On the Equations Defining Abelian Varieties. III *

D. Mumford (Cambridge, Mss.)

Contents
§ 10. Non-Degenerate Theta Functions ................................................. 215
§ 11. Satake’s Compactification ......................................................... 228
§ 12. Analytic Theta Functions ......................................................... 236

§ 10. Non-Degenerate Theta Functions

The third part of this paper is devoted (1) to a complete description of the boundary of the moduli space for abelian varieties described in § 9, and (2) to connecting our theory with the classical theory of theta functions. We begin by defining a theta function in a coordinate-free manner and investigating how and under what non-degeneracy restrictions we can construct a tower of abelian varieties having this as its theta function. Our goal is to find an inverse to the moduli map $\Theta$ described in § 9. Fix

i) a 2g-dimensional vector space $V$ over $\mathbb{Q}_2$;
ii) a skew-symmetric bi-multiplicative map:

\[ e: V \times V \rightarrow \{2^n\text{-th roots of 1 in } k\}, \]

\[ e(\alpha, \beta) = e(\alpha, \gamma) \cdot e(\beta, \gamma) \]
\[ e(\alpha, \beta \cdot \gamma) = e(\alpha, \beta) \cdot e(\alpha, \gamma); \]

iii) a maximal isotropic lattice $\Lambda \subset V$ (i.e., a compact, open subgroup such that $e(\alpha, \beta) = 1$, all $\alpha, \beta \in \Lambda$, maximal with this property);
iv) a quadratic character

\[ e_*: \frac{1}{2} \Lambda/\Lambda \rightarrow \{\pm 1\} \]

such that

\[ e_*(\alpha + \beta) e_*(\alpha) e_*(\beta) = e(\alpha, \beta)^2, \]

all $\alpha, \beta \in \frac{1}{2} \Lambda$.

* Part I of this paper has been published in Inventiones math. Vol. 1, pp. 287—354 and part II in Vol. 3, pp. 75—135.
We assume, however, that via a suitable isomorphism $V \cong Q^2$, $A \cong Z^2$, and $e, e_\kappa$ have the form defined in § 9. In fact, this is nearly always the case: if we write

$$e_\kappa(x) = (-1)^{\kappa(x)}$$

where $Q$ is a quadratic form on $\frac{1}{2} A$ with values in the field $F_2 = \{0, 1\}$, then $Q$ has an Art invariant $A(Q) \in F_2$. It is not hard to show that $(V, A, e, e_\kappa)$ has the required form only if $A(Q) = 0$. We leave this point to the reader.

**Definition 1.** A theta-function $\Theta$ on $V$ is a map $\Theta : V \to k$ satisfying

i) $\Theta(x + y) = e(x)(y) \cdot e(y) \Theta(x)$,

ii) $\Theta(-x) = \Theta(x)$, all $x \in V$,

iii) $\prod_{i=1}^{4} \Theta(x_\gamma) = 2^{-\kappa} \sum_{x \in A} e(x) \cdot \prod_{i=1}^{4} \Theta(x) + \gamma + \eta)

if $\gamma = -\frac{1}{2} \sum x_i$, $x_1, \ldots, x_4 \in V$ arbitrary.

If we let

$$S_0 = \{x \mid \Theta(x) + 0\} = \text{support}(\Theta),$$

then $S_0$ is a union of cosets of $A$. The structure of $S_0$ is a "fine" property of $\Theta$, so we introduce:

**Definition 2.** The coarse support $S_1$ of $\Theta$ is:

$$S_1 = \{x \mid \Theta(x) + 0, \text{for some } \eta \in \frac{1}{2} A\}.$$

We will see in § 11 that the coarse support $S_1$ of a theta function is either all of $V$, or $\frac{1}{2} A + W$ where $W \subset V$ is a proper subvector space. This is the essential difference between good and bad theta functions.

Note that $S_0 = -S_0$ and $S_1 = -S_1$. We always assume, in what follows, that $\Theta \neq 0$, i.e., $S_0 \neq \emptyset$.

1. If $x_1 \notin S_1$, $x_2$, $x_3$, $x_4 \in S_0$, then $2x_1 + x_2 + x_3 + x_4 \notin S_0$.

**Proof.** Use the quartic relation on $\Theta$, with $x_1 = 2x_1 + x_2 + x_3 + x_4$, $x_2 = x_2$, $x_3 = x_3$, $x_4 = x_4$, $y = -x_1 - x_2 - x_3 - x_4$. Q.E.D.

2. $0 \in S_1$.

**Proof.** Assume $0 \notin S_1$. Take any $y \in S_0$. Apply (1) with $x_2 = x_3 = y$, $x_4 = -y$ and we get a contradiction. Q.E.D.

3. $x, y \in S_0 \Rightarrow \frac{1}{2}(x + y) \in S_1$.

**Proof.** Apply (1) with $x_1 = \frac{1}{2}(x + y)$, $x_2 = x$, $x_3 = -y$ and $x_4 = -x$. Q.E.D.
Because of (2.), there is an \( \eta_0 \in \frac{1}{4} A \) such that \( \Theta(\eta_0) \neq 0 \). Fix one such \( \eta_0 \).

4. \((0) \subseteq (S_0 + \eta_0) \subseteq (2S_0 + A) \subseteq (4S_0 + A) \subseteq \cdots \).

**Proof.** By (3), if \( x \in S_0 \), then \( \frac{1}{2}(x + \eta_0) \in S_1 \), so \( x + \eta_0 \in 2S_0 + A \). This gives the 1st inclusion. This also shows that \( 2x \in 4S_0 + A \). Hence if \( y \in 2^l S_0 \), so \( y = 2^k \cdot x \), \( x \in S_0 \), then \( 2^k \cdot x \in 2^{k+1} S_0 + A \). This gives the rest of the inclusions. \( Q.E.D. \)

**Definition 3.**

\[ S_\infty = \bigcup_{k \geq 1} [2^k S_0 + A] . \]

5. \( S_\infty \) is a group.

**Proof.** Let \( x, y \in S_\infty \). Now \( x, y \in (2^l \cdot S_0 + A) \) for some \( l \geq l_0 \). Then \( x = 2^l \cdot x_0 + \eta, y = 2^l \cdot y_0 + \zeta, x_0, y_0 \in S_0 \) and \( \eta, \zeta \in A \). Therefore by (3), \( \frac{1}{2}(x_0 + y_0) \in S_1 \), hence \( 2^l(x_0 + y_0) \in 2^{l+1} \cdot S_0 + A \). Therefore \( x + y \in (2^{l+1} S_0 + A) \subseteq S_\infty \). \( Q.E.D. \)

6. \( S_\infty = W + A \), for some subvector space \( W \subseteq V \).

**Proof.** This is easily seen to be equivalent to asserting that \( S_\infty / A \) is a divisible subgroup of \( V / A \). But if \( x \in 2^k \cdot S_0 + A \), then \( x = 2^k \cdot x_0 + \eta \), \( x_0 \in S_0 \), \( \eta \in A \), hence \( x - \eta \in 2^{k-1} S_0 \subseteq 2 \cdot S_\infty \), i.e., the image of \( x \) in \( S_\infty / A \) is divisible by \( 2 \). \( Q.E.D. \)

**Definition 4.** A theta function is **non-degenerate** if equivalently:

(a) \( S_\infty = V \).

(a') \( S_\infty = \frac{1}{2} A \).

(a'') For all sufficiently large \( n \), \( 2^n \cdot S_0 + A \supset \frac{1}{2} A \).

(a'') For all sufficiently large \( n \), and \( x \in 2^{-n} A \), there is an \( \eta \in 2^{-n} A \) such that \( \Theta(x + \eta) \neq 0 \).

The next step is to form, via the function \( \Theta \), a sequence of graded rings:

**Definition 5.** If \( M \) is a vector space of \( k \)-valued functions on \( V \), let

\[ \mathcal{S}(M) = \bigoplus_{n=0}^{\infty} \mathcal{S}_n(M) , \]

where \( \mathcal{S}_0(M) = k, \mathcal{S}_1(M) = M, \) and \( \mathcal{S}_n(M), \) for \( n \geq 2 \), is the vector space of functions on \( V \) spanned by the products \( f_{i_1} \cdots f_{i_n}, \) \( (f_{i_j} \in M, \) all \( j \)).

Another convenient notation is the following:

\[ M^{*} = \left\{ \text{set of functions } x \mapsto f(\alpha/2), \right\} . \]

In particular, let

\[ M_{2^k} = \text{span of the functions } \Theta_{(\beta)}, \quad \text{all } \beta \in 2^{-k} A \]
where

\[ \Theta_{(\beta)}(x) = e(\beta/2, x) \cdot \Theta(x - \beta). \]

The corresponding rings \( \mathcal{S}(M_{2k}) \) will be the heart of our analysis. These are only half of the rings we need, however. To define the others, choose a decomposition:

\[ A = A_1 \oplus A_2 \]

such that \( Q_2 \cdot A_1 = V_1 \) is an isotropic subspace under \( e \), and such that \( e_\ast(\alpha/2) = 1 \) for all \( \alpha \in A_1 \) or \( A_2 \). This exists because if we choose coordinates \( V \cong Q^2 \times \) such that \( A_1, e, e_\ast \) take their standard forms, then \( A_1 = Z_2^2 \times \{0\}, A_2 = \{0\} \times Z_2^2 \) have these properties. In terms of \( A_1 \) and \( A_2 \), we now define a kind of "dual" theta-function \( \phi \). It is to satisfy the equations:

\[ \sum_{\zeta \in \frac{1}{4} A_1/A_1} e(\alpha, \zeta) \cdot \Theta(\alpha + \beta + \zeta) \cdot \Theta(\alpha - \beta + \zeta) = \phi(\alpha) \cdot \phi(\beta) \]

all \( \alpha, \beta \in V \). In fact, if we let \( \Phi(\alpha, \beta) \) denote the left-hand side of this equation, then the quartic equations on \( \Theta \) are equivalent to:

\[ \Phi(\alpha, \beta) \cdot \Phi(\gamma, \delta) = \Phi(\alpha, \delta) \cdot \Phi(\gamma, \beta) \]

for all \( \alpha, \beta, \gamma, \delta \in V \) (cf. proof of Lemma 2, § 8). This, plus the elementary fact \( \Phi(\alpha, \beta) = \Phi(\beta, \alpha) \) implies that one and (up to scalars) only one such \( \phi \) exists. Notice that \( \phi \) satisfies the equations:

(i) \( \phi(\alpha + \beta) = f_\ast(\beta) \cdot e(\beta, \alpha) \cdot \phi(\alpha) \), for all \( \alpha \in V, \beta \in \frac{1}{4} A_1 + A_2 \), if \( f_\ast(\beta_1 + \beta_2) = e(\frac{1}{4} \beta_1, \beta_2) (\beta_i \in A_i) \).

(ii) \( \phi(-\alpha) = \phi(\alpha) \), all \( \alpha \in V \),

as well as certain quartic equations. Now let

\[ M_{2k+1} = \text{span of the functions } \phi_{(\beta)}, \quad \beta \in 2^{-k-1} \cdot A \]

where

\[ \phi_{(\beta)}(x) = e(\beta, x) \cdot \phi(x - \beta). \]

**Proposition 1.** 1. \( \mathcal{S}_2(M_{2k}) \subseteq M_{2k+1} \), equality holding if and only if

for all \( \beta \in 2^{-k-1} A, \exists \gamma \in 2^{-k} A \) such that \( \phi(\beta + \gamma) = 0 \).

