COMPATIBILITY OF LOCAL AND GLOBAL LANGLANDS CORRESPONDENCES

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Abstract. We prove the compatibility of local and global Langlands correspondences for $GL_n$, which was proved up to semisimplification in [HT]. More precisely, for the $n$-dimensional $l$-adic representation $R_l(\Pi)$ of the Galois group of a CM-field $L$ attached to a conjugate self-dual regular algebraic cuspidal automorphic representation $\Pi$, which is square integrable at some finite place, we show that Frobenius semisimplification of the restriction of $R_l(\Pi)$ to the decomposition group of a prime $v$ of $L$ not dividing $l$ corresponds to $\Pi_v$ by the local Langlands correspondence.

Introduction

This paper is a continuation of [HT]. Let $L$ be an (imaginary) CM field and let $\Pi$ be a regular algebraic cuspidal automorphic representation of $GL_n(\mathbb{A}_L)$ which is conjugate self-dual ($\Pi \circ c \cong \Pi^\vee$) and square integrable at some finite place. In [HT] it is explained how to attach to $\Pi$ and an arbitrary rational prime $l$ (and an isomorphism $\iota : \mathbb{Q}^{nc}_l \cong \mathbb{C}$) a continuous semisimple representation $R_l(\Pi) : \text{Gal} (L^{ac}/L) \rightarrow GL_n(\mathbb{Q}^{nc}_l)$ which is characterised as follows. For every finite place $v$ of $L$ not dividing $l$

$$r R_l(\Pi)|_{W_{L_v}} = \text{rec}(\Pi_v^\vee | \det |^{\frac{1-n}{2}})^{ss},$$

where rec denotes the local Langlands correspondence and ss denotes the semisimplification (see [HT] for details). In that book it is also shown that $\Pi_v$ is tempered for all finite places $v$.

In this paper we strengthen this result to completely identify $R_l(\Pi)|_{I_v}$ for $v \not| l$. In particular, we prove the following theorem.

Theorem A. If $v \not| l$ then the Frobenius semisimplification of $R_l(\Pi)|_{W_{L_v}}$ is the $l$-adic representation attached to $r^{-1}\text{rec}(\Pi_v^\vee | \det |^{\frac{1-n}{2}})$.

As $R_l(\Pi)$ is semisimple and $\text{rec}(\Pi_v^\vee | \det |^{\frac{1-n}{2}})$ is indecomposable if $\Pi_v$ is square integrable, we obtain the following corollary.

Date: December 17, 2004.
The first author was partially supported by NSF grant number DMS-0100090.
Corollary B. If $\Pi_v$ is square integrable at a finite place $v \nmid l$, the representation $R_l(\Pi)$ is irreducible.

Using base change it is easy to reduce to the case that $\Pi_v$ has an Iwahori fixed vector. We descend $\Pi$ to an automorphic representation $\pi$ of a unitary group $G$ which locally at $v$ looks like $GL_n$ and at infinity looks like $U(n-1,1) \times U(n,0)^{[L:Q]/2-1}$. Then we realise $R_l(\Pi)$ in the cohomology of a Shimura variety $X$ associated to $G$ with Iwahori level structure at $v$. More precisely, for some $l$-adic sheaf $\mathcal{L}$, the $\pi^p$-isotypic component of $H^{n-1}(X, \mathcal{L})$ is, up to semisimplification and some twist, $R_l(\Pi)^a$ (for some $a \in \mathbb{Z}_{>0}$). We show that $X$ has semistable reduction and use the results of [HT] to calculate the cohomology of the (smooth, projective) strata of the reduction of $X$ above $p$ as a virtual $G(\mathbb{A}^{\infty,p}) \times \tilde{F}_p$-module (where $F$ denotes Frobenius). This description and the temperedness of $\Pi_v$ shows that the $\pi^p$-isotypic component of the cohomology of any strata is concentrated in the middle dimension. This implies that the $\pi^p$-isotypic component of the Rapoport-Zink weight spectral sequence degenerates at $E_1$, which allows us to calculate the action of inertia at $v$ on $H^{n-1}(X, \mathcal{L})$.

In the special case that $\Pi_v$ is a twist of a Steinberg representation and $\Pi_\infty$ has trivial infinitesimal character, the above theorem presumably follows from the results of Ito [I].

Acknowledgements. The authors are grateful to Tetsushi Ito, who asked a very helpful question.

1. The main theorem

We write $F^{ac}$ for an algebraic closure of a field $F$. Let $l$ be a rational prime and fix an isomorphism $\iota : Q_l^{ac} \cong \mathbb{C}$.

Suppose that $p \neq l$ is another rational prime. Let $K/Q_p$ be a finite extension. We will let $\mathcal{O}_K$ denote the ring of integers of $K$, $\wp_K$ the unique maximal ideal of $\mathcal{O}_K$, $\mathcal{v}_K$ the canonical valuation $K^\times \to \mathbb{Z}$, $k(\wp_K)$ the residue field $\mathcal{O}_K/\wp_K$ and $|\cdot|_K$ the absolute value normalised by $|x|_K = (\#k(\wp_K))^{-\mathcal{v}_K(x)}$. We will let $\text{Frob}_{\mathcal{v}_K}$ denote the geometric Frobenius element of $\text{Gal}(k(\mathcal{v}_K)^{ac}/k(\mathcal{v}_K))$. We will let $I_{\mathcal{v}_K}$ denote the kernel of the natural surjection $\text{Gal}(K^{ac}/K) \to \text{Gal}(k(\mathcal{v}_K)^{ac}/k(\mathcal{v}_K))$. We will let $W_K$ denote the preimage under $\text{Gal}(K^{ac}/K) \to \text{Gal}(k(\mathcal{v}_K)^{ac}/k(\mathcal{v}_K))$ of $\text{Frob}_{\mathcal{v}_K}$ endowed with a topology by decreeing that $I_K$ with its usual topology is an open subgroup of $W_K$. Local class field theory provides a canonical isomorphism $\text{Art}_K : K^\times \cong W_K^{ab}$, which takes uniformisers to lifts of $\text{Frob}_{\mathcal{v}_K}$.

Let $\Omega$ be an algebraically closed field of characteristic 0 and of the same cardinality as $\mathbb{C}$. (Thus in fact $\Omega \cong \mathbb{C}$.) By a Weil-Deligne representation of $W_K$ over $\Omega$ we mean a finite dimensional $\Omega$-vector space $V$ together with a homomorphism $r : W_K \to GL(V)$ with open kernel and an element $N \in \text{End}(V)$ which satisfies

$$r(\sigma)Nr(\sigma)^{-1} = |\text{Art}_K^{-1}(\sigma)|_K N.$$
We sometimes denote a Weil-Deligne representation by \((V, r, N)\) or simply \((r, N)\).

We call \((V, r, N)\) Frobenius semisimple if \(r\) is semisimple. If \((V, r, N)\) is any Weil-Deligne representation we define its Frobenius semisimplification \((V, r, N)^{F-ss} = (V, r^{ss}, N)\) as follows. Choose a lift \(\phi\) of \(\text{Frob}_{v_K}\) to \(W_K\). Let \(r(\phi) = su = us\) where \(s \in GL(V)\) is semisimple and \(u \in GL(V)\) is unipotent. For \(n \in \mathbb{Z}\) and \(\sigma \in I_K\) set \(r^{ss}(\phi^n \sigma) = s^n r(\sigma)\). This is independent of the choices, and gives a Frobenius semisimple Weil-Deligne representation.

One of the main results of [HT] is that, given a choice of \((#k(v_K))^{1/2} \in \Omega\), there is a bijection \(\text{rec}\) (the local Langlands correspondence) from isomorphism classes of irreducible smooth representations of \(GL_n(K)\) over \(\Omega\) to isomorphism classes of \(n\)-dimensional Frobenius semisimple Weil-Deligne representations of \(W_K\), and that this bijection is natural in a number of respects. (See [HT] for details.)

We will call a Weil-Deligne representation of \(W_K\) over \(\mathbb{Q}_{\text{ac}}\) bounded if for some (and hence all) \(\sigma \in W_K - I_K\) all the eigenvalues of \(r(\sigma)\) are \(l\)-adic units. There is an equivalence of categories between bounded Weil-Deligne representations of \(W_K\) over \(\mathbb{Q}_{\text{ac}}\) and continuous representations of \(\text{Gal}(\mathbb{K}_{\text{ac}}/\mathbb{K})\) on finite dimensional \(\mathbb{Q}_{\text{ac}}\)-vector spaces as follows. Fix a lift \(\phi \in W_K\) of \(\text{Frob}_{v_K}\) and a continuous homomorphism \(t : I_K \rightarrow \mathbb{Z}_l\). Send a Weil-Deligne representation \((V, r, N)\) to \((V, \rho)\), where \(\rho\) is the unique continuous representation of \(\text{Gal}(\mathbb{K}_{\text{ac}}/\mathbb{K})\) on \(V\) such that

\[
\rho(\phi^n \sigma) = r(\phi^n \sigma) \exp(t(\sigma)N)
\]

for all \(n \in \mathbb{Z}\) and \(\sigma \in I_K\). Up to natural isomorphism this functor is independent of the choices of \(t\) and \(\phi\). We will write \(\text{WD}(V, \rho)\) for the Weil-Deligne representation corresponding to a continuous representation \((V, \rho)\). If \(\text{WD}(V, \rho) = (V, r, N)\), then have \(\rho|^{ss}_{W_K} \cong r^{ss}\). (See [T], §4 and [D], §8 for details.)

Now suppose that \(L\) is a finite, imaginary CM extension of \(\mathbb{Q}\). Let \(c \in \text{Aut}(L)\) denote complex conjugation. Suppose that \(\Pi\) is a cuspidal automorphic representation of \(GL_n(\mathbb{A}_L)\) such that

- \(\Pi \circ c \cong \Pi^\vee\);
- \(\Pi_{\infty}\) has the same infinitesimal character as some algebraic representation over \(\mathbb{C}\) of the restriction of scalars from \(L\) to \(\mathbb{Q}\) of \(GL_n\);
- and for some finite place \(x\) of \(L\) the representation \(\Pi_{x}\) is square integrable.

