Compatibility of Local and Global Langlands Correspondences

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Abstract. We prove the compatibility of local and global Langlands correspondences for $GL_n$, which was proved up to semisimplification in [HT]. More precisely, for the $n$-dimensional $l$-adic representation $R_l(\Pi)$ of the Galois group of a CM-field $L$ attached to a conjugate self-dual regular algebraic cuspidal automorphic representation $\Pi$, which is square integrable at some finite place, we show that Frobenius semisimplification of the restriction of $R_l(\Pi)$ to the decomposition group of a prime $v$ of $L$ not dividing $l$ corresponds to $\Pi_v$ by the local Langlands correspondence.

Introduction

This paper is a continuation of [HT]. Let $L$ be an (imaginary) CM field and let $\Pi$ be a regular algebraic cuspidal automorphic representation of $GL_n(\mathbb{A}_L)$ which is conjugate self-dual ($\Pi \circ c \cong \Pi^\vee$) and square integrable at some finite place. In [HT] it is explained how to attach to $\Pi$ and an arbitrary rational prime $l$ (and an isomorphism $\iota : \mathbb{Q}_l^{pc} \cong \mathbb{C}$) a continuous semisimple representation

$$R_l(\Pi) : \text{Gal}(L^{ac}/L) \rightarrow GL_n(\mathbb{Q}_l^{pc})$$

which is characterised as follows. For every finite place $v$ of $L$ not dividing $l$

$$\iota R_l(\Pi)|_{W_{L_v}} = \text{rec}(\Pi_v^\vee | \det \frac{1}{2} - n)^{ss},$$

where $\text{rec}$ denotes the local Langlands correspondence and $ss$ denotes the semisimplification (see [HT] for details). In that book it is also shown that $\Pi_v$ is tempered for all finite places $v$.

In this paper we strengthen this result to completely identify $R_l(\Pi)|_{I_v}$ for $v \nmid l$. In particular, we prove the following theorem.

**Theorem A.** If $v \nmid l$ then the Frobenius semisimplification of $R_l(\Pi)|_{W_{L_v}}$ is the $l$-adic representation attached to $\iota^{-1}\text{rec}(\Pi_v^\vee | \det \frac{1}{2} - n)$.

As $R_l(\Pi)$ is semisimple and $\text{rec}(\Pi_v^\vee | \det \frac{1}{2} - n)$ is indecomposable if $\Pi_v$ is square integrable, we obtain the following corollary.

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Corollary B. If $\Pi_v$ is square integrable at a finite place $v \nmid l$, the representation $R_l(\Pi)$ is irreducible.

Using base change it is easy to reduce to the case that $\Pi_v$ has an Iwahori fixed vector. We descend $\Pi$ to an automorphic representation $\pi$ of a unitary group $G$ which locally at $v$ looks like $GL_n$ and at infinity looks like $U(n-1,1) \times U(n,0)[L,\mathbb{Q}]/2^{-1}$. Then we realise $R_l(\Pi)$ in the cohomology of a Shimura variety $X$ associated to $G$ with Iwahori level structure at $v$. More precisely, for some $l$-adic sheaf $\mathcal{L}$, the $\pi^l$-isotypic component of $H^{n-1}(X, \mathcal{L})$ is, up to semisimplification and some twist, $R_l(\Pi)^a$ (for some $a \in \mathbb{Z}_{>0}$). We show that $X$ has semistable reduction and use the results of [IT] to calculate the cohomology of the (smooth, projective) strata of the reduction of $X$ above $p$ as a virtual $G(A_{\infty,p}) \times \widehat{\mathbb{F}_p}$-module (where $F$ denotes Frobenius). This description and the temperedness of $\Pi_v$ shows that the $\pi^l$-isotypic component of the cohomology of any strata is concentrated in the middle dimension. This implies that the $\pi^l$-isotypic component of the Rapoport-Zink weight spectral sequence degenerates at $E_1$, which allows us to calculate the action of inertia at $v$ on $H^{n-1}(X, \mathcal{L})$.

In the special case that $\Pi_v$ is a twist of a Steinberg representation and $\Pi_{\infty}$ has trivial infinitesimal character, the above theorem presumably follows from the results of Ito [I].

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1. The main theorem

We write $F^{ac}$ for an algebraic closure of a field $F$. Let $l$ be a rational prime and fix an isomorphism $i : \mathbb{Q}_{l}^{ac} \cong \mathbb{C}$.

Suppose that $p \neq l$ is another rational prime. Let $K/\mathbb{Q}_p$ be a finite extension. We will let $\mathcal{O}_K$ denote the ring of integers of $K$, $v_K$ the unique maximal ideal of $\mathcal{O}_K$, $v_K$ the canonical valuation $K^\times \to \mathbb{Z}$, $k(v_K)$ the residue field $\mathcal{O}_K/v_K$ and $| \cdot |_K$ the absolute value normalised by $|x|_K = (\# k(v_K))^{-v_K(x)}$. We will let $\text{Frob}_{v_K}$ denote the geometric Frobenius element of $\text{Gal}(k(v_K)^{ac}/k(v_K))$. We will let $I_{v_K}$ denote the kernel of the natural surjection $\text{Gal}(K^{ac}/K) \to \text{Gal}(k(v_K)^{ac}/k(v_K))$. We will let $W_K$ denote the preimage under $\text{Gal}(K^{ac}/K) \to \text{Gal}(k(v_K)^{ac}/k(v_K))$ of $\text{Frob}_{v_K}$. endowed with a topology by decreeing that $I_K$ with its usual topology is an open subgroup of $W_K$. Local class field theory provides a canonical isomorphism $\text{Art}_K : K^\times \cong W_K^{ab}$, which takes uniformisers to lifts of $\text{Frob}_{v_K}$.

Let $\Omega$ be an algebraically closed field of characteristic 0 and of the same cardinality as $\mathbb{C}$. (Thus in fact $\Omega \cong \mathbb{C}$.) By a Weil-Deligne representation of $W_K$ over $\Omega$ we mean a finite dimensional $\Omega$-vector space $V$ together with a homomorphism $r : W_K \to GL(V)$ with open kernel and an element $N \in \text{End}(V)$ which satisfies

$$r(\sigma)Nr(\sigma)^{-1} = |\text{Art}_K^{-1}(\sigma)|_K N.$$
We sometimes denote a Weil-Deligne representation by \((V, r, N)\) or simply \((r, N)\).

We call \((V, r, N)\) Frobenius semisimple if \(r\) is semisimple. If \((V, r, N)\) is any Weil-Deligne representation we define its Frobenius semisimplification \((V, r, N)^{F-ss}\) as follows. Choose a lift \(\phi\) of \(\text{Frob}_v K\) to \(W K\). Let \(r(\phi) = su = us\) where \(s \in GL(V)\) is semisimple and \(u \in GL(V)\) is unipotent. For \(n \in \mathbb{Z}\) and \(\sigma \in I_K\) set \(r^{ss}(\phi^n \sigma) = s^n r(\sigma)\). This is independent of the choices, and gives a Frobenius semisimple Weil-Deligne representation.

One of the main results of [HT] is that, given a choice of \((#k(v_K))^{1/2} \in \Omega\), there is a bijection \(\text{rec}\) (the local Langlands correspondence) from isomorphism classes of irreducible smooth representations of \(GL_n(K)\) over \(\Omega\) to isomorphism classes of \(n\)-dimensional Frobenius semisimple Weil-Deligne representations of \(W_K\), and that this bijection is natural in a number of respects. (See [HT] for details.)

We will call a Weil-Deligne representation of \(W_K\) over \(\mathbb{Q}_{ac}\) bounded if for some (and hence all) \(\sigma \in W_K - I_K\) all the eigenvalues of \(r(\sigma)\) are \(l\)-adic units. There is an equivalence of categories between bounded Weil-Deligne representations of \(W_K\) over \(\mathbb{Q}_{ac}\) and continuous representations of \(\text{Gal}(K_{ac}/K)\) on finite dimensional \(\mathbb{Q}_{ac}\)-vector spaces as follows. Fix a lift \(\phi \in W_K\) of \(\text{Frob}_v K\) and a continuous homomorphism \(t : I_K \rightarrow \mathbb{Z}_l\). Send a Weil-Deligne representation \((V, r, N)\) to \((V, \rho)\), where \(\rho\) is the unique continuous representation of \(\text{Gal}(K_{ac}/K)\) on \(V\) such that

\[\rho(\phi^n \sigma) = r(\phi^n \sigma) \exp(t(\sigma)N)\]

for all \(n \in \mathbb{Z}\) and \(\sigma \in I_K\). Up to natural isomorphism this functor is independent of the choices of \(t\) and \(\phi\). We will write \(\text{WD}(V, \rho)\) for the Weil-Deligne representation corresponding to a continuous representation \((V, \rho)\). If \(\text{WD}(V, \rho) = (V, r, N)\), then have \(\rho|_{W_K}^{ss} \cong r^{ss}\). (See [T], §4 and [D], §8 for details.)

Now suppose that \(L\) is a finite, imaginary CM extension of \(\mathbb{Q}\). Let \(c \in \text{Aut}(L)\) denote complex conjugation. Suppose that \(\Pi\) is a cuspidal automorphic representation of \(GL_n(\mathbb{A}_L)\) such that

- \(\Pi \circ c \cong \Pi^\vee\);
- \(\Pi_{\infty}\) has the same infinitesimal character as some algebraic representation over \(\mathbb{C}\) of the restriction of scalars from \(L\) to \(\mathbb{Q}\) of \(GL_n\);
- and for some finite place \(x\) of \(L\) the representation \(\Pi_x\) is square integrable.

(In this paper ‘square integrable’ (resp. ‘tempered’) will mean the twist by a character of a pre-unitary representation which is square integrable (resp. tempered).) In [HT] (see theorem C in the introduction of [HT]) it is shown that there is a unique continuous semisimple representation

\[R_l(\Pi) : \text{Gal}(L^{ac}/L) \rightarrow GL_n(\mathbb{Q}_{l}^{ac})\]
such that for each finite place $v \not| l$ of $L$
\[
\text{rec}(\Pi_v^0| \det | \frac{1-n}{2}) = (iR_l(\Pi)|_{W_{L_v}}, N)
\]
for some $N$. Moreover it is shown that $\Pi_v$ is tempered for all finite places $v$ of $L$, which completely determines the $N$ (see lemma 1.3 below). If $n = 1$ both these assertions are true without the assumption that $\Pi \circ c \cong \Pi^0$.

