On the Equations Defining Abelian Varieties. III

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§ 10. Non-Degenerate Theta Functions

The third part of this paper is devoted (1) to a complete description of the boundary of the moduli space for abelian varieties described in § 9, and (2) to connecting our theory with the classical theory of theta functions. We begin by defining a theta function in a coordinate-free manner and investigating how and under what non-degeneracy restrictions we can construct a tower of abelian varieties having this as its theta function. Our goal is to find an inverse to the moduli map $\Theta$ described in § 9.

Fix

o) an algebraically closed field $k$, $\text{char}(k)+2$;

i) a $2g$-dimensional vector space $V$ over $\mathbb{Q}$;

ii) a skew-symmetric bi-multiplicative map:

$$e: V \times V \to \{2^{n\text{-th roots of 1 in } k}\},$$

i.e.,

$$e(\alpha, \alpha) = 1$$
$$e(\alpha \cdot \beta, \gamma) = e(\alpha, \gamma) \cdot e(\beta, \gamma)$$
$$e(\alpha, \beta \cdot \gamma) = e(\alpha, \beta) \cdot e(\alpha, \gamma);$$

iii) a maximal isotropic lattice $\Lambda \subset V$ (i.e., a compact, open subgroup such that $e(\alpha, \beta) = 1$, all $\alpha, \beta \in \Lambda$, maximal with this property);

iv) a quadratic character

$$e_*: \frac{1}{2} \Lambda/\Lambda \to \{-1, 1\}$$

such that

$$e_*(\alpha + \beta) e_*(\alpha) e_*(\beta) = e(\alpha, \beta)^2;$$

all $\alpha, \beta \in \frac{1}{2} \Lambda$.

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We assume, however, that via a suitable isomorphism \( V \cong \mathbb{Q}_2^2 \), \( \Lambda \cong \mathbb{Z}_2^2 \), and \( e, \ e'_e \) have the form defined in \( \S \, 9 \). In fact, this is nearly always the case: if we write

\[ e_e(z) = (-1)^{Q(z)} \]

where \( Q \) is a quadratic form on \( \frac{1}{2} \Lambda / \Lambda \) with values in the field \( F_2 = \{0, 1\} \), then \( Q \) has an Art invariant \( A(Q) \in F_2 \). It is not hard to show that \( (V, \Lambda, e, e'_e) \) has the required form only if \( A(Q) = 0 \). We leave this point to the reader.

**Definition 1.** A \( \theta \)-\textit{function} \( \Theta \) on \( V \) is a map \( \Theta : V \to k \) satisfying

i) \( \Theta(\alpha + \beta) = e_e(\beta/2) \cdot e(\beta/2, \alpha) \Theta(\alpha) \), all \( \alpha \in V, \beta \in \Lambda \),

ii) \( \Theta(-\alpha) = \Theta(\alpha) \), all \( \alpha \in V \),

iii) \( \prod_{\eta \in \frac{1}{2} \Lambda / \Lambda} \Theta(\alpha) = 2^{-\gamma} \sum_{\eta \in \frac{1}{2} \Lambda / \Lambda} e(\gamma, \eta) \cdot \prod_{i=1}^{4} \Theta(\alpha_i + \gamma + \eta) \)

if \( \gamma = -\frac{1}{2} \sum \alpha_i, \alpha_1, \ldots, \alpha_4 \in V \) arbitrary.

If we let

\[ S_0 = \{ \alpha | \Theta(\alpha) \neq 0 \} = \text{support} \, (\Theta), \]

then \( S_0 \) is a union of cosets of \( \Lambda \). The structure of \( S_0 \) is a "fine" property of \( \Theta \), so we introduce:

**Definition 2.** The coarse support \( S_1 \) of \( \Theta \) is:

\[ S_1 = \{ \alpha | \Theta(\alpha + \eta) \neq 0 \text{ for some } \eta \in \frac{1}{2} \Lambda \}. \]

We will see in \( \S \, 11 \) that the coarse support \( S_1 \) of a \( \theta \)-function is either all of \( V \), or \( \frac{1}{2} \Lambda + W \) where \( W \subseteq V \) is a proper subvectorspace. This is the essential difference between good and bad \( \theta \)-functions.

Note that \( S_0 = -S_0 \) and \( S_1 = -S_1 \). We always assume, in what follows, that \( \Theta \neq 0 \), i.e., \( S_0 \neq \emptyset \).

1. If \( x_1 \notin S_1, x_2, x_3, x_4 \in S_0 \), then \( 2x_1 + x_2 + x_3 + x_4 \notin S_0 \).

**Proof.** Use the quartic relation on \( \Theta \), with \( x_1 = 2x_1 + x_2 + x_3 + x_4 \), \( x_2 = x_2, x_3 = x_3, x_4 = x_4, \gamma = -x_1 - x_2 - x_3 - x_4 \). \( \quad Q.E.D. \)

2. \( 0 \in S_1 \).

**Proof.** Assume \( 0 \notin S_1 \). Take any \( y \in S_0 \). Apply (1) with \( x_2 = x_3 = y, x_4 = -y \) and we get a contradiction. \( \quad Q.E.D. \)

3. \( x, y \in S_0 \Rightarrow \frac{1}{2}(x+y) \in S_1 \).

**Proof.** Apply (1) with \( x_1 = \frac{1}{2}(x+y), x_2 = x, x_3 = -y \) and \( x_4 = -x \). \( \quad Q.E.D. \)
Because of (2.), there is an \( \eta_0 \geq \frac{1}{4} A \) such that \( \Theta(\eta_0) \geq 0 \). Fix one such \( \eta_0 \).

4. \((0) \in (S_0 + \eta_0) \subset (2S_0 + A) \subset (4S_0 + A) \subset \cdots \).

Proof. By (3), if \( x \in S_0 \), then \( \frac{1}{2}(x + \eta_0) \in S_1 \), so \( x + \eta_0 \in 2S_0 + A \). This gives the 1st inclusion. This also shows that \( 2x \in 4S_0 + A \). Hence if \( y = 2^k \cdot x \), \( x \in S_0 \), then \( 2^k \cdot x \in 2^{k+1} S_0 + A \). This gives the rest of the inclusions. \( Q.E.D. \)

Definition 3.

\[ S_\infty = \bigcup_{k \geq 1} [2^k S_0 + A] . \]

5. \( S_\infty \) is a group.

Proof. Let \( x, y \in S_\infty \). Now \( x, y \in (2^l \cdot S_0 + A) \) for some \( l \geq l_0 \). Then \( x = 2^l \cdot x_0 + \eta, y = 2^l \cdot y_0 + \zeta \), \( x_0, y_0 \in S_0 \) and \( \eta, \zeta \in A \). Therefore by (3), \( \frac{1}{2}(x_0 + y_0) \in S_1 \), hence \( 2^l(x_0 + y_0) \in 2^{l+1} \cdot S_0 + A \). Therefore \( x + y \in (2^{l+1} S_0 + A) \subset S_\infty \). \( Q.E.D. \)

6. \( S_\infty = W + A \), for some subvector space \( W \subset V \).

Proof. This is easily seen to be equivalent to asserting that \( S_\infty / A \) is a divisible subgroup of \( V / A \). But if \( x \in 2^k \cdot S_0 + A \), then \( x = 2^k \cdot x_0 + \eta \), \( x_0 \in S_0 \), \( \eta \in A \), hence \( x - \eta \in 2 \cdot 2^{k-1} S_0 \subset 2 \cdot S_\infty \), i.e., the image of \( x \) in \( S_\infty / A \) is divisible by 2. \( Q.E.D. \)

Definition 4. A theta function is non-degenerate if equivalently:

(a) \( S_\infty = V \).

(a') \( S_\infty \geq \frac{1}{4} A \).

(a'') For all sufficiently large \( n \), \( 2^n \cdot S_0 + A \geq \frac{1}{4} A \).

(a') For all sufficiently large \( n \), and \( x \in 2^{-n-1} A \), there is an \( \eta \geq 2^{-n} A \) such that \( \Theta(x + \eta) \geq 0 \).

The next step is to form, via the function \( \Theta \), a sequence of graded rings:

Definition 5. If \( M \) is a vector space of \( k \)-valued functions on \( V \), let

\[ \mathcal{S}(M) = \bigoplus_{n=0}^{\infty} \mathcal{S}_n(M) , \]

where \( \mathcal{S}_0(M) = k, \mathcal{S}_1(M) = M, \) and \( \mathcal{S}_n(M), \) for \( n \geq 2 \), is the vector space of functions on \( V \) spanned by the products \( f_1 \cdots f_n \), \( (f_j \in M, \) all \( j) \).

Another convenient notation is the following:

\[ M^* = \left\{ \text{set of functions } x \mapsto f(x/2) , \right\} \]

In particular, let

\[ M_{2^k} = \text{span of the functions } \Theta(\beta) , \quad \text{all } \beta \in 2^{-k} A \]
where

$$\Theta_{(\alpha)}(\beta) = e(\beta/2, \alpha) \cdot \Theta(\alpha - \beta).$$

The corresponding rings $\mathcal{R}(M_{2k})$ will be the heart of our analysis. These are only half of the rings we need, however. To define the others, choose a decomposition:

$$A = A_1 \oplus A_2$$

such that $Q_2 \cdot A_1 = V_i$ is an isotropic subspace under $e$, and such that $e_*(\alpha/2) = 1$ for all $\alpha \in A_1$ or $A_2$. This exists because if we choose coordinates $V \equiv \mathbb{Q}_2^e$ such that $A_i$, $e_i$ take their standard forms, then $A_1 = Z_2^* \times \{0\}$, $A_2 = \{0\} \times Z_2^*$ have these properties. In terms of $A_1$ and $A_2$, we now define a kind of “dual” theta-function $\phi$. It is to satisfy the equations:

$$\sum_{\zeta \in A_1/A_1} e(x, \zeta) \cdot \Theta(x + \beta + \zeta) \cdot \Theta(x - \beta + \zeta) = \phi(x) \cdot \phi(\beta)$$

all $x, \beta \in V$. In fact, if we let $\Phi(x, \beta)$ denote the left-hand side of this equation, then the quartic equations on $\Theta$ are equivalent to:

$$\Phi(x, \beta) \cdot \Phi(y, \delta) = \Phi(x, \delta) \cdot \Phi(y, \beta)$$

for all $x, \beta, y, \delta \in V$ (cf. proof of Lemma 2, § 8). This, plus the elementary fact $\Phi(x, \beta) = \Phi(\beta, x)$ implies that one and (up to scalars) only one such $\phi$ exists. Notice that $\phi$ satisfies the equations:

(i) $\phi(x + \beta) = f_\ast(\beta) \cdot e(\beta, x) \cdot \phi(x)$, for all $x \in V$, $\beta \in \frac{1}{2} A_1 + A_2$, if

$$f_\ast(\frac{1}{2} \beta_1 + \beta_2) = e(\frac{1}{2} \beta_1, \beta_2)(\beta_1 \in A_i).$$

(ii) $\phi(-x) = \phi(x)$, all $x \in V$,

as well as certain quartic equations. Now let

$$M_{2k+1} = \text{span of the functions } \phi_{(\beta)}, \quad \beta \in 2^{-k-1} A$$

where

$$\phi_{(\beta)}(x) = e(\beta, x) \cdot \phi(x - \beta).$$

**Proposition 1.** 1. $\mathcal{R}_1(M_{2k}) \subseteq M_{2k+1}$, equality holding if and only if for all $\beta \in 2^{-k-1} A$, $\exists \gamma \in 2^{-k} A$ such that $\phi(\beta + \gamma) = 0$.

