Compatibility of Local and Global
Langlands Correspondences

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COMPATIBILITY OF LOCAL AND GLOBAL LANGLANDS CORRESPONDENCES

RICHARD TAYLOR AND TERUYOSHI YOSHIDA

Abstract. We prove the compatibility of local and global Langlands correspondences for $GL_n$, which was proved up to semisimplification in [HT]. More precisely, for the $n$-dimensional $l$-adic representation $R_l(\Pi)$ of the Galois group of a CM-field $L$ attached to a conjugate self-dual regular algebraic cuspidal automorphic representation $\Pi$, which is square integrable at some finite place, we show that Frobenius semisimplification of the restriction of $R_l(\Pi)$ to the decomposition group of a prime $v$ of $L$ not dividing $l$ corresponds to $\Pi_v$ by the local Langlands correspondence.

Introduction

This paper is a continuation of [HT]. Let $L$ be an (imaginary) CM field and let $\Pi$ be a regular algebraic cuspidal automorphic representation of $GL_n(\mathbb{A}_L)$ which is conjugate self-dual ($\Pi \circ c \cong \Pi^\vee$) and square integrable at some finite place. In [HT] it is explained how to attach to $\Pi$ and an arbitrary rational prime $l$ (and an isomorphism $\iota : \mathbb{Q}_{l}^{ec} \xrightarrow{\sim} \mathbb{C}$) a continuous semisimple representation $R_l(\Pi) : \text{Gal}(L^{ac}/L) \rightarrow GL_n(\mathbb{Q}_{l}^{ec})$ which is characterised as follows. For every finite place $v$ of $L$ not dividing $l$

\[ \iota R_l(\Pi)|_{W_{L_v}}^{ss} = \text{rec}(\Pi_v^\vee|\det|^{\frac{1-n}{2}})^{ss}, \]

where rec denotes the local Langlands correspondence and ss denotes the semisimplification (see [HT] for details). In that book it is also shown that $\Pi_v$ is tempered for all finite places $v$.

In this paper we strengthen this result to completely identify $R_l(\Pi)|_{I_v}$ for $v \not\mid l$. In particular, we prove the following theorem.

Theorem A. If $v \not\mid l$ then the Frobenius semisimplification of $R_l(\Pi)|_{W_{L_v}}$ is the $l$-adic representation attached to $\iota^{-1}\text{rec}(\Pi_v^\vee|\det|^{\frac{1-n}{2}})$.

As $R_l(\Pi)$ is semisimple and $\text{rec}(\Pi_v^\vee|\det|^{\frac{1-n}{2}})$ is indecomposable if $\Pi_v$ is square integrable, we obtain the following corollary.

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Corollary B. If $\Pi_v$ is square integrable at a finite place $v \nmid l$, the representation $R_l(\Pi)$ is irreducible.

Using base change it is easy to reduce to the case that $\Pi_v$ has an Iwahori fixed vector. We descend $\Pi$ to an automorphic representation $\pi$ of a unitary group $G$ which locally at $v$ looks like $GL_n$ and at infinity looks like $U(n-1,1) \times U(n,0)^{[L:Q]/2-1}$. Then we realise $R_l(\Pi)$ in the cohomology of a Shimura variety $X$ associated to $G$ with Iwahori level structure at $v$. More precisely, for some $l$-adic sheaf $\mathcal{L}$, the $\pi^p$-isotypic component of $H^{n-1}(X, \mathcal{L})$ is, up to semisimplification and some twist, $R_l(\Pi)^a$ (for some $a \in \mathbb{Z}_{>0}$). We show that $X$ has semistable reduction and use the results of [IT] to calculate the cohomology of the (smooth, projective) strata of the reduction of $X$ above $p$ as a virtual $G(A^\infty \times F^\mathbb{Z})$-module (where $F$ denotes Frobenius). This description and the temperedness of $\Pi_v$ shows that the $\pi^p$-isotypic component of the cohomology of any strata is concentrated in the middle dimension. This implies that the $\pi^p$-isotypic component of the Rapoport-Zink weight spectral sequence degenerates at $E_1$, which allows us to calculate the action of inertia at $v$ on $H^{n-1}(X, \mathcal{L})$.

In the special case that $\Pi_v$ is a twist of a Steinberg representation and $\Pi_{\infty}$ has trivial infinitesimal character, the above theorem presumably follows from the results of Ito [I].

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1. The main theorem

We write $F^{ac}$ for an algebraic closure of a field $F$. Let $l$ be a rational prime and fix an isomorphism $\iota: \mathbb{Q}_{l}^{ac} \cong \mathbb{C}$.

Suppose that $p \neq l$ is another rational prime. Let $K/\mathbb{Q}_p$ be a finite extension. We will let $\mathcal{O}_K$ denote the ring of integers of $K$, $\wp_K$ the unique maximal ideal of $\mathcal{O}_K$, $v_K$ the canonical valuation $K^\times \to \mathbb{Z}$, $k(v_K)$ the residue field $\mathcal{O}_K/\wp_K$ and $|\cdot|^v_K$ the absolute value normalised by $|x|_K = (\#k(v_K))^{-v_K(x)}$. We will let $\text{Frob}_{v_K}$ denote the geometric Frobenius element of $\text{Gal}(k(v_K)^{ac}/k(v_K))$. We will let $I_{v_K}$ denote the kernel of the natural surjection $\text{Gal}(K^{ac}/K) \to \text{Gal}(k(v_K)^{ac}/k(v_K))$. We will let $W_K$ denote the preimage under $\text{Gal}(K^{ac}/K) \to \text{Gal}(k(v_K)^{ac}/k(v_K))$ of $\text{Frob}_{v_K}$ endowed with a topology by decreeing that $I_K$ with its usual topology is an open subgroup of $W_K$. Local class field theory provides a canonical isomorphism $\text{Art}_K : K^\times \cong W_K^{ab}$, which takes uniformisers to lifts of $\text{Frob}_{v_K}$.

Let $\Omega$ be an algebraically closed field of characteristic 0 and of the same cardinality as $\mathbb{C}$. (Thus in fact $\Omega \cong \mathbb{C}$.) By a Weil-Deligne representation of $W_K$ over $\Omega$ we mean a finite dimensional $\Omega$-vector space $V$ together with a homomorphism $r : W_K \to GL(V)$ with open kernel and an element $N \in \text{End}(V)$ which satisfies

$$r(\sigma)N r(\sigma)^{-1} = |\text{Art}_K^{-1}(\sigma)|_K N.$$
We sometimes denote a Weil-Deligne representation by \((V, r, N)\) or simply \((r, N)\).

We call \((V, r, N)\) Frobenius semisimple if \(r\) is semisimple. If \((V, r, N)\) is any Weil-Deligne representation we define its Frobenius semisimplification \(\left( V, r, N \right)^{F-ss} = (V, r^{ss}, N)\) as follows. Choose a lift \(\phi\) of \(\text{Frob}_{v_K}\) to \(\mathbb{W}_K\). Let \(r(\phi) = su = us\) where \(s \in GL(V)\) is semisimple and \(u \in GL(V)\) is unipotent. For \(n \in \mathbb{Z}\) and \(\sigma \in I_K\) set \(r(\phi^n \sigma) = s^n r(\sigma)\). This is independent of the choices, and gives a Frobenius semisimple Weil-Deligne representation.

One of the main results of [HT] is that, given a choice of \(# k(v_K) \leq \Omega\), there is a bijection rec (the local Langlands correspondence) from isomorphism classes of irreducible smooth representations of \(GL_n(K)\) over \(\Omega\) to isomorphism classes of \(n\)-dimensional Frobenius semisimple Weil-Deligne representations of \(W_K\), and that this bijection is natural in a number of respects. (See [HT] for details.)

We will call a Weil-Deligne representation of \(W_K\) over \(\mathbb{Q}_{ac}\) bounded if for some (and hence all) \(\sigma \in W_K - I_K\) all the eigenvalues of \(r(\sigma)\) are \(l\)-adic units. There is an equivalence of categories between bounded Weil-Deligne representations of \(W_K\) over \(\mathbb{Q}_{ac}\) and continuous representations of \(\text{Gal}(K_{ac}/K)\) on finite dimensional \(\mathbb{Q}_{ac}\)-vector spaces as follows. Fix a lift \(\phi \in W_K\) of \(\text{Frob}_{v_K}\) and a continuous homomorphism \(t: I_K \to \mathbb{Z}_l\). Send a Weil-Deligne representation \((V, r, N)\) to \((V, \rho)\), where \(\rho\) is the unique continuous representation of \(\text{Gal}(K_{ac}/K)\) on \(V\) such that

\[
\rho(\phi^n \sigma) = r(\phi^n \sigma) \exp(t(\sigma)N)
\]

for all \(n \in \mathbb{Z}\) and \(\sigma \in I_K\). Up to natural isomorphism this functor is independent of the choices of \(t\) and \(\phi\). We will write \(WD(V, \rho)\) for the Weil-Deligne representation corresponding to a continuous representation \((V, \rho)\). If \(WD(V, \rho) = (V, r, N)\), then have \(\rho^{ss}_{W_K} \cong r^{ss}\). (See [T], §4 and [D], §8 for details.)

Now suppose that \(L\) is a finite, imaginary CM extension of \(\mathbb{Q}\). Let \(c \in \text{Aut}(L)\) denote complex conjugation. Suppose that \(\Pi\) is a cuspidal automorphic representation of \(GL_n(\mathbb{A}_L)\) such that

- \(\Pi \circ c \cong \Pi^\vee\);
- \(\Pi_{\infty}\) has the same infinitesimal character as some algebraic representation over \(\mathbb{C}\) of the restriction of scalars from \(L\) to \(\mathbb{Q}\) of \(GL_n\);
- and for some finite place \(x\) of \(L\) the representation \(\Pi_x\) is square integrable.

(In this paper ‘square integrable’ (resp. ‘tempered’) will mean the twist by a character of a pre-unitary representation which is square integrable (resp. tempered).) In [HT] (see theorem C in the introduction of [HT]) it is shown that there is a unique continuous semisimple representation

\[
R_t(\Pi): \text{Gal}(L_{ac}/L) \to GL_n(\mathbb{Q}_{ac})
\]
such that for each finite place $v \nmid l$ of $L$

$$\text{rec}(\Pi_v^\vee | \det \frac{1-n}{2}) = (\iota R_l(\Pi)|_{W_{L_v}}, N)$$

for some $N$. Moreover it is shown that $\Pi_v$ is tempered for all finite places $v$ of $L$, which completely determines the $N$ (see lemma 1.3 below). If $n = 1$ both these assertions are true without the assumption that $\Pi \circ c \cong \Pi^\vee$.