The main theorem of this paper identifies the $N$ of $\text{WD}(R_l(\Pi)|_{\text{Gal}(L_{ac}^\infty/L_v)})$ with the above $N$. More precisely we prove the following.

**Theorem 1.1.** Keep the above notation and assumptions. Then for each finite place $v \not| l$ of $L$ there is an isomorphism
\[
i\text{WD}(R_l(\Pi)|_{\text{Gal}(L_{ac}^\infty/L_v)})|^{F-ss}_\sim \text{rec}(\Pi_v^0| \det | \frac{1-n}{2})
\]
of Weil-Deligne representations over $\mathbb{C}$.

As $R_l(\Pi)$ is semisimple and $\text{rec}(\Pi_v^0| \det | \frac{1-n}{2})$ is indecomposable if $\Pi_v$ is square integrable, we have the following corollary.

**Corollary 1.2.** If $\Pi_v$ is square integrable for a finite place $v \not| l$, then the representation $R_l(\Pi)$ is irreducible.

In the rest of this section we consider some generalities on Galois representations and Weil-Deligne representations. First consider Weil-Deligne representations over an algebraically closed field $\Omega$ of characteristic zero and the same cardinality as $\mathbb{C}$. For a finite extension $K'/K$ of $p$-adic fields, we define
\[
(V, r, N)|_{W_{K'}} = (V, r|_{W_{K'}}, N).
\]
If $(W, r)$ is a finite dimensional representation of $W_K$ with open kernel and if $s \in \mathbb{Z}_{\geq 1}$ we will write $\text{Sp}_s(W)$ for the Weil-Deligne representation
\[
(W^s, r|_{\text{Art}_K^{s-1}} \cong \cdots \cong r|_{\text{Art}_K^{-1}} |_{K} \oplus r, N)
\]
where $N : r|_{\text{Art}_K^{-1}} |_{K} \cong r|_{\text{Art}_K^{-1}} |_{K}$ for $i = 1, \ldots, s-1$. This defines $\text{Sp}_s(W)$ uniquely (up to isomorphism). If $W$ is irreducible then $\text{Sp}_s(W)$ is indecomposable and every indecomposable Weil-Deligne representation is of the form $\text{Sp}_s(W)$ for a unique $s$ and a unique irreducible $W$. If $\pi$ is an irreducible cuspidal representation of $GL_g(K)$ then $\text{rec}(\pi) = (r, 0)$ with $r$ irreducible. Moreover for any $s \in \mathbb{Z}_{\geq 1}$ we have (in the notation of section I.3 of [HT]) $\text{rec}(\text{Sp}_s(\pi)) = \text{Sp}_s(r)$.

If $q \in \mathbb{R}_{> 0}$, then by a **Weil $q$-number** we mean $\alpha \in \mathbb{Q}_{ac}$ such that for all $\sigma : \mathbb{Q}_{ac} \hookrightarrow \mathbb{C}$ we have $(\sigma \alpha)(\sigma \alpha) = q$. We will call a Weil-Deligne representation $(V, r, N)$ of $W_K$ **strictly pure of weight** $k \in \mathbb{R}$ if for some (and hence every) lift $\phi$ of $\text{Frob}_v|_K$, every eigenvalue $\alpha$ of $r(\phi)$ is a Weil $(\# k(\nu_K))^k$-number. In this case we must have $N = 0$. We will call $(V, r, N)$ **mixed** if it has an increasing filtration $\text{Fil}_i W$ with $\text{Fil}_i W V = V$ for $i >> 0$ and $= (0)$ for
compatibility of local and global Langlands correspondences

i << 0, such that the i-th graded piece is strictly pure of weight i. If \( (V, r, N) \) is mixed then there is a unique choice of filtration \( \text{Fil}_i W \), and \( N(\text{Fil}_i W V) \subset \text{Fil}_{i-2} W V \). Finally we will call \( (V, r, N) \) pure of weight \( k \) if it is mixed with all weights in \( k + \mathbb{Z} \) and if for all \( i \in \mathbb{Z}_{>0} \)

\[
N^i : \text{gr}_{k+i} W V \underset{k+i}{\sim} \text{gr}_{k-i} W V.
\]

If \( W \) is strictly pure of weight \( k \), then \( \text{Sp}_s(W) \) is pure of weight \( k -(s-1) \) for any \( s \in \mathbb{Z}_{>1} \).

(1) \( (V, r, N) \) is pure if and only if \( (V, r, N)_{F-ss} \) is.

(2) If \( L/K \) is a finite extension, then \( (V, r, N) \) is pure if and only if \( (V, r, N)|_W \) is pure.

(3) An irreducible smooth representation \( \pi \) of \( GL_n(K) \) has \( \sigma \pi \) tempered for all \( \sigma : \Omega \rightarrow \mathbb{C} \) if and only if \( \text{rec}(\pi) \) is pure of some weight.

(4) Given \( (V, r) \) with \( r \) semisimple, there is, up to equivalence, at most one choice of \( N \) which makes \( (V, r, N) \) pure.

(5) An irreducible smooth representation \( \pi \) of \( GL_n(K) \) has \( \sigma \pi \) tempered for all \( \sigma : \Omega \rightarrow \mathbb{C} \) if and only if \( \text{rec}(\pi) \) is pure of some weight.

Given \( (V, r) \) with \( r \) semisimple, there is, up to equivalence, at most one choice of \( N \) which makes \( (V, r, N) \) pure.

(6) Suppose that \( (V, r, N) \) is a Frobenius semisimple Weil-Deligne representation which is pure of weight \( k \). Suppose also that \( \text{Fil}^j V \) is a decreasing filtration of \( V \) by Weil-Deligne subrepresentations such that \( \text{Fil}^j V = (0) \) for \( j >> 0 \) and \( \text{Fil}^j V = V \) for \( j << 0 \). If for each \( j \)

\[
\bigwedge^i \text{gr}^j V \text{gr}^j V
\]

is pure of weight \( k \text{dim} \text{gr}^j V \), then

\[
V \cong \bigoplus_j \text{gr}^j V
\]

and each \( \text{gr}^j V \) is pure of weight \( k \).

Proof: The first two parts are straightforward (using the fact that the filtration \( \text{Fil}_i W \) is unique). For the third part recall that an irreducible smooth representation \( \text{Sp}_s(\pi_1) \boxplus \cdots \boxplus \text{Sp}_s(\pi_t) \) (see section I.3 of [HT]) is tempered if and only if the absolute values of the central characters of the \( \text{Sp}_s(\pi_i) \) are all equal.

Suppose that \( (V, r, N) \) is a Frobenius semisimple and pure of weight \( k \). As a \( W_K \)-module we can write uniquely \( V = \bigoplus_{i \in \mathbb{Z}} V_i \) where \( (V_i, r, 0) \) is strictly pure of weight \( k+i \). For \( i \in \mathbb{Z}_{>0} \) let \( V(i) \) denote the kernel of \( N^{i+1} : V_i \rightarrow V_{i-2} \). Then \( N : V_{i+2} \hookrightarrow V_i \) and \( V_i = NV_{i+2} \oplus V(i) \). Thus

\[
V = \bigoplus_{i \in \mathbb{Z}} \bigoplus_{j=0}^i N^j V(i),
\]
and for $0 \leq j \leq i$ the map $N^j : V(i) \to V_{i-2j}$ is injective. Also note that as a virtual $W_K$-module $[V(i)] = [V_i] - [V_{i+2} \otimes \text{Art}_K^{-1}]$. Thus if $r$ is semisimple then $(V, r)$ determines $(V, r, N)$ up to isomorphism. This establishes the fourth part.

Now consider the fifth part. If $W$ is a direct summand it is certainly pure of the same weight $k$ and $\wedge^{\dim W} W$ is then pure of weight $k \dim W$. Conversely if $W$ is pure of weight $k$ then

$$W = \bigoplus_{i \in \mathbb{Z}} \bigoplus_{j=0}^i N^j W(i),$$

where $W(i) = W \cap V(i)$. As a $W_K$-module we can decompose $V(i) = W(i) \oplus U(i)$. Setting

$$U = \bigoplus_{i \in \mathbb{Z}} \bigoplus_{j=0}^i N^j U(i),$$

we see that $V = W \oplus U$ as Weil-Deligne representations. Now suppose only that $\wedge^{\dim W} W$ is pure of weight $k \dim W$. Write

$$W \cong \bigoplus_j \text{Sp}_{s_j}(X_j)$$

where each $X_j$ is strictly pure of some weight $k + k_j + (s_j - 1)$. Then, looking at highest exterior powers, we see that $\sum j k_j = 0$. On the other hand as $V$ is pure we see that $k_j \leq 0$ for all $j$. We conclude that $k_j = 0$ for all $j$ and hence that $W$ is pure of weight $k$.

The final part follows from the fifth part by a simple inductive argument. □

Now let $L$ denote a number field. Write $| \cdot |_L$ for

$$\prod_x | \cdot |_L : \mathbb{A}^\times_L / L^\times \to \mathbb{R}^\times_0,$$

and write $\text{Art}_L$ for

$$\prod_x \text{Art}_{L_x} : \mathbb{A}_L^\times / L^\times \to \text{Gal}(L^{ac}/L)^{ab}.$$

We will call a continuous representation

$$R : \text{Gal}(L^{ac}/L) \to GL_n(\mathbb{Q}^{ac}_L)$$

pure of weight $k$ if for all but finitely many finite places $x$ of $L$ the representation $R$ is unramified at $x$ and every eigenvalue $\alpha$ of $R(\text{Frob}_x)$ is a Weil $(#k(x))^k$-number. Note that if $n = 1$ then $R$ is pure of weight $k$ if and only if for all $\iota : \mathbb{Q}^{ac}_L \hookrightarrow \mathbb{C}$ we have $|\iota R \circ \text{Art}_L|^2 = | \cdot |_L^{-k}$. In particular if $n = 1$ and $R$ is pure then $R|_{W_{L_x}}$ is strictly pure for all finite places $x$ of $L$.