2. $\mathcal{R}_2(M_{2k+1}) \subseteq M_{2k+2}$, equality holding if and only if for all $\beta \in 2^{-k-1} A$, $\exists \gamma \in 2^{-k} A$ such that $\Theta(\beta + \gamma) = 0$.

**Proof.** To compute $\mathcal{R}_2(M_{2k})$, note that it is spanned by the functions:

$$f(x) = \sum_{\eta \in \frac{1}{2} A_1/A_1} e\left(\eta, \frac{\beta_1 + \beta_2}{2}\right) \cdot \Theta_{(\beta_1 - \eta)}(x) \cdot \Theta_{(\beta_2 - \eta)}(x)$$
where \( \beta_i \in 2^{-k} A \). But
\[
f(x) = e \left( \frac{\beta_1 + \beta_2}{2}, \alpha \right) \cdot \sum_{\eta \in A_{1}/A_2} e \left( x - \frac{\beta_1 + \beta_2}{2}, \eta \right) \times \Theta(\alpha - \beta_1 + \eta) \cdot \Theta(\alpha - \beta_2 + \eta)
\]
\[
= e \left( \frac{\beta_1 + \beta_2}{2}, \alpha \right) \cdot \phi \left( x - \frac{\beta_1 + \beta_2}{2} \right) \cdot \phi \left( \frac{\beta_1 - \beta_2}{2} \right)
\]
\[
= \phi \left( \frac{\beta_1 + \beta_2}{2} \right) \cdot \phi \left( \frac{\beta_1 - \beta_2}{2} \right) \in M_{2k+1}.
\]

We get every \( \phi_{(\gamma)}, \gamma \in 2^{-k-1} A \), in this way, if and only if every such \( \gamma \) can be written:
\[
\gamma = \frac{\beta_1 + \beta_2}{2}, \quad \beta_i \in 2^{-k} A
\]
such that
\[
\phi \left( \frac{\beta_1 - \beta_2}{2} \right) \neq 0.
\]

This is exactly the condition in (1). To prove (2), first notice the identity:
\[
(\alpha) \sum_{\zeta \in A_2/A_2} e(\alpha, \zeta)^2 \cdot \phi(\alpha + \beta + \zeta) \cdot \phi(\alpha - \beta + \zeta)
\]
\[
= \sum_{\zeta \in A_2/A_2} e(\alpha, \zeta)^2 \cdot e(\alpha + \beta + \zeta, \eta) \cdot \Theta(2\alpha + 2\zeta + \eta) \cdot \Theta(2\beta + \eta)
\]
\[
= \sum_{\eta \in A_1/A_1} \Theta(2\alpha + \eta) \cdot \Theta(2\beta + \eta) \cdot e(\alpha + \beta, \eta) \cdot \left[ \sum_{\zeta \in A_2/A_2} e(2\zeta, \eta) \right]
\]
\[
= 2^s \cdot \Theta(2\alpha) \cdot \Theta(2\beta).
\]

Now \( \mathcal{S}_2(M_{2k+1})^* \) is spanned by the various functions:
\[
f(\alpha) = \sum_{\eta \in A_2/A_2} e(\eta, \beta_1 + \beta_2) \cdot \phi_{(\beta_1 - \eta)}(\alpha/2) \cdot \phi_{(\beta_2 - \eta)}(\alpha/2)
\]
where \( \beta_i \in 2^{-k-1} A \). But this \( f \) comes out as:
\[
f(\alpha) = 2^s \cdot \Theta_{(\beta_1 + \beta_2)}(\alpha) \cdot \Theta(\beta_1 - \beta_2) \in M_{2k+2}.
\]

(2) now follows just like (1). \( Q.E.D. \)

**Corollary.** If \( \Theta \) is non-degenerate, then for all \( k \gg 0 \),
\[
\mathcal{S}_2(M_{2k}) = M_{2k+1}
\]
\[
\mathcal{S}_2(M_{2k+1})^* = M_{2k+2}.
\]
Proof. The \(2^{nd}\) equality is clear, by condition (\(a''\)) of the definition of non-degenerate. As for the first, note that by formula (\(a\)) in the proof of the Proposition,

\[
2^{\ell} \Theta(x)^2 = \sum_{\zeta \in A_2/A_2} e(x, \zeta) \cdot \phi(x + \zeta) \cdot \phi(\zeta).
\]

Therefore, \(\Theta(x) \neq 0\) \(\Rightarrow\) \(\phi(x + \zeta) \neq 0\), some \(\zeta \in A_2/A_2\). Thus the non-degeneracy of \(\Theta\) implies the same for \(\phi\), and the \(1^{st}\) equality follows too.

Q.E.D.

In the following discussion, we shall assume that \(\Theta\) is non-degenerate. As usual, if \(R = \Sigma R_n\) is a graded ring, then \(\mathcal{R}(2)\) is the graded ring \(\Sigma R_{2n}\). The Corollary shows that there exists a \(k_0\) such that for all \(k \geq k_0\),

\[(\beta)\quad \mathcal{R}(M_k)(2) \cong \mathcal{R}(M_{k+1}).\]

In particular, the corresponding schemes

\[X = \text{Proj}(\mathcal{R}(M_k)),\]

for \(k \geq k_0\), are all canonically isomorphic. We shall prove eventually that this \(X\) is an abelian variety.

So far, we know that \(\mathcal{R}(M_k)\) is finitely generated over \(k\). Moreover, it has no nilpotents: if it did, it would have a homogeneous nilpotent element \(f \in \mathcal{R}(M_k)\). Then \(f \neq 0 \Rightarrow f(x) \neq 0\), some \(x \in V \Rightarrow f^N(x) \neq 0\), all \(N \Rightarrow f^N \neq 0\) in \(\mathcal{R}(M_k)\). Therefore, \(X\) is a reduced algebraic scheme over \(k\).

In fact, we can map

\[V/A \rightarrow X\]

by evaluating functions in \(\mathcal{R}(M_k)\) at points of \(V\). To be more precise, for all \(x \in V\), define a homogeneous prime ideal \(P(x) \subset \mathcal{R}(M_{2k})\) [resp. \(P(x) \subset \mathcal{R}(M_{2k+1})\)] by:

\[P(x) = \sum_n P_n(x)\]

\[P_n(x) = \{ f \in S_n(M_{2k}) | f(2^n x) = 0 \}\]

resp.

\[= \{ f \in S_n(M_{2k+1}) | f(2^n x) = 0 \}.\]

It is easy to check that for all \(k\), if the \(P(x)\) in \(\mathcal{R}(M_k)\) is intersected with \(\mathcal{R}(M_k)(2)\), the resulting ideal is equal to the \(P(x)\) in \(\mathcal{R}(M_{k+1})\) under the isomorphisms (\(\beta\)). For this reason, we omit a \(k\) in the notation \(P(x)\). Thus \(P(x)\) gives a well-defined point \(P(x) \in X\). It follows easily from the definition that:

a) \(P(x)\) is a \(k\)-rational point of \(X\),

b) \(P(x + \beta) = P(x)\), if \(\beta \in A\).
Moreover:

(3) \( \{ \overline{P}(x) | x \in V \} \) is dense in \( X \).

**Proof of (3).** Take \( 2k \geq k_0 \). If \( (3) \) were false, for large \( n \), there would be a non-zero function \( f \in \mathcal{S}_n(M_{2k}) \) that vanished at all \( \overline{P}(x) \)'s. But \( f(\overline{P}(x)) = 0 \iff f(2^k x) = 0 \), so \( f \) would vanish everywhere on \( V \), hence \( f = 0 \). \( \Box \).

One can do even more: for \( \alpha \in V \), I claim that there is an automorphism \( T_\alpha : X \to X \) such that \( T_\alpha(\overline{P}(\beta)) = \overline{P}(\alpha + \beta) \), all \( \beta \in V \). To construct \( T_\alpha \), let \( k_1 \) be the least integer such that \( 2^{k_1} \alpha \in A \). Define

\[
T_\alpha^* : \mathcal{S}(M_{2k}) \to \mathcal{S}(M_{2k})
\]

resp.:

\[
\mathcal{S}(M_{2k+1}) \to \mathcal{S}(M_{2k+1})
\]

by:

\[
T_\alpha^* f(\beta) = e(\beta, 2^k x) \cdot f(\beta + 2^k x), \quad \text{all } f \in \mathcal{S}_n(M_{2k})
\]

resp.

\[
e(\beta, 2^k x) \cdot f(\beta + 2^k x), \quad \text{all } f \in \mathcal{S}_n(M_{2k+1})
\]

(where we assume \( k \geq k_1 \)). To check that this is, indeed, an automorphism of \( \mathcal{S}(M_{2k}) \) [resp. \( \mathcal{S}(M_{2k+1}) \)], it suffices to check that \( T_\alpha^* \mathcal{S}_n(\Theta_{[\gamma]} \in M_{2k+1}) \), all \( \gamma \in 2^{-1} A \); and \( T_\alpha^* \mathcal{S}_n(\Theta_{[\gamma]} \in M_{2k+1}) \), all \( \gamma \in 2^{-k} A \). But, in fact, one computes:

\[
T_\alpha^* \Theta_{[\gamma]} = e_\alpha(2^{k-1} x) \cdot e(\gamma, 2^k x) \cdot \Theta_{[\gamma]}
\]

(\( \gamma \))

\[
T_\alpha^* \phi_{[\gamma]} = f_\alpha(2^k x) \cdot e(\gamma, 2^{k+1} x) \cdot \phi_{[\gamma]}
\]

Moreover, one finds that \( T_\alpha^* \), acting on \( \mathcal{S}(M_k) \), induces the same automorphism on \( \mathcal{S}(M_\lambda) \) (2) that you get by considering the \( T_\alpha^* \) acting on \( \mathcal{S}(M_{k+1}) \) and carrying it across via the isomorphisms \( (\beta) \) of \( \mathcal{S}(M_k) \) (2) and \( \mathcal{S}(M_{k+1}) \). Therefore, the \( T_\alpha^* \)'s all define one and the same automorphism \( T_\alpha \) of \( X \). Note that:

d) \( (T_\alpha^*)^{-1} (P(\beta)) = P(\alpha + \beta) \).

**Proof.** If \( f \in \mathcal{S}_n(M_{2k}) \) or \( \mathcal{S}_n(M_{2k+1}) \), then

\[
T_\alpha^* f \in P(\beta) \iff T_\alpha^* f(2^k x) = f(2^k x + 2^k \beta) = 0 \iff f \in P(\alpha + \beta),
\]

hence

\[
d') T_\alpha(\overline{P}(\beta)) = \overline{P}(\alpha + \beta).
\]

One checks also (via (\( \gamma \)) if you like) that:

\( e ) T_{\alpha_1 + \alpha_2} = T_{\alpha_1} \circ T_{\alpha_2}, \)

\( f ) T_\alpha = \text{id} \iff \alpha \in A, \)

so that \( T \) is a faithful action of the group \( V/A \) on the scheme \( X \).

