha} + 1 \leq \min\left\{\rho(E_\beta) \mid \alpha^* \leq \beta < \alpha\right\}$;

c) $M_{\alpha+1} = \text{Ult}_0(M_{\alpha^*}, E_\alpha)$ and

$$j_{\alpha^*,\alpha+1} : M_{\alpha^*} \to M_{\alpha+1}$$

is the associated embedding.

(4) If $0 < \beta < \eta$ is a limit ordinal then the set of α such that $\alpha <_{\mathcal{T}} \beta$ is cofinal in β and M_β is the limit of the M_α where $\alpha <_{\mathcal{T}} \beta$ relative to the embeddings; $j_{\alpha,\beta}$. □

Definition 4.7. Suppose that (M, δ) is a coarse premouse and that $\mathcal{T}$ is an iteration tree on (M, δ) with associated sequence,

$$\langle M_\alpha, E_\beta, j_{\gamma,\alpha} : \alpha < \eta, \beta + 1 < \eta, \gamma <_{\mathcal{T}} \alpha \rangle.$$

Suppose that $\theta \in \text{Ord}$. Then the iteration tree, $\mathcal{T}$, is a *(+θ)-iteration tree* if for all $\alpha + 1 < \eta$,

$$\sup\left\{\iota_{E_\beta} \mid \beta^* \leq \alpha < \beta\right\} + \theta \leq \rho(E_\alpha)$$

where for each $\beta + 1 < \eta$, β^* is the $\mathcal{T}$ predecessor of $\beta + 1$. □

Remark 4.8. By the definition of an iteration tree, if $\beta^* \leq \alpha < \beta$ then necessarily

$$\iota_{E_\beta} + 1 \leq \rho(E_\alpha).$$

Thus every iteration tree is a (+0)-iteration tree and every iteration tree of finite length is a (+1)-iteration tree. □

Definition 4.9. Suppose that (M, δ) is a coarse premouse. An *iteration strategy of order* $\omega_1 + 1$ for (M, δ) is a function I such that the following hold.

(1) Suppose that $\mathcal{T}$ is an iteration tree on (M, δ) of limit length such that $\text{LTH}(\mathcal{T}) \leq \omega_1$. Then $\mathcal{T} \in \text{dom}(I)$ and $I(\mathcal{T})$ is a maximal wellfounded branch of $\mathcal{T}$ of limit length.

(2) Suppose that $\mathcal{T}$ is an iteration tree on (M, δ) of limit length such that $\text{LTH}(\mathcal{T}) \leq \omega_1$. Suppose that for all limit $\eta < \text{LTH}(\mathcal{T})$, $I(\mathcal{T}|\eta) = \{\xi < \eta \mid \xi <_{\mathcal{T}} \eta\}$. Then $I(\mathcal{T})$ is a cofinal wellfounded branch of $\mathcal{T}$. □

Definition 4.10. Suppose that (M, δ) is a coarse premouse and that $\mathcal{T}$ is an iteration tree on (M, δ) with associated sequence,

$$\langle M_\alpha, E_\beta, j_{\gamma,\alpha} : \alpha < \eta, \beta + 1 < \eta, \gamma <_{\mathcal{T}} \alpha \rangle.$$

The iteration tree $\mathcal{T}$ is *strongly closed* if for all $\alpha + 1 < \eta$:

(1) $\mathcal{T}$ is a (+1)-iteration tree; and

(2) $\text{LTH}(E_\alpha)$ is strongly inaccessible in M_α and $\rho(E_\alpha) = \text{LTH}(E_\alpha)$ in M_α. □

Definition 4.11. Suppose that (M, δ) is a coarse premouse. A strongly closed iteration tree

$$\mathcal{T} = \langle M_\alpha, E_\beta, j_{\gamma,\alpha} : \alpha < \eta, \beta + 1 < \eta, \gamma <_{\mathcal{T}} \alpha \rangle$$

on (M, δ) is a 0-*strongly closed iteration tree* if for all $\alpha + 1 < \eta$,

$$\text{LTH}(E_\alpha) \leq j_{E_\alpha}(\kappa_{E_\alpha})$$

where for each $\alpha + 1 < \eta$,

$$j_{E_\alpha} : M_\alpha \to \text{Ult}_0(M_\alpha, E_\alpha)$$

is the ultrapower embedding (as computed in M_α). □

Definition 4.12. Suppose that (M, δ) is a coarse premouse and that $\mathcal{T}$ is a 0-strongly closed iteration tree on (M, δ) with associated sequence,

$$\langle M_\alpha, E_\beta, j_{\gamma,\alpha} : \alpha < \eta, \beta + 1 < \eta, \gamma <_{\mathcal{T}} \alpha \rangle.$$

Then

(1) $\mathcal{T}$ is *maximal* if $\text{LTH}(E_\beta) \leq \kappa_{E_\alpha}$ for all $\beta < \alpha^* < \alpha + 1 < \eta$.

(2) $\mathcal{T}$ is *strongly maximal* if $\kappa^*_{E_\beta} \leq \kappa_{E_\alpha}$ for all $\beta < \alpha^* < \alpha + 1 < \eta$.

(3) $\mathcal{T}$ is *non-overlapping* if $\kappa^*_{E_\beta} \leq \kappa_{E_\alpha}$ for all $\beta + 1 \leq_{\mathcal{T}} \alpha + 1 < \eta$. □

Definition 4.13 (Weak $(\omega_1 + 1)$-Iteration Hypothesis). Suppose that (M, δ) is a countable coarse premouse and that

$$\pi : M \to V_\Theta$$

is an elementary embedding. Then (M, δ) has an iteration strategy of order $\omega_1 + 1$ for 0-strongly closed maximal iteration trees on (M, δ). □

Definition 4.14 (Weak Unique Branch Hypothesis). Suppose that (V_Θ, δ) is a coarse premouse that $\mathcal{T}$ is a countable 0-strongly closed maximal iteration tree on (V_Θ, δ) of limit length. Then $\mathcal{T}$ has at most one cofinal wellfounded branch. □

Remark 4.15. The Weak $(\omega_1 + 1)$-Iteration Hypothesis and the Weak Unique Branch Hypothesis are special cases of the fundamental iteration hypotheses of [11]. The necessity of the restriction to strongly closed iteration trees for the Weak Unique Branch Hypothesis is given in Theorem 4.16. Note that 0-strongly closed iteration trees which are strongly maximal are necessarily non-overlapping. □

We give two counterexamples to the attempt of formulating variations of the iteration hypotheses above by weakening the requirement that the iteration trees be 0-strongly closed and maximal. The proofs are given in [20].

Theorem 4.16. *Suppose that there is a supercompact cardinal. Then there exist an extender E such that*

$$\nu_E = \left(2^{2^\kappa}\right)^{M_E}$$

where $\kappa = \kappa_E$ and $M_E = \mathrm{Ult}_0(V, E)$, and a 0-strongly closed strongly maximal iteration tree

$$\mathcal{T} = \langle M_\alpha, E_\beta, j_{\gamma,\alpha} : \alpha < \eta, \beta + 1 < \omega, \gamma <_{\mathcal{T}} \alpha\rangle$$

on M_E of length ω such that:

(1) $\kappa_{E_\alpha} > \kappa^*_E$ *for all $\alpha < \omega$,*

(2) $\mathcal{T}$ *has two wellfounded branches.* □

Theorem 4.17. *Suppose that there is a supercompact cardinal. Then there exist an extender E such that*

$$\nu_E = \left(2^{2^\kappa}\right)^{M_E}$$

where $\kappa = \kappa_E$ and $M_E = \mathrm{Ult}_0(V, E)$, and a 0-strongly closed strongly maximal iteration tree

$$\mathcal{T} = \langle M_\alpha, E_\beta, j_{\gamma,\alpha} : \alpha < \eta, \beta + 1 < \omega, \gamma <_{\mathcal{T}} \alpha\rangle$$

on M_E of length ω^2 such that:

(1) $\kappa_{E_\alpha} > \kappa^*_E$ *for all $\alpha < \omega^2$,*

(2) $\mathcal{T}$ *has only one cofinal branch and that branch is not wellfounded.* □

4.2 Martin-Steel extender sequences

An important precursor to the fine structural models of Mitchell-Steel of [12] are the Martin-Steel inner models of [11] and these represent the natural generalization of the definition of $L[U]$ to larger inner models.

Before giving the relevant definitions we note that replacing U by a single extender cannot work. Of course this requires being a bit careful about defining $L[E]$ where E is an extender.

Definition 4.18. Suppose $E = \langle E_a : a \in [\eta]^{<\omega}\rangle$ is an extender. Then $L[E]$ denotes $L[P_E]$ where $P_E = \{(a, B) \mid B \in E_a\}$. □

The following lemma shows that using just one extender cannot suffice to generate even an inner model with 2 measurable cardinals (if that extender is *short* in the sense that $\mathrm{LTH}(E) \leq \kappa^*_E$). This may seem surprising at first since a single extender, even with the requirement $\mathrm{LTH}(E) \leq \kappa^*_E$, *can* witness the existence of large cardinals far beyond the level of a single measurable cardinal.

Lemma 4.19. *Suppose that E is an extender such that $\mathrm{LTH}(E) \leq j_E(\kappa)$ where*

$$j_E : V \to M_E \cong \mathrm{Ult}_0(V, E)$$

is the ultrapower embedding. Let U be the normal ultrafilter on κ given by j_E. Then $L[E] = L[U]$. □

Using longer extenders does not really help but the requisite analysis is more involved since if there are two measurable cardinals then there is an extender E such that in $L[E]$ there is an inner model with two measurable cardinals and so

$$L[E] \neq L[U]$$

where U is the normal measure on κ_E given by E.

Theorem 4.20. *Suppose that F is an extender and $E = F|j_F(\xi)$ for some $\xi < j_F(\kappa_F)$ such that*

$$V_{\eta+\omega} \subset M_F \cong \mathrm{Ult}_0(V, F)$$

where $\eta = j_F(\xi)$. Then in $L[E]$ there is no inner model with a Woodin cardinal. □

Remark 4.21. If one drops the requirement that $E = F|\eta$ for some $\eta < \rho(F)$ (still requiring $\eta < j_F(\xi)$ for some $\xi < j_F(\kappa_F)$) then it is relatively consistent (from a proper class of measurable cardinals) that in all set-generic extensions of V, the following holds:

(1) For every set A, there exists an extender E such that $A \in L[E]$ and such that $\mathrm{LTH}(E) < j_E(\xi)$ for some $\xi < j_E(\kappa_E)$.

A natural conjecture is that if sufficient large cardinals exist in V, then (1) must hold outright in V. □

Thus one really needs to consider sequences of extenders and the Martin-Steel extender models are of the form $L[\tilde{E}]$ where

$$\tilde{E} \subseteq (\mathrm{Ord} \times \mathrm{Ord}) \times V$$

is a predicate defining a sequence of (total) extenders. The predicate $\tilde{E}$ is defined such that for all $(\alpha,\beta) \in \mathrm{dom}(\tilde{E})$, the set,

$$\left\{a \in V \mid ((\alpha,\beta), a) \in \tilde{E}\right\},$$

is an extender which we denote by E^α_β. In the case of the Martin-Steel inner models, the extender E^α_β is the extender derived from an elementary embedding

$$j : V \to M$$

such that $\mathcal{P}^\omega(\alpha) \subseteq M$ and such that $\alpha < j(\kappa)$.

For $(\alpha,\beta) \in \mathrm{dom}(\tilde{E})$, $\tilde{E}|(\alpha,\beta)$ is the extender sequence given by restricting $\tilde{E}$ to the set of all (η,γ) such that $(\eta,\gamma) <_{\mathcal{L}} (\alpha,\beta)$ in the lexicographical ordering of pairs of ordinals:

$$\tilde{E}|(\alpha,\beta) = \left\{((\eta,\gamma), a) \in \tilde{E} \mid (\eta,\gamma) <_{\mathcal{L}} (\alpha,\beta)\right\};$$

and $L[\tilde{E}|(\alpha,\beta)]$ is formally defined as $L[P]$ where P is obtained from $\tilde{E}|(\alpha,\beta)$ in the natural fashion as defined above in the case of a single extender.

Definition 4.22. An extender sequence,

$$\tilde{E} = \langle E^\alpha_\beta : (\alpha,\beta) \in \mathrm{dom}(\tilde{E})\rangle$$

is a *Martin-Steel extender sequence* if for each pair $(\alpha,\beta) \in \mathrm{dom}(\tilde{E})$:

(1) (Coherence) There exists an extender F such that:

   a) $\alpha < \rho(F)$ and $\rho(F)$ is strongly inaccessible.
   b) $E^\alpha_\beta = F|\alpha$.
   c) (shortness) $\alpha \leq j_F(\kappa_F)$.
   d) $j_F(\tilde{E})|(\alpha + 1, 0) = \tilde{E}|(\alpha,\beta)$.

(2) (Novelty) For all $\beta^* < \beta$, $(\alpha,\beta^*) \in \mathrm{dom}(\tilde{E})$ and

$$E^\alpha_{\beta^*} \cap L[\tilde{E}|(\alpha,\beta)] \neq E^\alpha_\beta \cap L[\tilde{E}|(\alpha,\beta)]$$

(3) (Initial Segment Condition) Suppose that

$$\kappa < \alpha^* < \alpha$$

where κ is the critical point associated to E^α_β.

Then there exists β^* such that $(\alpha^*,\beta^*) \in \mathrm{dom}(\tilde{E})$ and such that

$$E^{\alpha^*}_{\beta^*} \cap L[\tilde{E}|(\alpha^* + 1, 0)] = (E^\alpha_\beta|\alpha^*) \cap L[\tilde{E}|(\alpha^* + 1, 0)].$$ □

The Martin-Steel extender models are actually defined in [11] as $L[P]$ where P is a predicate defined from a sequence of sets of extenders. Such sequences are called Doddages and the approach of constructing extender models from Doddages has the advantage that the resulting inner model can be ordinal definable.

Definition 4.23. A *Doddage* is a sequence $\tilde{\mathcal{E}}$ such that

$$\mathrm{dom}(\tilde{\mathcal{E}}) \subseteq \mathrm{Ord} \times \mathrm{Ord}$$

and such that for all $(\alpha, \beta) \in \mathrm{dom}(\tilde{\mathcal{E}})$, $\tilde{\mathcal{E}}(\alpha, \beta)$ is a set of extenders of length α. □

Definition 4.24. Suppose that $\tilde{\mathcal{E}}$ is a Doddage. Then $L[\tilde{\mathcal{E}}]$ denotes $L[P_{\tilde{\mathcal{E}}}]$ where $P_{\tilde{\mathcal{E}}}$ is the set of all (α, β, s, a) such that

(1) $(\alpha, \beta) \in \mathrm{dom}(\tilde{\mathcal{E}})$,

(2) $s \in [\alpha]^{<\omega}$,

(3) $a \in E(s)$ for all $E \in \tilde{\mathcal{E}}(\alpha, \beta)$. □

Suppose $\tilde{\mathcal{E}}$ is a Doddage. For each $(\alpha, \beta) \in \mathrm{dom}(\tilde{\mathcal{E}})$ we denote $\tilde{\mathcal{E}}(\alpha, \beta)$ by $\mathcal{E}^{\alpha}_{\beta}$.

Definition 4.25. A Doddage,

$$\tilde{\mathcal{E}} = \langle \mathcal{E}^{\alpha}_{\beta} : (\alpha, \beta) \in \mathrm{dom}(\tilde{\mathcal{E}}) \rangle$$

is a *Martin-Steel Doddage* if for each pair $(\alpha, \beta) \in \mathrm{dom}(\tilde{\mathcal{E}})$ and for each extender $E \in \mathcal{E}^{\alpha}_{\beta}$,

(1) (Coherence) There exists an extender F such that

  a) $\alpha < \rho(F)$ and $\rho(F)$ is strongly inaccessible,
  b) $E = F|\alpha$,
  c) (shortness) $\alpha \leq j_F(\kappa_F)$,
  d) $j_F(\tilde{\mathcal{E}})|(\alpha + 1, 0) = \tilde{\mathcal{E}}|(\alpha, \beta)$.

(2) (Novelty) For all $\beta^* < \beta$, $(\alpha, \beta^*) \in \mathrm{dom}(\mathcal{E})$ and for all $E^* \in \mathcal{E}^{\alpha}_{\beta^*}$,

$$E^* \cap L[\tilde{\mathcal{E}}|(\alpha, \beta)] \neq E \cap L[\tilde{\mathcal{E}}|(\alpha, \beta)]$$

(3) (Initial Segment Condition) Suppose that

$$\kappa_E < \alpha^* < \alpha,$$

  Then there exists $(\alpha^*, \beta^*) \in \mathrm{dom}(\tilde{\mathcal{E}})$ and there exists $E^* \in \mathcal{E}^{\alpha^*}_{\beta^*}$ such that

$$E^* \cap L[\tilde{\mathcal{E}}|(\alpha^* + 1, 0)] = (E|\alpha^*) \cap L[\tilde{\mathcal{E}}|(\alpha^* + 1, 0)].$$ □

Definition 4.26. Suppose $\tilde{\mathcal{E}}$ is a Martin-Steel Doddage. Then $\tilde{\mathcal{E}}$ is *good* if for all $(\alpha, \beta) \in \mathrm{dom}(\tilde{\mathcal{E}})$, for all $E_0, E_1 \in \mathcal{E}^{\alpha}_{\beta}$, $E_0 \cap L[\tilde{\mathcal{E}}] = E_1 \cap L[\tilde{\mathcal{E}}]$. □

Theorem 4.27 (Martin-Steel). *Suppose that the Weak $(\omega_1 + 1)$-Iteration Hypothesis holds and that $\tilde{\mathcal{E}}$ is a Martin-Steel Doddage such that $\tilde{\mathcal{E}} \in V_\delta$ for some strongly inaccessible Mahlo cardinal δ. Then $\tilde{\mathcal{E}}$ is good.* □

Theorem 4.28 (Martin-Steel). *Suppose that the Weak $(\omega_1 + 1)$-Iteration Hypothesis holds and that there is a supercompact cardinal. Then there exists a Martin-Steel Doddage $\tilde{\mathcal{E}}$ such that there is a superstrong cardinal in $L[\tilde{\mathcal{E}}]$.* □

The following lemma follows from the definition of the coherence condition.

Lemma 4.29. *Suppose that $\tilde{\mathcal{E}}$ is a Martin-Steel Doddage, $(\alpha,\beta) \in \mathrm{dom}(\tilde{\mathcal{E}})$, and that F is an extender of minimum length which witnesses the coherence condition for $\tilde{\mathcal{E}}$ at (α,β). Then $\kappa_F = \iota_F$.* □

Remark 4.30. (1) Theorem 4.31, which is from [20], is the generalization of Kunen's theorem that $L[U]$ is uniquely specified by the measurable cardinal κ associated to U. We include the proof for the sake of completeness and because it provides a good introduction to the basic comparison arguments of inner model theory.

(2) The assumption that $(\tilde{\mathcal{E}}_0, \tilde{\mathcal{E}}_1) \in V_\delta$ for some strongly inaccessible Mahlo cardinal δ is only necessary because of how the Weak $(\omega_1 + 1)$-Iteration Hypothesis is formulated. Similarly for Theorem 4.27.

One really just needs that $(\tilde{\mathcal{E}}_0, \tilde{\mathcal{E}}_1) \in V_\delta$ for some strongly inaccessible δ such that

$$V_\delta \vDash \text{“}\tilde{\mathcal{E}}_0 \text{ and } \tilde{\mathcal{E}}_1 \text{ are Martin-Steel Doddages”}$$

which must hold if δ is strongly inaccessible and Mahlo.

Alternatively, one could just assume there is a proper class of strongly inaccessible cardinals. □

Theorem 4.31. *Suppose that the Weak $(\omega_1 + 1)$-Iteration Hypothesis holds. Suppose that $\tilde{\mathcal{E}}_0$ and $\tilde{\mathcal{E}}_1$ are Martin-Steel Doddages such that*

$$\mathrm{dom}(\tilde{\mathcal{E}}_0) = \mathrm{dom}(\tilde{\mathcal{E}}_1)$$

and such that $(\tilde{\mathcal{E}}_0, \tilde{\mathcal{E}}_1) \in V_\delta$ for some strongly inaccessible Mahlo cardinal δ. Then

$$L[\tilde{\mathcal{E}}_0] = L[\tilde{\mathcal{E}}_1],$$

and moreover for all $(\alpha,\beta) \in \mathrm{dom}(\tilde{\mathcal{E}}_0)$, for all $E_0 \in \tilde{\mathcal{E}}_0(\alpha,\beta)$, for all $E_1 \in \tilde{\mathcal{E}}_1(\alpha,\beta)$,

$$E_0 \cap L[\tilde{\mathcal{E}}_0] = E_1 \cap L[\tilde{\mathcal{E}}_1].$$

Proof. We sketch the proof. Fix δ such that

$$(\tilde{E}_0, \tilde{E}_1) \in V_\delta$$

and such that δ is a strongly inaccessible Mahlo cardinal.

It is convenient to fix some notation. Suppose that $\tilde{\mathcal{E}}$ and $\tilde{\mathcal{F}}$ are Martin-Steel Doddages such that $\mathrm{dom}(\tilde{\mathcal{E}}) = \mathrm{dom}(\tilde{\mathcal{F}})$. Define $\tilde{\mathcal{E}} \equiv \tilde{\mathcal{F}}$ if:

(1.1) $L[\tilde{\mathcal{E}}] = L[\tilde{\mathcal{F}}]$.

(1.2) For all $(\alpha,\beta) \in \mathrm{dom}(\tilde{\mathcal{E}})$, for all $E \in \tilde{\mathcal{E}}(\alpha,\beta)$, for all $F \in \tilde{\mathcal{F}}(\alpha,\beta)$,

$$E \cap L[\tilde{\mathcal{E}}] = F \cap L[\tilde{\mathcal{F}}].$$

Fix $(\tilde{\mathcal{E}}_0, \tilde{\mathcal{E}}_1, \delta)$ and suppose toward a contradiction that the theorem fails.

Suppose that (V_Θ, δ) is a premouse such that $\tilde{\mathcal{E}}_0 \in V_\delta$ and such that there exists a countable elementary substructure,

$$X \prec (V_\Theta, \delta)$$

such that (M, δ_M) has an $(\omega_1 + 1)$-iteration strategy for 0-strongly closed maximal iteration trees where (M, δ_M) is the transitive collapse of X.

Thus

$$V_\delta \vDash \tilde{\mathcal{E}}_0 \not\equiv \tilde{\mathcal{E}}_1.$$

Fix a countable elementary substructure,

$$X \prec (V_\Theta, \delta),$$

such that (M, δ_M) has an $(\omega_1 + 1)$-iteration strategy where (M, δ_M) is the transitive collapse of X.

By the elementarity of X we can suppose without loss of generality that $(\tilde{\mathcal{E}}_0, \tilde{\mathcal{E}}_1, \delta) \in X$. Let $(\tilde{\mathcal{E}}_0^M, \tilde{\mathcal{E}}_1^M) \in M$ be the image of $(\tilde{\mathcal{E}}_0, \tilde{\mathcal{E}}_1)$ under the collapsing map. Thus

$$M \cap V_{\delta_M} \vDash \tilde{\mathcal{E}}_0^M \not\equiv \tilde{\mathcal{E}}_1^M.$$

Fix an $(\omega_1 + 1)$-iteration strategy for (M, δ_M) and following this strategy we shall define two iteration trees

$$\mathcal{T} = \langle M_\alpha^{\mathcal{T}}, E_\beta^{\mathcal{T}}, j_{\gamma,\alpha}^{\mathcal{T}} : \alpha \leq \omega_1, \beta < \omega_1, \gamma <_{\mathcal{T}} \alpha \rangle$$

and

$$\mathcal{S} = \langle M_\alpha^{\mathcal{S}}, E_\beta^{\mathcal{S}}, j_{\gamma,\alpha}^{\mathcal{S}} : \alpha \leq \omega_1, \beta < \omega_1, \gamma <_{\mathcal{S}} \alpha \rangle$$

on (M, δ_M) each of length $\omega_1 + 1$ such that for all $\beta < \omega_1$, the predecessor of $\beta + 1$ relative to each of the two iteration trees is as small as possible for that iteration tree.

To define $\mathcal{S}$ and $\mathcal{T}$, we define a continuous increasing sequence

$$\langle (\beta_{\mathcal{S}}, \beta_{\mathcal{T}}) : \beta \leq \omega_1 \rangle$$

of pairs of ordinals and define $(\mathcal{S}|\beta_{\mathcal{S}}, \mathcal{T}|\beta_{\mathcal{T}})$ by induction on β with $(0_{\mathcal{S}}, 0_{\mathcal{T}}) = (0, 0)$. The limit stages are immediate. Therefore we can suppose that $\beta < \omega_1$ and that

$$j_{0,\beta_{\mathcal{S}}}^{\mathcal{S}} : M \to M_{\beta_{\mathcal{S}}}^{\mathcal{S}}$$

and

$$j_{0,\beta_{\mathcal{T}}}^{\mathcal{T}} : M \to M_{\beta_{\mathcal{T}}}^{\mathcal{T}}$$

are given. We define $((\beta+1)_{\mathcal{S}}, (\beta+1)_{\mathcal{T}})$ and at the same time we will define $E_{\beta_{\mathcal{S}}}^{\mathcal{S}}$ if $(\beta+1)_{\mathcal{S}} \neq \beta_{\mathcal{S}}$ and define $E_{\beta_{\mathcal{T}}}^{\mathcal{T}}$ if $(\beta+1)_{\mathcal{T}} \neq \beta_{\mathcal{T}}$. It is convenient to use the following notation. Suppose A, B are subsets of $\mathrm{Ord} \times \mathrm{Ord}$, then

$$A \leq_{\mathcal{L}} B$$

if $A = B$ or if A is an initial segment of B relative to the lexicographical order on $\mathrm{Ord} \times \mathrm{Ord}$.

Case 1. Suppose that there exists

$$(\eta, \gamma) \in j_{0,\beta_{\mathcal{S}}}^{\mathcal{S}}\left(\mathrm{dom}(\tilde{\mathcal{E}}_0^M)\right) \cap j_{0,\beta_{\mathcal{T}}}^{\mathcal{T}}\left(\mathrm{dom}(\tilde{\mathcal{E}}_0^M)\right)$$

such that

(2.1) $j_{0,\beta_{\mathcal{S}}}^{\mathcal{S}}\left(\mathrm{dom}(\tilde{\mathcal{E}}_0^M)\right)|(\eta, \gamma) = j_{0,\beta_{\mathcal{T}}}^{\mathcal{T}}\left(\mathrm{dom}(\tilde{\mathcal{E}}_0^M)\right)|(\eta, \gamma)$,

(2.2) there exist

$$E_{\mathcal{S}} \in \left(j_{0,\beta_{\mathcal{S}}}^{\mathcal{S}}\left(\tilde{\mathcal{E}}_0^M\right)\right)(\eta, \gamma) \cup \left(j_{0,\beta_{\mathcal{T}}}^{\mathcal{S}}\left(\tilde{\mathcal{E}}_1^M\right)\right)(\eta, \gamma)$$

and

$$E_{\mathcal{T}} \in \left(j_{0,\beta_{\mathcal{T}}}^{\mathcal{T}}\left(\tilde{\mathcal{E}}_0^M\right)\right)(\eta, \gamma) \cup \left(j_{0,\beta_{\mathcal{T}}}^{\mathcal{T}}\left(\tilde{\mathcal{E}}_1^M\right)\right)(\eta, \gamma)$$

such that

$$E_{\mathcal{S}} \cap M_{\beta_{\mathcal{S}}}^{\mathcal{S}} \cap M_{\beta_{\mathcal{T}}}^{\mathcal{T}} \neq E_{\mathcal{T}} \cap M_{\beta_{\mathcal{S}}}^{\mathcal{S}} \cap M_{\beta_{\mathcal{T}}}^{\mathcal{T}}.$$

Let (η, γ) be the least such pair (relative the lexicographical order) and define $E_{\beta_{\mathcal{S}}}^{\mathcal{S}}$ to be an extender in $M_{\beta_{\mathcal{S}}}^{\mathcal{S}}$ which witnesses the coherence condition for $E_{\mathcal{S}}$ relative to $j_{0,\beta_{\mathcal{S}}}^{\mathcal{S}}(\tilde{\mathcal{E}}_0^M)$ if

$$E_{\mathcal{S}} \in \left(j_{0,\beta_{\mathcal{S}}}^{\mathcal{S}}\left(\tilde{\mathcal{E}}_0^M\right)\right)(\eta, \gamma)$$

or witnesses coherence condition for $E_{\mathcal{S}}$ relative to $j_{0,\beta_{\mathcal{S}}}^{\mathcal{S}}(\tilde{\mathcal{E}}_1^M)$, with $\mathrm{LTH}(E_{\beta_{\mathcal{S}}}^{\mathcal{S}})$ as small as possible such that

$$\mathrm{LTH}(E_{\beta_{\mathcal{S}}}^{\mathcal{S}}) = \rho(E_{\beta_{\mathcal{S}}}^{\mathcal{S}})$$

and such that $\mathrm{LTH}(E_{\beta_{\mathcal{S}}}^{\mathcal{S}})$ is strongly inaccessible in $M_{\beta_{\mathcal{S}}}^{\mathcal{S}}$. Since both $j_{0,\beta_{\mathcal{S}}}^{\mathcal{S}}\left(\tilde{\mathcal{E}}_0^M\right)$ and $j_{0,\beta_{\mathcal{S}}}^{\mathcal{S}}\left(\tilde{\mathcal{E}}_1^M\right)$ are Martin-Steel Doddages in $M_{\beta_{\mathcal{S}}}^{\mathcal{S}}$ and since $\mathcal{E}_0, \mathcal{E}_1 \in V_\delta$, it follows that $E_{\beta_{\mathcal{S}}}^{\mathcal{S}}$ exists.

Similarly, define $E^{\mathcal{T}}_{\beta_{\mathcal{T}}}$ to be an extender in $M^{\mathcal{T}}_{\beta_{\mathcal{T}}}$ which witnesses the coherence condition for $E_{\mathcal{T}}$ relative to either $j^{\mathcal{T}}_{0,\beta_{\mathcal{T}}}(\tilde{\mathcal{E}}^M_0)$ if

$$E_{\mathcal{T}} \in \left(j^{\mathcal{T}}_{0,\beta_{\mathcal{T}}}\left(\tilde{\mathcal{E}}^M_0\right)\right)(\eta,\gamma)$$

or witnesses coherence condition for $E_{\mathcal{T}}$ relative to $j^{\mathcal{T}}_{0,\beta_{\mathcal{T}}}(\tilde{\mathcal{E}}^M_1)$ otherwise, with $\text{LTH}(E^{\mathcal{T}}_{\beta_{\mathcal{T}}})$ and small as possible such that

$$\text{LTH}(E^{\mathcal{T}}_{\beta_{\mathcal{T}}}) = \rho(E^{\mathcal{T}}_{\beta_{\mathcal{T}}})$$

and such that $\text{LTH}(E^{\mathcal{T}}_{\beta_{\mathcal{T}}})$ is strongly inaccessible in $M^{\mathcal{T}}_{\beta_{\mathcal{T}}}$. Exactly as above, since both $j^{\mathcal{T}}_{0,\beta_{\mathcal{T}}}\left(\tilde{\mathcal{E}}^M_0\right)$ and $j^{\mathcal{T}}_{0,\beta_{\mathcal{T}}}\left(\tilde{\mathcal{E}}^M_1\right)$ are Martin-Steel Doddages in $M^{\mathcal{T}}_{\beta_{\mathcal{T}}}$ and since $\mathcal{E}_0, \mathcal{E}_1 \in V_\delta$, it follows that $E^{\mathcal{T}}_{\beta_{\mathcal{T}}}$ exists.

Define $((\beta+1)_{\mathcal{S}}, (\beta+1)_{\mathcal{T}}) = (\beta_{\mathcal{S}}+1, \beta_{\mathcal{T}}+1)$.

Case 2. Otherwise. Then

$$j^{\mathcal{T}}_{0,\beta_{\mathcal{T}}}\left(\text{dom}(\tilde{\mathcal{E}}^M_0)\right) \not\leq_{\mathcal{L}} j^{\mathcal{S}}_{0,\beta_{\mathcal{S}}}\left(\text{dom}(\tilde{\mathcal{E}}^M_0)\right)$$

and

$$j^{\mathcal{S}}_{0,\beta_{\mathcal{S}}}\left(\text{dom}(\tilde{\mathcal{E}}^M_0)\right) \not\leq_{\mathcal{L}} j^{\mathcal{T}}_{0,\beta_{\mathcal{T}}}\left(\text{dom}(\tilde{\mathcal{E}}^M_0)\right).$$

Let $(\eta,\gamma)_{\mathcal{T}} = \min\left(j^{\mathcal{T}}_{0,\beta}\left(\text{dom}(\tilde{\mathcal{E}}^M_0)\right) \setminus j^{\mathcal{S}}_{0,\beta}\left(\text{dom}(\tilde{\mathcal{E}}^M_0)\right)\right)$ and let

$$(\eta,\gamma)_{\mathcal{S}} = \min\left(j^{\mathcal{S}}_{0,\beta}\left(\text{dom}(\tilde{\mathcal{E}}^M_0)\right)\right) \setminus j^{\mathcal{T}}_{0,\beta}\left(\text{dom}(\tilde{\mathcal{E}}^M_0)\right),$$

where in each case the minimum is relative to the lexicographical order.

Thus $(\eta,\gamma)_{\mathcal{S}} \neq (\eta,\gamma)_{\mathcal{T}}$. There are two subcases. If $(\eta,\gamma)_{\mathcal{S}} < (\eta,\gamma)_{\mathcal{T}}$ then let $E^{\mathcal{S}}_{\beta_{\mathcal{S}}} \in M^{\mathcal{S}}_{\beta_{\mathcal{S}}}$ be an extender which witnesses the coherence condition for some extender

$$E \in \left(j^{\mathcal{S}}_{0,\beta_{\mathcal{S}}}\left(\tilde{\mathcal{E}}^M_0\right)\right)((\eta,\gamma)_{\mathcal{S}})$$

with $\text{LTH}(E^{\mathcal{S}}_{\beta_{\mathcal{S}}})$ as small as possible such that

$$\text{LTH}(E^{\mathcal{S}}_{\beta_{\mathcal{S}}}) = \rho(E^{\mathcal{S}}_{\beta_{\mathcal{S}}})$$

and such that $\text{LTH}(E^{\mathcal{S}}_{\beta_{\mathcal{S}}})$ is strongly inaccessible in $M^{\mathcal{S}}_{\beta_{\mathcal{S}}}$. Exactly as above, since $j^{\mathcal{S}}_{0,\beta_{\mathcal{S}}}\left(\tilde{\mathcal{E}}^M_0\right)$ is a Martin-Steel Doddage in $M^{\mathcal{S}}_{\beta_{\mathcal{S}}}$ and since $\mathcal{E}_0 \in V_\delta$ it follows that $E^{\mathcal{S}}_{\beta_{\mathcal{S}}}$ exists.

Define $((\beta+1)_{\mathcal{S}}, (\beta+1)_{\mathcal{T}}) = (\beta_{\mathcal{S}}+1, \beta_{\mathcal{T}})$.

If $(\eta,\gamma)_{\mathcal{T}} < (\eta,\gamma)_{\mathcal{S}}$ then let $E^{\mathcal{T}}_{\beta_{\mathcal{T}}} \in M^{\mathcal{T}}_{\beta_{\mathcal{T}}}$ be an extender which witnesses the coherence condition for some extender

$$E \in \left(j^{\mathcal{T}}_{0,\beta_{\mathcal{T}}}\left(\tilde{\mathcal{E}}^M_0\right)\right)((\eta,\gamma)_{\mathcal{T}})$$

with $\text{LTH}(E^{\mathcal{T}}_{\beta_{\mathcal{T}}})$ as small as possible such that

$$\text{LTH}(E^{\mathcal{T}}_{\beta_{\mathcal{T}}}) = \rho(E^{\mathcal{T}}_{\beta_{\mathcal{T}}})$$

and such that $\text{LTH}(E^{\mathcal{T}}_{\beta_{\mathcal{T}}})$ is strongly inaccessible in $M^{\mathcal{T}}_{\beta_{\mathcal{T}}}$.

Define $((\beta+1)_{\mathcal{S}}, (\beta+1)_{\mathcal{T}}) = (\beta_{\mathcal{S}}, \beta_{\mathcal{T}}+1)$.