2. \( \mathcal{S}_2(M_{2k+1}) \subseteq M_{2k+2} \), equality holding if and only if for all \( \beta \in 2^{-k-1} A, \exists \gamma \in 2^{-k} A \) such that \( \Theta(\beta + \gamma) = 0 \).

**Proof.** To compute \( \mathcal{S}_2(M_{2k}) \), note that it is spanned by the functions:

\[ f(\alpha) = \sum_{\eta \in \frac{1}{4} A_1/A_1} e \left( \eta, \frac{\beta_1 + \beta_2}{2} \right) \cdot \Theta_{(\beta_1 - \eta)}(\alpha) \cdot \Theta_{(\beta_2 - \eta)}(\alpha) \]
where $\beta, e^{2^{-k}A}$. But
\[
f(\alpha) = e\left(\frac{\beta_1 + \beta_2}{2}, \alpha\right) \cdot \sum_{\eta \in \frac{\Lambda}{\Lambda_1}} e\left(\frac{\alpha - \beta_1 + \beta_2}{2}, \eta\right) \times
\Theta(\alpha - \beta_1 + \eta) \Theta(\alpha - \beta_2 + \eta)
\]
\[
= e\left(\frac{\beta_1 + \beta_2}{2}, \alpha\right) \cdot \phi\left(\frac{\alpha - \beta_1 + \beta_2}{2}\right) \cdot \phi\left(\frac{\beta_1 - \beta_2}{2}\right)
\]
\[
= \phi\left(\frac{\beta_1 + \beta_2}{2}\right) \cdot \phi\left(\frac{\beta_1 - \beta_2}{2}\right) \in M_{2k+1}.
\]

We get every $\phi_{\gamma^2}, \gamma \in 2^{-k-1}A$, in this way, if and only if every such $\gamma$ can be written:
\[
\gamma = \frac{\beta_1 + \beta_2}{2}, \quad \beta, e^{2^{-k}A}
\]

such that
\[
\phi\left(\frac{\beta_1 - \beta_2}{2}\right) \neq 0.
\]

This is exactly the condition in (1). To prove (2), first notice the identity:
\[
\sum_{ \zeta \in \frac{\Lambda}{\Lambda_2} } e(\alpha, \zeta) \cdot \phi(\alpha + \beta + \zeta) \cdot \phi(\alpha - \beta + \zeta)
\]
\[
= \sum_{ \zeta \in \frac{\Lambda}{\Lambda_2} } e(\alpha, \zeta) \cdot e(\alpha + \beta + \zeta, \eta) \cdot \Theta(2\alpha + 2\zeta + \eta) \cdot \Theta(2\beta + \eta)
\]
\[
= \sum_{ \zeta \in \frac{\Lambda}{\Lambda_2} } \Theta(2\alpha + \eta) \cdot \Theta(2\beta + \eta) \cdot e(\alpha + \beta, \eta) \cdot \left[ \sum_{ \zeta \in \frac{\Lambda}{\Lambda_2} } e(2\zeta, \eta) \right]
\]
\[
= 2^k \cdot \Theta(2\alpha) \cdot \Theta(2\beta).
\]

Now $\mathcal{S}_2(M_{2k+1})^*$ is spanned by the various functions:
\[
f(\alpha) = \sum_{\eta \in \frac{\Lambda}{\Lambda_2}} e(\eta, \beta_1 + \beta_2) \cdot \phi_{[\beta_1 - \eta]}(\alpha/2) \cdot \phi_{[\beta_2 - \eta]}(\alpha/2)
\]
where $\beta, e^{2^{-k-1}A}$. But this $f$ comes out as:
\[
f(\alpha) = 2^k \cdot \Theta_{[\beta_1 + \beta_2]}(\alpha) \cdot \Theta(\beta_1 - \beta_2) \in M_{2k+2}.
\]

(2) now follows just like (1). \textit{Q.E.D.}

\textbf{Corollary.} If $\Theta$ is non-degenerate, then for all $k \gg 0$,
\[
\mathcal{S}_2(M_{2k}) = M_{2k+1}
\]
\[
\mathcal{S}_2(M_{2k+1})^* = M_{2k+2}.
\]
Proof. The 2nd equality is clear, by condition (a'''') of the definition of non-degenerate. As for the first, note that by formula (a) in the proof of the Proposition,

\[ 2^2 \Theta(x)^2 = \sum_{\zeta \in A_2/A_2} e(x, \zeta) \cdot \phi(x + \zeta) \cdot \phi(\zeta). \]

Therefore, \( [\Theta(x) \neq 0] \Rightarrow [\phi(x + \zeta) \neq 0, \text{ some } \zeta \in A_2/A_2] \). Thus the non-degeneracy of \( \Theta \) implies the same for \( \phi \), and the 1st equality follows too. Q.E.D.

In the following discussion, we shall assume that \( \Theta \) is non-degenerate. As usual, if \( R = \Sigma R_n \) is a graded ring, then \( R(2) \) is the graded ring \( \Sigma R_{2n} \). The Corollary shows that there exists a \( k_0 \) such that for all \( k \geq k_0 \),

\[ \mathcal{S}(M_k)(2) \cong \mathcal{S}(M_{k+1}). \]

In particular, the corresponding schemes

\[ X = \text{Proj}(\mathcal{S}(M_k)), \]

for \( k \geq k_0 \), are all canonically isomorphic. We shall prove eventually that this \( X \) is an abelian variety.

So far, we know that \( \mathcal{S}(M_k) \) is finitely generated over \( k \). Moreover, it has no nilpotents: if it did, it would have a homogeneous nilpotent element \( f \in \mathcal{S}(M_k) \). Then \( f \neq 0 \Rightarrow f(x) \neq 0, \text{ some } x \in V \Rightarrow f^N(x) \neq 0, \text{ all } N \Rightarrow f^N \neq 0 \) in \( \mathcal{S}(M_k) \). Therefore, \( X \) is a reduced algebraic scheme over \( k \). In fact, we can map

\[ V/\Lambda \rightarrow X \]

by evaluating functions in \( \mathcal{S}(M_k) \) at points of \( V \). To be more precise, for all \( x \in V \), define a homogeneous prime ideal \( P(x) \in \mathcal{S}(M_{2k}) \) [resp. \( P(x) \in \mathcal{S}(M_{2k+1}) \)] by:

\[ P(x) = \sum_{n} P_n(x) \]

\[ P_n(x) = \{ f \in S_n(M_{2k}) \mid f(2^n x) = 0 \} \]

resp.

\[ = \{ f \in S_n(M_{2k+1}) \mid f(2^n x) = 0 \}. \]

It is easy to check that for all \( k \), if the \( P(x) \) in \( \mathcal{S}(M_k) \) is intersected with \( \mathcal{S}(M_k)(2) \), the resulting ideal is equal to the \( P(x) \) in \( \mathcal{S}(M_{k+1}) \) under the isomorphisms (b). For this reason, we omit a \( k \) in the notation \( P(x) \). Thus \( P(x) \) gives a well-defined point \( [P(x)] \in X \). It follows easily from the definition that:

a) \( P(x) \) is a \( k \)-rational point of \( X \),

b) \( P(x + \beta) = P(x) \), if \( \beta \in \Lambda \).
Moreover:

\( c \) \[ \{ \bar{P}(x) | x \in V \} \] is dense in \( X \).

**Proof of \( c \):** Take \( 2k \geq k_0 \). If \( c \) were false, for large \( n \), there would be a non-zero function \( f \in \mathcal{S}_n(M_{2k}) \) that vanished at all \( \bar{P}(x) \)'s. But \( f(\bar{P}(x)) = 0 \iff f(2^{k} x) = 0 \), so \( f \) would vanish everywhere on \( V \), hence \( f = 0 \). \( Q.E.D. \)

One can do even more: for \( \alpha \in V \), I claim that there is an automorphism \( T_{\alpha} : X \to X \) such that \( T_{\alpha}(\bar{P}(\beta)) = \bar{P}(\alpha + \beta) \), all \( \beta \in V \). To construct \( T_{\alpha} \), let \( k_1 \) be the least integer such that \( 2^{k_1} \alpha \in A \). Define

\[ T_{\alpha}^*: \mathcal{S}(M_{2k}) \to \mathcal{S}(M_{2k}) \]

resp.

\[ \mathcal{S}(M_{2k+1}) \to \mathcal{S}(M_{2k+1}) \]

by:

\[ T_{\alpha}^* f(\beta) = e(\beta, 2^{k_1 - 1} \alpha) \cdot f(\beta + 2^{k_1} \alpha), \quad \text{all } f \in S_n(M_{2k}) \]

resp.

\[ = e(\beta, 2^{k_1} \alpha) \cdot f(\beta + 2^{k_1} \alpha), \quad \text{all } f \in S_n(M_{2k+1}) \]

(where we assume \( k \geq k_1 \)). To check that this is, indeed, an automorphism of \( \mathcal{S}(M_{2k}) \) (resp. \( \mathcal{S}(M_{2k+1}) \)), it suffices to check that \( T_{\alpha}^* \theta(\gamma) \in M_{2k} \), all \( \gamma \in 2^{-k} A \); and \( T_{\alpha}^* \phi(\gamma) \in M_{2k+1} \), all \( \gamma \in 2^{-k_1} A \). But, in fact, one computes:

\[ T_{\alpha}^* \theta(\gamma) = e_*(2^{k_1 - 1} \alpha) \cdot e(\gamma, 2^{k_1} \alpha) \cdot \theta(\gamma) \]

\[ T_{\alpha}^* \phi(\gamma) = f_*(2^{k_1} \alpha) \cdot e(\gamma, 2^{k_1 - 1} \alpha) \cdot \phi(\gamma) \]

Moreover, one finds that \( T_{\alpha}^* \), acting on \( \mathcal{S}(M_k) \), induces the same automorphism on \( \mathcal{S}(M_1) \) (2) that you get by considering the \( T_{\alpha}^* \) acting on \( \mathcal{S}(M_{k+1}) \) and carrying it across via the isomorphisms \( (\beta) \) of \( \mathcal{S}(M_2) \) (2) and \( \mathcal{S}(M_{k+1}) \). Therefore, the \( T_{\alpha}^* \)'s all define one and the same automorphism \( T_{\alpha} \) of \( X \). Note that:

\[ d) \quad (T_{\alpha}^*)^{-1} (P(\beta)) = P(\alpha + \beta). \]

**Proof.** If \( f \in \mathcal{S}_n(M_{2k}) \) or \( \mathcal{S}_n(M_{2k+1}) \), then

\[ T_{\alpha}^* f(\beta) \iff T_{\alpha}^* f(2^{k} \beta) = 0 \iff f(2^{k} \alpha + 2^{k} \beta) = 0 \iff f \in P(\alpha + \beta), \]

hence

\[ d') \quad T_{\alpha} (P(\beta)) = P(\alpha + \beta). \]

One checks also (via \( \gamma \)) if you like that:

\[ c) \quad T_{\alpha_1 + \alpha_2} = T_{\alpha_1} \circ T_{\alpha_2}, \]

\[ f) \quad T_{\alpha} = \text{id.} \iff \alpha \in A, \]

so that \( T \) is a faithful action of the group \( V/A \) on the scheme \( X \).