(In this paper ‘square integrable’ (resp. ‘tempered’) will mean the twist by a character of a pre-unitary representation which is square integrable (resp. tempered).) In [HT] (see theorem C in the introduction of [HT]) it is shown that there is a unique continuous semisimple representation

\[
R_t(\Pi) : \text{Gal}(L^{ac}/L) \rightarrow GL_n(\mathbb{Q}_{\text{ac}}^{\text{ac}})
\]
such that for each finite place \( v \nmid l \) of \( L \)
\[
\text{rec}(\Pi_v^\vee | \det | \frac{1-n}{2}) = (iR_l(\Pi)|_{W_{L_v}^{ss}}, N)
\]
for some \( N \). Moreover it is shown that \( \Pi_v \) is tempered for all finite places \( v \) of \( L \), which completely determines the \( N \) (see lemma 1.3 below). If \( n = 1 \) both these assertions are true without the assumption that \( \Pi \circ \sigma \cong \Pi^\vee \).

The main theorem of this paper identifies the \( N \) of \( \text{WD}(R_l(\Pi)|_{\text{Gal}(L^{ac}_v/L_v)}) \) with the above \( N \). More precisely we prove the following.

**Theorem 1.1.** *Keep the above notation and assumptions. Then for each finite place \( v \nmid l \) of \( L \) there is an isomorphism
\[
i \text{WD}(R_l(\Pi)|_{\text{Gal}(L^{ac}_v/L_v)})^{F-ss} \cong \text{rec}(\Pi_v^\vee | \det | \frac{1-n}{2})
\]
of Weil-Deligne representations over \( \mathbb{C} \).*

As \( R_l(\Pi) \) is semisimple and \( \text{rec}(\Pi_v^\vee | \det | \frac{1-n}{2}) \) is indecomposable if \( \Pi_v \) is square integrable, we have the following corollary.

**Corollary 1.2.** *If \( \Pi_v \) is square integrable for a finite place \( v \nmid l \), then the representation \( R_l(\Pi) \) is irreducible.*

In the rest of this section we consider some generalities on Galois representations and Weil-Deligne representations. First consider Weil-Deligne representations over an algebraically closed field \( \Omega \) of characteristic zero and the same cardinality as \( \mathbb{C} \). For a finite extension \( K'/K \) of \( p \)-adic fields, we define
\[
(V, r, N)|_{W_{K'}} = (V, r|_{W_{K'}}, N).
\]
If \((V, r)\) is a finite dimensional representation of \( W_K \) with open kernel and if \( s \in \mathbb{Z}_{\geq 1} \) we will write \( \text{Sp}_s(W) \) for the Weil-Deligne representation
\[
(W^s, r|_{\text{Art}_K^{-1}|_K^s} \oplus \cdots \oplus r|_{\text{Art}_K^{-1}|_K}, N)
\]
where \( N : r|_{\text{Art}_K^{-1}|_K^s} \cong r|_{\text{Art}_K^{-1}|_K^s} \) for \( i = 1, \ldots, s-1 \). This defines \( \text{Sp}_s(W) \) uniquely (up to isomorphism). If \( W \) is irreducible then \( \text{Sp}_s(W) \) is indecomposable and every indecomposable Weil-Deligne representation is of the form \( \text{Sp}_s(W) \) for a unique \( s \) and a unique irreducible \( W \). If \( \pi \) is an irreducible cuspidal representation of \( GL_g(K) \) then \( \text{rec}(\pi) = (r, 0) \) with \( r \) irreducible. Moreover for any \( s \in \mathbb{Z}_{\geq 1} \) we have (in the notation of section I.3 of [HT])
\[
\text{rec}(\text{Sp}_s(\pi)) = \text{Sp}_s(r).
\]

If \( q \in \mathbb{R}_{>0} \), then by a *Weil \( q \)-number* we mean \( \alpha \in \mathbb{Q}^{ac} \) such that for all \( \sigma : \mathbb{Q}^{ac} \hookrightarrow \mathbb{C} \) we have \( (\sigma \alpha)(\sigma \alpha) = q \). We will call a Weil-Deligne representation \((V, r, N)\) of \( W_K \) strictly pure of weight \( k \in \mathbb{R} \) if for some (and hence every) lift \( \phi \) of \( \text{Frob}_{v_K} \), every eigenvalue \( \alpha \) of \( r(\phi) \) is a Weil \((#k(v_K))^k\)-number. In this case we must have \( N = 0 \). We will call \((V, r, N)\) *mixed* if it has an increasing filtration \( \text{Fil}_i^W \) with \( \text{Fil}_i^W V = V \) for \( i \gg 0 \) and \( = (0) \) for
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\( i << 0 \), such that the \( i \)-th graded piece is strictly pure of weight \( i \). If \((V, r, N)\) is mixed then there is a unique choice of filtration \( \text{Fil}^W_i \), and \( N(\text{Fil}^W_i V) \subset \text{Fil}^W_{i-2} V \). Finally we will call \((V, r, N)\) pure of weight \( k \) if it is mixed with all weights in \( k + \mathbb{Z} \) and if for all \( i \in \mathbb{Z}_{>0} \)

\[ N^i : \text{gr}^W_{k+i} V \sim \text{gr}^W_{k-i} V. \]

If \( W \) is strictly pure of weight \( k \), then \( \text{Sp}_s(W) \) is pure of weight \( k - (s - 1) \) for any \( s \in \mathbb{Z}_{\geq 1} \).

(1) \( (V, r, N) \) is pure if and only if \( (V, r, N)_{F\text{-ss}} \) is.

(2) If \( L/K \) is a finite extension, then \( (V, r, N) \) is pure if and only if \( (V, r, N)|_{W_L} \) is pure.

(3) An irreducible smooth representation \( \pi \) of \( GL_n(K) \) has \( \sigma \pi \) tempered for all \( \sigma : \Omega \hookrightarrow \mathbb{C} \) if and only if \( \text{rec}(\pi) \) is pure of some weight.

(4) Given \((V, r)\) with \( r \) semisimple, there is, up to equivalence, at most one choice of \( N \) which makes \((V, r, N)\) pure.

(5) If \((V, r, N)\) is a Frobenius semisimple Weil-Deligne representation which is pure of weight \( k \) and if \( W \subset V \) is a Weil-Deligne subrepresentation, then the following are equivalent:

\( \wedge^{\dim W} W \) is pure of weight \( k \dim W \),

(6) Suppose that \((V, r, N)\) is a Frobenius semisimple Weil-Deligne representation which is pure of weight \( k \). Suppose also that \( \text{Fil}^j V \) is a decreasing filtration of \( V \) by Weil-Deligne subrepresentations such that \( \text{Fil}^j V = (0) \) for \( j >> 0 \) and \( \text{Fil}^j V = V \) for \( j << 0 \). If for each \( j \)

\[ \wedge^{\dim \text{gr}^j V} \text{gr}^j V \]

is pure of weight \( k \dim \text{gr}^j V \), then

\[ V \cong \bigoplus \text{gr}^j V \]

and each \( \text{gr}^j V \) is pure of weight \( k \).

Proof: The first two parts are straightforward (using the fact that the filtration \( \text{Fil}^W_i \) is unique). For the third part recall that an irreducible smooth representation \( \text{Sp}_{s_1}(\pi_1) \boxplus \cdots \boxplus \text{Sp}_{s_t}(\pi_t) \) (see section I.3 of [HT]) is tempered if and only if the absolute values of the central characters of the \( \text{Sp}_{s_i}(\pi_i) \) are all equal.

Suppose that \((V, r, N)\) is Frobenius semisimple and pure of weight \( k \). As a \( W_K \)-module we can write uniquely \( V = \bigoplus_{i \in \mathbb{Z}} V_i \) where \((V_i, r, 0)\) is strictly pure of weight \( k+i \). For \( i \in \mathbb{Z}_{>0} \) let \( V(i) \) denote the kernel of \( N^{i+1} : V_i \to V_{i-2} \). Then \( N : V_{i+2} \hookrightarrow V_i \) and \( V_i = NV_{i+2} \oplus V(i) \). Thus

\[ V = \bigoplus_{i \in \mathbb{Z}} \bigoplus_{j=0}^i N^j V(i), \]
and for $0 \leq j \leq i$ the map $N^j : V(i) \to V_{i-2j}$ is injective. Also note that as a virtual $W_K$-module $[V(i)] = [V_i] - [V_{i+2} \otimes |\Art|_{K}^{-1}]$. Thus if $r$ is semisimple then $(V, r)$ determines $(V, r, N)$ up to isomorphism. This establishes the fourth part.

Now consider the fifth part. If $W$ is a direct summand it is certainly pure of the same weight $k$ and $\bigwedge^{\dim W} W$ is then pure of weight $k \dim W$. Conversely if $W$ is pure of weight $k$ then

$$W = \bigoplus_{i \in \mathbb{Z}} \bigoplus_{j=0}^{i} N^j W(i),$$

where $W(i) = W \cap V(i)$. As a $W_K$-module we can decompose $V(i) = W(i) \oplus U(i)$. Setting

$$U = \bigoplus_{i \in \mathbb{Z}} \bigoplus_{j=0}^{i} N^j U(i),$$

we see that $V = W \oplus U$ as Weil-Deligne representations. Now suppose only that $\bigwedge^{\dim W} W$ is pure of weight $k \dim W$. Write

$$W \cong \bigoplus_{j} \Sp_{s_j}(X_j)$$

where each $X_j$ is strictly pure of some weight $k + k_j + (s_j - 1)$. Then, looking at highest exterior powers, we see that $\sum j k_j = 0$. On the other hand as $V$ is pure we see that $k_j \leq 0$ for all $j$. We conclude that $k_j = 0$ for all $j$ and hence that $W$ is pure of weight $k$.

The final part follows from the fifth part by a simple inductive argument. □

Now let $L$ denote a number field. Write $| \cdot |_L$ for

$$\prod_x | \cdot |_L : \mathbb{A}_L^\times / L^\times \to \mathbb{R}_{>0}^\times,$$

and write $\Art_L$ for

$$\prod_x \Art_{L_x} : \mathbb{A}_L^\times / L^\times \to \Gal(L^{ac}/L)^{ab}.$$

We will call a continuous representation

$$R : \Gal(L^{ac}/L) \to GL_n(\mathbb{Q}_l^{ac})$$

pure of weight $k$ if for all but finitely many finite places $x$ of $L$ the representation $R$ is unramified at $x$ and every eigenvalue $\alpha$ of $R(\Frob_x)$ is a Weil $(\#k(x))^k$-number. Note that if $n = 1$ then $R$ is pure of weight $k$ if and only if for all $\iota : \mathbb{Q}_l^{ac} \hookrightarrow \mathbb{C}$ we have $|\iota R \circ \Art_L|^2 = |\iota|^k$. In particular if $n = 1$ and $R$ is pure then $R|_{W_L}$ is strictly pure for all finite places $x$ of $L$.