This completes the definition of $\mathcal{S}$ and $\mathcal{T}$. If at some stage β neither case applies then it follows that (interchanging $\mathcal{S}$ and $\mathcal{T}$ if necessary):

(3.1) $j^{\mathcal{S}}_{0,\beta_{\mathcal{T}}}\left(\text{dom}(\tilde{\mathcal{E}}^M_0)\right) \leq_{\mathcal{L}} j^{\mathcal{T}}_{0,\beta_{\mathcal{T}}}\left(\text{dom}(\tilde{\mathcal{E}}^M_0)\right)$,

(3.2) for all $(\eta,\gamma) \in j^{\mathcal{T}}_{0,\beta_{\mathcal{S}}}\left(\text{dom}(\tilde{\mathcal{E}}^M_0)\right)$,

$$E \cap M^{\mathcal{S}}_{\beta_{\mathcal{S}}} \cap M^{\mathcal{T}}_{\beta_{\mathcal{T}}} = F \cap M^{\mathcal{S}}_{\beta_{\mathcal{S}}} \cap M^{\mathcal{T}}_{\beta_{\mathcal{T}}}.$$

for all

$$E \in \left(j^{\mathcal{S}}_{0,\beta_{\mathcal{S}}}\left(\tilde{\mathcal{E}}^M_0\right)\right)(\eta,\gamma) \cup \left(j^{\mathcal{S}}_{0,\beta_{\mathcal{T}}}\left(\tilde{\mathcal{E}}^M_1\right)\right)(\eta,\gamma)$$

and for all

$$F \in \left(j^{\mathcal{T}}_{0,\beta_{\mathcal{T}}}\left(\tilde{\mathcal{E}}^M_0\right)\right)(\eta,\gamma) \cup \left(j^{\mathcal{T}}_{0,\beta_{\mathcal{T}}}\left(\tilde{\mathcal{E}}^M_1\right)\right)(\eta,\gamma).$$

If

$$j^{\mathcal{S}}_{0,\beta_{\mathcal{S}}}\left(\mathrm{dom}(\tilde{\mathcal{E}}^M_0)\right) = j^{\mathcal{T}}_{0,\beta_{\mathcal{T}}}\left(\mathrm{dom}(\tilde{\mathcal{E}}^M_0)\right),$$

then either

$$M^{\mathcal{S}}_{\beta_{\mathcal{S}}} \cap V_{j^{\mathcal{S}}_{0,\beta_{\mathcal{S}}}(\delta_M)} \vDash j^{\mathcal{S}}_{0,\beta_{\mathcal{S}}}(\tilde{\mathcal{E}}^M_0) \equiv j^{\mathcal{S}}_{0,\beta_{\mathcal{S}}}(\tilde{\mathcal{E}}^M_1)$$

or

$$M^{\mathcal{T}}_{\beta_{\mathcal{T}}} \cap V_{j^{\mathcal{T}}_{0,\beta_{\mathcal{T}}}(\delta_M)} \vDash j^{\mathcal{T}}_{0,\beta_{\mathcal{T}}}(\tilde{\mathcal{E}}^M_0) \equiv j^{\mathcal{T}}_{0,\beta_{\mathcal{T}}}(\tilde{\mathcal{E}}^M_1)$$

(depending on whether $j^{\mathcal{S}}_{0,\beta_{\mathcal{S}}}(\delta_M) \le j^{\mathcal{T}}_{0,\beta_{\mathcal{T}}}(\delta_M)$ or whether $j^{\mathcal{T}}_{0,\beta_{\mathcal{T}}}(\delta_M) \le j^{\mathcal{S}}_{0,\beta_{\mathcal{S}}}(\delta_M)$) and this contradicts the choice of $(M, \tilde{\mathcal{E}}^M_0, \tilde{\mathcal{E}}^M_1, \delta_M)$.

If $j^{\mathcal{S}}_{0,\beta_{\mathcal{S}}}\left(\mathrm{dom}(\tilde{\mathcal{E}}^M_0)\right)$ is a proper initial segment of $j^{\mathcal{T}}_{0,\beta_{\mathcal{T}}}\left(\mathrm{dom}(\tilde{\mathcal{E}}^M_0)\right)$ then it follows that

$$M^{\mathcal{S}}_{\beta_{\mathcal{S}}} \cap V_{j^{\mathcal{S}}_{0,\beta_{\mathcal{S}}}(\delta_M)} \vDash j^{\mathcal{S}}_{0,\beta_{\mathcal{S}}}(\tilde{\mathcal{E}}^M_0) \equiv j^{\mathcal{S}}_{0,\beta_{\mathcal{S}}}(\tilde{\mathcal{E}}^M_1)$$

and this again is a contradiction.

To see this latter claim fix $(\eta_0, \gamma_0) \in j^{\mathcal{T}}_{0,\beta_{\mathcal{T}}}\left(\mathrm{dom}(\tilde{\mathcal{E}}^M_0)\right)$ such that

$$j^{\mathcal{S}}_{0,\beta_{\mathcal{S}}}\left(\mathrm{dom}(\tilde{\mathcal{E}}^M_0)\right) = j^{\mathcal{T}}_{0,\beta_{\mathcal{T}}}\left(\mathrm{dom}(\tilde{\mathcal{E}}^M_0)\right)|(\eta_0, \gamma_0).$$

Since $j^{\mathcal{T}}_{0,\beta_{\mathcal{T}}}\left(\tilde{\mathcal{E}}^M_0\right)(\eta_0, \gamma_0)$ is defined it follows that

$$M^{\mathcal{T}}_{\beta} \cap V_{j^{\mathcal{T}}_{0,\beta}(\delta_M)} \vDash \text{“}(L[\tilde{\mathcal{E}}])^{\#} \text{ and } (L[\tilde{\mathcal{F}}])^{\#} \text{ exist”}$$

where $\tilde{\mathcal{E}} = j^{\mathcal{T}}_{0,\beta_{\mathcal{T}}}(\tilde{\mathcal{E}}^M_0)|(\eta_0, \gamma_0)$ and where $\tilde{\mathcal{F}} = j^{\mathcal{T}}_{0,\beta_{\mathcal{T}}}(\tilde{\mathcal{E}}^M_1)|(\eta_0, \gamma_0)$. Further since $(M^{\mathcal{T}}_{\beta_{\mathcal{T}}}, j^{\mathcal{T}}_{0,\beta_{\mathcal{T}}}(\delta_M))$ is iterable,

$$\left((L[\tilde{\mathcal{E}}])^{\#}\right)^{M^{\mathcal{T}}_{\beta_{\mathcal{T}}}} = (L[\tilde{\mathcal{E}}])^{\#}$$

and

$$\left((L[\tilde{\mathcal{F}}])^{\#}\right)^{M^{\mathcal{T}}_{\beta_{\mathcal{T}}}} = (L[\tilde{\mathcal{F}}])^{\#}.$$

Now by (3.2), it follows that

$$M^{\mathcal{S}}_{\beta_{\mathcal{S}}} \cap V_{j^{\mathcal{S}}_{0,\beta_{\mathcal{S}}}(\delta_M)} \vDash j^{\mathcal{S}}_{0,\beta_{\mathcal{S}}}(\tilde{\mathcal{E}}^M_0) \equiv j^{\mathcal{S}}_{0,\beta_{\mathcal{S}}}(\tilde{\mathcal{E}}^M_1)$$

as claimed. Therefore at every stage $\beta < \omega_1$, either Case 1 holds or Case 2 holds.

Note that for each extender, E, occurring in either $\mathcal{S}$ or $\mathcal{T}$, in the model from which E is chosen there exists λ such that

(4.1) $\lambda = |V_\lambda|$ and $\rho(E) = \mathrm{LTH}(E) = \lambda$,

(4.2) $\kappa_E = \iota_E$,

(4.3) λ is not a limit of inaccessible cardinals.

To see that (4.2) holds it suffices to see that if $\mathcal{E}$ is a Martin-Steel Doddage, $(\alpha,\beta) \in \mathrm{dom}(\mathcal{E})$ and if F is an extender which witnesses the coherence condition for $\mathcal{E}(\alpha,\beta)$ then necessarily $(\alpha,\beta) \in j_F(V_\kappa)$ where $\kappa = \kappa_F$.

This has two consequences. First, (4.1)–(4.3) imply that both $\mathcal{S}$ and $\mathcal{T}$ are non-overlapping; in fact, for all $\beta_1 < \beta_2$ if $\beta_1 + 1 <_{\mathcal{S}} \beta_2 + 1$ then $\mathrm{LTH}(E^{\mathcal{S}}_{\beta_1}) < \mathrm{CRT}(E^{\mathcal{S}}_{\beta_2})$, and similarly for $\mathcal{T}$. This is a slightly stronger condition. Second, by (4.2) both $\mathcal{S}$ and $\mathcal{T}$ are iteration trees involving only short extenders, and so (4.1)–(4.3) imply that both $\mathcal{S}$ and $\mathcal{T}$ are (+1)-iteration trees (which implies that they are each $(+\theta)$-iteration trees where θ is the least measurable cardinal of M). Therefore the iteration strategy fixed for (M, δ_M) must supply cofinal, wellfounded, branches at all limit stages $\beta \le \omega_1$.

We note that unlike the usual comparison arguments, it is not obviously the case that the lengths of the extenders in these iteration trees are nondecreasing, more precisely it is not obvious that for all $\beta_1 < \beta_2$, $\mathrm{LTH}(E^{\mathcal{S}}_{\beta_1}) \le \mathrm{LTH}(E^{\mathcal{S}}_{\beta_2})$. For example, suppose that $E^{\mathcal{S}}_{\beta_1}$ is chosen to witness the coherence condition relative to $j^{\mathcal{S}}_{0,\beta_1}(\tilde{\mathcal{E}}^M_0)$. Then there is no reason to expect that $E^{\mathcal{S}}_{\beta_1}$ coheres $j^{\mathcal{S}}_{0,\beta_1}(\tilde{\mathcal{E}}^M_1)$ and so at the next stage of the construction of $(\mathcal{S}, \mathcal{T})$ there may be an "earlier" disagreement.

We obtain a contradiction is the usual fashion. Let

$$Z \prec H(\omega_2)$$

be a countable elementary substructure such that $\{\mathcal{S}, \mathcal{T}\} \in Z$. Let

$$b_{\mathcal{S}} = \{\beta < \omega_1 \mid \beta <_{\mathcal{S}} \omega_1\}$$

and let $b_{\mathcal{T}} = \{\beta < \omega_1 \mid \beta <_{\mathcal{T}} \omega_1\}$. Thus $b_{\mathcal{S}}$ and $b_{\mathcal{T}}$ are each closed cofinal subsets of ω_1. Let $\beta_Z = Z \cap \omega_1$. The image of $(\mathcal{S}, \mathcal{T})$ under the transitive collapse of Z is $(\mathcal{S}|(\beta_Z + 1), \mathcal{T}|(\beta_Z + 1))$.

Let N be the transitive collapse of X and let

$$\pi : N \to H(\omega_2)$$

invert the transitive collapse. Thus $\beta_Z \in b_{\mathcal{S}} \cap b_{\mathcal{T}}$ and

(5.1) $\pi(M^{\mathcal{S}}_{\beta_Z}) = M^{\mathcal{S}}_{\omega_1}$ and $\pi|M^{\mathcal{S}}_{\beta_Z} = j^{\mathcal{S}}_{\beta_Z,\omega_1}$,

(5.2) $\pi(M^{\mathcal{S}}_{\beta_Z}) = M^{\mathcal{S}}_{\omega_1}$ and $\pi|M^{\mathcal{S}}_{\beta_Z} = j^{\mathcal{S}}_{\beta_Z,\omega_1}$.

We now come to the key points. Let $\alpha^{\mathcal{S}}_Z$ be such that $\beta_Z = (\alpha^{\mathcal{S}}_Z)^*$ computed relative to $<_{\mathcal{S}}$, and let $\alpha^{\mathcal{T}}_Z$ be such that $\beta_Z = (\alpha^{\mathcal{T}}_Z)^*$ computed relative to $<_{\mathcal{T}}$.

By (5.1)–(5.2) and since the iteration trees are non-overlapping:

(6.1) For all $\beta > \beta_Z$, $\mathrm{LTH}(E^{\mathcal{S}}_\beta) > \beta_Z$ and $\mathrm{LTH}(E^{\mathcal{T}}_\beta) > \beta_Z$;

(6.2) For all $\beta > \beta_Z$,

$$M^{\mathcal{S}}_\beta \cap V_{\beta_Z+\omega} = M^{\mathcal{S}}_{\beta_Z} \cap V_{\beta_Z+\omega}$$

and

$$M^{\mathcal{T}}_\beta \cap V_{\beta_Z+\omega} = M^{\mathcal{T}}_{\beta_Z} \cap V_{\beta_Z+\omega};$$

(6.3) Either

$$E^{\mathcal{S}}_{\alpha^{\mathcal{S}}_Z} \cap M^{\mathcal{S}}_{\beta_Z} \cap M^{\mathcal{T}}_{\beta_Z} = \left(E^{\mathcal{T}}_{\alpha^{\mathcal{T}}_Z}|\mathrm{LTH}(E^{\mathcal{S}}_{\alpha^{\mathcal{S}}_Z})\right) \cap M^{\mathcal{S}}_{\beta_Z} \cap M^{\mathcal{T}}_{\beta_Z},$$

or

$$E^{\mathcal{T}}_{\alpha^{\mathcal{T}}_Z} \cap M^{\mathcal{S}}_{\beta_Z} \cap M^{\mathcal{T}}_{\beta_Z} = \left(E^{\mathcal{S}}_{\alpha^{\mathcal{S}}_Z}|\mathrm{LTH}(E^{\mathcal{T}}_{\alpha^{\mathcal{T}}_Z})\right) \cap M^{\mathcal{S}}_{\beta_Z} \cap M^{\mathcal{T}}_{\beta_Z};$$

(6.4) For each α such that $\alpha^{\mathcal{S}}_Z < \alpha < \omega_1$, $\mathrm{LTH}(E^{\mathcal{S}}_{\alpha^{\mathcal{S}}_Z}) < \mathrm{LTH}(E^{\mathcal{S}}_\alpha)$,

(6.5) For each α such that $\alpha^{\mathcal{T}}_Z < \alpha < \omega_1$, $\mathrm{LTH}(E^{\mathcal{T}}_{\alpha^{\mathcal{T}}_Z}) < \mathrm{LTH}(E^{\mathcal{T}}_\alpha)$.

The third of these claims, (6.3), follows from (5.1) and (5.2) since both $\mathcal{S}$ and $\mathcal{T}$ are non-overlapping.

To see that (6.4) holds, suppose toward a contradiction that $\alpha^{\mathcal{S}}_Z < \alpha < \omega_1$ and that $\mathrm{LTH}(E^{\mathcal{S}}_{\alpha^{\mathcal{S}}_Z}) \geq \mathrm{LTH}(E^{\mathcal{S}}_\alpha)$. Let $\hat{\alpha}$ be such that

$$(\hat{\alpha})^* = \sup\{\beta \leq \alpha \mid \beta \in b_{\mathcal{S}}\},$$

and such that $\hat{\alpha} + 1 \in b_{\mathcal{S}}$, where $(\hat{\alpha})^*$ is computed relative to $<_{\mathcal{S}}$. Then $\hat{\alpha} \geq \alpha$ and $(\hat{\alpha})^* \geq \alpha^{\mathcal{S}}_Z + 1$ since $\alpha > \alpha^{\mathcal{S}}_Z$ and $\alpha^{\mathcal{S}}_Z + 1 \in b_{\mathcal{S}}$. But

$$\mathrm{CRT}(E^{\mathcal{S}}_{\hat{\alpha}}) < \min\left\{\rho(E^{\mathcal{S}}_\beta) \mid (\hat{\alpha})^* \leq \beta < \hat{\alpha}\right\} \leq \mathrm{LTH}(E^{\mathcal{S}}_\alpha) \leq \mathrm{LTH}(E^{\mathcal{S}}_{\alpha^{\mathcal{S}}_Z})$$

and since $\mathcal{S}$ is non-overlapping, $\mathrm{LTH}(E^{\mathcal{S}}_{\alpha^{\mathcal{S}}_Z}) \leq \mathrm{CRT}(E^{\mathcal{S}}_{\hat{\alpha}})$. This is a contradiction. The proof that (6.5) holds is similar as is the proof of (6.1). Finally (6.2) follows from (6.1) since each of the extenders, $E^{\mathcal{S}}_\beta$ and $E^{\mathcal{T}}_\beta$, is an extender of minimum possible length which witnesses the coherence condition for a Martin-Steel Doddage (such extenders cannot have length which is a limit of inaccessible cardinals).

We fix some notation. Suppose that $\beta \leq \omega_1$ and that $(\eta, \gamma) \in j^{\mathcal{S}}_{0,\beta}(\mathrm{dom}(\tilde{\mathcal{E}}_0))$. Let $\mathcal{M}^{\mathcal{S}}_{\tilde{\mathcal{E}}_0,\beta}(\eta, \gamma)$ denote the structure,

$$\left(L\left[j^{\mathcal{S}}_{0,\beta}\left(\tilde{\mathcal{E}}^M_0\right)|(\eta,\gamma)\right], j^{\mathcal{S}}_{0,\beta}\left(\tilde{\mathcal{E}}^M_0\right)|(\eta,\gamma) \cap L\left[j^{\mathcal{S}}_{0,\beta}\left(\tilde{\mathcal{E}}^M_0\right)|(\eta,\gamma)\right]\right),$$

and let $\mathcal{M}^{\mathcal{S}}_{\tilde{\mathcal{E}}_1,\beta}(\eta, \gamma)$ denote structure,

$$\left(L\left[j^{\mathcal{S}}_{0,\beta}\left(\tilde{\mathcal{E}}^M_1\right)|(\eta,\gamma)\right], j^{\mathcal{S}}_{0,\beta}\left(\tilde{\mathcal{E}}^M_1\right)|(\eta,\gamma) \cap L\left[j^{\mathcal{S}}_{0,\beta}\left(\tilde{\mathcal{E}}^M_1\right)|(\eta,\gamma)\right]\right).$$

Similarly, suppose that $\beta \leq \omega_1$, and that $(\eta, \gamma) \in j^{\mathcal{T}}_{0,\beta}(\mathrm{dom}(\tilde{\mathcal{E}}_0))$. Let $\mathcal{M}^{\mathcal{T}}_{\tilde{\mathcal{E}}_0,\beta}(\eta, \gamma)$ and $\mathcal{M}^{\mathcal{T}}_{\tilde{\mathcal{E}}_1,\beta}(\eta, \gamma)$ denote the analogous structures defined relative to $\mathcal{T}$.

Let $(\eta, \gamma)_{\mathcal{S}} \in j^{\mathcal{S}}_{0,\alpha^{\mathcal{S}}_Z}(\mathrm{dom}(\tilde{\mathcal{E}}_0))$ be the element involved in the definition of $E^{\mathcal{S}}_{\alpha^{\mathcal{S}}_Z}$. By (6.4) and the fact that the extenders $E^{\mathcal{S}}_\alpha$ are chosen of minimal length to witness the coherence condition:

(7.1) Suppose that $\alpha^{\mathcal{S}}_Z < \alpha < \omega_1$. Let (η, γ) be the element of $j^{\mathcal{S}}_{0,\alpha}(\mathrm{dom}(\tilde{\mathcal{E}}_0))$ involved in the definition of $E^{\mathcal{S}}_\alpha$. Then $\eta^{\mathcal{S}} < \eta$ where $(\eta^{\mathcal{S}}, \gamma^{\mathcal{S}}) = (\eta, \gamma)_{\mathcal{S}}$.

We claim that for all β such that $\alpha^{\mathcal{S}}_Z \leq \beta \leq \omega_1$:

(8.1) $j^{\mathcal{S}}_{0,\alpha^{\mathcal{S}}_Z}(\mathrm{dom}(\tilde{\mathcal{E}}_0))|(\eta, \gamma)_{\mathcal{S}} = j^{\mathcal{S}}_{0,\beta}(\mathrm{dom}(\tilde{\mathcal{E}}_0))|(\eta, \gamma)_{\mathcal{S}} = j^{\mathcal{T}}_{0,\beta}(\mathrm{dom}(\tilde{\mathcal{E}}_0))|(\eta, \gamma)_{\mathcal{S}}$;

(8.2) Let $(\eta^{\mathcal{S}}, \gamma^{\mathcal{S}}) = (\eta, \gamma)_{\mathcal{S}}$, then if $\alpha^{\mathcal{S}}_Z < \beta$,

$$j^{\mathcal{S}}_{0,\beta}(\mathrm{dom}(\tilde{\mathcal{E}}_0))|(\eta, \gamma)_{\mathcal{S}} = j^{\mathcal{S}}_{0,\beta}(\mathrm{dom}(\tilde{\mathcal{E}}_0))|(\eta^{\mathcal{S}} + 1, 0)$$

and

$$j^{\mathcal{T}}_{0,\beta}(\mathrm{dom}(\tilde{\mathcal{E}}_0))|(\eta, \gamma)_{\mathcal{S}} = j^{\mathcal{T}}_{0,\beta}(\mathrm{dom}(\tilde{\mathcal{E}}_0))|(\eta^{\mathcal{S}} + 1, 0);$$

(8.3) For all $(\eta^*, \gamma^*) \in j^{\mathcal{S}}_{0,\beta}(\mathrm{dom}(\tilde{\mathcal{E}}_0))|(\eta, \gamma)_{\mathcal{S}}$,

$$E \cap M^{\mathcal{S}}_\beta \cap M^{\mathcal{T}}_\beta = F \cap M^{\mathcal{S}}_\beta \cap M^{\mathcal{T}}_\beta,$$

for all

$$E \in \left(j^{\mathcal{S}}_{0,\beta}(\tilde{\mathcal{E}}_0)\right)(\eta^*, \gamma^*) \cup \left(j^{\mathcal{S}}_{0,\beta}(\tilde{\mathcal{E}}_1)\right)(\eta^*, \gamma^*)$$

and for all

$$F \in \left(j^{\mathcal{T}}_{0,\beta}(\tilde{\mathcal{E}}_0)\right)(\eta^*, \gamma^*) \cup \left(j^{\mathcal{T}}_{0,\beta}(\tilde{\mathcal{E}}_1)\right)(\eta^*, \gamma^*);$$

(8.4) $\mathcal{M}^{\mathcal{S}}_{\tilde{\mathcal{E}}_0,\beta}((\eta, \gamma)_{\mathcal{S}}) = \mathcal{M}^{\mathcal{S}}_{\tilde{\mathcal{E}}_1,\beta}((\eta, \gamma)_{\mathcal{S}}) = \mathcal{M}^{\mathcal{S}}_{\tilde{\mathcal{E}}_0,\alpha^{\mathcal{S}}_Z}((\eta, \gamma)_{\mathcal{S}}) = \mathcal{M}^{\mathcal{S}}_{\tilde{\mathcal{E}}_1,\alpha^{\mathcal{S}}_Z}((\eta, \gamma)_{\mathcal{S}})$;

(8.5) $\mathcal{M}^{\mathcal{T}}_{\tilde{\mathcal{E}}_0,\beta}((\eta, \gamma)_{\mathcal{S}}) = \mathcal{M}^{\mathcal{T}}_{\tilde{\mathcal{E}}_1,\beta}((\eta, \gamma)_{\mathcal{S}}) = \mathcal{M}^{\mathcal{T}}_{\tilde{\mathcal{E}}_0,\alpha^{\mathcal{S}}_Z}((\eta, \gamma)_{\mathcal{S}}) = \mathcal{M}^{\mathcal{T}}_{\tilde{\mathcal{E}}_0,\alpha^{\mathcal{S}}_Z}((\eta, \gamma)_{\mathcal{S}})$;

(8.6) $\mathcal{M}^{\mathcal{S}}_{\tilde{\mathcal{E}}_0,\alpha^{\mathcal{S}}_Z}((\eta, \gamma)_{\mathcal{S}}) = \mathcal{M}^{\mathcal{T}}_{\tilde{\mathcal{E}}_1,\alpha^{\mathcal{S}}_Z}((\eta, \gamma)_{\mathcal{S}})$;

(8.7) $\left(\mathcal{M}^{\mathcal{S}}_{\tilde{\mathcal{E}}_0,\alpha^{\mathcal{S}}_Z}((\eta, \gamma)_{\mathcal{S}})\right)^{\#} \in M^{\mathcal{S}}_\beta \cap M^{\mathcal{T}}_\beta$.

The only potential issue is (8.7); (8.1)–(8.6) follow from (6.1)–(6.5) and (7.1) by relatively standard arguments. The proof of (8.7) uses (8.1)–(8.6) and the definition of $\mathcal{S}$ and $\mathcal{T}$. There are two additional relevant points. First,

$$\omega_1 \subseteq M^{\mathcal{S}}_{\omega_1}$$

and so for all $a \in M^{\mathcal{S}}_{\omega_1}$, if

$$M^{\mathcal{S}}_{\omega_1} \vDash \text{“}a^{\#} \text{ exists”}$$

then $a^\# \in M^{\mathcal{S}}_{\omega_1}$ (and similarly for $M^{\mathcal{T}}_{\omega_1}$). Second, if $\tilde{\mathcal{E}}$ is a Martin-Steel Doddage and if $(\eta,\gamma) \in \mathrm{dom}(\tilde{\mathcal{E}})$ then since $\tilde{\mathcal{E}}(\eta,\gamma)$ is defined, necessarily $\left(L[\tilde{\mathcal{E}}|(\eta,\gamma)]\right)^\#$ exists.

Similarly, let $(\eta,\gamma)_{\mathcal{T}} \in j^{\mathcal{T}}_{0,\alpha^{\mathcal{T}}_Z}(\mathrm{dom}(\tilde{\mathcal{E}}_0))$ be the element involved in the definition of $E^{\mathcal{T}}_{\alpha^{\mathcal{T}}_Z}$. By (6.5), for all β such that $\alpha^{\mathcal{T}}_Z \leq \beta \leq \omega_1$;

(9.1) $j^{\mathcal{T}}_{0,\alpha^{\mathcal{T}}_Z}(\mathrm{dom}(\tilde{\mathcal{E}}_0))|(\eta,\gamma)_{\mathcal{T}} = j^{\mathcal{T}}_{0,\beta}(\mathrm{dom}(\tilde{\mathcal{E}}_0))|(\eta,\gamma)_{\mathcal{T}} = j^{\mathcal{S}}_{0,\beta}(\mathrm{dom}(\tilde{\mathcal{E}}_0))|(\eta,\gamma)_{\mathcal{T}}$;

(9.2) Let $(\eta^{\mathcal{T}},\gamma^{\mathcal{T}}) = (\eta,\gamma)_{\mathcal{T}}$, then if $\alpha^{\mathcal{T}}_Z < \beta$,
$$j^{\mathcal{S}}_{0,\beta}(\mathrm{dom}(\tilde{\mathcal{E}}_0))|(\eta,\gamma)_{\mathcal{T}} = j^{\mathcal{S}}_{0,\beta}(\mathrm{dom}(\tilde{\mathcal{E}}_0))|(\eta^{\mathcal{T}}+1,0),$$
and
$$j^{\mathcal{T}}_{0,\beta}(\mathrm{dom}(\tilde{\mathcal{E}}_0))|(\eta,\gamma)_{\mathcal{T}} = j^{\mathcal{T}}_{0,\beta}(\mathrm{dom}(\tilde{\mathcal{E}}_0))|(\eta^{\mathcal{T}}+1,0);$$

(9.3) For all $(\eta^*,\gamma^*) \in j^{\mathcal{T}}_{0,\beta}(\mathrm{dom}(\tilde{\mathcal{E}}_0))|(\eta,\gamma)_{\mathcal{T}}$,
$$E \cap M^{\mathcal{S}}_\beta \cap M^{\mathcal{T}}_\beta = F \cap M^{\mathcal{S}}_\beta \cap M^{\mathcal{T}}_\beta,$$
for all
$$E \in \left(j^{\mathcal{S}}_{0,\beta}(\tilde{\mathcal{E}}_0)\right)(\eta^*,\gamma^*) \cup \left(j^{\mathcal{S}}_{0,\beta}(\tilde{\mathcal{E}}_1)\right)(\eta^*,\gamma^*)$$
and for all
$$F \in \left(j^{\mathcal{T}}_{0,\beta}(\tilde{\mathcal{E}}_0)\right)(\eta^*,\gamma^*) \cup \left(j^{\mathcal{T}}_{0,\beta}(\tilde{\mathcal{E}}_1)\right)(\eta^*,\gamma^*);$$

(9.4) $\mathcal{M}^{\mathcal{S}}_{\tilde{\mathcal{E}}_0,\beta}((\eta,\gamma)_{\mathcal{T}}) = \mathcal{M}^{\mathcal{S}}_{\tilde{\mathcal{E}}_1,\beta}((\eta,\gamma)_{\mathcal{T}}) = \mathcal{M}^{\mathcal{S}}_{\tilde{\mathcal{E}}_0,\alpha^{\mathcal{T}}_Z}((\eta,\gamma)_{\mathcal{T}}) = \mathcal{M}^{\mathcal{S}}_{\tilde{\mathcal{E}}_1,\alpha^{\mathcal{T}}_Z}((\eta,\gamma)_{\mathcal{T}})$;

(9.5) $\mathcal{M}^{\mathcal{T}}_{\tilde{\mathcal{E}}_0,\beta}((\eta,\gamma)_{\mathcal{T}}) = \mathcal{M}^{\mathcal{T}}_{\tilde{\mathcal{E}}_1,\beta}((\eta,\gamma)_{\mathcal{T}}) = \mathcal{M}^{\mathcal{T}}_{\tilde{\mathcal{E}}_0,\alpha^{\mathcal{T}}_Z}((\eta,\gamma)_{\mathcal{T}}) = \mathcal{M}^{\mathcal{T}}_{\tilde{\mathcal{E}}_0,\alpha^{\mathcal{T}}_Z}((\eta,\gamma)_{\mathcal{T}})$;

(9.6) $\mathcal{M}^{\mathcal{S}}_{\tilde{\mathcal{E}}_0,\alpha^{\mathcal{T}}_Z}((\eta,\gamma)_{\mathcal{T}}) = \mathcal{M}^{\mathcal{T}}_{\tilde{\mathcal{E}}_1,\alpha^{\mathcal{T}}_Z}((\eta,\gamma)_{\mathcal{T}})$;

(9.7) $\left(\mathcal{M}^{\mathcal{T}}_{\tilde{\mathcal{E}}_0,\alpha^{\mathcal{T}}_Z}((\eta,\gamma)_{\mathcal{T}})\right)^\# \in M^{\mathcal{S}}_\beta \cap M^{\mathcal{T}}_\beta$.

Using (8.1)–(8.7) and (9.1)–(9.7), the argument is now very much like the standard arguments in a comparison proof.

By the definition of $\mathcal{S}$, $E^{\mathcal{S}}_{\alpha^{\mathcal{S}}_Z}$ witnesses in $M^{\mathcal{S}}_{\alpha^{\mathcal{S}}_Z}$ the coherence condition for $E^{\mathcal{S}}_{\alpha^{\mathcal{S}}_Z}|\eta^{\mathcal{S}}$ relative to either $j^{\mathcal{S}}_{0,\alpha^{\mathcal{S}}_Z}(\tilde{\mathcal{E}}_0)$ or $j^{\mathcal{S}}_{0,\alpha^{\mathcal{S}}_Z}(\tilde{\mathcal{E}}_1)$ where as in (8.2), $\eta^{\mathcal{S}}$ is the first coordinate of $(\eta,\gamma)_{\mathcal{S}}$.

Similarly, by the definition of $\mathcal{T}$, $E^{\mathcal{T}}_{\alpha^{\mathcal{T}}_Z}$ witnesses in $M^{\mathcal{T}}_{\alpha^{\mathcal{T}}_Z}$ the coherence condition for $E^{\mathcal{T}}_{\alpha^{\mathcal{T}}_Z}|\eta^{\mathcal{T}}$ relative to either $j^{\mathcal{T}}_{0,\alpha^{\mathcal{T}}_Z}(\tilde{\mathcal{E}}_0)$ or $j^{\mathcal{T}}_{0,\alpha^{\mathcal{T}}_Z}(\tilde{\mathcal{E}}_1)$ where as in (9.2), $\eta^{\mathcal{T}}$ is the first coordinate of $(\eta,\gamma)_{\mathcal{T}}$.

By (6.3), (8.1)–(8.7), and (9.1)–(9.7), and the novelty and initial segment conditions for Martin-Steel Doddages,
$$\eta^{\mathcal{S}} = \eta^{\mathcal{T}}$$
and $(\eta,\gamma)_{\mathcal{S}} = (\eta,\gamma)_{\mathcal{T}}$. This implies that both $E^{\mathcal{S}}_{\alpha^{\mathcal{S}}_Z}$ and $E^{\mathcal{T}}_{\alpha^{\mathcal{T}}_Z}$ were chosen according to (Case 1) in the construction of $\mathcal{S}$ and $\mathcal{T}$ and moreover the corresponding stages of the construction are the same, i.e., for some $\beta < \omega_1$,
$$(\beta_{\mathcal{S}},\beta_{\mathcal{T}}) = (\alpha^{\mathcal{S}}_Z,\alpha^{\mathcal{T}}_Z).$$
and $((\beta+1)_{\mathcal{S}},(\beta+1)_{\mathcal{T}}) = (\alpha^{\mathcal{S}}_Z+1,\alpha^{\mathcal{T}}_Z+1)$. But
$$(E^{\mathcal{S}}_{\alpha^{\mathcal{S}}_Z}|\eta) \cap M^{\mathcal{S}}_{\beta_{\mathcal{S}}} \cap M^{\mathcal{T}}_{\beta_{\mathcal{T}}} = (E^{\mathcal{T}}_{\alpha^{\mathcal{T}}_Z}|\eta) \cap M^{\mathcal{S}}_{\beta_{\mathcal{S}}} \cap M^{\mathcal{T}}_{\beta_{\mathcal{T}}}$$
where $\eta = \eta^{\mathcal{S}} = \eta^{\mathcal{T}}$, and this contradicts the disagreement which must have been satisfied in the definition of $(E^{\mathcal{S}}_{\beta_{\mathcal{S}}}, E^{\mathcal{T}}_{\beta_{\mathcal{T}}})$. □

4.3 Martin-Steel extender sequences with long extenders

Eliminating the shortness requirement, (1c) of Definition 4.22, in the definition of Martin-Steel extender sequences one obtains the natural extension of Martin-Steel extender sequences to the case of long extenders.

Definition 4.32. An extender sequence,

$$\tilde{E} = \langle E^{\alpha}_{\beta} : (\alpha, \beta) \in \mathrm{dom}(\tilde{E})\rangle$$

is a *generalized Martin-Steel extender sequence* if for each pair $(\alpha, \beta) \in \mathrm{dom}(\tilde{E})$:

(1) (Coherence) There exists an extender F such that

a) $\alpha < \rho(F)$ and $\rho(F)$ is strongly inaccessible,

b) $E^{\alpha}_{\beta} = F|\alpha$,

c) $j_F(\tilde{E})|(\alpha + 1, 0) = \tilde{E}|(\alpha, \beta)$.

(2) (Novelty) For all $\beta^* < \beta$, $(\alpha, \beta^*) \in \mathrm{dom}(\tilde{E})$ and

$$E^{\alpha}_{\beta^*} \cap L[\tilde{E}|(\alpha, \beta)] \neq E^{\alpha}_{\beta} \cap L[\tilde{E}|(\alpha, \beta)]$$

(3) (Initial Segment Condition) Suppose that

$$\kappa < \alpha^* < \alpha$$

where κ is the critical point associated to E^{α}_{β}.

Then there exists β^* such that $(\alpha^*, \beta^*) \in \mathrm{dom}(\tilde{E})$ and such that

$$E^{\alpha^*}_{\beta^*} \cap L[\tilde{E}|(\alpha^* + 1, 0)] = (E^{\alpha}_{\beta}|\alpha^*) \cap L[\tilde{E}|(\alpha^* + 1, 0)].$$ □

4.4 Fast club forcing

We fix some notation. For each strongly inaccessible cardinal δ, let $\mathbb{Q}_\delta$ be the following partial order (which adds a fast club at δ). Conditions are pairs (c, X) where c is a bounded closed subset of δ and X is a set of closed cofinal subsets of δ with $|X| < \delta$.

Suppose $(d, Y), (c, X) \in \mathbb{Q}_\delta$. Then $(d, Y) \leq (c, X)$ if the following hold.

(1) $c = d \cap (\sup(c) + 1)$ and $d \backslash c \subseteq \cap X$,

(2) $X \subseteq Y$.

Thus $\mathbb{Q}_\delta$ is $(<\delta)$-closed. Suppose $G \subset \mathbb{Q}_\delta$ is V-generic and let

$$C_G = \cup \{c \mid (c, X) \in G\}.$$

Then C_G is a closed cofinal subset of δ such that for all closed cofinal sets $D \subset \delta$ with $D \in V$, $C_G \backslash D$ is bounded in δ (so C_G is a fast club in δ).

Lemma 4.33. *Suppose κ is strongly inaccessible and $A \subseteq \kappa$. Suppose $G \subset \mathbb{Q}_\kappa$ is V-generic and in $V[G]$ there is a club $D \subseteq C_G$ such that*

$$D \cap \gamma \in L[A]$$

for all $\gamma < \kappa$. Then $V_\kappa \subset L[A]$.

Proof. Fix a term τ for D. By the homogeneity of $\mathbb{Q}_\kappa$, we can suppose

$$1 \Vdash \text{“}\tau \cap \gamma \in L[A] \text{ for all } \gamma < \kappa\text{”}$$

and that

$$1 \Vdash \text{“}\tau \text{ is closed, cofinal in } C_G\text{”}.$$

For each $\gamma < \kappa$, let D_γ be the set of $(c, X) \in \mathbb{Q}_\kappa$ such that

(1.1) $\gamma < \sup(c)$,

(1.2) for all $\alpha < \sup(c)$, either $(c, X) \Vdash$ "$\alpha \in \tau$" or $(c, X) \Vdash$ "$\alpha \notin \tau$",

(1.3) $\{\alpha < \sup(c) \mid (c, X) \Vdash \text{“}\alpha \in \tau\text{”}\}$ is cofinal in $\sup(c)$.

Thus for each $\gamma < \kappa$, D_γ is dense in $\mathbb{Q}_\kappa$. Further D_γ is $(<\kappa)$-closed. More precisely if

$$\langle (c_\alpha, X_\alpha) : \alpha < \eta \rangle$$

is a decreasing sequence in D_γ where $\eta < \kappa$, then

$$(c, X) \in D_\gamma$$

where

(2.1) $c = (\cup \{c_\alpha \mid \alpha < \eta\}) \cup \{\sup (\cup \{c_\alpha \mid \alpha < \eta\})\}$,

(2.2) $X = \cup \{X_\alpha \mid \alpha < \eta\}$.

Let $\mathbb{D} = \left\{D_\gamma \mid \gamma < \kappa\right\}$. Thus a filter $\mathcal{F} \subset \mathbb{Q}_\kappa$ is $\mathbb{D}$-generic if and only if for each $\gamma < \kappa$ there exists $(c, X) \in D_0 \cap \mathcal{F}$ such that $\gamma < \sup(c)$.