A remarkable consequence of all this is:

**Proposition 2.** If \( \Theta \) is non-degenerate, then \( \mathcal{S}(M_k) \) is an integral domain, for all \( k \).
Proof. We show first that $\mathcal{S}(M_4)$ is a domain if $k \geq k_0$. Since $\mathcal{S}(M_4)$ has no nilpotents, this is equivalent to showing that $X$ is irreducible. Now $V/A$ acts on $X$, so it permutes the various components of $X$, i.e., we have a homomorphism:

$$V/A \rightarrow S = \begin{cases} \text{gp. of permutations} \\ \text{of components of } X \end{cases}.$$ 

But $S$ is a finite group and $V/A$ is a divisible group. So $V/A$ must map each component $X_i$ into itself. On the other hand, the collection of points $P(x)$ forms a single orbit of the action of $V/A$ on $X$. Therefore, all these points $P(x)$ belong to a single component of $X$. Since they are also dense in $X$, $X$ can have only a single component. Therefore $\mathcal{S}(M_4)$ is a domain if $k \geq k_0$.

In general, suppose some $\mathcal{S}(M_l)$ were not a domain. Then there would be homogeneous elements $f \in \mathcal{S}(M_4)$, $g \in \mathcal{S}(M_4)$ such that $f \cdot g = 0$, $f \neq 0$, $g \neq 0$. Now $f^2$ and $g^2$ can be considered as elements of $\mathcal{S}(M_{k+1})$. Since $f \cdot g = 0$, we still have $f^2 \cdot g^2 = 0$. Also, since $\mathcal{S}(M_4)$ has no nilpotents, $f^2 \neq 0$ and $g^2 \neq 0$. Therefore $\mathcal{S}(M_{k+1})$ is not a domain either. Continuing in this way, we find that $\mathcal{S}(M_l)$ is not a domain for all $l \geq k$, which contradicts the first part of the proof. Q.E.D.

**Corollary 1.** The following are equivalent:

i) $\Theta$ is non-degenerate,
ii) $S_1 = V$, i.e., for all $x \in V$, $\exists \eta \in 1_1 \Lambda$ such that $\Theta(\langle \alpha + \eta \rangle) \neq 0$.
iii) For all $x \in 1_1 \Lambda$, $\exists \eta \in 1_1 \Lambda$ such that $\Theta(\langle \alpha + \eta \rangle) \neq 0$.

Proof. Clearly (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i). Now assume (i) holds. If $\Theta(\langle \alpha + \eta \rangle) = 0$, all $\eta \in 1_1 \Lambda$, then it would follow from the definition of $\Phi$ that $\Phi(\langle \alpha + \beta \rangle) \neq 0$, all $\beta \in V$. But this means that $\phi_{\langle \alpha + \eta \rangle} \cdot \phi_{\langle 0_1 \rangle} = 0$, i.e., one of the rings $\mathcal{S}(M_{k+1})$ is not domain. This contradicts the Prop., so (ii) must hold. Q.E.D.

**Corollary 2.** $\mathcal{S}(M_4) (2) \cong \mathcal{S}(M_{k+1})$, for all $k \geq 2$.

Proof. In view of Prop. 1, this follows from Cor. 1 provided that we check: $\forall x \in V$, $\exists \eta \in 1_1 \Lambda$ such that $\phi(\langle \alpha + \eta \rangle) \neq 0$. Looking back at the proof of the Cor. to Prop. 1, you see that this too follows from Cor. 1. Q.E.D.

To show that $X$ is actually an abelian variety, we could either define the group law explicitly, using the addition formula of §2, or else we can use only the action of $V/A$ on $X$ and combine this with general structure theorems on the automorphisms of a variety. Although the former is more elementary, we follow the latter approach as it is quicker.

$X$ is given to us together with a projective embedding. For example, $X = \text{Proj} (\mathcal{S}(M_4))$, so

$$X \subseteq \mathcal{P}(M_2).$$
Let $L_2$ be the invertible sheaf induced on $X$ via this embedding. If, via the isomorphism $X \cong \text{Proj} (\mathcal{O}(M_2))$, we embed $X$ in $\mathbb{P}(M_2)$, the induced sheaf $L_k$ is just:

$$L_k \cong L_2^{2k-2}.$$  

Let $\mathcal{P}$ denote the family of all invertible sheaves algebraically equivalent to $L_2$. We shall use the fact that $\text{Aut} (X, \mathcal{P})$, the group of automorphisms of the pair $X, \mathcal{P}$, is an algebraic group (Matsusaka [14], Grothendieck [15], p. 221–20). For all $x \in V/\mathcal{A}$, if $2^k x \in \mathcal{P}$, then $T_x$ is induced by an automorphism $T_x^*$ of $\mathcal{O}(M_{2k})$; therefore $T_x^*(L_{2k}) \cong L_{2k}$; therefore $T_x^*(L_2)$ differs from $L_2$ by an invertible sheaf of finite order; therefore $T_x^{-1}(\mathcal{P}) = \mathcal{P}$. In other words, the action of $V/\mathcal{A}$ on $X$ factors through an injective homomorphism:

$$V/\mathcal{A} \to \text{Aut} (X, \mathcal{P}).$$

Let $\mathcal{A}$ be the Zariski-closure of $V/\mathcal{A}$ in $\text{Aut} (X, \mathcal{P})$. Then $\mathcal{A}$ is connected since $V/\mathcal{A}$ is divisible and dense in $\mathcal{A}$ (cf. proof of Prop. 2), and $\mathcal{A}$ is commutative since $V/\mathcal{A}$ is commutative and dense in $\mathcal{A}$. Moreover, since the $V/\mathcal{A}$-orbit of $P_a$ is dense in $X$, the $\mathcal{A}$-orbit of $P_a$ must be an open dense set in $X$, i.e., $\mathcal{A}$ acts generically transitively on $X$. In fact, the morphism

$$\psi: \mathcal{A} \to X$$

$$\sigma \mapsto \sigma(P_a)$$

is an open immersion of $\mathcal{A}$ in $X$. This follows since the image $\psi(\mathcal{A})$ is always isomorphic to $\mathcal{A}/H$, $H$ = the stabilizer of $P_a$; and since $\mathcal{A}$ is commutative and acting faithfully on $X$, all stabilizers are trivial.

Next, we want to compute the dimension of $X$. I claim that the Hilbert polynomial of $(X, L_2)$ is given by:

**Proposition 3.** $\chi(L_2^2) = 4^k \cdot n^k$.

**Proof.** For $k$ large,

$$\chi(L_2^{2k}) = \dim \langle S_{2^{2k}}(M_2) \rangle$$

$$= \dim \langle M_{2+2k} \rangle.$$  

Now $M_{2(k+1)}$ is, by definition, the span of the $2^{2k(k+1)}$ functions $\Theta_{(k)}$, where $\beta$ runs over cosets of $2^{k-1} A/\mathcal{A}$. But these functions are linearly independent. To see this, look at the automorphisms $T_x^*$ of $\mathcal{O}(M_{2(k+1)})$, where $x \in 2^{k-1} A$. Use formulae (g) above and note that each $\Theta_{(k)}$ gives rise to a distinct set of eigenvalues for the $T_x^*$'s. Therefore, the $\Theta_{(k)}$'s could not be dependent unless one were identically zero, and this is not the case. Therefore

$$\dim M_{2(k+1)} = 4^k \cdot (2^{2k})^k.$$
This shows that $\chi(L^n) = 4^g \cdot n^g$ agree for an infinite set of values of $n$. Since both are polynomials, they are always equal. \textit{Q.E.D.}

\textbf{Corollary.} \dim X = g.

Returning to $A$, we find that $A$ is a commutative $g$-dimensional algebraic group containing a subgroup isomorphic to $(\mathbb{Q}_2/\mathbb{Z}_2)^{2g}$. From well-known structure theorems on algebraic groups, the only such $A$'s are abelian varieties. Therefore $A$ is complete, hence $A = X$, hence:

(I) $X$ is an abelian variety.

Moreover, in the course of proving this, we have also found that $V/A$ is acting on $X$ via translations, hence (comparing orders) we find:

(II) $x \mapsto \bar{p}(x)$ is a group isomorphism of $V/A$ with $\text{tor}_2(X)$.

Up to this point, identifying the various $\text{Proj} (\mathcal{S}(M_2))$'s has been useful. But to go further, it is more convenient now to drop these identifications. Therefore, now let

$$X_n = \text{Proj}(\mathcal{S}(M_{2n})).$$

This is a family of isomorphic abelian varieties. However, the most natural maps between them are given by the inclusions:

$$M_{2n+2} \subseteq M_{2n+1}$$
$$\mathcal{S}(M_{2n+2}) \subseteq \mathcal{S}(M_{2n+1})$$

inducing finite morphisms:

$$X_n \mapsto p_{n, n+1} X_{n+1}.$$ 

To check that $p$ is defined, we must know that $\mathcal{S}(M_{2n+2})$ is integrally dependent on $\mathcal{S}(M_{2n})$. But I claim:

$$\Theta(\gamma)^2 \cdot \Theta^{[2]}(\beta) = 2^{-8} \cdot \sum_{\eta \in \delta + A/A} e(\eta, \gamma) \cdot \Theta(\eta)^2 \cdot \Theta(\beta + \gamma - \eta) \cdot \Theta(\beta - \gamma + \eta).$$

[Proof. $\Theta(\gamma)^2 \cdot \Theta[M](\alpha)^2 = e(\beta, \alpha) \cdot \Theta(\gamma) \cdot \Theta(\beta - \alpha) \cdot \Theta(\alpha - \beta)$. By the quartic relations on $\Theta$, we get

$$= 2^{-8} e(\beta, \alpha) \sum_{\eta} e(-\gamma, \eta) \cdot \Theta(\eta)^2 \cdot \Theta(\beta - \alpha - \gamma + \eta) \cdot \Theta(\alpha - \beta - \gamma + \eta)$$

$$= 2^{-8} \sum_{\eta} e(\eta, \gamma) \cdot \Theta(\eta)^2 \cdot \Theta(\beta + \gamma - \eta)(\alpha) \cdot \Theta[8 - \gamma + \eta](\alpha). \text{ Q.E.D.}$$

Choose $\gamma \in \beta + 1/4 A$ so that $\Theta(\gamma) \neq 0$. Then if $\beta \in 2^{-n-1} A$, this equation shows that $\Theta^{[2]}(\beta) \in \mathcal{S}(M_{2n})$. This proves that $p$ is a finite morphism. Since $X_n$ and $X_{n+1}$ are abelian varieties, $p$ must be an isogeny.
Define prime ideals:

$$P^{(k)}(x) = \mathcal{S}(M_{2k})$$

$$P^{(k)}(x) = \sum_{n} P^{(k)}_{n}(x)$$

$$P^{(k)}_{n}(x) = \{ f \in \mathcal{S}_{n}(M_{2k}) | f(x) = 0 \}.$$ 

Then $P^{(k)}(x)$ defines a $k$-rational point $\psi_{k}(x) \in X_{k}$. We have

(a) $p(\psi_{k+1}(x)) = \psi_{k}(x)$.