The main theorem of this paper identifies the $N$ of $\text{WD}(R_l(\Pi)|_{\text{Gal}(L_{\overline{v}}^c/L_v)})$ with the above $N$. More precisely we prove the following.

**Theorem 1.1.** Keep the above notation and assumptions. Then for each finite place $v \nmid l$ of $L$ there is an isomorphism

$$
\iota \text{WD}(R_l(\Pi)|_{\text{Gal}(L_{\overline{v}}^c/L_v)})^{F-ss} \cong \text{rec}(\Pi_v^\vee | \det \frac{1-n}{2})
$$

of Weil-Deligne representations over $\mathbb{C}$.

As $R_l(\Pi)$ is semisimple and $\text{rec}(\Pi_v^\vee | \det \frac{1-n}{2})$ is indecomposable if $\Pi_v$ is square integrable, we have the following corollary.

**Corollary 1.2.** If $\Pi_v$ is square integrable for a finite place $v \nmid l$, then the representation $R_l(\Pi)$ is irreducible.

In the rest of this section we consider some generalities on Galois representations and Weil-Deligne representations. First consider Weil-Deligne representations over an algebraically closed field $\Omega$ of characteristic zero and the same cardinality as $\mathbb{C}$. For a finite extension $K'/K$ of $p$-adic fields, we define

$$(V, r, N)|_{W_{K'}} = (V, r|_{W_{K'}}, N).$$

If $(W, r)$ is a finite dimensional representation of $W_K$ with open kernel and if $s \in \mathbb{Z}_{\geq 1}$ we will write $\text{Sp}_s(W)$ for the Weil-Deligne representation

$$(W^s, \ r|_{\text{Art}^{-1}_K|_K^1} \oplus \cdots \oplus \ r|_{\text{Art}^{-1}_K|_K}, \ N)$$

where $N : r|_{\text{Art}^{-1}_K|_K^i} \rightarrow r|_{\text{Art}^{-1}_K|_K}$ for $i = 1, \ldots, s - 1$. This defines $\text{Sp}_s(W)$ uniquely (up to isomorphism). If $W$ is irreducible then $\text{Sp}_s(W)$ is indecomposable and every indecomposable Weil-Deligne representation is of the form $\text{Sp}_s(W)$ for a unique $s$ and a unique irreducible $W$. If $\pi$ is an irreducible cuspidal representation of $GL_g(K)$ then $\text{rec}(\pi) = (r, 0)$ with $r$ irreducible. Moreover for any $s \in \mathbb{Z}_{\geq 1}$ we have (in the notation of section I.3 of [HT])

$$\text{rec}(\text{Sp}_s(\pi)) = \text{Sp}_s(r).$$

If $q \in \mathbb{R}_{>0}$, then by a *Weil $q$-number* we mean $\alpha \in \mathbb{Q}_{ac}$ such that for all $\sigma : \mathbb{Q}_{ac} \hookrightarrow \mathbb{C}$ we have $(\sigma \alpha)(\sigma \alpha) = q$. We will call a Weil-Deligne representation $(V, r, N)$ of $W_K$ *strictly pure of weight* $k \in \mathbb{R}$ if for some (and hence every) lift $\phi$ of $\text{Frob}_{v_K}$, every eigenvalue $\alpha$ of $r(\phi)$ is a Weil $(\# k(v_K))^k$-number. In this case we must have $N = 0$. We will call $(V, r, N)$ *mixed* if it has an increasing filtration $\text{Fil}_i^W$ with $\text{Fil}_i^W V = V$ for $i >> 0$ and $= (0)$ for
$i \ll 0$, such that the $i$-th graded piece is strictly pure of weight $i$. If $(V, r, N)$ is mixed then there is a unique choice of filtration $\text{Fil}_i^W$, and $N(\text{Fil}_i^W V) \subset \text{Fil}_{i-2}^W V$. Finally we will call $(V, r, N)$ pure of weight $k$ if it is mixed with all weights in $k + \mathbb{Z}$ and if for all $i \in \mathbb{Z}_{>0}$

$$N^i : \text{gr}_{k+i}^W V \xrightarrow{\sim} \text{gr}_{k-i}^W V.$$  

If $W$ is strictly pure of weight $k$, then $\text{Sp}_{s}(W)$ is pure of weight $k - (s - 1)$ for any $s \in \mathbb{Z}_{\geq 1}$.

(It is generally conjectured that if $X$ is a proper smooth variety over a $p$-adic field $K$, then $\text{WD}(H^i(X \times_K K^{ac}, \mathbb{Q}^{ac})$ is pure of weight $i$ in the above sense.)

**Lemma 1.3.**

1. $(V, r, N)$ is pure if and only if $(V, r, N)^{F-ss}$ is.
2. If $L/K$ is a finite extension, then $(V, r, N)$ is pure if and only if $(V, r, N)|_{W_L}$ is pure.
3. An irreducible smooth representation $\pi$ of $GL_n(K)$ has $\sigma\pi$ tempered for all $\sigma : \Omega \hookrightarrow \mathbb{C}$ if and only if $\text{rec}(\pi)$ is pure of some weight.
4. Given $(V, r)$ with $r$ semisimple, there is, up to equivalence, at most one choice of $N$ which makes $(V, r, N)$ pure.
5. An irreducible smooth representation $\pi$ of $GL_n(K)$ has $\sigma\pi$ tempered for all $\sigma : \Omega \hookrightarrow \mathbb{C}$ if and only if $\text{rec}(\pi)$ is pure of some weight.
6. Given $(V, r, N)$ with $r$ semisimple, there is, up to equivalence, at most one choice of $N$ which makes $(V, r, N)$ pure.

(1) $(V, r, N)$ is pure if and only if $(V, r, N)^{F-ss}$ is.

(2) If $L/K$ is a finite extension, then $(V, r, N)$ is pure if and only if $(V, r, N)|_{W_L}$ is pure.

(3) An irreducible smooth representation $\pi$ of $GL_n(K)$ has $\sigma\pi$ tempered for all $\sigma : \Omega \hookrightarrow \mathbb{C}$ if and only if $\text{rec}(\pi)$ is pure of some weight.

(4) Given $(V, r)$ with $r$ semisimple, there is, up to equivalence, at most one choice of $N$ which makes $(V, r, N)$ pure.

(5) If $(V, r, N)$ is a Frobenius semisimple Weil-Deligne representation which is pure of weight $k$ and if $W \subset V$ is a Weil-Deligne subrepresentation, then the following are equivalent:
   a. $\bigwedge^{\dim W} V$ is pure of weight $k \dim W$,
   b. $W$ is pure of weight $k$,
   c. $W$ is a direct summand of $V$.

(6) Suppose that $(V, r, N)$ is a Frobenius semisimple Weil-Deligne representation which is pure of weight $k$. Suppose also that $\text{Fil}^W V$ is a decreasing filtration of $V$ by Weil-Deligne subrepresentations such that $\text{Fil}^W V = (0)$ for $j >> 0$ and $\text{Fil}^W V = V$ for $j << 0$. If for each $j$

$$\bigwedge^{\dim \text{gr}^W V} \text{gr}^W V$$

is pure of weight $k \dim \text{gr}^W V$, then

$$V \cong \bigoplus_j \text{gr}^W V$$

and each $\text{gr}^W V$ is pure of weight $k$.

**Proof:** The first two parts are straightforward (using the fact that the filtration $\text{Fil}_i^W$ is unique). For the third part recall that an irreducible smooth representation $Sp_{s_1}(\pi_1) \boxplus \cdots \boxplus Sp_{s_t}(\pi_t)$ (see section I.3 of [HT]) is tempered if and only if the absolute values of the central characters of the $Sp_{s_i}(\pi_i)$ are all equal.

Suppose that $(V, r, N)$ is Frobenius semisimple and pure of weight $k$. As a $W_K$-module we can write uniquely $V = \bigoplus_{i \in \mathbb{Z}} V_i$ where $(V_i, r, 0)$ is strictly pure of weight $k+i$. For $i \in \mathbb{Z}_{\geq 0}$ let $V(i)$ denote the kernel of $N^{i+1} : V_i \rightarrow V_{i-2}$. Then $N : V_{i+2} \hookrightarrow V_i$ and $V_i = NV_{i+2} \oplus V(i)$. Thus

$$V = \bigcup_{i \in \mathbb{Z}} \bigcup_{j=0}^i N^j V(i),$$
and for $0 \leq j \leq i$ the map $N^j : V(i) \to V_{i-2j}$ is injective. Also note that as a virtual $W_K$-module $[V(i)] = [V_i] - [V_{i+2} \otimes \Art^{-1}_K]$. Thus if $r$ is semisimple then $(V, r)$ determines $(V, r, N)$ up to isomorphism. This establishes the fourth part.

Now consider the fifth part. If $W$ is a direct summand it is certainly pure of the same weight $k$ and $\bigwedge^{\dim W} W$ is then pure of weight $k \dim W$. Conversely if $W$ is pure of weight $k$ then

$$W = \bigoplus_{i \in \mathbb{Z}} \bigoplus_{j=0}^{i} N^j W(i),$$

where $W(i) = W \cap V(i)$. As a $W_K$-module we can decompose $V(i) = W(i) \oplus U(i)$. Setting

$$U = \bigoplus_{i \in \mathbb{Z}} \bigoplus_{j=0}^{i} N^j U(i),$$

we see that $V = W \oplus U$ as Weil-Deligne representations. Now suppose only that $\bigwedge^{\dim W} W$ is pure of weight $k \dim W$. Write

$$W \cong \bigoplus_j \Sp_{s_j}(X_j)$$

where each $X_j$ is strictly pure of some weight $k + k_j + (s_j - 1)$. Then, looking at highest exterior powers, we see that $\sum_j k_j = 0$. On the other hand as $V$ is pure we see that $k_j \leq 0$ for all $j$. We conclude that $k_j = 0$ for all $j$ and hence that $W$ is pure of weight $k$.