If $\mathcal{F}$ is a $\mathbb{D}$-generic filter let $D_{\mathcal{F}}$ be the interpretation of τ by $\mathcal{F}$. Thus $D_{\mathcal{F}}$ is closed cofinal in κ and for all $\gamma < \kappa$, $D_{\mathcal{F}} \cap \gamma \in L[A]$. The key claim is the following.

(3.1) For each $B \subseteq \kappa$, there exists a pair $(\mathcal{F}_0, \mathcal{F}_1)$ of $\mathbb{D}$-generic filters such that if

$$\langle \eta_\alpha : \alpha < \kappa \rangle$$

is the increasing enumeration of $D_{\mathcal{F}_0} \cap D_{\mathcal{F}_1}$ then for all $\alpha < \kappa$, $\alpha \in B$ if and only if

$$\min \{\eta \in D_{\mathcal{F}_0} \mid \eta_\alpha < \eta\} < \min \{\eta \in D_{\mathcal{F}_1} \mid \eta_\alpha < \eta\}.$$

Since for all $\gamma < \kappa$, $(D_{\mathcal{F}_0} \cap \gamma, D_{\mathcal{F}_1} \cap \gamma) \in L[A]$, (3.1) implies that for all $\gamma < \kappa$, $B \cap \gamma \in L[A]$ and the lemma follows.

The proof of (3.1) follows by noting the following. Suppose $(c_0, X_0) \in \mathbb{Q}_\kappa$ and that either $(c_0, X_0) \in \mathbb{D}$ or $c_0 = \emptyset$. Then for each $\eta < \kappa$ such that $\sup(c_0) < \eta$, there exists $(c_1, X_1) \in \mathbb{D}$ such that

(4.1) $(c_1, X_1) < (c_0, X_0)$,

(4.2) $\eta < \sup(c_1)$,

(4.3) $c_1 \cap \eta = c_0$.

One uses this to construct decreasing sequences

$$\langle (c^0_\alpha, X^0_\alpha) : \alpha < \kappa \rangle$$

and

$$\langle (c^1_\alpha, X^1_\alpha) : \alpha < \kappa \rangle$$

of conditions in D_0 by induction on α such that for all α the following hold.

(5.1) $c^0_0 \cap c^1_0 = \emptyset$.

(5.2) $c^0_{\alpha+1} \cap c^1_{\alpha+1} = c^0_\alpha \cap c^1_\alpha$.

(5.3) If $\alpha > 0$ and α is a limit then

a) $c^0_\alpha = \cup \left\{c^0_\beta \mid \beta < \alpha\right\} \cup \sup \left(\cup \left\{c^0_\beta \mid \beta < \alpha\right\}\right)$,

b) $c^1_\alpha = \cup \left\{c^1_\beta \mid \beta < \alpha\right\} \cup \sup \left(\cup \left\{c^1_\beta \mid \beta < \alpha\right\}\right)$,

c) $\max(c^0_\alpha) = \max(c^1_\alpha)$,

d) if α is the η-th nonzero limit ordinal then $\eta \in B$ if and only if
$$\min(c^0_{\alpha+1}\backslash c^0_\alpha) < \min(c^1_{\alpha+1}\backslash c^1_\alpha).$$

The filters

(6.1) $\mathcal{F}_0$ generated by $\left\{(c^0_\alpha, X^0_\alpha) : \alpha < \kappa\right\}$,

(6.2) $\mathcal{F}_1$ generated by $\left\{(c^1_\alpha, X^1_\alpha) : \alpha < \kappa\right\}$,

witness (3.1) since:

(7.1) $D_{\mathcal{F}_0} \cap D_{\mathcal{F}_1} = \left\{\max(c^0_\alpha) \mid \alpha \text{ is a nonzero limit ordinal}\right\}$.

(7.2) $D_{\mathcal{F}_0} \cap D_{\mathcal{F}_1} = \left\{\max(c^1_\alpha) \mid \alpha \text{ is a nonzero limit ordinal}\right\}$. □

4.5 Weakly Σ_2-definable inner models

Definition 4.34. A sequence
$$N = \langle N_\alpha : \alpha \in \mathrm{Ord}\rangle$$
is *weakly Σ_2-definable* if there is a formula $\phi(x)$ such that:

(1) For all $\beta < \eta_1 < \eta_2 < \eta_3$, if $(N_\phi)^{V_{\eta_1}}|\beta = (N_\phi)^{V_{\eta_3}}|\beta$ then
$$(N_\phi)^{V_{\eta_1}}|\beta = (N_\phi)^{V_{\eta_2}}|\beta = (N_\phi)^{V_{\eta_3}}|\beta;$$

(2) For all $\beta \in \mathrm{Ord}$, $N|\beta = (N_\phi)^{V_\eta}|\beta$ for all sufficiently large η,

where for all γ, $(N_\phi)^{V_\gamma} = \{a \in V_\gamma \mid V_\gamma \models \phi[a]\}$. □

Definition 4.35. Suppose that $N \subset V$ is an inner model and $N \models \mathrm{ZFC}$. Then N is *weakly Σ_2-definable* if the sequence
$$\langle N \cap V_\alpha : \alpha \in \mathrm{Ord}\rangle$$
is weakly Σ_2-definable. □

Remark 4.36. Inner models N which are Σ_2-definable are weakly Σ_2-definable and so as a special case HOD, being Σ_2-definable, is weakly Σ_2-definable.

More generally, for each $\alpha \in \mathrm{Ord}$, let T_α be the Σ_2-theory of V with parameters from V_α. Then the sequence
$$\langle T_\alpha : \alpha \in \mathrm{Ord}\rangle$$
is weakly Σ_2-definable. □

Remark 4.37. The increasing enumeration $\langle \delta_\alpha : \alpha \in \mathrm{Ord}\rangle$ of all supercompact cardinals is weakly Σ_2-definable. □

Definition 4.38. Suppose that N is a transitive inner model of ZFC which is weakly Σ_2-definable and $V_\delta \prec_{\Sigma_2} V$. Then $(N)^{V_\delta}$ denotes the union of the sequence
$$\langle N^*_\alpha : \alpha < \delta\rangle = (N_\phi)^{V_\delta}$$
where ϕ is a formula which witnesses that
$$\langle N \cap V_\alpha : \alpha \in \mathrm{Ord}\rangle$$
is weakly Σ_2-definable. □

Remark 4.39. This is well-defined in the sense that it does not depend on the choice of the formula ϕ which witnesses that $\langle N \cap V_\alpha : \alpha \in \mathrm{Ord}\rangle$ is weakly Σ_2-definable. $\square$

Definition 4.40. A cardinal κ is a *strong cardinal* if for every λ there is an elementary embedding

$$j : V \to M$$

such that $\mathrm{CRT}(j) = \kappa$, $j(\kappa) > \lambda$, and such that $V_\lambda \subset M$. $\square$

Lemma 4.41. *Suppose that*

$$N = \langle N_\alpha : \alpha \in \mathrm{Ord}\rangle$$

is weakly Σ_2-definable and δ is a strong cardinal. Then $N \cap V_\delta = (N)^{V_\delta}$.

Proof. Let $\phi(x)$ be a formula which witnesses that N is weakly Σ_2-definable.

Assume toward a contradiction that $N|\delta \neq (N)^{V_\delta}$. Then there exists $\eta > \delta$ and $\beta < \delta$ such that

$$N|\beta = (N_\phi)^{V_\eta}|\beta \neq (N_\phi)^{V_\delta}|\beta.$$

Since δ is a strong cardinal, $V_\delta \prec_{\Sigma_2} V$ and so there exists $\beta < \eta_0 < \delta$ such that

$$N|\beta = (N_\phi)^{V_{\eta_0}}|\beta.$$

But then

(1.1) $\beta < \eta_0 < \delta < \eta$,

(1.2) $(N_\phi)^{V_{\eta_0}}|\beta = (N_\phi)^{V_\eta}$,

(1.3) $(N_\phi)^{V_{\eta_0}}|\beta \neq (N_\phi)^{V_\delta}|\beta$,

which is a contradiction. $\square$

Lemma 4.42. *Suppose that N is a transitive inner model of* ZFC*, N is weakly Σ_2-definable, δ is an extendible cardinal, and that*

$$V_\delta \subset N.$$

Then $V = N$.

Proof. Let ϕ be a formula which witnesses that

$$\langle N \cap V_\alpha : \alpha \in \mathrm{Ord}\rangle$$

is weakly Σ_2-definable. Since δ is a strong cardinal, by Lemma 4.41,

$$\langle V_\alpha : \alpha < \delta\rangle = (N_\phi)^{V_\delta}.$$

Since δ is an extendible cardinal, for a proper class of κ,

$$V_\delta \prec V_\kappa$$

and so for a proper class of κ,

$$\langle V_\alpha : \alpha < \kappa\rangle = (N_\phi)^{V_\kappa}.$$

Therefore

$$\langle V_\alpha : \alpha \in \mathrm{Ord}\rangle = \langle N \cap V_\alpha : \alpha \in \mathrm{Ord}\rangle$$

and so $V = N$. $\square$

Theorem 4.43. *Suppose that there is an extendible cardinal. Then there is a class-generic extension $V[G]$ of V in which the following hold.*

(1) $V[G] = (\mathrm{HOD})^{V[G]}$.

(2) $V[G]_\gamma = V_\gamma$ *where* γ *is the least strongly inaccessible cardinal of* V.

(3) *Every extendible cardinal of* V *is an extendible cardinal in* $V[G]$.

(4) *Suppose* $\mathbb{E} \subset \mathrm{Ord}$ *and* δ *are such that the following hold.*

(a) $L[\mathbb{E}]$ *is weakly* Σ_2*-definable.*

(b) δ *is an extendible cardinal in* $V[G]$.

(c) *Let* $X \subset \delta$ *be the set of all* $\kappa < \delta$ *such that there is an elementary embedding,*

$$j : V[G]_{\lambda+1} \to V[G]_{j(\lambda)+1}$$

with $\mathrm{CRT}(j) = \kappa$ *and* $j(\kappa) = \delta$, *where* λ *is the least strongly inaccessible cardinal above* κ. *Then there exists* $Y \subset X$ *such that* $Y \cap \xi \in L[\mathbb{E}]$ *for all* $\xi < \delta$ *and such that*

$$\sup(Y) = \sup(X) = \delta.$$

Then $L[\mathbb{E}] = V[G]$.

Proof. Let G be V-generic for the backward Easton iteration

$$\langle \mathbb{P}_\alpha : \alpha \in \mathrm{Ord} \rangle$$

where the following hold for each α.

(1.1) If α is strongly inaccessible and Mahlo in $V^{\mathbb{P}_\alpha}$ then

$$\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \mathbb{B} * \mathbb{Q}$$

where $\mathbb{B}$ adds a Cohen generic subset to α^+ and $\mathbb{Q}$ is the fast-club forcing $\mathbb{Q}_\gamma$ defined in $V^{\mathbb{P}_\alpha * \mathbb{B}}$ with $\gamma = \alpha$.

(1.2) If $\alpha = \beta + 1$ and β is strongly inaccessible and Mahlo in $V^{\mathbb{P}_\beta}$ then

$$\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \mathbb{H}$$

where $\mathbb{H}$ codes $(G_\alpha, V_{\alpha+1}, \langle \mathbb{P}_\xi : \xi \leq \alpha \rangle)$ into the powerset function before the next strongly inaccessible cardinal above. The set being coded is naturally a set of ordinals by the definition of of $\mathbb{P}_{\beta+1}$ as the iteration $\mathbb{P}_\beta * \mathbb{B} * \mathbb{Q}$, and so $\mathbb{H}$ can be chosen canonically.

(1.3) Otherwise $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha$.

By standard lifting arguments, every extendible cardinal of V remains extendible in $V[G]$.

We note that the following must hold in $V[G]$ where for each strongly inaccessible Mahlo cardinal γ of $V[G]$, C_γ is the fast club added by $G_{\gamma+1}$.

(2.1) Suppose that

$$\pi : V[G]_{\kappa+1} \to V[G]_{\pi(\kappa)+1}$$

is an elementary embedding such that $\mathrm{CRT}(\pi) < \kappa$ and such that κ is strongly inaccessible in $V[G]$. Let $\gamma = \mathrm{CRT}(\pi)$. Then $\pi(C_\gamma) = C_{\pi(\gamma)}$ and

$$C_{\pi(\gamma)} \cap \gamma = C_\gamma.$$

We have:

(3.1) $X \subset \delta$ is the set of all $\kappa < \delta$ such that there is an elementary embedding,

$$j : V[G]_{\lambda+1} \to V[G]_{j(\lambda)+1}$$

with $\mathrm{CRT}(j) = \kappa$ and $j(\kappa) = \delta$ where λ is the least strongly inaccessible cardinal above κ.

Therefore by (2.1)

(4.1) $X \subset C_\delta$, where C is the fast-club added by G at stage δ.

Thus:

(5.1) Y is a cofinal subset of C_δ such that $Y \cap \xi \in L[\mathbb{E}]$ for all $\xi < \delta$.

Since $\mathbb{E}$ is weakly Σ_2-definable in $V[G]$ and since δ is a strong cardinal in $V[G]$, by Lemma 4.41:

(6.1) $L[\mathbb{E}] \cap V[G]_\delta = (L[\mathbb{E}])^{V[G]_\delta}$.

Further since δ is strongly inaccessible and Mahlo in $V[G]$,

(7.1) $V[G]_\delta \subset V[G|\delta]$.

Therefore by Lemma 4.33 and (5.1)

$$V[G]_\delta \subset L[\mathbb{E}].$$

But then by Lemma 4.42, $V[G] = L[\mathbb{E}]$. □

Theorem 4.43 has quite a number of implications which constrain the possibilities for defining weak extender models for supercompactness which generalize L.

We end with this section with two theorems which deal with generalized Martin-Steel extender sequences. The first theorem is a corollary of the proof of Theorem 4.43 and the basic argument is given in [24]. The second theorem is a corollary of Theorem 4.43.

Theorem 4.44. *Suppose that* $V = \text{HOD}$ *and that there is an extendible cardinal. Then there is a generalized Martin-Steel extender sequence* $\tilde{E}$ *such that* $\tilde{E}$ *is* Σ_2*-definable,*

$$V = L[\tilde{E}],$$

and such that for each $(\alpha, \beta) \in \text{dom}(\tilde{E})$*,*

$$\alpha \leq \kappa^*_{E^\alpha_\beta} + 1.$$ □

Theorem 4.44 could just simply indicate that one needs additional conditions in the definition of generalized Martin-Steel extender sequences beyond the Novelty Condition and the Initial Segment Condition. The following variation of Theorem 4.43 essentially rules this out.

Theorem 4.45. *Assume that there is an extendible cardinal. Then there is a class-generic extension* $V[G]$ *of* V *in which the following hold.*

(1) $V[G] = (\text{HOD})^{V[G]}$.

(2) *Every extendible cardinal of* V *is an extendible cardinal in* $V[G]$.

(3) *Suppose that* $\tilde{E}$ *is a generalized Martin-Steel extender sequence such that* $\tilde{E}$ *is* Σ_2*-definable and such that*

$$V[G] \neq L[\tilde{E}].$$

Then for all $(\alpha, \beta) \in \text{dom}(\tilde{E})$*, if* $\kappa_{E^\alpha_\beta}$ *is an extendible cardinal in* $V[G]$ *then*

$$\alpha \leq \kappa^*_{E^\alpha_\beta} + 1.$$

Proof. Let $V[G]$ be the generic extension given by Theorem 4.43.

Suppose $(\alpha, \beta) \in \text{dom}(\tilde{E})$, $\kappa_{E^\alpha_\beta}$ is an extendible cardinal of $V[G]$, and that

$$\alpha > \kappa^*_{E^\alpha_\beta} + 1.$$

Let $\delta = \kappa_{E^\alpha_\beta}$ and $X \subset \delta$ be the set of all $\kappa < \delta$ such that there is an elementary embedding,

$$j : V[G]_{\lambda+1} \to V[G]_{j(\lambda)+1}$$

with $\mathrm{CRT}(j) = \kappa$ and $j(\kappa) = \delta$, where λ is the least strongly inaccessible cardinal above κ.

Let

$$Y = \left\{ \kappa_{E_\eta^{\delta+1}} \mid (\delta + 1, \eta) \in \mathrm{dom}(\tilde{E}) \text{ and } \delta = \kappa^*_{E_\eta^{\delta+1}} \right\}.$$

By the Novelty and Initial Segment Condition,

(1.1) $\sup(Y) = \delta$.

By the Coherence Condition, $Y \subset X$. Therefore by Lemma 4.33, the Coherence Condition again, and the proof of Theorem 4.43, $V[G] = L[\tilde{E}]$. □

5 The comparison obstruction

Definition 5.1. A cardinal κ is *superstrong* if there is an elementary embedding

$$j : V \to M$$

such that $\mathrm{CRT}(j) = \kappa$ and such that $V_{j(\kappa)} \subset M$. □

Theorem 4.45 arguably rules out any direct generalization of Kunen's $L[U]$ at the level of one measurable cardinal to the levels past superstrong. The point is that if $\tilde{E}$ is a generalized Martin-Steel extender sequence such that

$$\alpha \leq \kappa^*_{E^\alpha_\beta} + 1$$

for all $(\alpha,\beta) \in \mathrm{dom}(\tilde{E})$ then for all $(\alpha,\beta) \in \mathrm{dom}(\tilde{E})$, if E is the $L[\tilde{E}]$-extender given by E^α_β, then in $L[\tilde{E}]$,

$$\rho(E) \leq \nu_E \leq \kappa^*_E.$$

Therefore a new approach is needed and a reasonable candidate is the family of *partial extender models*, first defined by Mitchell–Steel, [12].

5.1 Partial extender models

Recall that a transitive set M is *rudimentarily closed* if

(1) for all $a, b \in M$, $\{a, b\} \in M$, and $\cup a \in M$,

(2) for all $a \in M$, if $b \subset [a]^n$ for some $n < \omega$ and b is Σ_0-definable with parameters from M, then $b \in M$.

The property that a transitive set M be rudimentary closed is formally defined as being closed under the functions generated by the following schemes, these are the *rudimentary functions*, Jensen [4].

(1) $f(a_0, \ldots, a_n) = a_i$.

(2) $f(a_0, \ldots, a_n) = a_i \backslash a_j$.

(3) $f(a_0, \ldots, a_n) = \left\{a_i, a_j\right\}$.

(4) $f(a_0, \ldots, a_n) = h(g_0(a_0, \ldots, a_n), \ldots, g_m(a_0, \ldots, a_n))$.

(5) $f(a_0, \ldots, a_n) = \cup\{g(b, a_1, \ldots, a_n) \mid b \in a_0\}$.

Definition 5.2. Suppose P is a set. Then $J_\alpha[P]$ is defined by induction on α as follows, [4].

(1) $J_0[P] = \emptyset$.

(2) $J_{\alpha+1}[P] = M$ where M is the smallest transitive rudimentarily closed set such that $J_\alpha[P] \in M$ and such that for each $b \in M$, $P \cap b \in M$.

(3) $J_\alpha[P] = \cup\left\{J_\beta[P] \mid \beta < \alpha\right\}$ if $\alpha > 0$ and α is a limit ordinal. □

Lemma 5.3. *Suppose $P \in V$, $\alpha \in \mathrm{Ord}$, and*

$$J_\alpha[P] \vDash \mathrm{ZF}\backslash\mathrm{Powerset}.$$

Then

$$J_\alpha[P] \vDash \text{Axiom of Choice}.$$ □

Definition 5.4. Suppose that $P \in V$ and $\alpha \in \mathrm{Ord}$. Then $J_\alpha[P]$ is *strongly acceptable* if for all $\beta < \alpha$ and for all $\kappa < \beta$, if

$$\mathcal{P}(\kappa) \cap J_\beta[P] \neq \mathcal{P}(\kappa) \cap J_{\beta+1}[P]$$

then $|J_\beta[P]| \leq \kappa$ in $J_{\beta+1}[P]$. □

Definition 5.5. E is an *partial extender* if E is an M-extender for transitive set such that $M \vDash \mathrm{ZFC}\backslash\mathrm{Powerset}$. □

Definition 5.6. Suppose $\mathbb{E} = \langle E_\alpha : \alpha \in \mathrm{dom}(\mathbb{E})\rangle$ is a sequence of partial extenders and that for all $\alpha \in \mathrm{dom}(\mathbb{E})$, $\mathrm{LTH}(E_\alpha) \leq \alpha$. Then for all $\eta \in \mathrm{Ord}$,

$$J^{\mathbb{E}}_\eta = J_\eta[P_{\mathbb{E}}]$$

where $P_{\mathbb{E}} = \{(\alpha, a, x) \mid \alpha \in \mathrm{dom}(\mathbb{E}), (a, x) \in E_\alpha\}$. □

Definition 5.7. Suppose that M is transitive,

$$M \vDash \mathrm{ZFC}\backslash\mathrm{Powerset}$$

and that E is an M-extender. Let

$$j_E : M \to N \cong \mathrm{Ult}_0(M, E)$$

be the ultrapower embedding. Then:

(1) $\kappa_E = \mathrm{CRT}(j_E)$ and $\kappa^*_E = j_E(\kappa_E)$.

(2) An ordinal $\xi < \mathrm{LTH}(E)$ is a *generator* of E if for all $f \in M$, for all $a \in [\xi]^{<\omega}$,

$$j_E(f)(a) \neq \xi.$$

(3) $\nu_E = \sup\{\xi + 1 \mid \xi \text{ is a generator of } E\}$; ν_E is the *natural length* of E.

(4) The M-extender E is a *short extender* if $\nu_E \leq j_E(\kappa_E)$ and E is a *long extender* if $j_E(\kappa_E) < \nu_E$.

(5) ι_E is the least cardinal γ of M such that $\nu_E \leq j_E(\gamma)$.

(6) F is the *Jensen completion* [8] of $E|\nu_E$ if F is the M-extender of length η given by j_E where

$$\eta = ((j_E(\iota_E))^+)^N.$$

(7) ν^*_E is the least $\theta \leq \nu_E$ such that $E|\theta \notin N$. □

In the following definition, the requirement that $J^{\mathbb{E}}_\alpha \vDash \mathrm{ZFC}\backslash\mathrm{Powerset}$ follows from the indexing requirement, but we repeat it for emphasis.

Definition 5.8. Suppose that $\mathbb{E}$ is a partial extender sequence and $\alpha \in \mathrm{dom}(\mathbb{E})$. Then $\mathbb{E}$ is a *good partial extender sequence at* α if the following hold where E is the partial extender $\mathbb{E}_\alpha$.

(1) $J^{\mathbb{E}}_\alpha$ is strongly acceptable and $J^{\mathbb{E}}_\alpha \vDash \mathrm{ZFC}\backslash\mathrm{Powerset}$.

(2) E is a $J^{\mathbb{E}}_\alpha$-extender.

(3) (Indexing) E is the Jensen completion of $E|\nu_E$ and $\alpha = \mathrm{LTH}(E)$.

(4) (Coherence) Let

$$j_E : J^{\mathbb{E}}_\alpha \to \mathrm{Ult}_0(J^{\mathbb{E}}_\alpha, E)$$

be the elementary embedding given by E. Then

$$j_E(\mathbb{E}|\alpha)|(\alpha + 1) = \mathbb{E}|\alpha.$$ □

[8]The Jensen completion was suggested by Sy Friedman as an alternative to the indexing scheme of Mitchell-Steel [12], and Jensen [5] was the first to develop the detailed fine-structure theory based on this indexing scheme.

5.2 Comparison by least disagreement

We consider a fairly general class of structures and we shall use the following definition repeatedly.

Definition 5.9. Suppose $M \models$ ZFC, M is transitive, $\mathbb{E}$ is a sequence of partial extenders from M, and $\delta < \lambda < \mathrm{Ord}^M$. Then δ is *witnessed* by the partial extenders on the sequence $\mathbb{E}$ to be λ-supercompact in M if there exists $\alpha \in \mathrm{dom}(\mathbb{E})$ such that

(1) E is an M-extender,

(2) $\kappa_E = \delta$ and $\lambda \leq \iota_E$,

(3) $j_E[\lambda] \in M_E$,

where E is the partial extender given by $\mathbb{E}_\alpha$ and where

$$j_E : M \to M_E \cong \mathrm{Ult}_0(M, E)$$

is the ultrapower embedding. □

We consider transitive structures of the form

$$(\mathcal{M}, \mathbb{E}) \models \mathrm{ZFC}$$

such that the following hold for all $\beta \in \mathrm{dom}(\mathbb{E})$ such that E is an $\mathcal{M}$-extender, where E is the partial extender given by $\mathbb{E}_\beta$.

Suppose that $\kappa_E < \iota_E$,

$$j_E : \mathcal{M} \to \mathcal{M}_E \cong \mathrm{Ult}_0(\mathcal{M}, E)$$

is the ultrapower embedding, and let $\iota = \iota_E$. Then:

(1) $\iota < \kappa_E^*$.

(2) (**First Supercompactness Condition**) Suppose that $j_E[\iota] \notin \mathcal{M}_E$ and let $\delta \leq \iota$ be least such that $j_E[\delta] \notin \mathcal{M}_E$. Then the following hold.

  a) Suppose that $\delta < \iota$ and that κ_E is supercompact in $\mathcal{M}$. Then $(\mathrm{cof}(\delta))^{\mathcal{M}} < \kappa_E$ and $\iota = (\delta^+)^{\mathcal{M}}$.

  b) Suppose $\delta = \iota$. Then ι is a limit cardinal of $\mathcal{M}$.

(3) (**Second Supercompactness Condition**) Suppose that $j_E[\iota] \in \mathcal{M}_E$. Then for some $\xi \in \mathrm{Ord}^{\mathcal{M}}$:

  a) (**Largest Generator Condition**) $\nu_E < j_E(\iota)$ and $\nu_E = \xi + 1$.

  b) (**First Initial Segment Condition**) $E|\eta \in \mathcal{M}_E$ for all $\eta < \xi$.

  c) (**Second Initial Segment Condition**) if $E|\xi \notin \mathcal{M}_E$ then $(\mathrm{cof}(\xi))^{\mathcal{M}_E} < j_E(\kappa_E)$.

(4) (**Coherence Condition**) $\mathcal{M}|\beta = \mathcal{M}_E|\beta$ and $\beta = \sup(j_E[\gamma]) = j_E(\gamma)$, $\gamma = (\iota^+)^{\mathcal{M}}$.

(5) (**Suitability Condition**) No $\delta < \kappa_E$ is $(<\kappa_E)$-supercompact in $\mathcal{M}$.

Thus we are assuming that Jensen indexing is being used and that $\mathcal{M}|\beta$ makes sense. If $\mathcal{M}$ is of the form of $L[\mathbb{E}]$ then this is immediate, but we are not assuming that $\mathcal{M}$ has this form.

We really have in mind that

$$\mathcal{M} = (J_\alpha[P], P \cap J_\alpha[P]),$$

for some set $P \in V$, $J_\alpha[P]$ is strongly acceptable, and that

$$\mathbb{E} = P|\mathrm{dom}(\mathbb{E}).$$

But there is no need to be so explicit at this stage. With notation as above and by any reasonable notion of coherence

$$\mathcal{M}|\beta = \mathrm{Ult}_0(\mathcal{M}, E)|\beta.$$

Further β is a successor cardinal in $\mathrm{Ult}_0(\mathcal{M}, E)$ and so $\mathrm{Ult}_0(\mathcal{M}, E)|\beta$ makes perfect sense by setting

$$\mathrm{Ult}_0(\mathcal{M}, E)|\beta = (H(\beta))^{\mathrm{Ult}_0(\mathcal{M},E)}$$

if $\mathcal{M}$ were simply a transitive set, and making the obvious adjustments for the additional predicates of $\mathcal{M}$ if $\mathcal{M}$ itself is a structure.

We assume that as part of the structure $(\mathcal{M}, \mathbb{E})$, there is a wellordering $<_{\mathcal{M}}$ of length $\mathrm{Ord}^{\mathcal{M}}$ such that for all uncountable regular cardinals γ of $\mathcal{M}$,

$$<_{\mathcal{M}} \cap (H(\gamma))^{\mathcal{M}}$$

is a wellordering of $(H(\gamma))^{\mathcal{M}}$ in length γ.

Thus we are really considering structures

$$(\mathcal{M}, \mathbb{E}) \vDash \mathrm{ZFC} + \mathrm{GCH}$$

where $\mathcal{M}$ itself is a structure with additional predicates including the wellordering, $<_{\mathcal{M}}$. All of this we suppress to simplify notation.

Therefore for every element $a \in \mathcal{M}$, a is definable in the structure

$$(\mathcal{M}, \mathbb{E})$$

from ordinal parameters, and this will be an important feature for us.

Remark 5.10. The requirement (3) combined with (4) implies:

(1) $\nu_E \leq \nu^*_E + 1$,

(2) $\nu_E = \nu^*_E$ if and only if ν^*_E is not a limit of generators.

This is a very natural version of a weak initial segment condition, see Definition 6.31 on page 86, and it would be a reasonable requirement to impose on all the partial extenders on the sequence $\mathbb{E}$ but we will not need this for our abstract treatment.

We do not impose the weak initial segment condition (which would imply in requirement (2) that $\delta = \iota$) and instead use the more complicated requirements listed above (which are slightly more general than we need in [25]) because we need in [25] to apply our main negative theorem, Theorem 5.35.

There are fairly general arguments, see Remark 5.13, that for the structures one is ultimately interested in for this account, one can always require the weak initial segment condition to hold whenever ι_E is a successor cardinal except in the situation where

$$\iota_E = (\delta^+)^{\mathcal{M}}$$

and $(\mathrm{cof}(\delta))^{\mathcal{M}} < \kappa_E$.

This accounts for the formulation of the First Supercompactness Condition. The sequences defined in [25] allow for more complicated failures of the weak initial segment condition if κ_E is not already ι_E-supercompact at the stage where E is indexed.

It is because of the coding constraints of Section 4 that one must allow failures of the weak initial segment condition. □

For the remainder of this section, writing $(\mathcal{M}, \mathbb{E}) \vDash \mathrm{ZFC}$ indicates that $(\mathcal{M}, \mathbb{E})$ is a transitive structure satisfying the conditions specified above, though for emphasis, we will also occasionally explicitly add the hypothesis of transitivity.

Definition 5.11. Suppose that $(\mathcal{M}, \mathbb{E}) \vDash \mathrm{ZFC}$.

(1) $(\mathcal{M}, \mathbb{E})$ is *finitely generated* if for some $a \in \mathcal{M}$, every element $b \in \mathcal{M}$ is definable in $(\mathcal{M}, \mathbb{E})$ from a.

(2) $X \prec (\mathcal{M}, \mathbb{E})$ is *finitely generated* if for some $a \in \mathcal{M}$, X is the set of all $b \in \mathcal{M}$ such that b is definable in $(\mathcal{M}, \mathbb{E})$ from a. □

Clearly, $X \prec (\mathcal{M}, \mathbb{E})$ is finitely generated if and only if $(\mathcal{M}_X, \mathbb{E}_X)$ is finitely generated where $(\mathcal{M}_X, \mathbb{E}_X)$ is the transitive collapse of X. Further since every element $a \in \mathcal{M}$ is definable in the structure

$$(\mathcal{M}, \mathbb{E})$$

from ordinal parameters, every $a \in \mathcal{M}$ belongs to a $\subseteq$-least finitely generated elementary substructure of $(\mathcal{M}, \mathbb{E})$.

We need an abstract notion of *backgrounding*. A rather weak version is defined below and suffices for our purposes.

Definition 5.12. Suppose $(\mathcal{M}, \mathbb{E}) \vDash$ ZFC and that $(\mathcal{M}, \mathbb{E})$ is transitive.

(1) $(\mathcal{M}, \mathbb{E})$ is *weakly backgrounded at κ* if for all $\mathcal{M}$-extenders E given by $\mathbb{E}$ with $\kappa = \kappa_E$, if $\kappa_E < \gamma$, if

$$j_E[\gamma] \in \mathcal{M}_E \cong \mathrm{Ult}_0(\mathcal{M}, E),$$

and if U is the normal measure on $(\mathcal{P}_\kappa(\gamma))^M$ given by E, then κ is a cardinal in V which is γ-supercompact in V and there is a normal fine κ-complete ultrafilter U^* on $\mathcal{P}_\kappa(\gamma)$ such that $U = U^* \cap \mathcal{M}$.

(2) $(\mathcal{M}, \mathbb{E})$ is *weakly backgrounded* if $(\mathcal{M}, \mathbb{E})$ is weakly backgrounded at κ for all $\kappa \in \mathrm{Ord}^{\mathcal{M}}$. □

Remark 5.13. Suppose M is a transitive set and $M \vDash$ ZFC. Following Hamkins [3], for each uncountable regular cardinal κ of M and for each cardinal γ of M, M satisfies the *κ-approximation property* at γ if for all $A \subset \gamma$, if $A \cap \sigma \in M$ for all $\sigma \in M$ with $|\sigma|^M < \kappa$ then $A \in M$.

A very conservative version of a backgrounded construction is as follows and here we are motivating the formulation of the First Supercompactness Condition, the other conditions are strongly motivated by current constructions.

The final model $(\mathcal{M}^\infty, \mathbb{E}^\infty)$ is constructed as a limit of approximations $(\mathcal{M}_\alpha, \mathbb{E}_\alpha)$, constructed at some ordinal stage α, where in passing from $(\mathcal{M}_\alpha, \mathbb{E}_\alpha)$ to $(\mathcal{M}_{\alpha+1}, \mathbb{E}_{\alpha+1})$ one only adds an extender in the following situation and for this discussion we set

$$(\mathcal{M}, \mathbb{E}_{\mathcal{M}}) = (\mathcal{M}_\alpha, \mathbb{E}_\alpha).$$

There exists an elementary embedding

$$j : V \to M$$

such that:

(1) $(\mathcal{M}, \mathbb{E}_{\mathcal{M}}) \vDash$ ZFC.

(2) $V_{j(\lambda)+1} \subset M$, $\mathrm{CRT}(j) < \lambda < j(\mathrm{CRT}(j))$, λ is strongly inaccessible.

(3) $j(\mathcal{M}, \mathbb{E}_{\mathcal{M}})|j(\lambda) = (\mathcal{M}, \mathbb{E}_{\mathcal{M}})|j(\lambda)$.

(4) There exists $\theta < \lambda$ such that $(F|\theta) \cap \mathcal{M} \notin \mathcal{M}$ where F is the V-extender of length λ given by j.

Let $\theta < \lambda$ be least such that $(F|\theta) \cap \mathcal{M} \notin \mathcal{M}$ where F is the V-extender of length λ given by j, let E be the $\mathcal{M}$-extender given by $(F|\theta) \cap \mathcal{M}$, and let

$$\mathcal{N} = j(\mathcal{M})|(\sup(j[(\iota_E^+)^{\mathcal{M}}])) = \mathcal{M}_E|j_E((\iota_E^+)^{\mathcal{M}})$$

where

$$j_E : (\mathcal{M}, \mathbb{E}_{\mathcal{M}}) \to (\mathcal{M}_E, \mathbb{E}_{\mathcal{M}_E}) \cong \mathrm{Ult}_0((\mathcal{M}, \mathbb{E}_{\mathcal{M}}), E)$$

is the ultrapower embedding.

The natural step would be to add $\mathrm{Ord}^{\mathcal{N}}$ to $\mathrm{dom}(\mathbb{E}_{\mathcal{M}})$ with E as the next extender. The coding constraints of Section 4 strongly suggest that one should only do this if (in essence) there exists $\sigma \in \mathcal{N}$ such that

- $|\sigma|^{\mathcal{N}} < \kappa^*_E$ and $E|\sigma \notin \mathcal{M}$.

Therefore this is the requirement which must be (in essence) satisfied in order to change $\mathbb{E}_{\mathcal{M}}$ in defining the next approximation to $\mathcal{M}^\infty$. That this suffices is by strong acceptability:

- Adding a new bounded subset of κ^*_E must (lead to the) *collapse* of κ^*_E in generating the next (sound) approximation to the final model.

The issue arises when one can be sure that the required set σ exists. We claim that if *no* such set σ exists then necessarily:

$$j(\iota_E) = \sup(j[\iota_E]) = \theta.$$

We verify this. First note that if $\sup(j[\iota_E]) = j(\iota_E)$ then by the definition of θ, necessarily $\theta = j(\iota_E)$. This is because if $\sup(j[\iota_E]) = j(\iota_E)$ then necessarily ι_E is a limit cardinal in $\mathcal{M}$.

Now suppose that $\sup(j[\iota_E]) < j(\iota_E)$. We claim:

- $\mathcal{M}$ must satisfy the κ_E-approximation property for all $\gamma < \iota_E$.

Suppose $A \subset \gamma$ and $A \cap \sigma \in \mathcal{M}$ for all $\sigma \in \mathcal{M}$ with $|\sigma|^{\mathcal{M}} < \kappa_E$. Applying j, $j(A) \cap \tau \in \mathcal{M}$ for all $\tau \in \mathcal{M}$ with $|\tau|^{\mathcal{M}} < j(\kappa_E)$ (since $j(\mathcal{M}) = \mathcal{M}$). Further $E|j(\gamma) \in \mathcal{M}$ and so $j[\gamma] \in \mathcal{M}$. Thus $j(A) \cap j[\gamma] \in \mathcal{M}$ and this implies that $A \in \mathcal{M}$.

Since $\mathcal{M}$ has that κ_E-approximation property at γ for all $\gamma < \iota_E$ and since $j(\mathcal{M}) = \mathcal{M}$:

- $\mathcal{M}$ has the $j(\kappa_E)$-approximation property at γ for all $\gamma < j(\iota_E)$.

Therefore if $\theta < j(\iota_E)$ $\mathcal{M}$ has the $j(\kappa_E)$ approximation property at $|\theta|^{\mathcal{M}}$ and it follows easily that σ exists. If $\theta = j(\iota_E)$ then $j[\iota_E] \in \mathcal{M}$ and so arguing as above, $\mathcal{M}$ has the κ_E-approximation property at ι_E. The only potential issue here is if

$$\iota_E = (\iota^+)^{\mathcal{M}}.$$

But then $\theta \geq j(\iota)$ and so $E|j(\iota) \in \mathcal{M}$ and this implies $j[\iota_E] \in \mathcal{M}$.