(b) $x \mapsto \psi_{k}(x)$ defines an isomorphism

$$\frac{V}{2^k A} \xrightarrow{\cong} \text{tor}_{2}(X_{k}).$$

(b) here follows from conclusion (II) above, noticing how we have reinterpreted the ideal $P(x)$. In fact, if we call $X$ the common abelian variety to which all the $X_{k}$'s were previously identified, then $\overline{P}(x) \in X$ corresponds exactly to $\psi_{k}(2^k x) \in X_{k}$. Therefore $\psi_{k}(x) = 0 \iff P(2^{-k} x) = 0 \iff 2^{-k} x \in A$. Moreover, this shows that via these identifications, we get a morphism:

$$\begin{array}{ccc}
X & \xrightarrow{\overline{P}} & \overline{P}(x) \\
\downarrow & & \downarrow \\
X_{k+1} & \xrightarrow{\psi_{k+1}(2^{k+1} x)} & \psi_{k+1}(2^{k+1} x) \\
\downarrow & & \downarrow \\
X_{k} & \xrightarrow{\psi_{k}(2^{k+1} x)} & \psi_{k}(2^{k+1} x) \\
\downarrow & & \downarrow \\
X & \xrightarrow{\overline{P}(2^k x)} & \overline{P}(2^k x). \\
\end{array}$$

This map, from $X$ to $X$, agrees with $2 \delta$ at all points $\overline{P}(x)$. Therefore it is equal to $2 \delta$. In particular:

(c) The degree of $p$ is $2^{2^x}$ and $\text{Ker}(p) = \text{Ker}(2 \delta)$. It follows that all the $X_{n}$'s generate a single 2-tower. Call this $X = \{ X_{n} \}_{n \in S}$, and let $X_{n} = X_{\alpha_{n}}, \alpha_{n} \in S$. Moreover, these $\alpha_{n}$'s are a cofinal set in $S$, by (c). In view of (a)

$$\alpha \mapsto \{ \psi_{k}(x) \}$$

defines a homomorphism

$$\psi : V \rightarrow V(X),$$

and (b) implies that $\psi$ is an isomorphism. More, (b) shows that the compact open subgroups $2^k A$ and $T(\alpha)$ correspond to each other under $\psi$.

This 2-tower is polarized too. Let $L_{k}$ be the sheaf $\sigma(1)$ on $X_{k}$ coming from its presentation as $\text{Proj}(\mathcal{S}(M_{2k}))$. Since the $p^{*}$'s come from gradation preserving homomorphisms of the $\mathcal{S}(M_{2k})$'s it follows that $p^{*}(L_{k}) \cong L_{k+1}$. To check that $L_{k}$ is totally symmetric, we need the inverse on $X_{k}$:
Let \( i^* (f) (x) = f(-x) \), all \( f \in \mathcal{S}(M_2) \).

Then \( i^* \) defines an involution

\[ i: X_k \rightarrow X_k \]

such that \( i(\psi_k(x)) = \psi_k(-x) \).

Therefore \( i \) agrees with the inverse of \( X_k \) on all points \( \psi_k(x) \), hence \( i \) is the inverse of \( X_k \).

Since \( i \) is induced at all by an automorphism \( i^* \) of \( \mathcal{S}(M_2) \), it follows that \( L_2 \) is at least a symmetric sheaf. Since

\[ \{ \psi_k(x) | x \in 2k^{-1} A / 2A \} = \text{Kernel of } 2 \delta \text{ in } X_k, \]

\( L_2 \) is totally symmetric if and only if \( i^* \) is the identity in \( \mathcal{S}(M_2) / P^{(i)}(x) \), all \( x \in 2k^{-1} A \). This means that for all \( f \in M_2 \), \( i^* f = f \in P^{(i)}(x) \), i.e., \( f(x) = f(-x) \). But \( M_2 \) is spanned by \( \Theta_{\{\beta\}} \)’s, \( \beta \in 2^{-k} A \), and if \( \beta \in 2^{-k} A, x \in 2k^{-1} A \), then:

\[ \Theta_{\{\beta\}}(-x) = e \left( \frac{\beta}{2}, -x \right) \Theta(-x, -\beta) = e \left( \frac{\beta}{2}, x \right) \Theta(x, -\beta) = \Theta_{\{\beta\}}(x). \]

Therefore all the \( L \)'s are totally symmetric and the \( \{X_2, L_2\} \) extends to a polarized 2-tower \( \mathcal{F} = \{X_2, L_2\} \). We shall leave it to the reader to check the key fact that \( \psi \) is symplectic:

(d) \( e(\psi, \alpha, \psi, \beta) = e(\alpha, \beta) \), all \( \alpha, \beta \in V \).

Recapitulating this whole section so far, we have defined an arrow:

\[ \Xi: \begin{cases} \text{Given a non-degenerate} \Theta \text{ on } V \rightarrow \text{construct a polarized } \\
\text{theta function } \Theta \text{ on } V \} \rightarrow \text{2-tower } \mathcal{F} = \{X_2, L_2\}, \end{cases} \]

\[ \text{plus a symplectic isomorphism } \psi: V \rightarrow V(X), \]

Now, on \( V \) we have the vector space of functions spanned by all the \( \Theta_{\{\beta\}} \)'s. On \( V(X) \), we have the vector space of all theta functions \( \mathcal{H}[\Gamma(\mathcal{F})] \) of the tower \( \mathcal{F} \).

**Proposition 4.** Via \( \psi \), these vector spaces are equal:

\[ \text{Span of } \Theta_{\{\beta\}} = \{ \Theta_{\{\beta\}} \psi | \psi \in \Gamma(\mathcal{F}) \}. \]

Moreover, \( \Theta \) itself is the unique function \( f \) (up to scalars) of the form \( \delta_{\{\alpha\}} \psi \) satisfying the functional equation:

\[ f(x + \beta) = e(\beta/2) \cdot e(\beta/2, x) \cdot f(x), \quad \text{all } x \in V, \beta \in A. \]

**Key Corollary 1.** If \( V = Q^2 \), \( A = Z^2 \), and \( e, e_0 \) have the standard forms of \( \S 9 \), then \( \Theta \) is exactly the theta function \( \Theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} \psi \) associated to the
triple \((X, \mathcal{F}, \psi^-1)\) in \(\S 9\). In other words, \(\Xi\) is an inverse to the map \(\Theta\) of \(\S 9\).

**Proof of Prop. 4.** Let \(\alpha \in 2^{-k_1} A\) and let \(k \geq k_1\). Define \(T_a^* : \mathcal{F}(M_{2k}) \to \mathcal{F}(M_{2k})\) slightly differently from before:

\[
T_a^* f(\beta) = e\left(\beta, \frac{\alpha}{2}\right)^nf(\beta + \alpha), \quad \text{all } f \in S_0(M_{2k}).
\]

Note \(T_a^{-1}(P(\psi(\beta)) = P(\psi_a(x + \beta)).\) Let \(T_a : X_a \to X_a\) be the automorphism induced by \(T_a^*\). Then \(T_a(\psi(\beta)) = \psi_a(x + \beta),\) hence \(T_a\) is translation by the point \(\psi(x),\) i.e.,

\[
T_a = T_{\psi_a}(x).
\]

Moreover, \(T_a^*\) also induces a compatible isomorphism:

\[
g_a(x) : L_a \to T_{\psi_a}(x) L_a.
\]

For all \(k \geq k_1\), these are compatible, so the totality of pairs

\[
g(x) = \{(\psi(x), g_k(x) \mid k \geq k_1)\}
\]

is a point of \(\mathcal{F}(\mathcal{F})\).

\((*)\) \(g(x) = e[\psi(x)],\) i.e., \(g(x)\) is the canonical element of \(\mathcal{F}(\mathcal{F})\) over the point \(\psi(x)\) in \(\Gamma(X)\).

**Proof of (*).** This requires checking 2 things: (i) \(g(x)\) is a symmetric element of \(\mathcal{F}(\mathcal{F}),\) i.e., \(\delta_{-1} g(x) = g(x)^{-1},\) and (ii) \(g(2x) = g(x)^2.\) In terms of \(T_a^*,\) this is the same as:

(i) \(\iota^* \circ T_a^* = (T_a^*)^{-1} \circ \iota^*\).

(ii) \(T_{2a}^* = T_a^* \circ T_a^*\).

These are both immediate. \(Q.E.D.\)

Next, notice that \(M_{2k} \cong \Gamma(X_k, L_k).\) In fact, there is a canonical map \(M_{2k} \to \Gamma(X_k, L_k);\) it is injective, since the ring \(\mathcal{F}(M_{2k})\) has no nilpotents, and only nilpotent elements of \(\mathcal{F}(M_{2k})\) define trivial sections of \(L_k^*;\) but it is easy to check that both \(\dim M_{2k}\) and \(\dim \Gamma(X_k, L_k)\) are equal to \(2^{2k};\) therefore \(M_{2k} \cong \Gamma(X_k, L_k).\) Therefore,

\[
\Gamma(\mathcal{F}) = \lim_k \Gamma(X_k, L_k) \cong \bigcup_k M_{2k} = \left\{\text{Span of all the functions } \Theta_\beta \mid \beta \in V\right\}.
\]

Now let \(f\) be some linear combination of the \(\Theta_\beta\). Say \(f \in M_{2k}\). Let \(f\) define \(s \in \Gamma(X_k, L_k)\). I claim that:

\((*)\) \(f(x) = \Theta_\beta(\psi x), \quad \text{all } \alpha \in V.\)

16 Inventiones math., Vol. 3
Taking a larger $k_1$ if necessary, we may suppose that $\alpha \in 2^{-k_1}A$. By
definition, $\vartheta_{(-\alpha)} \varphi$ is the "value" at the origin of $X_k$, of the section
of $L_{A_k}$ obtained via the map:
\[
\Gamma(X_k, L_{A_k}) \xrightarrow{\vartheta_{(-\alpha)}} \Gamma(X_k, T_{\vartheta_{(-\alpha)}} L_{A_k}) \xrightarrow{T_{\vartheta_{(-\alpha)}}} \Gamma(X_k, L_{A_k}).
\]

This means that we simply apply the automorphism $(T_*)^{-1}$ of $M_{2k}$ to $f$, and take
the value at the origin. But $T_\alpha = T^*_{-\alpha} - 1$, and $(T_\alpha f) (\psi) = f(\alpha)$,
so (**) is proven. Thus the span of the $\Theta_{\{\psi\}}$’s is the same as the space of
functions $\vartheta(x) \varphi \in \varphi$, $\varphi \in \Gamma(\mathcal{F})$.

As for the final assertion of the Proposition, on the one hand, $\Theta$ does
satisfy the functional equation there; and, from the general theory of the
space $\vartheta[\Gamma(\mathcal{F})]$ in § 8, we know that this functional equation has only a
1-dimensional set of solutions in $\vartheta[\Gamma(\mathcal{F})] \varphi$. Q.E.D.

**Corollary 2.** All $g$-dimensional principally polarized abelian varieties $X$
are isomorphic to $\text{Proj}(\mathcal{F}(M_2))$, where $M_2$ is the span of the $\Theta_{\{\psi\}}$’s,
$\beta \in \frac{1}{2}A$, for some non-degenerate theta function $\Theta$ on $V$.

**Proof.** Just take $\Theta$ to be the $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ attached to $X$ as in § 9, and carried
over to a function on $V$ by a suitable isomorphism of $V$ and $V(X)$. Q.E.D.

**Corollary 3.** The open set $M_\alpha \subset \overline{M}_\alpha$, which in § 9 represents
the moduli functor $\mathcal{M}_\alpha$, is the open set whose geometric points represent
non-degenerate theta functions, i.e.,
\[
E = \left\{ \text{set of all systems of coset representatives} \right\}.
\]
\[
r : \frac{1}{2} \mathbb{Z}_2^2 / \frac{1}{2} \mathbb{Z}_2^2 \rightarrow \frac{1}{2} \mathbb{Z}_2^2
\]

For all $r \in E$, let
\[
U_r = \left\{ \text{open set in } \overline{M}_\alpha \text{ defined by } X_0 = 0, \text{ all } \alpha \in \text{Image}(r) \right\}.
\]

Then
\[
M_\alpha = \bigcup_{r \in E} U_r.
\]

§ 11. Satake's Compactification

In this section, I want to analyze the degenerate theta functions $\Theta$
on $V$, in the sense of § 10. In particular, they all come from lower dimen-
sional non-degenerate theta-functions via "cusps". This will show that
the whole moduli scheme $\overline{M}_\alpha$ is a disjoint union of copies of the $M_\alpha$’s
for dimensions $g$ and lower i.e., that $\overline{M}_\alpha$ is the Satake compactification
of $M_\alpha$.