The final part follows from the fifth part by a simple inductive argument. □

Now let $L$ denote a number field. Write $| \cdot |_L$ for

$$\prod_x | \cdot |_{L_x} : \mathbb{A}^\infty_L / L^\times \to \mathbb{R}_{>0}^\times,$$

and write $\Art_L$ for

$$\prod_x \Art_{L_x} : \mathbb{A}^\infty_L / L^\times \to \Gal(L^{ac} / L)^{ab}.$$

We will call a continuous representation

$$R : \Gal(L^{ac} / L) \to GL_n(Q^{ac}_l)$$

pure of weight $k$ if for all but finitely many finite places $x$ of $L$ the representation $R$ is unramified at $x$ and every eigenvalue $\alpha$ of $R(\Frob_x)$ is a Weil $(\# k(x))^k$-number. Note that if $n = 1$ then $R$ is pure of weight $k$ if and only if for all $\iota : Q^{ac}_l \hookrightarrow \mathbb{C}$ we have $|\iota R \circ \Art_L|^2 = |\cdot |_{L}^{-k}$. In particular if $n = 1$ and $R$ is pure then $R|_{W_{L_x}}$ is strictly pure for all finite places $x$ of $L$.

We have the following lemma.
Lemma 1.4. Suppose that $M/L$ is a finite extension of number fields. Suppose also that $R : \text{Gal} (L^{ac}/L) \longrightarrow GL_n (\mathbb{Q}^{ac}_l)$ is a continuous semisimple representation which is pure of weight $k$. Suppose that $S : \text{Gal} (M^{ac}/M) \longrightarrow GL_n (\mathbb{Q}^{ac}_l)$ is another continuous representation with $S^{ss} \cong R|^{a}_{\text{Gal}(M^{ac}/M)}$ for some $a \in \mathbb{Z}_{>0}$. Suppose finally that $w$ is a place of $M$ above a finite place $v$ of $L$. If $\text{WD}(S|_{\text{Gal}(M^{ac}/w)})$ is pure of weight $k$, then $\text{WD}(R|_{\text{Gal}(L^{ac}/v)})$ is also pure of weight $k$.

Proof: Write

$$R|_{\text{Gal}(M^{ac}/M)} = \bigoplus_i R_i$$

where each $R_i$ is irreducible. Then $R_i$ is pure of weight $k \dim R_i$ and so that the top exterior power $\bigwedge^{\dim R_i} \text{WD}(R_i|_{\text{Gal}(M^{ac}/w)})$ is also pure of weight $k \dim R_i$. Lemma 1.3(6) tells us that

$$\text{WD}(S|_{\text{Gal}(M^{ac}/w)})^{ss} \cong \left( \bigoplus_i \text{WD}(R_i|_{\text{Gal}(M^{ac}/w)})^{ss} \right)^{a} \cong (\text{WD}(R|_{\text{Gal}(M^{ac}/w)})^{ss})^{a},$$

and that $\text{WD}(R|_{\text{Gal}(L^{ac}/v)})^{ss}$ is pure of weight $w$. Applying lemma 1.3(1) and (2), we see that $\text{WD}(R|_{\text{Gal}(L^{ac}/v)})$ is also pure. $\square$

2. Shimura varieties

In this section we study the geometry of integral models of Shimura varieties of the type considered in [HT], but with Iwahori level. It may be viewed as a generalisation of the work of Deligne-Rapoport [DR] in the case of modular curves.

In this section,

- let $E$ be an imaginary quadratic field, $F^{+}$ a totally real field and set $F = EF^{+}$;
- let $p$ be a rational prime which splits $p = uu^{c}$ in $E$;
- and let $w = w_1, ..., w_r$ be the primes of $F$ above $u$;
- and let $B$ be a division algebra with centre $F$ such that
  - $\dim_F B = n^2$,
  - $B^{op} \cong B \otimes_{F^{c}} F$,
  - at every place $x$ of $F$ either $B_x$ is split or a division algebra,
  - if $n$ is even then the number of finite places of $F^{+}$ above which $B$ is ramified is congruent to $1 + \frac{n}{2} [F^{+} : \mathbb{Q}]$ modulo 2.
Pick a positive involution $\ast$ on $B$ with $\ast|_F = c$. Let $V = B$ as a $B \otimes_F B^{\text{op}}$-module. For $\beta \in B^{\ast = -1}$ define a pairing

$$(\ , \ ) : V \times V \longrightarrow \mathbb{Q}$$

$$(x_1, x_2) \longmapsto \text{tr}_{F/\mathbb{Q}} \text{tr}_{B/F}(x_1 \beta x_2^\ast).$$

Also define an involution $\#$ on $B$ by $x^\# = \beta x^\ast \beta^{-1}$ and a reductive group $G/\mathbb{Q}$ by setting, for any $\mathbb{Q}$-algebra $R$, the group $G(R)$ equal to the set of $(\lambda, g) \in R^\times \times (B^{\text{op}} \otimes_{\mathbb{Q}} R)^\times$ such that

$$gg^\# = \lambda.$$ 

Let $\nu : G \to \mathbb{G}_m$ denote the multiplier character sending $(\lambda, g)$ to $\lambda$. Note that if $x$ is a rational prime which splits $x = yy_c$ in $E$ then

$$G(\mathbb{Q}_x) \xrightarrow{\sim} (B^g_{\text{op}})^\times \times \mathbb{Q}_x^\times \quad (\lambda, g) \longmapsto (g_y, \lambda).$$

We can and will assume that

- if $x$ is a rational prime which does not split in $E$ the $G \times \mathbb{Q}_x$ is quasisplit;
- the pairing $(\ , \ )$ on $V \otimes_{\mathbb{Q}} \mathbb{R}$ has invariants $(1, n - 1)$ at one embedding $\tau : F^+ \hookrightarrow \mathbb{R}$ and invariants $(0, n)$ at all other embeddings $F^+ \hookrightarrow \mathbb{R}$.

(See section I.7 of [HT] for details.)

Let $U$ be an open compact subgroup of $G(\mathbb{A}^\infty)$. Define a functor $\mathfrak{X}_U$ from the category of pairs $(S, s)$, where $S$ is a connected locally noetherian $F$-scheme and $s$ is a geometric point of $S$, to the category of sets, sending $(S, s)$ to the set of isogeny classes of four-tuples $(A, \lambda, i, \eta)$ where

- $A/S$ is an abelian scheme of dimension $[F^+ : \mathbb{Q}]n^2$;
- $i : B \hookrightarrow \text{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ such that $\text{Lie} A \otimes_{(E \otimes \mathcal{O}_S), 1 \otimes 1} \mathcal{O}_S$ is locally free over $\mathcal{O}_S$ of rank $n$ and the two actions of $F^+$ coincide;
- $\lambda : A \to A^\vee$ is a polarisation such that for all $b \in B$ we have $\lambda \circ i(b) = i(b^\ast)^\vee \circ \lambda$;
- $\eta$ is a $\pi_1(S, s)$-invariant $U$-orbit of isomorphisms of $B \otimes_{\mathbb{Q}} \mathbb{A}^\infty$-modules $\eta : V \otimes_{\mathbb{Q}} \mathbb{A}^\infty \to VA_s$ which take the standard pairing $(\ , \ )$ on $V$ to a $(A^\infty)^\times$-multiple of the $\lambda$-Weil pairing on $VA_s$.

Here $VA_s = \left( \lim_{N \to \infty} A[N](k(s)) \right) \otimes_{\mathbb{Z}} \mathbb{Q}$ is the adelic Tate module. For the precise notion of isogeny class see section III.1 of [HT]. If $s$ and $s'$ are both geometric points of a connected locally noetherian $F$-scheme $S$ then $\mathfrak{X}_U(S, s)$ and $\mathfrak{X}_U(S, s')$ are in canonical bijection. thus we may think of $\mathfrak{X}_U$ as a functor from connected locally noetherian $F$-schemes to sets. We
may further extend it to a functor from all locally noetherian $F$-schemes to sets by setting

$$\mathcal{X}_U\left(\prod_i S_i\right) = \prod_i \mathcal{X}_U(S_i).$$

If $U$ is sufficiently small (i.e. for some finite place $x$ of $\mathbb{Q}$ the projection of $U$ to $G(\mathbb{Q}_x)$ contains no element of finite order except 1) then $\mathcal{X}_U$ is represented by a smooth projective variety $X_U/F$ of dimension $n-1$. The inverse system of the $X_U$ for varying $U$ has a natural action of $G(\mathbb{A}^\infty)$.

Choose a maximal $\mathbb{Z}(p)$-order $\mathcal{O}_B$ of $B$ with $\mathcal{O}_B^* = \mathcal{O}_B$. Also fix an isomorphism $\mathcal{O}_{B,w}^{op} \cong M_n(\mathcal{O}_{F,w})$, and let $\varepsilon \in B_w$ denote the element corresponding to the diagonal matrix $(1,0,0,...,0) \in M_n(\mathcal{O}_{F,w})$. We decompose $G(\mathbb{A}^\infty)$ as

$$G(\mathbb{A}^\infty) = G(\mathbb{A}^{\infty,p}) \times \left(\prod_{i=2}^{r} (\mathcal{O}_{B,w_i}^{op})^\times\right) \times GL_n(F_w) \times \mathbb{Q}_p^\times. \quad (1)$$

Let $\varpi$ denote a uniformiser for $\mathcal{O}_{F,w}$. For $m = (m_2,...,m_r) \in \mathbb{Z}_{\geq 0}^{r-1}$, set

$$U_p^w(m) = \prod_{i=2}^{r} \ker \left( (\mathcal{O}_{B,w_i}^{op})^\times \to (\mathcal{O}_{B,w_i}^{op}/w_i^{m_i})^\times \right) \subset \prod_{i=2}^{r} (\mathcal{O}_{B,w_i}^{op})^\times.$$

Let $B_n$ denote the Borel subgroup of $GL_n$ consisting of upper triangular matrices and let $N_n$ denote its unipotent radical. Let $Iw_{n,w}$ denote the subgroup of $GL_n(\mathcal{O}_{F,w})$ consisting of matrices which reduce modulo $w$ to $B_n(k(w))$. We will consider the following open subgroups of $G(\mathbb{Q}_p)$:

$$\begin{align*}
Ma(m) &= U_p^w(m) \times GL_n(\mathcal{O}_{F,w}) \times \mathbb{Z}_p^\times, \\
Iw(m) &= U_p^w(m) \times Iw_{n,w} \times \mathbb{Z}_p^\times.
\end{align*}$$

Let $U^p$ be an open compact subgroup of $G(\mathbb{A}^{\infty,p})$. Write $U_0$ (resp. $U$) for $U^p \times Ma(m)$ (resp. $U^p \times Iw(m)$).