A remarkable consequence of all this is:

**Proposition 2.** If \( \Theta \) is non-degenerate, then \( \mathcal{S}(M_\lambda) \) is an integral domain, for all \( k \).
Proof. We show first that $\mathcal{S}(M_k)$ is a domain if $k \geq k_0$. Since $\mathcal{S}(M_k)$ has no nilpotents, this is equivalent to showing that $X$ is irreducible. Now $V/A$ acts on $X$, so it permutes the various components of $X$, i.e., we have a homomorphism:

$$V/A \rightarrow S = \left\{ \text{gp. of permutations} \right\} \cup \left\{ \text{of components of } X \right\}.$$  

But $S$ is a finite group and $V/A$ is a divisible group. So $V/A$ must map each component $X_i$ into itself. On the other hand, the collection of points $\{P(x)\}$ forms a single orbit of the action of $V/A$ on $X$. Therefore, all these points $\{P(x)\}$ belong to a single component of $X$. Since they are also dense in $X$, $X$ can have only one single component. Therefore $\mathcal{S}(M_k)$ is a domain if $k \geq k_0$.

In general, suppose some $\mathcal{S}(M_k)$ were not a domain. Then there would be homogeneous elements $f \in \mathcal{S}(M_k)$, $g \in \mathcal{S}(M_k)$ such that $f \cdot g = 0$, $f \neq 0$, $g \neq 0$. Now $f^2$ and $g^2$ can be considered as elements of $\mathcal{S}(M_{k+1})$. Since $f \cdot g = 0$, we still have $f^2 \cdot g^2 = 0$. Also, since $\mathcal{S}(M_k)$ has no nilpotents, $f^2 \neq 0$ and $g^2 \neq 0$. Therefore $\mathcal{S}(M_{k+1})$ is not a domain either. Continuing in this way, we find that $\mathcal{S}(M_l)$ is not a domain for all $l \geq k$, which contradicts the first part of the proof. Q.E.D.

**Corollary 1.** The following are equivalent:

i) $\Theta$ is non-degenerate,

ii) $S_1 = V$, i.e., for all $\alpha \in V$, $\exists \eta \in \frac{1}{2}A$ such that $\Theta(\alpha + \eta) \neq 0$.

iii) For all $\alpha \in \frac{1}{2}A$, $\exists \eta \in \frac{1}{2}A$ such that $\Theta(\alpha + \eta) \neq 0$.

**Proof.** Clearly (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i). Now assume (i) holds. If $\Theta(\alpha + \eta) = 0$, all $\eta \in \frac{1}{2}A$, then it would follow from the definition of $\phi$ that $\phi(\alpha + \beta) \cdot \phi(\beta) = 0$, all $\beta \in V$. But this means that $\phi_{[\alpha] \cdot \phi_{[\beta]} = 0$, i.e., one of the rings $\mathcal{S}(M_{k+1})$ is not domain. This contradicts the Prop., so (ii) must hold. Q.E.D.

**Corollary 2.** $\mathcal{S}(M_k) (2) \cong \mathcal{S}(M_{k+1})$, for all $k \geq 2$.

**Proof.** In view of Prop. 1, this follows from Cor. 1 provided that we check: $\forall \alpha \in V$, $\exists \eta \in \frac{1}{2}A$ such that $\phi(\alpha + \eta) \neq 0$. Looking back at the proof of the Cor. to Prop. 1, you see that this too follows from Cor. 1. Q.E.D.

To show that $X$ is actually an abelian variety, we could either define the group law explicitly, using the addition formula of § 2, or else we can use only the action of $V/A$ on $X$ and combine this with general structure theorems on the automorphisms of a variety. Although the former is more elementary, we follow the latter approach as it is quicker.

$X$ is given to us together with a projective embedding. For example, $X = \text{Proj} (\mathcal{S}(M_2))$, so

$$X \subset P(M_2).$$
Let $L_2$ be the invertible sheaf induced on $X$ via this embedding. If, via the isomorphism $X \cong \text{Proj}(\mathcal{O}(M_2))$, we embed $X$ in $P(M_2)$, the induced sheaf $L_k$ is just:

$$L_k \cong L_2^{2^{k-2}}.$$ 

Let $\mathcal{O}$ denote the family of all invertible sheaves algebraically equivalent to $L_2$. We shall use the fact that $\text{Aut}(X, \mathcal{O})$, the group of automorphisms of the pair $X$, $\mathcal{O}$, is an algebraic group (Matsusaka [14], Grothendieck [15], p. 221–20). For all $x \in V/A$, if $2^k x \in A$, then $T_x$ is induced by an automorphism $T^*_x$ of $\mathcal{O}(M_{2^k})$; therefore $T^*_x(L_{2^k}) \cong L_{2^k}$; therefore $T^*_x(L_2)$ differs from $L_2$ by an invertible sheaf of finite order; therefore $T^{-1}_x(\mathcal{O}) = \mathcal{O}$. In other words, the action of $V/A$ on $X$ factors through an injective homomorphism:

$$V/A \to \text{Aut}(X, \mathcal{O}).$$

Let $A$ be the Zariski-closure of $V/A$ in $\text{Aut}(X, \mathcal{O})$. Then $A$ is connected since $V/A$ is divisible and dense in $A$ (cf. proof of Prop. 2), and $A$ is commutative since $V/A$ is commutative and dense in $A$. Moreover, since the $V/A$-orbit of $P_0$ is dense in $X$, the $A$-orbit of $P_0$ must be an open dense set in $X$, i.e., $A$ acts generically transitively on $X$. In fact, the morphism

$$\psi: A \to X$$

$$\sigma \mapsto \sigma(P_0)$$

is an open immersion of $A$ in $X$. This follows since the image $\psi(A)$ is always isomorphic to $A/H$, $H = \text{the stabilizer of } P_0$; and since $A$ is commutative and acting faithfully on $X$, all stabilizers are trivial.

Next, we want to compute the dimension of $X$. I claim that the Hilbert polynomial of $(X, L_2)$ is given by:

**Proposition 3.** $\chi(L_2^k) = 4^k \cdot n^k$.

**Proof.** For $k$ large,

$$\chi(L_2^{2^k}) = \dim \left( S_{2^{2^k}}(M_2) \right)$$

$$= \dim \left( M_{2^{2^k}} \right).$$

Now $M_{2^{2^k}}$ is, by definition, the span of the $2^{2^k k^2}$ functions $\Theta_{(\ell)}$, where $\ell$ runs over cosets of $2^{k-1} A/A$. But these functions are linearly independent. To see this, look at the automorphisms $T^*_x$ of $\mathcal{O}(M_{2^{2^k} k^2})$, where $x \in 2^{k-1} A$. Use formulae (5) above and note that each $\Theta_{(\ell)}$ gives rise to a distinct set of eigenvalues for the $T^*_x$'s. Therefore, the $\Theta_{(\ell)}$'s could not be dependent unless one were identically zero, and this is not the case. Therefore

$$\dim M_{2^{2^k} k^2} = 4^k \cdot (2^{2^k})^k.$$
This shows that $\chi(L^2_2)$ and $4^e \cdot n^e$ agree for an infinite set of values of $n$. Since both are polynomials, they are always equal. \textit{Q.E.D.}

**Corollary.** $\dim X = g$.

Returning to $A$, we find that $A$ is a commutative $g$-dimensional algebraic group containing a subgroup isomorphic to $(Q_2/Z_2)^2$. From well-known structure theorems on algebraic groups, the only such $A$'s are abelian varieties. Therefore $A$ is complete, hence $A = X$, hence:

(I) $X$ is an abelian variety.

Moreover, in the course of proving this, we have also found that $V/A$ is acting on $X$ via translations, hence (comparing orders) we find:

(II) $x \mapsto \bar{p}(x)$ is a group isomorphism of $V/A$ with $\text{tor}_2(X)$.

Up to this point, identifying the various $\text{Proj}(\mathcal{O}(M))$'s has been useful. But to go further, it is more convenient now to drop these identifications. Therefore, now let

$$X_n = \text{Proj}(\mathcal{O}(M_{2n})).$$

This is a family of isomorphic abelian varieties. However, the most natural maps between them are given by the inclusions:

$$M_{2n} \subset M_{2n+2}$$

$$\mathcal{O}(M_{2n}) \subset \mathcal{O}(M_{2n+2})$$

inducing finite morphisms:

$$X_n \leftarrow X_{n+1}.$$

To check that $p$ is defined, we must know that $\mathcal{O}(M_{2n+2})$ is integrally dependent on $\mathcal{O}(M_{2n})$. But I claim:

$$\Theta(\gamma)^2 \cdot \Theta_{[\beta]} = 2^{-8} \cdot \sum_{\eta \in A/A} e(\eta, \gamma) \Theta(\eta)^2 \cdot \Theta_{[\beta + \gamma - \eta]} \cdot \Theta_{[\beta - \gamma - \eta]}.$$

[Proof. $\Theta(\gamma)^2 \cdot \Theta_{[\beta]}(x)^2 = e(\beta, x) \Theta(\gamma) \Theta(\beta - x) \Theta(x - \beta)$.

By the quartic relations on $\Theta$, we get

$$= 2^{-8} e(\beta, x) \sum_{\eta} e(-\gamma, \eta) \Theta(\eta)^2 \Theta(\beta - x - \gamma + \eta) \Theta(x - \beta - \gamma + \eta)$$

$$= 2^{-8} \sum_{\eta} e(\eta, \gamma) \Theta(\eta)^2 \cdot \Theta_{[\beta + \gamma - \eta]}(\alpha) \cdot \Theta_{[\beta - \gamma - \eta]}(\alpha). \ Q.E.D.]$$

Choose $\gamma \in \beta + 1/A$ so that $\Theta(\gamma) \neq 0$. Then if $\beta \in 2^{-n-1} A$, this equation shows that $\Theta_{[\beta]}(\gamma) \in \mathcal{O}(M_{2n})$. This proves that $p$ is a finite morphism. Since $X_n$ and $X_{n+1}$ are abelian varieties, $p$ must be an isogeny.
Define prime ideals:

\[ P^{(k)}(a) \subset \mathcal{S}(M_{2k}) \]

\[ P^{(k)}(a) = \sum_{n} P_{n}^{(k)}(a) \]

\[ P_{n}^{(k)}(a) = \{ f \in \mathcal{S}_n(M_{2k}) | f(a) = 0 \}. \]

Then \( P^{(k)}(a) \) defines a \( k \)-rational point \( \psi_k(a) \in X_k \). We have

(a) \( p(\psi_{k+1}(a)) = \psi_k(a) \).

(b) \( a \mapsto \psi_k(a) \) defines an isomorphism

\[ V/2^k A \longrightarrow \text{tor}_2(X_k). \]

(b) follows from conclusion (II) above, noticing how we have reinterpreted the ideal \( P(a) \). In fact, if we call \( X \) the common abelian variety to which all the \( X_k \)'s were previously identified, then \( P(a) \in X \) corresponds exactly to \( \psi_k(2^k a) \in X_k \). Therefore \( \psi_k(a) = 0 \iff P(2^{-k} a) = 0 \iff 2^{-k} a \in A \). Moreover, this shows that via these identifications, we get a morphism:

\[
\begin{array}{c}
X \\
\downarrow \psi_k \\
X_k \\
\downarrow \psi_k(2^k a) \\
2^k \psi_k(2^k a) \\
\downarrow \\
X_k \\
\downarrow \\
X \\
\end{array}
\]

This map, from \( X \) to \( X \), agrees with \( 2 \delta \) at all points \( P(a) \). Therefore it is equal to \( 2 \delta \). In particular:

(c) The degree of \( p \) is \( 2^{2^k} \) and \( \text{Ker}(p) = \text{Ker}(2 \delta) \). It follows that all the \( X_n \)'s generate a single 2-tower. Call this \( X = \{ X_n \}_{n \in S} \), and let \( X_n = X_{\alpha_n} \), \( \alpha_n \in S \). Moreover, these \( \alpha_n \)'s are a cofinal set in \( S \), by (c). In view of (a)

\[ \alpha \mapsto \{ \psi_k(a) \} \]

defines a homomorphism

\[ \psi: V \to V(X), \]

and (b) implies that \( \psi \) is an isomorphism. More, (b) shows that the compact open subgroups \( 2^k A \) and \( T(\alpha_k) \) correspond to each other under \( \psi \).