This implies that $\mathcal{M}$ has the $j(\kappa_E)$-approximation property at $j(\iota_E) \geq \theta$ and so again σ must exist.

This verifies the claim above that if no such a set σ exists then necessarily:

$$j(\iota_E) = \sup(j[\iota_E]) = \theta.$$

Now suppose that σ does not exist, κ_E is witnessed to be $({<}\lambda)$-supercompact in $\mathcal{M}$ by $\mathbb{E}_{\mathcal{M}}$, *and* that $(\mathcal{M}, \mathbb{E}_{\mathcal{M}})$ is weakly backgrounded. Thus $\mathcal{M}$ has the κ_E-approximation property at *all* $\gamma < \lambda$ and so $\mathcal{M}$ has the $j(\kappa_E)$-approximation property at all $\gamma < j(\lambda)$. This implies that the following must hold.

(1) $\mathrm{cof}(\iota_E) < \kappa_E$.

(2) There exists $\sigma \in \mathcal{M}|j((\iota^+_E)^{\mathcal{M}})$ such that $|\sigma|^{\mathcal{M}} < \kappa^*_E$ and such that $E|\sigma \notin \mathcal{M}$

Now again the coding constraints of Section 4 strongly suggest that if one changes $\mathbb{E}_{\mathcal{M}}$ then one should use $E^*|\theta^*$ where $E^* = F \cap \mathcal{M}$ and where $\theta^* < j((\iota^+_E)^{\mathcal{M}})$ is least such that (2) is witnessed to hold by some set

$$\sigma \in \mathrm{Ult}_0(\mathcal{M}, E^*|\theta^*).$$

Thus since no such σ exists in $\mathcal{N}$, which implies that $\theta^* > \theta$, necessarily $\theta^* = \xi + 1$ for some ξ which is a generator of E and this puts one in the situation corresponding to the First Supercompactness Condition. □

We define a fairly general notion of iteration.

Definition 5.14. Suppose $(\mathcal{M}, \mathbb{E}) \vDash$ ZFC and $(\mathcal{M}, \mathbb{E})$ is transitive. A *semi-iteration* of $(\mathcal{M}, \mathbb{E})$ is a continuous (linearly) directed system

$$((\mathcal{N}_\alpha, \mathbb{F}_\alpha), \pi_{\alpha,\beta}, E_\alpha : \alpha < \beta \leq \eta)$$

(with $\eta > 0$) such that the following hold for all $\alpha < \eta$ and for all $\alpha < \beta < \eta$.

(1) $(\mathcal{N}_0, \mathbb{F}_0) = (\mathcal{M}, \mathbb{E})$ and $\mathcal{N}_\alpha$ is transitive for all $\alpha \leq \eta$.

(2) E_α is an $\mathcal{N}_\alpha$-extender, $\mathcal{N}_{\alpha+1} = \mathrm{Ult}_0(\mathcal{N}_\alpha, E_\alpha)$, and

$$\pi_{\alpha,\alpha+1} : \mathcal{N}_\alpha \to \mathcal{N}_{\alpha+1}$$

is the ultrapower embedding.

(3) (**Suitability Condition**) No $\delta < \kappa_{E_\alpha}$ is $(<\kappa_{E_\alpha})$-supercompact in $\mathcal{N}_\alpha$.

(4) (**Non-overlapping Condition**) $\iota_{E_\alpha} < \kappa^*_{E_\alpha} \leq \kappa_{E_\beta}$.

(5) (**First Supercompactness Condition**) Suppose that $\pi_{\alpha,\alpha+1}[\iota_{E_\alpha}] \notin \mathcal{N}_{\alpha+1}$ and let $\delta \leq \iota_{E_\alpha}$ be least such that $\pi_{\alpha,\alpha+1}[\delta] \notin \mathcal{N}_{\alpha+1}$. Then the following hold.

a) Suppose $\delta < \iota_{E_\alpha}$ and that κ_{E_α} is $(\delta^+)^{\mathcal{N}_\alpha}$-supercompact in $\mathcal{N}_\alpha$. Then $(\mathrm{cof}(\delta))^{\mathcal{N}_\alpha} < \kappa_{E_\alpha}$ and $\iota_{E_\alpha} = (\delta^+)^{\mathcal{N}_\alpha}$.

b) Suppose $\delta = \iota_{E_\alpha}$. Then ι_{E_α} is a limit cardinal of $\mathcal{N}_\alpha$.

(6) (**Second Supercompactness Condition**) Suppose that $\pi_{\alpha,\alpha+1}[\iota_{E_\alpha}] \in \mathcal{N}_{\alpha+1}$ and that $\kappa_{E_\alpha} < \iota_{E_\alpha}$. Then there exists a generator ξ of E_α such that:

a) (**Generator Condition**) Either $\nu^*_{E_\alpha} = \xi$ or $\nu^*_{E_\alpha} = \xi + 1$.

b) (**Initial Segment Condition**) If $\nu^*_{E_\alpha} = \xi$ then ξ is a limit of generators and

$$(\mathrm{cof}(\xi))^{\mathcal{N}_{\alpha+1}} < \pi_{\alpha,\alpha+1}(\kappa_{E_\alpha}).$$

(7) (**Third Supercompactness Condition**) Suppose that $\pi_{\alpha,\alpha+1}[\iota_{E_\alpha}] \in \mathcal{N}_{\alpha+1}$, ι_{E_α} is a limit of strongly inaccessible cardinals in $\mathcal{N}_\alpha$ and let ι be the least cardinal of $\mathcal{N}_{\alpha+1}$ with $\nu^*_{E_\alpha} < \iota < \pi_{\alpha+1}(\iota_{E_\alpha})$ such that

$$\mathcal{N}_{\alpha+1}|\iota \vDash \mathrm{ZFC}.$$

Suppose that E_α has a generator υ such that

$$\nu^*_{E_\alpha} < \upsilon < \iota$$

and let υ_0 be the least such generator. Then there exist a transitive $(\mathcal{N}, \mathbb{E}_\mathcal{N}) \vDash \mathrm{ZFC}$ and an $\mathcal{N}$-extender F such that:

a) For all $a \in [\mathrm{LTH}(F)]^{<\omega}$, $F_a \in \mathcal{N}$,

b) $\mathcal{N}_{\alpha+1}|\iota = \mathrm{Ult}_0(\mathcal{N}, F)|\iota$, $\kappa_{E_\alpha} < \kappa_F$, $j_F(\kappa_F) = \kappa^*_{E_\alpha}$, and $j_F(\iota_F) = \iota$.

c) No $\delta < \kappa_F$ is $(<\kappa_F)$-supercompact in $\mathcal{N}$.

d) For some $\gamma \leq \iota_F$, $\upsilon_0 = \sup(j_F[\gamma])$, and either

$$j_F[\gamma] \in \mathrm{Ult}_0(\mathcal{N}, F)$$

or $\gamma = (\delta^+)^\mathcal{N} = \iota_F$ and $(\mathrm{cof}(\delta))^\mathcal{N} < \kappa_F$.

(8) (**Closeness Condition**) For all $a \in [\mathrm{LTH}(E_\alpha)]^{<\omega}$, $(E_\alpha)_a \in \mathcal{N}_\alpha$. □

Remark 5.15. Note that with notation as in the statement of the Third Supercompactness Condition, υ_0 cannot be a limit of generators of E_α. Further by the Second Supercompactness Condition there exists a generator ξ of E_α such that either $\nu^*_{E_\alpha} = \xi$ or $\nu^*_{E_\alpha} = \xi + 1$ and necessarily υ_0 is just the least generator υ of E_α such that $\upsilon > \xi$. Therefore if γ witnesses the requirement (7d) then one of the following must hold.

(1) $\gamma = \kappa_F$.

(2) $\nu^*_{E_\alpha} = \xi + 1$ and $\gamma = \left(|\hat{\xi}|^+\right)^\mathcal{N}$ where $j_F(\hat{\xi}) = \xi$.

(3) $\nu^*_{E_\alpha} = \xi$ and $\gamma = \left(|\hat{\xi}|^+\right)^{\mathcal{N}}$ where $j_F(\hat{\xi}) = \xi$.

The point here is that since $\upsilon_0 = \sup(j_F[\gamma])$, $|\xi|^{\mathcal{N}_\alpha}$ must be in the range of j_F. Also note that in the case where $\nu^*_{E_\alpha} = \xi$,

$$(\mathrm{cof}(\xi))^{\mathcal{N}_{\alpha+1}} < \kappa^*_{E_\alpha}$$

and so $j_F(\hat{\xi}) = \sup(j_F[\hat{\xi}])$. □

Remark 5.16. These conditions are motivated by the elementary embeddings produced by iteration trees. The proof of the main theorem, Theorem 5.35, would be a bit simpler if we eliminated the Third Supercompactness Condition and required as part of the Second Supercompactness Condition that

$$\nu_{E_\alpha} = \nu^*_{E_\alpha} + 1$$

if $\nu^*_{E_\alpha+1}$ is a limit of generators and

$$\nu_{E_\alpha} = \nu^*_{E_\alpha}$$

otherwise.

This is true for the iteration embeddings (of ZFC structures) which can be generated by (maximal) iteration trees at the finite levels of supercompact, such as those in [24].

However at the infinite levels of supercompact, this stronger condition can fail. But in the proof of Theorem 5.35, this potential failure is handled by the Third Supercompactness Condition.

The reason the stronger condition can fail is that E_α might originate as the last extender of an active structure which occurs as a model in the iteration tree *before* the stage where E_α is chosen. In this case the active structure with E_α as the last extender is the model at the stage where E_α is chosen and moreover this model is a semi-iterate of that earlier model.

Finally if ι_{E_α} is not a successor cardinal then there can exist many cardinals between $\nu^*_{E_\alpha}$ and the Jensen index of E_α. In this case the identity $\nu_E \le \nu^*_E + 1$ need not be preserved under semi-iterations and so $\nu_{E_\alpha} \le \nu^*_{E_\alpha} + 1$ might fail. □

We isolate in two definitions, Definition 5.17 and Definition 5.24, the key assumptions that we shall need. Our position based on the results of [24] is that these should follow under very general assumptions from any theory of weakly background structures for which comparison can be proved through iterations by least disagreement. In fact we shall only need Definition 5.24 but Definition 5.17 provides a clearer context for motivating both the definitions.

Definition 5.17. Suppose that $(\mathcal{M}, \mathbb{E}) \vDash \mathrm{ZFC}$ and that $(\mathcal{M}, \mathbb{E})$ is transitive. Then $(\mathcal{M}, \mathbb{E})$ satisfies *comparison* if for all

$$X \prec (\mathcal{M}, \mathbb{E})$$

and all

$$Y \prec (\mathcal{M}, \mathbb{E})$$

the following hold where $(\mathcal{M}_X, \mathbb{E}_X)$ is the transitive collapse of X and $(\mathcal{M}_Y, \mathbb{E}_Y)$ is the transitive collapse of X.

Suppose that X and Y are finitely generated, $(\mathcal{M}_X, \mathbb{E}_X) \neq (\mathcal{M}_Y, \mathbb{E}_Y)$, and

$$X \cap \mathbb{R} = Y \cap \mathbb{R}.$$

Suppose that neither $(\mathcal{M}_X, \mathbb{E}_X)$ or $(\mathcal{M}_Y, \mathbb{E}_Y)$ is a semi-iterate of the other. Then there exists semi-iterations,

$$((\mathcal{N}^X_\alpha, \mathbb{F}^X_\alpha), \pi^X_{\alpha,\beta}, E^X_\alpha : \alpha < \beta \le \eta_X)$$

of $(\mathcal{M}_X, \mathbb{E}_X)$, and

$$((\mathcal{N}^Y_\alpha, \mathbb{F}^Y_\alpha), \pi^Y_{\alpha,\beta}, E^Y_\alpha : \alpha < \beta \le \eta_Y)$$

of $(\mathcal{M}_Y, \mathbb{E}_Y)$ such that:

(1) $(\mathcal{N}^X_{\eta_X}, \mathbb{F}^X_{\eta_X}) = (\mathcal{N}^Y_{\eta_Y}, \mathbb{F}^Y_{\eta_Y})$.

(2) (**First Disagreement Condition**) $E^X_0 \neq E^Y_0$.

(3) (**Second Disagreement Condition**) Suppose that $\iota_{E^X_0} < \lambda$, $\iota_{E^Y_0} < \lambda$, and that

$$\mathcal{P}(\lambda) \cap \mathcal{M}_X = \mathcal{P}(\lambda) \cap \mathcal{M}_Y.$$

Then

$$\pi^X_{0,\eta_X}|\mathcal{P}(\lambda) \neq \pi^Y_{0,\eta_Y}|\mathcal{P}(\lambda).$$ □

Remark 5.18. (1) The *larger* the structure $(\mathcal{M}, \mathbb{E})$ the stronger the requirement that comparison hold is.

For example if every element of $\mathcal{M}$ is definable in $(\mathcal{M}, \mathbb{E})$ then there are no non-trivial finitely generated $X \prec (\mathcal{M}, \mathbb{E})$ and comparison holds vacuously. However if $\mathrm{cof}(\mathrm{Ord}^{\mathcal{M}}) > \omega$ then $X \in \mathcal{M}$ for *every* finitely generated $X \prec (\mathcal{M}, \mathbb{E})$.

(2) We comment briefly on why the requirements specified in Definition 5.17 are reasonable.

Condition (2) is clearly the result of comparison through least disagreement where the semi-iterations are given by the cofinal branches of the maximal iteration trees.

Finally the last condition, (3), lies at the core of comparison by least disagreement. Having this *provably* fail (while maintaining $E^X_0 \neq E^Y_0$) would seem to require an entirely new approach to inner model theory.

In fact we could weaken (3) for our purposes and add the assumption that λ is strongly inaccessible in $\mathcal{M}_X$ with $\iota_{E^X_0} < \lambda$ and $\iota_{E^Y_0} < \lambda$. □

Remark 5.19. Suppose that with notation as in Definition 5.17, $(\mathcal{M}_Y, \mathbb{E}_Y)$ is a semi-iterate of $(\mathcal{M}_X, \mathbb{E}_X)$.

More precisely suppose that

$$\pi : (\mathcal{M}_X, \mathbb{E}_X) \to (\mathcal{N}, \mathbb{E}_{\mathcal{N}}) = (\mathcal{M}_Y, \mathbb{E}_Y)$$

is given by a semi-iteration of $(\mathcal{M}_X, \mathbb{E}_X)$.

One can show by appealing to the fact that $(\mathcal{M}_Y, \mathbb{E}_Y)$ is finitely generated, that the semi-iteration giving π must have finite length and moreover that it must be an internal iteration with each extender being the extender generated by a single ultrafilter. Thus these cases of X and Y are really rather special. □

The following theorem is a corollary of the main theorem of [24] and results of [25] but the only relevant result of [25] is one which allows one to exploit the Weak Unique Branch Hypothesis (which only allows short extenders in the iteration trees) versus a slightly stronger iteration hypothesis.

Recall that κ is m-extendible, where $m < \omega$, if there is an elementary embedding

$$j : V_{\kappa+m} \to V_{j(\kappa)+m}$$

such that $\mathrm{CRT}(j) = \kappa$.

Theorem 5.20 (Weak Unique Branch Hypothesis). *Assume that for each $m < \omega$, there is a proper class of m-extendible cardinals. Then there exists a partial extender sequence*

$$\mathbb{E} = \langle \mathbb{E}_\alpha : \alpha \in \mathrm{dom}(\mathbb{E}) \rangle$$

such that the following hold.

(1) *$L[\mathbb{E}]$ is weakly backgrounded and $L[\mathbb{E}]$ is weakly Σ_2-definable.*

(2) *$(L_\alpha[\mathbb{E}], \mathbb{E}|\alpha)$ satisfies comparison for each ordinal α such that $(L_\alpha[\mathbb{E}], \mathbb{E}|\alpha) \vDash$ ZFC.*

(3) *For each* ξ *and for each* $m < \omega$*, there exists* $\alpha \in \mathrm{dom}(\mathbb{E})$ *such that*

   (a) $\alpha > \xi$*,*

   (b) $\mathbb{E}_\alpha$ *is an* $L[\mathbb{E}]$*-extender which witnesses that* κ *is m-extendible in* $L[\mathbb{E}]$ *where* $\kappa = \mathrm{CRT}(\mathbb{E}_\alpha)$*.*

(4) $L[\mathbb{E}] \vDash$ *"The Weak Unique Branch Hypothesis".* □

Thus one also gets an equivalence.

Theorem 5.21. *The following are equivalent.*

(1) *There exists a countable transitive set* $M \vDash \mathrm{ZFC}$ *such that*

   (a) $M \vDash$ *"For each* $m < \omega$*, there is a proper class of m-extendible cardinals".*

   (b) $M \vDash$ *"The Weak Unique Branch Hypothesis".*

(2) *There exists a countable transitive* $(\mathcal{M}, \mathbb{E}) \vDash \mathrm{ZFC}$ *such that*

   (a) $(\mathcal{M}, \mathbb{E}) \vDash$ *"For each* $m < \omega$*, there is a proper class of m-extendible cardinals".*

   (b) $(\mathcal{M}, \mathbb{E}) \vDash$ *"The Weak Unique Branch Hypothesis".*

   (c) $(\mathcal{M}, \mathbb{E})|\alpha$ *satisfies comparison for each* α *such that* $(\mathcal{M}, \mathbb{E})|\alpha \vDash \mathrm{ZFC}$*.* □

We need a version of Definition 5.17 for pairs.

Definition 5.22. Suppose that $(\mathcal{M}_0, \mathbb{E}_0) \vDash \mathrm{ZFC}$ and that $(\mathcal{M}_1, \mathbb{E}_1) \vDash \mathrm{ZFC}$. Suppose each structure is transitive and κ is a regular cardinal of both structures. Then the pair

$$((\mathcal{M}_0, \mathbb{E}_0), (\mathcal{M}_1, \mathbb{E}_1))$$

is a *coherent pair at* κ if

$$(\kappa^+)^{\mathcal{M}_0} = (\kappa^+)^{\mathcal{M}_1}$$

and

$$(\mathcal{M}_0, \mathbb{E}_0)|(\kappa^+)^{\mathcal{M}_0} = (\mathcal{M}_1, \mathbb{E}_1)|(\kappa^+)^{\mathcal{M}_1}.$$ □

Definition 5.23. Suppose that

$$((\mathcal{M}_0, \mathbb{E}_0), (\mathcal{M}_1, \mathbb{E}_1))$$

is a coherent pair at κ. A *semi-iteration* at κ of the (ordered) pair,

$$((\mathcal{M}_0, \mathbb{E}_0), (\mathcal{M}_1, \mathbb{E}_1))$$

is a continuous (linearly) directed system

$$((\mathcal{N}_\alpha, \mathbb{F}_\alpha), \pi_{\alpha,\beta}, E_\alpha : \alpha < \beta \leq \eta)$$

such that the following hold for all $\alpha < \beta < \eta$.

(1) $(\mathcal{N}_0, \mathbb{F}_0) \in \{(\mathcal{M}_0, \mathbb{E}_0), (\mathcal{M}_1, \mathbb{E}_1)\}$ and

$$((\mathcal{N}_\alpha, \mathbb{F}_\alpha), \pi_{\alpha,\beta}, E_\alpha : \alpha < \beta \leq \eta)$$

   is a semi-iteration of $(\mathcal{N}_0, \mathbb{F}_0)$.

(2) If $\mathcal{N}_0 = \mathcal{M}_1$ then $\kappa < \iota$ for some $\iota \in \mathrm{SP}(E_0)$. □

Definition 5.24. Suppose that $(\mathcal{M}, \mathbb{E}) \vDash$ ZFC, $(\mathcal{M}, \mathbb{E})$ is transitive, κ is a measurable cardinal in V, U is a normal measure on κ, and $U \cap \mathcal{M} \in \mathcal{M}$. Let

$$(\mathcal{M}_U, \mathbb{E}_U) = \mathrm{Ult}_0\left((\mathcal{M}, \mathbb{E}), U\right)$$

and suppose that

$$((\mathcal{M}, \mathbb{E}), (\mathcal{M}_U, \mathbb{E}_U))$$

is a coherent pair at κ. Then $(\mathcal{M}_U, \mathbb{E}_U)$ satisfies *comparison backed up by* $(\mathcal{M}, \mathbb{E})$ at κ if the following hold.

Suppose $X \prec (\mathcal{M}, \mathbb{E})$, X is finitely generated, $U \cap \mathcal{M} \in X$,

$$(\mathcal{M}_X, \mathbb{E}_X)$$

is the transitive collapse of X, κ_X is the image of κ under the transitive collapse, and

$$(\mathcal{M}_U^X, \mathbb{E}_U^X)$$

is the image of $(X \cap \mathcal{M}_U, X \cap \mathbb{E}_U)$ under the transitive collapse. Suppose that $(\mathcal{M}_U^X, \mathbb{E}_U^X)$ is not a semi-iterate of $(\mathcal{M}_X, \mathbb{E}_X)$. Then there exist semi-iterations,

$$((\mathcal{N}_\alpha^0, \mathbb{F}_\alpha^0), \pi_{\alpha,\beta}^0, E_\alpha^0 : \alpha < \beta \leq \eta_0)$$

of $(\mathcal{M}_X, \mathbb{E}_X)$, and

$$((\mathcal{N}_\alpha^1, \mathbb{F}_\alpha^1), \pi_{\alpha,\beta}^1, E_\alpha^1 : \alpha < \beta \leq \eta_1)$$

of the pair $\left((\mathcal{M}_X, \mathbb{E}_X), (\mathcal{M}_U^X, \mathbb{E}_U^X)\right)$ at κ_X such that:

(1) $\left(\mathcal{N}_{\eta_0}^0, \mathbb{F}_{\eta_0}^0\right) = \left(\mathcal{N}_{\eta_1}^1, \mathbb{F}_{\eta_1}^1\right)$.

(2) (**First Disagreement Condition**) $E_0^0 \neq E_0^1$.

(3) (**Second Disagreement Condition**) Suppose that $\iota_{E_0^0} < \lambda$, $\iota_{E_0^1} < \lambda$, and that

$$\mathcal{P}(\lambda) \cap \mathcal{N}_0^0 = \mathcal{P}(\lambda) \cap \mathcal{N}_0^1.$$

Then

$$\pi_{0,\eta_0}^0 | \mathcal{P}(\lambda) \neq \pi_{0,\eta_1}^1 | \mathcal{P}(\lambda).$$

□

Remark 5.25. We will only use condition (3) in the situation where λ is strongly inaccessible in $\mathcal{M}_X = \mathcal{N}_0^0$ with

$$\max(\iota_{E_0^0}, \iota_{E_0^1}) < \lambda$$

and much more. □

Remark 5.26. The semi-iteration of the coherent pair $\left((\mathcal{M}_X, \mathbb{E}_X), (\mathcal{M}_U^X, \mathbb{E}_U^X)\right)$ is not like the iteration of a phalanx in [12]. It really is closer to a semi-iteration of $\mathcal{M}_X$ where U_X is allowed to be the initial extender. But even that is not completely accurate since the next extender can act on $\mathcal{M}_U^X$ and yet have critical point strictly below $j_U^X(\kappa_X)$ where

$$j_U^X : (\mathcal{M}_X, \mathbb{E}_X) \to \mathrm{Ult}_0\left((\mathcal{M}_X, \mathbb{E}_X), U_X\right) \cong (\mathcal{M}_U^X, \mathbb{E}_U^X)$$

is the ultrapower embedding. □

The following lemma shows that the requirement in Definition 5.24 that

$$U \cap \mathcal{M} \in \mathcal{M}$$

is necessarily satisfied in many cases. This lemma is a weak variation of the Universality Theorem, Theorem 3.26.

Lemma 5.27. *Suppose that* $(\mathcal{M}, \mathbb{E}) \vDash$ ZFC, $(\mathcal{M}, \mathbb{E})$ *is weakly backgrounded,* $\delta < \mathrm{Ord}^{\mathcal{M}}$*, and that* δ *is witnessed by the* $\mathcal{M}$*-extenders on the sequence* $\mathbb{E}$ *to be supercompact in* $\mathcal{M}$*. Suppose* $\delta < \kappa < \mathrm{Ord}^{\mathcal{M}}$ *and that* U *is a* δ*-complete ultrafilter on* κ*. Then* $U \cap \mathcal{M} \in \mathcal{M}$*.*

Proof. Let $\lambda = |V_{\kappa+\omega} \cap \mathcal{M}|^{\mathcal{M}}$ and let μ be a δ-complete normal fine ultrafilter on $\mathcal{P}_\delta(\lambda)$ such that

(1.1) $\mathcal{M} \cap \mathcal{P}_\delta(\lambda) \in \mu$,

(1.2) $\mu \cap \mathcal{M} \in \mathcal{M}$.

The ultrafilter μ must exist since $\mathcal{M}$ is weakly backgrounded and since δ is witnessed by the $\mathcal{M}$-extenders on the sequence $\mathbb{E}$ to be supercompact in $\mathcal{M}$.

Fix a bijection

$$\pi : \lambda \to V_{\kappa+\omega} \cap \mathcal{M}$$

with $\pi \in \mathcal{M}$ and let I be the set of all $\sigma \in \mathcal{P}_\delta(\lambda) \cap M$ such that for each $\xi < \kappa$ there exists $\eta < \lambda$ such that

(2.1) $\eta \in \sigma$,

(2.2) $\pi(\eta)$ is a δ-compete ultrafilter in κ in $\mathcal{M}$,

(2.3) for all $A \in \mathcal{P}(\kappa) \cap \pi[\sigma]$, $A \in \pi(\eta)$ if and only if $\xi \in A$.

The key point is that $I \in \mu$. This is easily verified by working in $\mathcal{M}$ and using that in $\mathcal{M}$, $\mu \cap \mathcal{M}$ is a δ-complete normal fine ultrafilter on $\mathcal{P}_\delta(\lambda)$.

Define

$$f : I \to \lambda$$

by $f(\sigma) = \eta$ such that

(3.1) $\pi(\eta)$ is a δ-complete ultrafilter on κ in $\mathcal{M}$,

(3.2) $\eta \in \sigma$,

(3.3) $\pi(\eta) \cap \pi[\sigma] = U \cap \pi[\sigma]$.

Since $I \in \mu$, there must exist $\eta_0 < \lambda$ such that

$$\{\sigma \in I \mid f(\sigma) = \eta_0\} \in \mu.$$

Thus $\pi(\eta_0) = U \cap \mathcal{M}$ and this proves the lemma. □

As a corollary of Lemma 5.27, we obtain the following strong version of Theorem 3.40.

Theorem 5.28. *Suppose that* δ *is an extendible cardinal and that* $\kappa \geq \delta$ *is a measurable cardinal. Then* κ *is a measurable cardinal in* HOD*.*

Proof. By Lemma 3.37, we can reduce to the case that κ is not ω-strongly measurable in HOD. But then by Theorem 3.39, HOD is a weak extender model for δ is supercompact and so by (the proof of) Lemma 5.27, κ is a measurable cardinal in HOD. □

We prove three easy lemmas and the latter two are quite useful. These require a definition. For this definition and these three lemmas, the notation $(\mathcal{M}, \mathbb{E})$ and $(\mathcal{N}, \mathbb{F})$ indicates that the structures are transitive.

Definition 5.29. Suppose that $(\mathcal{M}, \mathbb{E}) \vDash$ ZFC and that

$$\pi : (\mathcal{M}, \mathbb{E}) \to (\mathcal{N}, \mathbb{F})$$

is an elementary embedding which is cofinal. Then π is *close* to $(\mathcal{M}, \mathbb{E})$ if for each $X \in \mathcal{M}$ and each $a \in \pi(X)$,

$$\{Z \in \mathcal{P}(X) \cap \mathcal{M} \mid a \in \pi(Z)\} \in \mathcal{M}. \qquad \square$$

The following lemma which is essentially immediate from the definition of close embedding, identifies a useful feature of close embeddings. This feature is a weak form of coherence.

Lemma 5.30. *Suppose that* $(\mathcal{M}, \mathbb{E}) \vDash \mathrm{ZFC}$ *and that*

$$\pi : (\mathcal{M}, \mathbb{E}) \to (\mathcal{N}, \mathbb{F})$$

is an elementary embedding which is close to $(\mathcal{M}, \mathbb{E})$*. Suppose* $\iota < \mathrm{Ord}^{\mathcal{M}}$ *and*

$$\pi[\iota] \in \mathcal{N}.$$

Then $\mathcal{P}(\iota) \cap \mathcal{M} = \mathcal{P}(\iota) \cap \mathcal{N}$*.*

Proof. Clearly $\mathcal{P}(\iota) \cap \mathcal{M} \subseteq \mathcal{P}(\iota) \cap \mathcal{N}$. Now suppose $A \in \mathcal{P}(\iota) \cap \mathcal{N}$. Then $\pi[A] \in \mathcal{N}$. Let

$$a = (\pi[A], \pi[\iota])$$

and let $X \in \mathcal{M}$ be a transitive set such that $a \in \pi(X)$. Since π is close to $\mathcal{M}$, $U \in \mathcal{M}$ where

$$U = \{Z \in \mathcal{P}(X) \cap \mathcal{M} \mid a \in \pi(Z)\}.$$

Thus $A \in \mathrm{Ult}_0(\mathcal{M}, U) \subseteq \mathcal{M}$ and so $\mathcal{P}(\iota) \cap \mathcal{N} \subseteq \mathcal{P}(\iota) \cap \mathcal{M}$. □

Lemma 5.31. *Suppose that* $(\mathcal{M}, \mathbb{E}) \vDash \mathrm{ZFC}$ *and that*

$$\pi : (\mathcal{M}, \mathbb{E}) \to (\mathcal{N}, \mathbb{F})$$

is an elementary embedding which is given by a semi-iteration of $(\mathcal{M}, \mathbb{E})$*. Then* π *is close to* $(\mathcal{M}, \mathbb{E})$*.*

Proof. The key point is that the composition of close embeddings is close. We verify this.

Suppose that

$$\pi_0 : (\mathcal{M}_0, \mathbb{E}_0) \to (\mathcal{M}_1, \mathbb{E}_1)$$

and

$$\pi_1 : (\mathcal{M}_1, \mathbb{E}_1) \to (\mathcal{M}_2, \mathbb{E}_2)$$

are each close embeddings. Fix $Y \in \mathcal{M}_0$ and $a \in \pi_1 \circ \pi_0(Y)$. We must show that

$$\{Z \in \mathcal{P}(Y) \cap \mathcal{M}_0 \mid a \in \pi_1 \circ \pi_0(Z)\} \in \mathcal{M}_0.$$

Let

$$W = \{Z \in \mathcal{P}(\pi_0(Y)) \cap \mathcal{M}_1 \mid a \in \pi_1(Z)\}.$$

Then $W \in \mathcal{M}_1$ and in $\mathcal{M}_1$, W is an ultrafilter on $\pi_0(Y)$.

Let

$$W^* = \{Z \in \mathcal{P}(\mathcal{P}(\mathcal{P}(Y))) \cap \mathcal{M}_0 \mid W \in \pi_0(Z)\}.$$

Then $W^* \in \mathcal{M}_0$ and in $\mathcal{M}_0$, W^* is an ultrafilter on $\beta(Y)$, the space of all ultrafilters on Y.

Fix $Z \in \mathcal{P}(Y) \cap \mathcal{M}_0$. Then

$$a \in \pi_1 \circ \pi_0(Z)$$

if and only if

$$\pi_0(Z) \in W.$$

Let

$$Z^* = \{U \in \mathcal{P}(\mathcal{P}(Y)) \cap \mathcal{M}_0 \mid Z \in U\}.$$

Thus $\pi_0(Z) \in W$ if and only if $W \in \pi_0(Z^*)$. But $W \in \pi_0(Z^*)$ if and only if $Z^* \in W^*$.

Therefore $a \in \pi_1 \circ \pi_0(Z)$ if and only if $Z^* \in W^*$, and this implies

$$\{Z \in \mathcal{P}(Y) \cap \mathcal{M}_0 \mid a \in \pi_1 \circ \pi_0(Z)\} \in \mathcal{M}_0.$$

This proves $\pi_1 \circ \pi_0$ is close to $(\mathcal{M}_0, \mathbb{E}_0)$.

The lemma now follows easily by induction of the length of semi-iterations. □

The next lemma is an abstract version of the uniqueness of iteration embeddings.

Lemma 5.32. *Suppose that* $(\mathcal{M}, \mathbb{E}) \vDash$ ZFC *and is finitely generated. Suppose that*

$$\pi_0 : (\mathcal{M}, \mathbb{E}) \to (\mathcal{N}, \mathbb{F})$$

and

$$\pi_1 : (\mathcal{M}, \mathbb{E}) \to (\mathcal{N}, \mathbb{F})$$

are elementary embeddings each of which is close to $(\mathcal{M}, \mathbb{E})$*. Then* $\pi_0 = \pi_1$*.*

Proof. Let $\xi \in \mathcal{M}_0 \cap \mathrm{Ord}$ be such that every element of $\mathcal{M}$ is definable in $(\mathcal{M}, \mathbb{E})$ from ξ. It suffices to show that

$$\pi_0(\xi) = \pi_1(\xi).$$

Let $\xi_0 = \pi_0(\xi)$ and let $\xi_1 = \pi_1(\xi)$. Assume toward a contradiction that $\xi_0 < \xi_1$. Let

$$U = \{Z \subset \xi \mid \xi_0 \in \pi_1(Z)\}.$$

Thus $U \in \mathcal{M}$. Let

$$j_U : (\mathcal{M}, \mathbb{E}) \to (\mathcal{M}_U, \mathbb{E}_U)$$

be the ultrapower embedding given by U and let

$$k_U : (\mathcal{M}_U, \mathbb{E}_U) \to (\mathcal{N}, \mathbb{F})$$

be the factor embedding such that $\pi_1 = k_U \circ j_U$. Let ξ_0^U be the element of $\mathcal{M}_U$ such that $k_U(\xi_0^U) = \xi_0$.

Let $(\mathcal{N}_X, \mathbb{F}_X)$ be the transitive collapse of X where X is the set of all $a \in \mathcal{N}$ such that a is definable in $(\mathcal{N}, \mathbb{F})$ from ξ_0. Then

$$(\mathcal{N}_X, \mathbb{F}_X) = (\mathcal{M}, \mathbb{E}).$$

But $X \subset k_U[\mathcal{M}_U]$ since $\xi_0 = k_U(\xi_0^U)$ and since $\pi_1 = k_U \circ j_U$.

Thus we have:

(1.1) $\xi_0^U < j_U(\xi)$ since $k_U(\xi_0^U) = \xi_0 < \xi_1 = \pi_1(\xi) = k_U \circ j_U(\xi)$.

(1.2) Let X_U be the set of all $a \in \mathcal{M}_U$ such that a is definable in $(\mathcal{M}_U, \mathbb{E}_U)$ from ξ_0^U, and let $(\mathcal{M}_{X_U}, \mathbb{E}_{X_U})$ be the transitive collapse of X. Then

$$(\mathcal{M}_{X_U}, \mathbb{E}_{X_U}) = (\mathcal{M}, \mathbb{E})$$

and necessarily ξ is the image of ξ_0^U under the transitive collapse of X_U.

Let

$$\pi_U : (\mathcal{M}, \mathbb{E}) \to (\mathcal{M}_U, \mathbb{E}_U)$$

invert the transitive collapse of X_U. Thus $\pi_U(\xi) = \xi_0^U < j_U(\xi)$ and there is a canonical elementary embedding

$$j : \mathrm{Ult}_0(\mathcal{M}, U) \to \mathrm{Ult}_0(\mathcal{M}_U, \pi_U(U)).$$

Now one can generate an illfounded iteration of $\mathcal{M}$ of length ω which is induced by a linear iteration of a rank initial segment of $\mathcal{M}$, and this is a contradiction. □

Remark 5.33. Suppose $M \vDash$ ZFC is a transitive set in which there is a supercompact cardinal with Ord^M as small as possible. Then there is a linear iteration of M in length ω, by ultrapowers, such that the direct limit is not wellfounded.

In fact, any linear iteration

$$\mathcal{T} = \langle (M_i, U_i), j_{i,k} : i < k < \omega \rangle$$

by ultrapowers such that

(1) for all $\alpha < \mathrm{Ord}^M$ there exists $i < \omega$ such that $\lambda_i > j_{0,\alpha}(\alpha)$, where $U_i \in M_i$ is a normal fine κ_i-complete ultrafilter on $(\mathcal{P}_{\kappa_i}(\lambda_i))^{M_i}$ and where $\kappa_i = \mathrm{CRT}(j_{U_i})$,

(2) there exists $\kappa < \mathrm{Ord}^M$ such that $\mathrm{CRT}(j_{U_i}) \leq j_{0,i}(\kappa)$ for all $i < \omega$,

must have ill-founded direct limit.

This is by the minimality of Ord^M (taking a generic collapse and then appealing to $\underset{\sim}{\Sigma}^1_1$-absoluteness) and since by (1)–(2), Ord^M is in the wellfounded part of the direct limit.

Thus in the proof of Lemma 5.32, it is critical that the linear iteration of length ω have the simple form of being induced by a linear iteration of a rank initial segment of $\mathcal{M}$. □

Lemma 5.34. *Suppose $(\mathcal{M}, \mathbb{E}) \vDash \mathrm{ZFC}$ is finitely generated, $U \in \mathcal{M}$, and that in $\mathcal{M}$, U is a κ-complete normal ultrafilter on κ. Let*

$$(\mathcal{M}_U, \mathbb{E}_U) = \mathrm{Ult}_0((\mathcal{M}, \mathbb{E}), U).$$

Then the following are equivalent.

