On the Equations Defining Abelian Varieties. III

Inventiones math. 3, 215—244 (1967)

On the Equations Defining Abelian Varieties. III*

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Contents

§ 10. Non-Degenerate Theta Functions

The third part of this paper is devoted (1) to a complete description of the boundary of the moduli space for abelian varieties described in § 9, and (2) to connecting our theory with the classical theory of theta functions. We begin by defining a theta function in a coordinate-free manner and investigating how and under what non-degeneracy restrictions we can construct a tower of abelian varieties having this as its theta function. Our goal is to find an inverse to the moduli map Θ described in § 9. Fix

o) an algebraically closed field k, char $(k)\neq 2$;

i) a $2g$-dimensional vector space V over $\boldsymbol{Q}_2$;

ii) a skew-symmetric bi-multiplicative map:

$$e\colon\ V\times V\longrightarrow\{2^n\text{-th roots of 1 in } k\},$$

i.e.,

$$e(\alpha,\alpha)=1$$
$$e(\alpha\cdot\beta,\gamma)=e(\alpha,\gamma)\cdot e(\beta,\gamma)$$
$$e(\alpha,\beta\cdot\gamma)=e(\alpha,\beta)\cdot e(\alpha,\gamma);$$

iii) a maximal isotropic lattice $\Lambda\subset V$ (i.e., a compact, open subgroup such that $e(\alpha,\beta)=1$, all $\alpha,\beta\in\Lambda$, maximal with this property);

iv) a quadratic character

$$e_*\colon\ \tfrac{1}{2}\Lambda/\Lambda\longrightarrow\{\pm 1\}$$

such that

$$e_*(\alpha+\beta)\,e_*(\alpha)\,e_*(\beta)=e(\alpha,\beta)^2,$$

all $\alpha,\beta\in\frac{1}{2}\Lambda$.

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* Part I of this paper has been published in Inventiones math. Vol. 1, pp. 287—354 and part II in Vol. 3, pp. 75—135.

We assume, however, that via a suitable isomorphism $V \cong \mathbf{Q}_2^{2g}$, $\Lambda \cong \mathbf{Z}_2^{2g}$, and e, e_* have the form defined in § 9. In fact, this is nearly always the case: if we write

$$e_*(\alpha) = (-1)^{Q(\alpha)}$$

where Q is a quadratic form on $\frac{1}{2}\Lambda/\Lambda$ with values in the field $\mathbf{F}_2 = \{0, 1\}$, then Q has an Arf invariant $\Delta(Q) \in \mathbf{F}_2$. It is not hard to show that (V, Λ, e, e_*) has the required form only if $\Delta(Q) = 0$. We leave this point to the reader.

Definition 1. A *theta-function* Θ on V is a map $\Theta: V \to k$ satisfying

i) $\Theta(\alpha+\beta) = e_*(\beta/2) \cdot e(\beta/2, \alpha)\, \Theta(\alpha)$, all $\alpha \in V$, $\beta \in \Lambda$,

ii) $\Theta(-\alpha) = \Theta(\alpha)$, all $\alpha \in V$,

iii) $\prod_{i=1}^{4} \Theta(\alpha_i) = 2^{-g} \sum_{\eta \in \frac{1}{2}\Lambda/\Lambda} e(\gamma, \eta) \cdot \prod_{i=1}^{4} \Theta(\alpha_i + \gamma + \eta)$

if $\gamma = -\frac{1}{2}\sum_{1}^{4} \alpha_i$, $\alpha_1, \ldots, \alpha_4 \in V$ arbitrary.

If we let

$$S_0 = \{\alpha \mid \Theta(\alpha) \neq 0\} = \text{support}(\Theta),$$

then S_0 is a union of cosets of Λ. The structure of S_0 is a "fine" property of Θ, so we introduce:

Definition 2. The *coarse support* S_1 of Θ is:

$$S_1 = \{\alpha \mid \Theta(\alpha+\eta) \neq 0, \text{ for some } \eta \in \tfrac{1}{2}\Lambda\}.$$

We will see in § 11 that the coarse support S_1 of a theta function is either all of V, or $\frac{1}{2}\Lambda + W$ where $W \subset V$ is a proper subvectorspace. This is the essential difference between good and bad theta functions.

Note that $S_0 = -S_0$ and $S_1 = -S_1$. We always assume, in what follows, that $\Theta \not\equiv 0$, i.e., $S_0 \neq \phi$.

1. If $x_1 \notin S_1$, $x_2, x_3, x_4 \in S_0$, then $2x_1 + x_2 + x_3 + x_4 \notin S_0$.

Proof. Use the quartic relation on Θ, with $\alpha_1 = 2x_1 + x_2 + x_3 + x_4$, $\alpha_2 = x_2$, $\alpha_3 = x_3$, $\alpha_4 = x_4$, $\gamma = -x_1 - x_2 - x_3 - x_4$. *Q.E.D.*

2. $0 \in S_1$.

Proof. Assume $0 \notin S_1$. Take any $y \in S_0$. Apply (1.) with $x_2 = x_3 = y$, $x_4 = -y$ and we get a contradiction. *Q.E.D.*

3. $x, y \in S_0 \Rightarrow \frac{1}{2}(x+y) \in S_1$.

Proof. Apply (1.) with $x_1 = \frac{1}{2}(x+y)$, $x_2 = x$, $x_3 = -y$ and $x_4 = -x$. *Q.E.D.*

Because of (2.), there is an $\eta_0 \in \frac{1}{2}\Lambda$ such that $\Theta(\eta_0) \neq 0$. Fix one such η_0.

4. $(0) \subseteq (S_0+\eta_0) \subseteq (2S_0+\Lambda) \subseteq (4S_0+\Lambda) \subseteq \cdots$.