This 2-tower is polarized too. Let \( L_k \) be the sheaf \( o(1) \) on \( X_k \) coming from its presentation as \( \text{Proj}(\mathcal{S}(M_{2k})) \). Since the \( p \)'s come from gradation preserving homomorphisms of the \( \mathcal{S}(M_{2k}) \)'s it follows that \( p^*(L_k) \cong L_{k+1} \). To check that \( L_k \) is totally symmetric, we need the inverse on \( X_k \):
Let $i^*(f)(\chi) = f(-\chi)$, all $f \in \mathcal{S}(M_{2k})$.

Then $i^*$ defines an involution

$$i: X_k \to X_k$$

such that $i(\psi_k(\chi)) = \psi_k(-\chi)$.

Therefore $i$ agrees with the inverse of $X_k$ on all points $\psi_k(\chi)$, hence $i$ is inverse of $X_k$.

Since $i$ is induced at all by an automorphism $i^*$ of $\mathcal{S}(M_{2k})$, it follows that $L_k$ is at least a symmetric sheaf. Since

$$\{\psi_k(\chi) | \chi \in 2^{k-1} / 2^k \} = \text{Kernel of } 2 \delta \text{ in } X_k,$$

$L_k$ is totally symmetric if and only if $i^*$ is the identity in $\mathcal{S}(M_{2k}) / P^{(k)}(\chi)$, all $\chi \in 2^{k-1} / 2^k$. This means that for all $f \in M_{2k}$, $i^* f - f \in P^{(k)}(\chi)$, i.e., $f(\chi) = f(-\chi)$. But $M_{2k}$ is spanned by $\Theta_{(\beta)}$'s, $\beta \in 2^{-k} \Lambda$, and if $\beta \in 2^{-k} \Lambda$, $\chi \in 2^{k-1} / 2^k$, then:

$$\Theta_{(\beta)}(-\chi) = e\left(\frac{\beta}{2}, -\chi\right) \Theta(-\chi - \beta) = e\left(\frac{\beta}{2}, \chi\right) \Theta(\chi - \beta) = \Theta_{(\beta)}(\chi).$$

Therefore all the $L$'s are totally symmetric and $\{X_k, L_k\}$ extends to a polarized 2-tower $\mathcal{T} = \{X_k, L_k\}$. We shall leave it to the reader to check the key fact that $\psi$ is symplectic:

(d) $e_{\chi}(\psi_\chi, \psi_\beta) = e(\chi, \beta)$, all $\chi, \beta \in V$.

Recapitulating this whole section so far, we have defined an arrow:

$$\Xi: \left\{ \begin{array}{l} \text{Given a non-degenerate theta function } \Theta \text{ on } V \\ \text{construct a polarized 2-tower } \mathcal{T} = \{X_k, L_k\}, \\ \text{plus a symplectic isomorphism } \\ \psi: V \cong V(X) \end{array} \right\}.$$ 

Now, on $V$ we have the vector space of functions spanned by all the $\Theta_{(\beta)}$'s. On $V(X)$, we have the vector space of all theta functions $\mathcal{S}[\Gamma(\mathcal{F})]$ of the tower $\mathcal{T}$.

**Proposition 4.** Via $\psi$, these vector spaces are equal:

Span of $\Theta_{(\beta)}$'s = $\langle \Theta_{(\beta)} \psi | \beta \in \Gamma(\mathcal{F}) \rangle$.

Moreover, $\Theta$ itself is the unique function $f$ (up to scalars) of the form $\delta_{(\alpha)} \psi$ satisfying the functional equation:

$$f(\chi + \beta) = e_{\chi}(\beta/2) \cdot e(\beta/2, \chi) \cdot f(\chi), \quad \text{all } \chi \in V, \beta \in \Lambda.$$ 

**Key Corollary 1.** If $V = \mathbb{Q}_2^x$, $\Lambda = \mathbb{Z}_2^x$, and $e$, $e_\bullet$ have the standard forms of § 9, then $\Theta$ is exactly the theta function $\mathcal{S}\left[\begin{array}{c} 0 \\ 0 \end{array}\right] \circ \psi$ associated to the
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triple \((X, \mathcal{F}, \psi^{-1})\) in § 9. In other words, \(\Xi\) is an inverse to the map \(\Theta\) of § 9.

**Proof of Prop. 4.** Let \(x \in 2^{-k_1} A\) and let \(k \geq k_1\). Define \(T^*_a : \mathcal{G}(M_{2k}) \rightarrow \mathcal{G}(M_{2k})\) slightly differently from before:

\[
T^*_a f(\beta) = e \left( \beta, \frac{y}{2} \right) \cdot f(\beta + x), \quad \text{all } f \in S_x(M_{2k}).
\]

Note \(T^*_a \Psi^{-1}(P(\beta)) = P^{(k)}(x + \beta)\). Let \(T^*_a : X_k \rightarrow X_k\) be the automorphism induced by \(T^*_a\). Then \(T^*_a(\psi_\beta(\beta)) = \psi_\beta(x + \beta)\), hence \(T^*_a\) is translation by the point \(\psi_\beta(x)\), i.e.,

\[
T^*_a = T^*_a\psi_\beta(x).
\]

Moreover, \(T^*_a\) also induces a compatible isomorphism:

\[
g_k(x) : L_k \cong T^*_a\psi_\beta(x) L_k.
\]

For all \(k \geq k_1\), these are compatible, so the totality of pairs

\[
g(x) = \{ (\psi_\beta(x), g_k(x) | k \geq k_1) \}
\]

is a point of \(\mathcal{G}(\mathcal{F})\).

\((\ast)\) \(g(x) = \sigma(\psi(x))\), i.e., \(g(x)\) is the canonical element of \(\mathcal{G}(\mathcal{F})\) over the point \(\psi(x)\) in \(\mathcal{V}(\mathcal{F})\).

**Proof of \(\ast\).** This requires checking 2 things: (i) \(g(x)\) is a symmetric element of \(\mathcal{G}(\mathcal{F})\), i.e., \(\delta_{-1} g(x) = g(x)^{-1}\), and (ii) \(g(2x) = g(x)^2\). In terms of \(T^*_a\), this is the same as:

(i) \(t^* \circ T^*_a = (T^*_a)^{-1} \circ t^*\).

(ii) \(T^*_a = T^*_a \circ T^*_a\).

These are both immediate. Q.E.D.

Next, notice that \(M_{2k} \cong \Gamma(X_k, L_k)\). In fact, there is a canonical map \(M_{2k} \rightarrow \Gamma(X_k, L_k)\); it is injective, since the ring \(\mathcal{G}(M_{2k})\) has no nilpotents, and only nilpotent elements of \(\mathcal{G}(M_{2k})\) define trivial sections of \(L_k\); but it is easy to check that both \(\dim M_{2k}\) and \(\dim \Gamma(X_k, L_k)\) are equal to \(2^{2k}\); therefore \(M_{2k} \cong \Gamma(X_k, L_k)\). Therefore,

\[
\Gamma(\mathcal{F}) = \lim_k \Gamma(X_k, L_k) \cong \bigcup_k M_{2k} = \left\{ \text{Span of all the} \right\} \left\{ \beta \in V \right\}
\]

Now let \(f\) be some linear combination of the \(\Theta_{[\beta]}\). Say \(f \in M_{2k}\). Let \(f\) define \(s \in \Gamma(X_k, L_k)\). I claim that:

\((\ast)\) \(f(x) = \theta_{[\beta]}(\psi x), \quad \text{all } x \in V\).

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Taking a larger $k_1$ if necessary, we may suppose that $x \in 2^{-k_1}A$. By definition, $(\theta_{\psi})$ at $\psi x$ is the "value" of $x_1$ at the origin of $X_{\psi}$, of the section $L_{\psi}$ obtained via the map:

$$
\Gamma(X_{\psi_1}, L_{\psi_2}) \xrightarrow{\psi_{\psi_1}(\cdot)} \Gamma(X_{\psi_2}, T^*_{\psi_2}(\cdot)) \xrightarrow{T_{\psi_2}^-} \Gamma(X_{\psi_1}, L_{\psi_1}).
$$

This means that we simply apply the automorphism $(T^*_{\psi})^{-1}$ of $M_{2k}$ to $f$, and take the value at the origin. But $T^*_{\psi} = T^*_{\psi}^{-1}$, and $(T^*_{\psi} f)(0) = f(0)$, so (\star) is proven. Thus the span of the $\Theta_{\{\psi\}}$'s is the same as the space of functions $\delta[\Gamma]\circ \psi, \psi \in \Gamma(\mathcal{F})$.

As for the final assertion of the Proposition, on the one hand, $\Theta$ does satisfy the functional equation there; and, from the general theory of the space $\delta[\Gamma(\mathcal{F})]$ in §8, we know that this functional equation has only a 1-dimensional set of solutions in $\delta[\Gamma(\mathcal{F})]\circ \psi$. Q.E.D.

**Corollary 2.** All $g$-dimensional principally polarized abelian varieties $X$ are isomorphic to $\text{Proj}(\mathcal{F}(\mathcal{M}_{\xi}))$, where $\mathcal{M}_{\xi}$ is the span of the $\Theta_{\{\psi\}}$'s, $\beta \in \frac{1}{2}A_2$, for some non-degenerate theta function $\Theta$ on $V$.

Proof. Just take $\Theta$ to be the $\theta \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ attached to $X$ as in §9, and carried over to a function on $V$ by a suitable isomorphism of $V$ and $V(\xi)$. Q.E.D.

**Corollary 3.** The open set $\mathcal{M}_{\infty} \subset \bar{\mathcal{M}}_{\infty}$, which in §9 represents the moduli functor $\mathcal{M}_{\infty}$, is the open set whose geometric points represent non-degenerate theta functions, i.e.,

$$
E = \left\{ \text{set of all systems of coset representatives} \right\},
$$

$$
\begin{bmatrix}
\frac{1}{2}Z_2^2 \\
\frac{1}{2}Z_2^2
\end{bmatrix} \rightarrow \begin{bmatrix}
\frac{1}{2}Z_2^2 \\
\frac{1}{2}Z_2^2
\end{bmatrix}
$$

For all $r \in E$, let

$$
U_r = \left\{ \text{open set in } \bar{\mathcal{M}}_{\infty} \text{ defined by } X_r = 0, \text{ all } x \in \text{Image}(r) \right\}.
$$

Then

$$
\mathcal{M}_{\infty} = \bigcup_{r \in E} U_r.
$$

§11. Satake's Compactification

In this section, I want to analyze the degenerate theta functions $\Theta$ on $V$, in the sense of §10. In particular, they all come from lower dimensional non-degenerate theta-functions via "cusps". This will show that the whole moduli scheme $\bar{\mathcal{M}}_{\infty}$ is a disjoint union of copies of the $\mathcal{M}_{\infty}$'s for dimensions $g$ and lower i.e., that $\bar{\mathcal{M}}_{\infty}$ is the Satake compactification of $\mathcal{M}_{\infty}$.\footnote{Added in Proof. A closer study has shown that $\bar{\mathcal{M}}_{\infty}$ is not normal along $\bar{\mathcal{M}}_{\infty} - \mathcal{M}_{\infty}$. Its normalization is Satake's compactification.}
Return to the discussion at the beginning of §10: let \( V, \Lambda, e, e^* \) be given as before. First, I want to describe a way of forming degenerate theta functions on \( V \) out of theta functions on lower dimensional spaces.