We have the following lemma.
Lemma 1.4. Suppose that $M/L$ is a finite extension of number fields. Suppose also that
\[ R : \text{Gal} \left( L^{ac}/L \right) \longrightarrow GL_n(\mathbb{Q}_{\ell}) \]
is a continuous semisimple representation which is pure of weight $k$. Suppose that
\[ S : \text{Gal} \left( M^{ac}/M \right) \longrightarrow GL_n(\mathbb{Q}_{\ell}) \]
is another continuous representation with $S^{ss} \cong R|_{\text{Gal}(M^{ac}/M)}$ for some $a \in \mathbb{Z}_{>0}$. Suppose finally that $w$ is a place of $M$ above a finite place $v$ of $L$. If $\text{WD}(S|_{\text{Gal}(M^{ac}/M_w)})$ is pure of weight $k$, then $\text{WD}(R|_{\text{Gal}(L^{ac}/L_v)})$ is also pure of weight $k$.

Proof: Write
\[ R|_{\text{Gal}(M^{ac}/M)} = \bigoplus_i R_i \]
where each $R_i$ is irreducible. Then $R_i$ is pure of weight $k \dim R_i$ and so that the top exterior power $\bigwedge^{\dim R_i} \text{WD}(R_i|_{\text{Gal}(M^{ac}/M_w)})$ is also pure of weight $k \dim R_i$. Lemma 1.3(6) tells us that
\[ \text{WD}(S|_{\text{Gal}(M^{ac}/M_w)})^{F-ss} \cong \left( \bigoplus_i \text{WD}(R_i|_{\text{Gal}(M^{ac}/M_w)})^{F-ss} \right)^{a} \cong (\text{WD}(R|_{\text{Gal}(M^{ac}/M_w)})^{F-ss})^{a}, \]
and that $\text{WD}(R|_{\text{Gal}(M^{ac}/M_w)})^{F-ss}$ is pure of weight $w$. Applying lemma 1.3(1) and (2), we see that $\text{WD}(R|_{\text{Gal}(L^{ac}/L_v)})$ is also pure. □

2. Shimura varieties

In this section we study the geometry of integral models of Shimura varieties of the type considered in [HT], but with Iwahori level. It may be viewed as a generalisation of the work of Deligne-Rapoport [DR] in the case of modular curves.

In this section,

- let $E$ be an imaginary quadratic field, $F^{+}$ a totally real field and set $F = EF^{+}$;
- let $p$ be a rational prime which splits $p = uu^c$ in $E$;
- and let $w = w_1, ... , w_r$ be the primes of $F$ above $w$;
- and let $B$ be a division algebra with centre $F$ such that
  - $\dim_F B = n^2$,
  - $B^{op} \cong B \otimes_{F,c} F$,
  - at every place $x$ of $F$ either $B_x$ is split or a division algebra,
  - if $n$ is even then the number of finite places of $F^+$ above which $B$ is ramified is congruent to $1 + \frac{n}{2}[F^+: \mathbb{Q}]$ modulo 2.
Pick a positive involution $*$ on $B$ with $*|_F = c$. Let $V = B$ as a $B \otimes_F B^{\text{op}}$-module. For $\beta \in B^{*-1}$ define a pairing

$$(x_1, x_2) : V \times V \longrightarrow \mathbb{Q}$$

$$\mapsto \text{tr}_{F/\mathbb{Q}} \text{tr}_{B/F}(x_1 \beta x_2^*)$$

Also define an involution $\#$ on $B$ by $x^\# = \beta x^* \beta^{-1}$ and a reductive group $G/\mathbb{Q}$ by setting, for any $\mathbb{Q}$-algebra $R$, the group $G(R)$ equal to the set of

$$(\lambda, g) \in R^\times \times (B^{\text{op}} \otimes_{\mathbb{Q}} R)^\times$$

such that

$$gg^\# = \lambda.$$

Let $\nu : G \to \mathbb{G}_m$ denote the multiplier character sending $(\lambda, g)$ to $\lambda$. Note that if $x$ is a rational prime which splits $x = yy_c$ in $E$ then

$$G(\mathbb{Q}_x) \sim \sim \sim \sim (B^{\text{op}}_y)^\times \times \mathbb{Q}_x^\times$$

$$(\lambda, g) \sim \sim \sim \sim (g_y, \lambda).$$

We can and will assume that

- if $x$ is a rational prime which does not split in $E$ the $G \times \mathbb{Q}_x$ is quasisplit;
- the pairing $(\ , \ )$ on $V \otimes_{\mathbb{Q}} \mathbb{R}$ has invariants $(1, n-1)$ at one embedding $\tau : F^+ \hookrightarrow \mathbb{R}$ and invariants $(0, n)$ at all other embeddings $F^+ \hookrightarrow \mathbb{R}$.

(See section I.7 of [HT] for details.)

Let $U$ be an open compact subgroup of $G(\mathbb{A}^\infty)$. Define a functor $\mathcal{X}_U$ from the category of pairs $(S, s)$, where $S$ is a connected locally noetherian $F$-scheme and $s$ is a geometric point of $S$, to the category of sets, sending $(S, s)$ to the set of isogeny classes of four-tuples $(A, \lambda, i, \eta)$ where

- $A/S$ is an abelian scheme of dimension $[F^+ : \mathbb{Q}] n^2$;
- $i : B \hookrightarrow \text{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ such that $\text{Lie } A \otimes_{(E \otimes \mathbb{Q}) \otimes \mathbb{O}_S} \mathbb{O}_S$ is locally free over $\mathbb{O}_S$ of rank $n$ and the two actions of $F^+$ coincide;
- $\lambda : A \to A^\vee$ is a polarisation such that for all $b \in B$ we have $\lambda \circ i(b) = i(b^*)^\vee \circ \lambda$;
- $\eta$ is a $\pi_1(S, s)$-invariant $U$-orbit of isomorphisms of $B \otimes_{\mathbb{Q}} \mathbb{A}^\infty$-modules $\eta : V \otimes_{\mathbb{Q}} \mathbb{A}^\infty \to VA_s$ which take the standard pairing $(\ , \ )$ on $V$ to a $(\mathbb{A}^\infty)^\times$-multiple of the $\lambda$-Weil pairing on $VA_s$.

Here $VA_s = \left( \lim_{N} A[N](k(s)) \right) \otimes_{\mathbb{Z}} \mathbb{Q}$ is the adelic Tate module. For the precise notion of isogeny class see section III.1 of [HT]. If $s$ and $s'$ are both geometric points of a connected locally noetherian $F$-scheme $S$ then $\mathcal{X}_U(S, s)$ and $\mathcal{X}_U(S, s')$ are in canonical bijection. thus we may think of $\mathcal{X}_U$ as a functor from connected locally noetherian $F$-schemes to sets. We
may further extend it to a functor from all locally noetherian $F$-schemes to sets by setting

\[ \mathfrak{X}_U \left( \prod_i S_i \right) = \prod_i \mathfrak{X}_U(S_i). \]

If $U$ is sufficiently small (i.e. for some finite place $x$ of $Q$ the projection of $U$ to $G(Q_x)$ contains no element of finite order except 1) then $\mathfrak{X}_U$ is represented by a smooth projective variety $X_U/F$ of dimension $n-1$. The inverse system of the $X_U$ for varying $U$ has a natural action of $G(\mathbb{A}^\infty)$.

Choose a maximal $\mathbb{Z}_{(p)}$-order $\mathcal{O}_B$ of $B$ with $\mathcal{O}^*_B = \mathcal{O}_B$. Also fix an isomorphism $\mathcal{O}^\op_{B,w} \cong M_n(\mathcal{O}_{F,w})$, and let $\varepsilon \in B_w$ denote the element corresponding to the diagonal matrix $(1,0,0,\ldots,0) \in M_n(\mathcal{O}_{F,w})$. We decompose $G(\mathbb{A}^\infty)$ as

\[ G(\mathbb{A}^\infty) = G(\mathbb{A}^\infty,p) \times \left( \prod_{i=2}^r (B_{w_i}^\op)^{\times} \right) \times GL_n(F_w) \times \mathbb{Q}_p^{\times}. \]

Let $\varpi$ denote a uniformiser for $\mathcal{O}_{F,w}$. For $m = (m_2,\ldots,m_r) \in \mathbb{Z}_{\geq 0}^{r-1}$, set

\[ U_p^w(m) = \prod_{i=2}^r \ker \left( (\mathcal{O}_{B,w_i}^\op)^{\times} \to (\mathcal{O}_{B,w_i}^\op/B_{w_i}^\op)^{\times} \right) \subset \prod_{i=2}^r (B_{w_i}^\op)^{\times}. \]

Let $B_n$ denote the Borel subgroup of $GL_n$ consisting of upper triangular matrices and let $N_n$ denote its unipotent radical. Let $I_{w,n,w}$ denote the subgroup of $GL_n(\mathcal{O}_{F,w})$ consisting of matrices which reduce modulo $w$ to $B_n(k(w))$. We will consider the following open subgroups of $G(\mathbb{Q}_p)$:

\[ \text{Ma}(m) = U_p^w(m) \times GL_n(\mathcal{O}_{F,w}) \times \mathbb{Z}_p^{\times}, \]
\[ \text{Iw}(m) = U_p^w(m) \times I_{w,n,w} \times \mathbb{Z}_p^{\times}. \]

Let $U_p$ be an open compact subgroup of $G(\mathbb{A}^\infty,p)$. Write $U_0$ (resp. $U$) for $U_p \times \text{Ma}(m)$ (resp. $U_p \times \text{Iw}(m)$).

We recall that in section III.4 [HT] integral model of $X_{U_0}$ over $\mathcal{O}_{F,w}$ is defined. It represents the functor $\mathfrak{X}_{U_0}$ from locally noetherian $\mathcal{O}_{F,w}$-schemes to sets. As above, $\mathfrak{X}_{U_0}$ is initially defined on the category of connected locally noetherian $\mathcal{O}_{F,w}$-schemes with a geometric point to sets. It sends $(S,s)$ to the set of prime-to-$p$ isogeny classes of $(r+3)$-tuples $(A,\lambda,i,\eta^p,\alpha_i)$, where

- $A/S$ is an abelian scheme of dimension $[F^+:Q]n^2$;
- $i: \mathcal{O}_B \hookrightarrow \text{End} (A) \otimes \mathbb{Z}_p$ such that $\text{Lie} A \otimes (\mathcal{O}_{F,u} \otimes \mathbb{Z}_p, \mathcal{O}_S)$, $1 \otimes 1 \mathcal{O}_S$ is locally free of rank $n$ and the two actions of $\mathcal{O}_F$ coincide;
- $\lambda: A \to A^\vee$ is a prime-to-$p$ polarisation such that for all $b \in \mathcal{O}_B$ we have $\lambda \circ i(b) = i(b^*)^\vee \circ \lambda$;
• $\pi^p$ is a $\pi_1(S, s)$-invariant $U^p$-orbit of isomorphisms of $B \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p}$-modules $\eta: V \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p} \to V^p A_s$ which take the standard pairing $(\ , \ )$ on $V$ to a $(\mathbb{A}^{\infty,p})^\times$-multiple of the $\lambda$-Weil pairing on $V^p A_s$;

• for $2 \leq i \leq r$, $\alpha_i: (w_i^{-m_i} \mathcal{O}_{B,w_i}/\mathcal{O}_{B,w_i})S \sim A[w_i^{m_i}]$ is an isomorphism of $S$-schemes with $\mathcal{O}_B$-actions;

Then $X_{U_0}$ is smooth and projective over $\mathcal{O}_{F,w}$ ([HT], page 109). As $U^p$ varies, the inverse system of the $X_{U_0}$'s has an action of $G(\mathbb{A}^{\infty,p})$.