Proof. By (3), if $x \in S_0$, then $\frac{1}{2}(x+\eta_0) \in S_1$, so $x+\eta_0 \in 2S_0+\Lambda$. This gives the 1st inclusion. This also shows that $2x \in 4S_0+\Lambda$. Hence if $y \in 2^k S_0$, so $y=2^k \cdot x$, $x \in S_0$, then $2^k \cdot x \in 2^{k+1} S_0+\Lambda$. This gives the rest of the inclusions. *Q.E.D.*

Definition 3.

$$S_\infty = \bigcup_{k \geqq 1}^{\infty} [2^k S_0+\Lambda].$$

5. S_∞ is a group.

Proof. Let $x, y \in S_\infty$. Now $x, y \in (2^l \cdot S_0+\Lambda)$ for some $l \geqq l_0$. Then $x=2^l \cdot x_0+\eta$, $y=2^l \cdot y_0+\zeta$, $x_0, y_0 \in S_0$ and $\eta, \zeta \in \Lambda$. Therefore by (3), $\frac{1}{2}(x_0+y_0) \in S_1$, hence $2^l(x_0+y_0) \in 2^{l+1} \cdot S_0+\Lambda$. Therefore $x+y \in (2^{l+1} S_0+\Lambda) \subset S_\infty$. *Q.E.D.*

6. $S_\infty = W+\Lambda$, for some subvectorspace $W \subset V$.

Proof. This is easily seen to be equivalent to asserting that S_∞/Λ is a divisible subgroup of V/Λ. But if $x \in 2^k \cdot S_0+\Lambda$, then $x=2^k \cdot x_0+\eta$, $x_0 \in S_0$, $\eta \in \Lambda$, hence $x-\eta \in 2\{2^{k-1} S_0\} \subset 2 \cdot S_\infty$, i.e., the image of x in S_∞/Λ is divisible by 2. *Q.E.D.*

Definition 4. A theta function is *non-degenerate* if equivalently:

(a) $S_\infty = V$.

(a′) $S_\infty \supset \frac{1}{2}\Lambda$.

(a″) For all sufficiently large n, $2^n \cdot S_0+\Lambda \supset \frac{1}{2}\Lambda$.

(a‴) For all sufficiently large n, and $\alpha \in 2^{-n-1}\Lambda$, there is an $\eta \in 2^{-n}\Lambda$ such that $\Theta(\alpha+\eta) \neq 0$.

The next step is to form, via the function Θ, a sequence of graded rings:

Definition 5. If M is a vector space of k-valued functions on V, let

$$\mathscr{S}(M) = \bigoplus_{n=0}^{\infty} \mathscr{S}_n(M),$$

where $\mathscr{S}_0(M)=k$, $\mathscr{S}_1(M)=M$, and $\mathscr{S}_n(M)$, for $n \geqq 2$, is the vector spac of functions on V spanned by the products $f_{i_1} \cdots f_{i_n}$, $(f_{i_j} \in M$, all $j)$. Another convenient notation is the following:

$$M^* = \begin{Bmatrix} \text{set of functions } \alpha \mapsto f(\alpha/2), \\ \text{all } f \in M \end{Bmatrix}.$$

In particular, let

$$M_{2k} = \text{span of the functions } \Theta_{[\beta]}, \qquad \text{all } \beta \in 2^{-k}\Lambda$$

where

$$\Theta_{[\beta]}(\alpha)=e(\beta/2,\alpha)\cdot\Theta(\alpha-\beta).$$

The corresponding rings $\mathscr{S}(M_{2k})$ will be the heart of our analysis. These are only half of the rings we need, however. To define the others, choose a decomposition:

$$\Lambda=\Lambda_1\oplus\Lambda_2$$

such that $\boldsymbol{Q}_2\cdot\Lambda_i=V_i$ is an isotropic subspace under e, and such that $e_*(\alpha/2)=1$ for all $\alpha\in\Lambda_1$ or Λ_2. This exists because if we choose coordinates $V\cong\boldsymbol{Q}_2^{2g}$ such that Λ, e, e_* take their standard forms, then $\Lambda_1=\boldsymbol{Z}_g^2\times\{0\}$, $\Lambda_2=\{0\}\times\boldsymbol{Z}_g^2$ have these properties. In terms of Λ_1 and Λ_2, we now define a kind of "dual" theta-function ϕ. It is to satisfy the equations:

$$\sum_{\zeta\in\frac{1}{2}\Lambda_1/\Lambda_1} e(\alpha,\zeta)\cdot\Theta(\alpha+\beta+\zeta)\cdot\Theta(\alpha-\beta+\zeta)=\phi(\alpha)\cdot\phi(\beta)$$

all $\alpha,\beta\in V$. In fact, if we let $\Phi(\alpha,\beta)$ denote the left-hand side of this equation, then the quartic equations on Θ are equivalent to:

$$\Phi(\alpha,\beta)\cdot\Phi(\gamma,\delta)=\Phi(\alpha,\delta)\cdot\Phi(\gamma,\beta)$$

for all $\alpha,\beta,\gamma,\delta\in V$ (cf. proof of Lemma 2, § 8). This, plus the elementary fact $\Phi(\alpha,\beta)=\Phi(\beta,\alpha)$ implies that one and (up to scalars) only one such ϕ exists. Notice that ϕ satisfies the equations:

(i) $\phi(\alpha+\beta)=f_*(\beta)\cdot e(\beta,\alpha)\cdot\phi(\alpha)$, for all $\alpha\in V$, $\beta\in\frac{1}{2}\Lambda_1+\Lambda_2$, if $f_*(\frac{1}{2}\beta_1+\beta_2)=e(\frac{1}{2}\beta_1,\beta_2)$ $(\beta_i\in\Lambda_i)$.