**Definition 1.** A cusp is a subspace \( W \subset V \) such that \( W^{\perp} \subset W \), i.e., if \( \alpha \in V \) has the property \( e(\alpha, \beta) = 1 \), all \( \beta \in W \), then \( \alpha \in W \).

Given a cusp \( W \), let:

\[
\tilde{V} = W/W^{\perp} \\
\tilde{\Lambda} = \Lambda \cap W/\Lambda \cap W^{\perp} \\
\tilde{e} = \text{induced skew-symmetric pairing, } \tilde{V} \times \tilde{V} \to \mathbb{G}^*.
\]

**Lemma.** \( \tilde{\Lambda} \) is a maximal isotropic lattice in \( \tilde{V} \), (for \( \tilde{e} \)).

**Proof.** Notice that \( \Lambda/\Lambda \cap W \) is a free \( \mathbb{Z}_2 \)-module. Therefore the sequence:

\[
0 \to \Lambda \cap W \to \Lambda \to \Lambda / \Lambda \cap W \to 0
\]

splits, and \( \Lambda = \Lambda_1 \oplus (\Lambda \cap W) \) for some sub \( \mathbb{Z}_2 \)-module \( \Lambda_1 \). Let \( V_1 = \mathcal{Q}_2 \cdot \Lambda_1 \), so \( V = V_1 \oplus W \). Now I claim:

\[
(\Lambda \cap W)^\perp = \Lambda + W^{\perp}.
\]

[In fact, let \( \alpha \in V \) satisfy \( e(\alpha, \beta) = 1 \), all \( \beta \in \Lambda \cap W \). Since \( V_1 \) and \( W \) are dual vector spaces via \( e \), there is a \( \gamma \in W^{\perp} \) such that \( e(\alpha, \beta) = e(\gamma, \beta) \) all \( \beta \in V_1 \). But then \( \alpha - \gamma \) is orthogonal to both \( V_1 \) and \( \Lambda \cap W \), hence orthogonal to \( \Lambda \), hence \( \alpha - \gamma \in \Lambda \). Thus \( \alpha \in W^{\perp} + \Lambda \).]

Now to show \( \tilde{\Lambda} \) is maximal isotropic, let \( \alpha \in W \) have an image \( \tilde{\alpha} \) in \( \tilde{V} \) perpendicular to \( \tilde{\Lambda} \), i.e., \( \alpha \in (W \cap \Lambda)^\perp \). By (•), \( \alpha = \alpha_1 + \alpha_2 \), where \( \alpha_1 \in \Lambda \), \( \alpha_2 \in W^{\perp} \). But then \( \alpha_1 = \alpha - \alpha_2 \in W \). Therefore \( \alpha_1 \in W \cap \Lambda \) so \( \tilde{\alpha} = \tilde{\alpha}_1 \in \tilde{\Lambda} \). 

Q.E.D.

**Definition 2.** A cusp with origin is a cusp \( W \subset V \), plus an element \( \eta_0 \in \frac{1}{2} \Lambda \) such that

i) \( e_*(\alpha) = e(\alpha, \eta_0)^2 \), all \( \alpha \in W^{\perp} \cap (\frac{1}{2} \Lambda) \).

ii) \( e_*(\eta_0) = 1 \).

It is not hard to check that every cusp has at least one origin: we leave this to the reader. Given a cusp with origin, look at the map

\[
\alpha \mapsto e_*(\alpha) \cdot e(\alpha, \eta_0)^2
\]

where \( \alpha \in \frac{1}{2} \Lambda \cap W \). If \( \beta \in \frac{1}{2} \Lambda \cap W^{\perp} \), then

\[
e_*(\alpha + \beta) \cdot e(\alpha + \beta, \eta_0)^2 = e_*(\alpha) \cdot e_*(\beta) \cdot e(\alpha, \beta)^2 \cdot e(\alpha, \eta_0)^2 \cdot e(\beta, \eta_0)^2
\]

\[= e_*(\alpha) \cdot e(\alpha, \eta_0)^2.
\]
Thus there is a quadratic form $\tilde{e}_a : \frac{1}{2}\tilde{A}/\tilde{A} \to \{\pm 1\}$ such that

\begin{align*}
\tilde{e}_a(\tilde{x}) &= e_a(x) \cdot e(x, \eta_0)^2, \\
&\quad \forall x \in \frac{1}{2}A \cap W.
\end{align*}

It is not hard to check that the new data $(\tilde{V}, \tilde{A}, \tilde{e}, \tilde{e}_a)$ has the standard form required in §10 (i.e., that the associated Arf-invariant is 0). We leave this to the reader also.

Now let $\tilde{\Theta}$ be a theta-function on $\tilde{V}$. 

**Definition 3.** For all $a \in V$, let

\[
T_{\tilde{V}, 0} \Theta(a) = \begin{cases} 
0 & \text{if } a \notin \eta_0 + W + A \\
\tilde{e}_a \left( \frac{\eta_1}{2} \right) e \left( \frac{\eta_1}{2}, \eta_0 \right) e \left( \frac{\eta_0 + \eta_1}{2}, \eta \right) \tilde{\Theta}(\tilde{a}_0) & \text{if } a = \eta_0 + \eta_1 + a_0, \quad \eta_1 \in A, \quad a_0 \in W.
\end{cases}
\]

**Proposition 1.** The above $T_{\tilde{V}, 0} \tilde{\Theta}$ is well-defined (note that the $a \in V$ may be decomposed in more than way as $a = \eta_0 + \eta_1 + a_0$), and is a theta-function on $V$.

The proof of this Proposition is a ghastly but wholly straightforward set of computations. It took me several hours to do every bit and as I was no wiser at the end — except that I knew the definition was correct — I shall omit details here. Our main result is:

**Theorem.** Let $\Theta$ be any theta-function on $V$, and let $W$ be the subspace of $V$ such that $S_0 = W + A$ (cf. §10). Then $W$ is a cusp, and if $\eta_0$ is any origin for $W$, $\Theta$ is equal to $T_{\tilde{V}, 0} \tilde{\Theta}$ for some non-degenerate theta-function $\tilde{\Theta}$ on $\tilde{V}$. In particular, $W$ is characterized by:

\[
\text{coarse support}(\Theta) = W + \frac{1}{2}A.\]

The proof of this theorem will be based on the $\Theta \leftrightarrow \mu$ correspondence, given in Lemma 1, §8. Before taking up the proof of the Theorem, we want to give this correspondence a more intrinsic formulation. Let $V = W_1 \oplus W_2$, where $W_i$ are maximal isotropic subspaces, such that

i) $A = A_1 \oplus A_2$, $A_i = A \cap W_i$.

ii) $e_a(x/2) = 1$, all $a$ in $A_i$ or in $A_2$.

Then

a) Define a measure $\mu$ on $W_1$, from a theta function $\Theta$ on $V$ via

\[
\mu(\alpha_1 + 2^{n_1}A_1) = 2^{-n_1} \sum_{\alpha_2 \in 2^{n_2}A_2/A_2} e \left( \alpha_1, \alpha_2, A_2 \right) \cdot \Theta(\alpha_1 + \alpha_2).
\]

b) Define a theta function $\Theta$ on $V$, from a measure $\mu$ on $W_1$, via

\[
\Theta(\alpha_1 + \alpha_2) = e \left( \alpha_1, \frac{\alpha_2}{2} \right) \int_{A_2} e(\alpha_2, \beta) \cdot d\mu(\beta).
\]
Our proof will be based on the fact that any finitely additive measure \( \mu \) (on the algebra of compact open subsets of \( W_1 \)) has a support, i.e., a smallest closed set \( S \) such that:

\[
\mu(U) = 0, \quad \text{all compact open } U's \text{ in } W_1 - S.
\]

**Proof.** Say \( S_A \) and \( S_B \) are closed sets such that \( \mu(U) = 0 \) if \( U \subseteq W_1 - S_A \) or \( U \subseteq W_1 - S_B \). Then let \( U \subseteq W_1 - (S_A \cap S_B) \) be a compact open set. We must decompose \( U \) into \( U_A \cup U_B \), where \( U_A \subseteq W_1 - S_A \) and \( U_B \subseteq W_1 - S_B \), and \( U_A \) and \( U_B \) are compact and open. For all \( x \in U \cap S_A \), note that \( x \notin S_A \), so we can find a compact, open neighborhood \( U_x \) of \( x \) such that

\[
U_x = U \cap (W_1 - S_B).
\]

Since \( U \cap S_A \) is compact, it can be covered by a finite set of these \( U_x \)'s: say

\[
U \cap S_A \subseteq \bigcup_{x_1} U_{x_1} \cup \cdots \cup U_{x_n}.
\]

Let \( U_B = U_{x_1} \cup \cdots \cup U_{x_n} \). By construction \( U_B \subseteq U \cap (W_1 - S_B) \) and \( U_B \) is compact and open. Let \( U_A = U - U_B \). Then \( U_A \) is also compact and open and since \( U_B \supseteq U \cap S_B \), it follows that \( U_A \subseteq U \cap (W_1 - S_B) \). By assumption on \( S_A \) and \( S_B \), we have \( \mu(U_A) = 0 \) and \( \mu(U_B) = 0 \). Therefore \( \mu(U) = 0 \). This shows that the family of sets:

\[
\mathcal{S} = \{ S \text{ closed in } W_1 \mid \mu(U) = 0 \text{ for all compact open sets } U \subseteq W_1 - S \}
\]

is closed under finite intersections. Now let

\[
S^* = \bigcap_{S \in \mathcal{S}} S.
\]

I claim \( S^* \in \mathcal{S} \) too. Let \( U \subseteq W_1 - S^* \) be a compact open set. Since

\[
W_1 - S^* = \bigcup_{S \in \mathcal{S}} (W_1 - S),
\]

it follows that \( U \) is covered by the open sets \( U \cap (W_1 - S) \), where \( S \in \mathcal{S} \). Since \( U \) is compact, it can be covered by a finite number of such open sets:

\[
U \subseteq (W_1 - S_1) \cup \cdots \cup (W_1 - S_n)
\]

where \( S_1, \ldots, S_n \in \mathcal{S} \). Now let \( T \in \mathcal{S} \) be a closed set contained in all these \( S_i \). Then \( U \subseteq W_1 - T \). But \( T \in \mathcal{S} \) means that this implies \( \mu(U) = 0 \). So \( \mu(U) = 0 \) whenever \( U \subseteq W_1 - S^* \), i.e., \( S^* \in \mathcal{S} \) too. Q.E.D.