We have the following lemma.
Lemma 1.4. Suppose that $M/L$ is a finite extension of number fields. Suppose also that
\[ R : \text{Gal}(L^{ac}/L) \rightarrow GL_n(\mathbb{Q}^{ac}_\ell) \]
is a continuous semisimple representation which is pure of weight $k$. Suppose that
\[ S : \text{Gal}(M^{ac}/M) \rightarrow GL_{an}(\mathbb{Q}^{ac}_\ell) \]
is another continuous representation with $S^{ss} \cong R|_{\text{Gal}(M^{ac}/M)}^a$ for some $a \in \mathbb{Z}_{>0}$. Suppose finally that $w$ is a place of $M$ above a finite place $v$ of $L$. If $\text{WD}(S|_{\text{Gal}(M^{ac}/M_w)})$ is pure of weight $k$, then $\text{WD}(R|_{\text{Gal}(L^{ac}_w/L_v)})$ is also pure of weight $k$.

**Proof:** Write
\[ R|_{\text{Gal}(M^{ac}/M)} = \bigoplus_i R_i \]
where each $R_i$ is irreducible. Then det $R_i$ is pure of weight $k \dim R_i$ and so that the top exterior power $\bigwedge^{\dim R_i} \text{WD}(R_i|_{\text{Gal}(M^{ac}/M_w)})$ is also pure of weight $k \dim R_i$. Lemma 1.3(6) tells us that
\[ \text{WD}(S|_{\text{Gal}(M^{ac}/M_w)})^{F,ss} \cong \left( \bigoplus_i \text{WD}(R_i|_{\text{Gal}(M^{ac}/M_w)})^{F,ss} \right)^a \cong \left( \text{WD}(R|_{\text{Gal}(M^{ac}/M_w)})^{F,ss} \right)^a, \]
and that $\text{WD}(R|_{\text{Gal}(M^{ac}/M_w)})^{F,ss}$ is pure of weight $w$. Applying lemma 1.3(1) and (2), we see that $\text{WD}(R|_{\text{Gal}(L^{ac}_w/L_v)})$ is also pure. \(\square\)

2. Shimura varieties

In this section we study the geometry of integral models of Shimura varieties of the type considered in [HT], but with Iwahori level. It may be viewed as a generalisation of the work of Deligne-Rapoport [DR] in the case of modular curves.

In this section,

- let $E$ be an imaginary quadratic field, $F^+$ a totally real field and set $F = EF^+$;
- let $p$ be a rational prime which splits $p = uu^c$ in $E$;
- and let $w = w_1, ..., w_r$ be the primes of $F$ above $u$;
- and let $B$ be a division algebra with centre $F$ such that
  - $\dim_F B = n^2$,
  - $B^{op} \cong B \otimes_{F,c} F$,
  - at every place $x$ of $F$ either $B_x$ is split or a division algebra,
  - if $n$ is even then the number of finite places of $F^+$ above which $B$ is ramified is congruent to $1 + \frac{n}{2}[F^+ : \mathbb{Q}]$ modulo 2.
Pick a positive involution \(*\) on \(B\) with \(*|_F = c\). Let \(V = B\) as a \(B \otimes_F B^{\text{op}}\)-module. For \(\beta \in B^{*-1}\) define a pairing

\[
(\ , \ ) : V \times V \longrightarrow Q \\
(x_1, x_2) \longmapsto \text{tr}_{F/Q} \text{tr}_{B/F} (x_1 \beta x_2^*).
\]

Also define an involution \(#\) on \(B\) by \(x^# = \beta x^* \beta^{-1}\) and a reductive group \(G/Q\) by setting, for any \(\mathbb{Q}\)-algebra \(R\), the group \(G(R)\) equal to the set of

\[
(\lambda, g) \in R^\times \times (B^{\text{op}} \otimes_\mathbb{Q} R)^\times
\]

such that

\[
g g^# = \lambda.
\]

Let \(\nu : G \rightarrow \mathbb{G}_m\) denote the multiplier character sending \((\lambda, g)\) to \(\lambda\). Note that if \(x\) is a rational prime which splits \(x = yy^c\) in \(E\) then

\[
G(\mathbb{Q}_x) \sim (B^y)^\times \times \mathbb{Q}_x^\times \\
(\lambda, g) \longmapsto (g y, \lambda).
\]

We can and will assume that

- if \(x\) is a rational prime which does not split in \(E\) the \(G \times \mathbb{Q}_x\) is quasisplit;
- the pairing \((\ , \ )\) on \(V \otimes_\mathbb{Q} \mathbb{R}\) has invariants \((1, n - 1)\) at one embedding \(\tau : F^+ \hookrightarrow \mathbb{R}\) and invariants \((0, n)\) at all other embeddings \(F^+ \hookrightarrow \mathbb{R}\).

(See section I.7 of [HT] for details.)

Let \(U\) be an open compact subgroup of \(G(\mathbb{A}^\infty)\). Define a functor \(\mathcal{X}_U\) from the category of pairs \((S, s)\), where \(S\) is a connected locally noetherian \(F\)-scheme and \(s\) is a geometric point of \(S\), to the category of sets, sending \((S, s)\) to the set of isogeny classes of four-tuples \((A, \lambda, i, \eta)\) where

- \(A/S\) is an abelian scheme of dimension \([F^+ : \mathbb{Q}]n^2\);
- \(i : B \hookrightarrow \text{End}(A) \otimes_\mathbb{Z} \mathbb{Q}\) such that \(\text{Lie} A \otimes (E \otimes_\mathbb{Q} \mathcal{O}_S) \cdot 1 \otimes 1 \mathcal{O}_S\) is locally free over \(\mathcal{O}_S\) of rank \(n\) and the two actions of \(F^+\) coincide;
- \(\lambda : A \rightarrow A^\vee\) is a polarisation such that for all \(b \in B\) we have \(\lambda \circ i(b) = i(b^*)^\vee \circ \lambda\);
- \(\pi\) is a \(\pi_1(S, s)\)-invariant \(U\)-orbit of isomorphisms of \(B \otimes_\mathbb{Q} \mathbb{A}^\infty\)-modules \(\eta : V \otimes_\mathbb{Q} \mathbb{A}^\infty \rightarrow VA_s\) which take the standard pairing \((\ , \ )\) on \(V\) to a \((\mathbb{A}^\infty)^\times\)-multiple of the \(\lambda\)-Weil pairing on \(VA_s\).

Here \(VA_s = \left(\lim_N A[N](k(s))\right) \otimes_\mathbb{Z} \mathbb{Q}\) is the adelic Tate module. For the precise notion of isogeny class see section III.1 of [HT]. If \(s\) and \(s'\) are both geometric points of a connected locally noetherian \(F\)-scheme \(S\) then \(\mathcal{X}_U(S, s)\) and \(\mathcal{X}_U(S, s')\) are in canonical bijection. thus we may think of \(\mathcal{X}_U\) as a functor from connected locally noetherian \(F\)-schemes to sets. We
may further extend it to a functor from all locally noetherian $F$-schemes to sets by setting
\[ \mathbb{X}_U \left( \prod_i S_i \right) = \prod_i \mathbb{X}_U(S_i). \]

If $U$ is sufficiently small (i.e. for some finite place $x$ of $\mathbb{Q}$ the projection of $U$ to $G(\mathbb{Q}_x)$ contains no element of finite order except 1) then $\mathbb{X}_U$ is represented by a smooth projective variety $X_U/F$ of dimension $n - 1$. The inverse system of the $X_U$ for varying $U$ has a natural action of $G(\mathbb{A}^\infty)$.

Choose a maximal $\mathbb{Z}_{(p)}$-order $\mathcal{O}_B$ of $B$ with $\mathcal{O}_B^* = \mathcal{O}_B$. Also fix an isomorphism $\mathcal{O}_{B,w}^{op} \cong M_n(\mathcal{O}_{F,w})$, and let $\varepsilon \in B_w$ denote the element corresponding to the diagonal matrix $(1, 0, 0, ..., 0) \in M_n(\mathcal{O}_{F,w})$. We decompose $G(\mathbb{A}^\infty)$ as

\[ G(\mathbb{A}^\infty) = G(\mathbb{A}^{\infty,p}) \times \left( \prod_{i=2}^r (\mathcal{O}_{B,w_i}^{op})^\times \right) \times GL_n(F_w) \times \mathbb{Q}_p^\times. \]

Let $\varpi$ denote a uniformiser for $\mathcal{O}_{F,w}$. For $m = (m_2, ..., m_r) \in \mathbb{Z}_{\geq 0}^{r-1}$, set
\[ U_p^w(m) = \prod_{i=2}^r \ker \left( (\mathcal{O}_{B,w_i}^{op})^\times \to (\mathcal{O}_{B,w_i}^{op}/w_i^{m_i})^\times \right) \subset \prod_{i=2}^r (\mathcal{O}_{B,w_i}^{op})^\times. \]

Let $B_n$ denote the Borel subgroup of $GL_n$ consisting of upper triangular matrices and let $N_n$ denote its unipotent radical. Let $Iw_{n,w}$ denote the subgroup of $GL_n(\mathcal{O}_{F,w})$ consisting of matrices which reduce modulo $w$ to $B_n(k(w))$. We will consider the following open subgroups of $G(\mathbb{Q}_p)$:
\[ \text{Ma}(m) = U_p^w(m) \times GL_n(\mathcal{O}_{F,w}) \times \mathbb{Z}_p^\times \]
\[ \text{Iw}(m) = U_p^w(m) \times Iw_{n,w} \times \mathbb{Z}_p^\times. \]

Let $U^p$ be an open compact subgroup of $G(\mathbb{A}^{\infty,p})$. Write $U_0$ (resp. $U$) for $U^p \times \text{Ma}(m)$ (resp. $U^p \times \text{Iw}(m)$).