(ii) $\phi(-\alpha)=\phi(\alpha)$, all $\alpha\in V$,

as well as certain quartic equations. Now let

$$M_{2k+1}=\text{span of the functions } \phi_{[\beta]},\qquad \beta\in 2^{-k-1}\cdot\Lambda$$

where

$$\phi_{[\beta]}(\alpha)=e(\beta,\alpha)\cdot\phi(\alpha-\beta).$$

Proposition 1. 1. $\mathscr{S}_2(M_{2k})\subseteq M_{2k+1}$, *equality holding if and only if for all* $\beta\in 2^{-k-1}\Lambda$, $\exists\gamma\in 2^{-k}\Lambda$ *such that* $\phi(\beta+\gamma)\neq 0$.

2. $\mathscr{S}_2(M_{2k+1})^*\subseteq M_{2k+2}$, *equality holding if and only if for all* $\beta\in 2^{-k-1}\Lambda$, $\exists\gamma\in 2^{-k}\Lambda$ *such that* $\Theta(\beta+\gamma)\neq 0$.

Proof. To compute $\mathscr{S}_2(M_{2k})$, note that it is spanned by the functions:

$$f(\alpha)=\sum_{\eta\in\frac{1}{2}\Lambda_1/\Lambda_1} e\left(\eta,\frac{\beta_1+\beta_2}{2}\right)\cdot\Theta_{[\beta_1-\eta]}(\alpha)\cdot\Theta_{[\beta_2-\eta]}(\alpha)$$

where $\beta_i \in 2^{-k}\Lambda$. But

$$f(\alpha) = e\left(\frac{\beta_1+\beta_2}{2}, \alpha\right) \cdot \sum_{\eta \in \frac{1}{2}\Lambda_1/\Lambda_1} e\left(\alpha - \frac{\beta_1+\beta_2}{2}, \eta\right) \times$$
$$\times \Theta(\alpha - \beta_1 + \eta)\, \Theta(\alpha - \beta_2 + \eta)$$
$$= e\left(\frac{\beta_1+\beta_2}{2}, \alpha\right) \cdot \phi\left(\alpha - \frac{\beta_1+\beta_2}{2}\right) \cdot \phi\left(\frac{\beta_1-\beta_2}{2}\right)$$
$$= \phi_{\left[\frac{\beta_1+\beta_2}{2}\right]}(\alpha) \cdot \phi\left(\frac{\beta_1-\beta_2}{2}\right) \in M_{2k+1}.$$

We get every $\phi_{[\gamma]}$, $\gamma \in 2^{-k-1}\Lambda$, in this way, if and only if every such γ can be written:

$$\gamma = \frac{\beta_1+\beta_2}{2}, \qquad \beta_i \in 2^{-k}\Lambda$$

such that

$$\phi\left(\frac{\beta_1-\beta_2}{2}\right) \neq 0.$$

This is exactly the condition in (1). To prove (2), first notice the identity:

$$(\alpha) \quad \sum_{\zeta \in \frac{1}{2}\Lambda_2/\Lambda_2} e(\alpha, \zeta)^2 \cdot \phi(\alpha+\beta+\zeta) \cdot \phi(\alpha-\beta+\zeta)$$
$$= \sum_{\substack{\zeta \in \frac{1}{2}\Lambda_2/\Lambda_2 \\ \eta \in \frac{1}{2}\Lambda_1/\Lambda_1}} e(\alpha, \zeta)^2 \cdot e(\alpha+\beta+\zeta, \eta) \cdot \Theta(2\alpha+2\zeta+\eta) \cdot \Theta(2\beta+\eta)$$
$$= \sum_{\eta \in \frac{1}{2}\Lambda_1/\Lambda_1} \Theta(2\alpha+\eta) \cdot \Theta(2\beta+\eta) \cdot e(\alpha+\beta, \eta) \cdot \Big[\sum_{\zeta \in \frac{1}{2}\Lambda_2/\Lambda_2} e(2\zeta, \eta)\Big]$$
$$= 2^g \cdot \Theta(2\alpha) \cdot \Theta(2\beta).$$

Now $\mathscr{S}_2(M_{2k+1})^*$ is spanned by the various functions:

$$f(\alpha) = \sum_{\eta \in \frac{1}{2}\Lambda_2/\Lambda_2} e(\eta, \beta_1+\beta_2) \cdot \phi_{[\beta_1-\eta]}(\alpha/2) \cdot \phi_{[\beta_2-\eta]}(\alpha/2)$$

where $\beta_i \in 2^{-k-1}\Lambda$. But this f comes out as:

$$f(\alpha) = 2^g \cdot \Theta_{[\beta_1+\beta_2]}(\alpha) \cdot \Theta(\beta_1-\beta_2) \in M_{2k+2}.$$

(2) now follows just like (1). *Q.E.D.*

Corollary. *If Θ is non-degenerate, then for all $k \geq 0$,*

$$\mathscr{S}_2(M_{2k}) = M_{2k+1}$$
$$\mathscr{S}_2(M_{2k+1})^* = M_{2k+2}.$$

Proof. The 2nd equality is clear, by condition (a‴) of the definition of non-degenerate. As for the first, note that by formula (α) in the proof of the Proposition,

$$2^g\,\Theta(\alpha)^2 = \sum_{\zeta\in\frac{1}{2}\Lambda_2/\Lambda_2} e(\alpha,\zeta)\cdot\phi(\alpha+\zeta)\cdot\phi(\zeta).$$

Therefore, $[\Theta(\alpha)\neq 0]\Rightarrow[\phi(\alpha+\zeta)\neq 0$, some $\zeta\in\frac{1}{2}\Lambda_2]$. Thus the non-degeneracy of Θ implies the same for ϕ, and the 1st equality follows too. *Q.E.D.*

In the following discussion, we shall assume that Θ is non-degenerate. As usual, if $R=\Sigma R_n$ is a graded ring, then $R(2)$ is the graded ring ΣR_{2n}. The Corollary shows that there exists a k_0 such that for all $k\geqq k_0$,

$$(\beta)\qquad \mathscr{S}(M_k)(2)\cong\mathscr{S}(M_{k+1}).$$

In particular, the corresponding schemes

$$X=\operatorname{Proj}(\mathscr{S}(M_k)),$$

for $k\geqq k_0$, are all canonically isomorphic. We shall prove eventually that this X is an abelian variety.