**Proposition.** Let \( \mu \) be a non-zero even Gaussian measure on \( W_1 \) (i.e., \( \mu \) has the property (A) of Lemma 1, § 8). Then the support \( S \) of \( \mu \) is a sub-vector space of \( W_1 \).
Proof. Notice that if $\mu_1$, $\mu_2$ are 2 measures on $W_1$, and $\mu_1 \times \mu_2$ is the induced measure on $W_1 \times W_1$, then

$$\text{Support}(\mu_1 \times \mu_2) = \text{Support}(\mu_1) \times \text{Support}(\mu_2).$$

Let $\xi: W_1 \times W_1 \to W_1 \times W_1$ be the map $\xi((x, y)) = (x + y, x - y)$. By definition, a Gaussian measure $\mu$ is associated to a second measure $\nu$ such that

$$\xi_* (\mu \times \mu) = \nu \times \nu.$$

Therefore, if $S' = \text{Support} (\nu)$, it follows that $\xi(S \times S) = S' \times S'$. In particular

$$\alpha \in S \iff (\alpha, \alpha) \in S \times S$$
$$\iff (2\alpha, 0) = \xi((\alpha, \alpha)) \in S' \times S'.$$

Since $S$ is non-empty, $0 \in S'$, and $\alpha \in S' \iff 2\alpha \in S'$, i.e., $S' = 2S$. Therefore $0 \in S$ too, and we find:

$$\alpha \in S \iff (\alpha, 0) \in S \times S$$
$$\iff (\alpha, \alpha) = \xi((\alpha, 0)) \in S' \times S'$$
$$\iff \alpha \in S'.$$

Therefore $S = S'$ also. Finally,

$$\alpha, \beta \in S \Rightarrow (\alpha, \beta) \in S \times S$$
$$\Rightarrow (\alpha + \beta, \alpha - \beta) \in S' \times S'$$
$$\Rightarrow \alpha + \beta, \alpha - \beta \in S' = S.$$

Thus $S$ is a closed subgroup of $W_1$, such that $S = 2S$. Therefore $S$ is a subvectorspace over $Q_2$. Q.E.D.

Corollary. For all $\gamma_2 \in W_2$, all theta functions $\Theta$ on $V$,

$$\text{Support}(\Theta) \subseteq \{ \alpha | e(\alpha, \gamma_2) = 1 \} \Rightarrow \Theta(\alpha + \lambda \gamma_2) = e \left( \alpha, \frac{\lambda \gamma_2}{2} \right) \Theta(\alpha),$$

all $\lambda \in Q_2$.

Proof. The assumption on the support of $\Theta$ implies (cf. (a) above) that $\mu(\alpha_1 + 2^n \lambda_1) = 0$ if $e(\alpha_1, \gamma_2) = 1$. Therefore,

$$\text{Support}(\mu) \subseteq \{ \alpha_1 \in W_1 | e(\alpha_1, \gamma_2) = 1 \}.$$

Since this support is a vector space,

$$\text{Support}(\mu) \subseteq W_1 \cap (Q_2 \cdot \gamma_2)\perp.$$
Let $H$ denote the hyperplane $W_1 \cap (Q_2 \cdot \gamma_2)^\perp$. Then
\[
\Theta(x_1 + x_2) = e\left(\frac{x_2}{2}, \frac{x_2}{(x_1 + x_2) \cap H}\right) e(x_2, \beta) \cdot d\mu(\beta).
\]
Thus
\[
\Theta(x_1 + x_2 + x_2) = e\left(\frac{x_2 + x_2}{2}, \frac{x_2 + x_2}{(x_1 + x_2) \cap H}\right) e(x_2 + x_2, \beta) \cdot d\mu(\beta)
\]
and since $e(\lambda \beta, \beta) = 1$ when $\beta \in H$, this comes out
\[
e\left(\frac{x_2 + x_2}{2}\right) \cdot e(x_2, \beta) \cdot d\mu(\beta) = e\left(\frac{x_2 + x_2}{2}\right) \cdot \Theta(x_1 + x_2). \quad Q.E.D.
\]
In fact, I claim that the same Corollary holds for all $\gamma \in V$, not just for $\gamma \in W_2$. This can be seen by noting that for any $\gamma \in V$, there is a symplectic automorphism $T: V \to V$ such that $T(\gamma) = \lambda$, i.e., $T \in \text{Sp}(V, \frac{1}{2}A)$, such that $T^{-1}(\gamma) \in W_2$. Going back to the action of the symplectic group introduced in §9, we see that:

\[
\begin{cases}
\text{If } \Theta \text{ is a theta-function, then so is } \Theta', \text{ where} \\
\Theta'(x) = e(\eta/2, \alpha) \Theta(Tx - T\eta)
\end{cases}
\]
where $\eta \in \frac{1}{2}A$ satisfies
\[
e(x/2) \cdot e(x/2) = e(\eta, \alpha), \quad \text{all } x \in A.
\]

Now assume $\text{Supp}(\Theta) \subseteq \{x \mid e(x, \gamma) = 1\}$. Then
\[
\text{Supp}(\Theta') = \eta + T^{-1}(\text{Supp}(\Theta))
\]
\[
\subseteq \eta + \{x \mid e(x, T^{-1} \gamma) = 1\}
\]
\[
\subseteq \{x \mid e(x, T^{-1} \gamma) = 1\} \quad \text{(if } n \gg 0\text{)}.
\]

Therefore, by the Corollary
\[
\Theta'(x + \lambda T^{-1} \gamma) = e\left(\frac{x + \lambda}{2}, \frac{T^{-1} \gamma}{2}\right) \Theta'(x), \quad \text{all } \lambda \in Q_2,
\]
from which
\[
\Theta(x + \lambda \gamma) = e\left(\frac{x + \lambda}{2}\right) \cdot \Theta(x)
\]
follows immediately. We are now ready for the Proof itself:

Proof of Theorem. We know that the support of $\Theta$ meets $\frac{1}{2}A$ (cf. §10): choose $\eta_0 \in \text{Supp}(\Theta) \cap \frac{1}{2}A$. Then:
\[
\text{Supp}(\Theta) + \eta_0 \subseteq W + A
\]
(§ 10, assertion (4.) at the beginning). Therefore, if \( \gamma \in W^\perp \cap (2A) \) it follows that \( e(x, \gamma) = 1 \), all \( x \in \text{Supp}(\Theta) \). But then by Corollary above — as generalized —

\[
\Theta(x + \lambda \cdot \gamma) = e\left(x, \frac{\lambda \gamma}{2}\right) \cdot \Theta(x), \quad \text{all } \lambda \in \mathbb{Q}.
\]

This shows that

\[
(\ast) \quad \Theta(x + \gamma) = e\left(x, \frac{\gamma}{2}\right) \cdot \Theta(x), \quad \text{all } \gamma \in W^\perp.
\]

In particular, \( \Theta(\eta_0 + \gamma) \neq 0 \), all \( \gamma \in W^\perp \), hence \( W^\perp + \eta_0 \subseteq W + A + \eta_0 \). Therefore \( W^\perp \subseteq W \), i.e., \( W \) is a cusp.

Now suppose we take an arbitrary point \( x \) in the Support of \( \Theta \). We know that \( x \) can be written as:

\[
x = \eta_0 + \eta_1 + \alpha_0, \quad \eta_1 \in A, \quad \alpha_0 \in W.
\]

But then:

\[
\Theta(x) = e_* \left( \frac{\eta_1}{2}, \eta_0 + \alpha_0 \right) \cdot \Theta(\eta_0 + \alpha_0)
\]

\[
= e_* \left( \frac{\eta_1}{2}, \eta_0 \right) \cdot \Theta\left(\frac{\eta_0 + \eta_1}{2}, \alpha_0 \right) \cdot e\left(\frac{\eta_0}{2}, \alpha_0 \right) \cdot \Theta(\eta_0 + \alpha_0).
\]

Define a function \( \tilde{\Theta} \) on \( W \) by

\[
\tilde{\Theta}(x) = e\left(x, \frac{\eta_0}{2}\right) \cdot \Theta(x + \eta_0).
\]

If \( \gamma \in W^\perp \), we compute (using \( \ast \)):

\[
\tilde{\Theta}(x + \gamma) = e\left(x + \gamma, \frac{\eta_0}{2}\right) \cdot \Theta(x + \eta_0 + \gamma)
\]

\[
= e\left(x + \eta_0, \frac{\gamma}{2}\right) \cdot e\left(x, \frac{\eta_0}{2}\right) \cdot \Theta(\eta_0 + \gamma)
\]

\[
= \tilde{\Theta}(x).
\]

This shows that \( \tilde{\Theta} \) is, in reality, a function on \( \tilde{W} = W \cap W^\perp \), and that \( \Theta \) is exactly the function \( T_{W_n, \eta_0} \Theta \) obtained from \( \tilde{\Theta} \) via Definition 3.

To check that \( \eta_0 \) is an origin for \( W \), look at \( \ast \) when \( \gamma^\perp \in W \cap A \). Then:

\[
e\left(x, \frac{\gamma}{2}\right) \cdot \Theta(x) = \Theta(x + \gamma) = e_* \left( \frac{\gamma}{2}, \frac{x}{2} \right) \cdot \Theta(x)
\]

hence

\[
e_* \left( \frac{\gamma}{2} \right) = e(x, \gamma) \quad \text{if } \Theta(x) \neq 0.
\]
So
\[ e_\ast \left( \frac{\gamma}{2} \right) = e(\eta_0, \nu), \quad \text{all } \gamma \in W^\perp \cap \Lambda. \]
Moreover, using
\[ \Theta(\eta_0) = \Theta(-\eta_0 + 2\eta_0) = e_\ast(\eta_0) \Theta(-\eta_0) \]
and
\[ \Theta(-\eta_0) = \Theta(\eta_0) = 0, \]
we conclude that \( e_\ast(\eta_0) = 1 \) too.

The fact that \( \tilde{\Theta} \) is again a theta-function is simply a matter of applying the calculations of Prop. 1 in reverse and is quite straightforward. We omit this. The final point is that \( \tilde{\Theta} \) is non-degenerate. But since \( S_\infty \equiv W \), we know that for all \( \alpha \in W \), \( \alpha = 2^k \beta + \eta_1 \), where \( \Theta(\beta) \neq 0, \eta_1 \in \Lambda \). Then \( \beta = \eta_0 + \eta_2 + \eta_3 + \eta_4 \), \( \eta_2 \in \Lambda \), \( \beta_0 \in W \), and \( \Theta(\beta_0) = 0 \). Since
\[ \alpha - 2^k \beta_0 = \eta_1 + 2^k \eta_0 + 2^k \eta_2 \in W \cap \Lambda, \]
this shows that for all \( \alpha \in W \), \( \alpha = 2^k \beta_0 + \eta_3 \), where \( \tilde{\Theta}(\beta_0) \neq 0, \eta_3 \in W \cap \Lambda \). This means exactly that the \( S_\infty \) for \( \tilde{\Theta} \) is all of \( \tilde{V} \), i.e., \( \tilde{\Theta} \) is non-degenerate. Q.E.D.

The main Theorem can now be reformulated to give a Satake-like decomposition of \( \tilde{M}_\infty \). More precisely, for each integer \( g \geq 0 \), let \( \tilde{M}_\infty(g) = \text{Proj} \) defined in § 9, Def. 3 with indices \( \alpha \in \mathcal{O}_g \). \( \tilde{M}_\infty(g) \) is the open set in \( \tilde{M}_\infty \) whose geometric points are the non-degenerate theta functions.