We recall that in section III.4 [HT] integral model of $X_{U_0}$ over $\mathcal{O}_{F,w}$ is defined. It represents the functor $\mathcal{X}_{U_0}$ from locally noetherian $\mathcal{O}_{F,w}$-schemes to sets. As above, $\mathcal{X}_{U_0}$ is initially defined on the category of connected locally noetherian $\mathcal{O}_{F,w}$ schemes with a geometric point to sets. It sends $(S,s)$ to the set of prime-to-$p$ isogeny classes of $(r+3)$-tuples $(A,\lambda,i,\tau_p,\alpha_i)$, where

- $A/S$ is an abelian scheme of dimension $[F^+ : \mathbb{Q}]n^2$;
- $i : \mathcal{O}_B \hookrightarrow \text{End}(A) \otimes_{\mathbb{Z}} \mathbb{Z}(p)$ such that $\text{Lie} A \otimes (\mathcal{O}_{F,u} \otimes_{\mathbb{Z}_p} \mathcal{O}_S), 1 \otimes 1 \mathcal{O}_S$ is locally free of rank $n$ and the two actions of $\mathcal{O}_F$ coincide;
- $\lambda : A \to A^\vee$ is a prime-to-$p$ polarisation such that for all $b \in \mathcal{O}_B$ we have $\lambda \circ i(b) = i(b^*)^\vee \circ \lambda;$

Lemma 2.1. If \( G \) is a \( \mathbb{A}^{\infty,p} \)-module with \( \lambda \)-Weil pairing on \( V^p A_s \); then for \( 2 \leq i \leq r \), \( \alpha_i : (w_i^{-m_i} \mathcal{O}_{B,w_i}/\mathcal{O}_{B,w_i})_S \sim A[w_i^{m_i}] \) is an isomorphism of \( S \)-schemes with \( \mathcal{O}_B \)-actions;

Then \( X_{U_0} \) is smooth and projective over \( \mathcal{O}_{F,w} \) ([HT], page 109). As \( U^p \) varies, the inverse system of the \( X_{U_0} \)'s has an action of \( G(\mathbb{A}^{\infty,p}) \).

Given an \((r+3)\)-tuple as above we will write \( G_A \) for \( \varepsilon A[w^\infty] \) a Barsotti-Tate \( \mathcal{O}_{F,w} \)-module. Over a base in which \( p \) is nilpotent it is one dimensional. If \( A \) denotes the universal abelian scheme over \( X_{U_0} \), we will write \( \mathcal{G} \) for \( G_A \). This \( \mathcal{G} \) is compatible, i.e. the two actions of \( \mathcal{O}_{F,w} \) on \( \text{Lie} \mathcal{G} \) coincide (see [HT]).

Write \( \overline{X}_{U_0} \) for the special fibre \( X_{U_0} \times \text{Spec} \mathcal{O}_{F,w} \text{Spec} k(w) \). For \( 0 \leq h \leq n - 1 \), we let \( \overline{X}_{U_0}^h \) denote the reduced closed subscheme of \( \overline{X}_{U_0} \) whose closed geometric points \( s \) are those for which the maximal etale quotient of \( \mathcal{G}_s \) has \( \mathcal{O}_{F,w} \)-height at most \( h \), and let

\[
\overline{X}_{U_0}^h = \overline{X}_{U_0}^h - \overline{X}_{U_0}^{h-1}
\]

(where we set \( \overline{X}_{U_0}^{-1} = 0 \)). Then \( \overline{X}_{U_0}^h \) is smooth of pure dimension \( h \) (corollary III.4.4 of [HT]), and on it there is a short exact sequence

\[
(0) \rightarrow \mathcal{G}^0 \rightarrow \mathcal{G} \rightarrow \mathcal{G}^{\text{et}} \rightarrow (0)
\]

where \( \mathcal{G}^0 \) is a formal Barsotti-Tate \( \mathcal{O}_{F,w} \)-module and \( \mathcal{G}^{\text{et}} \) is an etale Barsotti-Tate \( \mathcal{O}_{F,w} \)-module with \( \mathcal{O}_{F,w} \)-height \( h \).

Lemma 2.1. If \( 0 \leq h \leq n - 1 \) then the Zariski closure of \( \overline{X}_{U_0}^h \) contains \( \overline{X}_{U_0}^0 \).

Proof: This is ‘well known’, but for lack of a reference we give a proof. Let \( x \) be a closed geometric point of \( \overline{X}_{U_0}^0 \). By lemma II.4.1 of [HT] the formal completion of \( \overline{X}_{U_0} \times \text{Spec} k(w)^{ac} \) at \( x \) is isomorphic to the equicharacteristic universal deformation ring of \( \mathcal{G}_x \). According to the proof of proposition 4.2 of [Dr] this is \( \text{Spf} k(w)^{ac}[[T_1, \ldots, T_{n-1}]] \) and we can choose the \( T_i \) and a formal parameter \( S \) on the universal deformation of \( \mathcal{G}_x \) such that

\[
[w](S) \equiv w S + \sum_{i=1}^{n-1} T_i S^{#k(w)^i} + S^{#k(w)^n} \quad \text{(mod } S^{#k(w)^n+1})\]

Thus we get a morphism

\[
\text{Spec } k(w)^{ac}[[T_1, \ldots, T_{n-1}]] \rightarrow \overline{X}_{U_0}
\]

lying over \( x : k(w)^{ac} \rightarrow \overline{X}_{U_0} \), such that, if \( k \) denotes the algebraic closure of the field of fractions of \( k(w)^{ac}[[T_1, \ldots, T_{n-1}]]/(T_1, \ldots, T_{n-h-1}) \), then the induced map

\[
\text{Spec } k \rightarrow \overline{X}_{U_0}
\]
factors through $\overline{X}^{(h)}_{U_0}$. Thus $x$ is in the closure of $\overline{X}^{(h)}_{U_0}$, and the lemma follows. □

Now we define the functor $\mathcal{X}_U$. Again we initially define it as a functor from the category of connected locally noetherian schemes with a geometric point to sets, but then (as above) we extend it to a functor from locally noetherian schemes to sets. The functor $\mathcal{X}_U$ will send $(S, s)$ to the set of prime-to-$p$ isogeny classes of $(r + 4)$-tuples $(A, \lambda, i, \overline{p}, \mathcal{C}, \alpha_i)$, where $(A, \lambda, i, \overline{p}, \alpha_i)$ is as in the definition of $gX_{U_0}$ and $\mathcal{C}$ is a chain of isogenies of compatible Barsotti-Tate $\mathcal{O}_{F,w}$-modules, each of degree $\#k(w)$ and with composite equal to the canonical map $\mathcal{G} \to \mathcal{G}/\mathcal{G}[w]$. There is a natural transformation of functors $\mathcal{X}_U \to \mathcal{X}_{U_0}$.

**Lemma 2.2.** The functor $\mathcal{X}_U$ is represented by a scheme $X_U$ which is finite over $X_{U_0}$. The scheme $X_U$ has some irreducible components of dimension $n$.

**Proof:** By denoting the kernel of $\mathcal{G}_0 \to \mathcal{G}_j$ by $\mathcal{K}_j \subset \mathcal{G}[w]$, we can view the above chain as a flag

$$0 = \mathcal{K}_0 \subset \mathcal{K}_1 \subset \mathcal{K}_2 \subset \cdots \subset \mathcal{K}_{n−1} \subset \mathcal{K}_n = \mathcal{G}[w]$$

of closed finite flat subgroup schemes with $\mathcal{O}_{F,w}$-action, with each $\mathcal{K}_j/\mathcal{K}_{j−1}$ having order $\#k(w)$. Let $\mathcal{H}$ denote the sheaf of Hopf algebras over $X_{U_0}$ defining $\mathcal{G}[w]$. Then $\mathcal{X}_U$ is represented by a closed subscheme $X_U$ of the Grassmanian of chains of locally free direct summands of $\mathcal{H}$. (The closed conditions require that the subsheaves are sheaves of ideals defining a flag of closed subgroup schemes with the desired properties.) Thus $X_U$ is projective over $\mathcal{O}_{F,w}$. At each closed geometric point $s$ of $X_{U_0}$ the number of possible $\mathcal{O}_{F,w}$-submodules of $\mathcal{G}[w]_s \cong \mathcal{G}[w]_s^0 \times \mathcal{G}[w]_s^{et}$ is finite, so $X_U$ is finite over $X_{U_0}$. To see that $X_U$ has some components of dimension $n$ it suffices to note that on the generic fibre the map to $X_{U_0}$ is finite etale. □

We say an isogeny $\mathcal{G} \to \mathcal{G}'$ of one-dimensional compatible Barsotti-Tate $\mathcal{O}_{F,w}$-modules over a scheme $S$ of characteristic $p$ has **connected kernel** if it induces the zero map on $\text{Lie} \mathcal{G}$. We will denote the relative Frobenius map by $F : \mathcal{G} \to \mathcal{G}^{(p)}$ and let $f = [k(w) : \mathbb{F}_p]$, and then $F^f : \mathcal{G} \to \mathcal{G}(\#k(w))$ is an isogeny of compatible Barsotti-Tate $\mathcal{O}_{F,w}$-modules of degree $\#k(w)$ and has connected kernel.

We have the following rigidity lemma.

**Lemma 2.3.** Let $W$ denote the ring of integers of the completion of the maximal unramified extension of $F_w$. Suppose that $R$ is an Artinian local $W$-algebra with residue field $k(w)^{ac}$. Suppose also that

$$\mathcal{C} : \mathcal{G}_0 \to \mathcal{G}_1 \to \cdots \to \mathcal{G}_n = \mathcal{G}_0$$

is a chain of isogenies of degree $\#k(w)$ of one-dimensional compatible formal Barsotti-Tate $\mathcal{O}_{F,w}$-modules of $\mathcal{O}_{F,w}$-height $n$ with composite equal to multiplication by $\varpi_w$. If every isogeny $\mathcal{G}_{i−1} \to \mathcal{G}_i$ has connected kernel (for $i = 1, ..., n$) then $R$ is a $k(w)^{ac}$-algebra and $\mathcal{C}$
is the pull-back of a chain of Barsotti-Tate $\mathcal{O}_{F,w}$-modules over $k(w)^{ac}$, with all the isogenies isomorphic to $F^i$.