So far, we know that $\mathscr{S}(M_k)$ is finitely generated over k. Moreover, it has no nilpotents: if it did, it would have a homogeneous nilpotent element $f\in\mathscr{S}_n(M_k)$. Then $f\neq 0\Rightarrow f(\alpha)\neq 0$, some $\alpha\in V\Rightarrow f^N(\alpha)\neq 0$, all $N\Rightarrow f^N\neq 0$ in $\mathscr{S}_{nN}(M_k)$. Therefore, X is a reduced algebraic scheme over k. In fact, we can map

$$V/\Lambda\to X$$

by evaluating functions in $\mathscr{S}(M_k)$ at points of V. To be more precise, for all $\alpha\in V$, define a homogeneous prime ideal $P(\alpha)\subset\mathscr{S}(M_{2k})$ [resp. $P(\alpha)\subset\mathscr{S}(M_{2k+1})$] by:

$$P(\alpha)=\sum_n P_n(\alpha)$$

$$P_n(\alpha)=\{f\in S_n(M_{2k})\mid f(2^k\alpha)=0\}$$

resp.

$$=\{f\in S_n(M_{2k+1})\mid f(2^k\alpha)=0\}.$$

It is easy to check that for all k, if the $P(\alpha)$ in $\mathscr{S}(M_k)$ is intersected with $\mathscr{S}(M_k)$ (2), the resulting ideal is equal to the $P(\alpha)$ in $\mathscr{S}(M_{k+1})$ under the isomorphisms (β). For this reason, we omit a k in the notation $P(\alpha)$. Thus $P(\alpha)$ gives a well-defined point $\bar{P}(\alpha)\in X$. It follows easily from the definition that:

a) $\bar{P}(\alpha)$ *is a k-rational point of* X,

b) $\bar{P}(\alpha+\beta)=\bar{P}(\alpha)$, *if* $\beta\in\Lambda$.

Moreover:

c) $\{\bar{P}(\alpha)\,|\,\alpha\in V\}$ *is dense in* X.

Proof of c. Take $2k\geqq k_0$. If (c) were false, for large n, there would be a non-zero function $f\in\mathcal{S}_n(M_{2k})$ that vanished at all $\bar{P}(\alpha)$'s. But $f(\bar{P}(\alpha))=0\Leftrightarrow f(2^k\alpha)=0$, so f would vanish everywhere on V, hence $f=0$. *Q.E.D.*

One can do even more: for $\alpha\in V$, I claim that there is an automorphism $T_\alpha\colon X\to X$ such that $T_\alpha(\bar{P}(\beta))=\bar{P}(\alpha+\beta)$, all $\beta\in V$. To construct T_α, let k_1 be the least integer such that $2^{k_1}\alpha\in\Lambda$. Define

$$T_\alpha^*\colon\quad \mathcal{S}(M_{2k})\to\mathcal{S}(M_{2k})$$

resp.:

$$\mathcal{S}(M_{2k+1})\to\mathcal{S}(M_{2k+1})$$

by:

$$T_\alpha^* f(\beta)=e(\beta,2^{k-1}\alpha)^n\cdot f(\beta+2^k\alpha),\qquad \text{all } f\in S_n(M_{2k})$$

resp.

$$=e(\beta,2^k\alpha)^n\cdot f(\beta+2^k\alpha),\qquad \text{all } f\in S_n(M_{2k+1})$$

(where we assume $k\geqq k_1$). To check that this is, indeed, an automorphism of $\mathcal{S}(M_{2k})$ [resp. $\mathcal{S}(M_{2k+1})$], it suffices to check that $T_\alpha^*\,\Theta_{[\gamma]}\in M_{2k}$, all $\gamma\in 2^{-k}\Lambda$; and $T_\alpha^*\,\phi_{[\gamma]}\in M_{2k+1}$, all $\gamma\in 2^{-k-1}\Lambda$. But, in fact, one computes:

$$(\gamma)\qquad \begin{aligned} T_\alpha^*\,\Theta_{[\gamma]}&=e_*(2^{k-1}\alpha)\cdot e(\gamma,2^k\alpha)\cdot\Theta_{[\gamma]}\\ T_\alpha^*\,\phi_{[\gamma]}&=f_*(2^k\alpha)\cdot e(\gamma,2^{k+1}\alpha)\cdot\phi_{[\gamma]}.\end{aligned}$$

Moreover, one finds that T_α^*, acting on $\mathcal{S}(M_k)$, induces the same automorphism on $\mathcal{S}(M_k)\,(2)$ that you get by considering the T_α^* acting on $\mathcal{S}(M_{k+1})$ and carrying it across via the isomorphisms (β) of $\mathcal{S}(M_k)\,(2)$ and $\mathcal{S}(M_{k+1})$. Therefore, the T_α^*'s all define one and the same automorphism T_α of X. Note that:

d) $(T_\alpha^*)^{-1}(P(\beta))=P(\alpha+\beta)$.