If \( h < g \), we define a vast number of closed immersions
\[ i_w: \tilde{M}_\infty(h) \to \tilde{M}_\infty(g) \]
as follows: let \( W \subseteq \mathcal{O}_g \) be a cusp such that \( 2h = \dim (W/W^\perp) \). For each such \( W \), choose an origin \( \eta_0 \in \frac{1}{2} \mathbb{Z}_2^g \), and a symplectic isomorphism:
\[ \phi: \mathcal{O}_g \cong W/W^\perp \]
such that
\[ \phi(\mathbb{Z}_2^h) = W \cap \Lambda/W^\perp \cap \Lambda, \]
\[ \chi(\frac{1}{2} a_1 \cdot a_2) = \tilde{e}_\ast(\frac{1}{2} \phi(a)), \quad \text{all } a \in \mathbb{Z}_2^h. \]
Then \( i_w \) is defined by the homomorphism of the homogeneous coordinate ring:
\[ i_w^\ast(X^{(\alpha)}) = \begin{cases} 0 & \text{if } \alpha \notin \eta_0 + W + \mathbb{Z}_2^g \\ e_\ast \left( \frac{\eta_1}{2} \right) \cdot e \left( \frac{\eta_1}{2}, \eta_0 \right) \cdot e \left( \eta_0 + \frac{\eta_1}{2}, \alpha \right) \cdot X^{(\alpha)}(x_0) & \text{if } \alpha = \eta_0 + a_0 + \eta_1, \ a_0 \in W, \ \eta_1 \in \mathbb{Z}_2^g. \end{cases} \]
(Here $X_1^g$, $X_2^h$ are the coordinates used to define $\overline{M}_\infty(g)$, $\overline{M}_\infty(h)$ respectively). Then we get the restatement:

**Main Theorem.**

\[
\overline{M}_\infty(g) = \left\{ \text{disjoint union of the locally closed subschemes } i_w(M_\infty(h)) \right\},
\]

the union being taken over all cusps $W \subseteq \mathcal{Q}_\mathbb{C}^\infty$.

§ 12. Analytic Theta Functions

In this section, we work over the field $\mathbb{C}$ of complex numbers. We have two purposes: (a) to sketch an approach to the classical theory of $\Theta$-functions, analogous to our theory of algebraic $\Theta$-functions, and (b) to use this to compute our algebraic $\Theta$-functions via the classical ones, when $k = \mathbb{C}$.

We will make use of the following lemma:

**Lemma 1.** Let $X$ be a compact Kähler manifold. Then the operator

\[
\frac{1}{2\pi i} \delta \overline{\delta}
\]

defines a surjection:

\[
\left\{ \text{functions on } X \right\} \to \left\{ \text{real closed } C^\infty(1,1) \text{-forms } \Omega \text{ on } X, \text{ with 0 cohomology class} \right\}
\]

with kernel consisting only of constants.

**Corollary.** Let $L$ be an analytic line bundle on $X$. Let $c_1(L) \in H^1(X, \mathbb{C})$ be its first Chern class. Then for all real closed $C^\infty(1,1)$-forms $\Omega$ whose cohomology class equals $c_1(L)$, there is one (up to a constant) only Hermitian structure $\|\|$ on $L$ whose associated curvature form is $\Omega$.

The lemma is standard and we omit the proof. The Corollary can be proven by choosing one Hermitian structure $\|\|_0$ on $L$: let $\Omega_0$ be its curvature form. Then any other Hermitian structure on $L$ is given by $\rho \cdot \|\|_0$, where $\rho$ is a positive real $C^\infty$ function on $X$: and its curvature form $\Omega$ is

\[
\Omega = \frac{1}{2\pi i} \delta \overline{\delta} \log \rho + \Omega_0.
\]

Now use the Lemma and everything comes out. **Q.E.D.**

In particular, when $X$ is an abelian variety, an analytic line bundle $L$ on $X$ has one and (up to a constant) only one Hermitian structure $\|\|$ whose curvature form $\Omega$ is a translation-invariant $(1,1)$-form. In what follows, we will always put this Hermitian structure on line bundles on abelian varieties. In this case, $\Omega$ is determined by its value at the origin.
Now let \( \hat{X} \) be the universal covering space of \( X \). \( \hat{X} \) is a complex vector space, and if

\[
p: \hat{X} \rightarrow X
\]

is the canonical homomorphism, \( dp \) induces a canonical identification between \( \hat{X} \) and the tangent space of \( X \) at the origin (or at any other point). Therefore, any translation-invariant real 2-form \( \Omega \) on \( X \) defines and is defined by a real-linear skew-symmetric form:

\[
E: \hat{X} \times \hat{X} \rightarrow \mathbb{R}.
\]

\( E \) is a \((1, 1)\)-form if and only if \( E(ix, iy) = E(x, y) \), all \( x, y \in X \). Moreover, let \( A = \text{kernel}(p) \). \( A \) is a lattice in \( X \), canonically isomorphic to \( H_1(X, \mathbb{Z}) \). Since the first chern class of a line bundle is integral, if \( E \) represents \( c_1(L) \), then \( E \) must take integral values on \( A \times A \):

\[
E(A \times A) \subseteq \mathbb{Z}.
\]

If we lift \( L \) to \( \hat{X} \), we have a situation in which the following lemma applies:

**Lemma 2.** Let \( Y \) be a complex vector space, and let \( L_1, L_2 \) be 2 analytic-Hermitian line bundles on \( Y \). Then a holomorphic-unitary isomorphism \( \phi: L_1 \rightarrow L_2 \) exists if and only if the curvature forms of \( L_1, L_2 \) are equal; if so, \( \phi \) is unique up to a scalar of absolute value 1.

**Proof.** Standard methods.

In particular, let \( Y = \hat{X} \), and let \( M = p^*(L) \) be induced from an abelian variety. Give \( L \) and hence \( M \) the Hermitian structure with constant curvature form \( E \). The above lemma has 2 applications:

(I) Construction of a nilpotent group \( \mathcal{G} \): If \( x \in X \), and \( T_x \) denotes translation by \( x \), then the lemma shows that \( M \) and \( T_x^* M \) are holomorphic-unitary isomorphic. If

\[
\mathcal{G}(M) = \{(x, \phi) | \phi \text{ a holo.-unit. isom. of } M \text{ with } T_x^* M\},
\]

then \( \mathcal{G}(M) \) is, as before, a group lying in an exact sequence:

\[
1 \rightarrow C^*_1 \rightarrow \mathcal{G}(M) \rightarrow X \rightarrow 0
\]

\( (C^*_1 = \text{complex numbers of absolute value 1}) \).

(II) Construction of canonical "trivialization" of \( M \): Let \( 1 \) denote the trivial analytic line bundle over \( X \) with canonical section 1. To put a Hermitian structure on \( 1 \), we may set \( \|1\| = \) any positive real \( C^\infty \)-function. For example, let

\[
\|1\|(x) = e^{-x^2/2H(x, x)}
\]
where $H$ is a Hermitian form on $X$. The corresponding curvature form $E: \hat{X} \times \hat{X} \to \mathbb{R}$ is easily checked to equal $\text{Im}(H)$. But

$$H \mapsto E = \text{Im}(H)$$

sets up an isomorphism:

$$\begin{align*}
\left\{ \text{hermitian forms on } X \right\} & \cong \left\{ \text{real skew-symmetric forms } E \text{ on } X \right\}, \\
& \text{such that } E(i \, x, i \, y) = E(x, y)
\end{align*}$$

so for each $L$ on $X$ with translation-invariant curvature form, we have a unique Hermitian structure on $\mathbf{1}$ of the above type so that $\mathbf{1} \cong L$. In particular, we get a canonical

$$\mathbf{1} \cong M.$$

We can now develop a theory along similar lines to our algebraic theory. For example, if $H$ is positive definite, then let:

$\mathcal{H}$ = Hilbert space of $L^2$-holomorphic sections of $M$ over $\hat{X}$.

Then $\mathcal{G}(M)$ has a natural unitary representation on $\mathcal{H}$, it is irreducible, and it turns out to be the only irreducible unitary representation of $\mathcal{G}(M)$ in which $C_\mathbb{C} \subset \mathcal{G}(M)$ acts by its natural character. This is the situation described by CARTIER [2], and studied by CARTIER and many others, e.g., MACKEY, FOCK, WEIL etc. Exactly as in § 1, $\mathcal{G}(M)$ governs the “descent” of the Hermitian bundle $M$ to the abelian variety $X$, (or to other ones $X' = [\hat{X}]$ another lattice), and the “descent” of holomorphic sections of $M$ to holomorphic sections of its descended form. Thus we get:

**Proposition 1.** There is a $1-1$ correspondence between

1. Hermitian-analytic line bundles $L'$ on $X$ such that $p^* L' \cong M$,
2. subgroups $K \subset \mathcal{G}(M)$, such that $K \cap C_\mathbb{C} \equiv \{1\}$ whose image in $\hat{X}$ is $\Lambda = \ker (p: \hat{X} \to X)$.

Moreover, the holomorphic sections of $M$ of the form $p^* (s'), s' \in \Gamma(X, L')$, are exactly those sections $s$ which are invariant under $K$, i.e.,

$$s = T^* x (\phi(s)), \quad \text{all } (x, \phi) \in K.$$

**Proof.** Straightforward.

Finally, via the canonical trivialization of $M$, holomorphic sections of $M$ correspond to holomorphic functions on $\hat{X}$: thus each section $s \in \Gamma(X, L)$ defines a holomorphic function on $\hat{X}$. These are the classical theta-functions.

As far as moduli are concerned, the simplest and most basic result is the following: we set out to classify triples consisting of —
1. a complex vector space $Y$, of dimension 2;
2. an analytic, Hermitian line bundle $M$ on $Y$, with curvature form $E=\text{Im } H$, $H$ positive definite.
3. Parametrized lattices in $Y$, i.e., monomorphisms

$$\alpha: \mathbb{Z}^g \rightarrow Y$$

such that

$$E(\alpha x, \alpha y) = \langle x_1 \cdot y_2 - x_2 \cdot y_1 \rangle$$

if

$$x=(x_1, x_2), \quad y=(y_1, y_2).$$

Such triples arise if we start with a principally polarized abelian variety $(X, L)$, together with a symplectic isomorphism:

$$\beta: \mathbb{Z}^g \sim H_1(X, \mathbb{Z}).$$

Namely, let $Y=\hat{X}, M=p^* L$ with canonical Hermitian structure, and let $\beta$ define $\alpha$ via the natural maps $H_1(X, \mathbb{Z}) \cong \text{Ker } (p: \hat{X} \rightarrow X) \subset \hat{X}$. Conversely, the triple $(Y, M, \alpha)$ determines $X$ and $\beta$, and $L$ up to replacing $L$ by $T^*_x L$, some $x \in X$.

Let $\mathcal{S}=\text{SIEGEL's } g \times g$ upper half-plane. Then the moduli result is:

**Proposition 2.** There is a natural bijection between the set of isomorphism classes of triples $(Y, M, \alpha)$ and $\mathcal{S}$. In this bijection, $\tau \in \mathcal{S}$ corresponds to

$$Y=\mathbb{C}^g,$$

$$M=1 \text{ with hermitian structure } \|1\|(x) = e^{-2\pi \cdot \beta \cdot \tau},$$

$$\alpha((x_1, x_2)) = x_1 + \tau \cdot x_2$$

where $B=(\text{Im } \tau)^{-1}$.