Given an $(r + 3)$-tuple as above we will write $G_A$ for $\varepsilon A[w^\infty]$ a Barsotti-Tate $\mathcal{O}_{F,w}$-module. Over a base in which $p$ is nilpotent it is one dimensional. If $A$ denotes the universal abelian scheme over $X_{U_0}$, we will write $\mathcal{G}$ for $G_A$. This $\mathcal{G}$ is compatible, i.e. the two actions of $\mathcal{O}_{F,w}$ on $\text{Lie} \mathcal{G}$ coincide (see [HT]).

Write $\overline{X}_{U_0}$ for the special fibre $X_{U_0} \times_{\text{Spec} \mathcal{O}_{F,w}} \text{Spec} k(w)$. For $0 \leq h \leq n - 1$, we let $\overline{X}_{U_0}^{[h]}$ denote the reduced closed subscheme of $\overline{X}_{U_0}$ whose closed geometric points $s$ are those for which the maximal étale quotient of $\mathcal{G}_s$ has $\mathcal{O}_{F,w}$-height at most $h$, and let

$$\overline{X}_{U_0}^{(h)} = \overline{X}_{U_0}^{[h]} - \overline{X}_{U_0}^{[h - 1]}$$

(where we set $\overline{X}_{U_0}^{[-1]} = \emptyset$). Then $\overline{X}_{U_0}^{(h)}$ is smooth of pure dimension $h$ (corollary III.4.4 of [HT]), and on it there is a short exact sequence

$$(0) \to \mathcal{G}^0 \to \mathcal{G} \to \mathcal{G}^{\text{et}} \to (0)$$

where $\mathcal{G}^0$ is a formal Barsotti-Tate $\mathcal{O}_{F,w}$-module and $\mathcal{G}^{\text{et}}$ is an étale Barsotti-Tate $\mathcal{O}_{F,w}$-module with $\mathcal{O}_{F,w}$-height $h$.

**Lemma 2.1.** If $0 \leq h \leq n - 1$ then the Zariski closure of $\overline{X}_{U_0}^{(h)}$ contains $\overline{X}_{U_0}^{(0)}$.

**Proof:** This is ‘well known’, but for lack of a reference we give a proof. Let $x$ be a closed geometric point of $\overline{X}_{U_0}^{(0)}$. By lemma II.4.1 of [HT] the formal completion of $\overline{X}_{U_0} \times \text{Spec} k(w)^{ac}$ at $x$ is isomorphic to the equicharacteristic universal deformation ring of $\mathcal{G}_x$. According to the proof of proposition 4.2 of [Dr] this is $\text{Spf} k(w)^{ac}[T_1, \ldots, T_{n - 1}]$ and we can choose the $T_i$ and a formal parameter $S$ on the universal deformation of $\mathcal{G}_x$ such that

$$[\varpi] S \equiv \varpi_w S + \sum_{i=1}^{n-1} T_i S^{\#k(w)^i} + S^{\#k(w)^n} \pmod{S^{\#k(w)^{n + 1}}}.$$ 

Thus we get a morphism

$$\text{Spec} k(w)^{ac}[T_1, \ldots, T_{n - 1}] \to \overline{X}_{U_0}$$

lying over $x : k(w)^{ac} \to \overline{X}_{U_0}$, such that, if $k$ denotes the algebraic closure of the field of fractions of $k(w)^{ac}[T_1, \ldots, T_{n - 1}]/(T_1, \ldots, T_{n - h - 1})$, then the induced map

$$\text{Spec} k \to \overline{X}_{U_0}$$
factors through $\overline{X}_{U_0}^{(h)}$. Thus $x$ is in the closure of $\overline{X}_{U_0}^{(h)}$, and the lemma follows. □

Now we define the functor $\mathcal{X}_U$. Again we initially define it as a functor from the category of connected locally noetherian schemes with a geometric point to sets, but then (as above) we extend it to a functor from locally noetherian schemes to sets. The functor $\mathcal{X}_U$ will send $(S, s, (A, \lambda, i, \overline{p}, o, \alpha_i))$ to the set of prime-to-$p$ isogeny classes of $(r + 4)$-tuples $(A, \lambda, i, \overline{p}, o, \alpha_i)$, where $(A, \lambda, i, \overline{p}, o, \alpha_i)$ is as in the definition of $gX_{U_0}$ and $C$ is a chain of isogenies
\[
C : \mathcal{G} = \mathcal{G}_0 \to \mathcal{G}_1 \to \cdots \to \mathcal{G}_n = \mathcal{G}/\mathcal{G}[w]
\]
of compatible Barsotti-Tate $\mathcal{O}_{F, w}$-modules, each of degree $\#(w)$ and with composite equal to the canonical map $\mathcal{G} \to \mathcal{G}/\mathcal{G}[w]$. There is a natural transformation of functors $\mathcal{X}_U \to \mathcal{X}_{U_0}$.

**Lemma 2.2.** The functor $\mathcal{X}_U$ is represented by a scheme $X_U$ which is finite over $X_{U_0}$. The scheme $X_U$ has some irreducible components of dimension $n$.

**Proof:** By denoting the kernel of $\mathcal{G}_0 \to \mathcal{G}_j$ by $K_j \subset \mathcal{G}[w]$, we can view the above chain as a flag
\[
0 = K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_{n-1} \subset K_n = \mathcal{G}[w]
\]
of closed finite flat subgroup schemes with $\mathcal{O}_{F, w}$-action, with each $K_j/K_{j-1}$ having order $\#(w)$. Let $\mathcal{H}$ denote the sheaf of Hopf algebras over $X_{U_0}$ defining $\mathcal{G}[w]$. Then $\mathcal{X}_U$ is represented by a closed subscheme $X_U$ of the Grassmanian of chains of locally free direct summands of $\mathcal{H}$. (The closed conditions require that the subsheaves are sheaves of ideals defining a flag of closed subgroup schemes with the desired properties.) Thus $X_U$ is projective over $\mathcal{O}_{F, w}$. At each closed geometric point $s$ of $X_{U_0}$ the number of possible $\mathcal{O}_{F, w}$-submodules of $\mathcal{G}[w]_s \cong \mathcal{G}[w]_s^0 \times \mathcal{G}[w]_s^{et}$ is finite, so $X_U$ is finite over $X_{U_0}$. To see that $X_U$ has some components of dimension $n$ it suffices to note that on the generic fibre the map to $X_{U_0}$ is finite etale. □

We say an isogeny $\mathcal{G} \to \mathcal{G}'$ of one-dimensional compatible Barsotti-Tate $\mathcal{O}_{F, w}$-modules over a scheme $S$ of characteristic $p$ has **connected kernel** if it induces the zero map on Lie $\mathcal{G}$. We will denote the relative Frobenius map by $F : \mathcal{G} \to \mathcal{G}^{(p)}$ and let $f = [k(w) : F_p]$, and then $F^f : \mathcal{G} \to \mathcal{G}((\#(w)))$ is an isogeny of compatible Barsotti-Tate $\mathcal{O}_{F, w}$-modules of degree $\#(w)$ and has connected kernel.

We have the following rigidity lemma.

**Lemma 2.3.** Let $W$ denote the ring of integers of the completion of the maximal unramified extension of $F_w$. Suppose that $R$ is an Artinian local $W$-algebra with residue field $k(w)^{ac}$. Suppose also that
\[
C : \mathcal{G}_0 \to \mathcal{G}_1 \to \cdots \to \mathcal{G}_n = \mathcal{G}_0
\]
is a chain of isogenies of degree $\#(w)$ of one-dimensional compatible formal Barsotti-Tate $\mathcal{O}_{F, w}$-modules of $\mathcal{O}_{F, w}$-height $n$ with composite equal to multiplication by $\varpi_w$. If every isogeny $\mathcal{G}_i \to \mathcal{G}_i$ has connected kernel (for $i = 1, \ldots, n$) then $R$ is a $k(w)^{ac}$-algebra and $C$
is the pull-back of a chain of Barsotti-Tate $\mathcal{O}_{F,w}$-modules over $k(w)^{ac}$, with all the isogenies isomorphic to $F^f$.

Proof: As the composite of the $n$ isogenies induces multiplication by $\varpi_w$ on the tangent space, $\varpi_w = 0$ in $R$, i.e. $R$ is a $k(w)^{ac}$-algebra. Choose a parameter $T_i$ for $\mathcal{G}_i$ over $R$. With respect to these choices, let $f_i(T_i) \in R[[T_i]]$ represent $\mathcal{G}_{i-1} \rightarrow \mathcal{G}_i$. We can write $f_i(T_i) = g_i(T_i^{nh_i})$ with $h_i \in \mathbb{Z}_{\geq 0}$ and $g'_i(0) \neq 0$. (See [F], chapter I, §3, Theorem 2.) As $\mathcal{G}_{i-1} \rightarrow \mathcal{G}_i$ has connected kernel, $f'_i(0) = 0$ and $h_i > 0$. As $f_i$ commutes with the action $[r]$ for all $r \in \mathcal{O}_{F,w}$, we have $\pi^{nh_i} = \pi$ for all $\pi \in k(w)$, hence $h_i$ is a multiple of $f = [k(w) : \mathbb{F}_p]$. Reducing modulo the maximal ideal of $R$ we see that $h_i \leq f$ and in fact $h_i = f$ and $g'_i(0) \in R^\times$. Thus $\mathcal{G}_i \cong \mathcal{G}_0^{(\#k(w)^n)}$ in such a way that the isogeny $\mathcal{G}_0 \rightarrow \mathcal{G}_i$ is identified with $F^{f_i}$. In particular $\mathcal{G}_0 \cong \mathcal{G}_0^{(\#k(w)^m)}$ and hence $\mathcal{G}_0 \cong \mathcal{G}_0^{(\#k(w)^n m)}$ for any $m \in \mathbb{Z}_{\geq 0}$. As $R$ is Artinian some power of the absolute Frobenius on $R$ factors through $k(w)^{ac}$. Thus $\mathcal{G}_0$ is a pull-back from $k(w)^{ac}$ and the lemma follows. $\square$

Now let $Y_{U,i}$ denote the closed subscheme of $\overline{X}_U = X_U \times_{\text{Spec} \mathcal{O}_{F,w}} \text{Spec} k(w)$ over which $\mathcal{G}_{i-1} \rightarrow \mathcal{G}_i$ has connected kernel.