Proof. If $f\in\mathcal{S}_n(M_{2k})$ or $\mathcal{S}_n(M_{2k+1})$, then

$$T_\alpha^* f\in P(\beta)\ \Leftrightarrow\ T_\alpha^* f(2^k\beta)=0\ \Leftrightarrow\ f(2^k\alpha+2^k\beta)=0\ \Leftrightarrow\ f\in P(\alpha+\beta),$$

hence

d′) $T_\alpha(\bar{P}(\beta))=\bar{P}(\alpha+\beta)$.

One checks also (via (γ) if you like) that:

e) $T_{\alpha_1+\alpha_2}=T_{\alpha_1}\circ T_{\alpha_2}$,

f) $T_\alpha=\mathrm{id.}\Leftrightarrow\alpha\in\Lambda$,

so that T is a faithful action of the group V/Λ on the scheme X.

A remarkable consequence of all this is:

Proposition 2. *If* Θ *is non-degenerate, then* $\mathcal{S}(M_k)$ *is an integral domain, for all* k.

Proof. We show first that $\mathscr{S}(M_k)$ is a domain if $k \geqq k_0$. Since $\mathscr{S}(M_k)$ has no nilpotents, this is equivalent to showing that X is irreducible. Now V/Λ acts on X, so it permutes the various components of X, i.e., we have a homomorphism:

$$V/\Lambda \to S = \begin{Bmatrix} \text{gp. of permutations} \\ \text{of components of } X \end{Bmatrix}.$$

But S is a *finite* group and V/Λ is a *divisible* group. So V/Λ must map each component X_i into itself. On the other hand, the collection of points $\{\bar{P}(\alpha)\}$ forms a single orbit of the action of V/Λ on X. Therefore, all these points $\{\bar{P}(\alpha)\}$ belong to a single component of X. Since they are also dense in X, X can have only a single component. Therefore $\mathscr{S}(M_k)$ is a domain if $k \geqq k_0$.

In general, suppose some $\mathscr{S}(M_k)$ were not a domain. Then there would be homogeneous elements $f \in \mathscr{S}_n(M_k)$, $g \in \mathscr{S}_m(M_k)$ such that $f \cdot g = 0$, $f \neq 0$, $g \neq 0$. Now f^2 and g^2 can be considered as elements of $\mathscr{S}(M_{k+1})$. Since $f \cdot g = 0$, we still have $f^2 \cdot g^2 = 0$. Also, since $\mathscr{S}(M_k)$ has no nilpotents, $f^2 \neq 0$ and $g^2 \neq 0$. Therefore $\mathscr{S}(M_{k+1})$ is not a domain either. Continuing in this way, we find that $\mathscr{S}(M_l)$ is not a domain for all $l \geqq k$, which contradicts the first part of the proof. *Q.E.D.*

Corollary 1. *The following are equivalent:*

i) *Θ is non-degenerate,*

ii) *$S_1 = V$, i.e., for all $\alpha \in V$, $\exists \eta \in \frac{1}{2}\Lambda$ such that $\Theta(\alpha+\eta) \neq 0$.*

iii) *For all $\alpha \in \frac{1}{4}\Lambda$, $\exists \eta \in \frac{1}{2}\Lambda$ such that $\Theta(\alpha+\eta) \neq 0$.*

Proof. Clearly (ii)$\Rightarrow$(iii)$\Rightarrow$(i). Now assume (i) holds. If $\Theta(\alpha+\eta)=0$, all $\eta \in \frac{1}{2}\Lambda$, then it would follow from the definition of ϕ that $\phi(\alpha+\beta) \times \phi(\beta) = 0$, all $\beta \in V$. But this means that $\phi_{[-\alpha]} \cdot \phi_{[0]} = 0$, i.e., one of the rings $\mathscr{S}(M_{2k+1})$ is not domain. This contradicts the Prop., so (ii) must hold. *Q.E.D.*

Corollary 2. $\mathscr{S}(M_k)\,(2) \cong \mathscr{S}(M_{k+1})$, *for all $k \geqq 2$.*

Proof. In view of Prop. 1, this follows from Cor. 1 provided that we check: $\forall \alpha \in V$, $\exists \eta \in \frac{1}{2}\Lambda$ such that $\phi(\alpha+\eta) \neq 0$. Looking back at the proof of the Cor. to Prop. 1, you see that this too follows from Cor. 1. *Q.E.D.*

To show that X is actually an abelian variety, we could either define the group law explicitly, using the addition formula of § 2, or else we can use only the action of V/Λ on X and combine this with general structure theorems on the automorphisms of a variety. Although the former is more elementary, we follow the latter approach as it is quicker.

X is given to us together with a projective embedding. For example, $X = \operatorname{Proj}(\mathscr{S}(M_2))$, so

$$X \subset \boldsymbol{P}(M_2).$$

Let L_2 be the invertible sheaf induced on X via this embedding. If, via the isomorphism $X \cong \mathrm{Proj}\,(\mathscr{S}(M_k))$, we embed X in $\boldsymbol{P}(M_k)$, the induced sheaf L_k is just:

$$L_k \cong L_2^{2k-2}.$$

Let $\mathscr{P}$ denote the family of all invertible sheaves algebraically equivalent to L_2. We shall use the fact that Aut $(X, \mathscr{P})$, the group of automorphisms of the pair X, $\mathscr{P}$, is an algebraic group (MATSUSAKA [*14*], GROTHENDIECK [*15*], p. 221–20). For all $\alpha \in V/\Lambda$, if $2^k \alpha \in \Lambda$, then T_α is induced by an automorphism T_α^* of $\mathscr{S}(M_{2k})$; therefore $T_\alpha^*(L_{2k}) \cong L_{2k}$; therefore $T_\alpha^*(L_2)$ differs from L_2 by an invertible sheaf of finite order; therefore $T_\alpha^{-1}(\mathscr{P}) = \mathscr{P}$. In other words, the action of V/Λ on X factors through an injective homomorphism:

$$V/\Lambda \to \mathrm{Aut}\,(X, \mathscr{P}).$$

Let A be the Zariski-closure of V/Λ in Aut $(X, \mathscr{P})$. Then A is connected since V/Λ is divisible and dense in A (cf. proof of Prop. 2), and A is commutative since V/Λ is commutative and dense in A. Moreover, since the V/Λ-orbit of $\bar{P}_0$ is dense in X, the A-orbit of $\bar{P}_0$ must be an open dense set in X, i.e., A acts generically transitively on X. In fact, the morphism

$$\begin{aligned} \psi:\ & A \to X \\ & \sigma \mapsto \sigma(\bar{P}_0) \end{aligned}$$

is an open immersion of A in X. This follows since the image $\psi(A)$ is always isomorphic to A/H, H = the stabilizer of $\bar{P}_0$; and since A is commutative and acting faithfully on X, all stabilizers are trivial.

Next, we want to compute the dimension of X. I claim that the Hilbert polynomial of (X, L_2) is given by:

Proposition 3. $\chi(L_2^n) = 4^g \cdot n^g$.

Proof. For k large,

$$\begin{aligned} \chi(L_2^{2^{2k}}) &= \dim\big(S_{2^{2k}}(M_2)\big) \\ &= \dim(M_{2+2k}). \end{aligned}$$

Now $M_{2(k+1)}$ is, by definition, the span of the $2^{2g(k+1)}$ functions $\Theta_{[\beta]}$, where β runs over cosets of $2^{-k-1}\Lambda/\Lambda$. But these functions are linearly independent. To see this, look at the automorphisms T_α^* of $\mathscr{S}(M_{2(k+1)})$, where $\alpha \in 2^{-k-1}\Lambda$. Use formulae ($\gamma$) above and note that each $\Theta_{[\gamma]}$ gives rise to a distinct set of eigenvalues for the T_α^*'s. Therefore, the $\Theta_{[\gamma]}$'s could not be dependent unless one were identically zero, and this is not the case. Therefore

$$\dim M_{2(k+1)} = 4^g \cdot (2^{2k})^g.$$

This shows that $\chi(L_2^n)$ and $4^g \cdot n^g$ agree for an infinite set of values of n. Since both are polynomials, they are always equal. *Q.E.D.*

Corollary. dim $X=g$.

Returning to A, we find that A is a commutative g-dimensional algebraic group containing a subgroup isomorphic to $(\mathbf{Q}_2/\mathbf{Z}_2)^{2g}$. From well-known structure theorems on algebraic groups, the only such A's are abelian varieties. Therefore A is complete, hence $A=X$, hence:

(I) *X is an abelian variety.*

Moreover, in the course of proving this, we have also found that V/Λ is acting on X via translations, hence (comparing orders) we find:

(II) $\alpha \mapsto \bar{P}(\alpha)$ *is a group isomorphism of* V/Λ *with* $\mathrm{tor}_2(X)$.

Up to this point, identifying the various $\mathrm{Proj}(\mathscr{S}(M_k))$'s has been useful. But to go further, it is more convenient now to drop these identifications. Therefore, now let

$$X_n = \mathrm{Proj}(\mathscr{S}(M_{2n})).$$

This is a family of isomorphic abelian varieties. However, the most natural maps between them are given by the inclusions:

$$M_{2n} \subset M_{2n+2}$$

$$\mathscr{S}(M_{2n}) \subset \mathscr{S}(M_{2n+2})$$

inducing finite morphisms:

$$X_n \xleftarrow{\;p\;} X_{n+1}.$$

To check that p is defined, we must know that $\mathscr{S}(M_{2n+2})$ is integrally dependent on $\mathscr{S}(M_{2n})$. But I claim:

$$\Theta(\gamma)^2 \cdot \Theta_{[\beta]}^2 = 2^{-g} \cdot \sum_{\eta \in \frac{1}{2}\Lambda/\Lambda} e(\eta, \gamma)\, \Theta(\eta)^2 \cdot \Theta_{[\beta+\gamma-\eta]} \cdot \Theta_{[\beta-\gamma+\eta]}.$$

[*Proof.* $\Theta(\gamma)^2 \cdot \Theta_{[\beta]}(\alpha)^2 = e(\beta, \alpha)\, \Theta(\gamma)\, \Theta(\gamma)\, \Theta(\beta-\alpha)\, \Theta(\alpha-\beta)$.

By the quartic relations on Θ, we get

$$= 2^{-g} e(\beta, \alpha) \sum_{\eta} e(-\gamma, \eta)\, \Theta(\eta)^2\, \Theta(\beta-\alpha-\gamma+\eta)\, \Theta(\alpha-\beta-\gamma+\eta)$$

$$= 2^{-g} \sum_{\eta} e(\eta, \gamma)\, \Theta(\eta)^2 \cdot \Theta_{[\beta+\gamma-\eta]}(\alpha) \cdot \Theta_{[\beta-\gamma+\eta]}(\alpha). \quad Q.E.D.]$$

Choose $\gamma \in \beta + \frac{1}{2}\Lambda$ so that $\Theta(\gamma) \neq 0$. Then if $\beta \in 2^{-n-1}\Lambda$, this equation shows that $\Theta_{[\beta]}^2 \in \mathscr{S}(M_{2n})$. This proves that p is a finite morphism. Since X_n and X_{n+1} are abelian varieties, p must be an isogeny.