**Proof:** As the composite of the $n$ isogenies induces multiplication by $\varpi_w$ on the tangent space, $\varpi_w = 0$ in $R$, i.e. $R$ is a $k(w)^{ac}$-algebra. Choose a parameter $T_i$ for $\mathcal{G}_i$ over $R$. With respect to these choices, let $f_i(T_i) \in R[[T_i]]$ represent $\mathcal{G}_{i-1} \to \mathcal{G}_i$. We can write $f_i(T_i) = g_i(T_i^{p^{h_i}})$ with $h_i \in \mathbb{Z}_{\geq 0}$ and $g_i'(0) \neq 0$. (See [F], chapter I, §3, Theorem 2.) As $\mathcal{G}_{i-1} \to \mathcal{G}_i$ has connected kernel, $f'_i(0) = 0$ and $h_i > 0$. As $f_i$ commutes with the action $[r]$ for all $r \in \mathcal{O}_{F,w}$, we have $\varpi_i^{p^{h_i}} = \pi$ for all $\pi \in k(w)$, hence $h_i$ is a multiple of $f = [k(w) : \mathbb{F}_p]$. Reducing modulo the maximal ideal of $R$ we see that $h_i \leq f$ and so in fact $h_i = f$ and $g_i'(0) \in R^\times$. Thus $\mathcal{G}_i \cong \mathcal{G}_0^{(\#k(w)^n)}$ in such a way that the isogeny $\mathcal{G}_0 \to \mathcal{G}_i$ is identified with $F^{f_i}$. In particular $\mathcal{G}_0 \cong \mathcal{G}_0^{(\#k(w)^{nm})}$ and hence $\mathcal{G}_0 \cong \mathcal{G}_0^{(\#k(w)^{nm})}$ for any $m \in \mathbb{Z}_{\geq 0}$. As $R$ is Artinian some power of the absolute Frobenius on $R$ factors through $k(w)^{ac}$. Thus $\mathcal{G}_0$ is a pull-back from $k(w)^{ac}$ and the lemma follows. \(\square\)

Now let $Y_{U,i}$ denote the closed subscheme of $\overline{X}_U = X_U \times \text{Spec} \mathcal{O}_{F,w} \text{Spec} k(w)$ over which $\mathcal{G}_{i-1} \to \mathcal{G}_i$ has connected kernel.

**Proposition 2.4.**

1. $X_U$ has pure dimension $n$ and semistable reduction over $\mathcal{O}_{F,w}$, that is, for all closed points $x$ of the special fibre $X_U \times_{\text{Spec} \mathcal{O}_{F,w}} \text{Spec} k(w)$, there exists an etale morphism $V \to X_U$ with $x \in \text{Im} V$ and an etale $\mathcal{O}_{F,w}$-morphism:

   $$V \to \text{Spec} \mathcal{O}_{F,w}[T_1, \ldots, T_n]/(T_1 \cdots T_m - \varpi_w)$$

   for some $1 \leq m \leq n$, where $\varpi_w$ is a uniformizer of $\mathcal{O}_{F,w}$.

2. $X_U$ is regular and the natural map $X_U \to X_{U_0}$ is finite and flat.

3. Each $Y_{U,i}$ is smooth over $\text{Spec} k(w)$ of pure dimension $n - 1$, $\overline{X}_U = \bigcup_{i=1}^n Y_{U,i}$ and, for $i \neq j$ the schemes $Y_{U,i}$ and $Y_{U,j}$ share no common connected component. In particular, $X_U$ has strictly semistable reduction.

**Proof:** In this proof we will make repeated use of the following version of Deligne’s homogeneity principle ([DR]). Write $W$ for the ring of integers of the completion of the maximal unramified extension of $F_w$. In what follows, if $s$ is a closed geometric point of an $\mathcal{O}_{F,w}$-scheme $X$ locally of finite type, then we write $\mathcal{O}_{X,s}$ for the completion of the strict Henselisation of $X$ at $s$, i.e. $\mathcal{O}_{X,s} = \mathcal{O}_{X \times \text{Spec} W,s}$. Let $\mathcal{P}$ be a property of complete noetherian local $W$-algebras such that if $X$ is a $\mathcal{O}_{F,w}$-scheme locally of finite type then the set of closed geometric points $s$ of $X$ for which $\mathcal{O}_{X,s}$ has property $\mathcal{P}$ is Zariski open. Also let $X \to X_{U_0}$ be a finite morphism with the following properties

1. If $s$ is a closed geometric point of $\overline{X}_{U_0}^{(h)}$ then, up to isomorphism, $\mathcal{O}_{X,s}$ does not depend on $s$ (but only on $h$).

2. There is a unique geometric point of $X$ above any geometric point of $\overline{X}_{U_0}^{(0)}$. 
If \( \mathcal{O}_{X,s}^\wedge \) has property \( P \) for every geometric point of \( X \) over \( \overline{X}_U^{(0)} \), then \( \mathcal{O}_{X,s}^\wedge \) has property \( P \) for every closed geometric point of \( X \). Indeed, if we let \( Z \) denote the closed subset of \( X \) where \( P \) does not hold, then its image in \( X_U^{(0)} \) is closed and is either empty or contains some \( \overline{X}_U^{(h)} \). In the latter case, by lemma 2.1, it also contains \( \overline{X}_U^{(0)} \), which is impossible. Thus \( Z \) must be empty.

Note that both \( X = X_U \) and \( X = Y_{U,i} \) satisfy the above condition (ii) for the homogeneity principle, by letting \( R = k(w)^{ac} \) in lemma 2.3.

(1): The dimension of \( \mathcal{O}_{X_U,s}^\wedge \) as \( s \) runs over geometric points of \( X_U \) above \( \overline{X}_U^{(0)} \) is constant, say \( m \). Applying the homogeneity principle to \( X = X_U \) with \( P \) being ‘dimension \( m \)’, we see that \( X_U \) has pure dimension \( m \). By lemma 2.2 we must have \( m = n \) and \( X_U \) has pure dimension \( n \).

Now we will apply the above homogeneity principle to \( X = X_U \) taking \( P \) to be ‘isomorphic to \( W[[T_1, \ldots, T_n]]/(T_1 \cdots T_m - \varpi_w) \) for some \( m \leq n \). By a standard argument (see e.g. the proof of proposition 4.10 of [Y]) the set of points with this property is open and if all closed geometric points of \( X_U \) have this property \( P \) then \( X_U \) is semistable of pure dimension \( n \).

Let \( s \) be a geometric point of \( X_U \) over a point of \( \overline{X}_U^{(0)} \). Choose a basis \( e_i \) of \( \text{Lie} \mathcal{G}_i \) over \( \mathcal{O}_{X_U,s}^\wedge \) such that \( e_n \) maps to \( e_0 \) under the isomorphism \( \mathcal{G}_n = \mathcal{G}_0[\mathcal{G}_0[w] \cong \mathcal{G}_0 \) induced by \( \varpi_w \). With respect to these bases let \( X_i \in \mathcal{O}_{X_U,s}^\wedge \) represent the linear map \( \text{Lie} \mathcal{G}_{i-1} \to \text{Lie} \mathcal{G}_i \). Then

\[
X_1 \cdots X_n = \varpi_w.
\]

Moreover it follows from lemma 2.3 that \( \mathcal{O}_{X_U,s}^\wedge/(X_1, \ldots, X_n) = k(w)^{ac} \). (Because, by lemma III.4.1 of [HT], \( \mathcal{O}_{X_U,s}^\wedge \) is the universal deformation space of \( \mathcal{G}_s \). Hence by lemma 2.3, \( \mathcal{O}_{X_U,s}^\wedge \) is the universal deformation space for the chain

\[
\mathcal{G}_s \xrightarrow{F^f} \mathcal{G}_s^{\#(k(w))} \xrightarrow{F^f} \cdots \xrightarrow{F^f} \mathcal{G}_s^{\#(k(w)^n)} \cong \mathcal{G}_s/\mathcal{G}_s[\varpi_w].
\]

Thus we get a surjection

\[
W[[X_1, \ldots, X_n]]/(X_1 \cdots X_n - \varpi_w) \twoheadrightarrow \mathcal{O}_{X_U,s}^\wedge
\]

and as \( \mathcal{O}_{X_U,s}^\wedge \) has dimension \( n \) this map must be an isomorphism.

(2): We see at once that \( X_U \) is regular. Then [AK] V, 3.6 tells us that \( X_U \to X_U^{(0)} \) is flat.

(3): We apply the homogeneity principle to \( X = Y_{U,i} \) taking \( P \) to be ‘formally smooth of dimension \( n - 1 \)’. If \( s \) is a geometric point of \( Y_{U,i} \) above \( \overline{X}_U^{(0)} \) then we see that \( \mathcal{O}_{Y_{U,i,s}}^\wedge \) is cut out in \( \mathcal{O}_{X_U,s}^\wedge \cong W[[X_1, \ldots, X_n]]/(X_1 \cdots X_n - \varpi_w) \) by the single equation \( X_i = 0 \). (We are using the parameters \( X_i \) defined above.) Thus

\[
\mathcal{O}_{Y_{U,i,s}}^\wedge \cong k(w)^{ac}[[X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n]]
\]
is formally smooth of dimension \(n - 1\). We deduce that \(Y_{U,i}\) is smooth of pure dimension \(n - 1\).

As our \(G/X_U\) is one-dimensional, over a closed point, at least one of the isogenies \(G_{i-1} \rightarrow G_i\) must have connected kernel, which shows that \(X_U = \bigcup_i Y_{U,i}\). Suppose \(Y_{U,i}\) and \(Y_{U,j}\) share a connected component \(Y\) for some \(i \neq j\). Then \(Y\) would be finite flat over \(X_{U_0}\) and so the image of \(Y\) would meet \(X_{U_0}^{(n-1)}\). This is impossible as above a closed point of \(X_{U_0}^{(n-1)}\) one isogeny among the chain can have connected kernel. Thus, for \(i \neq j\) the closed subschemes \(Y_{U,i}\) and \(Y_{U,j}\) have no connected component in common. □

By the strict semistability, if we write, for \(S \subset \{1, ..., n\}\),

\[ Y_{U,S} = \bigcap_{i \in S} Y_{U,i}, \quad Y_{U,S}^0 = Y_{U,S} - \bigcup_{T \supsetneq S} Y_{U,T} \]

then \(Y_{U,S}\) is smooth over \(\text{Spec } k(w)\) of pure dimension \(n - \#S\) and \(Y_{U,S}^0\) are disjoint for different \(S\). With respect to the finite flat map \(X_U \rightarrow X_{U_0}\), the inverse image of \(X_{U_0}^{[h]}\) is exactly the locus where at least \(n - h\) of the isogenies have connected kernel, i.e. \(\bigcup_{\#S \geq n-h} Y_{U,S}\).