**Proposition 2.4.**

1. $X_U$ has pure dimension $n$ and semistable reduction over $\mathcal{O}_{F,w}$, that is, for all closed points $x$ of the special fibre $X_U \times_{\text{Spec} \mathcal{O}_{F,w}} \text{Spec} k(w)$, there exists an etale morphism $V \rightarrow X_U$ with $x \in \text{Im} V$ and an etale $\mathcal{O}_{F,w}$-morphism:

$$V \rightarrow \text{Spec} \mathcal{O}_{F,w}[T_1, \ldots, T_n]/(T_1 \cdots T_m - \varpi_w)$$

for some $1 \leq m \leq n$, where $\varpi_w$ is a uniformizer of $\mathcal{O}_{F,w}$.

2. $X_U$ is regular and the natural map $X_U \rightarrow X_{U_0}$ is finite and flat.

3. Each $Y_{U,i}$ is smooth over $\text{Spec} k(w)$ of pure dimension $n - 1$, $\overline{X}_U = \bigcup_{i=1}^n Y_{U,i}$ and, for $i \neq j$ the schemes $Y_{U,i}$ and $Y_{U,j}$ share no common connected component. In particular, $X_U$ has strictly semistable reduction.

Proof: In this proof we will make repeated use of the following version of Deligne’s homogeneity principle ([DR]). Write $W$ for the ring of integers of the completion of the maximal unramified extension of $F_w$. In what follows, if $s$ is a closed geometric point of an $\mathcal{O}_{F,w}$-scheme $X$ locally of finite type, then we write $\mathcal{O}_{X,s}$ for the completion of the strict Henselisation of $X$ at $s$, i.e. $\mathcal{O}_{X,s} = \mathcal{O}_{X,s} \times_{\text{Spec} W,s}$. Let $P$ be a property of complete noetherian local $W$-algebras such that if $X$ is a $\mathcal{O}_{F,w}$-scheme locally of finite type then the set of closed geometric points $s$ of $X$ for which $\mathcal{O}_{X,s}$ has property $P$ is Zariski open. Also let $X \rightarrow X_{U_0}$ be a finite morphism with the following properties

(i) If $s$ is a closed geometric point of $\overline{X}_{U_0}$ then, up to isomorphism, $\mathcal{O}_{X,s}$ does not depend on $s$ (but only on $h$).

(ii) There is a unique geometric point of $X$ above any geometric point of $\overline{X}_{U_0}$.
If $O_{X,s}^\wedge$ has property $P$ for every geometric point of $X$ over $\overline{X}_{U_0}^{(0)}$, then $O_{X,s}^\wedge$ has property $P$ for every closed geometric point of $X$. Indeed, if we let $Z$ denote the closed subset of $X$ where $P$ does not hold, then its image in $X_{U_0}$ is closed and is either empty or contains some $\overline{X}_{U_0}^{(h)}$. In the latter case, by lemma 2.1, it also contains $\overline{X}_{U_0}^{(0)}$, which is impossible. Thus $Z$ must be empty.

Note that both $X = X_U$ and $X = Y_{U,i}$ satisfy the above condition (ii) for the homogeneity principle, by letting $R = k(w)^{ac}$ in lemma 2.3.

(1): The dimension of $O_{X_U,s}^\wedge$ as $s$ runs over geometric points of $X_U$ above $\overline{X}_{U_0}^{(0)}$ is constant, say $m$. Applying the homogeneity principle to $X = X_U$ with $P$ being ‘dimension $m$’, we see that $X_U$ has pure dimension $m$. By lemma 2.2 we must have $m = n$ and $X_U$ has pure dimension $n$.

Now we will apply the above homogeneity principle to $X = X_U$ taking $P$ to be ‘isomorphic to $W[[T_1, ..., T_n]]/(T_1 \cdots T_m - \varpi_w)$ for some $m \leq n'$. By a standard argument (see e.g. the proof of proposition 4.10 of [Y]) the set of points with this property is open and if all closed geometric points of $X_U$ have this property $P$ then $X_U$ is semistable of pure dimension $n$.

Let $s$ be a geometric point of $X_U$ over a point of $\overline{X}_{U_0}^{(0)}$. Choose a basis $e_i$ of Lie $G_i$ over $O_{X_U,s}^\wedge$ such that $e_n$ maps to $e_0$ under the isomorphism $G_n = G_0[G_0[w] \to G_0$ induced by $\varpi_w$. With respect to these bases let $X_i \in O_{X_U,s}^\wedge$ represent the linear map Lie $G_{i-1} \to$ Lie $G_i$. Then

$$X_1 \cdots X_n = \varpi_w.$$  

Moreover it follows from lemma 2.3 that $O_{X_U,s}^\wedge/(X_1, ..., X_n) = k(w)^{ac}$. (Because, by lemma III.4.1 of [HT], $O_{X_U,s}^\wedge$ is the universal deformation space of $G_s$. Hence by lemma 2.3, $O_{X_U,s}^\wedge$ is the universal deformation space for the chain

$$G_s \xrightarrow{F^f} G_s^{\# k(w)} \xrightarrow{F^f} \cdots \xrightarrow{F^f} G_s^{\# k(w)^n} \cong G_s[G_s[\varpi_w].)$$

Thus we get a surjection

$$W[[X_1, ..., X_n]]/(X_1 \cdots X_n - \varpi_w) \to O_{X_U,s}^\wedge$$

and as $O_{X_U,s}^\wedge$ has dimension $n$ this map must be an isomorphism.

(2): We see at once that $X_U$ is regular. Then [AK] V, 3.6 tells us that $X_U \to X_{U_0}$ is flat.

(3): We apply the homogeneity principle to $X = Y_{U,i}$ taking $P$ to be ‘formally smooth of dimension $n - 1$’. If $s$ is a geometric point of $Y_{U,i}$ above $\overline{X}_{U_0}^{(0)}$ then we see that $O_{Y_{U,i},s}^\wedge$ is cut out in $O_{X_U,s}^\wedge \cong W[[X_1, ..., X_n]]/(X_1 \cdots X_n - \varpi_w)$ by the single equation $X_i = 0$. (We are using the parameters $X_i$ defined above.) Thus

$$O_{Y_{U,i},s}^\wedge \cong k(w)^{ac}[[X_1, ..., X_{i-1}, X_{i+1}, ..., X_n]]$$
is formally smooth of dimension $n - 1$. We deduce that $Y_{U,i}$ is smooth of pure dimension $n - 1$.

As our $G/X_U$ is one-dimensional, over a closed point, at least one of the isogenies $G_{i-1} \to G_i$ must have connected kernel, which shows that $X_U = \bigcup_i Y_{U,i}$. Suppose $Y_{U,i}$ and $Y_{U,j}$ share a connected component $Y$ for some $i \neq j$. Then $Y$ would be finite flat over $X_{U_0}$ and so the image of $Y$ would meet $X_{U_0}^{(n-1)}$. This is impossible as above a closed point of $X_{U_0}^{(n-1)}$ one isogeny among the chain can have connected kernel. Thus, for $i \neq j$ the closed subschemes $Y_{U,i}$ and $Y_{U,j}$ have no connected component in common. \hfill $\square$

By the strict semistability, if we write, for $S \subset \{1, ..., n\}$,

$$Y_{U,S} = \bigcap_{i \in S} Y_{U,i}, \quad Y_{U,S}^0 = Y_{U,S} - \bigcup_{T \supset S} Y_{U,T}$$

then $Y_{U,S}$ is smooth over $\text{Spec} \ k(w)$ of pure dimension $n - \#S$ and $Y_{U,S}^0$ are disjoint for different $S$. With respect to the finite flat map $X_U \to X_{U_0}$, the inverse image of $X_{U_0}^{(h)}$ is exactly the locus where at least $n - h$ of the isogenies have connected kernel, i.e., $\bigcup_{\#S \geq n-h} Y_{U,S}$.

Hence the inverse image of $X_{U_0}^{(h)}$ is equal to $\bigcup_{\#S = n-h} Y_{U,S}^0$. Also note that the inverse system of $Y_{U,S}^0$ for varying $U^p$ is stable by the action of $G(\mathbb{A}^{\infty,p})$.

Next we will relate the open strata $Y_{U,S}^0$ to the Igusa varieties of the first kind defined in [HT]. For $0 \leq h \leq n - 1$ and $m \in \mathbb{Z}_{\geq 0}$, we write $I_{U^p,m}^{(h)}$ for the Igusa varities of the first kind defined on page 121 of [HT]. We also define an Iwahori-Igusa variety of the first kind

$$I_{U}^{(h)} / X_{U_0}^{(h)}$$

as the moduli space of chains of isogenies

$$G^\text{et} = G_0 \to G_1 \to \cdots \to G_h = G^\text{et} / G^\text{et}[w]$$

eq \text{etale Barsotti-Tate } \mathcal{O}_{F,w}\text{-modules, each isogeny having degree } \#k(w) \text{ and with composite equal to the natural map } G^\text{et} \to G^\text{et}/G^\text{et}[w]. \text{ Then } I_{U}^{(h)} \text{ is finite etale over } X_{U_0}^{(h)}, \text{ and as the Igusa variety } I_{U^p,(1,m)}^{(h)} \text{ classifies the isomorphisms}

$$\alpha^\text{et}_1 : (w^{-1}\mathcal{O}_{F,w}/\mathcal{O}_{F,w})^h_{X_{U_0}^{(h)}} \to G^\text{et}[w],$$

the natural map

$$I_{U^p,(1,m)}^{(h)} \to I_{U}^{(h)}$$

is finite etale and Galois with Galois group $B_h(k(w))$. Hence we can identify $I_{U}^{(h)}$ with $I_{U^p,(1,m)}^{(h)}/B_h(k(w))$. Note that the system $I_{U}^{(h)}$ for varying $U^p$ naturally inherits the action of $G(\mathbb{A}^{\infty,p})$.