Define prime ideals:

$$P^{(k)}(\alpha) \subset \mathcal{S}(M_{2k})$$

via

$$P^{(k)}(\alpha) = \sum_n P_n^{(k)}(\alpha)$$

$$P_n^{(k)}(\alpha) = \{f \in \mathcal{S}_n(M_{2k}) \mid f(\alpha) = 0\}.$$

Then $P^{(k)}(\alpha)$ defines a k-rational point $\psi_k(\alpha) \in X_k$. We have

(a) $p(\psi_{k+1}(\alpha)) = \psi_k(\alpha)$.

(b) $\alpha \mapsto \psi_k(\alpha)$ defines an isomorphism

$$V/2^k \Lambda \xrightarrow{\approx} \operatorname{tor}_2(X_k).$$

(b) here follows from conclusion (II) above, noticing how we have reinterpreted the ideal $P(\alpha)$. In fact, if we call X the common abelian variety to which all the X_k's were previously identified, then $\bar{P}(\alpha) \in X$ corresponds exactly to $\psi_k(2^k \alpha) \in X_k$. Therefore $\psi_k(\alpha) = 0 \Leftrightarrow \bar{P}(2^{-k}\alpha) = 0 \Leftrightarrow 2^{-k}\alpha \in \Lambda$. Moreover, this shows that via these identifications, we get a morphism:

$$\begin{array}{cc}
X & \bar{P}(\alpha) \\
\wr\| & \downarrow \\
X_{k+1} & \psi_{k+1}(2^{k+1}\alpha) \\
{\scriptstyle p}\downarrow & \downarrow \\
X_k & \psi_k(2^{k+1}\alpha) \\
\wr\| & \downarrow \\
X & \bar{P}(2\alpha) = 2\bar{P}(\alpha).
\end{array}$$

This map, from X to X, agrees with 2δ at all points $\bar{P}(\alpha)$. Therefore it is equal to 2δ. In particular:

(c) The degree of p is 2^{2g} and $\operatorname{Ker}(p) = \operatorname{Ker}(2\delta)$. It follows that all the X_n's generate a single 2-tower. Call this $\underset{\leftarrow}{X} = \{X_\alpha\}_{\alpha \in S}$, and let $X_n = X_{\alpha_n}$, $\alpha_n \in S$. Moreover, these α_n's are a cofinal set in S, by (c). In view of (a)

$$\alpha \mapsto \{\psi_k(\alpha)\}$$

defines a homomorphism

$$\psi\colon V \to V(\underset{\leftarrow}{X}),$$

and (b) implies that ψ is an isomorphism. More, (b) shows that the compact open subgroups $2^k \Lambda$ and $T(\alpha_k)$ correspond to each other under ψ.

This 2-tower is polarized too. Let L_k be the sheaf $\underline{o}(1)$ on X_k coming from its presentation as $\operatorname{Proj}(\mathcal{S}(M_{2k}))$. Since the p's comes from gradation preserving homomorphisms of the $\mathcal{S}(M_{2k})$'s it follows that $p^*(L_k) \cong L_{k+1}$. To check that L_k is totally symmetric, we need the inverse on X_k:

Let $\iota^*(f)(\alpha)=f(-\alpha)$, all $f\in\mathcal{S}(M_{2k})$.

Then ι^* defines an involution

$$\iota\colon X_k\longrightarrow X_k$$

such that $\iota(\psi_k(\alpha))=\psi_k(-\alpha)$.

Therefore ι agrees with the inverse of X_k on all points $\psi_k(\alpha)$, hence $\iota=$ inverse of X_k.

Since ι is induced at all by an automorphism ι^* of $\mathcal{S}(M_{2k})$, it follows that L_k is at least a symmetric sheaf. Since

$$\{\psi_k(\alpha)\,|\,\alpha\in 2^{k-1}\Lambda/2^k\Lambda\}=\text{Kernel of } 2\delta \text{ in } X_k,$$

L_k is totally symmetric if and only if ι^* is the identity in $\mathcal{S}(M_{2k})/P^{(k)}(\alpha)$, all $\alpha\in 2^{k-1}\Lambda$. This means that for all $f\in M_{2k}$, $\iota^*f-f\in P_1^{(k)}(\alpha)$, i.e., $f(\alpha)=f(-\alpha)$. But M_{2k} is spanned by $\Theta_{[\beta]}$'s, $\beta\in 2^{-k}\Lambda$, and if $\beta\in 2^{-k}\Lambda$, $\alpha\in 2^{k-1}\Lambda$, then:

$$\Theta_{[\beta]}(-\alpha)=e\left(\frac{\beta}{2},\,-\alpha\right)\Theta(-\alpha-\beta)=e\left(\frac{\beta}{2},\,\alpha\right)\Theta(\alpha-\beta)=\Theta_{[\beta]}(\alpha).$$

Therefore all the L_n's are totally symmetric and $\{X_n, L_n\}$ extends to a polarized 2-tower $\mathcal{T}=\{X_\alpha, L_\alpha\}$. We shall leave it to the reader to check the key fact that ψ is symplectic:

(d) $e_\lambda(\psi\alpha,\psi\beta)=e(\alpha,\beta)$, all $\alpha,\beta\in V$.

Recapitulating this whole section so far, we have defined an arrow:

$$\Xi\colon \left\{\begin{matrix}\text{Given a non-degenerate}\\ \text{theta function } \Theta \text{ on } V\end{matrix}\right\}\longrightarrow\left\{\begin{matrix}\text{construct a polarized}\\ \text{2-tower } \mathcal{T}=\{X_\alpha, L_\alpha\},\\ \text{plus a symplectic isomorphism}\\ \psi\colon V\xrightarrow{\approx} V(\underset{\leftarrow}{X})\end{matrix}\right\}.$$

Now, on V we have the vector space of functions spanned by all the $\Theta_{[\beta]}$'s. On $V(\underset{\leftarrow}{X})$, we have the vector space of all theta functions $\vartheta[\Gamma(\mathcal{T})]$ of the tower $\mathcal{T}$.