Hence the inverse image of \(X_{U_0}^{[h]}\) is equal to \(\bigcup_{\#S = n-h} Y_{U,S}^0\). Also note that the inverse system of \(Y_{U,S}^0\) for varying \(U^p\) is stable by the action of \(G(\mathbb{A}^{\infty,p})\).

Next we will relate the open strata \(Y_{U,S}^0\) to the Igusa varieties of the first kind defined in [HT]. For \(0 \leq h \leq n - 1\) and \(m \in \mathbb{Z}_{\geq 0}\), we write \(I_{U,1}^{(h)}\) for the Igusa varieties of the first kind defined on page 121 of [HT]. We also define an *Iwahori-Igusa variety of the first kind*

\[ I_U^{(h)} / X_U^{(h)} \]

as the moduli space of chains of isogenies

\[ G^{\text{et}} = G_0 \rightarrow G_1 \rightarrow \cdots \rightarrow G_h = G^{\text{et}} / G^{\text{et}}[w] \]

of etale Barsotti-Tate \(O_{F,w}\)-modules, each isogeny having degree \(#k(w)\) and with composite equal to the natural map \(G^{\text{et}} \rightarrow G^{\text{et}} / G^{\text{et}}[w]\). Then \(I_U^{(h)}\) is finite etale over \(X_U^{(h)}\), and as the Igusa variety \(I_{U,1}^{(h)}\) classifies the isomorphisms

\[ \alpha_1^{(h)} : (w^{-1}O_{F,w} / O_{F,w})_U^{(h)} \rightarrow G^{\text{et}}[w] \]

the natural map

\[ I_{U,1}^{(h)} \rightarrow I_U^{(h)} \]

is finite etale and Galois with Galois group \(B_h(k(w))\). Hence we can identify \(I_U^{(h)}\) with \(I_{U,1}^{(h)}/B_h(k(w))\). Note that the system \(I_U^{(h)}\) for varying \(U^p\) naturally inherits the action of \(G(\mathbb{A}^{\infty,p})\).

**Lemma 2.5.** For \(S \subset \{1, ..., n\}\) with \(\#S = n - h\), there exists a finite map of \(X_U^{(h)}\)-schemes

\[ \varphi : Y_{U,S}^0 \rightarrow I_U^{(h)} \]
which is bijective on the geometric points.

Proof: The map is defined in a natural way from the chain of isogenies $C$ by passing to the etale quotient $G^e$, and it is finite as $Y_{U,S}^0$ (resp. $I_{U}^{(h)}$) is finite (resp. finite etale) over $\overline{X}_{U_0}$. Let $s$ be a closed geometric point of $I_{U}^{(h)}$ with a chain of isogenies

$$G^e_s = G^e_0 \rightarrow \cdots \rightarrow G^e_i = G^e_i / G^e_s [w].$$

For $1 \leq i \leq n$ let $j(i)$ denote the number of elements of $S$ which are less than or equal to $i$. Set $G_i = (G^e_s)_{[\# k(w)/i]} \times G^e_{i-j(i)}$. If $i \not\in S$, define an isogeny $G_{i-1} \rightarrow G_i$ to be the identity times the given isogeny $G^e_{i-1} \rightarrow G^e_i$. If $i \in S$, define an isogeny $G_{i-1} \rightarrow G_i$ to be $F^j$ times the identity. Then

$$G_0 \rightarrow \cdots \rightarrow G_n$$

defines the unique geometric point of $Y_{U,S}^0$ above $s$. □

Now recall from [HT] III.2, that for an irreducible algebraic representation $\xi$ of $G$ over $\mathbb{Q}_l^{ac}$, one can associate a lisse $\mathbb{Q}_l^{ac}$-sheaf $\mathcal{L}_\xi/X_U$ for every $U$ such that $X_U$ is defined, and the action of $G(\mathbb{A}^{\infty,p})$ extends to $\mathcal{L}_\xi$. The sheaf $\mathcal{L}_\xi$ is extended to the integral models and Igusa varieties, and on $I_{U_p,(1,m)}^{(h)}$ and $Y_{U,S}^0$ they are the pull back of $\mathcal{L}_\xi$ on $\overline{X}_{U_0}^{(h)}$.

Corollary 2.6. For every $i \in \mathbb{Z}_{\geq 0}$, we have isomorphisms

$$H^i_c(Y_{U,S}^0 \times_{k(w)} k(w)^{ac}, \mathcal{L}_\xi) \sim H^i_c(I_{U}^{(h)} \times_{k(w)} k(w)^{ac}, \mathcal{L}_\xi) \sim H^i_c(I_{U_p,(1,m)}^{(h)} \times_{k(w)} k(w)^{ac}, \mathcal{L}_\xi) B_{h}(k(w))$$

that are compatible with the actions of $G(\mathbb{A}^{\infty,p})$ when we vary $U_p$.

Proof: By lemma 2.5, for any lisse $\mathbb{Q}_l^{ac}$-sheaf $\mathcal{F}$ on $I_{U}^{(h)}$, we have $\mathcal{F} \cong \varphi_* \varphi^* \mathcal{F}$ by looking at the stalks at all geometric points. As $\varphi$ is finite the first isomorphism follows. The second isomorphism follows easily as $I_{U_p,(1,m)}^{(h)} \rightarrow I_{U}^{(h)}$ is finite etale and Galois with Galois group $B_{h}(k(w))$. □

In the next section, we will be interested in the $G(\mathbb{A}^{\infty,p}) \times \text{Frob}_{w}^{Z}$-modules

$$H^i(Y_{w(m),S}, \mathcal{L}_\xi) = \lim_{U_p} H^i(Y_{U,S} \times_{k(w)} k(w)^{ac}, \mathcal{L}_\xi).$$

Here we relate the alternating sum of these modules to the cohomology of Igusa varieties. We will define the elements of $\text{Groth}(G(\mathbb{A}^{\infty,p}) \times \text{Frob}_{w}^{Z})$ (we write $\text{Groth}(G)$ for the
Grothendieck group of admissible $G$-modules) as follows:

$$
[H(Y_{Iw(m)}, S, \mathcal{L}_\xi)] = \sum_i (-1)^{n-\#S-i} H^i(Y_{Iw(m)}, S, \mathcal{L}_\xi),
$$

$$
[H_c(Y^0_{Iw(m)}, S, \mathcal{L}_\xi)] = \sum_i (-1)^{n-\#S-i} \text{lim}_{U^p} H^i_c(Y^0_{U,S} \times_{k(w)} k(w)^{ac}, \mathcal{L}_\xi),
$$

$$
[H_c(I^{(h)}_{Iw(m)}, \mathcal{L}_\xi)] = \sum_i (-1)^{h-i} \text{lim}_{U^p} H^i_c(I^{(h)}_{U \times_{k(w)} k(w)} \times_{k(w)} k(w)^{ac}, \mathcal{L}_\xi).
$$

Then, because

$$
Y_{U,S} = \bigcup_{T \supset S} Y^0_{U,T}
$$

for each $U = U^p \times Iw(m)$, we have equalities

$$
[H(Y_{Iw(m)}, S, \mathcal{L}_\xi)] = \sum_{T \supset S} (-1)^{(n-\#S)-(n-\#T)} [H_c(Y^0_{Iw(m)}, T, \mathcal{L}_\xi)]
$$

$$
= \sum_{T \supset S} (-1)^{(n-\#S)-(n-\#T)} [H_c(I^{(n-\#T)}_{Iw(m)}, \mathcal{L}_\xi)].
$$

As there are $\binom{n-\#S}{h}$ subsets $T$ with $\#T = n-h$ and $T \supset S$, we conclude:

**Lemma 2.7.** We have an equality

$$
[H(Y_{Iw(m)}, S, \mathcal{L}_\xi)] = \sum_{h=0}^{n-\#S} (-1)^{n-\#S-h} \binom{n-\#S}{h} [H_c(I^{(h)}_{Iw(m)}, \mathcal{L}_\xi)]
$$

in the Grothendieck group of admissible $G(\mathbb{A}_\infty^p) \times \text{Frob}^{Z_{Iw}}$-modules over $\mathbb{Q}_l^{ac}$.

3. **Proof of the main theorem**

We now return to the situation in theorem 1.1. Recall that $L$ is an imaginary CM field and that $\Pi$ is a cuspidal automorphic representation of $GL_n(\mathbb{A}_L)$ such that

- $\Pi \circ c \cong \Pi^\vee$;
- $\Pi_\infty$ has the same infinitesimal character as some algebraic representation over $\mathbb{C}$ of the restriction of scalars from $L$ to $\mathbb{Q}$ of $GL_n$;
- and for some finite place $x$ of $L$ the representation $\Pi_x$ is square integrable.

Recall also that $v$ is a place of $L$ above a rational prime $p$, that $l \neq p$ is a second rational prime and that $\iota : \mathbb{Q}_l^{ac} \cong \mathbb{C}$. Recall finally that $R_l(\Pi)$ is the $l$-adic representation associated to $\Pi$.

Choose a quadratic CM extension $L'/L$ in which $v$ and $x$ split. Choose places $v' \neq x'$ of $L'$ above $v$ and $x$ respectively. Also choose an imaginary quadratic field $E'$ and a totally real field $F'$ such that

...
• $[F^+: \mathbb{Q}]$ is even;
• $F = EF^+$ is soluble and Galois over $L'$;
• $p$ splits as $uv^c$ in $E$;
• there is a place $w$ of $F$ above $u$ and $v'$ such that $\Pi_{F,w}$ has an Iwahori fixed vector;
• $x$ lies above a rational prime which splits in $E$ and $x'$ splits in $F$.

Denote by $\Pi_F$ the base change of $\Pi$ to $GL_n(\mathbb{A}_F)$. Note that the component of $\Pi_F$ at a place above $x'$ is square integrable and hence $\Pi_F$ is cuspidal.