**Lemma 2.5.** For $S \subset \{1, ..., n\}$ with $\#S = n - h$, there exists a finite map of $X_{U_0}^{(h)}$-schemes

$$\varphi : Y_{U,S}^0 \to I_{U}^{(h)}$$
which is bijective on the geometric points.

Proof: The map is defined in a natural way from the chain of isogenies \( \mathcal{C} \) by passing to the etale quotient \( \mathcal{G}^{et} \), and it is finite as \( Y_{U,S}^{0} \) (resp. \( I_{U}^{(h)} \)) is finite (resp. finite etale) over \( \mathcal{X}_{U_{0}^{0}}^{(h)} \). Let \( s \) be a closed geometric point of \( I_{U}^{(h)} \) with a chain of isogenies

\[
\mathcal{G}_{s}^{et} = \mathcal{G}_{0}^{et} \to \cdots \to \mathcal{G}_{h}^{et} = \mathcal{G}_{s}^{et} / \mathcal{G}_{s}^{et} \{w\}.
\]

For \( 1 \leq i \leq n \) let \( j(i) \) denote the number of elements of \( S \) which are less than or equal to \( i \). Set \( \mathcal{G}_{i} = (\mathcal{G}_{s}^{et})(\# k(w)^{j(i)} \times \mathcal{G}_{t-i}^{et}(i)). \) If \( i \notin S \), define an isogeny \( \mathcal{G}_{i-1} \to \mathcal{G}_{i} \) to be the identity times the given isogeny \( \mathcal{G}_{i}^{et} \to \mathcal{G}_{i}^{et} \). If \( i \in S \), define an isogeny \( \mathcal{G}_{i-1} \to \mathcal{G}_{i} \) to be \( F^{j} \) times the identity. Then

\[
\mathcal{G}_{0} \to \cdots \to \mathcal{G}_{n}
\]
defines the unique geometric point of \( Y_{U,S}^{0} \) above \( s \). □

Now recall from [HT] III.2, that for an irreducible algebraic representation \( \xi \) of \( G \) over \( \mathbb{Q}^{ac} \), one can associate a lisse \( \mathbb{Q}^{ac} \)-sheaf \( \mathcal{L}_{\xi} / X_{U} \) for every \( U \) such that \( X_{U} \) is defined, and the action of \( G(\mathbb{A}^{\infty,p}) \) extends to \( \mathcal{L}_{\xi} \). The sheaf \( \mathcal{L}_{\xi} \) is extended to the integral models and Igusa varieties, and on \( I_{U_{p}(1,m)}^{(h)} \) and \( Y_{U_{p},S}^{0} \) they are the pull back of \( \mathcal{L}_{\xi} \) on \( \mathcal{X}_{U_{0}^{0}}^{(h)} \).

Corollary 2.6. For every \( i \in \mathbb{Z}_{\geq 0} \), we have isomorphisms

\[
H^{i}_{c}(Y_{U,S}^{0} \times k(w)^{ac}, \mathcal{L}_{\xi}) \xrightarrow{\sim} H^{i}_{c}(I_{U}^{(h)} \times k(w)^{ac}, \mathcal{L}_{\xi})
\]

\[
\xrightarrow{\sim} H^{i}_{c}(I_{U_{p}(1,m)}^{(h)} \times k(w)^{ac}, \mathcal{L}_{\xi}) B_{h}(k(w))
\]

that are compatible with the actions of \( G(\mathbb{A}^{\infty,p}) \) when we vary \( U_{p} \).

Proof: By lemma 2.5, for any lisse \( \mathbb{Q}^{ac} \)-sheaf \( \mathcal{F} \) on \( I_{U}^{(h)} \), we have \( \mathcal{F} \cong \varphi_{*} \varphi^{*} \mathcal{F} \) by looking at the stalks at all geometric points. As \( \varphi \) is finite the first isomorphism follows. The second isomorphism follows easily as \( I_{U_{p}(1,m)}^{(h)} \to I_{U}^{(h)} \) is finite etale and Galois with Galois group \( B_{h}(k(w)) \). □

In the next section, we will be interested in the \( G(\mathbb{A}^{\infty,p}) \times \text{Frob}_{w}^{\mathbb{F}} \)-modules

\[
H^{i}(Y_{1 w(m),S}; \mathcal{L}_{\xi}) = \lim_{U_{p}} H^{i}(Y_{U,S} \times k(w)^{ac}; \mathcal{L}_{\xi}).
\]

Here we relate the alternating sum of these modules to the cohomology of Igusa varieties. We will define the elements of Groth \( (G(\mathbb{A}^{\infty,p}) \times \text{Frob}_{w}^{\mathbb{F}}) \) (we write Groth \( (G) \) for the
Grothendieck group of admissible $G$-modules) as follows:

$$
\left[ H(Y_{Iw(m)}, S, \mathcal{L}_\xi) \right] = \sum_i (-1)^{n-\#S-i} H^i(Y_{Iw(m)}, S, \mathcal{L}_\xi),
$$

$$
\left[ H_c(Y^0_{Iw(m)}, S, \mathcal{L}_\xi) \right] = \sum_i (-1)^{n-\#S-i} \lim_{U^p} H^i_c(Y^0_{U,S} \times_{k(w)} k(w)^{ac}, \mathcal{L}_\xi),
$$

$$
\left[ H_c(I^{(h)}_{Iw(m)}, \mathcal{L}_\xi) \right] = \sum_i (-1)^{h-i} \lim_{U^p} H^i_c(I^{(h)}_{U} \times_{k(w)} k(w)^{ac}, \mathcal{L}_\xi).
$$

Then, because

$$
Y_{U,S} = \bigcup_{T \supseteq S} Y^0_{U,T}
$$

for each $U = U^p \times Iw(m)$, we have equalities

$$
\left[ H(Y_{Iw(m)}, S, \mathcal{L}_\xi) \right] = \sum_{T \supseteq S} (-1)^{(n-\#S)-(n-\#T)} \left[ H_c(Y^0_{Iw(m)}, T, \mathcal{L}_\xi) \right],
$$

$$
= \sum_{T \supseteq S} (-1)^{(n-\#S)-(n-\#T)} \left[ H_c(I^{(n-\#T)}_{Iw(m)}, \mathcal{L}_\xi) \right].
$$

As there are $\binom{n-\#S}{h}$ subsets $T$ with $\#T = n - h$ and $T \supseteq S$, we conclude:

**Lemma 2.7.** We have an equality

$$
\left[ H(Y_{Iw(m)}, S, \mathcal{L}_\xi) \right] = \sum_{h=0}^{n-\#S} (-1)^{n-\#S-h} \binom{n-\#S}{h} \left[ H_c(I^{(h)}_{Iw(m)}, \mathcal{L}_\xi) \right]
$$

in the Grothendieck group of admissible $G(A^{\infty,p}) \times \text{Frob}_w$-modules over $\mathbb{Q}_{l}^{ac}$.

3. Proof of the main theorem

We now return to the situation in theorem 1.1. Recall that $L$ is an imaginary CM field and that $\Pi$ is a cuspidal automorphic representation of $GL_n(A_L)$ such that

- $\Pi \circ c \cong \Pi^\vee$;
- $\Pi_\infty$ has the same infinitesimal character as some algebraic representation over $\mathbb{C}$ of the restriction of scalars from $L$ to $\mathbb{Q}$ of $GL_n$;
- and for some finite place $x$ of $L$ the representation $\Pi_x$ is square integrable.

Recall also that $v$ is a place of $L$ above a rational prime $p$, that $l \neq p$ is a second rational prime and that $\iota: \mathbb{Q}_{l}^{ac} \cong \mathbb{C}$. Recall finally that $R_l(\Pi)$ is the $l$-adic representation associated to $\Pi$.

Choose a quadratic CM extension $L'/L$ in which $v$ and $x$ split. Choose places $v' \neq x'$ of $L'$ above $v$ and $x$ respectively. Also choose an imaginary quadratic field $E$ and a totally real field $F^+$ such that
• $[F^+: \mathbb{Q}]$ is even;
• $F = EF^+$ is soluble and Galois over $L'$;
• $p$ splits as $uw^*$ in $E$;
• there is a place $w$ of $F$ above $u$ and $v'$ such that $\Pi_{F,w}$ has an Iwahori fixed vector;
• $x$ lies above a rational prime which splits in $E$ and $x'$ splits in $F$.

Denote by $\Pi_F$ the base change of $\Pi$ to $GL_n(\mathbb{A}_F)$. Note that the component of $\Pi_F$ at a place above $x'$ is square integrable and hence $\Pi_F$ is cuspidal.

Choose a division algebra $B$ with centre $F$ as in the previous section and satisfying

• $B_x$ is split for all places $x \neq z, z^c$ of $F$.

Also choose $\gamma$ and $G$ as in the previous section. Then it follows from theorem VI.2.9 and lemma VI.2.10 of [HT] that we can find

• a character $\psi : \mathbb{A}_E^\times \to \mathbb{C}^\times$,
• an irreducible algebraic representation $\xi$ of $G$ over $\mathbb{Q}_l^{ac}$,
• and an automorphic representation $\pi$ of $G(\mathbb{A})$,

such that

• $\pi_\infty$ is cohomological for $\iota \xi$,
• $\psi$ is unramified above $p$,
• $\psi^{[1]}|_{E_{\infty}^\times}$ is the inverse of the restriction of $\iota \xi$ to $E_{\infty}^\times \subset G(\mathbb{R})$,
• $\psi^{[1]}$ is the restriction of the central character of $\Pi_F$ to $\mathbb{A}_E^\times$,
• and if $x$ is a rational prime which splits $yy^{-}$ in $E$ then $\pi_x = (\bigotimes_{z \mid y} JL^{-1}(\Pi_z)) \otimes \psi_y$ as a representation of $(B_{p}^{op})^\times \times \mathbb{Q}_l^\times \cong (\bigotimes_{z \mid y} (B_{p}^{op})^\times) \times \mathbb{Q}_l^\times$.