(1) *$(\mathcal{M}_U, \mathbb{E}_U)$ is a semi-iterate of $(\mathcal{M}, \mathbb{E})$.*

(2) *No $\delta < \kappa$ is witnessed to be $(<\kappa)$-supercompact in $\mathcal{M}$ by $\mathbb{E}$.*

Proof. Clearly (2) implies (1) and the witness is the semi-iteration

$$((\mathcal{N}_\alpha, \mathbb{F}_\alpha), \pi_{\alpha,\beta}, E_\alpha : \alpha < \beta \leq \eta)$$

of $(\mathcal{M}, \mathbb{E})$ where $\eta = 1$ and E_0 is the extender given by U.

Now suppose that (1) holds and that

$$\pi : (\mathcal{M}, \mathbb{E}) \to (\mathcal{M}_U, \mathbb{E}_U)$$

is given by a semi-iteration of $(\mathcal{M}_U, \mathbb{E}_U)$. Let

$$\pi_U : (\mathcal{M}, \mathbb{E}) \to (\mathcal{M}_U, \mathbb{E}_U)$$

be the ultrapower embedding.

By Lemma 5.32, $\pi = \pi_U$ and this implies (2) since $\kappa = \mathrm{CRT}(\pi_U)$. □

We now come to our main theorem. The fundamental idea is to simply use the basic arguments from, for example [12], for establishing that various extenders which belong to an iterable structure, must be on the sequence of an iterable structure. The definitions of a coherent pair and of comparison for such pairs were formulated by isolating very general features sufficient for the implementation of these arguments.

The situation here however is quite different because by universality (for example, Lemma 5.27) there can be extenders which belong to the structure whose associated critical point cannot be the critical point of *any* extender on the sequence (because of the Suitability Condition).

The issue then is exactly how is this potential conflict resolved. The theorem shows that the only resolution is through a failure of comparison based on least disagreement.

Theorem 5.35. *Suppose that δ is supercompact and that $\Omega > \delta$ is a a strongly inaccessible cardinal. Then there is no weakly backgrounded structure $(\mathcal{M}, \mathbb{E}) \vDash \mathrm{ZFC}$ such that the following hold.*

(1) *$\Omega = \mathrm{Ord}^{\mathcal{M}}$ and δ is witnessed by the $\mathcal{M}$-extenders on the sequence $\mathbb{E}$ to be supercompact in $\mathcal{M}$.*

(2) *There exists a measurable cardinal $\delta < \kappa < \Omega$ and a normal measure U on κ such that the following hold where*

$$(\mathcal{M}_U, \mathbb{E}_U) = \mathrm{Ult}_0\left((\mathcal{M}, \mathbb{E}), U\right).$$

(a) *$((\mathcal{M}, \mathbb{E}), (\mathcal{M}_U, \mathbb{E}_U))$ is a coherent pair at κ.*

(b) $U \cap \mathcal{M} \in \mathcal{M}$.

(c) $(\mathcal{M}_U, \mathbb{E}_U)$ *satisfies comparison backed up by* $(\mathcal{M}, \mathbb{E})$ *at* κ.

Proof. Assume toward a contradiction that $(\mathcal{M}, \mathbb{E})$ is weakly backgrounded and that $(\mathcal{M}, \mathbb{E})$, U, and κ satisfy (1) and (2). Note that by Lemma 5.27, the requirement (2b) follows from the assumption that $(\mathcal{M}, \mathbb{E})$ is weakly backgrounded.

Let

$$e_U : (\mathcal{M}, \mathbb{E}) \to (\mathcal{M}_U, \mathbb{E}_U)$$

be the ultrapower embedding as defined in $(\mathcal{M}, \mathbb{E})$ using $U \cap \mathcal{M}$. Let

$$X \prec (\mathcal{M}, \mathbb{E})$$

be the elementary substructure given by the set of all $a \in \mathcal{M}$ such that a is definable in $(\mathcal{M}, \mathbb{E})$ from $\{U \cap \mathcal{M}\}$.

(1.1) Let $\left(\mathcal{M}_U^X, \mathbb{E}_U^X\right)$ be the transitive collapse of $(X \cap \mathcal{M}_U, X \cap \mathbb{E}_U)$.

(1.2) Let $(\mathcal{M}_X, \mathbb{E}_X)$ be the transitive collapse of X and let κ_X be the image of κ under the transitive collapse of X.

(1.3) Let δ_X be the image of δ under the transitive collapse of X.

(1.4) Let U_X be the image of $U \cap \mathcal{M}$ under the transitive collapse of X.

(1.5) Let

$$e_U^X : (\mathcal{M}_X, \mathbb{E}_X) \to (\mathcal{M}_U^X, \mathbb{E}_U^X)$$

be the image of e_U under the transitive collapse of X.

By Lemma 5.34, $(\mathcal{M}_U^X, \mathbb{E}_U^X)$ is not a semi-iterate of $(\mathcal{M}_X, \mathbb{E}_X)$. Therefore, since $(\mathcal{M}_U, \mathbb{E}_U)$ satisfies comparison backed up by $(\mathcal{M}, \mathbb{E})$ at κ, there exist semi-iterations

$$((\mathcal{N}_\alpha^0, \mathbb{F}_\alpha^0), \pi_{\alpha,\beta}^0, E_\alpha^0 : \alpha < \beta \leq \eta_0)$$

of $(\mathcal{M}_X, \mathbb{E}_X)$, and

$$((\mathcal{N}_\alpha^1, \mathbb{F}_\alpha^1), \pi_{\alpha,\beta}^1, E_\alpha^1 : \alpha < \beta \leq \eta_1)$$

of the pair $\left((\mathcal{M}_X, \mathbb{E}_X), (\mathcal{M}_U^X, \mathbb{E}_U^X)\right)$ at κ_X such that:

(2.1) $\left(\mathcal{N}_{\eta_0}^0, \mathbb{F}_{\eta_0}^0\right) = \left(\mathcal{N}_{\eta_1}^1, \mathbb{F}_{\eta_1}^1\right)$.

(2.2) $E_0^0 \neq E_0^1$.

(2.3) Suppose that $\iota_{E_0^0} < \theta$, $\iota_{E_0^1} < \theta$, and that

$$\mathcal{N}_0^0 \cap \mathcal{P}(\theta) = \mathcal{N}_0^1 \cap \mathcal{P}(\theta).$$

Then $\pi_{0,\eta_0}^0 | \mathcal{P}(\theta) \neq \pi_{0,\eta_1}^1 | \mathcal{P}(\theta)$.

We prove the following.

(3.1) $(\mathcal{N}_0^1, \mathbb{F}_0^1) = \left(\mathcal{M}_U^X, \mathbb{E}_U^X\right)$.

(3.2) $\pi_{0,\eta_0}^0 = \pi_{0,\eta_1}^1 \circ e_U^X$.

Assume toward a contradiction that $(\mathcal{N}^1_0, \mathbb{F}^1_0) = (\mathcal{M}_X, \mathbb{E}_X)$. Then

$$\pi^0_{0,\eta_0} : (\mathcal{M}_X, \mathbb{E}_X) \to (\mathcal{N}^0_{\eta_0}, \mathbb{F}^0_{\eta_0})$$

and

$$\pi^1_{0,\eta_1} : (\mathcal{M}_X, \mathbb{E}_X) \to (\mathcal{N}^1_{\eta_0}, \mathbb{F}^1_{\eta_1})$$

are each embeddings of the finitely generated $(\mathcal{M}_X, \mathbb{E}_X)$ into the same structure and by Lemma 5.31, each embedding is close to $(\mathcal{M}_X, \mathbb{E}_X)$. Therefore by Lemma 5.32,

$$\pi^0_{0,\eta_0} = \pi^1_{0,\eta_1}$$

and this contradicts (2.3).

This proves (3.1). Thus $(\mathcal{N}^1_0, \mathbb{F}^1_0) = \left(\mathcal{M}^X_U, \mathbb{E}^X_U\right)$ and so

$$\pi^0_{0,\eta_0} : (\mathcal{M}_X, \mathbb{E}_X) \to (\mathcal{N}^0_{\eta_0}, \mathbb{F}^0_{\eta_0})$$

and

$$\pi^1_{0,\eta_1} \circ e^X_U : (\mathcal{M}_X, \mathbb{E}_X) \to (\mathcal{N}^1_{\eta_0}, \mathbb{F}^1_{\eta_1})$$

are each embeddings of the finitely generated $(\mathcal{M}_X, \mathbb{E}_X)$ into the same structure.

By Lemma 5.31, π^0_{0,η_0} is close to $(\mathcal{M}_X, \mathbb{E}_X)$ and π^1_{0,η_1} is close to $(\mathcal{M}^X_U, \mathbb{E}^X_U)$. But e_U is trivially close to $(\mathcal{M}_X, \mathbb{E}_X)$ and so since close embeddings are closed under compositions (see the proof of Lemma 5.31), $\pi^1_{0,\eta_1} \circ e^X_U$ is close to $(\mathcal{M}_X, \mathbb{E}_X)$. Therefore by Lemma 5.32, $\pi^0_{0,\eta_0} = \pi^1_{0,\eta_1} \circ e^X_U$. This proves (3.1) and (3.2).

By the Suitability Condition, Definition 5.14(3), of semi-iterations,

(4.1) $\kappa_{E^0_0} \leq \delta_X$,

(4.2) $\kappa_{E^1_0} \leq \delta_X$.

We next prove the following.

(5.1) $\kappa_{E^0_0} = \delta_X$, $\iota_{E^0_0} = \kappa_X$ and $\kappa_X \in \mathrm{SP}(E^0_0)$.

(5.2) $\kappa_{E^1_0} = \delta_X$ and $\iota_{E^1_0} > \kappa_X$.

Assume toward a contradiction that $\kappa_{E^0_0} \neq \delta_X$. Then $\kappa_{E^1_0} \neq \delta_X$ and by the three properties of semi-iterations specified as the Suitable Condition, the First Supercompactness Condition, and the Closeness Condition; both

$$\kappa_{E^0_0} \leq \iota_{E^0_0} < \delta_X$$

and

$$\kappa_{E^1_0} \leq \iota_{E^1_0} < \delta_X.$$

But then by (3.2), and for all sufficiently large

$$\theta < \delta_X,$$

we have:

(6.1) $\mathrm{SP}(E^0_0) \cup \mathrm{SP}(E^1_0) \subset \theta$,

(6.2) $\theta < \kappa_X$,

(6.3) $\pi^0_{0,\eta_0} | \mathcal{P}(\theta) = \pi^1_{0,\eta_1} | \mathcal{P}(\theta)$.

This contradicts (2.3). This proves that $\kappa_{E^0_0} = \delta_X$. Note that we have only used the much weaker version of the Second Disagreement Condition (see Definition 5.24) where one requires in addition that λ be strongly inaccessible in the models.

By (3.2), and since $\kappa_{E^0_0} = \delta_X$, necessarily $\kappa_{E^1_0} = \delta_X$. We now prove the rest of the claims in each of (5.1) and (5.2).

If $\iota_{E^1_0} \leq \kappa_X$ then $\mathcal{N}^1_0 = \mathcal{M}_X$ and so by (3.1), $\iota_{E^1_0} > \kappa_X$. We now use (3.1) and (3.2) to show that:

(7.1) $\kappa_X \in \text{SP}(E^0_0)$.

(7.2) $\iota_{E^0_0} = \kappa_X$.

This will finish the proof of (5.1) and (5.2).

Since $\iota_{E^1_0} > \kappa_X$, by the First Supercompactness Condition (5) in the definition of a semi-iteration, Definition 5.14 on page 54, necessarily

$$\pi^1_{0,\eta_1}[\kappa_X] \in \mathcal{N}^1_{\eta_1} = \mathcal{N}^0_{\eta_0}$$

and so by (3.1)–(3.2),

$$\pi^0_{0,\eta_0}[\kappa_X] \in \mathcal{N}^0_{\eta_0}.$$

But then by backwards induction,

(8.1) $\pi^0_{0,1}[\kappa_X] \in \mathcal{N}^0_1$.

Thus since $\kappa_{E^0_0} = \delta_X$, necessarily $\kappa_X \in \text{SP}(E^0_0)$. This proves (7.1).

Assume toward a contradiction that $\iota_{E^0_0} > \kappa_X$. Then again by the First Supercompactness Condition of semi-iterations,

$$\pi^0_{0,1}[\epsilon] \in \mathcal{N}^0_1$$

where

$$\epsilon = ((\kappa_X)^+)^{\mathcal{N}^0_0}.$$

By Lemma 5.30 and the closeness of $\pi^0_{0,1}$ to $\mathcal{N}^0_0$, this implies that

$$\mathcal{P}(\epsilon) \cap \mathcal{N}^0_0 = \mathcal{P}(\epsilon) \cap \mathcal{N}^0_1,$$

and so since $\text{CRT}(\pi^0_{1,\eta_0}) \geq \pi^1_{0,1}(\kappa_{E^0_0}) > \epsilon$,

$$\mathcal{P}(\epsilon) \cap \mathcal{N}^0_0 = \mathcal{P}(\epsilon) \cap \mathcal{N}^0_{\eta_0}.$$

Let G_0 be the $\mathcal{N}^0_0$-extender given by π^0_{0,η_0}. Then

$$G_0|\pi^0_{0,\eta_0}(\kappa_X) \in \mathcal{N}^0_{\eta_0} = \mathcal{N}^1_{\eta_1}$$

and so by (3.2),

$$U_X \in \mathcal{N}^1_{\eta_1}.$$

But we have that $\iota_{E^1_0} > \kappa_X$ and $\iota_{E^1_0} \geq \epsilon$. Therefore by the First Supercompactness Condition of semi-iterations,

$$\pi^1_{0,1}[\epsilon] \in \mathcal{N}^1_1$$

and so just as above

$$\mathcal{P}(\epsilon) \cap \mathcal{N}^1_0 = \mathcal{P}(\epsilon) \cap \mathcal{N}^1_{\eta_1}.$$

But

$$\mathcal{N}^0_{\eta_0} = \mathcal{N}^1_{\eta_1}$$

and so $U_X \in \mathcal{N}^1_0$ which is a contradiction since by (3.1), $\mathcal{N}^1_0 = \mathcal{M}^X_U = \text{Ult}_0(\mathcal{M}_X, U_X)$.

This proves (7.1) and (7.2), and finishes the proof of (5.1) and (5.2).

We continue with G_0 as specified above and let G_1 be the $\mathcal{N}^1_0$-extender given by π^1_{0,η_1}. Let ξ_0 be least such that $G_0|\xi_0 \notin \mathcal{N}^0_{\eta_0}$. Let ξ^0_0 be least such that $E^0_0|\xi^0_0 \notin \mathcal{N}^0_1$. We note (then prove) the following.

(9.1) $\xi_0 < \pi^0_{0,\eta_0}(\kappa_X)$.

(9.2) $\xi_0 = \pi^0_{1,\eta_0}(\xi^0_0)$.

(9.3) $\xi_0 = \pi^1_{0,\eta_1}(\kappa_X) + 1$.

(9.4) There exists $\kappa_0 \in \mathcal{N}_1^0$ such that $\pi^0_{1,\eta_0}(\kappa_0) = \pi^1_{0,\eta_1}(\kappa_X)$.

The last claim, (9.4), follows trivially from (9.1)–(9.3), and it is (9.4) that we need.

It is useful to note, while proving (9.1)–(9.3), that since

$$j^0_{0,\eta_0} = j^1_{0,\eta_1} \circ e^X_U,$$

and since $\kappa_X < \iota_{E^1_0}$, necessarily

(10.1) $G_0|j^1_{0,\eta_1}(\kappa_X) = G_1|j^1_{0,\eta_1}(\kappa_X)$,

(10.2) $G_1|j^1_{0,\eta_1}(\kappa_X) \in \mathcal{N}^1_{\eta_1} = \mathcal{N}^0_{\eta_0}$.

Further for all $A \in \mathcal{P}(\kappa_X) \cap \mathcal{M}_X = \mathcal{P}(\kappa_X) \cap \mathcal{M}^X_U$:

$$\begin{aligned} A \in U_X &\iff \kappa_X \in e^U_X(A) \\ &\iff j^1_{0,\eta_1}(\kappa_X) \in j^1_{0,\eta_1} \circ e^U_X(A) \\ &\iff j^1_{0,\eta_1}(\kappa_X) \in j^0_{0,\eta_0}(A). \end{aligned}$$

By our general assumptions, in particular the Second Supercompactness Condition in the definition of semi-iterations on page 55, together with (8.1), there is a generator ξ of E^0_0 such that:

(11.1) either $\nu^*_{E^0_0} = \xi$ or $\nu^*_{E^0_0} = \xi + 1$,

(11.2) $\xi < \pi^0_{0,1}(\iota_{E^0_0}) = \pi^0_{0,1}(\kappa_X)$,

(11.3) if $\nu^*_{E^0_0} = \xi$ then ξ is a limit of generators of E^0_0 and $(\mathrm{cof}(\xi))^{\mathcal{N}^0_1} < \pi^0_{0,1}\left(\kappa_{E^0_0}\right)$.

Therefore:

(12.1) $\xi^0_0 = \xi + 1$ or $\xi^0_0 = \xi$ and $(\mathrm{cof}(\xi))^{\mathcal{N}^0_1} < \pi^0_{0,1}\left(\kappa_{E^0_0}\right)$.

The key point is that by (12.1), if $\xi^0_0 = \xi$ then

$$\pi^0_{1,\eta_0}(\xi^0_0) = \sup(\pi^0_{1,\eta_0}[\xi^0_0]).$$

Therefore, since:

(13.1) $G_0|\mathrm{LTH}(H) = H$ where

- a) $H = \pi^0_{1,\eta_0}(E^0_0|\xi)$ if $\xi^0_0 = \xi + 1$, and
- b) H is the extender given by $\pi^0_{1,\eta_0}[E^0_0|\xi^0_0]$ if $\xi^0_0 = \xi$,

(13.2) π^0_{1,η_0} is close to $\mathcal{N}^0_1$;

necessarily

$$\pi^0_{1,\eta_0}(\xi^0_0) = \xi_0.$$

The claims (9.1)–(9.3) now follow from (3.1), (3.2), (5.1), and (5.2).

Let $\kappa_0 \in \mathcal{N}^0_1$ be such that

$$\pi^0_{1,\eta_0}(\kappa_0) = \pi^1_{0,\eta_1}(\kappa_X)$$

Thus κ_0 is strongly inaccessible in $\mathcal{N}^0_1$ and

$$\kappa_0 < \pi^0_{0,1}(\kappa_X) = \pi^0_{0,1}\left(\iota_{E^0_0}\right).$$

Further

(14.1) $\nu^*_{E^0_0} = \kappa_0 + 1$.

Let

$$\lambda_0 = ((\kappa_0)^+)^{\mathcal{N}_1^0}$$

and let

$$\lambda_1 = (\kappa_X^+)^{\mathcal{N}_0^1}.$$

Thus $\pi^0_{1,\eta_0}(\lambda_0) = \pi^1_{0,\eta_1}(\lambda_1)$. Let $\lambda = \pi^0_{1,\eta_0}(\lambda_0) = \pi^1_{0,\eta_1}(\lambda_1)$.

Let

$$\delta^* = \pi^0_{0,\eta_0}(\delta_X) = \pi^1_{0,\eta_1}(\delta_X) = \pi^1_{0,\eta_1} \circ e^X_U(\delta_X)$$

and let

$$Y^* \subset (\mathcal{P}_{\delta^*}(\lambda))^{\mathcal{N}^0_{\eta_0}} = (\mathcal{P}_{\delta^*}(\lambda))^{\mathcal{N}^1_{\eta_1}}$$

be the least Solovay set (see Lemma 3.4) which is definable in $\mathcal{N}^*|\iota^*$ where

$$\mathcal{N}^* = \mathcal{N}^0_{\eta_0} = \mathcal{N}^1_{\eta_1}$$

and where ι^* is the least strongly inaccessible cardinal of $\mathcal{N}^*$ above λ. For each $\theta < \lambda$, let $(Y^*)_\theta = \sigma$ if $\sigma \in Y^*$ and $\sup(\sigma) = \theta$. This is well-defined.

We prove:

(15.1) $\nu^*_{E^0_0} < \nu_{E^0_0}$.

Assume toward a contradiction that $\nu^*_{E^0_0} = \nu_{E^0_0}$. Let $1 \leq \eta < \eta_0$ be least such that

$$\pi^0_{1,\eta}(\lambda_0) \in \mathrm{SP}(E^0_\eta)$$

where here and below we set $\pi^0_{1,1}$ to be the identity. Since

$$\pi^0_{1,\eta_0}(\lambda_0) = \pi^1_{0,\eta_1}(\lambda_1)$$

and since

$$\sup(\pi^1_{0,\eta_1}[\lambda_1]) < \pi^1_{0,\eta_1}(\lambda_1)$$

it follows that η must exist. The relevant points are:

(16.1) $\sup(\pi^0_{1,\eta_0}[\lambda_0]) = \sup\left(\pi^0_{1,\eta_0}(\lambda_0) \cap \left\{\pi^0_{1,\eta_0}(f)(\pi^0_{1,\eta_0}(\kappa_0)) \mid f \in \mathcal{N}^0_1\right\}\right)$.

(16.2) $\sup(\pi^1_{0,\eta_1}[\lambda_1]) = \sup\left(\pi^1_{0,\eta_1}(\lambda_1) \cap \left\{\pi^1_{0,\eta_1}(f)(\pi^1_{0,\eta_1}(\kappa_X)) \mid f \in \mathcal{M}^X_U\right\}\right)$.

Note and this is a key point:

$$\nu_{E^0_0} = \kappa_0 + 1$$

and so

$$\mathcal{N}^0_1 = \left\{\pi^0_{0,1}(f)(a) \mid a \leq \kappa_0, f \in \mathcal{M}_X\right\}.$$

This implies

(17.1) $\sup(\pi^0_{1,\eta_0}[\lambda_0]) = \sup\left(\pi^0_{1,\eta_0}(\lambda_0) \cap \left\{\pi^0_{0,\eta_0}(f)(\pi^0_{1,\eta_0}(\kappa_0)) \mid f \in \mathcal{M}_X\right\}\right)$.

Thus by (16.1), (16.2), and (17.1),

(18.1) $\sup(\pi^0_{1,\eta_0}[\lambda_0]) = \sup(\pi^1_{0,\eta_1}[\lambda_1])$

By the choice of η,

(19.1) $\sup(\pi^0_{1,\eta_0}[\lambda_0]) = \sup(\pi^0_{\eta,\eta_0}[\pi^0_{1,\eta}(\lambda_0)])$.

We have that $\lambda_0 = (\kappa_0^+)^{\mathcal{N}_1^0}$ and that κ_0 is strongly inaccessible in $\mathcal{N}_1^0$. Therefore by the properties of semi-iterations and the choice of η,

$$\pi^0_{\eta,\eta+1}[\pi^0_{1,\eta}(\lambda_0)] \in \mathcal{N}^0_{\eta+1},$$

and this implies that

$$\pi^0_{\eta,\eta_0}[\pi^0_{1,\eta}(\lambda_0)] \in \mathcal{N}^0_{\eta_0}$$

since $\mathrm{CRT}(\pi^0_{\eta+1,\eta_0}) \geq \pi^0_{\eta,\eta+1}\left(\kappa_{E^0_\eta}\right) > \pi^0_{1,\eta}(\lambda_0)$.

Let

$$\theta_0 = \sup(\pi^0_{\eta,\eta_0}[\pi^0_{1,\eta}(\lambda_0)]) = \sup(\pi^0_{1,\eta_0}[\lambda_0]) = \sup(\pi^1_{0,\eta_1}[\lambda_1]).$$

Thus

(20.1) $(Y^*)_{\theta_0} = \pi^0_{\eta,\eta_0}[\pi^0_{1,\eta}(\lambda_0)]$,

(20.2) $(Y^*)_{\theta_0} = \pi^1_{0,\eta_1}[\lambda_1]$.

This is a contradiction since

$$\pi^0_{\eta,\eta_0}[\pi^0_{1,\eta}(\lambda_0)] \cap \pi^0_{0,\eta_0}(\delta_X) = \kappa_{E^0_\eta}$$

and

$$\pi^1_{0,\eta_1}[\lambda_1] \cap \pi^1_{0,\eta_1}(\delta_X) = \kappa_{E^1_0},$$

noting that $\kappa_{E^1_0} = \delta_X$ and

$$\pi^0_{0,\eta}(\delta_X) = \mathrm{CRT}(\pi^0_{\eta,\eta_0}) > \delta_X.$$

This proves (15.1).

Let υ_0 be the least generator of E^0_0 such that $\nu^*_{E^0_0} < \upsilon_0$ noting that since $\nu^*_{E^0_0} = \kappa_0 + 1$, $\nu^*_{E^0_0}$ cannot be a generator of E^0_0.

We can reduce to the case that

$$\upsilon_0 = ((\kappa_0)^+)^{\mathrm{Ult}_0(\mathcal{M}_X, E^*)}$$

where $E^* = E^0_0|\nu^*_{E^0_0}$, since otherwise,

$$((\kappa_0)^+)^{\mathrm{Ult}_0(\mathcal{M}_X,E^*)} = ((\kappa_0)^+)^{\mathrm{Ult}_0(\mathcal{M}_X,E^0_0)} = ((\kappa_0)^+)^{\mathcal{N}^0_1},$$

and we can simply repeat the proof of (15.1) to again obtain a contradiction.

By (7.2), $\iota_{E^0_0} = \kappa_X$ and so since κ_X is a measurable cardinal $\mathcal{M}_X$, $\iota_{E^0_0}$ is a limit of strongly inaccessible cardinals in $\mathcal{M}_X$.

Therefore $\pi^0_{0,1}(\iota_{E^0_0})$ is a limit of strongly inaccessible cardinals in $\mathcal{N}^0_1$. Let ι be the least cardinal of $\mathcal{N}^0_1$ with $\nu^*_{E^0_0} < \iota < \pi^0_{0,1}(\iota_{E^0_0})$ such that

$$\mathcal{N}^0_1|\iota \vDash \mathrm{ZFC}.$$

By the Third Supercompactness Condition of semi-iterations, and since $\upsilon_0 < \iota$, there exist a transitive $(\hat{\mathcal{N}}, \mathbb{E}_{\hat{\mathcal{N}}}) \vDash \mathrm{ZFC}$ and an $\hat{\mathcal{N}}$-extender F such that:

(21.1) for all $a \in [\mathrm{LTH}(F)]^{<\omega}$, $F_a \in \hat{\mathcal{N}}$,

(21.2) $\mathcal{N}^0_1|\iota = \mathrm{Ult}_0(\hat{\mathcal{N}}, F)|\iota$, $\kappa_{E^0_0} < \kappa_F$, $j_F(\kappa_F) = \kappa^*_{E^0_0}$, and $j_F(\iota_F) = \iota$.

(21.3) No cardinal of $\hat{\mathcal{N}}$ below κ_F is $(<\kappa_F)$-supercompact in $\hat{\mathcal{N}}$.

(21.4) For some $\gamma \leq \iota_F$, $\upsilon_0 = \sup(j_F[\gamma])$, and either

$$j_F[\gamma] \in \mathrm{Ult}_0(\hat{\mathcal{N}}, F)$$

or for some cardinal δ_F of $\hat{\mathcal{N}}$, $\gamma = (\delta_F^+)^{\hat{\mathcal{N}}} = \iota_F$ and $(\mathrm{cof}(\delta_F))^{\hat{\mathcal{N}}} < \kappa_F$.

Fix γ as given by (21.4). Since $\upsilon_0 = \sup(j_F[\gamma])$ and since

$$\upsilon_0 = ((\kappa_0)^+)^{\mathrm{Ult}_0(\mathcal{M}_X, E^*)}$$

where $E^* = E_0^0|\nu^*_{E_0^0}$, necessarily $\gamma = ((\hat{\kappa})^+)^{\hat{\mathcal{N}}}$ for some strongly inaccessible cardinal $\hat{\mathcal{N}}$ such that

$$\kappa_F < \hat{\kappa} < \iota_F.$$

The point here is that κ_0 must be in the range of j_F and so $\hat{\kappa}$ is the strongly inaccessible cardinal of $\mathcal{N}$ such that $j_F(\hat{\kappa}) = \kappa_0$. Therefore by (21.4)

(22.1) $j_F[\gamma] \in \mathrm{Ult}_0(\hat{\mathcal{N}}, F)$,

(22.2) $\pi^0_{1,\eta_0}[j_F[\gamma]] \in \mathcal{N}^0_{\eta_0}$,

noting that (22.1) implies (22.2).

Now we can just repeat the proof of (15.1) one last time and obtain a contradiction, finishing the proof of the theorem. Note:

(23.1) $\pi^0_{1,\eta_0}(\upsilon_0) = \sup(\pi^0_{1,\eta_0}[\upsilon_0])$.

(23.2) $\sup(\pi^0_{1,\eta_0}[\upsilon_0]) = \sup\left(\pi^0_{1,\eta_0}(\lambda_0) \cap \left\{\pi^0_{0,\eta_0}(f)(\pi^0_{1,\eta_0}(\kappa_0)) \mid f \in \mathcal{M}_X\right\}\right)$.

(23.3) $\sup(\pi^1_{0,\eta_1}[\lambda_1]) = \sup\left(\pi^1_{0,\eta_1}(\lambda_1) \cap \left\{\pi^1_{0,\eta_1}(f)(\pi^1_{0,\eta_1}(\kappa_X)) \mid f \in \mathcal{M}^X_U\right\}\right)$.

(23.4) $\sup(\pi^1_{0,\eta_1}[\lambda_1]) = \sup\left(\pi^1_{0,\eta_1}(\lambda_1) \cap \left\{\pi^1_{0,\eta_1} \circ e^X_U(f)(\pi^1_{0,\eta_1}(\kappa_X)) \mid f \in \mathcal{M}_X\right\}\right)$.

Thus

(24.1) $\sup(\pi^0_{1,\eta_0}[\upsilon_0]) = \sup(\pi^1_{0,\eta_1}[\lambda_1])$.

Let

$$\theta = \pi^0_{1,\eta_0}(\upsilon_0) = \sup(\pi^0_{1,\eta_0}[\upsilon_0]) = \sup(\pi^1_{0,\eta_1}[\lambda_1]).$$

Therefore:

(25.1) $(Y^*)_\theta = \pi^1_{0,\eta_1}[\lambda_1]$,

(25.2) $(Y^*)_\theta = \pi^0_{1,\eta_0}(j_F[\gamma]) = \pi^0_{1,\eta_0}[j_F[\gamma]]$,

since $\upsilon_0 = \sup(j_F[\gamma])$. But

$$\pi^1_{0,\eta_1}[\lambda_1] \cap \delta^* = \delta_X = \kappa_{E_0^1}$$

and

$$\pi^0_{1,\eta_0}[j_F[\gamma]] \cap \delta^* = \kappa_F > \delta_X.$$

This is again a contradiction. □

6 The amenability obstruction

We show that there is no inner model with a supercompact cardinal which is a fine structure model such that every level is an amenable sound structure. There is no abstraction of comparison or iterability involved here and we shall also prove versions where the amenability condition is significantly weakened. Thus these constraints apply to a much wider class of inner models than the comparison constraints of the previous section.

We actually show there is no such inner model (with amenable and sound levels) in which there is a cardinal κ which is $\kappa^{+\omega}$-supercompact.

The nonstrategic-extender models of [24], which reach the finite levels of supercompactness, are amenable and sound at every level, and so the extent of those constructions in reaching levels of the large cardinal hierarchy is best possible.

Further the variations on amenability that we consider include both the cases where at each level the predicate is only required to be amenable to an initial segment of the structure, or even more generally, simply specifies an ω-sequence of predicates each which is only required to be amenable to some initial segment of the structure.

These generalizations exclude a variety of natural attempts to extend the structures of [24] to the infinite levels of supercompactness.

Finally we shall show in Lemma 6.29 that these generalizations are all equivalent and moreover just corollaries of the theorem of Shelah, [16], that if the *Approachability Property* holds at $\kappa^{+(\omega+1)}$ then κ cannot be $\kappa^{+\omega}$-supercompact.

6.1 Soundness

We define an abstract notion of soundness. This is just the natural definition given, for example, the basic definitions of modern fine structure theory, and here we follow the basic framework of [12].

Definition 6.1. (1) Let $\mathcal{L}_{(\mathrm{gen})}$ be the language of set theory together with unary predicates $\dot{\mathbb{P}}$ and $\dot{P}$.

(2) Suppose that $\mathcal{M} = (J_\alpha[\mathbb{P}], \mathbb{P}|\alpha, \mathbb{P}_\alpha)$ and that $\mathbb{P}_\alpha \subseteq J_\alpha[\mathbb{P}]$. Then $\mathcal{M}$ defines an $\mathcal{L}_{(\mathrm{gen})}$-structure where $\dot{\mathbb{P}}$ interpreted by $\mathbb{P}|\alpha$ and $\dot{P}$ interpreted by $\mathbb{P}_\alpha$.

(3) Suppose $\mathcal{M}$ is a (transitive) $\mathcal{L}_{(\mathrm{gen})}$-structure. Then $\mathbb{P}_\mathcal{M}$ is the interpretation of $\dot{\mathbb{P}}$ and $P_\mathcal{M}$ is the interpretation of $\dot{P}$. □

Remark 6.2. We shall always assume that an $\mathcal{L}_{(\mathrm{gen})}$-structure is either of the form

$$\mathcal{M} = (J_\alpha[\mathbb{P}], \mathbb{P}|\alpha, \mathbb{P}_\alpha)$$

in the $\dot{\mathbb{P}}$-*active* case, or of the form

$$\mathcal{M} = (J_\alpha[\mathbb{P}], \mathbb{P}|\alpha)$$

in the $\dot{\mathbb{P}}$-*passive* case. In particular we are restricting to transitive structures. □

For the following definition we implicitly restrict to $\mathcal{L}_{(\mathrm{gen})}$-structures which are *weakly amenable*, see Remark 6.17. The case of more general structures will involve altering the definition of a $({}_{\mathcal{L}_{(\mathrm{gen})}})\Sigma_1$-formula, see the discussion after Remark 6.17.

Definition 6.3. $\mathcal{L}^+_{(\mathrm{gen})}$ is $\mathcal{L}_{(\mathrm{gen})}$ expanded by adding 3-ary predicates $\dot{T}_n$ for $1 \leq n < \omega$. Suppose θ is a formula of $\mathcal{L}^+_{(\mathrm{gen})}$.

(1) θ is $({}_{\mathcal{L}_{(\mathrm{gen})}})\Sigma_1$ if θ is a Σ_1-formula relative to $\mathcal{L}_{(\mathrm{gen})}$.

(2) θ is $({}_{\mathcal{L}_{(\text{gen})}})\Sigma_{n+1}$ if there is a Σ_1-formula $\phi(x_0, \ldots, x_m, x_{m+1}, x_{m+2})$ of $\mathcal{L}_{(\text{gen})}$ such that

$$\theta = \exists x_m \exists x_{m+1} \exists x_{m+2} \left(\dot{T}_n(x_m, x_{m+1}, x_{m+2}) \wedge \phi \right). \qquad \square$$

Definition 6.4. (1) For each formula $\phi(x_0, \ldots, x_n, x_{n+1})$ of $\mathcal{L}^+_{(\text{gen})}$ (with free occurrences of x_{n+1}), $\tau_\phi(x_0, \ldots, x_n)$ is the Skolem term given by ϕ and for each $1 \leq n < \omega$, $({}_{\mathcal{L}_{(\text{gen})}})\text{Sk}_n$ is the smallest collection of terms closed under composition and containing all the terms τ_ϕ where ψ is $({}_{\mathcal{L}_{(\text{gen})}})\Sigma_n$.

(2) A formula ψ is a *generalized* $({}_{\mathcal{L}_{(\text{gen})}})\Sigma_n$*-formula*, where $1 \leq n < \omega$, if for some $({}_{\mathcal{L}_{(\text{gen})}})\Sigma_n$-formula $\phi[x_0, \ldots, x_m]$, collection of terms closed under composition and containing all the terms τ_ϕ where ψ is $({}_{\mathcal{L}_{(\text{gen})}})\Sigma_n$. Thus for any $\mathcal{L}^+_{(\text{gen})}$-formula ϕ, the arity of τ_ϕ is m where m is largest such that x_{m+1} is a free variable of ψ (and τ_ϕ is defined only if $m \geq 1$).

(3) A formula ψ is a *generalized* $({}_{\mathcal{L}_{(\text{gen})}})\Sigma_n$*-formula*, where $1 \leq n < \omega$, if for some $({}_{\mathcal{L}_{(\text{gen})}})\Sigma_n$-formula $\phi[x_0, \ldots, x_m]$,

$$\psi = \phi(x_0, \ldots, x_m : \sigma_0, \ldots, \sigma_m)$$

where

a) for each $i \leq m$, $\sigma_i \in ({}_{\mathcal{L}_{(\text{gen})}})\text{Sk}_n$, and σ_i is free for x_i in ϕ,

b) $\phi(x_0, \ldots, x_m : \sigma_0, \ldots, \sigma_m)$ is the formula obtained from ψ by substituting σ_i for each free occurrence of x_i. $\square$

Suppose $\mathcal{M}$ is a $\mathcal{L}_{(\text{gen})}$-structure. By induction on $1 \leq n < \omega$, we define the interpretation of $\dot{T}_n$ in $\mathcal{M}$, denoted $T_n^{\mathcal{M}}$, and the n-th projectum of $\mathcal{M}$, denoted $\rho_n^{\mathcal{M}}$. Simultaneously we define the interpretations of τ_ϕ which we denote $\tau_\phi^{\mathcal{M}}$. To simplify notation a bit we adopt the following conventions.

(1) $\phi(x_0, \ldots, x_m)$ indicates the free variables of ψ are included in $\{x_0, \ldots, x_m\}$ and that x_m is a free variable of ϕ.