\footnote{Added in Proof. A closer study has shown that $\overline{M}_\alpha$ is not normal along $\overline{M}_\alpha - M_\alpha$. Its normalization is Satake's compactification.}
Return to the discussion at the beginning of §10: let \( V, A, e, e^* \) be given as before. First, I want to describe a way of forming degenerate theta functions on \( V \) out of theta functions on lower dimensional spaces.

**Definition 1.** A cusp is a subspace \( W \subset V \) such that \( W^\perp \subset W \), i.e., if \( \alpha \in V \) has the property \( e(\alpha, \beta) = 1 \), all \( \beta \in W \), then \( \alpha \in W \).

Given a cusp \( W \), let:
\[
\tilde{V} = W/W^\perp
\]
\[
\tilde{A} = A \cap W/A \cap W^\perp
\]
\( \tilde{e} \) is induced skew-symmetric pairing, \( \tilde{V} \times \tilde{V} \to k^* \).

**Lemma.** \( \tilde{A} \) is a maximal isotropic lattice in \( \tilde{V} \), (for \( \tilde{e} \)).

**Proof.** Notice that \( A/A \cap W \) is a free \( Z_2 \)-module. Therefore the sequence:
\[
0 \to A \cap W \to A \to A/A \cap W \to 0
\]
splits, and \( A = A_1 \oplus (A \cap W) \) for some sub \( Z_2 \)-module \( A_1 \). Let \( V_1 = Q_2 \cdot A_1 \), so \( V = V_1 \oplus W \). Now I claim:

\[(*) \quad (A \cap W)^\perp = A + W^\perp.\]

[In fact, let \( \alpha \in V \) satisfy \( e(\alpha, \beta) = 1 \), all \( \beta \in A \cap W \). Since \( V_1 \) and \( W \) are dual vector spaces via \( e \), there is a \( \gamma \in W^\perp \) such that \( e(\alpha, \beta) = e(\gamma, \beta) \) all \( \beta \in V_1 \). But then \( \alpha - \gamma \) is orthogonal to both \( V_1 \) and \( A \cap W \), hence orthogonal to \( A \), hence \( \alpha - \gamma \in A \). Thus \( \alpha \in W^\perp + A \).]

Now to show \( \tilde{A} \) is maximal isotropic, let \( \alpha \in W \) have an image \( \tilde{\alpha} \) in \( \tilde{V} \) perpendicular to \( \tilde{A} \), i.e., \( \alphabeta(\tilde{W} \cap A)^\perp \). By (*)&, \( \alpha = \alpha_1 + \alpha_2 \), where \( \alpha_1 \in A \), \( \alpha_2 \in W^\perp \). But then \( \alpha = \alpha - \alpha_2 \in W \). Therefore \( \alpha_1 \in W \cap A \) so \( \tilde{\alpha} = \tilde{\alpha}_1 \in \tilde{A} \). Q.E.D.

**Definition 2.** A cusp with origin is a cusp \( W \subset V \), plus an element \( \eta_0 \in \frac{1}{2} A \) such that

i) \( e_*(\alpha) = e(\alpha, \eta_0)^2 \), all \( \alpha \in W^\perp \cap (\frac{1}{2} A) \).

ii) \( e_*(\eta_0) = 1 \).

It is not hard to check that every cusp has at least one origin: we leave this to the reader. Given a cusp with origin, look at the map
\[
\alpha \mapsto e_*(\alpha) \cdot e(\alpha, \eta_0)^2
\]
where \( \alpha \in \frac{1}{2} A \cap W \). If \( \beta \in \frac{1}{2} A \cap W^\perp \), then
\[
e_*(\alpha + \beta) \cdot e(\alpha + \beta, \eta_0)^2 = e_*(\alpha) \cdot e_*(\beta) \cdot e(\alpha, \beta)^2 \cdot e(\alpha, \eta_0)^2 \cdot e(\beta, \eta_0)^2
\]
\[= e_*(\alpha) \cdot e(\alpha, \eta_0)^2.\]
Thus there is a quadratic form $\tilde{e}_\alpha : \frac{1}{2} \tilde{A}/\tilde{A} \to \{ \pm 1 \}$ such that
\[(\ast) \quad \tilde{e}_\alpha(\tilde{e}) = e_\alpha(\alpha) \cdot e(\alpha, \eta_0)^2, \quad \text{all } \alpha \in \frac{1}{2} A \cap W.
\]

It is not hard to check that the new data $(\tilde{V}, \tilde{A}, \tilde{e}, \tilde{e}_\alpha)$ has the standard form required in § 10 (i.e., that the associated Arf-invariant is 0). We leave this to the reader also.

Now let $\tilde{\Theta}$ be a theta-function on $\tilde{V}$.

**Definition 3.** For all $\alpha \in V$, let
\[
T_{W, \eta_0} \Theta(\alpha) = \begin{cases} 
0 & \text{if } \alpha \notin \eta_0 + W + A \\
\tilde{e}_\alpha \left( \frac{\eta_1}{2} \right) e \left( \frac{\eta_1}{2}, \eta_0 \right) e \left( \frac{\eta_0 + \eta_1}{2}, \alpha \right) \tilde{\Theta}(\tilde{\alpha}_0) & \text{if } \alpha = \eta_0 + \eta_1 + \alpha_0, \; \eta_1 \in A, \; \alpha_0 \in W.
\end{cases}
\]

**Proposition 1.** The above $T_{W, \eta_0} \tilde{\Theta}$ is well-defined (note that the $\alpha \in V$ may be decomposed in more than one way as $\alpha = \eta_0 + \eta_1 + \alpha_0$, and is a theta-function on $V$.

The proof of this Proposition is a ghastly but wholly straightforward set of computations. It took me several hours to do every bit and as I was no wiser at the end — except that I knew the definition was correct — I shall omit details here. Our main result is:

**Theorem.** Let $\Theta$ be any theta-function on $V$, and let $W$ be the subspace of $V$ such that $S_\infty = W + A$ (cf. § 10). Then $W$ is a cusp, and if $\eta_0$ is any origin for $W$, $\Theta$ is equal to $T_{W, \eta_0} \tilde{\Theta}$ for some non-degenerate theta-function $\tilde{\Theta}$ on $\tilde{W}$. In particular, $W$ is characterized by:

coarse support ($\Theta$) = $W + \frac{1}{2} A$.

The proof of this theorem will be based on the $\Theta \leftrightarrow \mu$ correspondence, given in Lemma 1, § 8. Before taking up the proof of the Theorem, we want to give this correspondence a more intrinsic formulation. Let $V = W_1 \oplus W_2$, where $W_i$ are maximal isotropic subspaces, such that

i) $A = A_1 \oplus A_2$, $A_i = A \cap W_i$.

ii) $e_\alpha(\alpha/2) = 1$, all $\alpha$ in $A_1$ or in $A_2$.

Then

a) Define a measure $\mu$ on $W_1$, from a theta function $\Theta$ on $V$ via

$$
\mu(\alpha_1 + 2^a A_1) = 2^{-a} \sum_{\alpha_2 \in 2^{-a} A_2} e \left( \frac{\alpha_1 + \alpha_2}{2} \right) \cdot \Theta(\alpha_1 + \alpha_2).
$$

b) Define a theta function $\Theta$ on $V$, from a measure $\mu$ on $W_1$, via

$$
\Theta(\alpha_1 + \alpha_2) = e \left( \frac{\alpha_2}{2}, \alpha_1 \right) \cdot \int_{A_2} e(\alpha_2, \beta) \cdot d\mu(\beta).
$$
On the Equations Defining Abelian Varieties. III

231

Our proof will be based on the fact that any finitely additive measure \( \mu \) (on the algebra of compact open subsets of \( W_1 \)) has a support, i.e., a smallest closed set \( S \) such that:

\[
\mu(U) = 0, \quad \text{all compact open } U's \text{ in } W_1 - S.
\]

**Proof.** Say \( S_A \) and \( S_B \) are closed sets such that \( \mu(U) = 0 \) if \( U \subset W_1 - S_A \) or \( U \subset W_1 - S_B \). Then let \( U \subset W_1 - (S_A \cap S_B) \) be a compact open set. We must decompose \( U \) into \( U_A \cup U_B \), where \( U_A \subset W_1 - S_A \), and \( U_B \subset W_1 - S_B \), and \( U_A \) and \( U_B \) are compact and open. For all \( x \in U \cap S_A \), note that \( x \not\in S_A \), so we can find a compact, open neighborhood \( U_x \) of \( x \) such that

\[
U_x \subset U \cap (W_1 - S_B).
\]

Since \( U \cap S_A \) is compact, it can be covered by a finite set of these \( U_x \)'s: say

\[
U \cap S_A \subset [U_{x_1} \cup \cdots \cup U_{x_n}].
\]

Let \( U_B = U_{x_1} \cup \cdots \cup U_{x_n} \). By construction \( U_B = U \cap (W_1 - S_B) \) and \( U_B \) is compact and open. Let \( U_A = U - U_B \). Then \( U_A \) is also compact and open and since \( U_B \supset U \cap S_B \), it follows that \( U_A \subset U \cap (W_1 - S_B) \). By assumption on \( S_A \) and \( S_B \), we have \( \mu(U_A) = 0 \) and \( \mu(U_B) = 0 \). Therefore \( \mu(U) = 0 \). This shows that the family of sets:

\[
\mathcal{S} = \{S \text{ closed in } W_1 | \mu(U) = 0 \text{ for all compact open sets } U \subset W_1 - S\}
\]

is closed under finite intersections. Now let

\[
S^* = \bigcap_{S \in \mathcal{S}} S.
\]

I claim \( S^* \in \mathcal{S} \) too. Let \( U \subset W_1 - S^* \) be a compact open set. Since

\[
W_1 - S^* = \bigcup_{S \in \mathcal{S}} (W_1 - S),
\]

it follows that \( U \) is covered by the open sets \( U \cap (W_1 - S) \), where \( S \in \mathcal{S} \). Since \( U \) is compact, it can be covered by a finite number of such open sets:

\[
U \subset (W_1 - S_1) \cup \cdots \cup (W_1 - S_n)
\]

where \( S_1, ..., S_n \in \mathcal{S} \). Now let \( T \in \mathcal{S} \) be a closed set contained in all these \( S_i \). Then \( U \subset W_1 - T \). But \( T \in \mathcal{S} \) means that this implies \( \mu(U) = 0 \). So \( \mu(U) = 0 \) whenever \( U \subset W_1 - S^* \), i.e., \( S^* \in \mathcal{S} \) too. Q.E.D.