Choose a division algebra $B$ with centre $F$ as in the previous section and satisfying

• $B_x$ is split for all places $x \neq z, z^c$ of $F$.

Also choose $\beta$ and $G$ as in the previous section. Then it follows from theorem VI.2.9 and lemma VI.2.10 of [HT] that we can find

• a character $\psi : \mathbb{A}_E^\times / E^\times \rightarrow \mathbb{C}^\times$,
• an irreducible algebraic representation $\xi$ of $G$ over $\mathbb{Q}_{lc}^\times$,
• and an automorphic representation $\pi$ of $G(\mathbb{A})$,

such that

• $\pi_\infty$ is cohomological for $i\xi$,
• $\psi$ is unramified above $p$,
• $\psi^c |_{E_\infty^\times}$ is the inverse of the restriction of $i\xi$ to $E_\infty^\times \subset G(\mathbb{R})$,
• $\psi^c / \psi$ is the restriction of the central character of $\Pi_F$ to $\mathbb{A}_E^\times$,
• and if $x$ is a rational prime which splits $yf^c$ in $E$ then $\pi_x = (\bigotimes_{z \mid y} JL^{-1}(\Pi_z)) \otimes \psi_y$ as a representation of $(B_y^{op})^\times \times \mathbb{Q}_y^\times \cong (\bigotimes_{z \mid y}(B_z^{op})^\times) \times \mathbb{Q}_y^\times$.

Here JL denotes the identity if $B_z$ is split and denotes the Jacquet-Langlands correspondence if $B_z$ is a division algebra. (See section I.3 of [HT].)

We will call two irreducible admissible representations $\pi'$ and $\pi''$ of $G(\mathbb{A}^\infty)$ nearly equivalent if $\pi'_x \cong \pi''_x$ for all but finitely many rational primes $x$. If $M$ is an admissible $G(\mathbb{A}^\infty)$-module and $\pi'$ is an irreducible admissible representation of $G(\mathbb{A}^\infty)$ then we define the $\pi'$-near isotypic component $M[\pi']$ of $M$ to be the largest $G(\mathbb{A}^\infty)$-submodule of $M$ all whose irreducible subquotients are nearly equivalent to $\pi'$. Then

$$M = \bigoplus M[\pi']$$

as $\pi'$ runs over near equivalence classes of irreducible admissible $G(\mathbb{A}^\infty)$-modules. (This follows from the following fact. Suppose that $A$ is a (commutative) polynomial algebra over $\mathbb{C}$ in countably many variables, and that $M$ is an $A$-module which is finitely generated over
We can write

\[ M = \bigoplus_m M_m, \]

where \( m \) runs over maximal ideals of \( A \) with residue field \( \mathbb{C} \).

We consider the Shimura varieties \( X_U/F \) for open compact subgroups \( U \) of \( G(\mathbb{A}_\infty) \) as in the last section. Then

\[ H^i(X, \mathcal{L}_\xi) = \lim_{\rightarrow} H^i(X_U \times_F F^{ac}, \mathcal{L}_\xi) \]

is a semisimple, admissible \( G(\mathbb{A}_\infty) \)-module with a commuting continuous action of the Galois group \( \text{Gal}(F^{ac}/F) \). (For details see III.2 of [HT].)

The following lemma follows from [HT], particularly corollary VI.2.3, corollary VI.2.7 and theorem VII.1.7.

**Lemma 3.1.** Keep the notation and assumptions above. (In particular we are assuming that \( \pi \) arises from a cuspidal automorphic representation \( \Pi \) of \( GL_n(\mathbb{A}_F) \).)

1. If \( i \neq n - 1 \) then \( H^i(X, \mathcal{L}_\xi)[\pi] = (0) \).
2. As \( G(\mathbb{A}_\infty) \times \text{Gal}(F^{ac}/F) \)-modules,

\[ H^{n-1}(X, \mathcal{L}_\xi)[\pi] = \bigoplus_{\pi'} \pi' \otimes R'_i(\Pi)^m(\pi') \otimes R_i(\psi), \]

where \( \pi' \) runs over irreducible admissible representations of \( G(\mathbb{A}_\infty) \) nearly equivalent to \( \pi \) and where \( m(\pi') \in \mathbb{Z}_{\geq 0} \), and \( R_i(\Pi) = R'_i(\Pi)^{ss} \).
3. \( m(\pi) > 0 \).
4. If \( m(\pi') > 0 \) then \( \pi'_p \cong \pi_p \).

If \( \pi' \) is an irreducible admissible representation of \( G(\mathbb{A}_\infty) \) we can decompose it as \( (\pi')^p \otimes (\prod_{k=2}^r \pi'_{w_k}) \otimes \pi'_0 \otimes \pi''_0 \) corresponding to the decomposition (1). If \( \pi'' \) is an irreducible admissible representation of \( G(\mathbb{A}_\infty, p) \) and \( N \) is an admissible \( G(\mathbb{A}_\infty, p) \)-module then we can define the \( \pi'' \)-near isotypic component of \( N \) in the same manner as we did for \( G(\mathbb{A}_\infty) \)-modules. If \( M \) is an admissible \( G(\mathbb{A}_\infty) \)-module and \( \pi' \) is an irreducible admissible representation of \( G(\mathbb{A}_\infty) \) then

\[ M^{Iw(\pi')}[(\pi')^p] = M[\pi']^{Iw(m)}. \]

We will write

\[ H^i(X_{Iw(\pi')}, \mathcal{L}_\xi) = \lim_{\rightarrow} H^i(X_{U^p \times Iw(\pi')} \times_F F^{ac}, \mathcal{L}_\xi) \cong H^i(X, \mathcal{L}_\xi)^{Iw(m)}. \]

It is a semisimple admissible \( G(\mathbb{A}_\infty, p) \)-module with a commuting continuous action of \( \text{Gal}(F^{ac}/F) \).

**Theorem 3.2.** Keep the above notation and assumptions. (In particular we are assuming that \( \pi \) arises from a cuspidal automorphic representation \( \Pi \) of \( GL_n(\mathbb{A}_F) \).) Let \( U^p \) be a
sufficiently small open compact subgroup of $G(\mathbb{A}^{\infty,p})$. Then

$$\text{WD}(H^{n-1}(X_{Iw(m)}, \mathcal{L}_\xi)[\pi^p]^U)$$

is pure.

**Proof:** As $X_U = X_{U^p \times Iw(m)}$ is strictly semistable by proposition 2.4, we can use the Rapoport-Zink weight spectral sequence [RZ] to compute $H^{n-1}(X_{Iw(m)}, \mathcal{L}_\xi)$. For $X_U$, it reads

$$E_1^{i,j}(U) = \bigoplus_{t \geq \text{max}(0,-i)} \bigoplus_{S = i+2t+1} H^{j-2t}(Y_{U,S} \times k(w)_{ac}, \mathcal{L}_\xi(-t)) \Rightarrow H^{i+j}(X_U \times_F F_w^{ac}, \mathcal{L}_\xi).$$

Passing to the limit with respect to $U^p$, it gives rise to the following spectral sequence of admissible $G(\mathbb{A}^{\infty,p}) \times \text{Frob}_w$-modules

$$E_1^{i,j}(Iw(m)) = \bigoplus_{t \geq \text{max}(0,-i)} \bigoplus_{S = i+2t+1} H^{j-2t}(Y_{Iw(m),S}, \mathcal{L}_\xi(-t)) \Rightarrow H^{i+j}(X_{Iw(m)}, \mathcal{L}_\xi).$$

Hence we get a spectral sequence of $\text{Frob}_w$-modules

$$(2) \quad E_1^{i,j}(Iw(m))[\pi^p]^{U^p} \Rightarrow H^{i+j}(X_{Iw(m)}, \mathcal{L}_\xi)[\pi^p]^{U^p}. $$

The sheaf $\mathcal{L}_\xi$ is pure, say of weight $w_\xi$. Thus the action of $\text{Frob}_w$ on $E_1^{i,j}$ is pure of weight $w_\xi + j$ by the Weil conjectures. The theory of weight spectral sequence ([RZ]) defines an operator

$$N : E_1^{i,j}(Iw(m))[\pi^p]^{U^p} \rightarrow E_1^{i+2j-2}(Iw(m))[\pi^p]^{U^p},$$

which induces the $N$ for $\text{WD}(H^{i+j}(X_{Iw(m)}, \mathcal{L}_\xi)[\pi^p]^{U^p})$ and has the property that

$$N^i : E_1^{i,j+i}(Iw(m))[\pi^p]^{U^p} \Rightarrow E_1^{i,j-i}(Iw(m))[\pi^p]^{U^p}$$

for all $i$. If the spectral sequence (2) degenerates at $E_1$, then it follows that the Weil-Deligne representation $\text{WD}(H^{n-1}(X_{Iw(m)}, \mathcal{L}_\xi)[\pi^p]^{U^p})$ is pure of weight $w_\xi + (n-1)$. Thus it suffices to show that

$$E_1^{i,j}(Iw(m))[\pi^p]^{U^p} = (0)$$

if $i + j \neq n - 1$, i.e. that

$$H^j(Y_{Iw(m),S}, \mathcal{L}_\xi)[\pi^p]^{U^p} = (0)$$

if $j \neq n - \#S$.