(2) Suppose $\phi(x_0, \ldots, x_m)$ is a formula, $m > 0$, and $s \in |\mathcal{M}|^{<\omega}$. We write $\mathcal{M} \vDash \phi[\bar{s}]$ to indicate both $|s| = m + 1$ and that

$$\mathcal{M} \vDash \phi[s_0, \ldots, s_m].$$

Definition 6.5. Suppose $\mathcal{M}$ is a $\mathcal{L}_{(\text{gen})}$-structure. Suppose $1 \leq n < \omega$.

(1) Suppose that $\phi(x_0, \ldots, x_{m+1})$ is a $({}_{\mathcal{L}_{(\text{gen})}})\Sigma_n$-formula and that $\tau_\phi(x_0, \ldots, x_{m+1})$ is the corresponding $({}_{\mathcal{L}_{(\text{gen})}})\text{Sk}_n$-term. Then for each

$$\langle a_i : i \leq m \rangle \in |\mathcal{M}|^{<\omega},$$

$\langle a_i : i \leq m \rangle \in \text{dom}(\tau_\phi^{\mathcal{M}})$ and for each $b = \tau_\phi^{\mathcal{M}}(a_0, \ldots, a_m)$ if

a) $\mathcal{M} \vDash \phi[a_0, \ldots, a_m, b]$,

b) for all $c <_{\mathcal{M}} b$, $\mathcal{M} \vDash (\neg\phi)[a_0, \ldots, a_m, c]$.

(2) For each $X \subseteq |\mathcal{M}|$,

$$\text{Th}_n^{\mathcal{M}}(X) = \{(\phi, s) \mid s \in X^{<\omega}, \psi \text{ is generalized } ({}_{\mathcal{L}_{(\text{gen})}})\Sigma_n, \mathcal{M} \vDash \phi[\bar{s}]\}.$$

(3) $\rho_n^{\mathcal{M}}$ is the least ordinal $\rho \leq \alpha$, such that $\text{Th}_n(\rho \cup \{q\}) \notin |\mathcal{M}|$ for some $q \in |\mathcal{M}|$, where $\alpha = |\mathcal{M}| \cap \text{Ord}$.

(4) $T_n^{\mathcal{M}}(\alpha, q, b)$ if and only if $\alpha < \rho_n^{\mathcal{M}}$, $q \in |\mathcal{M}|$, and $b = \text{Th}_n^{\mathcal{M}}(\alpha \cup \{q\})$. $\square$

Definition 6.6. Suppose $\mathcal{M}$ is a $\mathcal{L}_{(\text{gen})}$-structure. Suppose $X \subseteq |\mathcal{M}|$, $X \neq \emptyset$, $1 \leq n < \omega$, and $\rho_k^{\mathcal{M}} > 0$ for all $0 < k < n$.

(1) $S_n^{\mathcal{M}}(X) = \left\{\tau^{\mathcal{M}}(s) \mid s \in \mathrm{dom}(\tau_\phi^{\mathcal{M}}) \cap X^{<\omega} \text{ and } \tau \in ({}_{\mathcal{L}_{(\mathrm{gen})}})\mathrm{Sk}_n\right\}$.

(2) $\mathcal{H}_n^{\mathcal{M}}(X)$ is the $\mathcal{L}_{(\mathrm{gen})}$-structure given by the transitive collapse of

$$\left(S_n^{\mathcal{M}}(X), \mathbb{P}_{\mathcal{M}} \cap S_n^{\mathcal{M}}(X), P_{\mathcal{M}} \cap S_n^{\mathcal{M}}(X)\right).$$ □

Definition 6.7. Suppose $\mathcal{M}$ is an amenable $\mathcal{L}_{(\mathrm{gen})}$-structure. Then $\mathcal{M}$ is ω-*sound* if for each $k+1<\omega$, one of the following hold.

(1) $k > 0$ and $\rho_k^{\mathcal{M}} = 0$.

(2) There exists $a \in \mathcal{M}$ such that $\mathcal{M} = \mathcal{H}_{k+1}^{\mathcal{M}}(\rho_{k+1}^{\mathcal{M}} \cup \{a\})$. □

Remark 6.8. The definition of soundness here does not involve any notion of a standard parameter or any properties of the standard parameters such as solidity. Thus it is far weaker than the usual notions of soundness. □

Remark 6.9. We illustrate why sound structures are so useful. Suppose $\mathcal{M}$ is an amenable ω-sound, $\mathcal{L}_{(\mathrm{gen})}$-structure, $0 < k < \omega$, and

$$\rho_k^M > 0.$$

Let $\rho = \rho_k^{\mathcal{M}}$ and let $q \in \mathcal{M}$ be such that

$$\mathcal{M} = \mathcal{H}_k^{\mathcal{M}}(\rho, \{q\}).$$

Let $\mathcal{N} = (\mathcal{M}|\rho, T)$ where $T = \mathrm{Th}_k^{\mathcal{M}}(\rho \cup \{q\})$, naturally coded as a subset of $\mathcal{M}|\rho$.

(1) $\mathcal{N}$ is an amenable $\mathcal{L}_{(\mathrm{gen})}$-structure.

(2) $\mathcal{N}$ is ω-sound and $\rho_1^{\mathcal{N}} = \rho_{k+1}^{\mathcal{M}}$.

(3) Suppose $A \subset \mathcal{M}|\rho$. Then the following are equivalent.

a) A is $({}_{\mathcal{L}_{(\mathrm{gen})}})\Sigma_1$-definable in $\mathcal{N}$ from parameters.

b) A is $({}_{\mathcal{L}_{(\mathrm{gen})}})\Sigma_{k+1}$-definable in $\mathcal{M}$ from parameters. □

6.2 The amenability obstruction

Definition 6.10. Suppose that $\mathbb{P} \subset \mathrm{Ord} \times V$. Then $J[\mathbb{P}]$ is *amenable* if for all $\alpha \in \mathrm{dom}(\mathbb{P})$, the following hold.

(1) $(J_\alpha[\mathbb{P}], \mathbb{P}|\alpha) \vDash$ Comprehension

(2) $\mathbb{P}_\alpha \subseteq J_\alpha[\mathbb{P}]$.

(3) For all $\beta < \alpha$, $\mathbb{P}_\alpha \cap J_\beta[\mathbb{P}] \in J_\alpha[\mathbb{P}]$. □

Remark 6.11. The requirement that if $\alpha \in \mathrm{dom}(\mathbb{P})$ then

$$(J_\alpha[\mathbb{P}], \mathbb{P}|\alpha) \vDash \text{Comprehension}$$

just simplifies things conceptually and is not necessary. For example if $\alpha > \omega$, it implies that α is a limit ordinal.

Note that in the case where $\mathbb{P}$ is a good partial extender sequence, we require (see Definition 5.8) that if $\alpha \in \mathrm{dom}(\mathbb{P})$ then

$$(J_\alpha[\mathbb{P}], \mathbb{P}|\alpha) \vDash \mathrm{ZFC}\backslash\mathrm{Powerset},$$

which is of course a much stronger condition. □

We now include soundness and define when $J[\mathbb{P}]$ is amenable and sound. We shall define soundness more generally just after Remark 6.17 but that definition will be based on a reformulation of the definition of $({}_{\mathcal{L}_{(\mathrm{gen})}})\Sigma_n$-formulas together with a reformulation of Definition 6.7 to include the case when $\mathcal{M}$ is not amenable, see Definition 6.22.

Definition 6.12. Suppose that $\mathbb{P} \subset \mathrm{Ord} \times V$ and that $J[\mathbb{P}]$ is amenable. Then $J[\mathbb{P}]$ is *sound* if for each $\alpha \in \mathrm{Ord}$:

(1) $(J_\alpha[\mathbb{P}], \mathbb{P}|\alpha, \emptyset)$ is ω-sound,

(2) if $\alpha \in \mathrm{dom}(\mathbb{P})$ then $(J_\alpha[\mathbb{P}], \mathbb{P}|\alpha, \mathbb{P}_\alpha)$ is ω-sound. □

Of course, if $J[\mathbb{P}]$ is amenable then for each $\alpha \in \mathrm{dom}(\mathbb{P})$, the structure

$$(J_\alpha[\mathbb{P}], \mathbb{P}|\alpha, \emptyset)$$

is trivially ω-sound since $(J_\alpha[\mathbb{P}], \mathbb{P}|\alpha) \vDash$ Comprehension

The following lemma is immediate from the definitions.

Lemma 6.13. *Suppose that $\mathbb{P} \subset \mathrm{Ord} \times V$, $J[\mathbb{P}]$ is amenable, and $J[\mathbb{P}]$ is sound. Then* GCH *holds in* $J[\mathbb{P}]$. □

Remark 6.14. Assuming GCH, if κ is $\kappa^{+\omega}$-supercompact then necessarily κ is $\kappa^{+(\omega+1)}$-supercompact. Rephrased (and now not assuming GCH), if

$$j : V \to M$$

is an elementary embedding with critical point κ, the following are equivalent.

(1) $j[V_{\kappa+\omega}] \in M$.

(2) $j[V_{\kappa+\omega+1}] \in M$.

More generally for any set X and for any $\gamma < \kappa$, if

$$j[X] \in M$$

then $j[X^\gamma] \in M$. □

Lemma 6.15 (GCH). *Suppose that κ is $\kappa^{+\omega}$-supercompact and that $\mathbb{P} \subset V_\kappa$. Then there exist $\delta < \kappa$ and an elementary embedding*

$$\pi : \Big(H(\gamma), \mathbb{P} \cap H(\gamma)\Big) \to \Big(H(\pi(\gamma)), \mathbb{P} \cap H(\pi(\gamma))\Big)$$

such that

(1) $\pi \in V_\kappa$ *and* $\mathrm{CRT}(\pi) = \delta$.

(2) $\gamma = \delta^{+(\omega+1)}$ *and* $\pi(\gamma) = (\pi(\gamma))^{+(\omega+1)}$.

Proof. Since GCH holds, κ is $\kappa^{+(\omega+1)}$-supercompact. Let

$$j : V \to M$$

be an elementary embedding such that $\mathrm{CRT}(j) = \kappa$ and $M^\lambda \subset M$ where

$$\lambda = \kappa^{+(\omega+1)} = |V_{\kappa+\omega+1}| = |H(\kappa^{+(\omega+1)})|.$$

Let $N = j(M)$ and let

$$j(j) \circ j : V \to N$$

be the iteration embedding. Then

$$j|H(\kappa^{+(\omega+1)}) \in j(j) \circ j(V_\kappa) = N_{j(j(\kappa))}$$

witness that the conclusion of lemma holds in N at $j(j) \circ j(\kappa)$ for $j(j) \circ j(\mathbb{P})$. □

Theorem 6.16. *Suppose that $\mathbb{P} \subset \mathrm{Ord} \times V$ and that $J[\mathbb{P}]$ is amenable and sound. Then*

$$J[\mathbb{P}] \vDash \text{“There are no cardinals } \kappa \text{ which are } \kappa^{+\omega}\text{-supercompact”}.$$

Proof. We work in $(J[\mathbb{P}], \mathbb{P})$. Assume toward a contradiction that there exists κ such that κ is $\kappa^{+\omega}$-supercompact. Therefore by Lemma 6.13 and Lemma 6.15, there exist $\delta < \kappa$ and an elementary embedding

$$\pi : (J_\gamma[\mathbb{P}], \mathbb{P}|\gamma) \to (J_{\pi(\gamma)}[\mathbb{P}], \mathbb{P}|\pi(\gamma))$$

such that

(1.1) $\gamma = \delta^{+(\omega+1)}$ and $\mathrm{CRT}(\pi) = \delta$,

(1.2) $\pi(\gamma) = (\pi(\delta))^{+(\omega+1)}$.

Let $\eta = \sup(\pi[\gamma])$. The key point is that there can be no closed set $C \subset \eta$ such that

(2.1) $|C| < \pi(\delta^{+\omega})$,

(2.2) C is cofinal in η,

(2.3) $C \cap \xi \in J_\eta[\mathbb{P}]$ for all $\xi < \eta$.

Note that this claim implies that weak-$\square$ must fail at $\delta^{+\omega}$. In fact the relevant principle is $\mathcal{AP}_{\kappa^+}$ which is an even weaker principle. See Definition 6.25.

Assume toward a contradiction that C exists. Let

$$D = \{\xi < \gamma \mid \pi(\xi) \in C\}.$$

Thus D is ω-closed and by (2.3), D is cofinal in γ. Let $\xi_0 \in D$ be such that

$$|D \cap \xi_0| \geq \delta^{+\omega}.$$

Then $C \cap \pi(\xi_0)$ covers $\pi[D \cap \xi_0]$ and by (2.1),

$$\mathcal{P}(C \cap \pi(\xi_0)) \in J_\eta[\mathbb{P}].$$

This implies that $\pi[\delta^{+\omega}] \in J_\eta[\mathbb{P}]$ and so

$$\pi|H(\delta^{+\omega}) \in J_\eta[\mathbb{P}].$$

But then $\pi[\gamma]$ is definable from parameters in $J_\eta[\mathbb{P}]$ and this is a contradiction since

$$(J_\eta[\mathbb{P}], \mathbb{P}|\eta) \vDash \mathrm{ZFC} \backslash \mathrm{Powerset}.$$

This proves the claim.

Let $\alpha > \eta$ be least such that for some $0 < k < \omega$,

$$\rho_k^{\mathcal{M}} < \eta$$

where $\mathcal{M} = (J_\alpha[\mathbb{P}], \mathbb{P}|\alpha)$ if $\alpha \notin \mathrm{dom}(\mathbb{P})$ and

$$\mathcal{M} = (J_\alpha[\mathbb{P}], \mathbb{P}|\alpha, \mathbb{P}_\alpha)$$

otherwise. We assume $\alpha \in \mathrm{dom}(\mathbb{P})$. The case that $\alpha \notin \mathrm{dom}(\mathbb{P})$ is easier.

Fix k to be least such that $\rho_k^{\mathcal{M}} < \eta$. Thus

$$\rho_k^{\mathcal{M}} = \pi(\delta)^{+\omega}.$$

We first prove:

(3.1) $k > 1$.

Assume toward a contradiction that $k = 1$. By soundness, there exists $q \in J_\alpha[\mathbb{P}]$ such that

$$\mathcal{M} = \mathcal{H}_1^{\mathcal{M}}(\rho_1^{\mathcal{M}} \cup \{q\}) = S_1^{\mathcal{M}}(\rho_1^{\mathcal{M}} \cup \{q\}).$$

Thus there is a partial function

$$f : \pi(\delta)^{+\omega} \to \eta$$

such that f is a surjection and such that f is generalized $(_{\mathcal{L}_{(\mathrm{gen})}})\Sigma_1$-definable in $\mathcal{M}$ from parameters. Because $\mathcal{M}$ is amenable, we can reduce to the case that f is Σ_1-definable from parameters in the structure

$$\mathcal{M} = (J_\alpha[\mathbb{P}], \mathbb{P}|\alpha, \mathbb{P}_\alpha).$$

Fix a Σ_1-formula $\phi(x_0, x_1, x_2)$ and $c_0 \in J_\alpha[\mathbb{P}]$ such that

$$f = \{(a, b) \in J_\alpha[\mathbb{P}] \mid (J_\alpha[\mathbb{P}], \mathbb{P}|\alpha, \mathbb{P}_\alpha) \vDash \phi[a, b, c_0]\}$$

We can require that $\phi(x_0, x_1, x_2)$ has the form

$$(\exists \xi < \alpha)\psi\left[x_0, x_1, x_2, \mathbb{P}_\alpha \cap J_\xi[\mathbb{P}]\right]$$

where ψ is a $(_{\mathcal{L}_{(\mathrm{gen})}})\Sigma_1$-formula not mentioning $\dot{P}$, the predicate for $\mathbb{P}_\alpha$.

Fix $\pi(\delta)^{+\omega} < \alpha_0 < \alpha$ such that $c_0 \in J_{\alpha_0}[\mathbb{P}]$. For each $\alpha_0 < \beta < \alpha$, let f_β be the set of all $(a, b) \in J_\beta[\mathbb{P}]$ such that

$$(J_\beta[\mathbb{P}], \mathbb{P}|\beta) \vDash \psi[a, b, c_0, \mathbb{P}_\alpha \cap J_\xi[\mathbb{P}]]$$

for some $\xi < \beta$ such that $\mathbb{P}_\alpha \cap J_\xi[\mathbb{P}] \in J_\beta[\mathbb{P}]$.

Thus $f_\beta \subseteq f$, $f_\beta \in J_\alpha[\mathbb{P}]$, and

$$f = \cup\left\{f_\beta \mid \alpha_0 < \beta < \alpha\right\}$$

There are two cases.

Case 1: $\mathrm{cof}(\alpha) \neq \delta^{+(\omega+1)}$.

There must exist $\pi(\delta)^{+\omega} < \beta < \alpha$ such that f_β has range cofinal in η. But

$$f_\beta \in J_\alpha[\mathbb{P}].$$

This contradicts the choice of α.

Case 2: $\mathrm{cof}(\alpha) = \delta^{+(\omega+1)}$.

Let $\theta < \pi(\delta)^{+\omega}$ be least such that $f|\theta$ has cofinal range in η. Thus

$$\mathrm{cof}(\theta) = \mathrm{cof}(\eta) = \delta^{+(\omega+1)}.$$

Fix $X \subset \theta$ such that

(4.1) $|X| = \delta^{+(\omega+1)}$,

(4.2) $f|X$ has cofinal range in η.

We have $H(\pi(\delta)^{+\omega}) \subset J_\alpha[\mathbb{P}]$ and so clearly $X \in J_\alpha[\mathbb{P}]$. This implies there is an increasing cofinal continuous function

$$g : \delta^{+(\omega+1)} \to \eta$$

such that g is Σ_1-definable from parameters in $(J_\alpha[\mathbb{P}], \mathbb{P}|\alpha, \mathbb{P}_\alpha)$ and such that

$$g|\xi \in J_\alpha[\mathbb{P}]$$

for each $\xi < \delta^{+(\omega+1)}$. Let C be the range of g. Then C satisfies (2.1)–(2.3) which is a contradiction.

This proves (3.1). Let $n = k - 1 > 0$. Thus

$$\rho_n^{\mathcal{M}} > \eta$$

and either $\rho_n^{\mathcal{M}} = \alpha$ or $\rho_n^{\mathcal{M}}$ is a cardinal of $J_\alpha[\mathbb{P}]$. Let $\rho = \rho_n^{\mathcal{M}}$ and fix $q \in J_\alpha[\mathbb{P}]$ such that

$$\mathcal{M} = \mathcal{H}_n^{\mathcal{M}}(\rho \cup \{q\}).$$

The structure,

$$\mathcal{N} = (J_\rho[\mathbb{P}], T)$$

is amenable where $T = \mathrm{Th}_n^{\mathcal{M}}(\rho \cup \{q\})$ (naturally coded as a subset of ρ) since

$$(J_\rho[\mathbb{P}], \mathbb{P}|\rho) \vDash \text{Comprehension}.$$

In fact since ρ is a cardinal of $J_\alpha[\mathbb{P}]$, either

$$(J_\rho[\mathbb{P}], \mathbb{P}|\rho) \vDash \text{ZFC}\backslash\text{Powerset}$$

or

$$(J_\rho[\mathbb{P}], \mathbb{P}|\rho) \vDash \text{ZFC}\backslash\text{Replacement},$$

which is a much strong claim.

The key point is that

$$\rho_1^{\mathcal{N}} = \rho_{n+1}^{\mathcal{M}}$$

and that for some $p \in J_\rho[\mathbb{P}]$

$$\mathcal{N} = \mathcal{H}_1^{\mathcal{N}}(\rho_1^{\mathcal{N}} \cup \{p\})$$

We can now just repeat the proof of (3.1). □

One can weaken the notion of amenability quite a bit and still prove Theorem 6.16. The point here is that the definition of soundness does not require amenability though we will alter the basic definitions slightly when generalizing to non-amenable structures.

As we have already indicated, these generalizations rule out many natural approaches to extending the constructions of [24] in an attempt to reach the infinite levels of supercompactness.

Remark 6.17. Define $\mathbb{P}$ to be *weakly amenable* if for all $\alpha \in \mathrm{dom}(\mathbb{P})$, there exists a limit ordinal $\gamma < \alpha$ such that:

(1) $\mathbb{P}_\alpha \subseteq J_\gamma[\mathbb{P}]$.

(2) For all $\beta < \gamma$, $\mathbb{P}_\alpha \cap J_\beta[\mathbb{P}] \in J_\alpha[\mathbb{P}]$.

Then the proof of Theorem 6.16 adapts to prove the corresponding theorem for weakly amenable $\mathbb{P}$. This requires dealing with more cases since there are now two relevant cofinalities, $\mathrm{cof}(\alpha)$ and $\mathrm{cof}(\gamma)$. Note though that if $\gamma < \alpha$ and $\mathbb{P}_\alpha \notin J_\alpha[\mathbb{P}]$ then necessarily

$$\rho_1^{\mathcal{M}} \leq \gamma$$

where $\mathcal{M} = (J_\alpha[\mathbb{P}], \mathbb{P}|\alpha, \mathbb{P}_\alpha)$. Thus the additional cases only arise in proving (3.1) in the proof of Theorem 6.16 and the rest of the proof is exactly the same. We leave the details to the reader since there is a more general theorem, Theorem 6.30, which we shall obtain as a corollary of Theorem 6.16 and Lemma 6.29.

In fact all these theorems are really equivalent since we shall prove in Lemma 6.29 that if $J[\mathbb{P}]$ is ω-weakly amenable and sound as defined below then there exists $\mathbb{P}^*$ such that

$$J[\mathbb{P}] = J[\mathbb{P}^*]$$

and such that $J[\mathbb{P}^*]$ is amenable and sound. □

Remark 6.18. The backgrounding scenario described in Remark 5.13 illustrates how one might naturally be led to consider weakly amenable structures which are not amenable. We continue that discussion and focus *just* on the case where $\mathcal{M}$ is weakly backgrounded and $\mathbb{E}_{\mathcal{M}}$ witnesses that κ_E is λ-supercompact in $\mathcal{M}$. Notation now is as in Remark 5.13.

We have set

$$\mathcal{N} = \mathcal{M}|\sup(j[(\iota_E^+)^{\mathcal{M}}]) = \mathcal{M}|\sup(j_E(\iota_E^+)^{\mathcal{M}})$$

and the coding obstruction of Section 4 has been interpreted to require that one must (in general) have that there exists a set $\sigma \in \mathcal{N}$ such that $|\sigma|^{\mathcal{N}} < \kappa_E^*$ and such that $E \cap \sigma \notin \mathcal{M}$. Otherwise one cannot add E to the sequence to construct the next approximation to the final model.

If no such σ exists then we argued that $\mathrm{cof}(\iota_E) < \kappa_E$ which implies that

$$j(\iota_E) = \sup(j[\iota_E]).$$

Now replace $\mathcal{N}$ with

$$\mathcal{N}^* = \mathcal{M}|\left(j\left((\iota_E^+)^{\mathcal{M}}\right)\right).$$

Thus since $\mathcal{M}$ is weakly backgrounded and since $\mathbb{E}_{\mathcal{M}}$ witnesses that κ_E is supercompact in $\mathcal{M}$, there *does* exist

$$\sigma \in \mathcal{N}^*$$

such that $|\sigma|^{\mathcal{N}^*} < \kappa_E^*$ and such that $E \cap \sigma \notin \mathcal{M}$. The structure

$$(\mathcal{N}^*, E)$$

is weakly amenable since $\theta = j(\iota_E)$.

This suggests altering the indexing scheme to allow in this situation that E be added with index $\mathrm{Ord}^{\mathcal{N}^*}$ so that

$$(\mathcal{N}^*, \mathbb{E}_{\mathcal{N}^*}, E)$$

is that next approximation to the final model, and so this suggests developing an alternative fine-structural hierarchy which allows such generalized indexing schemes.

But this cannot work to reach the infinite levels of supercompact. □

One can further generalize by only requiring that for each $\alpha \in \mathrm{dom}(\mathbb{P})$, $\mathbb{P}_\alpha$ specifies an ω-sequence of predicates each of which is weakly amenable to $J_\alpha[\mathbb{P}]$.

Definition 6.19. Suppose that $\alpha \in \mathrm{dom}(\mathbb{P})$ and α is a limit ordinal. Then

$$(J_\alpha[\mathbb{P}], \mathbb{P}|\alpha, \mathbb{P}_\alpha)$$

is *ω-weakly amenable* if

$$\mathbb{P}_\alpha \subset \omega \times J_\alpha[\mathbb{P}]$$

and for each $n < \omega$, there exists a limit ordinal $\gamma_n \leq \alpha$ such that

(1) $(\mathbb{P}_\alpha)_n \subset J_{\gamma_n}[\mathbb{P}]$,

(2) $(\mathbb{P}_\alpha)_n \cap J_\xi[\mathbb{P}] \in J_\alpha[\mathbb{P}]$ for each $\xi < \gamma_n$. □

Definition 6.20. Suppose that $\mathbb{P} \subset \mathrm{Ord} \times V$. Then $\mathbb{P}$ is *ω-weakly amenable* if for all $\alpha \in \mathrm{dom}(\mathbb{P})$,

(1) $(J_\alpha[\mathbb{P}], \mathbb{P}|\alpha) \vDash$ Comprehension,

(2) $(J_\alpha[\mathbb{P}], \mathbb{P}|\alpha, \mathbb{P}_\alpha)$ is ω-weakly amenable. □

The notion of soundness is exactly as defined for the case of amenable $\mathbb{P}$ except that $({}_{\mathcal{L}_{(\mathrm{gen})}})\Sigma_n$-formulas are redefined as follows.

Definition 6.21. $\mathcal{L}^-_{(\mathrm{gen})}$ is $\mathcal{L}_{(\mathrm{gen})}$ reduced by eliminating $\dot{P}$. □

Definition 6.22. Suppose θ is a formula of ${\mathcal{L}_{(\mathrm{gen})}}^+$.

(1) θ is $({}_{\mathcal{L}_{(\mathrm{gen})}})\Sigma_1$ if there is a Σ_1-formula $\phi(x_0, x_1)$ of $\mathcal{L}^-_{(\mathrm{gen})}$ such that

$$\theta = \exists x_0 \left(\text{“}x_0 \subset \dot{P}\text{”} \wedge \text{“}x_0 \text{ is finite”} \wedge \phi\right).$$

(2) θ is $({}_{\mathcal{L}_{(\mathrm{gen})}})\Sigma_{n+1}$ if there is a Σ_1-formula $\phi(x_0, \ldots, x_m, x_{m+1}, x_{m+2})$ of $\mathcal{L}^-_{(\mathrm{gen})}$ such that

$$\theta = \exists x_m \exists x_{m+1} \exists x_{m+2} \left(\dot{T}_n(x_m, x_{m+1}, x_{m+2}) \wedge \phi\right). \qquad \square$$

For the amenable $\mathcal{L}_{(\mathrm{gen})}$-structures, this change in the definition of $({}_{\mathcal{L}_{(\mathrm{gen})}})\Sigma_n$-formulas amounts to simply replacing $\mathbb{P}$ with $\mathbb{P}^*$ where

$$\mathrm{dom}(\mathbb{P}) = \mathrm{dom}(\mathbb{P}^*)$$

and for each $\alpha \in \mathrm{dom}(\mathbb{P})$,

$$\mathbb{P}^*_\alpha = \left\{\mathbb{P}_\alpha \cap J_\xi[\mathbb{P}] \mid \xi < \alpha\right\}.$$

Therefore we could have simply made this part of our abstract definitions in Section 6.1 of $\mathcal{L}_{(\mathrm{gen})}$-structures and $({}_{\mathcal{L}_{(\mathrm{gen})}})\Sigma_n$-formulas.

With this change, we can naturally define when an $\mathcal{L}_{(\mathrm{gen})}$-structure $\mathcal{M}$ is ω-sound without restricting to the case that $\mathcal{M}$ is an amenable $\mathcal{L}_{(\mathrm{gen})}$-structures which we did in Definition 6.7. Thus we can define when $J[\mathbb{P}]$ is sound for an arbitrary class

$$\mathbb{P} \subset \mathrm{Ord} \times V,$$

and this we do in the following definition.

Definition 6.23. Suppose that $\mathbb{P} \subset \mathrm{Ord} \times V$. Then $J[\mathbb{P}]$ is *sound* if for each $\alpha \in \mathrm{Ord}$, $\mathcal{M}$ is ω-sound where $\mathcal{M} = (J_\alpha[\mathbb{P}], \mathbb{P}|\alpha, \emptyset)$ if $\alpha \notin \mathrm{dom}(\mathbb{P})$, and

$$\mathcal{M} = (J_\alpha[\mathbb{P}], \mathbb{P}|\alpha, \mathbb{P}_\alpha)$$

otherwise. $\square$

Lemma 6.24. *Suppose that* $\mathbb{P} \subset \mathrm{Ord} \times V$ *and* $J[\mathbb{P}]$ *is sound. Then* GCH *holds in* $J[\mathbb{P}]$. $\square$

6.3 Weak amenability and the Approachability Property

The analysis of $J[\mathbb{P}]$ which are ω-weakly amenable and sound involves the *Approachability Property*, $\mathcal{AP}$.

Definition 6.25 (Foreman-Magidor). Suppose that κ is an infinite cardinal. Then $\mathcal{AP}_{\kappa^+}$ holds if there is a sequence

$$\langle C_\alpha : \alpha < \kappa^+\rangle$$

such that for all limit $\alpha < \kappa^+$:

(1) C_α is a closed cofinal subset of α and $\mathrm{ordertype}(C_\alpha) \leq \kappa$.

(2) If $\mathrm{cof}(\alpha) < \kappa$ then $|C_\alpha| < \kappa$.

(3) For all $\beta < \alpha$, $C_\alpha \cap \beta = C_\gamma$ for some $\gamma < \alpha$. $\square$

Remark 6.26. (1) $\mathcal{AP}_{\kappa^+}$ holds for all regular cardinals κ assuming GCH and we are only really interested in the situation where GCH holds.

(2) The usual definition of $\mathcal{AP}_{\kappa^+}$ is slightly different. The definition above highlights the principle $\mathcal{AP}_{\kappa^+}$ as a very weak version of $\square_\kappa$.

Note that in clause (3) of the definition, if one required γ to be a limit ordinal whenever β is a limit point of C_α then one has a $\square_\kappa$ sequence.

(3) One advantage of formulating $\mathcal{AP}_{\kappa^+}$ as in Definition 6.25 is that it gives the notion of a witness for $\mathcal{AP}_{\kappa^+}$ and we shall use this freely. $\square$

The following lemma implicit in the introductory remarks of [2] gives the two equivalent formulations of $\mathcal{AP}_{\kappa^+}$. We also refer the reader to [2] for an historical perspective noting that the $\mathcal{AP}$ family of principles have origin in prior work of Shelah [16].

Lemma 6.27 (Foreman–Magidor:[2]). *Suppose that κ is an infinite cardinal. Then the following are equivalent.*

(1) *$\mathcal{AP}_{\kappa^+}$ holds.*

(2) *For all*

$$X \prec H(\kappa^{++})$$

if $|X| = \kappa$ and $\kappa \subset X$ then there exists a closed cofinal subset $C \subset X \cap \kappa^+$ such that

(a) ordertype$(C) \leq \kappa$,

(b) *if* $\mathrm{cof}(X \cap \kappa^+) < \kappa$ *then* $|C| < \kappa$,

(c) $C \cap \xi \in X$ *for all* $\xi < \sup(X \cap \kappa^+)$.

Proof. We first show that (1) implies (2). Assume that $\mathcal{AP}_{\kappa^+}$ holds and that

$$X \prec H(\kappa^{++})$$

is such that both $|X| = \kappa$ and $\kappa \subset X$. Thus there exists a sequence

$$\mathcal{C} = \langle C_\alpha : \alpha < \kappa^+\rangle$$

such that $\mathcal{C}$ witnesses $\mathcal{AP}_{\kappa^+}$ and such that $\mathcal{C} \in X$. The key point is that there is an enumeration

$$\langle Z_\theta : \theta < \kappa^+\rangle \in X$$

of bounded subsets of κ^+ and closed unbounded set $D \subset \kappa^+$ such that

(1.1) $D \in X$,

(1.2) for all $\xi \in D$, for all $\beta < \xi$, $C_\xi \cap \beta \in \{Z_\theta \mid \theta < \xi\}$.

Let $C = C_{\alpha_0}$ where $\alpha_0 = X \cap \kappa^+$. Then

(2.1) ordertype$(C) \leq \kappa$,

(2.2) if $\mathrm{cof}(X \cap \kappa^+) < \kappa$ then $|C| < \kappa$,

(2.3) $C \cap \xi \in X$ for all $\xi < X \cap \kappa^+$.

This proves (2). Now suppose (2) holds and let

$$\mathcal{X} = \langle X_\eta : \eta < \kappa^+\rangle$$

be a continuous elementary chain such that for all $\eta < \kappa^+$:

(3.1) $X_\eta \prec H(\kappa^{++})$ and $X_\eta \in X_{\eta+1}$.

(3.2) $\kappa \subset X_\eta$ and $|X_\eta| = \kappa$.

For each $\eta < \kappa^+$, let $\alpha_\eta = X_\eta \cap \kappa^+$ and let C_{α_η} be a closed cofinal subset of α_η such that:

(4.1) ordertype$(C_{\alpha_\eta}) \leq \kappa$.

(4.2) If $\mathrm{cof}(X_\eta \cap \kappa^+) < \kappa$ then $|C_{\alpha_\eta}| < \kappa$.

(4.3) $C_{\alpha_\eta} \cap \xi \in X_\eta$ for all $\xi < X \cap \kappa^+$.

These sets exist by (2). Since $\mathcal{X}$ is continuous, for each limit $\eta < \kappa^+$, and for each $\xi < \eta$,

$$C_{\alpha_\eta} \cap \xi \in X_\theta$$

for all sufficiently large $\theta < \eta$. Thus the sequence

$$\langle C_{\alpha_\eta} : \eta < \kappa^+ \rangle$$

can easily be expanded to a sequence

$$\langle C_\alpha : \alpha < \kappa^+ \rangle$$

which witnesses $\mathcal{AP}_{\kappa^+}$. The relevant point here is that for each $\kappa < \eta_0 < \eta_1 < \kappa^+$, one can always choose a sequence

$$\langle D_\xi : \alpha_{\eta_0} < \xi < \alpha_{\eta_1} \rangle$$

which witnesses $\mathcal{AP}_{\kappa^+}$ on the interval $(\alpha_{\eta_0}, \alpha_{\eta_1})$. □

The following easy lemma shows that GCH is in some sense equivalent to soundness. Lemma 6.29, which we prove below, shows that GCH together with $\mathcal{AP}$ is in the same sense equivalent to soundness and amenability.

Lemma 6.28. *Suppose that $\mathbb{P} \subset \mathrm{Ord} \times V$. Then the following are equivalent.*

(1) $J[\mathbb{P}] \vDash \mathrm{GCH}$.

(2) *There exists $\mathbb{P}^* \subset \mathrm{Ord} \times V$ such that:*

  (a) *$J[\mathbb{P}^*]$ is sound.*

  (b) $J[\mathbb{P}] = J[\mathbb{P}^*]$.

  (c) *$\mathbb{P}^*$ is Σ_2-definable in $(J[\mathbb{P}], \mathbb{P})$.* □

Lemma 6.29 also shows that Theorem 6.16 and its generalization to the case of ω-weak amenable $J[\mathbb{P}]$ are equivalent and moreover just corollaries of the theorem of [16] that if the Approachability Property holds at $\kappa^{+(\omega+1)}$ (this is $\mathcal{AP}_\lambda$ where $\lambda = \kappa^{+(\omega+1)}$) then κ cannot be $\kappa^{+\omega}$-supercompact.

Lemma 6.29. *Suppose that $\mathbb{P} \subset \mathrm{Ord} \times V$ and that $J[\mathbb{P}] \vDash \mathrm{GCH}$. Then the following are equivalent.*

(1) *For each uncountable cardinal κ of $J[\mathbb{P}]$, $\mathcal{AP}_{\kappa^+}$ holds in $J[\mathbb{P}]$.*

(2) *There exists $\mathbb{P}^* \subset \mathrm{Ord} \times V$ such that:*

  (a) *$J[\mathbb{P}^*]$ is amenable and sound.*

  (b) $J[\mathbb{P}] = J[\mathbb{P}^*]$.

  (c) *$\mathbb{P}^*$ is Σ_2-definable in $(J[\mathbb{P}], \mathbb{P})$.*

(3) *There exists $\mathbb{P}^* \subset \mathrm{Ord} \times V$ such that:*

  (a) *$J[\mathbb{P}^*]$ is ω-weakly amenable and sound.*

  (b) $J[\mathbb{P}] = J[\mathbb{P}^*]$.

  (c) *$\mathbb{P}^*$ is Σ_2-definable in $(J[\mathbb{P}], \mathbb{P})$.*

Proof. We work in $(J[\mathbb{P}], \mathbb{P})$. It suffices to prove that (1) implies (2) and that (3) implies (1).

We first assume (1) and prove (2). Suppose κ is an uncountable cardinal and let

$$C = \langle C_\alpha : \kappa < \alpha < \kappa^+ \rangle$$

be the $(J[\mathbb{P}], \mathbb{P})$-least witness that $\mathcal{AP}_{\kappa^+}$ holds.

Let
$$D = \{\kappa < \alpha < \kappa^+ \mid (J_\alpha[\mathbb{P}], \mathbb{P}|\alpha, C|\alpha) \prec (J_{\kappa^*}[\mathbb{P}], \mathbb{P}|\kappa^+, C)\}.$$
Given $\mathbb{P}^*|\kappa$ such that
$$J_\kappa[\mathbb{P}^*] = H(\kappa),$$
define $\mathbb{P}^*|\kappa^+$ to be the $(J[\mathbb{P}], \mathbb{P})$-least set
$$\mathbb{H} \subset \kappa^+ \times J[\mathbb{P}]$$
such that

(1.1) $\mathbb{H}|\kappa = \mathbb{P}^*|\kappa$.