We recall that in section III.4 [HT] integral model of $X_{U_0}$ over $\mathcal{O}_{F,w}$ is defined. It represents the functor $\mathbb{X}_{U_0}$ from locally noetherian $\mathcal{O}_{F,w}$-schemes to sets. As above, $\mathbb{X}_{U_0}$ is initially defined on the category of connected locally noetherian $\mathcal{O}_{F,w}$-schemes with a geometric point to sets. It sends $(S, s)$ to the set of prime-to-$p$ isogeny classes of $(r + 3)$-tuples $(A, \lambda, i, \varpi^p, \alpha_i)$, where

- $A/S$ is an abelian scheme of dimension $[F^+: \mathbb{Q}]n^2$;
- $i : \mathcal{O}_B \hookrightarrow \text{End}(A) \otimes_{\mathbb{Z}} \mathbb{Z}(p)$ such that $\text{Lie} A \otimes (\mathcal{O}_{F,u} \otimes \mathbb{Z}(p), \mathcal{O}_S) \otimes 1 \mathcal{O}_S$ is locally free of rank $n$ and the two actions of $\mathcal{O}_F$ coincide;
- $\lambda : A \to A^\vee$ is a prime-to-$p$ polarisation such that for all $b \in \mathcal{O}_B$ we have $\lambda \circ i(b) = i(b^\vee) \circ \lambda$;
• $\pi^p$ is a $\pi_1(S, s)$-invariant $U^p$-orbit of isomorphisms of $B \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p}$-modules $\eta : V \otimes_{\mathbb{Q}} \mathbb{A}^{\infty,p} \to V^p A_s$ which take the standard pairing $(\ , \ )$ on $V$ to a $\mathbb{A}^{\infty,p}$-multiple of the $\lambda$-Weil pairing on $V^p A_s$;

• for $2 \leq i \leq r$, $\alpha_i : (w_i^{-m_i} \mathcal{O}_{B,w_i}/\mathcal{O}_{B,w_i})s \sim A[w_i^{m_i}]$ is an isomorphism of $S$-schemes with $\mathcal{O}_B$-actions.

Then $X_{U_0}$ is smooth and projective over $\mathcal{O}_{F,w}$ ([HT], page 109). As $U^p$ varies, the inverse system of the $X_{U_0}$’s has an action of $G(\mathbb{A}^{\infty,p})$.

Given an $(r+3)$-tuple as above we will write $\mathcal{G}_A$ for $\varepsilon A[w^{\infty}]$ a Barsotti-Tate $\mathcal{O}_{F,w}$-module. Over a base in which $p$ is nilpotent it is one dimensional. If $\mathcal{A}$ denotes the universal abelian scheme over $X_{U_0}$, we will write $\mathcal{G}$ for $\mathcal{G}_A$. This $\mathcal{G}$ is compatible, i.e. the two actions of $\mathcal{O}_{F,w}$ on Lie $\mathcal{G}$ coincide (see [HT]).

Write $\overline{X}_{U_0}$ for the special fibre $X_{U_0} \times_{\mathrm{Spec} \mathcal{O}_{F,w}} \mathrm{Spec} k(w)$. For $0 \leq h \leq n - 1$, we let $\overline{X}^{[h]}_{U_0}$ denote the reduced closed subscheme of $\overline{X}_{U_0}$ whose closed geometric points $s$ are those for which the maximal etale quotient of $\mathcal{G}_s$ has $\mathcal{O}_{F,w}$-height at most $h$, and let

$$\overline{X}^{(h)}_{U_0} = \overline{X}^{[h]}_{U_0} - \overline{X}^{[h-1]}_{U_0}$$

(where we set $\overline{X}^{[1]}_{U_0} = \emptyset$). Then $\overline{X}^{(h)}_{U_0}$ is smooth of pure dimension $h$ (corollary III.4.4 of [HT]), and on it there is a short exact sequence

$$(0) \longrightarrow \mathcal{G}^0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}^{et} \longrightarrow (0)$$

where $\mathcal{G}^0$ is a formal Barsotti-Tate $\mathcal{O}_{F,w}$-module and $\mathcal{G}^{et}$ is an etale Barsotti-Tate $\mathcal{O}_{F,w}$-module with $\mathcal{O}_{F,w}$-height $h$.

**Lemma 2.1.** If $0 \leq h \leq n - 1$ then the Zariski closure of $\overline{X}^{(h)}_{U_0}$ contains $\overline{X}^{(0)}_{U_0}$.

**Proof:** This is ‘well known’, but for lack of a reference we give a proof. Let $x$ be a closed geometric point of $\overline{X}^{(0)}_{U_0}$. By lemma II.4.1 of [HT] the formal completion of $X_{U_0} \times_{\mathrm{Spec} k(w)^{ac}} x$ is isomorphic to the equicharacteristic universal deformation ring of $\mathcal{G}_x$. According to the proof of proposition 4.2 of [Dr] this is Spf $k(w)^{ac}[[T_1, \ldots, T_{n-1}]]$ and we can choose the $T_i$ and a formal parameter $S$ on the universal deformation of $\mathcal{G}_x$ such that

$$[w_s](S) \equiv w_s S + \sum_{i=1}^{n-1} T_i S^{k(w)^i} + S^{k(w)^n} \equiv (\mod S^{k(w)^n+1}).$$

Thus we get a morphism

$$\mathrm{Spec} k(w)^{ac}[[T_1, \ldots, T_{n-1}]] \longrightarrow \overline{X}_{U_0}$$

lying over $x : k(w)^{ac} \to \overline{X}_{U_0}$, such that, if $k$ denotes the algebraic closure of the field of fractions of $k(w)^{ac}[[T_1, \ldots, T_{n-1}]]/(T_1, \ldots, T_{n-1})$, then the induced map

$$\mathrm{Spec} k \longrightarrow \overline{X}_{U_0}$$
factors through $X_{U_0}^{(h)}$. Thus $x$ is in the closure of $X_{U_0}^{(h)}$, and the lemma follows. □

Now we define the functor $X_U$. Again we initially define it as a functor from the category of connected locally noetherian schemes with a geometric point to sets, but then (as above) we extend it to a functor from locally noetherian schemes to sets. The functor $X_U$ will send $(S, s)$ to the set of prime-to-$p$ isogeny classes of $(r + 4)$-tuples $(A, \lambda, i, \bar{p}, C, \alpha_i)$, where $(A, \lambda, i, \bar{p}^i, \alpha_i)$ is as in the definition of $gX_{U_0}$ and $C$ is a chain of isogenies
\[
C: G = G_0 \to G_1 \to \cdots \to G_n = G/G[w]
\]
of compatible Barsotti-Tate $O_{F,w}$-modules, each of degree $\#k(w)$ and with composite equal to the canonical map $G \to G/G[w]$. There is a natural transformation of functors $X_U \to X_{U_0}$.

**Lemma 2.2.** The functor $X_U$ is represented by a scheme $X_U$ which is finite over $X_{U_0}$. The scheme $X_U$ has some irreducible components of dimension $n$.

**Proof:** By denoting the kernel of $G_0 \to G_j$ by $\mathcal{K}_j \subset G[w]$, we can view the above chain as a flag
\[
0 = \mathcal{K}_0 \subset \mathcal{K}_1 \subset \mathcal{K}_2 \subset \cdots \subset \mathcal{K}_{n-1} \subset \mathcal{K}_n = G[w]
\]
of closed finite flat subgroup schemes with $O_{F,w}$-action, with each $\mathcal{K}_j/\mathcal{K}_{j-1}$ having order $\#k(w)$. Let $\mathcal{H}$ denote the sheaf of Hopf algebras over $X_{U_0}$ defining $G[w]$. Then $X_U$ is represented by a closed subscheme $X_U$ of the Grassmanian of chains of locally free direct summands of $\mathcal{H}$. (The closed conditions require that the subsheaves are sheaves of ideals defining a flag of closed subgroup schemes with the desired properties.) Thus $X_U$ is projective over $O_{F,w}$. At each closed geometric point $s$ of $X_{U_0}$ the number of possible $O_{F,w}$-submodules of $G[w]_s \cong G[w]_s^0 \times G[w]_s^{\text{et}}$ is finite, so $X_U$ is finite over $X_{U_0}$. To see that $X_U$ has some components of dimension $n$ it suffices to note that on the generic fibre the map to $X_{U_0}$ is finite etale. □

We say an isogeny $G \to G'$ of one-dimensional compatible Barsotti-Tate $O_{F,w}$-modules over a scheme $S$ of characteristic $p$ has **connected kernel** if it induces the zero map on Lie $G$. We will denote the relative Frobenius map by $F : G \to G^{(p)}$ and let $f = [k(w) : \mathbb{F}_p]$, and then $F^f : G \to G^{(#k(w))}$ is an isogeny of compatible Barsotti-Tate $O_{F,w}$-modules of degree $\#k(w)$ and has connected kernel.

We have the following rigidity lemma.

**Lemma 2.3.** Let $W$ denote the ring of integers of the completion of the maximal unramified extension of $F_w$. Suppose that $R$ is an Artinian local $W$-algebra with residue field $k(w)^{ac}$. Suppose also that
\[
\mathcal{C} : G_0 \to G_1 \to \cdots \to G_n = G_0
\]
is a chain of isogenies of degree $\#k(w)$ of one-dimensional compatible formal Barsotti-Tate $O_{F,w}$-modules of $O_{F,w}$-height $n$ with composite equal to multiplication by $\varpi_w$. If every isogeny $G_{i-1} \to G_i$ has connected kernel (for $i = 1, \ldots, n$) then $R$ is a $k(w)^{ac}$-algebra and $\mathcal{C}$
is the pull-back of a chain of Barsotti-Tate \(\mathcal{O}_{F,w}\)-modules over \(k(w)^{ac}\), with all the isogenies isomorphic to \(F^f\).