**Proposition.** Let \( \mu \) be a non-zero even Gaussian measure on \( W_1 \) (i.e., \( \mu \) has the property \( A \) of Lemma 1, § 8). Then the support \( S \) of \( \mu \) is a sub-vector space of \( W_1 \).
Proof. Notice that if \( \mu_1, \mu_2 \) are 2 measures on \( W_1 \), and \( \mu_1 \times \mu_2 \) is the induced measure on \( W_1 \times W_1 \), then

\[
\text{Support}(\mu_1 \times \mu_2) = \text{Support}(\mu_1) \times \text{Support}(\mu_2).
\]

Let \( \xi : W_1 \times W_1 \rightarrow W_1 \times W_1 \) be the map \( \xi((x, y)) = (x+y, x-y) \). By definition, a Gaussian measure \( \mu \) is associated to a second measure \( v \) such that

\[
\xi_* (\mu \times \mu) = v \times v.
\]

Therefore, if \( S' = \text{Support } (v) \), it follows that \( \xi(S \times S) = S' \times S' \). In particular

\[
\begin{align*}
\alpha \in S & \iff (\alpha, \alpha) \in S \times S \\
& \iff (2\alpha, 0) = \xi((\alpha, \alpha)) \in S' \times S'.
\end{align*}
\]

Since \( S \) is non-empty, \( 0 \in S' \), and \( \alpha \in S \iff 2\alpha \in S' \), i.e., \( S' = 2S \). Therefore \( 0 \in S \) too, and we find:

\[
\begin{align*}
\alpha \in S & \iff (\alpha, 0) \in S \times S \\
& \iff (\alpha, \alpha) = \xi((\alpha, 0)) \in S' \times S' \\
& \iff \alpha \in S'.
\end{align*}
\]

Therefore \( S = S' \) also. Finally,

\[
\begin{align*}
\alpha, \beta \in S & \Rightarrow (\alpha, \beta) \in S \times S \\
& \Rightarrow (\alpha + \beta, \alpha - \beta) \in S' \times S' \\
& \Rightarrow \alpha + \beta, \alpha - \beta \in S' = S.
\end{align*}
\]

Thus \( S \) is a closed subgroup of \( W_1 \), such that \( S = 2S \). Therefore \( S \) is a subvector space over \( Q_2 \). Q.E.D.

Corollary. For all \( \gamma_2 \in W_2 \), all theta functions \( \Theta \) on \( V \),

\[
\text{Support}(\Theta) \subset \{ \alpha \mid e(\alpha, \gamma_2) = 1 \} \Rightarrow \Theta(\alpha + \lambda \gamma_2) = e\left(\alpha, \frac{\lambda \gamma_2}{2}\right) \Theta(\alpha),
\]

all \( \lambda \in Q_2 \).

Proof. The assumption on the support of \( \Theta \) implies (cf. (a) above) that \( \mu(\alpha_1 + 2^a \alpha_1) = 0 \) if \( e(\alpha_1, \gamma_2) + 1 \). Therefore,

\[
\text{Support}(\mu) \subset \{ \alpha_1 \in W_1 \mid e(\alpha_1, \gamma_2) = 1 \}.
\]

Since this support is a vector space,

\[
\text{Support}(\mu) \subset W_1 \cap (Q_2 \cdot \gamma_2)^\perp.
\]
Let $H$ denote the hyperplane $W_1 \cap (Q_2 \cdot \gamma_2)^\perp$. Then
\[
\Theta(a_1 + a_2) = e\left(\frac{a_2}{2} \right) \int_{(a_1 + \Lambda_1) \cap H} e(a_2, \beta) \cdot d\mu(\beta).
\]
Thus
\[
\Theta(a_1 + a_2 + \lambda \gamma_2) = e\left(\frac{a_2 + \lambda \gamma_2}{2} \right) \int_{(a_1 + \Lambda_1) \cap H} e(a_2 + \lambda \gamma_2, \beta) \cdot d\mu(\beta)
\]
and since $e(\lambda \gamma_2, \beta) = 1$ when $\beta \in H$, this comes out
\[
eq e\left(\frac{\lambda \gamma_2}{2} \right) \cdot \left\{ e\left(\frac{a_2}{2} \right) \int_{(a_1 + \Lambda_1) \cap H} e(a_2, \beta) \cdot d\mu(\beta) \right\}
\]
\[
= e\left(\frac{\lambda \gamma_2}{2} \right) \cdot \Theta(a_1 + a_2).
\]
Q.E.D.

In fact, I claim that the same Corollary holds for all $\gamma \in V$, not just for $\gamma \in W_2$. This can be seen by noting that for any $\gamma \in V$, there is a symplectic automorphism $T: V \to V$ such that $T(\Lambda) = \Lambda$, i.e., $T \in \text{Sp}(V, \Lambda)$, such that $T^{-1}(\gamma) \in W_2$. Going back to the action of the symplectic group introduced in §9, we see that:

\[
\begin{cases}
\text{If } \Theta \text{ is a theta-function, then so is } \Theta', \text{ where} \\
\Theta'(a) = e(\eta/2, a) \Theta(Ta - T\eta)
\end{cases}
\]

where $\eta \in \frac{1}{2} \Lambda$ satisfies
\[
e(a/2) \cdot e_a(Ta/2) = e(\eta, a), \quad \text{all } a \in \Lambda.
\]

Now assume $\text{Supp}(\Theta) \subseteq \{x \mid e(x, \gamma) = 1\}$. Then
\[
\text{Supp}(\Theta') \subseteq \eta + T^{-1}(\text{Supp}(\Theta))
\]
\[
\subseteq \eta + \{x \mid e(x, T^{-1} \gamma) = 1\}
\]
\[
\subseteq \{x \mid e(x, 2^n T^{-1} \gamma) = 1\} \quad (\text{if } n \gg 0).
\]

Therefore, by the Corollary
\[
\Theta'(a + \lambda T^{-1} \gamma) = e\left(\frac{\lambda T^{-1} \gamma}{2} \right) \Theta'(a), \quad \text{all } \lambda \in Q_2,
\]
from which
\[
\Theta(a + \lambda \gamma) = e\left(\frac{\lambda \gamma}{2} \right) \cdot \Theta(a)
\]
follows immediately. We are now ready for the Proof itself:

**Proof of Theorem.** We know that the support of $\Theta$ meets $\frac{1}{2} \Lambda$ (cf. §10): choose $\eta_0 \in \text{Supp}(\Theta) \cap \frac{1}{2} \Lambda$. Then:
\[
\text{Supp}(\Theta) + \eta_0 \subseteq W + \Lambda
\]
(§ 10, assertion (4.) at the beginning). Therefore, if \( \gamma \in W^\perp \cap (2A) \) it follows that \( e(x, \gamma) = 1 \), all \( x \in \text{Supp}(\Theta) \). But then by Corollary above — as generalized —

\[
\Theta(x + \lambda \cdot \gamma) = e \left( x, \frac{\lambda \gamma}{2} \right) \cdot \Theta(x), \quad \text{all } \lambda \in \mathbb{Q}_2.
\]

This shows that

\[
(\ast) \quad \Theta(x + \gamma) = e \left( x, \frac{\gamma}{2} \right) \cdot \Theta(x), \quad \text{all } \gamma \in W^\perp.
\]

In particular, \( \Theta(\eta_0 + \gamma) \neq 0 \), all \( \gamma \in W^\perp \), hence \( W^\perp + \eta_0 \subseteq W + A + \eta_0 \). Therefore \( W^\perp \subseteq W \), i.e., \( W \) is a cusp.

Now suppose we take an arbitrary point \( x \) in the Support of \( \Theta \). We know that \( x \) can be written as:

\[
x = \eta_0 + \eta_1 + \alpha_0, \quad \eta_1 \in A, \; \alpha_0 \in W.
\]

But then:

\[
\Theta(x) = e_* \left( \frac{\eta_1}{2} \right) \cdot e \left( \frac{\eta_1}{2}, \eta_0 + \alpha_0 \right) \cdot \Theta(\eta_0 + \alpha_0)
\]

\[
= e_* \left( \frac{\eta_1}{2} \right) \cdot e \left( \frac{\eta_1}{2}, \eta_0 \right) \cdot e \left( \frac{\eta_0 + \eta_1}{2}, x \right) \cdot \left[ e \left( x, \frac{\eta_0}{2} \right) \cdot \Theta(\eta_0 + \alpha_0) \right].
\]

Define a function \( \tilde{\Theta} \) on \( W \) by

\[
\tilde{\Theta}(x) = e \left( x, \frac{\eta_0}{2} \right) \cdot \Theta(x + \eta_0).
\]

If \( \gamma \in W^\perp \), we compute (using \((\ast))\):

\[
\tilde{\Theta}(x + \gamma) = e \left( x + \gamma, \frac{\eta_0}{2} \right) \cdot \Theta(x + \eta_0 + \gamma)
\]

\[
= e \left( \gamma, \frac{\eta_0}{2} \right) \cdot e \left( x + \eta_0, \frac{\gamma}{2} \right) \cdot e \left( x, \frac{\eta_0}{2} \right) \cdot \Theta(x + \eta_0)
\]

\[
= \tilde{\Theta}(x).
\]

This shows that \( \tilde{\Theta} \) is, in reality, a function on \( \tilde{V} = W/W^\perp \), and that \( \Theta \) is exactly the function \( T_{W, \eta_0} \tilde{\Theta} \) obtained from \( \tilde{\Theta} \) via Definition 3.

To check that \( \eta_0 \) is an origin for \( W \), look at \((\ast)\) when \( \gamma \in W \cap A \). Then:

\[
e \left( x, \frac{\gamma}{2} \right) \cdot \Theta(x) = \Theta(x + \gamma) = e_* \left( \frac{\gamma}{2} \right) \cdot e \left( \frac{\gamma}{2}, x \right) \cdot \Theta(x)
\]

hence

\[
e_* \left( \frac{\gamma}{2} \right) = e(x, \gamma) \quad \text{if } \Theta(x) \neq 0.
\]
So
\[ e_\ast \left( \frac{\gamma}{2} \right) = e(\eta_0, \gamma), \quad \text{all } \gamma \in W^\perp \cap \Lambda. \]
Moreover, using
\[ \Theta(\eta_0) = \Theta(-\eta_0 + 2\eta_0) = e_\ast(\eta_0) \Theta(-\eta_0) \]
and
\[ \Theta(-\eta_0) = \Theta(\eta_0) \pm 0, \]
we conclude that \( e_\ast(\eta_0) = 1 \) too.

The fact that \( \Theta \) is again a theta-function is simply a matter of applying the calculations of Prop. 1 in reverse and is quite straightforward. We omit this. The final point is that \( \Theta \) is non-degenerate. But since \( S^\infty = W \), we know that for all \( \alpha \in W, \alpha = 2^k \beta + \eta_1 \), where \( \Theta(\beta) \neq 0, \eta_1 \in \Lambda \). Then \( \beta = \eta_0 + \eta_2 + \beta_0, \eta_2 \in \Lambda, \beta_0 \in W \), and \( \Theta(\beta_0) \neq 0 \). Since
\[ \alpha = 2^k \beta_0 = \eta_1 + 2^k \eta_0 + 2^k \eta_2 \in W \cap \Lambda, \]
this shows that for all \( \alpha \in W, \alpha = 2^k \beta_0 + \eta_3 \), where \( \Theta(\beta_0) \neq 0, \eta_3 \in W \cap \Lambda \). This means exactly that the \( S^\infty \) for \( \Theta \) is all of \( \bar{V} \), i.e., \( \Theta \) is non-degenerate. Q.E.D.

The main Theorem can now be reformulated to give a Satake-like decomposition of \( \bar{M}^\infty \). More precisely, for each integer \( g \geq 0 \), let \( \bar{M}^\infty(g) \) = the Proj defined in § 9, Def. 3 with indices \( \alpha \in Q^+_2 \).