(1.2) If $(J_\eta[\mathbb{H}], \mathbb{H}|\eta) \nvDash \text{ZFC}\backslash\text{Powerset}$ then $\mathbb{H}_\eta = \emptyset$;

and such that the following hold for all $\kappa < \eta < \kappa^+$ such that
$$(J_\eta[\mathbb{H}], \mathbb{H}|\eta) \vDash \text{ZFC}\backslash\text{Powerset}.$$

(2.1) κ is the largest cardinal of $J_\eta[\mathbb{H}]$.

(2.2) Suppose $\text{cof}(\eta) < \text{cof}(\kappa)$. Then $\mathbb{H}_\eta$ is a cofinal closed subset of η such that $\text{ordertype}(\mathbb{H}_\eta) = \text{cof}(\eta)$ and such that $\mathbb{H}_\eta \cap \xi \in J_\eta[\mathbb{H}]$ for all $\xi < \eta$.

(2.3) Suppose $\text{cof}(\eta) = \text{cof}(\kappa)$. Let $\eta^* \in D$ be the least element of D above η. Then there is a set $E \subset \kappa$ which codes
$$(J_{\eta^*}[\mathbb{P}], \mathbb{P}|\eta^*, C|\eta^*)$$
and there are increasing continuous cofinal functions
$$f : \text{cof}(\kappa) \to \eta$$
and
$$g : \text{cof}(\kappa) \to \kappa$$
such that
$$\mathbb{H}_\eta = \{(f(\xi), g(\xi) \cap E) \mid \xi < \text{cof}(\kappa)\}.$$

(2.4) Suppose that $\text{cof}(\eta) > \text{cof}(\kappa)$. Then $\eta \in D$,
$$J_\eta[\mathbb{P}] \subseteq J_\eta[\mathbb{H}]$$
and $\mathbb{H}_\eta = C_\eta$.

Note that in (2.3), since GCH holds, since κ is the largest cardinal of $J_\eta[\mathbb{H}]$, and since $\text{cof}(\eta) = \text{cof}(\kappa)$, for each $\xi < \eta$ and for each $\gamma < \text{cof}(\kappa)$,
$$(J_\xi[\mathbb{H}])^\gamma \in J_\eta[\mathbb{H}].$$
Therefore the set $\mathbb{H}_\eta$ is necessarily amenable to $J_\eta[\mathbb{H}]$ where $\mathbb{H}_\eta$ is as defined in (2.3) but for any choice of (f, g) and for any set $E \subset \kappa$.

Also note the following regarding (2.4). Suppose $\kappa < \eta < \kappa^+$ and
$$(J_\eta[\mathbb{H}], \mathbb{H}|\eta) \vDash \text{ZFC}\backslash\text{Powerset}.$$
Then either $\text{cof}(\eta) = \omega$ or the set
$$\left\{\xi < \eta \mid (J_\xi[\mathbb{H}], \mathbb{H}|\xi) \prec (J_\eta[\mathbb{H}], \mathbb{H}|\eta)\right\}$$
is closed and cofinal in η.

With these two observations it follows that for any choice of $\mathbb{P}^*|\kappa$ such that
$$J_\kappa[\mathbb{P}^*] = H(\kappa),$$
$\mathbb{H}$ exists satisfying (1.1), (1.2), and satisfying (2.1)–(2.4) for all $\kappa < \eta < \kappa^+$.

It follows that $J[\mathbb{P}^*]$ is amenable and sound, and that $J[\mathbb{P}] = J[\mathbb{P}^*]$. Clearly $\mathbb{P}^*$ is Σ_2-definable in $(J[\mathbb{P}], \mathbb{P})$. This proves that (1) implies (2).

We now assume (3). Clearly we can just reduce to the case that $\mathbb{P} = \mathbb{P}^*$. Fix

$$X \prec (J_{\kappa^+}[\mathbb{P}], \mathbb{P}|\kappa^+)$$

such that $\kappa \subset X$ and $|X| = \kappa$. We prove that there exists $C \subset X \cap \kappa^+$ such that

(3.1) C is closed cofinal in $X \cap \kappa^+$,

(3.2) ordertype$(C) \leq \kappa$ and if $\mathrm{cof}(X \cap \kappa^+) < \kappa$ then $|C| < \kappa$,

(3.3) $C \cap \xi \in X$ for each $\xi < X \cap \kappa^+$.

By Lemma 6.27, this implies $\mathcal{AP}_{\kappa^+}$ holds.

Let $\gamma = \mathrm{cof}(\kappa)$. We can reduce to the case that

$$\mathrm{cof}(X \cap \kappa^+) > \gamma$$

for otherwise the existence of C is immediate since for each $\alpha \in X \cap \kappa^+$,

$$\mathcal{P}_\gamma(\alpha) \subset X.$$

Let $X \cap \kappa^+ < \alpha < \kappa^+$ be least such that

$$\rho_n^{\mathcal{M}} = \kappa$$

for some $n < \omega$ where $\mathcal{M} = (J_\alpha[\mathbb{P}], \mathbb{P}|\alpha, \mathbb{P}_\alpha)$ if $\alpha \in \mathrm{dom}(\mathbb{P})$ and $\mathcal{M} = (J_\alpha[\mathbb{P}], \mathbb{P}|\alpha)$ if $\alpha \notin \mathrm{dom}(\mathbb{P})$.

We can further reduce to the case that $\alpha \in \mathrm{dom}(\mathbb{P})$ and $n = 1$ since otherwise we can reduce to the case that $\mathcal{M}$ is an amenable structure in which case by the proof of Theorem 6.16, C exists satisfying (3.1)–(3.3).

The structure $\mathcal{M}$ is 1-sound and so

$$\mathcal{M} = \mathcal{H}_1^{\mathcal{M}}(\kappa \cup \{p\})$$

for some $p \in \mathcal{M}$. Thus there is a partial surjection

$$f : \kappa \to X \cap \kappa^+$$

such that f is generalized $(_{\mathcal{L}_{(\mathrm{gen})}})\Sigma_1$-definable in $\mathcal{M}$ from p. Arguing as in the proof of Theorem 6.30, for some $m < \omega$, the partial function

$$f_{(m)} : \kappa \to X \cap \kappa^+$$

has cofinal range and is generalized $(_{\mathcal{L}_{(\mathrm{gen})}})\Sigma_1$-definable in $\mathcal{M}_{(m)}$ from p where

$$\mathcal{M}_{(m)} = (J_\alpha[\mathbb{P}], \mathbb{P}|\alpha, \mathbb{P}_\alpha|m)$$

where

$$\mathbb{P}_\alpha|m = \mathbb{P}_\alpha \cap (m \times J_\alpha[\mathbb{P}]).$$

Here $f_{(m)}$ is simply f as defined in $\mathcal{M}_{(m)}$.

Since $\mathrm{cof}(\kappa) < \mathrm{cof}(X \cap \kappa^+) < \kappa$, there exists $\xi < \kappa$ such that range$(f_{(m)}|\xi)$ is cofinal in $X \cap \kappa^+$. Therefore letting $\delta = \mathrm{cof}(X \cap \kappa^+)$, there exist $q \in \mathcal{M}$ and a cofinal continuous function

$$g : \delta \to X \cap \kappa^+$$

such that g is generalized $(_{\mathcal{L}_{(\mathrm{gen})}})\Sigma_1$-definable in $\mathcal{M}_{(m)}$. We assume that m is as small as possible over all possible choices of $\mathbb{H}$ yielding

$$\mathcal{M}_{\mathbb{H}} = (J_\alpha[\mathbb{P}], \mathbb{P}|\alpha, \mathbb{H}),$$

such that

$$\mathbb{H} \subset m \times J_\alpha[\mathbb{P}]$$

and such that for each $k < m$ there exists $\theta < \alpha$ such that

(4.1) $\mathbb{H}_k \subset J_\theta[\mathbb{P}]$,

(4.2) $\mathbb{H}_k \cap J_\beta[\mathbb{P}] \in J_\alpha[\mathbb{P}]$ for all $\beta < \alpha$;

and relative to all possible generalized $({}_{\mathcal{L}_{(\text{gen})}})\Sigma_1$-formulas with parameters. If $m = 0$ then $\mathcal{M}_{\mathbb{H}}$ is amenable, and by the proof of Theorem 6.16, C exists satisfying (3.1)–(3.3). Therefore we can reduce to the case that $m > 0$.

For each $k < m$, let $\theta_k = \sup((\mathbb{P}_\alpha)_k)$ and let I be the set of all finite sequences $\langle \beta_k : k < m \rangle$ such that $\beta_k < \theta_k$ for each $k < m$. For each $a \in I$, let

$$\mathcal{M}_a = (J_\alpha[\mathbb{P}], \mathbb{P}|\alpha, \mathbb{P}_a)$$

where $\mathbb{P}_a$ is the set of all (k, b) such that $k < m$ and $b \in (\mathbb{P}_\alpha)_k|\beta_k$, and $a = \langle \beta_k : k < m \rangle$. Let g_a be g as interpreted in the structure $\mathcal{M}_a$. Thus:

(5.1) The set I is directed under the order $a < b$ if $a_i < b_i$ for all $i < m$.

(5.2) For each $\eta < \delta$, there exists $a \in I$ such $g_b(\eta) = g(\eta)$ for all $b > a$.

We claim:

(6.1) For each $k < m$, $\text{cof}(\theta_k) = \delta$.

Let

$$s = \{i < m \mid \text{cof}(\theta_k) < \delta\}$$

and let

$$t = \{i < m \mid \text{cof}(\theta_k) > \delta\}\,.$$

For each $\eta < \delta$, let $a_\eta \in I$ be such that

$$g(\eta) = g_a(\eta)$$

for all $a > a_\eta$. There must exist a cofinal set $A_0 \subset \delta$ and $c \in I$ such that for all $\eta \in A_0$ and for all $i \in s$,

$$(a_\eta)_i < c_i.$$

Therefore by the minimality of m, it follows that $s = \emptyset$. Similarly there must exist $d \in I$ such that for all $\eta < \delta$ and for all $i \in t$,

$$(a_\eta)_i < d_i.$$

This implies $t = \emptyset$, again by the minimality of m. This proves (6.1).

Note that by (5.2) and (6.1),

(7.1) For each $\eta < \delta$, there exists $a \in I$ such that $g|\eta = g_a|\eta$.

There are two cases.

Case 1: $\text{cof}(\alpha) \geq \delta$.

For each $a \in I$ and for each $\sup(a) < \beta < \alpha$, let $g_{(\beta,a)}$ be g as interpreted in the structure

$$\mathcal{M}_{(\beta,a)} = (J_\beta[\mathbb{P}], \mathbb{P}|\beta, \mathbb{P}_a)$$

where as above $\mathbb{P}_a$ is the set of all (k, b) such that $k < m$ and $b \in (\mathbb{P}_\alpha)_k|\beta_k$ and

$$\langle \beta_k : k < m \rangle = a.$$

This all makes perfect sense even though we cannot require that

$$(J_\beta[\mathbb{P}], \mathbb{P}|\beta) \vDash \text{Comprehension}$$

which we have generally imposed at active stages.

Since $\text{cof}(\alpha) \geq \delta$ it follows that for each $\eta < \delta$ there exists $a \in I$ and $\beta < \alpha$ such that

$$g|\eta = g_{(\beta,a)}|\eta.$$

For each $a \in I$, $\mathcal{M}_{(\beta,a)} \in J_\alpha[\mathbb{P}]$, and so for each $\eta < \delta$,

$$g[\eta] \in J_\alpha[\mathbb{P}].$$

Therefore letting $C = g[\delta]$, C witnesses (3.1)–(3.3).

Case 2: $\mathrm{cof}(\alpha) < \delta$.

Since $\mathrm{cof}(\alpha) < \delta$ there must exist $\beta_0 < \alpha$ such that for all $k < m$,

(8.1) $\theta_k < \beta_0$,

(8.2) $(\mathbb{P})_k \cap J_\xi[\mathbb{P}] \in J_{\beta_0}[\mathbb{P}]$ for all $\xi < \theta_k$.

Again since $\mathrm{cof}(\alpha) < \delta$, there must exist limit ordinal β such that, $\beta_0 < \beta < \alpha$, an elmement $r \in J_\beta[\mathbb{P}]$, and a cofinal increasing continuous function

$$h : \delta \to X \cap \kappa^+$$

such that h is generalized $(_{\mathcal{L}_{(\mathrm{gen})}})\Sigma_1$-definable from r in the structure

$$(J_\beta[\mathbb{P}], \mathbb{P}|\beta, \mathbb{P}_\alpha|m).$$

For each $a \in I$, let h_a be h as interpreted in the structure

$$\mathcal{M}_{(\beta,a)} = (J_\beta[\mathbb{P}], \mathbb{P}|\beta, \mathbb{P}_a)$$

where $\mathbb{P}_a$ is as defined above.

Thus exactly as for g and $\mathcal{M}_{(m)}$:

(9.1) For each $\eta < \delta$, there exists $a \in I$ such $h_b(\eta) = h(\eta)$ for all $b > a$.

By (6.1) and (9.1):

(10.1) For each $\eta < \delta$, there exists $a \in I$ such that $h|\eta = h_a|\eta$.

Finally exactly as in Case 1, for each $a \in I$, $\mathcal{M}_{(\beta,a)} \in J_\alpha[\mathbb{P}]$, and so for each $\eta < \delta$,

$$h[\eta] \in J_\alpha[\mathbb{P}].$$

Therefore letting $C = h[\delta]$, C witnesses (3.1)–(3.3). □

As an immediate corollary we obtain the following generalization of Theorem 6.16.

Theorem 6.30. *Suppose that $\mathbb{P} \subset \mathrm{Ord} \times V$ and that $J[\mathbb{P}]$ is ω-weakly amenable and sound. Then*

$$J[\mathbb{P}] \vDash \text{“There are no cardinals } \kappa \text{ which are } \kappa^{+\omega}\text{-supercompact”}.$$

Proof. By Lemma 6.29 there exists $\mathbb{P}^* \subset \mathrm{Ord} \times V$ such that:

(1.1) $J[\mathbb{P}^*]$ is amenable and sound.

(1.2) $J[\mathbb{P}] = J[\mathbb{P}^*]$.

(1.3) $\mathbb{P}^*$ is Σ_2-definable in $(J[\mathbb{P}], \mathbb{P})$.

The theorem is an immediate corollary of this by Theorem 6.16. □

6.4 Good partial extender sequences revisited

Definition 6.31. Suppose that $\mathbb{E}$ is a good partial extender sequence and $\alpha \in \mathrm{dom}(\mathbb{E})$. Let E be the $J_\alpha[\mathbb{E}]$-extender given by $\mathbb{E}$ at α and let

$$j_E : J_\alpha[\mathbb{E}] \to \mathrm{Ult}_0(J_\alpha[\mathbb{E}], E)$$

be the ultrapower embedding. Then $\mathbb{E}$ satisfies the *weak initial segment condition* at α if

$$E|j_E(\eta) \in J_\alpha[\mathbb{E}]$$

for all $\eta < \iota_E$. □

The weak initial segment condition at α is equivalent to a level of supercompactness for the associated extender E.

Lemma 6.32. *Suppose that $\mathbb{E}$ is a good partial extender sequence and $\alpha \in \mathrm{dom}(\mathbb{E})$. Let E be the $J_\alpha[\mathbb{E}]$-extender given by $\mathbb{E}$ at α and let*

$$j_E : J_\alpha[\mathbb{E}] \to \mathrm{Ult}_0(J_\alpha[\mathbb{E}], E)$$

be the ultrapower embedding. Then the following are equivalent.

(1) $\mathbb{E}$ *satisfies the weak initial segment condition at* α.

(2) *Suppose $\iota < \iota_E$ is a cardinal of $J_\alpha[\mathbb{E}]$ and $\gamma = (\iota^+)^{J_\alpha[\mathbb{E}]}$. Then $j_E[\gamma] \in \mathrm{Ult}_0(J_\alpha[\mathbb{E}], E)$.*

Proof. We first assume (1) and prove (2). Let $\eta = j_E(\iota)$. Thus

$$E|j_E(\eta) \in J_\alpha[\mathbb{E}]$$

and

$$J_\alpha[\mathbb{E}] = \mathrm{Ult}_0(J_\alpha[\mathbb{E}], E)|\alpha.$$

By the elementarity of j_E,

$$\mathrm{Ult}_0(J_\alpha[\mathbb{E}], E) \vDash \mathrm{ZFC}\backslash\mathrm{Powerset}$$

and thus (2) holds.

Now assume (2) holds and fix $\eta < \iota_E$. Let $\iota = |\eta|^{J_\alpha[\mathbb{E}]}$ and let $\gamma = (\iota^+)^{J_\alpha[\mathbb{E}]}$. Thus by (2),

$$j_E[\gamma] \in \mathrm{Ult}_0(J_\alpha[\mathbb{E}], E).$$

But this implies that

$$j_E[J_\gamma[\mathbb{E}]] \in \mathrm{Ult}_0(J_\alpha[\mathbb{E}], E)$$

since

$$\mathrm{Ult}_0(J_\alpha[\mathbb{E}], E)|\alpha = J_\alpha[\mathbb{E}].$$

By the strong acceptability of $J_\alpha[\mathbb{E}]$,

$$\mathcal{P}(\iota) \cap J_\alpha[\mathbb{E}] \subset J_\gamma[\mathbb{E}]$$

and so it follows that

$$E|j_E(\eta) \in \mathrm{Ult}_0(J_\alpha[\mathbb{E}], E).$$

Finally by the strong acceptability of $\mathrm{Ult}_0(J_\alpha[\mathbb{E}], E)$ and since

$$\alpha = j_E(\lambda)$$

where $\lambda = (\iota_E^+)^{J_\alpha[\mathbb{E}]}$, this implies that $E|j_E(\eta) \in J_\alpha[\mathbb{E}]$. □

Lemma 6.33. *Suppose that $\mathbb{E}$ is a good partial extender sequence which satisfies the weak initial segment condition at all $\alpha \in \mathrm{dom}(\mathbb{E})$. Then for all $\alpha \in \mathrm{dom}(\mathbb{E})$, $\mathbb{E}_\alpha$ can be coded by a set E such that for some $\beta \leq \alpha$,*

(1) *Either* $(J_\beta[\mathbb{E}], \mathbb{E}|\beta) \vDash$ ZFC\Powerset *or* $(J_\beta[\mathbb{E}], \mathbb{E}|\beta) \vDash$ ZFC\Replacement.

(2) $E \subset J_\beta[\mathbb{E}]$ *and* $E \cap J_\xi[\mathbb{E}] \in J_\beta[\mathbb{E}]$ *for all* $\xi < \beta$.

(3) $(J_\beta[\mathbb{E}], E)$ *and* $(J_\alpha[\mathbb{E}], \mathbb{E}_\alpha)$ *are logically equivalent.* □

The main theorems from [24] actually yield amenable structures at the finite levels of supercompactness because of satisfying the weak initial segment condition.

These can easily be extended to the levels of ω-extendible cardinals where one obtains the close version of amenability indicated in the previous lemma. But since the focus of [24] is the finite levels of supercompact, we restrict the statement of Theorem 6.35 to such levels as well.

Remark 6.34. In some sense, Theorem 6.35 represents, modulo the iteration hypothesis, the strongest possible result for the extent of a fine-structural hierarchy of inner models where some version of amenability holds at all active stages.

Thus the finite levels of supercompactness emerges as a canonical and natural threshold within the large cardinal hierarchy beyond which a new approach is required for the construction of fine-structural inner models, [25]. □

The results of [24] combined with those of [25] yield the following theorem. Just as for Theorem 5.20, the only use here of the results of [25] is to reduce the iteration hypothesis to the Weak Unique Branch Iteration Hypothesis.

Theorem 6.35 (Weak Unique Branch Hypothesis). *Assume that for each* $m < \omega$*, there is a proper class of* m*-extendible cardinals. Then there exists a good partial extender sequence* $\mathbb{E} = \langle \mathbb{E}_\alpha : \alpha \in \mathrm{dom}(\mathbb{E}) \rangle$ *such that the following hold.*

(1) $J[\mathbb{E}]$ *is weakly backgrounded and* $L[\mathbb{E}]$ *is weakly* Σ_2*-definable.*

(2) $J[\mathbb{E}]$ *satisfies comparison.*

(3) *For each* ξ *and for each* $m < \omega$*, there exists* $\alpha \in \mathrm{dom}(\mathbb{E})$ *such that*

  (a) $\alpha > \xi$,

  (b) $\mathbb{E}_\alpha$ *is a* $J[\mathbb{E}]$*-extender which witnesses that* κ *is* m*-extendible in* $J[\mathbb{E}]$ *where* $\kappa = \kappa_{\mathbb{E}_\alpha}$.

(4) $\mathbb{E}$ *satisfies the weak initial segment condition at all* $\alpha \in \mathrm{dom}(\mathbb{E})$. □

6.5 Weak extender models and comparison

The definition of comparison, Definition 5.17, can naturally be generalized to arbitrary inner models. There are many possible versions and a natural rather weak version is as follows where the notion of close embedding is as given in Definition 5.29 but applied to all elementary embeddings, $j : M \to N$, where M and N are transitive models of ZFC.

Definition 6.36. Suppose that N is a weak extender model for δ is supercompact and that

$$N \vDash \text{“}V = \mathrm{HOD}\text{”}.$$

Then N satisfies *weak comparison* if for all $X, Y \prec_{\Sigma_2} N$ the following hold where N_X is the transitive collapse of X and N_Y is the transitive collapse of Y.

Suppose that N_X and N_Y are finitely generated models of ZFC, $N_X \neq N_Y$, and

$$N_X \cap \mathbb{R} = N_Y \cap \mathbb{R}.$$

Then there exist a transitive set N^* and elementary embeddings

$$\pi_X : N_X \to N^*$$

and

$$\pi_Y : N_Y \to N^*$$

such that π_X is close to N_X and π_Y is close to N_Y. □

Remark 6.37. The elementary embeddings witnessing weak comparison are not required to have any special form. Thus suppose there is an elementary embedding

$$\pi : N_X \to N_Y$$

such that π is close to N_X and that N^* is the ultrapower of N_Y by a countably complete non-principal ultrafilter of N. Then there trivially exist elementary embeddings

$$\pi_X : N_X \to N^*$$

and

$$\pi_Y : N_Y \to N^*$$

such that π_X is close to N_X and π_Y is close to N_Y.

Thus one can require that the elementary embeddings witnessing weak comparison each be nontrivial. □

The conclusion of weak comparison is downward absolute to N and moreover the definition of weak comparison can be applied with $N = V$ provided there is a supercompact cardinal in V and that $V = \text{HOD}$.

Thus the following is a natural test question for the existence of a generalization of L at the level of supercompact cardinals based on anything like the current methodology for the construction of such inner models.

Question 6.38. *Suppose that there is a supercompact cardinal and that* $V = \text{HOD}$. *Can weak comparison hold?*

The results we have discussed arguably show that for the construction of fine structural extender models which are weakly backgrounded, going beyond the level of ω-extendible cardinals requires:

(1) Allowing the weak initial segment condition to fail.

(2) Allowing levels which are not even ω-weakly amenable.

(3) Altering how comparison is proved if one can provably (from some large cardinal hypothesis) reach the level a weak extender model for supercompactness

Therefore we cannot just use good partial extender sequences to generate these models with anything remotely like the current methodology. A natural alternative is to augment extender models with their iteration strategies. These are *strategic-extender* models.

7 Epilogue

The emerging picture based on the results we have surveyed in the previous sections is that in generalizing inner models to level of weak extender models for supercompactness, one must not only have failures of the weak initial segment condition but moreover that the hierarchy cannot consist of models constructed from *just* a sequence of partial extenders.

The only alternative at present is an inner model theory based on a *strategic* premice–this is a hierarchy of structures constructed from both a partial extender sequence and the iteration strategy for the initial segments of the structure–for backgrounded structures this would be the iteration strategy inherited from V assuming (some version of) the Weak Unique Branch Hypothesis.

Of course if one must pass to the strategic hierarchy then the strategic-extender structures can no longer be *layered* and so a fundamentally new approach to the strategic-extender hierarchy is required compared to the current approach, [15]. This is all the subject of [25].

Remark 7.1. From the broader perspective here is the picture which is emerging.

(1) At the lowest levels, reaching past measurable cardinals, the fine-structure models can be simply defined (at reasonable closure points) and there is no distinction between the extender and strategic-extender hierarchies.

(2) Ascending to levels below that of one Woodin cardinal there is still no distinction (again at reasonable closure points) between the extender and strategic-extender hierarchies.

(3) Passing one Woodin cardinal and up to the finite levels of supercompactness, the extender and strategic-extender hierarchies strongly diverge but both exist.

(4) Reaching the infinite levels of supercompactness requires a complete failure of amenability and moreover at some point past the finite levels of supercompact, the extender hierarchy fails and one is left with just the strategic-extender hierarchy. □

The following summarizes, in more detail and within the context of the obstructions identified in the previous sections, the approach of [25] and the key issue is what happens with the iterability problem.

The coding obstruction is handled by ensuring that at all extender-active stages, the projectum is at most the image of the critical point. This strategy has already been used in unpublished work of Steel and Neeman-Steel. This is discussed at length in [24].

This approach also facilitates the comparison proof and so there are a number of reasons to take this path.

The amenability obstruction is handled because in meeting the coding obstruction one must allow the weak initial segment condition to fail, and so one is forced to allow structures which are not even ω-weakly amenable at their extender-active stages. These structures arise naturally in the backgrounded construction and previous approaches would attempt to circumvent them.

The comparison obstruction is handled because one compares (in the general case), suitably iterable structures against backgrounded structures restricting to the situation where the backgrounded structure *does not move*. This is forced by passing into the hierarchy of strategic-extender structures. Steel has independently realized that this methodology is the key to extending the fine-structure theory of nonstrategic-extender models to the hierarchy of strategic-extender models, and he has developed the machinery for this.

Finally one is forced into the strategic-extender hierarchy in order to connect with AD^+ for the proof of iterability (which is an induction). The proof of iterability is only possible by connecting with the general theory of AD^+-models and that connection does not exist in the nonstrategic case. This connection is through the HOD's as computed within the AD^+-models. These models are already known (in many cases) to be strategic-extender models and known under fairly general conditions to *never* be nonstrategic-extender (or pure extender) models.

The fundamental reason for the necessity of this methodology is the following. If one assumes iterability, for example the Weak Unique Branch Hypothesis, and that there is a huge cardinal then there is nothing which prevents the construction of the *much simpler* nonstrategic-extender models up to the level where one violates the comparison obstruction, Theorem 5.35.

Therefore the Weak Unique Branch Hypothesis *must be false* and the construction is vacuous. Further in this case, the only credible possibility which remains is that iterability is proved by induction and *not* on the basis of some general iteration hypothesis for V.

Verifying this is in fact what happens is the main task ahead [25]. In particular it is necessary to verify that there are no further hidden obstructions and that can only be done by carefully working through all the details.

As conjectured in [20], one can formulate the axiom, $V = \text{Ultimate-}L$, based on the strategic-extender models, without referring to the detailed fine-structure theory of these models, or even using the definition of the structures yielding the levels of the models.

The main point here is that in the context of a proper class of Woodin cardinals, there are naturally defined approximations to Ultimate-L and the collection is rich enough to make a definiton of the axiom, $V = \text{Ultimate-}L$, possible without specifying the detailed level-by-level definition of Ultimate-L.

The approximations form a hierarchy and it has been verified that for an initial segment of the hierarchy, the approximations *are* strategic-extender models. The conjecture of course is that all the approximations are strategic-extender models and there is quite a bit of evidence for this conjecture. However this is not the key issue.

The key issue is whether the axiom $V = \text{Ultimate-}L$ formulated in terms of these approximations must necessarily hold in *some* weak extender model for supercompactness assuming that there is an extendible cardinal. Presumably any proof of this must yield as a corollary that these approximations are all strategic-extender models.

Before giving the requisite preliminary definitions, we note that the situation is analogous to being able to formulate the axiom, $V = L$, without specifying the definition of L. This is easily done as illustrated by the following lemma.

Lemma 7.2. *The following are equivalent.*

(1) $V = L$.

(2) *For each Σ_2-sentence ϕ, if $V \vDash \phi$ then there exists a countable ordinal α such that*

$$N \vDash \phi$$

where $N = \cap\left\{M \mid M \vDash \text{ZFC}\backslash\text{Powerset} \text{ and } \text{Ord}^M = \alpha\right\}$. □

We recall the definition of the universally Baire sets, [1].

Definition 7.3 (Feng-Magidor-Woodin). A set $A \subseteq \mathbb{R}$ is *universally Baire* if for all topological spaces Ω and for all continuous functions $\pi : \Omega \to \mathbb{R}$, the preimage of A by π has the property of Baire in the space Ω. □

If there is a proper class of Woodin cardinals then the collection of the universally Baire sets has very strong closure properties. Large cardinal hypotheses are necessary for this since for example, if $V = L$ then every set $A \subset \mathbb{R}$ is the image of a universally Baire set by a continuous function, $f : \mathbb{R} \to \mathbb{R}$.

Theorem 7.4. *Suppose that there is a proper class of Woodin cardinals and that $A \subseteq \mathbb{R}$ is universally Baire. Then every set*

$$B \in \mathcal{P}(\mathbb{R}) \cap L(A, \mathbb{R})$$

is universally Baire. □

Theorem 7.4 combined with the seminal Martin-Steel Theorem [10], which shows that if $A \subset \mathbb{R}$ is universally Baire and there is a Woodin cardinal with a measurable cardinal above, then A is determined, one obtains the following theorem which is central to analyzing the structure of the universally Baire sets.

The axiom, AD^+, is a technical variation of the axiom, AD, which asserts that all sets $A \subset \mathbb{R}$ are determined. While it remains an interesting open question whether AD^+ and AD are equivalent (over $\mathrm{ZF} + \mathrm{DC}_{\mathbb{R}}$), the AD-models of interest are all AD^+-models.

Theorem 7.5. *Suppose that there is a proper class of Woodin cardinals and that $A \subseteq \mathbb{R}$ is universally Baire. Then*

$$L(A, \mathbb{R}) \vDash \mathrm{AD}^+.$$ □

Definition 7.6. Suppose that $A \subseteq \mathbb{R}$ is universally Baire. Then $\Theta^{L(A,\mathbb{R})}$ is the supremum of the ordinals α such that there is a surjection, $\pi : \mathbb{R} \to \alpha$, such that $\pi \in L(A, \mathbb{R})$. □

If $A \subset \mathbb{R}$ is universally Baire and there is a proper class of Woodin cardinal then $\Theta^{L(A,\mathbb{R})}$ is a measure of the complexity of A.

The connection with inner models for large cardinals begins with the following theorem.

Theorem 7.7. *Suppose that there is a proper class of Woodin cardinals and that A is universally Baire. Then $\Theta^{L(A,\mathbb{R})}$ is a Woodin cardinal in* $\mathrm{HOD}^{L(A,\mathbb{R})}$. □

For the formulation of $V =$ Ultimate-L we give and the analysis we shall do, it is convenient to use the following notation from the theory of AD^+. The definition of the *Solovay Sequence* originates in [17].

Definition 7.8 ($\mathrm{ZF} + \mathrm{AD}^+$). (1) Θ denotes the supremum of the set of $\alpha \in \mathrm{Ord}$ such that there is a surjection $\pi : \mathbb{R} \to \alpha$.

(2) (Solovay Sequence) $\langle \Theta_\alpha : \alpha \leq \Omega \rangle$ is the sequence defined by induction on α as follows.

a) Θ_0 is the supremum of the set of $\xi \in \mathrm{Ord}$ such that there is a surjection $\pi : \mathbb{R} \to \xi$ such that π is OD.

b) $\Theta_{\alpha+1}$ the supremum of the set of $\xi \in \mathrm{Ord}$ such that there is a surjection $\pi : \mathcal{P}(\Theta_\alpha) \to \xi$ such that π is OD.

c) $\Theta_\alpha = \sup \left\{ \Theta_\beta \mid \beta < \alpha \right\}$ if α is a nonzero limit ordinal.

d) $\Theta = \Theta_\Omega$. □

Remark 7.9. Assume AD^+ and that $V = L(\mathcal{P}(\mathbb{R}))$. Let

$$\langle \Theta_\alpha : \alpha \leq \Omega \rangle$$

be the Solovay sequence. Suppose that $\alpha \leq \Omega$ and that either $\alpha = 0$ or α is not a limit ordinal. Then the following hold.

(1) Θ_α is a Woodin cardinal in HOD.

(2) Let δ be the largest Suslin cardinal such that $\delta < \Theta_\alpha$. Then δ is a strong cardinal in $\mathrm{HOD} \cap V_{\Theta_\alpha}$.

Thus if $V = L(A, \mathbb{R})$ and the largest Suslin cardinal is on the Solovay sequence then it must be both a limit of Woodin cardinals and a strong cardinal in $\mathrm{HOD}^{L(A,\mathbb{R})} \cap V_\Theta$. □

We fix some notation to simplify various statements.

Definition 7.10. Assume there is a proper class of Woodin cardinals.

(1) Γ^∞ is the set of all universally Baire sets.

(2) $\Gamma \lhd \Gamma^\infty$ if the following hold.

a) $\Gamma \subsetneq \Gamma^\infty$ and $\Gamma = \mathcal{P}(\mathbb{R}) \cap L(\Gamma, \mathbb{R})$,

b) $L(\Gamma, \mathbb{R}) \vDash \neg \mathrm{AD}_{\mathbb{R}}$. □

We note the following lemma from the basic theory of AD^+. With notation as in this lemma, $\Theta^{L(\Gamma,\mathbb{R})}$ is a Woodin cardinal in $\mathrm{HOD}^{L(\Gamma,\mathbb{R})}$ and δ is a strong cardinal in

$$\mathrm{HOD}^{L(\Gamma,\mathbb{R})}|\Theta^{L(\Gamma,\mathbb{R})},$$

where δ is the largest Suslin cardinal of $L(\Gamma, \mathbb{R})$.

Lemma 7.11. *Suppose there is a proper class of Woodin cardinals and that $\Gamma \lhd \Gamma^\infty$. Then there is a largest Suslin cardinal in $L(\Gamma, \mathbb{R})$.* □

The following theorems are from [23]. These theorems connect aspects of the large cardinal structure of the HOD of an AD^+ which satisfies $V = L(\mathcal{P}(\mathbb{R}))$, with the structure of the Suslin cardinals in that determinacy model.

Theorem 7.12. *Suppose there is a proper class of Woodin cardinals, $\Gamma \subsetneq \Gamma^\infty$, and that*

$$\Gamma = \mathcal{P}(\mathbb{R}) \cap L(\Gamma, \mathbb{R}).$$

Then $\Gamma \lhd \Gamma^\infty$ if and only if $\Theta^{L(\Gamma,\mathbb{R})}$ is a Woodin cardinal in $\mathrm{HOD}^{L(\Gamma,\mathbb{R})}$. □

Theorem 7.13. *Suppose there is a proper class of Woodin cardinals and that $\Gamma \lhd \Gamma^\infty$. Let δ be the largest Suslin cardinal of $L(\Gamma, \mathbb{R})$. Then the following are equivalent.*

(1) *δ is a limit of Woodin cardinals in* $\mathrm{HOD}^{L(\Gamma,\mathbb{R})}$.

(2) $\delta < \Omega^{L(\Gamma,\mathbb{R})}$ *and* $\delta = (\Theta_\delta)^{L(\Gamma,\mathbb{R})}$. □

Assume there are infinitely many Woodin cardinals with a measurable cardinal above them all (for example assume there is a proper class of Woodin cardinals). Then for many universally Baire sets $A \subset \mathbb{R}$, the inner model,

$$\mathrm{HOD}^{L(A,\mathbb{R})},$$

has been verified to be a strategic-extender model. The natural conjecture is that (assuming there are infinitely many Woodin cardinals with a measurable cardinal above) this must be true for *all* universally Baire sets.

This suggests how to formulate the axiom $V = \text{Ultimate-}L$ and the following is the formulation of the axiom $V = \text{Ultimate-}L$ implicitly defined in [20], except that the large cardinal hypothesis is altered.

Definition 7.14 ($V = \text{Ultimate-}L$). (1) There is a proper class of Woodin cardinals.

(2) For each Σ_2-sentence ϕ, if ϕ holds in V then there exists a universally Baire set $A \subseteq \mathbb{R}$ such that

$$\mathrm{HOD}^{L(A,\mathbb{R})} \vDash \phi$$ □

The following version of the axiom is given in [22].

Axiom 1. (1) There is a proper class of Woodin cardinals.

(2) There is a proper class of strong cardinals.

(3) For each Σ_4-sentence ϕ, if ϕ holds in V then there exists $\Gamma \lhd \Gamma^\infty$ such that

$$\mathrm{HOD}^{L(\Gamma,\mathbb{R})} \cap V_\Theta \vDash \phi$$

where $\Theta = \Theta^{L(\Gamma,\mathbb{R})}$. □

This version of the axiom implies the following intermediate version which therefore became an elegant candidate for the formulation of V = Ultimate-L.

Axiom 2. (1) There is a strong cardinal which is a limit of Woodin cardinals.

(2) For each Σ_3-sentence ϕ, if ϕ holds in V then there exists a universally Baire set $A \subseteq \mathbb{R}$ such that

$$\mathrm{HOD}^{L(A,\mathbb{R})} \cap V_\Theta \vDash \phi$$

where $\Theta = \Theta^{L(A,\mathbb{R})}$. □

The main motivation behind these variations was the intuition that if V = Ultimate-L then the models,

$$\mathrm{HOD}^{L(\Gamma,\mathbb{R})} \cap V_\Theta$$

where $\Theta = \Theta^{L(\Gamma,\mathbb{R})}$ and $\Gamma \lhd \Gamma^\infty$, should resemble V if Γ is sufficiently closed. We shall see below however that this is very likely impossible.

A more pragmatic motivation for considering these kinds of variations is simply that with more reflection and a stronger large cardinal hypothesis, the consequences, such as those indicated in Theorem 7.26 below, are easier to obtain.