We first recall some notation from [HT]. For $h = 0, \ldots, n-1$ let $P_h$ denote the maximal parabolic in $GL_n$ consisting of matrices $g \in GL_n$ with $g_{ij} = 0$ for $i > n - h$ and $j \leq n - h$. Also let $N_h$ denote the unipotent radical of $P_h$, let $P_h^{op}$ denote the opposite parabolic and let $N_h^{op}$ denote the unipotent radical of $P_h^{op}$. Let $D_{F_p,n-h}$ denote the division algebra with centre $F_w$ and Hasse invariant $1/(n-h)$. If $\pi'$ is a square integrable representation of $GL_{n-h}(F_w)$, let $\varphi_{\pi'}$ denote a pseudo-coefficient for $\pi'$ as in section I.3 of [HT]. (Note that this depends on the choice of a Haar measure, but in the formulae below this choice will always be cancelled by the choice of an associated Haar measure on $D_{F_p,n-h}$. See [HT] for details.)
If we introduce the limit of cohomology groups of Igusa varieties for varying level structure at $p$ as in (see p.136 of [HT]):

$$[H_c(I^{(h)}_F, \mathcal{L}_\xi)] = \sum_i (-1)^{h-i} \lim_{U_p,m} H^i_c(I^{(h)}_{U_p,m} \times_{k(w)} k(w)^{ac}, \mathcal{L}_\xi),$$

then the second isomorphism of corollary 2.6 and theorem V.5.4 of [HT] tell us that

$$n[H_c(I_{Iw(m)}^{(h)}, \mathcal{L}_\xi)] = n[H_c(I^{(h)}, \mathcal{L}_\xi)]_{U_p(m) \times Iw,h,w}^w = \sum_i (-1)^{n-1-i} \text{Red}^{(h)}[H^i(X, \mathcal{L}_\xi)]_{U_p(m)}^w$$

in Groth $(G(\mathbb{A}^{\infty,p}) \times \text{Frob}_w^Z)$, where

$$\text{Red}^{(h)} : \text{Groth} (GL_n(F_w) \times \mathbb{Q}_p^\times) \rightarrow \text{Groth} (\text{Frob}_w^Z)$$

is the composite of the normalised Jacquet functor

$$J_{N^{(w)}} : \text{Groth} (GL_n(F_w) \times \mathbb{Q}_p^\times) \rightarrow \text{Groth} (GL_{n-h}(F_w) \times GL_h(F_w) \times \mathbb{Q}_p^\times)$$

with the functor

$$\text{Groth} (GL_{n-h}(F_w) \times GL_h(F_w) \times \mathbb{Q}_p^\times) \rightarrow \text{Groth} (\text{Frob}_w^Z)$$

which sends $[\alpha \otimes \beta \otimes \gamma]$ to

$$\sum_\phi \text{vol}(D_{F_w,n-h}^{X}/F_w^{\infty})^{-1} \text{tr}(\varphi_{\text{Sp}_{n-h}(\phi)})(\dim \beta)^{Iw,h,w} \left[ \text{rec} \left( \phi^{-1} \mid \frac{i+n}{w^2} (\gamma \mathbb{Z}_p \circ N_{F_w/E_u})^{-1} \right) \right],$$

where the sum is over characters $\phi$ of $F_w^{\infty}/\mathcal{O}_{F_w}^{\infty}$. (We just took the $Iw,h,w$-invariant part of the $\text{Red}_1^{(h)}$, which is defined on p.182 of [HT]. Note that $\text{Frob}_w$ acts on $H_c(I^{(h)}, \mathcal{L}_\xi)$ as

$$(1, p^{-[k(w):\mathbb{F}_p]}, -1, 1, 1) \in G(\mathbb{A}^{\infty,p}) \times (\mathbb{Q}_p^\times/\mathbb{Z}_p^\times) \times \mathbb{Z} \times GL_h(F_w) \times \prod_{i=2}^r (B^{\text{op}}_{w_i}^\times),$$

where we have identified $D_{F_w,n-h}^{X}/\mathcal{O}_{F_w,n-h}^{\infty}$ with $\mathbb{Z}$ via $w(\det).$)

In particular, by lemma 3.1(1), we have an equality in Groth (Frob$_w^Z$):

$$n[H_c(I_{Iw(m)}^{(h)}, \mathcal{L}_\xi)]_{[\pi_p]^{U_p}} = \text{Red}^{(h)}[H^{n-1}(X, \mathcal{L}_\xi)]_{[\pi_p]^{U_p}}.$$
Combining this with lemma 2.7, we get
\[ n(\dim \Pi_{F,w}^{\text{Iw},w}) \left[ H(Y_{\text{Iw}(m)},S,\mathcal{L}_\xi)[\pi^p]^{U_p} \right] \]
\[ = (\dim H^{n-1}(X,\mathcal{L}_\xi)^{\text{Iw}(m)}[\pi^p]^{U_p}) \sum_{h=0}^{n-\#S} (-1)^{n-\#S-h} \binom{n-\#S}{h} \text{Red}(h)[\Pi_{F,w} \otimes \psi_u]. \]

As \( \Pi_{F,w} \) is tempered, it is a full normalised induction of the form
\[ n-\text{Ind}^{GL_n(F_w)}_{P(F_w)}(\text{Sp}_{s_1}(\pi_1) \otimes \cdots \otimes \text{Sp}_{s_\ell}(\pi_\ell)), \]
where \( \pi_i \) is an irreducible cuspidal representation of \( GL_{g_i}(F_w) \) and \( P \) is a parabolic subgroup of \( GL_n \) with Levi component \( GL_{g_1} \times \cdots \times GL_{g_\ell} \). As \( \Pi_{F,w} \) has an Iwahori fixed vector, we must have \( g_i = 1 \) and \( \pi_i \) unramified for all \( i \). Note that, for this type of representation (full induced from square integrable \( \text{Sp}_{s_i}(\pi_i) \) with \( \pi_i \) an unramified character of \( F_w^\times \)),
\[ \dim (n-\text{Ind}^{GL_n(F_w)}_{P(F_w)}(\text{Sp}_{s_1}(\pi_1) \otimes \cdots \otimes \text{Sp}_{s_\ell}(\pi_\ell)))^{\text{Iw},w} \]
\[ = \#P(k(w)) GL_n(k(w))/B_n(k(w)) = \frac{n}{\prod_j s_j!}. \]

We can compute \( \text{Red}(h)[\Pi_{F,w} \otimes \psi_u] \) using lemma I.3.9 of [HT] (but note the typo there — “positive integers \( h_1,\ldots,h_t \)” should read “non-negative integers \( h_1,\ldots,h_t \)” ). Putting \( V_i = \text{rec}(\pi_i^{-1}|_{\psi_u \circ N_{F_w,E}})^{-1} \), we see that
\[ \text{Red}(h)[\Pi_{F,w} \otimes \psi_u] = \sum_i \dim (n-\text{Ind}^{GL_n(F_w)}_{P'(F_w)}(\text{Sp}_{s_i+n-h}(\pi_i) |^{n-h} \otimes \bigotimes_{j \neq i} \text{Sp}_{s_j}(\pi_j)))^{\text{Iw},w} [V_i] \]
\[ = \sum_i \frac{h!}{(s_i+n-h)!(\prod_{j \neq i} s_j)!} [V_i] \]
where the sum runs only over those \( i \) for which \( s_i \geq n-h \), and \( P' \subset GL_h \) is a parabolic subgroup. Thus
\[ \frac{n}{\prod_j s_j!} \left[ H(Y_{\text{Iw}(m)},S,\mathcal{L}_\xi)[\pi^p]^{U_p} \right] \]
\[ = D \sum_{h=0}^{n-\#S} (-1)^{n-\#S-h} \binom{n-\#S}{h} \sum_{i: s_i \geq n-h} \frac{h!}{(s_i+h-n)!(\prod_{j \neq i} s_j)!} [V_i] \]
\[ = D \sum_{i=1}^{t} \binom{n-\#S}{s_i-\#S} \prod_{j \neq i} s_j! \sum_{h=n-s_i}^{n-\#S} (-1)^{n-\#S-h} \binom{s_i-\#S}{h+s_i-n} [V_i] \]
\[ = D \sum_{s_i=\#S}^{n-\#S} \frac{(n-\#S)!}{\prod_{j \neq i} s_j!} [V_i], \]
where \( D = \dim H^{n-1}(X,\mathcal{L}_\xi)^{\text{Iw}(m)}[\pi^p]^{U_p} \), and so
\[ n\binom{n}{\#S} \left[ H(Y_{\text{Iw}(m)},S,\mathcal{L}_\xi)[\pi^p]^{U_p} \right] = (\dim H^{n-1}(X,\mathcal{L}_\xi)^{\text{Iw}(m)}[\pi^p]^{U_p}) \sum_{s_i=\#S} [V_i]. \]
As $\Pi_{F,w}$ is tempered, $\text{rec}(\Pi_{F,w}^\vee \otimes (\psi_u \circ N_{F,w/E_u}))|\det|^{\frac{1-n}{2}}$ is pure of weight $w_\xi + (n - 1)$. Hence

$$V_i = \text{rec}(\pi_i^{-1} | 1 - \# S^w (\psi_u \circ N_{F,w/E_u})^{-1} | \frac{\# S - n}{2})$$

is strictly pure of weight $w_\xi + (n - \# S)$. The Weil conjectures then tell us that

$$H^j(Y_{Iw(m)}, S, L_\xi)[\pi^p]^U_p = (0)$$

for $j \neq n - \# S$. The theorem follows. □

We can now conclude the proof of theorem 1.1. Choose $k$ so that $|\chi_{\Pi}| = | | \frac{n(k+n-1)}{L} |$ where $\chi_{\Pi}$ is the central character of $\Pi$. Set

$$V = H^{n-1}(X_{Iw(m)}, L_\xi)[\pi^p]^U_p \otimes R_l(\psi)^{-1},$$

a continuous representation of $\text{Gal}(F^{ac}/F)$. We know that

1. $V^{ss} \cong R_l(\Pi)^{a}_{\text{Gal}(F^{ac}/F)}$ for some $a \in \mathbb{Z}_{>0}$,
2. $V$ is pure of weight $k$ (proposition III.2.1 of [HT] and a computation of the determinant),
3. $\text{WD}(V|_{\text{Gal}(F^{ac}/F_{w})})$ is pure of weight $k$ (use theorem 3.2 and a computation of the determinant).

Thus lemma 1.4 tells us that $\text{WD}(R_l(\Pi)|_{\text{Gal}(L^{ac}_{w}/L_{w})})^{F-ss}$ is pure. On the other hand, as $\Pi_v$ is tempered (corollary VII.1.11 of [HT]), $\text{rec}(\Pi_v^\vee |\det|^{\frac{1-n}{2}})$ is pure by lemma 1.3(3). As the representation of the Weil group in $\text{rec}(\Pi_v^\vee |\det|^{\frac{1-n}{2}})$ and $\text{WD}(R_l(\Pi)|_{\text{Gal}(L^{ac}_{w}/L_{w})})^{F-ss}$ are equivalent, we deduce from lemma 1.3(4) that

$$\text{WD}(R_l(\Pi)|_{\text{Gal}(L^{ac}_{w}/L_{w})})^{F-ss} \cong \text{rec}(\Pi_v^\vee |\det|^{\frac{1-n}{2}}),$$

as desired.

References


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