\( M^\infty(g) \) = the open set in \( \bar{M}^\infty(g) \) whose geometric points are the non-degenerate theta functions.

If \( h < g \), we define a vast number of closed immersions
\[ i_w: M^\infty(h) \hookrightarrow M^\infty(g) \]
as follows: let \( W \subseteq Q^+_2 \) be a cusp such that \( 2h = \dim (W/W^\perp) \). For each such \( W \), choose an origin \( \eta_0 \in \frac{1}{2} Z^+_2 \), and a symplectic isomorphism:
\[ \phi: Q^+_2 \overset{\cong}{\rightarrow} W/W^\perp \]
such that
\[ \phi(Z^+_2) = W \cap \Lambda/W^\perp \cap \Lambda, \]
\[ \chi(\tfrac{1}{2} a_1 \cdot a_2) = e_\ast \left( \frac{1}{2} \phi(a) \right), \quad \text{all } a \in Z^+_2. \]
Then \( i_w \) is defined by the homomorphism of the homogeneous coordinate ring:
\[ i^*_w(X^\ast)(\alpha) = \begin{cases} 0 & \text{if } \alpha \neq \eta_0 + W + Z^+_2 \\ e_\ast \left( \frac{\eta_1}{2} \right) e \left( \frac{\eta_1}{2}, \eta_0 \right) e \left( \eta_0 + \eta_1 + \alpha, \cdot \right) X^\ast(\alpha) & \text{if } \alpha = \eta_0 + \alpha_0 + \eta_1, \quad \alpha_0 \in W, \eta_1 \in Z^+_2. \end{cases} \]
(Here \(X_1^{(g)}, X_2^{(h)}\) are the coordinates used to define \(\bar{M}_{\infty}(g), \bar{M}_{\infty}(h)\) respectively). Then we get the restatement:

**Main Theorem.**

\[
\bar{M}_{\infty}(g) = \left\{ \text{disjoint union of the locally closed subschemes } i_w(M_{\infty}(h)) \right\},
\]

the union being taken over all cusps \(W \subseteq \mathbb{Q}_2^+\).

---

**§ 12. Analytic Theta Functions**

In this section, we work over the field \(C\) of complex numbers. We have 2 purposes: (a) to sketch an approach to the classical theory of \(\Theta\)-functions, analogous to our theory of algebraic \(\Theta\)-functions, and (b) to use this to compute our algebraic \(\Theta\)-functions via the classical ones, when \(k = C\).

We will make use of the following lemma:

**Lemma 1.** Let \(X\) be a compact Kähler manifold. Then the operator

\[
\frac{1}{2\pi i} \delta \bar{\delta}
\]

defines a surjection:

\[
\left\{ \text{functions on } X \right\} \rightarrow \left\{ \text{real closed } C^\infty (1,1)\text{-forms } \Omega \text{ on } X, \text{ with } 0 \text{ cohomology class} \right\}
\]

with kernel consisting only of constants.

**Corollary.** Let \(L\) be an analytic line bundle on \(X\). Let \(c_1(L) \in H^2(X, C)\) be its first Chern class. Then for all real closed \(C^\infty (1,1)\)-forms \(\Omega\) whose cohomology class equals \(c_1(L)\), there is one and (up to a constant) only one Hermitian structure \(\|\|\) on \(L\) whose associated curvature form is \(\Omega\).

The lemma is standard and we omit the proof. The Corollary can be proven by choosing one Hermitian structure \(\|0\|\) on \(L\): let \(\Omega_0\) be its curvature form. Then any other Hermitian structure on \(L\) is given by \(\rho \cdot \|0\|\), where \(\rho\) is a positive real \(C^\infty\) function on \(X\): and its curvature form \(\Omega\) is

\[
\Omega = \frac{1}{2\pi i} \delta \bar{\delta} \log \rho + \Omega_0.
\]

Now use the Lemma and everything comes out. **Q.E.D.**

In particular, when \(X\) is an abelian variety, an analytic line bundle \(L\) on \(X\) has one and (up to a constant) only one Hermitian structure \(\|\|\) whose curvature form \(\Omega\) is a translation-invariant \((1,1)\)-form. In what follows, we will always put this Hermitian structure on line bundles on abelian varieties. In this case, \(\Omega\) is determined by its value at the origin.
Now let \( \hat{X} \) be the universal covering space of \( X \). \( \hat{X} \) is a complex vector space, and if

\[
p: \hat{X} \to X
\]

is the canonical homomorphism, \( dp \) induces a canonical identification between \( \hat{X} \) and the tangent space of \( X \) at the origin (or at any other point). Therefore, any translation-invariant real 2-form \( \Omega \) on \( X \) defines and is defined by a real-linear skew-symmetric form:

\[
E: \hat{X} \times \hat{X} \to \mathbb{R}.
\]

\( E \) is a \((1, 1)\)-form if and only if \( E(ix, iy) = E(x, y) \), all \( x, y \in X \). Moreover, let \( A = \text{kernel} (p) \). \( A \) is a lattice in \( X \), canonically isomorphic to \( H_1 (X, \mathbb{Z}) \). Since the first Chern class of a line bundle is integral, if \( E \) represents \( c_1 (L) \), then \( E \) must take integral values on \( A \times A \):

\[
E(A \times A) \subseteq \mathbb{Z}.
\]

If we lift \( L \) to \( \hat{X} \), we have a situation in which the following lemma applies:

**Lemma 2.** Let \( Y \) be a complex vector space, and let \( L_1, L_2 \) be 2 analytic-Hermitian line bundles on \( Y \). Then a holomorphic-unitary isomorphism \( \phi: L_1 \to L_2 \) exists if and only if the curvature forms of \( L_1, L_2 \) are equal; if so, \( \phi \) is unique up to a scalar of absolute value 1.

**Proof.** Standard methods.

In particular, let \( Y = \hat{X} \), and let \( M = p^* (L) \) be induced from an abelian variety. Give \( L \) and hence \( M \) the Hermitian structure with constant curvature form \( E \). The above lemma has 2 applications:

(I) Construction of a nilpotent group \( \mathcal{G} \): If \( x \in X \), and \( T_x \) denotes translation by \( x \), then the lemma shows that \( M \) and \( T_x^* M \) are holomorphic-unitary isomorphic. If

\[
\mathcal{G}(M) = \{ (x, \phi) | \phi \text{ a holo.-unit. isom. of } M \text{ with } T_x^* M \},
\]

then \( \mathcal{G}(M) \) is, as before, a group lying in an exact sequence:

\[
1 \to C^*_1 \to \mathcal{G}(M) \to X \to 0
\]

\( (C^*_1 = \text{complex numbers of absolute value } 1) \).

(II) Construction of canonical “trivialization” of \( M \): Let \( 1 \) denote the trivial analytic line bundle over \( X \) with canonical section 1. To put a Hermitian structure on \( 1 \), we may set \( \| 1 \| = \text{any positive real } C^\infty\)-function. For example, let

\[
\| 1 \| (x) = e^{-x^2/2H(x, x)}
\]
where $H$ is a Hermitian form on $X$. The corresponding curvature form $E: \hat{X} \times \hat{X} \to \mathbb{R}$ is easily checked to equal $\text{Im}(H)$. But

$$H \mapsto E = \text{Im}(H)$$

sets up an isomorphism:

$$\left\{ \text{hermitian forms on } X \right\} \sim \left\{ \text{real skew-symmetric forms } E \text{ on } X \left| \begin{array}{c}
E(i\, x, i\, y) = E(x, y) \\
\end{array} \right. \right\}.$$ 

so for each $L$ on $X$ with translation-invariant curvature form, we have a unique Hermitian structure on $1$ of the above type so that $1 \cong L$. In particular, we get a canonical

$$1 \cong M.$$ 

We can now develop a theory along similar lines to our algebraic theory. For example, if $H$ is positive definite, then let:

$\mathcal{H}$ = Hilbert space of $L^2$-holomorphic sections of $M$ over $\hat{X}$.

Then $\mathcal{G}(M)$ has a natural unitary representation on $\mathcal{H}$, it is irreducible, and it turns out to be the only irreducible unitary representation of $\mathcal{G}(M)$ in which $C^*_1 \subset \mathcal{G}(M)$ acts by its natural character. This is the situation described by Cartier [2], and studied by Cartier and many others, e.g., Mackey, Fock, Weil etc. Exactly as in §1, $\mathcal{F}(M)$ governs the “descent” of the Hermitian bundle $M$ to the abelian variety $X$, (or to other ones $X'=[\hat{X}/\text{another lattice}]$, and the “descent” of holomorphic sections of $M$ to holomorphic sections of its descended form. Thus we get:

**Proposition 1.** There is a $1\!-\!1$ correspondence between

1. Hermitian-analytic line bundles $L'$ on $X$ such that $p^* L' \cong M$,

2. subgroups $K \subset \mathcal{G}(M)$, such that $K \cap C^*_1 = \{1\}$ whose image in $\hat{X}$ is $A = \text{ker}(p: \hat{X} \to X)$.

Moreover, the holomorphic sections of $M$ of the form $p^* (s'), s' \in \Gamma(X, L')$, are exactly those sections $s$ which are invariant under $K$, i.e.,

$$s = T^*_x (\phi (s)), \quad \text{all } (x, \phi) \in K.$$ 

**Proof.** Straightforward.

Finally, via the canonical trivialization of $M$, holomorphic sections of $M$ correspond to holomorphic functions on $\hat{X}$: thus each section $s \in \Gamma(X, L)$ defines a holomorphic function on $\hat{X}$. These are the classical theta-functions.

As far as moduli are concerned, the simplest and most basic result is the following: we set out to classify triples consisting of —
1. a complex vector space $Y$, of dimension 2;
2. an analytic, Hermitian line bundle $M$ on $Y$, with curvature form $E = \text{Im} \ H$, $H$ positive definite.
3. Parametrized lattices in $Y$, i.e., monomorphisms 
\[ \alpha: \mathbb{Z}^g \rightarrow Y \]
such that 
\[ E(x, y) = (x_1 \cdot y_2 - x_2 \cdot y_1) \]
if 
\[ x = (x_1, x_2), \quad y = (y_1, y_2). \]

Such triples arise if we start with a principally polarized abelian variety $(X, L)$, together with a symplectic isomorphism: 
\[ \beta: \mathbb{Z}^g \sim H_4(X, \mathbb{Z}). \]

Namely, let $Y = \hat{X}$, $M = p^* L$ with canonical Hermitian structure, and let $\beta$ define $\alpha$ via the natural maps $H_4(X, \mathbb{Z}) \cong \text{Ker} (p: X \rightarrow \hat{X}) \subset \hat{X}$. Conversely, the triple $(Y, M, \alpha)$ determines $X$ and $\beta$, and $L$ up to replacing $L$ by $T_{x}^* L$, some $x \in X$.

Let $\mathcal{S} = \text{Siegel's g x g upper half-plane}$. Then the moduli result is:

**Proposition 2.** There is a natural bijection between the set of isomorphism classes of triples $(Y, M, \alpha)$ and $\mathcal{S}$. In this bijection, $\tau \in \mathcal{S}$ corresponds to 
\[ Y = \mathbb{C}^g, \]
\[ M = 1 \quad \text{with hermitian structure} \]
\[ \|1\| (x) = e^{-\frac{\pi}{2} \langle x, B \cdot x \rangle}, \]
\[ \alpha((x_1, x_2)) = x_1 + \tau \cdot x_2 \]
where $B = (\text{Im} \ \tau)^{-1}$.