Here $JL$ denotes the identity if $B_z$ is split and denotes the Jacquet-Langlands correspondence if $B_z$ is a division algebra. (See section I.3 of [HT].)

We will call two irreducible admissible representations $\pi'$ and $\pi''$ of $G(\mathbb{A}^\infty)$ nearly equivalent if $\pi'_x \cong \pi''_x$ for all but finitely many rational primes $x$. If $M$ is an admissible $G(\mathbb{A}^\infty)$-module and $\pi'$ is an irreducible admissible representation of $G(\mathbb{A}^\infty)$ then we define the $\pi'$-near isotypic component $M[\pi']$ of $M$ to be the largest $G(\mathbb{A}^\infty)$-submodule of $M$ all whose irreducible subquotients are nearly equivalent to $\pi'$. Then

$$M = \bigoplus M[\pi']$$

as $\pi'$ runs over near equivalence classes of irreducible admissible $G(\mathbb{A}^\infty)$-modules. (This follows from the following fact. Suppose that $A$ is a (commutative) polynomial algebra over $\mathbb{C}$ in countably many variables, and that $M$ is an $A$-module which is finitely generated over
Then we can write
\[ M = \bigoplus_m M_m, \]
where \( m \) runs over maximal ideals of \( A \) with residue field \( \mathbb{C} \).

We consider the Shimura varieties \( X_U/F \) for open compact subgroups \( U \) of \( G(\mathbb{A}^\infty) \) as in the last section. Then
\[ H^i(X, \mathcal{L}_\xi) = \lim_{\rightarrow} H^i(X_U \times_F F^{ac}, \mathcal{L}_\xi) \]
is a semisimple, admissible \( G(\mathbb{A}^\infty) \)-module with a commuting continuous action of the Galois group \( \text{Gal}(F^{ac}/F) \). (For details see III.2 of [HT].)

The following lemma follows from [HT], particularly corollary VI.2.3, corollary VI.2.7 and theorem VII.1.7.

**Lemma 3.1.** Keep the notation and assumptions above. (In particular we are assuming that \( \pi \) arises from a cuspidal automorphic representation \( \Pi \) of \( GL_n(\mathbb{A}_F) \).)

1. If \( i \neq n - 1 \) then \( H^i(X, \mathcal{L}_\xi)[\pi] = (0) \).
2. As \( G(\mathbb{A}^\infty) \times \text{Gal}(F^{ac}/F) \)-modules,
\[ H^{n-1}(X, \mathcal{L}_\xi)[\pi] = \bigoplus_{\pi'} \pi' \otimes R'_i(\Pi)^{m(\pi')} \otimes R_i(\psi), \]
where \( \pi' \) runs over irreducible admissible representations of \( G(\mathbb{A}^\infty) \) nearly equivalent to \( \pi \) and where \( m(\pi') \in \mathbb{Z}_{\geq 0} \), and \( R_i(\Pi) = R'_i(\Pi)^{ss} \).
3. \( m(\pi) > 0 \).
4. If \( m(\pi') > 0 \) then \( \pi'_p \cong \pi_p \).

If \( \pi' \) is an irreducible admissible representation of \( G(\mathbb{A}^\infty) \) we can decompose it as \( (\pi')^p \otimes (\prod_{i=2}^{\infty} \pi_i^{(\ell_i)}) \otimes \pi_0 \otimes \pi_{p,0} \), corresponding to the decomposition (1). If \( \pi'' \) is an irreducible admissible representation of \( G(\mathbb{A}^{\infty,p}) \) and \( N \) is an admissible \( G(\mathbb{A}^{\infty,p}) \)-module then we can define the \( \pi'' \)-near isotypic component of \( N \) in the same manner as we did for \( G(\mathbb{A}^\infty) \)-modules. If \( M \) is an admissible \( G(\mathbb{A}^\infty) \)-module and \( \pi' \) is an irreducible admissible representation of \( G(\mathbb{A}^\infty) \) then
\[ M^{1w(m)}[(\pi')^p] = M[\pi']^{1w(m)}. \]

We will write
\[ H^i(X_{1w(m)}, \mathcal{L}_\xi) = \lim_{\rightarrow} H^i(X_{U^p \times 1w(m)} \times_F F^{ac}, \mathcal{L}_\xi) \cong H^i(X, \mathcal{L}_\xi)^{1w(m)}. \]

It is a semisimple admissible \( G(\mathbb{A}^{\infty,p}) \)-module with a commuting continuous action of \( \text{Gal}(F^{ac}/F) \).

**Theorem 3.2.** Keep the above notation and assumptions. (In particular we are assuming that \( \pi \) arises from a cuspidal automorphic representation \( \Pi \) of \( GL_n(\mathbb{A}_F) \).) Let \( U^p \) be a
sufficiently small open compact subgroup of $G(K_{\infty,p})$. Then

$$\text{WD}(H^{n-1}(X_{Iw(m)}, \mathcal{L}_\xi)[\pi^p]^{UP})$$

is pure.

**Proof:** As $X_U = X_{UP \times Iw(m)}$ is strictly semistable by proposition 2.4, we can use the Rapoport-Zink weight spectral sequence [RZ] to compute $H^{n-1}(X_{Iw(m)}, \mathcal{L}_\xi)$. For $X_U$, it reads

$$E_{1}^{i,j}(U) = \bigoplus_{t \geq \max(0,-i)} \bigoplus_{\#S = i + 2t + 1} H^{j-2t}(Y_{Iw,m} \times k(w)^{ac}, \mathcal{L}_\xi(-t)) \Rightarrow H^{i+j}(X_U \times F_w^{ac}, \mathcal{L}_\xi).$$

Passing to the limit with respect to $U^p$, it gives rise to the following spectral sequence of admissible $G(F) \times \text{Frob}_w$-modules

$$E_{1}^{i,j}(Iw(m)) = \bigoplus_{t \geq \max(0,-i)} \bigoplus_{\#S = i + 2t + 1} H^{j-2t}(Y_{Iw(m),S}, \mathcal{L}_\xi(-t)) \Rightarrow H^{i+j}(X_{Iw(m),S}, \mathcal{L}_\xi).$$

Hence we get a spectral sequence of $\text{Frob}_w$-modules

$$E_{1}^{i,j}(Iw(m))[\pi^p]^{UP} \Rightarrow H^{i+j}(X_{Iw(m),S}, \mathcal{L}_\xi)[\pi^p]^{UP}. \tag{2}$$

The sheaf $\mathcal{L}_\xi$ is pure, say of weight $w_\xi$. Thus the action of $\text{Frob}_w$ on $E_{1}^{i,j}$ is pure of weight $w_\xi + j$ by the Weil conjectures. The theory of weight spectral sequence ([RZ]) defines an operator

$$N : E_{1}^{i,j}(Iw(m))[\pi^p]^{UP}(1) \to E_{1}^{i+2j-2}(Iw(m))[\pi^p]^{UP},$$

which induces the $N$ for $\text{WD}(H^{i+j}(X_{Iw(m),S}, \mathcal{L}_\xi)[\pi^p]^{UP})$ and has the property that

$$N^i : E_{1}^{i,j+i}(Iw(m))[\pi^p]^{UP}(i) \to E_{1}^{i,j-i}(Iw(m))[\pi^p]^{UP}$$

for all $i$. If the spectral sequence (2) degenerates at $E_1$, then it follows that the Weil-Deligne representation $\text{WD}(H^{n-1}(X_{Iw(m),S}, \mathcal{L}_\xi)[\pi^p]^{UP})$ is pure of weight $w_\xi + (n-1)$. Thus it suffices to show that

$$E_{1}^{i,j}(Iw(m))[\pi^p]^{UP} = (0)$$

if $i + j \neq n - 1$, i.e. that

$$H^{j}(Y_{Iw(m),S}, \mathcal{L}_\xi)[\pi^p]^{UP} = (0)$$

if $j \neq n - \#S$.

We first recall some notation from [HT]. For $h = 0, \ldots, n-1$ let $P_h$ denote the maximal parabolic in $GL_n$ consisting of matrices $g \in GL_n$ with $g_{ij} = 0$ for $i > n-h$ and $j \leq n-h$. Also let $N_h$ denote the unipotent radical of $P_h$, let $P_{h,op}$ denote the opposite parabolic and let $N_{h,op}$ denote the unipotent radical of $P_{h,op}$. Let $D_{F_w,n-h}$ denote the division algebra with centre $F_w$ and Hasse invariant $1/(n-h)$. If $\pi'$ is a square integrable representation of $GL_{n-h}(F_w)$, let $\varphi_{\pi'}$ denote a pseudo-coefficient for $\pi'$ as in section I.3 of [HT]. (Note that this depends on the choice of a Haar measure, but in the formulae below this choice will always be cancelled by the choice of an associated Haar measure on $D_{F_w,n-h}^\times$. See [HT] for details.)
If we introduce the limit of cohomology groups of Igusa varieties for varying level structure at \( p \) as in (see p.136 of [HT]):

\[
[H_c(I^{(h)}, \mathcal{L}_\xi)] = \sum_i (-1)^{h-i} \lim_{U_p,m} H^i_c(I^{(h)}_{U_p,m} \times_{k(w)} k(w)^{ac}, \mathcal{L}_\xi),
\]

then the second isomorphism of corollary 2.6 and theorem V.5.4 of [HT] tell us that

\[
n[H_c(I^{(h)}_{Iw(m)}, \mathcal{L}_\xi)] = n[H_c(I^{(h)}, \mathcal{L}_\xi)]_{U_p(m) \times Iw,h,w}^{U_w(m)}
= \sum_i (-1)^{n-i} \text{Red}^{(h)}[H^i(X, \mathcal{L}_\xi)]_{U_p(m)}^{U_w(m)}
\]

in Groth \((G(\mathbb{A}^{\infty,p}) \times \text{Frob}_w^\mathbb{Z})\), where

\[
\text{Red}^{(h)} : \text{Groth} (GL_n(F_w) \times \mathbb{Q}_p^\times) \longrightarrow \text{Groth} (\text{Frob}_w^\mathbb{Z})
\]

is the composite of the normalised Jacquet functor

\[
J_{X_w}^{\mathbb{A}^\infty} : \text{Groth} (GL_n(F_w) \times \mathbb{Q}_p^\times) \longrightarrow \text{Groth} (GL_{n-h}(F_w) \times GL_h(F_w) \times \mathbb{Q}_p^\times)
\]