On the other hand, one clearly wants a version which can hold in a weak extender model for supercompactness.

For Σ_2-sentences there is no difference in formulating V = Ultimate-L in terms of reflecting to $\mathrm{HOD}^{L(A,\mathbb{R})}$ versus reflecting to

$$\mathrm{HOD}^{L(A,\mathbb{R})} \cap V_\Theta$$

where $\Theta = \Theta^{L(A,\mathbb{R})}$.

Lemma 7.15. *Suppose that there is a proper class of Woodin cardinals. Then the following are equivalent.*

(1) *For each Σ_2-sentence ϕ, if ϕ holds in V then there exists a universally Baire set $A \subseteq \mathbb{R}$ such that*

$$\mathrm{HOD}^{L(A,\mathbb{R})} \vDash \phi.$$

(2) *For each Σ_2-sentence ϕ, if ϕ holds in V then there exists a universally Baire set $A \subseteq \mathbb{R}$ such that*

$$\mathrm{HOD}^{L(A,\mathbb{R})} \cap V_\Theta \vDash \phi$$

where $\Theta = \Theta^{L(A,\mathbb{R})}$.

Proof. Let δ_A be the largest Suslin cardinal of $L(A,\mathbb{R})$. By the general theory of AD^+, there exists a set $T \subset \delta_A$ such that in $L(A,\mathbb{R})$ every set is OD with parameters from $\{T\} \cup \mathbb{R}$.

This implies by Vopenka's Theorem adapted to $L(A,\mathbb{R})$, that there exists a set $X \subset \Theta^{L(A,\mathbb{R})}$ such that

$$\mathrm{HOD}^{L(A,\mathbb{R})} = L[X].$$

Let T_0 be the theory, ZFC\Replacement together with Σ_1-Replacement and the sentence which asserts that for all $Z \subset \mathrm{Ord}$, $Z^\#$ exists. Then for all $\alpha \in \mathrm{Ord}$, if

$$\mathrm{HOD}^{L(A,\mathbb{R})} \cap V_\alpha \vDash T_0,$$

necessarily $\alpha \leq \Theta^{L(A,\mathbb{R})}$. The lemma follows easily from this. □

Lemma 7.15 is false for Π_2-sentences and this claim follows easily from the proof of Lemma 7.15 since that proof shows that if for every set $Y \subset \text{Ord}$, $Y^\#$ exists, then there is a Π_2-sentence which holds in V and in $\text{HOD}^{L(A,\mathbb{R})} \cap V_\Theta$ where $\Theta = \Theta^{L(A,\mathbb{R})}$, but which cannot hold in $\text{HOD}^{L(A,\mathbb{R})}$.

This suggests the kinds of variations in the formulation of $V = \text{Ultimate-}L$ indicated above and the following lemma motivates the second formulation, given above as Axiom 2, since it shows that Axiom 1 (even weakened to just Σ_3-sentences) implies Axiom 2.

Lemma 7.16. *Suppose there is a proper class of Woodin cardinals, $\Gamma \lhd \Gamma^\infty$, and that for all $A \in \Gamma$, $(A, \mathbb{R})^\# \in L(\Gamma, \mathbb{R})$. Then there exists $A \in \Gamma$ such that for all Σ_3-sentences ϕ, if*

$$\text{HOD}^{L(\Gamma,\mathbb{R})} \cap V_{\Theta_\Gamma} \vDash \phi$$

then

$$\text{HOD}^{L(A,\mathbb{R})} \cap V_{\Theta_A} \vDash \phi$$

where $\Theta_A = \Theta^{L(A,\mathbb{R})}$ and $\Theta_\Gamma = \Theta^{L(\Gamma,\mathbb{R})}$.

Proof. Let δ be the largest Suslin cardinal of $L(\Gamma, \mathbb{R})$ and fix $\delta < \eta_0 < \Theta_\Gamma$ such that

$$\text{HOD}^{L(\Gamma,\mathbb{R})} \cap V_{\eta_0} \prec \text{HOD}^{L(\Gamma,\mathbb{R})} \cap V_{\Theta_\Gamma}$$

Note that η_0 exists since Θ_Γ is strongly inaccessible in $\text{HOD}^{L(\Gamma,\mathbb{R})}$. Let $A \in \Gamma$ be such that

$$\max(\delta, \eta_0) < \Theta_A$$

where $\Theta_A = \Theta^{L(A,\mathbb{R})}$. By the Moschovakis Coding Lemma, [13], δ is the largest Suslin cardinal of $L(A, \mathbb{R})$ and so by the general theory of AD^+ (and in particular by the proof that δ is a strong cardinal in $\text{HOD}^{L(\Gamma,\mathbb{R})} \cap V_{\Theta_\Gamma}$), it follows that

$$\text{HOD}^{L(A,\mathbb{R})} \cap V_{\Theta_A} = \text{HOD}^{L(\Gamma,\mathbb{R})} \cap V_{\Theta_A}.$$

This implies that for all Σ_3-sentences ϕ, if

$$\text{HOD}^{L(\Gamma,\mathbb{R})} \cap V_{\Theta_\Gamma} \vDash \phi$$

then

$$\text{HOD}^{L(A,\mathbb{R})} \cap V_{\Theta_A} \vDash \phi$$

and so A witnesses the lemma. □

The following lemma also holds for the variation of $V = \text{Ultimate-}L$ given above as Axiom 2, and the proof is the same.

Lemma 7.17. *Suppose there is a proper class of Woodin cardinals, $\Gamma \lhd \Gamma^\infty$, and that for all $A \in \Gamma$, $(A, \mathbb{R})^\# \in L(\Gamma, \mathbb{R})$. Suppose that*

$$\text{HOD}^{L(\Gamma,\mathbb{R})} \cap V_{\Theta_\Gamma} \vDash \text{“There is a proper class of Woodin cardinals”}$$

Then

$$\text{HOD}^{L(\Gamma,\mathbb{R})} \cap V_{\Theta_\Gamma} \vDash \text{“}V = \text{Ultimate-}L\text{”}$$

where $\Theta_\Gamma = \Theta^{L(\Gamma,\mathbb{R})}$.

Proof. Fix a Σ_2-sentence ϕ such that

$$\text{HOD}^{L(\Gamma,\mathbb{R})} \cap V_{\Theta_\Gamma} \vDash \phi.$$

By Lemma 7.16, there exists $A \in \Gamma$ such that

$$\text{HOD}^{L(A,\mathbb{R})} \cap V_{\Theta_A} \vDash \phi.$$

Thus by the Δ^2_1-Basis Theorem, there exists in $L(\Gamma, \mathbb{R})$ a Δ^2_1-set Z which codes $(X, \mathbb{R})^\#$ where $X \subseteq \mathbb{R}$ is such that

$$\text{HOD}^{L(X,\mathbb{R})} \cap V_{\Theta_X} \vDash \phi$$

and $\Theta_X = \Theta^{L(X,\mathbb{R})}$. Since Σ^2_1 has the scale property in $L(\Gamma, \mathbb{R})$,

$$(V_{\omega+1} \cap \mathrm{HOD}^{L(\Gamma,\mathbb{R})}, Z \cap \mathrm{HOD}^{L(\Gamma,\mathbb{R})}) \prec (V_{\omega+1}, Z)$$

and again by the scale property for Σ^2_1, $Z \cap \mathrm{HOD}^{L(\Gamma,\mathbb{R})}$ is universally Baire in

$$\mathrm{HOD}^{L(\Gamma,\mathbb{R})} \cap V_{\Theta_\Gamma}.$$

Therefore $X \cap \mathrm{HOD}^{L(\Gamma,\mathbb{R})}$ witnesses the necessary instance of reflection. □

Remark 7.18. Suppose there is a proper class of Woodin cardinals which are limits of Woodin cardinals. Then by the results of Sargsyan, there exists $\Gamma \lhd \Gamma^\infty$, such that for all $A \in \Gamma$, $(A, \mathbb{R})^\# \in L(\Gamma, \mathbb{R})$ and such that

$$\delta_\Gamma = (\Theta_{\delta_\Gamma})^{L(\Gamma,\mathbb{R})}$$

where δ_Γ is the largest Suslin cardinal of $L(\Gamma, \mathbb{R})$. □

The reason for not simply declaring Axiom 2 as the axiom V = Ultimate-L is a recent result which shows (assuming what seem to be extremely plausible assumptions) that if $L(A, \mathbb{R}) \vDash \mathrm{AD}^+$ then

$$\mathrm{HOD}^{L(A,\mathbb{R})} \cap V_\Theta \vDash \text{“There are no supercompact cardinals”}$$

where $\Theta = \Theta^{L(A,\mathbb{R})}$. In fact one obtains that no cardinal $\kappa < \Theta^{L(A,\mathbb{R})}$ of $\mathrm{HOD}^{L(A,\mathbb{R})}$ is λ-supercompact where λ is the least $L(A, \mathbb{R})$-cardinal above κ.

The “plausible assumptions” concern the representation of the rank initial segments of $\mathrm{HOD}^{L(A,\mathbb{R})}$ below the largest Suslin cardinal of $L(A, \mathbb{R})$ as the direct limit of structures in an appropriate hierarchy of strategic-extender structures.

The restriction to rank initial segments of $\mathrm{HOD}^{L(A,\mathbb{R})}$ below the largest Suslin cardinal of $L(A, \mathbb{R})$ suffices here since:

(1) If $U \in \mathrm{HOD}^{L(A,\mathbb{R})}$ is a countably complete uniform ultrafilter in $\mathrm{HOD}^{L(A,\mathbb{R})}$ on some ordinal γ, then necessarily $\gamma < \Theta^{L(A,\mathbb{R})}$.

(2) If $\Theta = \Theta^{L(A,\mathbb{R})}$ then

$$\left(L_\delta(A, \mathbb{R}), \mathrm{HOD}^{L(A,\mathbb{R})} \cap L_\delta(\mathbb{R})\right) \prec_{\Sigma_1} \left(L_\Theta(A, \mathbb{R}), \mathrm{HOD}^{L(A,\mathbb{R})} \cap L_\Theta(\mathbb{R})\right)$$

where δ is the largest Suslin cardinal of $L(A, \mathbb{R})$. This is a corollary of the proof that δ is a strong cardinal in $\mathrm{HOD}^{L(A,\mathbb{R})}|\Theta$.

Such a representation would yield the following conjecture which is all one needs. Define that a set $X \subset \mathcal{P}(Y)$ *generates a countably complete filter* if $\cap\sigma \neq \emptyset$ for each countable set $\sigma \subset X$.

Definition 7.19 (**HOD-Ultrafilter Conjecture**)**.** Suppose that $A \subset \mathbb{R}$, $L(A, \mathbb{R}) \vDash \mathrm{AD}^+$, $U \in \mathrm{HOD}^{L(A,\mathbb{R})}$, and

$$\mathrm{HOD}^{L(A,\mathbb{R})} \vDash \text{“}U \text{ is a countably complete ultrafilter”}.$$

Then U generates a countably complete filter. □

The HOD-Ultrafilter Conjecture implies that ω_1 must be the least measurable cardinal in $\mathrm{HOD}^{L(A,\mathbb{R})}$. This is already known, and that analysis yields the following theorem.

Theorem 7.20. *Suppose that $A \subset \mathbb{R}$ and $L(A, \mathbb{R}) \vDash \mathrm{AD}^+$. Then ω_1^V is the least measurable cardinal in $\mathrm{HOD}^{L(A,\mathbb{R})}$ and every ultrafilter $U \in \mathrm{HOD}^{L(A,\mathbb{R})}$ on ω_1^V which is countably complete in $\mathrm{HOD}^{L(A,\mathbb{R})}$ generates a countably complete filter.* □

The proof of Theorem 7.20 highlights a subtle aspect of the HOD-Ultrafilter Conjecture, even if one restricts to just ultrafilters on ω_1^V. There are countable sets

$$\sigma \subset \mathcal{P}(\omega_1) \cap \mathrm{HOD}^{L(A,\mathbb{R})}$$

which *cannot* be covered by countable sets $\tau \in \mathrm{HOD}$. This is because the cardinal successor of ω_1^V in $\mathrm{HOD}^{L(A,\mathbb{R})}$ has countable cofinality.

The following theorem from [23] provides some evidence for the HOD-Ultrafilter Conjecture. The strength of this evidence is arguable since B can be chosen so that $\Theta^{L(A,\mathbb{R})}$ is the *only* Woodin cardinal of $\mathrm{HOD}_B^{L(A,\mathbb{R})}$.

Theorem 7.21. *Suppose that $A \subset \mathbb{R}$ and $L(A,\mathbb{R}) \vDash \mathrm{AD}^+$. Then there exists $B \subset \mathbb{R}$ such that $B \in L(A,\mathbb{R})$ and such that for all $U \in \mathrm{HOD}_B^{L(A,\mathbb{R})}$ for which*

$$\mathrm{HOD}_B^{L(A,\mathbb{R})} \vDash \text{“}U \text{ is a countably complete ultrafilter”},$$

the filter generated by U is countably complete. □

Assuming $V \vDash \mathrm{AD}$, if N is an inner model of ZFC then Θ is always a limit of strongly inaccessible cardinals of N which have cofinality ω in V. This shows that if

$$V \vDash \mathrm{AD}^+ + \text{“}V = L(\mathcal{P}(\mathbb{R}))\text{”}$$

and if the HOD-Ultrafilter Conjecture holds in V then there can be no supercompact cardinals in $\mathrm{HOD} \cap V_\Theta$. The basic argument is given in the proof of Theorem 7.24.

A much tighter connection between the HOD-Ultrafilter Conjecture and the degree to which supercompactness can occur in the model,

$$\mathrm{HOD}^{L(A,\mathbb{R})} \cap V_\Theta$$

where $\Theta = \Theta^{L(A,\mathbb{R})}$, follows from the following theorem from [23].

Theorem 7.22. *Suppose that $L(A,\mathbb{R}) \vDash \mathrm{AD}^+$, $\kappa < \Theta^{L(A,\mathbb{R})}$ is a cardinal of $\mathrm{HOD}^{L(A,\mathbb{R})}$, and*

$$\lambda = (|\kappa|^+)^{L(A,\mathbb{R})}.$$

Then there is a countable set $\sigma \subset \lambda$ such that $\sigma \not\subset \tau$ for any set $\tau \in \mathrm{HOD}^{L(A,\mathbb{R})}$ such that $\mathrm{ordertype}(\tau) < \kappa$. □

Remark 7.23. The proof of Theorem 7.22 actually shows that if $N \subset L(A,\mathbb{R})$ is an inner model of ZFC (containing the ordinals) and if $\omega < \lambda < \Theta^{L(A,\mathbb{R})}$ is a cardinal of $L(A,\mathbb{R})$, then λ is a limit of strongly inaccessible cardinals of N which have *countable* cofinality in $L(A,\mathbb{R})$ (and hence have countable cofinality in V). □

Theorem 7.22 combined with the HOD-Ultrafilter Conjecture yields the following theorem. For this theorem, the distinction between $\mathrm{HOD}^{L(A,\mathbb{R})}$ and $\mathrm{HOD}^{L(A,\mathbb{R})} \cap V_\Theta$, where $\Theta = \Theta^{L(A,\mathbb{R})}$, is not relevant.

An immediate corollary of Theorem 7.24 is that assuming the HOD-Ultrafilter Conjecture holds for $L(A,\mathbb{R})$ then

$$\mathrm{HOD}^{L(A,\mathbb{R})} \cap V_\Theta \vDash \text{“There are no supercompact cardinals”}$$

where $\Theta = \Theta^{L(A,\mathbb{R})}$.

Theorem 7.24. *Suppose $A \subset \mathbb{R}$, $L(A,\mathbb{R}) \vDash \mathrm{AD}^+$, and that the HOD-Ultrafilter Conjecture holds for $L(A,\mathbb{R})$. Suppose κ is a cardinal of $\mathrm{HOD}^{L(A,\mathbb{R})}$ and $\lambda = (|\kappa|^+)^{L(A,\mathbb{R})}$. Then*

$$\mathrm{HOD}^{L(A,\mathbb{R})} \vDash \text{“}\kappa \text{ is not } \lambda\text{-supercompact.”}$$

Proof. Assume not and let $U \in \mathrm{HOD}^{L(A,\mathbb{R})}$ be such that

$$\mathrm{HOD}^{L(A,\mathbb{R})} \cap V_\Theta \vDash \text{“}U \text{ is a } \kappa\text{-complete fine ultrafilter on } \mathcal{P}_\kappa(\lambda)\text{”}.$$

Since the HOD-Ultrafilter Conjecture holds for $L(A, \mathbb{R})$, U generates a countably complete filter. Therefore for all countable $\sigma \subset \lambda$ there must exist

$$\tau \in \mathrm{HOD}^{L(A,\mathbb{R})}$$

such that $\sigma \subset \tau$ and such that $\mathrm{ordertype}(\tau) < \kappa$. This contradicts Theorem 7.22. □

Thus the models,

$$\mathrm{HOD}^{L(\Gamma,\mathbb{R})} \cap V_{\Theta_\Gamma}$$

where $\Theta_\Gamma = \Theta^{L(\Gamma,\mathbb{R})}$ and $\Gamma \lhd \Gamma^\infty$, very likely *cannot* resemble V in context of large cardinals no matter how Γ is chosen. Any resemblance is limited to the level of Σ_2-sentences.

By Theorem 7.12, allowing Wadge initial segments $\Gamma \subset \Gamma^\infty$ for which

$$L(\Gamma, \mathbb{R}) \vDash \mathrm{AD}_\mathbb{R}$$

(equivalently, which do not satisfy $\Gamma \lhd \Gamma^\infty$) cannot help which was the original point for focusing on $\Gamma \lhd \Gamma^\infty$. Here again, plausible assumptions give a much stronger result, specifically that there can be no strong cardinal in $\mathrm{HOD}^{L(\Gamma,\mathbb{R})}|\Theta_\Gamma$.

The next theorem is from [23] and highlights a very useful consequence of the axiom V = Ultimate-L. This a typical consequence of the axiom V = Ultimate-L which is easier to obtain if one assumes a version with stronger large cardinal assumptions and with more reflection as in [22].

Theorem 7.25 (V = Ultimate-L). *For each cardinal κ, if $V[G]$ is a set-generic extension of V then there exists an elementary embedding*

$$\pi : (H(\kappa^+))^V \to N$$

such that $(\pi, N) \in V$ and such that $N \in \mathrm{HOD}^{V[G]}$. □

The following theorem from [23] summarizes some of the key consequences of the axiom V = Ultimate-L where the Generic-Multiverse is the generic-multiverse generated by V, [21].

These are proved in [22], assuming Theorem 7.32 and Theorem 7.25, but only for the somewhat stronger formulation of V = Ultimate-L which is given there.

In light of the necessity of the revision of the formulation of V = Ultimate-L, those theorems of [22] are not really relevant now.

Theorem 7.26 (V = Ultimate-L). (1) CH *holds.*

(2) V = HOD.

(3) *V is the minimum universe of the Generic-Multiverse.* □

The conclusions (2)–(3) of Theorem 7.26 actually follow from Theorem 7.25 by fairly general arguments and we briefly sketch how. These arguments are from [22].

We first prove the following corollary of Theorem 7.25 which is a very strong version Theorem 7.26(2). We then use Theorem 7.27 to prove Theorem 7.26(3), which we isolate as Theorem 7.28 below.

Theorem 7.27 (V = Ultimate-L). *Suppose $V[G]$ is a set generic extension of V. Then*

$$V \subseteq (\mathrm{HOD})^{V[G]}.$$

Proof. Fix a partial order $\mathbb{P} \in V$ such that G is V-generic for $\mathbb{P}$ and let $\delta = |\mathbb{P}|^V$. We prove that for all regular cardinals $\kappa > \delta$, $(\mathcal{P}(\kappa))^V \subset (\mathrm{HOD})^{V[G]}$ and this will show that $V \subseteq (\mathrm{HOD})^{V[G]}$.

Fix a regular cardinal $\kappa > \delta$ and let $\langle S_\alpha : \alpha < \kappa \rangle \in V$ be a partition of the set

$$S = \left\{ \alpha < \kappa \mid (\mathrm{cof}(\alpha))^V = \omega \right\}$$

into stationary sets such that there is a closed unbounded set $C_0 \subset \kappa$ such that $C_0 \in V$ and such that for each $\eta \in C_0 \cap S$,

$$\eta \in \cup \left\{ S_\xi \mid \xi < \eta \right\}.$$

Note that if $C \subseteq \kappa$ is a closed cofinal set with $C \in V[G]$ then there must exist a closed cofinal set $D \subseteq C$ such that $D \in V$. Therefore each S_α is a stationary set in $V[G]$.

By Theorem 7.25 there exists an elementary embedding

$$\pi : (H(\kappa^+))^V \to N$$

such that $N \in (\mathrm{HOD})^{V[G]}$ and such that $(\pi, N) \in V$. Let

$$\langle T_\beta : \beta < \pi(\kappa) \rangle = \pi(\langle S_\alpha : \alpha < \kappa \rangle).$$

Working in $V[G]$, define

$$Z = \left\{ \beta < \pi(\kappa) \mid T_\beta \cap C \neq \emptyset \text{ for all closed cofinal sets } C \subset \sup(\pi[\kappa]) \right\}.$$

Thus $Z \in (\mathrm{HOD})^{V[G]}$ since $\langle T_\beta : \beta < \pi(\kappa) \rangle \in (\mathrm{HOD})^{V[G]}$. Note that for all $\xi \in S$, $\pi(\xi) = \sup(\pi[\xi])$. Therefore since each set S_α is a stationary subset of κ in $V[G]$,

$$\pi[\kappa] \subseteq Z$$

and so since for each $\eta \in C_0 \cap S$,

$$\eta \in \cup \left\{ S_\xi \mid \xi < \eta \right\},$$

necessarily,

$$Z = \pi[\kappa].$$

For each $X \in N$, let $X^* = \{\alpha < \kappa \mid \pi(\alpha) \in X\}$. Since $N \in (\mathrm{HOD})^{V[G]}$ and since $\pi[\kappa] \in (\mathrm{HOD})^{V[G]}$,

$$\{X^* \mid X \in N\} \subset (\mathrm{HOD})^{V[G]}.$$

Finally $(\mathcal{P}(\kappa))^V \subset \mathrm{dom}(\pi)$ and so

$$(\mathcal{P}(\kappa))^V = \{X^* \mid X \in N\}$$

which implies that $(\mathcal{P}(\kappa))^V \subset (\mathrm{HOD})^{V[G]}$. □

Theorem 7.28 (V = Ultimate-L). *V is the minimum universe of the Generic-Multiverse.*

Proof. Suppose that $V[G] = V_0[G_0]$, $G \subset \mathbb{P}$ is V-generic for some partial order $\mathbb{P} \in V$, and $G_0 \subset \mathbb{P}_0$ is V_0-generic for some partial order $\mathbb{P}_0 \in V_0$. We must prove that $V \subseteq V_0$.

Fix a cardinal $\delta \in V$ such that $|\mathbb{P}|^V < \delta$ and such that $|\mathbb{P}_0|^{V_0} < \delta$. The key points are that in V,

$$\mathrm{RO}(\mathbb{P} \times \mathrm{Coll}(\omega, \delta)) \cong \mathrm{RO}(\mathrm{Coll}(\omega, \delta)),$$

and that in V_0,

$$\mathrm{RO}(\mathbb{P}_0 \times \mathrm{Coll}(\omega, \delta)) \cong \mathrm{RO}(\mathrm{Coll}(\omega, \delta)).$$

Suppose $g \subset \mathrm{Coll}(\omega, \delta)$ is $V[G]$-generic. Therefore by the homogeneity of $\mathrm{Coll}(\omega, \delta)$,

$$(\mathrm{HOD})^{V[g]} = (\mathrm{HOD})^{V[G][g]} = (\mathrm{HOD})^{V_0[G_0][g]} = (\mathrm{HOD})^{V_0[g]} \subseteq V_0.$$

By Theorem 7.27, $V \subseteq (\mathrm{HOD})^{V[g]}$ and so $V \subseteq (\mathrm{HOD})^{V_0[g]} \subset V_0$. □

Remark 7.29. Usuba [19] has proved a remarkable theorem. If sufficient large cardinals exist in V then the Generic-Multiverse has a unique minimum element.

Thus arguably any candidate for the axiom V = Ultimate-L must imply that V is the minimum universe of the Generic-Multiverse. □

The problem of whether V = Ultimate-L implies the Ω Conjecture is more subtle and this is because of the restriction to Σ_2-sentences in the formulation of the axiom. The stronger versions given as Axiom 1 and Axiom 2 each imply the Ω Conjecture.

What one seems to need in order to prove the Ω Conjecture from V = Ultimate-L is the following conjecture which also follows from the previously discussed "plausible assumptions".

Definition 7.30 (**Θ_0 Conjecture).** Suppose $L(A,\mathbb{R}) \models \mathrm{AD}^+$. Then $(\Theta_0)^{L(A,\mathbb{R})}$ is the least Woodin cardinal of $\mathrm{HOD}^{L(A,\mathbb{R})}$. □

The following theorem from [23] provides strong evidence for the Θ_0 Conjecture. Note that Θ_0 is same allowing x has a parameter for any $x \in \mathbb{R}$; more precisely, if $\pi : \mathbb{R} \to \alpha$ is a surjection which is OD_x for some $x \in \mathbb{R}$, then $\alpha < \Theta_0$.

Theorem 7.31. *Suppose that $A \subset \mathbb{R}$ and that $L(A,\mathbb{R}) \models \mathrm{AD}^+$. Then for a Turing cone of x, $(\Theta_0)^{L(A,\mathbb{R})}$ is the least Woodin cardinal of* $\mathrm{HOD}_x^{L(A,\mathbb{R})}$. □

Theorem 7.32 (V = Ultimate-L). *Assume the Θ_0 Conjecture and let λ be the least Woodin cardinal. Then there is a partial order $\mathbb{P} \in V_{\lambda+1}$ such that if $G \subset \mathbb{P}$ is V-generic then in $V[G]$ every Δ^2_1 subset of $\mathbb{R}$ is universally Baire.*

Proof. We just sketch the proof which requires basic elements of the theory of AD^+.

It suffices to prove that if $\kappa > \lambda$ and $|V_\kappa| = \kappa$ then there is a partial order $\mathbb{P} \in V_{\lambda+1}$ such that if $G \subset \mathbb{P}$ is V-generic then in $V[G]$ every Δ^2_1 subset of $\mathbb{R}$ is $(<\kappa)$-universally Baire.

This is expressible by a Π_2-sentence, ψ. Assume toward a contradiction that $(\neg\psi)$ holds. Then since V = Ultimate-L holds, there exists a universally Baire set $A \subset \mathbb{R}$ such that the following hold where $\Theta_A = \Theta^{L(A,\mathbb{R})}$.

(1.1) $\mathrm{HOD}^{L(A,\mathbb{R})} \cap V_{\Theta_A} \models (\neg\psi)$.

(1.2) The Θ_0 Conjecture holds for $L(A,\mathbb{R})$.

Since Σ^2_1 has the scale property in $L(A,\mathbb{R})$, every set $Z \subset \mathbb{R}$ which is Σ^2_1-definable in $L(A,\mathbb{R})$, is the projection of a tree T such that $T \in \mathrm{HOD}^{L(A,\mathbb{R})}$.

Let δ_A be the largest Suslin cardinal of $L(A,\mathbb{R})$. Thus δ_A is $(<\Theta_A)$-strong in $\mathrm{HOD}^{L(A,\mathbb{R})}$ and therefore $(\Theta_0)^{L(A,\mathbb{R})} < \delta_A$ since $(\Theta_0)^{L(A,\mathbb{R})}$ is the least Woodin cardinal in $\mathrm{HOD}^{L(A,\mathbb{R})}$.

Let $G \subset \mathrm{Coll}(\omega_1,\mathbb{R})$ be $L(A,\mathbb{R})$-generic. A key point is that by Vopenka's Theorem and the definition of $(\Theta_0)^{L(A,\mathbb{R})}$, $\mathrm{HOD}^{L(A,\mathbb{R})}[G]$ is a generic extension of $\mathrm{HOD}^{L(A,\mathbb{R})}$ for a partial order of size at most $(\Theta_0)^{L(A,\mathbb{R})}$ in $\mathrm{HOD}^{L(A,\mathbb{R})}$.

For each $\lambda < \delta_A$ there is a unique normal fine countably complete ultrafilter, U_λ, in $L(A,\mathbb{R})$ on $\mathcal{P}_{\omega_1}(\lambda)$. Thus every set of reals which is Δ^2_1-definable in $L(A,\mathbb{R})$ is $(<\delta_A)$-universally Baire in $\mathrm{HOD}^{L(A,\mathbb{R})}[G]$, appealing to the closure of
$$\mathrm{HOD}^{L(A,\mathbb{R})} \cap \mathcal{P}(\mathrm{Ord})$$
under the ultrapowers maps π_λ as computed in $L(A,\mathbb{R})$ using the ultrafilters U_λ. We view π_λ as acting on all sets of ordinals where the ultrapowers are computed using all functions in $L(A,\mathbb{R})$.

We have that δ_A is $(<\Theta_A)$-strong in $\mathrm{HOD}^{L(A,\mathbb{R})}$ and this implies that δ_A is $(<\Theta_A)$-strong in $\mathrm{HOD}^{L(A,\mathbb{R})}[G]$. Thus every set $Z \subset \mathbb{R}$ which is Δ^2_1-definable in $L(A,\mathbb{R})$, is universally Baire in
$$\mathrm{HOD}^{L(A,\mathbb{R})}[G] \cap V_{\Theta_A}.$$
But this includes all the sets $Z \subset \mathbb{R}$ which are Δ^2_1-definable in
$$\mathrm{HOD}^{L(A,\mathbb{R})} \cap V_{\Theta_A}$$
and this proves that
$$\mathrm{HOD}^{L(A,\mathbb{R})} \cap V_{\Theta_A} \models \psi,$$
which is a contradiction. □

The proof of Theorem 7.32 adapts to prove the following more striking version of that theorem (assuming there is a strong cardinal). This requires the following variation of the Θ_0 Conjecture which is really a strong version of Theorem 7.13.

Definition 7.33 (**Θ_α Conjecture).** Suppose that $A \subset \mathbb{R}$,

$$L(A, \mathbb{R}) \vDash \mathrm{AD}^+,$$

and $\gamma < \Theta^{L(A,\mathbb{R})}$. Then the following are equivalent.

(1) γ is a limit of Woodin cardinals in $\mathrm{HOD}^{L(A,\mathbb{R})}$ and no cardinal κ is $(<\gamma)$-strong in $\mathrm{HOD}^{L(A,\mathbb{R})}$.

(2) $\gamma = (\Theta_\alpha)^{L(A,\mathbb{R})}$ for some limit ordinal $\alpha > 0$. □

Remark 7.34. (1) The Θ_0 Conjecture characterizes Θ_0 in HOD, whereas the Θ_α Conjecture characterizes in HOD, all the Θ_α where $\alpha > 0$ and α is a limit ordinal.

(2) By [23], assuming AD^+ and that $V = L(\mathcal{P}(\mathbb{R}))$, if $\alpha > 0$ is a limit ordinal then Θ_α cannot be a limit of HOD-cardinals which are $(<\Theta_\alpha)$-strong in HOD. Thus Θ_α is a Woodin cardinal in HOD if and only if $\alpha = 0$ or α is not a limit ordinal. This implies Theorem 7.12. □

We note the following theorem, [23].

Theorem 7.35 (Ω Conjecture). *Suppose there is a proper class of Woodin cardinals. Then there is a partial order $\mathbb{P}$ such that if $G \subset \mathbb{P}$ is V-generic then in $V[G]$,*

$$V(\mathbb{R}^{V[G]}) \vDash \mathrm{AD}^+. \quad \square$$

The conclusion of Theorem 7.36 (augmented with the Θ_0 Conjecture) is simply a much stronger version of the conclusion of Theorem 7.35, showing that one can require $\mathbb{P}$ to be homogeneous and in addition both that λ_0 is $\Theta = \Theta_0$ in $V(\mathbb{R}^{V[G]})$ and that V_{λ_0} is *exactly* $\mathrm{HOD}|\Theta_0$ as computed in the $L(\mathcal{P}(\mathbb{R}))$ of $V(\mathbb{R}^{V[G]})$, where λ_0 is the least Woodin cardinal of V.

Theorem 7.36 (V = Ultimate-L). *Assume the Θ_α Conjecture and suppose that there is a strong cardinal. Let λ be the least strong cardinal. Then there is a homogeneous partial order $\mathbb{P} \in V_{\lambda+1}$ such that if $G \subset \mathbb{P}$ is V-generic then in $V[G]$ the following hold where*

$$\Gamma_G = (\Gamma^\infty)^{V[G]}$$

and where $\mathbb{R}_G = \mathbb{R}^{V[G]}$.

(1) $V(\Gamma_G, \mathbb{R}_G) \vDash \mathrm{AD}_\mathbb{R}$ + "Θ *is regular" and* $\Gamma_G = \mathcal{P}(\mathbb{R}_G) \cap V(\Gamma_G, \mathbb{R}_G)$.

(2) $\lambda = \Theta^{L(\Gamma_G,\mathbb{R}_G)}$ *and* $V_\lambda = \mathrm{HOD}^{L(\Gamma_G,\mathbb{R}_G)} \cap V[G]_\lambda$. □

The conclusion of Theorem 7.36 is actually equivalent to V = Ultimate-L assuming that there is a strong cardinal and that the Θ_α Conjecture holds.

To obtain the Ω Conjecture from V = Ultimate-L and the Θ_0 Conjecture, we use the following lemma which is a special case of Lemma 217 on page 315 in [20].

Lemma 7.37. *Suppose that there is a proper class of Woodin cardinals and that for every set $Z \subset \mathbb{R}$, if Z is Δ^2_1-definable then Z is universally Baire. Then*

$$\mathrm{HOD} \vDash \text{"The } \Omega \text{ Conjecture"} \quad \square$$

Theorem 7.38 (V = Ultimate-L). *Assume the Θ_0 Conjecture. Then the Ω Conjecture holds.*

Proof. By Theorem 7.32, there is a partial order $\mathbb{P}$ such that if $G \subset \mathbb{P}$ is V-generic then in $V[G]$ every Δ^2_1 subset of $\mathbb{R}$ is universally Baire. By Lemma 7.37,

$$(\mathrm{HOD})^{V[G]} \vDash \text{"The } \Omega \text{ Conjecture"}$$

and by Theorem 7.27,

$$V \subseteq (\mathrm{HOD})^{V[G]}.$$

Therefore $(\mathrm{HOD})^{V[G]}$ must be a generic extension of V. Finally the Ω Conjecture is absolute between set-generic extensions and so the Ω Conjecture holds in V. □

Finally, a very natural open question is the following where weak comparison is as defined in Definition 6.36. This question makes sense by Theorem 7.26(2).

Question 7.39. *Does* $V =$ Ultimate-L *imply weak comparison?* □

We now very briefly consider the Ultimate-L Conjecture and we begin by noting the following lemma. This lemma explains why in the formulation of the Ultimate-L Conjecture it is reasonable to require N be a weak extender model for the supercompactness of δ, versus just requiring that N be a weak extender model for the supercompactness of *some* cardinal.

Lemma 7.40. *Suppose that N is a weak extender model for the supercompactness of κ, N is weakly Σ_2-definable, and that $\delta > \kappa$ is an extendible cardinal. Then N is a weak extender model for the supercompactness of δ.*

Proof. Let $\lambda > \delta$ be such that $V_\lambda \prec_{\Sigma_4} V$ and let

$$j : V_{\lambda+1} \to V_{j(\lambda)+1}$$

be an elementary embedding such that $\mathrm{CRT}(j) = \delta$ and $j(\delta) > \lambda$. Thus:

(1.1) $N \cap V_\lambda = (N)^{V_\lambda}$.

(1.2) $V_\lambda \vDash$ "N is a weak extender model for κ is supercompact".

(1.3) $V_{j(\lambda)} \vDash$ "N is a weak extender model for κ is supercompact".

(1.4) $(N)^{V_{j(\lambda)}} \cap V_\lambda = N \cap V_\lambda$ where $(N)^{V_{j(\lambda)}}$ is N as computed in $V_{j(\lambda)}$. This well-defined by the elementarity of j.

Therefore by (the proof of) the Universality Theorem, Theorem 3.26, for each $\delta < \gamma < \lambda$,

$$j|(V_\gamma \cap N) \in (N)^{V_{j(\lambda)}}$$

and so for each $\delta < \gamma < \lambda$, there exists a δ-complete normal fine ultrafilter U on $\mathcal{P}_\delta(\gamma)$ such that

(2.1) $N \cap \mathcal{P}_\delta(\gamma) \in U$,

(2.2) $U \cap N \in N$.

This implies that N is a weak extender model for δ is supercompact. □

We end with the following conjecture which is the minor variation of the version of the Ultimate-L Conjecture given in [24] obtained by dropping one clause[9]. Proving either conjecture would show in a decisive fashion the transcendence of the strategic-extender hierarchy.

The same applies to the weaker conjecture where one drops the requirement that $N \subseteq \mathrm{HOD}$ or the requirement that N be weakly Σ_2-definable. Here, for example, one could simply conjecture that if κ is strongly inaccessible and δ is an extendible cardinal in V_κ then there exists $N \in V_{\kappa+1}$ such that

$$N \vDash \text{"}V = \text{Ultimate-}L\text{"}$$

and such that relative to V_κ, N is a weak extender model for the supercompactness of δ.

[9]which asserts that there is an extender sequence $\tilde{E}$ of length δ in V whose restriction to N both belongs to N and witnesses in N that δ is a Woodin cardinal.

Conjecture 7.41. *Suppose that δ is an extendible cardinal. Then there exists a weak extender model N for the supercompactness of δ such that:*

(1) *N is weakly Σ_2-definable and $N \subset \mathrm{HOD}$,*

(2) $N \vDash$ "$V = \text{Ultimate-}L$". □

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