The final topic I want to discuss is the relation between the classical and algebraic theories. Let's start with:

- $X$ = abelian variety;
- $L$ = symmetric, ample, degree 1 sheaf on $X$. [Assume for simplicity that $L$ is so chosen among its translates $T_{x}^* L$, $x \in X$, that its unique section is even; equivalently, that the Arf invariant of $Q$, where $e^*_Q(x) = (-1)^{Q(x)}$, is 0.]

Let
\[ L = \text{line bundle on } X \text{ whose holomorphic sections are } L; \]
\[ \hat{X} = \text{universal covering space of } X; \]
\[ V_2(X) = \text{2-Tate group of } X. \]
Also, let $A = \text{inverse image in } \hat{X} \text{ of } \text{tor}_2 \{X\}, \text{ i.e.,}$

$$\bigcup_a 2^{-a} \cdot A, \text{ if } A = \text{Ker}(p: \hat{X} \to X).$$

Then we have canonical maps:

Note that $A$ is dense in both $V_2(X)$ and $X$. We have "trivialized" $L$ when it is pulled up to $V_2(X)$ or to $X$, in § 8 and just above. Thus we have 2 distinct trivializations of $L$ on $A$. The main result is that these differ by an elementary factor:

**Theorem 3.** Let $1$ denote the trivial complex line bundle on $A$. Then the following diagram commutes:

$$\begin{array}{c}
\text{(L, pulled back to } A) \\
\text{algebraic trivialization} \\
\text{classical trivialization} \\
\text{multiplication by } a \cdot e^{-H(\cdot, \cdot)/2} \\
\downarrow \\
\text{1}
\end{array}$$

where $a \in C^*$ and $E = \text{Im}(H)$ is the curvature form of $L$.

**Proof:** Let $M_1 = p_1^* L$ be the induced line bundle on $V_2(X)$ or $\hat{X}$. Let $\psi: M_1 \to 1$ be the classical trivialization. The algebraic trivialization of $M_1$ is based on finding a distinguished collection of isomorphisms

$$\varphi_a: M_1 \to T_a^* M_1,$$

all $a \in V_2(X)$. In fact, let $i = \text{inverse map in all our groups, and let } \rho: M_i \to T_i^* M_i \text{ be the isomorphism induced by the symmetry of } L. \text{ Then, for all elements } 2a \in V_2(X), \varphi_{2a} \text{ is characterized by the existence of } \varphi_a \text{ satisfying:}$

i) \(\varphi_{2a} = T_a^* \varphi_a \circ \varphi_a\)

ii) \(i^* \varphi_a \circ \rho = T_a^* [\rho \circ \varphi_a^{-1}]\),

iii) \(\varphi_a \text{ is induced by an algebraic isomorphism }\)

$$\varphi_a: (2^a L) \xrightarrow{\sim} (2^a L) \xrightarrow{T_{\rho_i(a)}^a L}$$
for some $n$, i.e., via the factorization:

\[
\begin{array}{c}
\begin{array}{c}
X \\
2^{n} \delta
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
V_2(X) \\
p_1
\end{array}
\end{array}
\xrightarrow{2^{n} \delta}
\begin{array}{c}
\begin{array}{c}
X \\
p_1
\end{array}
\end{array}
\]

But introduce, for all $a \in X$, isomorphisms $\psi_a$ from $M_2$ to $T_aM_2$ via:

\[
M_2 \xrightarrow{\psi} 1 \xrightarrow{=} T_a^* 1 \xrightarrow{=} T_a^* M
\]

where

\[
f_a(x) = e^{2iH(x, a) + H(\sigma, a)/2}.
\]

Also introduce

\[
\rho': M_2 \xrightarrow{\psi} 1 \xrightarrow{\text{canonical identification}} T_a^* 1 \xrightarrow{=} T_a^* M.
\]

One checks easily that $\psi_a$ and $\rho'$ are holomorphic and unitary isomorphisms. Therefore $\rho$ and $\rho'$ can differ only by a constant: and since both are the identity at $0 \in X$, $\rho = \rho'$. Moreover, if $a \in 2^{-n}A$, then the algebraic isomorphism $\phi_a^*: (2^* \delta)^* L \xrightarrow{\sim} (2^* \delta)^* T_{p_1(0)}^* L$, referred to in (iii) above, induces an isomorphism $\phi_a^*: M_2 \xrightarrow{\sim} T_a^* M_2$ via the factorization

\[
\begin{array}{c}
\begin{array}{c}
X \\
2^{n} \delta
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
V_2(X) \\
p_1
\end{array}
\end{array}
\xrightarrow{2^{n} \delta}
\begin{array}{c}
\begin{array}{c}
X \\
p_1
\end{array}
\end{array}
\]

Since $\phi_a^*$ is also holomorphic and unitary, it differs from $\psi_a$ only by a constant. Next, note that $\{f_a\}$ satisfy the identities:

i). $f_{a}(x) = f_{a}(x + a) \cdot f_{a}(x)$,

ii). $f_{a}(-x) = f_{a}(x - a)^{-1}$.

These translate readily into the identities on the $\{\psi_a\}$:

i). $\psi_a = T_a^* \psi_a \circ \psi_a$.

ii). $T_a^* \psi_a \circ \rho = T_{-a}^*[\rho \circ \psi_a^{-1}]$.

Finally, i), ii), plus the fact that $\phi_a^*$ induces $\psi_a$, shows that $\psi_a$ and $\phi_a^*$ induce the same isomorphism of $L$ on $A_2$, with $T_a^*(L)$ on $A_2$, all $a \in A_2$.

Finally, to compare the 2 trivializations, start with the unit section 1 of $L$ on $A_2$. This goes over, via the algebraic trivialization, to a section $s$ of $L$ on $A_2$ such that, for all $a \in A_2$,

\[
s(a) = \phi_a(0)[s(0)]
\]
\( \phi_a(0) \) is the induced isomorphism from the fibre \( L_0 \) or \((M_1)_0 \) to the fibre \( L_{p_1(a)} \) or \((M_1)_a \). But under the classical trivialization \( \psi \), \( \psi_a(0) \) corresponds to the isomorphism of fibres:

\[
\begin{array}{ccc}
\mathbf{1}_0 & \xrightarrow{\text{mult. by } e^{\pi i/2 H(s,a)}} & \mathbf{1}_0 \\
\| & & \| \\
C & & C.
\end{array}
\]

Therefore, the section \( s \) goes over, under the classical trivialization, to a section of \( 1 \) which, if it has value \( x \) at 0, has value

\[ a \cdot e^{\pi i/2 H(s,a)} \]

at \( a \). All in all, the section 1 of \( 1 \) has gone into the section

\[ g(a) = a \cdot e^{\pi i/2 H(s,a)} \]

of 1. \( \Box \).

**Corollary.** If the unique section \( s \) of \( L \) (up to scalars) defines

a) the holomorphic function \( \Theta \) on \( \hat{X} \) via the classical trivialization,

b) the 2-adic theta-function \( \Theta \) on \( V_2(X) \) via the algebraic trivialization, then

\[ \Theta(x) = a \cdot e^{\frac{\pi}{2} H(x,x)} \cdot \Theta_a(x) \]

all \( x \in A \).

To calculate \( \Theta \) and hence \( \Theta_a \) by analytic means, we must know the "descent data"

\[ K = \mathcal{G}(M_2) \]

that defines \( L \) on \( X \). Let \( e_\#: \frac{1}{2} \mathbb{Z}/\mathbb{Z} \to \{ \pm 1 \} \) be the quadratic character defined by \( L \). Then, as we saw in § 8, the descent data for the pull-back \( M_1 \) of \( L \) is the group:

\[ \{(x, \phi) \mid x \in A \cdot \mathbb{Z}_2, \phi = e_\#(\frac{1}{2} x) \cdot \phi_x \}. \]

In view of the proof of the theorem, this implies that

\[ K = \{(x, \psi) \mid x \in A, \psi = e_\#(\frac{1}{2} x) \cdot \psi_x \}. \]

(Notation as in proof of Theorem). Now a \( K \)-invariant section \( s \) of \( M_2 \)

is one which satisfies \( T_\#(s) = \phi(s) \), all \((a, \phi) \in K \). Going back to the definition of \( \psi_a \), one sees that if \( f = \psi(s) \) is the function on \( X \) corresponding to \( s \), then \( f \) is \( K \)-invariant if and only if

\[ f(x + a) = e_\#(\frac{1}{2} a) f_\#(x) \cdot f(x) \]

all \( x \in \hat{X}, a \in A \). From this it follows that \( \Theta_a \) must be the unique holomorphic function satisfying \((\ast)\).
To go further and write down this $\Theta_\kappa$ as an infinite series, it is convenient to introduce coordinates. Let
\[ i: \mathbb{Z}^n \rightarrow A \] be a symplectic isomorphism.

Coordinatize $\hat{X}$ via
\[ \hat{X} \cong C^\ell \]
so that $i((n_1, 0)) = n_1$, and let $\tau$ be the $g \times g$ matrix defined by
\[ i((0, n_2)) = \tau \cdot n_2. \]

Because of our assumption on $e_\kappa^l$, hence on $e_\kappa$, if we choose coordinates correctly, we can assume that
\[ e_\kappa \left[ i((n_1, n_2)) \right] = (-1)^{n_1 \cdot n_2}. \]

As we saw in Prop. 2, if we now express:
\[ H(z, \bar{z}) = z \cdot B \cdot \bar{z} \]
then $B = (\text{Im } \tau)^{-1}$. Finally, set
\[ \Theta_\kappa(z) = e^{\frac{1}{2}z \cdot B \cdot \bar{z}} \cdot \sum_{n \in \mathbb{Z}^\ell} e^{2\pi i \left( \frac{1}{2}n \cdot \tau \cdot n + n \cdot z \right)}. \]

It is easy to check that this is a holomorphic function satisfying (*). Therefore, this is the sought-for theta-function. Combining this with the Corollary, we find
\[ \Theta_\kappa(z) = e^{\frac{1}{2}z \cdot B \cdot \bar{z}} \cdot \sum_{n \in \mathbb{Z}^\ell} e^{2\pi i \left( \frac{1}{2}n \cdot \tau \cdot n + n \cdot z \right)} \quad \text{all } z \in \bigcup_{k} 2^{-k} A. \]

If
\[ z = i((z_1, z_2)), \quad z_k \in \bigcup_{k} 2^{-k} \cdot (Z^\ell), \]
then after rearranging, one finds
\[ \Theta_\kappa(z_1, z_2) = e^{-x_i z_1 \cdot z_2} \cdot \sum_{n \in \mathbb{Z}^\ell} e^{2\pi i \left( \frac{1}{2}n \cdot \tau \cdot n + n \cdot a_1 \right)}. \]

The function so defined clearly extends to a locally constant function defined for all $\alpha_1, \alpha_2 \in Q^\ell$: it is the sought-for algebraic theta function defined in § 8. Comparing this with the formula in Lemma 1, § 8, expressing $\Theta_\kappa$ in terms of the finitely additive measure $\mu$ on $Q^\ell$, we also get an analytic description for $\mu$:
\[
\begin{cases}
\mu \text{ is countably additive}, \\
\mu = \sum_{x \in D} e^{i \delta_x}, \\
\delta_x = \text{delta measure at } x, \\
D = \bigcup_{k} 2^{-k} Z^\ell.
\end{cases}
\]

17 Inventiones math., Vol. 3
References


Department of Mathematics
Harvard University
Cambridge, Massachusetts

(Received February 20, 1967)