with the functor

\[
\text{Groth} (GL_{n-h}(F_w) \times GL_h(F_w) \times \mathbb{Q}_p^\times) \longrightarrow \text{Groth} (\text{Frob}_w^\mathbb{Z})
\]

which sends \([\alpha \otimes \beta \otimes \gamma] \rightarrow \sum \text{vol}(D_{F_w,n-h}^X/F_w^X) \alpha) \otimes \gamma \otimes \alpha \text{Sp}_{n-h}(\phi)) (\dim \beta^{Iw,h,w}) \text{rec}(\phi^{-1} | \frac{\text{w}^{12}}{\text{w}^2} (\gamma \mathbb{Z}_p^\times \circ N_{F_w/E_w}^{-1})],
\]

where the sum is over characters \( \phi \) of \( F_w^X/O_{F_w}^X \). (We just took the \( Iw,h,w \)-invariant part of the \( \text{Red}^{(h)}_1 \), which is defined on p.182 of [HT]. Note that \( \text{Frob}_w \) acts on \( H^i_c(I^{(h)}, \mathcal{L}_\xi) \) as

\[
(1, p^{-[k(w):\mathbb{F}_p]}, -1, 1, 1) \in G(\mathbb{A}^{\infty,p}) \times (\mathbb{Q}_p^\times / \mathbb{Z}_p^\times) \times \mathbb{Z} \times GL_h(F_w) \times \prod_{i=2}^r (B_{w_i}^{ap})^\times,
\]

where we have identified \( D_{F_w,n-h}^X/O_{D_{F_w,n-h}}^X \) with \( \mathbb{Z} \) via \( w(\det) \).

In particular, by lemma 3.1(1), we have an equality in Groth \((\text{Frob}_w^\mathbb{Z})\):

\[
n[H_c(I^{(h)}_{Iw(m)}, \mathcal{L}_\xi)[\pi^p]^{U_p}] = \text{Red}^{(h)}[H^{n-1}(X, \mathcal{L}_\xi)]_{U_p(m)}^{U_p}(m)\pi^p[U_p].
\]

Moreover \( H^{n-1}(X, \mathcal{L}_\xi)^{Iw(m)}[\pi^p]^{U_p} \) is \( \pi_w \otimes \pi_{p,0} \)-isotypic as a \( GL_n(F_w) \times \mathbb{Q}_p^\times \)-module by lemma 3.1(4). As \( \pi_w = \Pi_{F,w} \) has an Iwahori fixed vector and \( \pi_{p,0} = \psi_u \) is unramified,

\[
(\dim \Pi_{F,w}^{Iw,w}) [H^{n-1}(X, \mathcal{L}_\xi)^{Iw(m)}[\pi^p]^{U_p}] = (\dim H^{n-1}(X, \mathcal{L}_\xi)^{Iw(m)}[\pi^p]^{U_p}) [\Pi_{F,w} \otimes \psi_u],
\]

and

\[
n(\dim \Pi_{F,w}^{Iw,w}) [H_c(I^{(h)}_{Iw(m)}, \mathcal{L}_\xi)[\pi^p]^{U_p}] = (\dim H^{n-1}(X, \mathcal{L}_\xi)^{Iw(m)}[\pi^p]^{U_p}) \text{Red}^{(h)}[\Pi_{F,w} \otimes \psi_u].
\]
Combining this with lemma 2.7, we get
\[ \dim \Pi_{F,w}^{Iw,n,w} H(Y_{Iw(m),S}, L_{\xi})[\pi_p]^{U_p} \]
\[ = (\dim H^{n-1}(X, L_{\xi})^{Iw(m)}[\pi_p]^{U_p}) \sum_{h=0}^{n-\#S} (-1)^{n-\#S-h} \left( \frac{n - \#S}{h} \right) \text{Red}^h[\Pi_{F,w} \otimes \psi_u]. \]

As \( \Pi_{F,w} \) is tempered, it is a full normalised induction of the form
\[ \text{n-Ind}_{P(F_w)}^{\text{GL}_n(F_w)}(\text{Sp}_{s_1}^{\pi_1} \otimes \cdots \otimes \text{Sp}_{s_t}^{\pi_t}), \]
where \( \pi_i \) is an irreducible cuspidal representation of \( \text{GL}_{g_i}(F_w) \) and \( P \) is a parabolic subgroup of \( \text{GL}_n \) with Levi component \( \text{GL}_{s_1} \times \cdots \times \text{GL}_{s_t} \). As \( \Pi_{F,w} \) has an Iwahori fixed vector, we must have \( g_i = 1 \) and \( \pi_i \) unramified for all \( i \). Note that, for this type of representation (full induced from square integrables \( \text{Sp}_{s_i}^{\pi_i} \) with \( \pi_i \) an unramified character of \( F_w^X \)),
\[ \dim (\text{n-Ind}_{P(F_w)}^{\text{GL}_n(F_w)}(\text{Sp}_{s_1}^{\pi_1} \otimes \cdots \otimes \text{Sp}_{s_t}^{\pi_t}))^{Iw,n,w} \]
\[ = \#P(k(w)) \text{GL}_n(k(w))/B_n(k(w)) = \frac{n!}{\prod j s_j!}. \]

We can compute \( \text{Red}^h[\Pi_{F,w} \otimes \psi_u] \) using lemma I.3.9 of [HT] (but note the typo there — “positive integers \( h_1, \ldots, h_t \)” should read “non-negative integers \( h_1, \ldots, h_t \)” ). Putting \( V_i = \text{rec}(\pi_i^{-1} | \psi_u) \big/ \text{N}_{F_w/E_w}^{-1} \), we see that
\[ \text{Red}^h[\Pi_{F,w} \otimes \psi_u] = \sum_i \dim (\text{n-Ind}_{P'(F_w)}^{\text{GL}_n(F_w)}(\text{Sp}_{s_i}^{\pi_i} | \otimes_{j \neq i} \text{Sp}_{s_j}(\pi_j)))^{Iw,k,w}[V_i] \]
\[ = \sum_i \frac{h!}{(s_i + h - n)! \prod_{j \neq i} s_j!} [V_i] \]
where the sum runs only over those \( i \) for which \( s_i \geq n - h \), and \( P' \subset \text{GL}_h \) is a parabolic subgroup. Thus
\[ n \frac{n!}{\prod_j s_j!} H(Y_{Iw(m),S}, L_{\xi})[\pi_p]^{U_p} \]
\[ = D \sum_{h=0}^{n-\#S} (-1)^{n-\#S-h} \binom{n - \#S}{h} \sum_{i: s_i \geq n-h} \frac{h!}{(s_i + h - n)! \prod_{j \neq i} s_j!} [V_i] \]
\[ = D \sum_{i=1}^{t} \frac{(n - \#S)!}{(s_i - \#S)! \prod_{j \neq i} s_j!} \sum_{h=n-s_i}^{\#S} (-1)^{n-\#S-h} \binom{s_i - \#S}{h + s_i - n} [V_i] \]
\[ = D \sum_{s_i = \#S}^{\#S} \frac{(n - \#S)!}{\prod_{j \neq i} s_j!} [V_i], \]
where \( D = \dim H^{n-1}(X, L_{\xi})^{Iw(m)}[\pi_p]^{U_p} \), and so
\[ n^{\frac{n}{\#S}} H(Y_{Iw(m),S}, L_{\xi})[\pi_p]^{U_p} = (\dim H^{n-1}(X, L_{\xi})^{Iw(m)}[\pi_p]^{U_p}) \sum_{s_i = \#S} \binom{n}{\#S} [V_i]. \]
As $\Pi_{F,w}$ is tempered, $\text{rec}(\Pi_{F,w}^\vee \otimes (\psi_u \circ N_{F_w/E_u}))|\det |_{w}^{\frac{1-n}{2}}$ is pure of weight $w\xi + (n - 1)$. Hence

$$V_i = \text{rec}(\pi_{i}^{-1} | \frac{1-#S}{w} (\psi_u \circ N_{F_w/E_u})^{-1} | \frac{#S-n}{w} )$$

is strictly pure of weight $w\xi + (n - #S)$. The Weil conjectures then tell us that

$$H^j(Y_{Iw(m),S}, \mathcal{L}_\xi)[\pi_p]^{UP} = (0)$$

for $j \neq n - #S$. The theorem follows. □

We can now conclude the proof of theorem 1.1. Choose $k$ so that $|\chi_\Pi| = |\frac{n(k+n-1)}{L}|$ where $\chi_\Pi$ is the central character of $\Pi$. Set

$$V = H^{n-1}(X_{Iw(m),L}\xi)[\pi_p]^{UP} \otimes R_t(\psi)^{-1},$$

a continuous representation of $\text{Gal}(F_{ac}/F)$. We know that

1. $V^{ss} \simeq R_t(\Pi)|_{\text{Gal}(F_{ac}/F)}^a$ for some $a \in \mathbb{Z}_{>0}$,
2. $V$ is pure of weight $k$ (proposition III.2.1 of [HT] and a computation of the determinant),
3. $\text{WD}(V|_{\text{Gal}(F_{ac}/F_{w})})$ is pure of weight $k$ (use theorem 3.2 and a computation of the determinant).

Thus lemma 1.4 tells us that $\text{WD}(R_t(\Pi)|_{\text{Gal}(L_{w}^{ac}/L_{w})})^{F_{ss}}$ is pure. On the other hand, as $\Pi_v$ is tempered (corollary VII.1.11 of [HT]), $\text{rec}(\Pi_{v}^{\vee} | \det |_{L_{w}^{ac}/L_{w}}^{\frac{1-n}{2}})$ is pure by lemma 1.3(3). As the representation of the Weil group in $\text{rec}(\Pi_{v}^{\vee} | \det |_{L_{w}^{ac}/L_{w}}^{\frac{1-n}{2}})$ and $\text{WD}(R_t(\Pi)|_{\text{Gal}(L_{w}^{ac}/L_{w})})^{F_{ss}}$ are equivalent, we deduce from lemma 1.3(4) that

$$\text{WD}(R_t(\Pi)|_{\text{Gal}(L_{w}^{ac}/L_{w})})^{F_{ss}} \simeq \text{rec}(\Pi_{v}^{\vee} | \det |_{L_{w}^{ac}/L_{w}}^{\frac{1-n}{2}}),$$

as desired.

References


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