**Proof:** As the composite of the \(n\) isogenies induces multiplication by \(w\) on the tangent space, \(\varpi_w = 0\) in \(R\), i.e. \(R\) is a \(k(w)^{ac}\)-algebra. Choose a parameter \(T_i\) for \(G_i\) over \(R\). With respect to these choices, let \(f_i(T_i) \in R[[T_i]]\) represent \(G_{i-1} \to G_i\). We can write \(f_i(T_i) = g_i(T_i^{p^{h_i}})\) with \(h_i \in \mathbb{Z}_{\geq 0}\) and \(g_i'(0) \neq 0\). (See [F], chapter I, §3, Theorem 2.) As \(G_{i-1} \to G_i\) has connected kernel, \(f_i'(0) = 0\) and \(h_i > 0\). As \(f_i\) commutes with the action \([r]\) for all \(r \in \mathcal{O}_{F,w}\), we have \(f^{h_i} = \pi\) for all \(\pi \in k(w)\), hence \(h_i\) is a multiple of \(f = [k(w) : \mathbb{F}_p]\). Reducing modulo the maximal ideal of \(R\) we see that \(h_i \leq f\) and so in fact \(h_i = f\) and \(g_i'(0) \in R^\times\). Thus \(G_i \cong G_0^{(\#k(w)^f)}\) in such a way that the isogeny \(G_0 \to G_i\) is identified with \(F^{f_i}\). In particular \(G_0 \cong G_0^{(\#k(w)^n)}\) and hence \(G_0 \cong G_0^{(\#k(w)^{nm})}\) for any \(m \in \mathbb{Z}_{\geq 0}\). As \(R\) is Artinian some power of the absolute Frobenius on \(R\) factors through \(k(w)^{ac}\). Thus \(G_0\) is a pull-back from \(k(w)^{ac}\) and the lemma follows. \(\square\)

Now let \(U_{i,\varpi}\) denote the closed subscheme of \(\overline{X}_U = X_U \times_{\text{Spec} \mathcal{O}_{F,w}} \text{Spec} k(w)\) over which \(G_{i-1} \to G_i\) has connected kernel.

**Proposition 2.4.**  
(1) \(X_U\) has pure dimension \(n\) and semistable reduction over \(\mathcal{O}_{F,w}\), that is, for all closed points \(x\) of the special fibre \(X_U \times_{\text{Spec} \mathcal{O}_{F,w}} \text{Spec} k(w)\), there exists an etale morphism \(V \to X_U\) with \(x \in \text{Im} V\) and an etale \(\mathcal{O}_{F,w}\)-morphism:

\[
V \to \text{Spec} \mathcal{O}_{F,w}[T_1, \ldots, T_n]/(T_1 \cdots T_n - \varpi_w)
\]

for some \(1 \leq m \leq n\), where \(\varpi_w\) is a uniformizer of \(\mathcal{O}_{F,w}\).

(2) \(X_U\) is regular and the natural map \(X_U \to X_{U_0}\) is finite and flat.

(3) Each \(U_{i,\varpi}\) is smooth over \(\text{Spec} k(w)\) of pure dimension \(n - 1\), \(\overline{X}_U = \bigcup_{i=1}^n Y_{U,i}\) and, for \(i \neq j\) the schemes \(Y_{U,i}\) and \(Y_{U,j}\) share no common connected component. In particular, \(X_U\) has strictly semistable reduction.

**Proof:** In this proof we will make repeated use of the following version of Deligne’s homogeneity principle ([DR]). Write \(W\) for the ring of integers of the completion of the maximal unramified extension of \(F_w\). In what follows, if \(s\) is a closed geometric point of an \(\mathcal{O}_{F,w}\)-scheme \(X\) locally of finite type, then we write \(\mathcal{O}_{X,s}^h\) for the completion of the strict Henselisation of \(X\) at \(s\), i.e. \(\mathcal{O}_{X,s}^h = \mathcal{O}_{X,s}^h \times_{\text{Spec} \mathcal{W},s}\). Let \(P\) be a property of complete noetherian local \(W\)-algebras such that if \(X\) is a \(\mathcal{O}_{F,w}\)-scheme locally of finite type then the set of closed geometric points \(s\) of \(X\) for which \(\mathcal{O}_{X,s}^h\) has property \(P\) is Zariski open. Also let \(X \to X_{U_0}\) be a finite morphism with the following properties

(i) If \(s\) is a closed geometric point of \(\overline{X}_{U_0}^h\) then, up to isomorphism, \(\mathcal{O}_{X,s}^h\) does not depend on \(s\) (but only on \(h\)).

(ii) There is a unique geometric point of \(X\) above any geometric point of \(\overline{X}_{U_0}^h\).
If \( O_{X,s}^\wedge \) has property \( P \) for every geometric point of \( X \) over \( X_{U_0}^{(0)} \), then \( O_{X,s}^\wedge \) has property \( P \) for every closed geometric point of \( X \). Indeed, if we let \( Z \) denote the closed subset of \( X \) where \( P \) does not hold, then its image in \( X_{U_0} \) is closed and is either empty or contains some \( X_{U_0}^{(h)} \). In the latter case, by lemma 2.1, it also contains \( X_{U_0}^{(0)} \), which is impossible. Thus \( Z \) must be empty.

Note that both \( X = X_U \) and \( X = Y_{U,i} \) satisfy the above condition (ii) for the homogeneity principle, by letting \( R = k(w)^{ac} \) in lemma 2.3.

(1): The dimension of \( O_{X,U,s}^\wedge \) as \( s \) runs over geometric points of \( X_U \) above \( X_{U_0}^{(0)} \) is constant, say \( m \). Applying the homogeneity principle to \( X = X_U \) with \( P \) being ‘dimension \( m \)’, we see that \( X_U \) has pure dimension \( m \). By lemma 2.2 we must have \( m = n \) and \( X_U \) has pure dimension \( n \).

Now we will apply the above homogeneity principle to \( X = X_U \) taking \( P \) to be ‘isomorphic to \( W[[T_1, ..., T_n]]/(T_1 \cdots T_m - w) \) for some \( m \leq n \). By a standard argument (see e.g. the proof of proposition 4.10 of [Y]) the set of points with this property is open and if all closed geometric points of \( X_U \) have this property \( P \) then \( X_U \) is semistable of pure dimension \( n \).

Let \( s \) be a geometric point of \( X_U \) over a point of \( X_{U_0}^{(0)} \). Choose a basis \( e_i \) of \( \text{Lie} \mathcal{G}_i \) over \( O_{X,U,s}^\wedge \), such that \( e_n \) maps to \( e_0 \) under the isomorphism \( \mathcal{G}_n = \mathcal{G}_0/\mathcal{G}_0[w] \cong \mathcal{G}_0 \) induced by \( w \). With respect to these bases let \( X_i \in O_{X,U,s}^\wedge \) represent the linear map \( \text{Lie} \mathcal{G}_{i-1} \to \text{Lie} \mathcal{G}_i \).

Then

\[
X_1 \cdots X_n = w.
\]

Moreover it follows from lemma 2.3 that \( O_{X,U,s}^\wedge/(X_1, ..., X_n) = k(w)^{ac} \). (Because, by lemma III.4.1 of [HT], \( O_{X,U_0,s}^\wedge \) is the universal deformation space of \( \mathcal{G}_s \). Hence by lemma 2.3, \( O_{X,U,s}^\wedge \) is the universal deformation space for the chain

\[
\mathcal{G}_s \xrightarrow{F^i} \mathcal{G}_s^{(k(w))} \xrightarrow{F^i} \cdots \xrightarrow{F^i} \mathcal{G}_s^{(k(w)^n)} \cong \mathcal{G}_s/\mathcal{G}_s[w].
\]

Thus we get a surjection

\[
W[[X_1, ..., X_n]]/(X_1 \cdots X_n - w) \twoheadrightarrow O_{X,U,s}^\wedge
\]

and as \( O_{X,U,s}^\wedge \) has dimension \( n \) this map must be an isomorphism.

(2): We see at once that \( X_U \) is regular. Then [AK] V, 3.6 tells us that \( X_U \to X_{U_0} \) is flat.

(3): We apply the homogeneity principle to \( X = Y_{U,i} \) taking \( P \) to be ‘formally smooth of dimension \( n - 1 \)’. If \( s \) is a geometric point of \( Y_{U,i} \) above \( X_{U_0}^{(0)} \), then we see that \( O_{Y_{U,i},s}^\wedge \) is cut out in \( O_{X,U,s}^\wedge \cong W[[X_1, ..., X_n]]/(X_1 \cdots X_n - w) \) by the single equation \( X_i = 0 \). (We are using the parameters \( X_i \) defined above.) Thus

\[
O_{Y_{U,i},s}^\wedge \cong k(w)^{ac}[[X_1, ..., X_{i-1}, X_{i+1}, ..., X_n]]
\]
is formally smooth of dimension \( n - 1 \). We deduce that \( Y_{U,i} \) is smooth of pure dimension \( n - 1 \).

As our \( G/X_U \) is one-dimensional, over a closed point, at least one of the isogenies \( G_{i-1} \to G_i \) must have connected kernel, which shows that \( X_U = \bigcup_i Y_{U,i} \). Suppose \( Y_{U,i} \) and \( Y_{U,j} \) share a connected component \( Y \) for some \( i \neq j \). Then \( Y \) would be finite flat over \( X_{U_0} \) and so the image of \( Y \) would meet \( X_{U_0}^{(n-1)} \). This is impossible as above a closed point of \( X_{U_0}^{(n-1)} \) one isogeny among the chain can have connected kernel. Thus, for \( i \neq j \) the closed subschemes \( Y_{U,i} \) and \( Y_{U,j} \) have no connected component in common. \( \square \)

By the strict semistability, if we write, for \( S \subset \{1, \ldots, n\} \),

\[
Y_{U,S} = \bigcap_{i \in S} Y_{U,i}, \quad Y_{U,S}^0 = Y_{U,S} - \bigcup_{T \supset S} Y_{U,T}
\]

then \( Y_{U,S} \) is smooth over \( \text{Spec } k(w) \) of pure dimension \( n - \#S \) and \( Y_{U,S}^0 \) are disjoint for different \( S \). With respect to the finite flat map \( X_U \to X_{U_0} \), the inverse image of \( X_{U_0}^{(h)} \) is exactly the locus where at least \( n - h \) of the isogenies have connected kernel, i.e. \( \bigcup_{\#S = n-h} Y_{U,S} \).

Hence the inverse image of \( X_{U_0}^{(h)} \) is equal to \( \bigcup_{\#S = n-h} Y_{U,S}^0 \). Also note that the inverse system of \( Y_{U,S}^0 \) for varying \( U^p \) is stable by the action of \( G(\mathbb{A}^{\infty,p}) \).