The final topic I want to discuss is the relation between the classical and algebraic theories. Let's start with:

$X$ = abelian variety;
$L$ = symmetric, ample degree 1 sheaf on $X$. [Assume for simplicity that $L$ is so chosen among its translates $T^*_x L, x \in X_2$, that its unique section is even; equivalently, that the Arf invariant of $Q$, where $e_{\phi}^* (x) = (-1)^{Q(x)}$, is 0.]

Let

$L$ = line bundle on $\hat{X}$ whose holomorphic sections are $L$;
$\hat{X}$ = universal covering space of $X$;
$V_2(X) = 2$-Tate group of $X$. 

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Also, let $A_2$ = inverse image in $\hat{X}$ of $\text{tor}_2(X)$, i.e.,

$$\bigcup_a 2^{-a} \cdot A, \quad \text{if} \quad A = \text{Ker}(p: \hat{X} \to X).$$

Then we have canonical maps:

Note that $A_2$ is dense in both $V_2(X)$ and $X$. We have "trivialized" $L$ when it is pulled up to $V_2(X)$ or to $X$, in §8 and just above. Thus we have 2 distinct trivializations of $L$ on $A_2$. The main result is that these differ by an elementary factor:

**Theorem 3.** Let $1$ denote the trivial complex line bundle on $A_2$. Then the following diagram commutes:

$$(L, \text{pulled back to } A_2) \quad \text{algebraic trivialization} \quad \text{multiplication by } a \cdot e^{\alpha H(a, a)/2} \quad \text{cl assical trivialization} \quad 1$$

where $a \in \mathbb{C}^*$ and $E = \text{Im}(H)$ is the curvature form of $L$.

**Proof:** Let $M_1 = p_1^* L$ = induced line bundle on $V_2(X)$ or $\hat{X}$. Let $\psi: M_2 \to 1$ be the classical trivialization. The algebraic trivialization of $M_1$ is based on finding a distinguished collection of isomorphisms

$$\varphi_a: M_1 \to T_a^* M_1,$$

all $a \in V_2(X)$. In fact, let $\tau$ = inverse map in all our groups, and let $\rho: M_1 \to T^* M_1$ be the isomorphism induced by the symmetry of $L$. Then, for all elements $2a \in V_2(X)$, $\varphi_{2a}$ is characterized by the existence of $\varphi_a$ satisfying:

i) $\varphi_{2a} = T_a^* \varphi_a \circ \varphi_a$

ii) $\tau^* \varphi_a \circ \rho = T_a^* \varphi_a^{-1}$

iii) $\varphi_a$ is induced by an algebraic isomorphism

$$\varphi'_a: (2^a \delta)^* L \xrightarrow{\sim} (2^a \delta)^* (T_{p_1(a)} L)$$
for some \( n \), i.e., via the factorization:

\[
\begin{array}{c}
V_2(X) \xrightarrow{p_1} \xrightarrow{2^{-n}} \xrightarrow{2^n \delta} X.
\end{array}
\]

But introduce, for all \( a \in X \), isomorphisms \( \psi_a \) from \( M_2 \) to \( T^*_a M_2 \) via:

\[
\begin{array}{c}
M_2 \xrightarrow{\sim} T^*_a 1 \xrightarrow{\sim} T^*_a M \xrightarrow{\text{mult. by } f_a(x)} T^*_a 1 \xrightarrow{\sim} T^*_a M
\end{array}
\]

where

\[
f_a(x) = e^{i(H(x, a) + H(a, a)/2)}.
\]

Also introduce

\[
\rho': M_2 \xrightarrow{\sim} 1 \xrightarrow{\text{canonical identification}} i^* 1 \xrightarrow{\sim} i^* M.
\]

One checks easily that \( \psi_a \) and \( \rho' \) are holomorphic and unitary isomorphisms. Therefore \( \rho \) and \( \rho' \) can differ only by a constant: and since both are the identity at \( 0 \in X \), \( \rho = \rho' \). Moreover, if \( a \in 2^{-n} A \), then the algebraic isomorphism \( \phi'^*_a: (2^n \delta)^* L \xrightarrow{\sim} (2^n \delta)^* T^*_a(0) L \), referred to in (iii) above, induces an isomorphism \( \phi'^*_a: M_2 \to T^*_a M_2 \) via the factorization

\[
\begin{array}{c}
P_1 \xrightarrow{2^{-n}} \xrightarrow{2^n \delta} X.
\end{array}
\]

Since \( \phi'^*_a \) is also holomorphic and unitary, it differs from \( \psi_a \) only by a constant. Next, note that \( \{ f_a \} \) satisfy the identities:

\( i' \): \( f_a(x) = f_a(x + a) \cdot f_a(x) \),

\( ii' \): \( f_a(-x) = f_a(x) \cdot f_a^{-1}(x) \).

These translate readily into the identities on the \( \{ \psi_a \} \):

\( i'' \): \( \psi_{2a} = T^*_a \psi_a \psi_a \cdot \psi_a \).

\( ii'' \): \( i^* \psi_a \circ \rho = T^*_a[\rho \circ \psi_a^{-1}] \).

Finally, \( i'' \), \( ii'' \), plus the fact that \( \phi'^*_a \) induces \( \psi_a \), shows that \( \psi_a \) and \( \phi_a \) induce the same isomorphism of \( L \) on \( A_2 \), with \( T^*_a(L) \) on \( A_2 \), all \( a \in A_2 \).

Finally, to compare the 2 trivializations, start with the unit section 1 of \( L \) on \( A_2 \). This goes over, via the algebraic trivialization, to a section \( s \) of \( L \) on \( A_2 \) such that, for all \( a \in A_2 \),

\[
s(a) = \phi_a(0)[s(0)].
\]
(i.e., $\phi_a(0)$ is the induced isomorphism from the fibre $L_0$ or $(M_1)_0$ to the fibre $L_{p_1(a)}$ or $(M_1)_a$). But under the classical trivialization $\psi$, $\psi_a(0)$ corresponds to the isomorphism of fibres:

\[
\begin{array}{ccc}
I_0 & \xrightarrow{\text{mult. by } e^{x^2/2H(a,a)}} & I_0 \\
\| & & \| \\
C & & C.
\end{array}
\]

Therefore, the section $s$ goes over, under the classical trivialization, to a section of $I$ which, if it has value $\alpha$ at $0$, has value $\alpha \cdot e^{x^2/2H(a,a)}$ at $a$. All in all, the section $1$ of $I$ has gone into the section $g(a) = \alpha \cdot e^{x^2/2H(a,a)}$ of $I$. Q.E.D.

**Corollary.** If the unique section $s$ of $L$ (up to scalars) defines

a) the holomorphic function $\Theta_h$ on $\hat{X}$ via the classical trivialization,

b) the $2$-adic theta-function $\Theta_a$ on $V_2(X)$ via the algebraic trivialization,

then

\[\Theta_h(x) = \alpha \cdot e^{x^2/2H(x,x)} \cdot \Theta_a(x)\]

all $x \in \Lambda_2$.

To calculate $\Theta_h$ and hence $\Theta_a$ by analytic means, we must know the “descent data”

\[K \subset \mathcal{G}(M_2)\]

that defines $L$ on $X$. Let $e_\ast : \frac{1}{2} \mathbb{A}/\mathbb{A} \to \{\pm 1\}$ be the quadratic character defined by $L$. Then, as we saw in §8, the descent data for the pull-back $M_1$ of $L$ is the group:

\[\{(x, \phi) \mid x \in \Lambda \cdot \mathbb{T}, \phi = e_\ast(\frac{1}{2} x) \cdot \phi_a\}.\]

In view of the proof of the theorem, this implies that

\[K = \{(x, \psi) \mid x \in \Lambda, \psi = e_\ast(\frac{1}{2} x) \cdot \psi_a\}.\]

(Notation as in proof of Theorem). Now a $K$-invariant section $s$ of $M_2$ is one which satisfies $T^a_\ast(s) = \phi(s)$, all $(a, \phi) \in K$. Going back to the definition of $\psi_a$, one sees that if $f = \psi(s)$ is the function on $X$ corresponding to $s$, then $f$ is $K$-invariant if and only if

\[f(x + a) = e_\ast(\frac{1}{2} a) f_a(x) \cdot f(x)\]

all $x \in \hat{X}, a \in \Lambda$. From this it follows that $\Theta_a$ must be the unique holomorphic function satisfying (●).
To go further and write down this $\Theta_h$ as an infinite series, it is convenient to introduce coordinates. Let
\[ i : \mathbb{Z}^{2g} \rightarrow A \] be a symplectic isomorphism.

Coordinatize $\hat{X}$ via
\[ \hat{X} \cong \mathbb{C}^g \]
so that $i((n_1, 0)) = n_1$, and let $\tau$ be the $g \times g$ matrix defined by
\[ i((0, n_2)) = \tau \cdot n_2. \]

Because of our assumption on $e^t_+$, hence on $e_+$, if we choose coordinates correctly, we can assume that
\[ e_+ \left( \frac{1}{2} i((n_1, n_2)) \right) = (-1)^{n_1 \cdot n_2}. \]

As we saw in Prop. 2, if we now express:
\[ H(z, \bar{z}) = z \cdot B \cdot \bar{z} \]
then $B = (\text{Im } \tau)^{-1}$. Finally, set
\[ \Theta_h(z) = e^{\frac{1}{2} z \cdot B \cdot \bar{z}} \sum_{n \in \mathbb{Z}^g} e^{2\pi i \frac{1}{2} (n^t \cdot \tau \cdot n + i n^t \cdot z)}. \]

It is easy to check that this is a holomorphic function satisfying (*). Therefore, this is the sought-for theta-function. Combining this with the Corollary, we find
\[ \Theta_d(z) = e^{\frac{1}{2} i z \cdot \bar{z}} \sum_{n \in \mathbb{Z}^g} e^{2\pi i \frac{1}{2} (n^t \cdot \tau \cdot n + i n^t \cdot z)} \quad \text{all } z \in \bigcup_n 2^{-k} A. \]

If
\[ z = i((x_1, x_2)), \quad x_i \in \bigcup_k 2^{-k} \cdot (\mathbb{Z}^g), \]
then after rearranging, one finds
\[ \Theta_d(x_1, x_2) = e^{-x_1^t \delta_{x_1} - x_2^t \delta_{x_2}} \sum_{n \in \mathbb{Z}^g} e^{2\pi i \frac{1}{2} (n^t \cdot \tau \cdot n + i n^t \cdot x_1)}. \]

The function so defined clearly extends to a locally constant function defined for all $x_1, x_2 \in \mathbb{Q}^g$: it is the sought-for algebraic theta function defined in § 8. Comparing this with the formula in Lemma 1, § 8, expressing $\Theta_d$ in terms of the finitely additive measure $\mu$ on $\mathbb{Q}^g$, we also get an analytic description for $\mu$:

\[ \begin{cases} 
\mu \text{ is countably additive,} \\
\mu = \sum_{x \in D} e^{i x_1^t \cdot \tau \cdot x} \cdot \delta_x, \\
\delta_x = \text{delta measure at } x, \\
D = \bigcup_k 2^{-k} \mathbb{Z}^g.
\end{cases} \]

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