Next we will relate the open strata \( Y_{U,S}^0 \) to the Igusa varieties of the first kind defined in [HT]. For \( 0 \leq h \leq n - 1 \) and \( m \in \mathbb{Z}_{\geq 0}^t \), we write \( I_{U^p,m}^{(h)} \) for the Igusa varieties of the first kind defined on page 121 of [HT]. We also define an \textit{Iwahori-Igusa variety of the first kind}

\[
I_U^{(h)} / X_{U_0}^{(h)}
\]

as the moduli space of chains of isogenies

\[
G_{\text{et}} = G_0 \to G_1 \to \cdots \to G_h = G_{\text{et}} / G_{\text{et}}[w]
\]

of etale Barsotti-Tate \( \mathcal{O}_{F,w} \)-modules, each isogeny having degree \( \#k(w) \) and with composite equal to the natural map \( G_{\text{et}} \to G_{\text{et}} / G_{\text{et}}[w] \). Then \( I_U^{(h)} \) is finite etale over \( X_{U_0}^{(h)} \), and as the Igusa variety \( I_{U^p,(1,m)}^{(h)} \) classifies the isomorphisms

\[
\alpha_1^{(h)} : (w^{-1} \mathcal{O}_{F,w} / \mathcal{O}_{F,w})_{X_{U_0}^{(h)}}^h \to G_{\text{et}}[w],
\]

the natural map

\[
I_{U^p,(1,m)}^{(h)} \to I_U^{(h)}
\]

is finite etale and Galois with Galois group \( B_h(k(w)) \). Hence we can identify \( I_U^{(h)} \) with \( I_{U^p,(1,m)}^{(h)} / B_h(k(w)) \). Note that the system \( I_U^{(h)} \) for varying \( U^p \) naturally inherits the action of \( G(\mathbb{A}^{\infty,p}) \).

**Lemma 2.5.** For \( S \subset \{1, \ldots, n\} \) with \( \#S = n-h \), there exists a finite map of \( X_{U_0}^{(h)} \)-schemes

\[
\varphi : Y_{U,S}^0 \to I_U^{(h)}
\]
which is bijective on the geometric points.

Proof: The map is defined in a natural way from the chain of isogenies \( C \) by passing to the etale quotient \( G^e \), and it is finite as \( Y_{U,S} \) (resp. \( I^h \)) is finite (resp. finite etale) over \( \overline{\mathbb{X}}_{U_0}^{(h)} \). Let \( s \) be a closed geometric point of \( I^h \) with a chain of isogenies

\[
G^e_s = G^e_0 \to \cdots \to G^e_h = G^e_s / G^e_s [w].
\]

For \( 1 \leq i \leq n \) let \( j(i) \) denote the number of elements of \( S \) which are less than or equal to \( i \). Set \( G_i = (G^e_s)^{(#^{k(w)\in i(i)})} \times G^e_{i-j(i)} \). If \( i \not\in S \), define an isogeny \( G_{i-1} \to G_i \) to be the identity times the given isogeny \( G^e_{i-1} \to G^e_i \). If \( i \in S \), define an isogeny \( G_{i-1} \to G_i \) to be \( F^j \) times the identity. Then

\[
G_0 \to \cdots \to G_n
\]
defines the unique geometric point of \( Y_{U,S}^0 \) above \( s \). □

Now recall from [HT] III.2, that for an irreducible algebraic representation \( \xi \) of \( G \) over \( \mathbb{Q}^a_c \), one can associate a lisse \( \mathbb{Q}^a_c \)-sheaf \( \mathcal{L}_\xi / X_U \) for every \( U \) such that \( X_U \) is defined, and the action of \( G(\mathbb{A}^\infty_p) \) extends to \( \mathcal{L}_\xi \). The sheaf \( \mathcal{L}_\xi \) is extended to the integral models and Igusa varieties, and on \( I^{(h)}_{U,p,1,m} \) and \( Y_{U,S}^0 \) they are the pull back of \( \mathcal{L}_\xi \) on \( \overline{\mathbb{X}}_{U_0}^{(h)} \).

**Corollary 2.6.** For every \( i \in \mathbb{Z}_{\geq 0} \), we have isomorphisms

\[
H^i_c(Y_{U,S} \times k(w) k(w)^{ac}, \mathcal{L}_\xi) \sim H^i_c(I^{(h)}_U \times k(w) k(w)^{ac}, \mathcal{L}_\xi) \sim H^i_c(I^{(h)}_{U,p,1,m} \times k(w) k(w)^{ac}, \mathcal{L}_\xi) B_h(k(w))
\]

that are compatible with the actions of \( G(\mathbb{A}^\infty_p) \) when we vary \( U^p \).

**Proof:** By lemma 2.5, for any lisse \( \mathbb{Q}^a_c \)-sheaf \( \mathcal{F} \) on \( I^{(h)}_U \), we have \( \mathcal{F} \cong \varphi_* \varphi^* \mathcal{F} \) by looking at the stalks at all geometric points. As \( \varphi \) is finite the first isomorphism follows. The second isomorphism follows easily as \( I^{(h)}_{U,p,1,m} \to I^{(h)}_U \) is finite etale and Galois with Galois group \( B_h(k(w)) \). □

In the next section, we will be interested in the \( G(\mathbb{A}^\infty_p) \times \text{Frob}^\mathbb{Z}_w \)-modules

\[
H^i(Y_{1w(m),S}, \mathcal{L}_\xi) = \lim_{U^p} H^i(Y_{U,S} \times k(w) k(w)^{ac}, \mathcal{L}_\xi).
\]

Here we relate the alternating sum of these modules to the cohomology of Igusa varieties. We will define the elements of Groth \( (G(\mathbb{A}^\infty_p) \times \text{Frob}^\mathbb{Z}_w) \) (we write Groth \( (G) \) for the
Grothendieck group of admissible $G$-modules) as follows:

$$[H(Y_{Iw(m)},S, \mathcal{L}_\xi)] = \sum_i (-1)^{n-\#S-i} H^i(Y_{Iw(m)},S, \mathcal{L}_\xi),$$

$$[H_c(Y^0_{Iw(m)},S, \mathcal{L}_\xi)] = \sum_i (-1)^{n-\#S-i} \lim_{U^p} H^i_c(Y^0_{U,S} \times_{k(w)} k(w)^{ac}, \mathcal{L}_\xi),$$

$$[H_c(I^{(h)}_{Iw(m)}, \mathcal{L}_\xi)] = \sum_i (-1)^{h-i} \lim_{U^p} H^i_c(I^{(h)}_{U} \times_{k(w)} k(w)^{ac}, \mathcal{L}_\xi).$$

Then, because $Y_{U,S} = \bigcup_{T \supseteq S} Y_{U,T}$ for each $U = U^p \times Iw(m)$, we have equalities

$$[H(Y_{Iw(m)},S, \mathcal{L}_\xi)] = \sum_{T \supseteq S} (-1)^{(n-\#S)-(n-\#T)} [H_c(Y^0_{Iw(m)},T, \mathcal{L}_\xi)]$$

$$= \sum_{T \supseteq S} (-1)^{(n-\#S)-(n-\#T)} [H_c(I^{(n-\#T)}_{Iw(m)}, \mathcal{L}_\xi)].$$

As there are $\binom{n-\#S}{h}$ subsets $T$ with $\#T = n - h$ and $T \supseteq S$, we conclude:

**Lemma 2.7.** We have an equality

$$[H(Y_{Iw(m)},S, \mathcal{L}_\xi)] = \sum_{h=0}^{n-\#S} (-1)^{n-\#S-h} \binom{n-\#S}{h} [H_c(I^{(h)}_{Iw(m)}, \mathcal{L}_\xi)]$$

in the Grothendieck group of admissible $G(\mathbb{A}_\infty^\infty) \times \text{Frob}_w$-modules over $\mathbb{Q}^{ac}$.

### 3. Proof of the main theorem

We now return to the situation in theorem 1.1. Recall that $L$ is an imaginary CM field and that $\Pi$ is a cuspidal automorphic representation of $GL_n(\mathbb{A}_L)$ such that

- $\Pi \circ c \cong \Pi^\vee$;
- $\Pi_\infty$ has the same infinitesimal character as some algebraic representation over $\mathbb{C}$ of the restriction of scalars from $L$ to $\mathbb{Q}$ of $GL_n$;
- and for some finite place $x$ of $L$ the representation $\Pi_x$ is square integrable.

Recall also that $v$ is a place of $L$ above a rational prime $p$, that $l \neq p$ is a second rational prime and that $\iota : \mathbb{Q}^{ac} \overset{\sim}{\to} \mathbb{C}$. Recall finally that $R_l(\Pi)$ is the $l$-adic representation associated to $\Pi$.

Choose a quadratic CM extension $L'/L$ in which $v$ and $x$ split. Choose places $v' \neq x'$ of $L'$ above $v$ and $x$ respectively. Also choose an imaginary quadratic field $E$ and a totally real field $F^+$ such that
• \([F^+:\mathbb{Q}]\) is even;
• \(F = EF^+\) is soluble and Galois over \(L'\);
• \(p\) splits as \(uv'\) in \(E\);
• there is a place \(w\) of \(F\) above \(u\) and \(v'\) such that \(\Pi_{F,w}\) has an Iwahori fixed vector;
• \(x\) lies above a rational prime which splits in \(E\) and \(x'\) splits in \(F\).

Denote by \(\Pi_F\) the base change of \(\Pi\) to \(GL_n(\mathbb{A}_F)\). Note that the component of \(\Pi_F\) at a place above \(x'\) is square integrable and hence \(\Pi_F\) is cuspidal.

Choose a division algebra \(B\) with centre \(F\) as in the previous section and satisfying

• \(B_x\) is split for all places \(x \neq z, z'\) of \(F\).

Also choose \(\alpha, \beta\) and \(G\) as in the previous section. Then it follows from theorem VI.2.9 and lemma VI.2.10 of [HT] that we can find

• a character \(\psi : \mathbb{A}_E^\times / E^\times \to \mathbb{C}^\times\),
• an irreducible algebraic representation \(\xi\) of \(G\) over \(\mathbb{Q}_{\text{al}}\),
• and an automorphic representation \(\pi\) of \(G(\mathbb{A})\),

such that

• \(\pi_\infty\) is cohomological for \(\iota\xi\),
• \(\psi\) is unramified above \(p\),
• \(\psi|_{E^\times_\infty}\) is the inverse of the restriction of \(\iota\xi\) to \(E^\times_\infty \subset G(\mathbb{R})\),
• \(\psi^c/\psi\) is the restriction of the central character of \(\Pi_F\) to \(\mathbb{A}_E^\times\),
• and if \(x\) is a rational prime which splits \(yy'\) in \(E\) then \(\pi_x = (\otimes_{z|y} \text{JL}^{-1}(\Pi_z)) \otimes \psi_y\)
as a representation of \((B_y^p)^\times \times \mathbb{Q}_x^\times \cong (\otimes_{z|y} (B_z^p)^\times) \times \mathbb{Q}_x^\times\).

Here \(\text{JL}\) denotes the identity if \(B_z\) is split and denotes the Jacquet-Langlands correspondence if \(B_z\) is a division algebra. (See section I.3 of [HT].)

We will call two irreducible admissible representations \(\pi'\) and \(\pi''\) of \(G(\mathbb{A}^\infty)\) nearly equivalent if \(\pi'_x \cong \pi''_x\) for all but finitely many rational primes \(x\). If \(M\) is an admissible \(G(\mathbb{A}^\infty)\)-module and \(\pi'\) is an irreducible admissible representation of \(G(\mathbb{A}^\infty)\) then we define the \(\pi'\)-near isotypic component \(M[\pi']\) of \(M\) to be the largest \(G(\mathbb{A}^\infty)\)-submodule of \(M\) all whose irreducible subquotients are nearly equivalent to \(\pi'\). Then

\[M = \bigoplus M[\pi']\]
as \(\pi'\) runs over near equivalence classes of irreducible admissible \(G(\mathbb{A}^\infty)\)-modules. (This follows from the following fact. Suppose that \(A\) is a (commutative) polynomial algebra over \(\mathbb{C}\) in countably many variables, and that \(M\) is an \(A\)-module which is finitely generated over
Then we can write
\[ M = \bigoplus_m M_m, \]
where \( m \) runs over maximal ideals of \( A \) with residue field \( C \).

We consider the Shimura varieties \( X_U/F \) for open compact subgroups \( U \) of \( G(\A) \) as in the last section. Then
\[ H^i(X, \mathcal{L}_\xi) = \lim_{\rightarrow U} H^i(X_U \times_F F^{ac}, \mathcal{L}_\xi) \]
is a semisimple, admissible \( G(\A) \)-module with a commuting continuous action of the Galois group \( \text{Gal}(F^{ac}/F) \). (For details see III.2 of \([HT]\).)

The following lemma follows from \([HT]\), particularly corollary VI.2.3, corollary VI.2.7 and theorem VII.1.7.

**Lemma 3.1.** Keep the notation and assumptions above. (In particular we are assuming that \( \pi \) arises from a cuspidal automorphic representation \( \Pi \) of \( GL_n(\A_F) \).)

1. If \( i \neq n-1 \) then \( H^i(X, \mathcal{L}_\xi)[\pi] = (0) \).
2. As \( G(\A) \times \text{Gal}(F^{ac}/F) \)-modules,
\[ H^{n-1}(X, \mathcal{L}_\xi)[\pi] = \bigoplus_{\pi'} \pi' \otimes R'_i(\Pi)^{m(\pi')} \otimes R_i(\psi), \]
where \( \pi' \) runs over irreducible admissible representations of \( G(\A) \) nearly equivalent to \( \pi \) and where \( m(\pi') \in \mathbb{Z}_{\geq 0} \), and \( R_i(\Pi) = R'_i(\Pi)^{ss} \).
3. \( m(\pi) > 0 \).
4. If \( m(\pi') > 0 \) then \( \pi'_p \cong \pi_p \).

If \( \pi' \) is an irreducible admissible representation of \( G(\A) \) we can decompose it as \( (\pi')^p \otimes \left( \prod_{i=2}^r \pi'_i \right)^{m(\pi')} \otimes \pi'_p, \) corresponding to the decomposition (1). If \( \pi'' \) is an irreducible admissible representation of \( G(\A^{\infty}) \) and \( N \) is an admissible \( G(\A^{\infty}) \)-module then we can define the \( \pi'' \)-near isotypic component of \( N \) in the same manner as we did for \( G(\A) \)-modules. If \( M \) is an admissible \( G(\A) \)-module and \( \pi' \) is an irreducible admissible representation of \( G(\A) \) then
\[ M^{1w(m)}[(\pi')^p] = M[\pi]^{1w(m)} \]
We will write
\[ H^i(X_{1w(m)}, \mathcal{L}_\xi) = \lim_{\rightarrow U^p} H^i(X_{U^p \times 1w(m)} \times F^{ac}, \mathcal{L}_\xi) \cong H^i(X, \mathcal{L}_\xi)^{1w(m)}. \]
It is a semisimple admissible \( G(\A^{\infty}) \)-module with a commuting continuous action of \( \text{Gal}(F^{ac}/F) \).

**Theorem 3.2.** Keep the above notation and assumptions. (In particular we are assuming that \( \pi \) arises from a cuspidal automorphic representation \( \Pi \) of \( GL_n(\A_F) \).) Let \( U^p \) be a
sufficiently small open compact subgroup of $G(k_{\infty,p})$. Then
\[ \text{WD}(H^{n-1}(X_{Iw(m)}, \mathcal{L}_\xi)[\pi^p]^{U_p}) \]
is pure.

Proof: As $X_U = X_{U_P \times Iw(m)}$ is strictly semistable by proposition 2.4, we can use the Rapoport-Zink weight spectral sequence [RZ] to compute $H^{n-1}(X_{Iw(m)}, \mathcal{L}_\xi)$. For $X_U$, it reads
\[ E_1^{i,j}(U) = \bigoplus_{t \geq \max(0,-i)} \bigoplus_{\#S = i+2t+1} H^{j-2t}(Y_{U,S} \times_k k)^{ac}, \mathcal{L}_\xi(-t)) \Rightarrow H^{i+j}(X_U \times_F F_w^{ac}, \mathcal{L}_\xi). \]

Passing to the limit with respect to $U^p$, it gives rise to the following spectral sequence of admissible $G(k_{\infty,p}) \times \text{Frob}_w^Z$-modules
\[ E_1^{i,j}(Iw(m)) = \bigoplus_{t \geq \max(0,-i)} \bigoplus_{\#S = i+2t+1} H^{j-2t}(Y_{Iw(m),S}, \mathcal{L}_\xi(-t)) \Rightarrow H^{i+j}(X_{Iw(m)}, \mathcal{L}_\xi). \]

Hence we get a spectral sequence of $\text{Frob}_w^Z$-modules
\[ (2) \quad E_1^{i,j}(Iw(m))[\pi^p]^{U_p} \Rightarrow H^{i+j}(X_{Iw(m)}, \mathcal{L}_\xi)[\pi^p]^{U_p}. \]
The sheaf $\mathcal{L}_\xi$ is pure, say of weight $w_\xi$. Thus the action of $\text{Frob}_w$ on $E_1^{i,j}$ is pure of weight $w_\xi + j$ by the Weil conjectures. The theory of weight spectral sequence ([RZ]) defines an operator
\[ N : E_1^{i,j}(Iw(m))[\pi^p]^{U_p}(1) \to E_1^{i+2,j-2}(Iw(m))[\pi^p]^{U_p}, \]
which induces the $N$ for $\text{WD}(H^{i+j}(X_{Iw(m)}, \mathcal{L}_\xi)[\pi^p]^{U_p})$ and has the property that
\[ N^i : E_1^{i,j+i}(Iw(m))[\pi^p]^{U_p}(i) \cong E_1^{i,j-i}(Iw(m))[\pi^p]^{U_p} \]
for all $i$. If the spectral sequence (2) degenerates at $E_1$, then it follows that the Weil-Deligne representation $\text{WD}(H^{n-1}(X_{Iw(m)}, \mathcal{L}_\xi)[\pi^p]^{U_p})$ is pure of weight $w_\xi + (n-1)$. Thus it suffices to show that
\[ E_1^{i,j}(Iw(m))[\pi^p]^{U_p} = (0) \]
if $i + j \neq n - 1$, i.e. that
\[ H^j(Y_{Iw(m),S}, \mathcal{L}_\xi)[\pi^p]^{U_p} = (0) \]
if $j \neq n - \#S$.

We first recall some notation from [HT]. For $h = 0, ..., n-1$ let $P_h$ denote the maximal parabolic in $GL_n$ consisting of matrices $g \in GL_n$ with $g_{ij} = 0$ for $i > n - h$ and $j \leq n - h$. Also let $N_h$ denote the unipotent radical of $P_h$, let $P_h^{op}$ denote the opposite parabolic and let $N_h^{op}$ denote the unipotent radical of $P_h^{op}$. Let $D_{F_{w,n-h}}$ denote the division algebra with centre $F_w$ and Hasse invariant $1/(n-h)$. If $\pi'$ is a square integrable representation of $GL_{n-h}(F_w)$, let $\varphi_{\pi'}$ denote a pseudo-coefficient for $\pi'$ as in section I.3 of [HT]. (Note that this depends on the choice of a Haar measure, but in the formulae below this choice will always be cancelled by the choice of an associated Haar measure on $D_{F_{w,n-h}}^X$. See [HT] for details.)
If we introduce the limit of cohomology groups of Igusa varieties for varying level structure at $p$ as in (see p.136 of [HT]):

$$[H_c(I^{(h)}, \mathcal{L}_\xi)] = \sum_i (-1)^{h-i} \lim_{U_p, m} H^i_c(I^{(h)}_{U_p, m} \times_{k(w)} k(w)^{ac}, \mathcal{L}_\xi),$$

then the second isomorphism of corollary 2.6 and theorem V.5.4 of [HT] tell us that

$$n[H_c(I^{(h)}_{Iw(m)}, \mathcal{L}_\xi)] = n[H_c(I^{(h)}, \mathcal{L}_\xi)]_{U_p(m) \times Iw,h,w} = \sum_i (-1)^{n-1-i} \text{Red}^{(h)} [H^i(X, \mathcal{L}_\xi)_{U_p(m)}]$$

in $\text{Groth}(G(A^{\infty,p}) \times \text{Frob}_w^Z)$, where

$$\text{Red}^{(h)} : \text{Groth}(GL_n(F_w) \times \mathbb{Q}_p^X) \longrightarrow \text{Groth} (\text{Frob}_w^Z)$$

is the composite of the normalised Jacquet functor

$$J_{N,w} : \text{Groth}(GL_n(F_w) \times \mathbb{Q}_p^X) \longrightarrow \text{Groth}(GL_{n-h}(F_w) \times GL_h(F_w) \times \mathbb{Q}_p^X)$$

with the functor

$$\text{Groth}(GL_{n-h}(F_w) \times GL_h(F_w) \times \mathbb{Q}_p^X) \longrightarrow \text{Groth}(\text{Frob}_w^Z)$$

which sends $[\alpha \otimes \beta \otimes \gamma]$ to

$$\sum_\phi \text{vol}(D_{F_w,n-h}^X/F_w^X)^{-1} \text{tr}(\varphi_{Sp_{n-h}(\phi)})(\dim \beta_{Iw,h,w}) \left[ \text{rec}\left(\phi^{-1} \mid \frac{1}{p^n}(\gamma_{Z_p} \circ \mathbb{N}_{F_w/E_w})^{-1}\right)\right],$$

where the sum is over characters $\phi$ of $F_w^X/O_{F_w}^X$. (We just took the $Iw_{h,w}$-invariant part of the $\text{Red}^{(h)}_1$, which is defined on p.182 of [HT]. Note that $\text{Frob}_w$ acts on $H_c(I^{(h)}, \mathcal{L}_\xi)$ as

$$(1, p^{-[k(w):F_p]}, -1, 1, 1) \in G(A^{\infty,p}) \times (\mathbb{Q}_p^X/O_{F_w}^X) \times \mathbb{Z} \times GL_h(F_w) \times \prod_{i=2}^r (B_{F_i}^{w,m})^X,)$$

where we have identified $D_{F_w,n-h}/O_{F_w}^Z$ with $Z$ via $w(\text{det})$.)

In particular, by lemma 3.1(1), we have an equality in Groth($\text{Frob}_w^Z$):

$$n[H_c(I^{(h)}_{Iw(m)}, \mathcal{L}_\xi)]_{[\pi]\text{Rec}^{U_p}]} = n[H_c(I^{(h)}, \mathcal{L}_\xi)]_{[\pi]^{U_p}]} = \text{Red}^{(h)} [H^{n-1}(X, \mathcal{L}_\xi)_{U_p(m)}[\pi]^{U_p}] = \text{Red}^{(h)} [\text{Rec}^{U_p}]_{[\pi]^{U_p}}]$$

Moreover $H^{n-1}(X, \mathcal{L}_\xi)_{U_p(m)}[\pi]^{U_p}$ is $\pi_w \otimes \pi_{p,0}$-isotypic as a $GL_n(F_w) \times \mathbb{Q}_p^X$-module by lemma 3.1(4). As $\pi_w = \Pi_{F,w}$ has an Iwahori fixed vector and $\pi_{p,0} = \psi_u$ is unramified,

$$(\dim \Pi_{F,w}^{Iw,m}) [H^{n-1}(X, \mathcal{L}_\xi)_{U_p(m)}[\pi]^{U_p}] = (\dim H^{n-1}(X, \mathcal{L}_\xi)^{Iw(m)}[\pi]^{U_p}) [\Pi_{F,w} \otimes \psi_u],$$

and

$$n(\dim \Pi_{F,w}^{Iw,m}) [H_c(I^{(h)}_{Iw(m)}, \mathcal{L}_\xi)]_{[\pi]^{U_p}]} = (\dim H^{n-1}(X, \mathcal{L}_\xi)^{Iw(m)}[\pi]^{U_p}) \text{Red}^{(h)} [\Pi_{F,w} \otimes \psi_u].$$
Combining this with lemma 2.7, we get

\[ n(\dim \Pi_{F,w}^{1_w,w}) [H(Y_{1w(m)}, S, \mathcal{L}_\xi)|\pi^p]_{U^p} \]

\[ = (\dim H^{n-1}(X, \mathcal{L}_\xi)^{1w(m)}[\pi^p]_{U^p}) \sum_{h=0}^{n-\#S} (-1)^{n-\#S-h} \binom{n-\#S}{h} \text{Red}(h) [\Pi_{F,w} \otimes \psi_u]. \]

As \( \Pi_{F,w} \) is tempered, it is a full normalised induction of the form

\[ \text{n-Ind}_{P(F_w)}^{GL_n(F_w)}(\text{Sp}_{s_1}(\pi_1) \otimes \cdots \otimes \text{Sp}_{s_\ell}(\pi_\ell)), \]

where \( \pi_i \) is an irreducible cuspidal representation of \( GL_{g_i}(F_w) \) and \( P \) is a parabolic subgroup of \( GL_n \) with Levi component \( GL_{s_1 g_1} \times \cdots \times GL_{s_\ell g_\ell} \). As \( \Pi_{F,w} \) has an Iwahori fixed vector, we must have \( g_i = 1 \) and \( \pi_i \) unramified for all \( i \). Note that, for this type of representation (full induced from square integrable \( \text{Sp}_{s_i}(\pi_i) \) with \( \pi_i \) an unramified character of \( F_w^{x_1} \)),

\[ \dim(n-\text{Ind}_{P(F_w)}^{GL_n(F_w)}(\text{Sp}_{s_1}(\pi_1) \otimes \cdots \otimes \text{Sp}_{s_\ell}(\pi_\ell)))^{1w_{n,w}} = \#P(k(w))|GL_n(k(w))/B_n(k(w)) = \frac{n!}{\prod_j s_j!}. \]

We can compute \( \text{Red}(h)[\Pi_{F,w} \otimes \psi_u] \) using lemma I.3.9 of [HT] (but note the typo there — “positive integers \( h_1, \ldots, h_t \)” should read “non-negative integers \( h_1, \ldots, h_t \)”). Putting \( V_i = \text{rec}(\pi_i^{-1}|_{\psi_u \circ N_{F_w/E_w}^{-1}}) \), we see that

\[ \text{Red}(h)[\Pi_{F,w} \otimes \psi_u] = \sum_i \dim(n-\text{Ind}_{P'(F_w)}^{GL_{s_1 h+n}(\pi_i)} |^{n-h}_{j \neq i} \bigotimes_{j \neq i} \text{Sp}_{s_j}(\pi_j))^{1w_{h,w}} [V_i] \]

\[ = \sum_i \frac{h!}{(s_i + h - n)! \prod_{j \neq i} s_j!} [V_i] \]

where the sum runs only over those \( i \) for which \( s_i \geq n - h \), and \( P' \subset GL_h \) is a parabolic subgroup. Thus

\[ n \frac{n!}{\prod_j s_j!} [H(Y_{1w(m)}, S, \mathcal{L}_\xi)|\pi^p]_{U^p} \]

\[ = D \sum_{h=0}^{n-\#S} (-1)^{n-\#S-h} \binom{n-\#S}{h} \sum_{i: s_i \geq n - h} \frac{h!}{(s_i + h - n)! \prod_{j \neq i} s_j!} [V_i] \]

\[ = D \sum_{i=1}^{t} \frac{(n - #S)!}{(s_i - #S)! \prod_{j \neq i} s_j!} \sum_{h=n-s_i}^{n-\#S} (-1)^{n-\#S-h} \binom{s_i - #S}{h + s_i - n} [V_i] \]

\[ = D \sum_{s_i = #S}^{n} \frac{(n - #S)!}{\prod_{j \neq i} s_j!} [V_i], \]

where \( D = \dim H^{n-1}(X, \mathcal{L}_\xi)^{1w(m)}[\pi^p]_{U^p} \), and so

\[ n^{\#S} [H(Y_{1w(m)}, S, \mathcal{L}_\xi)|\pi^p]_{U^p} = (\dim H^{n-1}(X, \mathcal{L}_\xi)^{1w(m)}[\pi^p]_{U^p}) \sum_{s_i = #S} [V_i]. \]
As $\Pi_{F,w}$ is tempered, $\text{rec}(\Pi_{F,w}^\vee \otimes (\psi_u^\vee \circ N_{F_w/E_w})) | \det^{1-n} \in \mathbb{Z}$ is pure of weight $w_\xi + (n - 1)$.

Hence

$$V_i = \text{rec}(\pi_i^{-1} | \det^{1-\#S} (\psi_u \circ N_{F_w/E_w})^{-1} | \det^{\#S-n} | \det^{1-n} )$$

is strictly pure of weight $w_\xi + (n - \#S)$. The Weil conjectures then tell us that

$$H^j(Y_{Iw(m),S}, L_{\xi})[\pi_p]^U_P = (0)$$

for $j \neq n - \#S$. The theorem follows. $\square$

We can now conclude the proof of theorem 1.1. Choose $k$ so that $|\chi_\Pi| = |\chi_{\Pi}^{n+1}(L)|^2$ where $\chi_\Pi$ is the central character of $\Pi$. Set

$$V = H^{n-1}(X_{Iw(m), L_{\xi}}, L_{\xi})[\pi_p]^U_P \otimes R_l(\psi)^{-1},$$

a continuous representation of $\text{Gal}(F^{ac}/F)$. We know that

1. $V^{ss} \simeq R_l(\Pi)^{a}_{\text{Gal}(F^{ac}/F)}$ for some $a \in \mathbb{Z}_{>0}$,
2. $V$ is pure of weight $k$ (proposition III.2.1 of [HT] and a computation of the determinant),
3. $\text{WD}(V|_{\text{Gal}(F^{ac}/F)_{w}})$ is pure of weight $k$ (use theorem 3.2 and a computation of the determinant).

Thus lemma 1.4 tells us that $\text{WD}(R_l(\Pi)|_{\text{Gal}(L_0^{ac}/L_0)})^{F_{ss}}$ is pure. On the other hand, as $\Pi_v$ is tempered (corollary VII.1.11 of [HT]), $\text{rec}(\Pi_v^\vee | \det^{1-n} )$ is pure by lemma 1.3(3). As the representation of the Weil group in $\text{rec}(\Pi_v^\vee | \det^{1-n} )$ and $\text{WD}(R_l(\Pi)|_{\text{Gal}(L_0^{ac}/L_0)})^{F_{ss}}$ are equivalent, we deduce from lemma 1.3(4) that

$$\text{WD}(R_l(\Pi)|_{\text{Gal}(L_0^{ac}/L_0)})^{F_{ss}} \simeq \text{rec}(\Pi_v^\vee | \det^{1-n} ),$$

as desired.

References


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