Proposition 4. *Via ψ, these vector spaces are equal:*

Span of $\Theta_{[\beta]}$'s $=\{\vartheta_{[s]}\circ\psi\,|\,s\in\Gamma(\mathcal{T})\}$.

Moreover, Θ itself is the unique function f (up to scalars) of the form $\vartheta_{[s]}\circ\psi$ satisfying the functional equation:

$$f(\alpha+\beta)=e_*(\beta/2)\cdot e(\beta/2,\alpha)\cdot f(\alpha),\qquad \text{all } \alpha\in V,\ \beta\in\Lambda.$$

Key Corollary 1. *If $V=\mathbf{Q}_2^{2g}$, $\Lambda=\mathbf{Z}_2^{2g}$, and e, e_* have the standard forms of § 9, then Θ is exactly the theta function $\vartheta\begin{bmatrix}0\\0\end{bmatrix}\circ\psi$ associated to the* *triple* $(\underset{\sim}{X}, \mathcal{T}, \psi^{-1})$ *in* § 9. *In other words,* Ξ *is an inverse to the map* Θ *of* § 9.

Proof of Prop. 4. Let $\alpha \in 2^{-k_1}\Lambda$ and let $k \geqq k_1$. Define $T_\alpha^*: \mathcal{S}(M_{2k}) \to \mathcal{S}(M_{2k})$ slightly differently from before:

$$T_\alpha^* f(\beta) = e\left(\beta, \frac{\alpha}{2}\right)^n \cdot f(\beta+\alpha), \qquad \text{all } f \in S_n(M_{2k}).$$

Note $T_\alpha^{*-1}(P^{(k)}(\beta)) = P^{(k)}(\alpha+\beta)$. Let $T_\alpha: X_k \to X_k$ be the automorphism induced by T_α^*. Then $T_\alpha(\psi_k(\beta)) = \psi_k(\alpha+\beta)$, hence T_α is translation by the point $\psi_k(\alpha)$, i.e.,

$$T_\alpha = T_{\psi_k(\alpha)}.$$

Moreover, T_α^* also induces a compatible isomorphism:

$$g_k(\alpha): L_k \xrightarrow{\sim} T_{\psi_k(\alpha)}^* L_k.$$

For all $k \geqq k_1$, these are compatible, so the totality of pairs

$$g(\alpha) = \{(\psi_k(\alpha), g_k(\alpha) \mid k \geqq k_1\}$$

is a point of $\mathcal{G}(\mathcal{T})$.

(*) $g(\alpha) = \sigma[\psi(\alpha)]$, i.e., $g(\alpha)$ is the canonical element of $\mathcal{G}(\mathcal{T})$ over the point $\psi(\alpha)$ in $V(\underset{\sim}{X})$.

Proof of *. This requires checking 2 things: (i) $g(\alpha)$ is a symmetric element of $\mathcal{G}(\mathcal{T})$, i.e., $\delta_{-1} g(\alpha) = g(\alpha)^{-1}$, and (ii) $g(2\alpha) = g(\alpha)^2$. In terms of T_α^*, this is the same as:

(i) $\iota^* \circ T_\alpha^* = (T_\alpha^*)^{-1} \circ \iota^*$.

(ii) $T_{2\alpha}^* = T_\alpha^* \circ T_\alpha^*$.

These are both immediate. *Q.E.D.*

Next, notice that $M_{2k} \cong \Gamma(X_k, L_k)$. In fact, there is a canonical map $M_{2k} \to \Gamma(X_k, L_k)$; it is injective, since the ring $\mathcal{S}(M_{2k})$ has no nilpotents, and only nilpotent elements of $\mathcal{S}_n(M_{2k})$ define trivial sections of L_k^n; but it is easy to check that both dim M_{2k} and dim $\Gamma(X_k, L_k)$ are equal to 2^{2kg}; therefore $M_{2k} \cong \Gamma(X_k, L_k)$. Therefore,

$$\Gamma(\mathcal{T}) = \varinjlim_k \Gamma(X_k, L_k) \cong \bigcup_k M_{2k} = \left\{\begin{matrix}\text{Span of } all \text{ the} \\ \text{functions } \Theta_{[\beta]} \\ \beta \in V\end{matrix}\right\}.$$

Now let f be some linear combination of the $\Theta_{[\beta]}$. Say $f \in M_{2k_1}$. Let f define $s \in \Gamma(X_{k_1}, L_{k_1})$. I claim that:

$$(*) \qquad f(\alpha) = \vartheta_{[s]}(\psi\alpha), \qquad \text{all } \alpha \in V.$$

Taking a larger k_1 if necessary, we may suppose that $\alpha \in 2^{-k_1}\Lambda$. By definition, $\vartheta_{[s]}$ at $\psi\alpha$ is the "value" at the origin of X_{k_1} of the section of L_{k_1} obtained via the map:

$$\Gamma(X_{k_1}, L_{k_1}) \xrightarrow[g_{k_1}(-\alpha)]{\sim} \Gamma(X_{k_1}, T^*_{\psi_{k_1}(-\alpha)} L_{k_1}) \xrightarrow[T^*_{\psi_{k_1}(\alpha)}]{\sim} \Gamma(X_{k_1}, L_{k_1}).$$

This means that we simply apply the automorphism $(T^*_{-\alpha})^{-1}$ of M_{2k} to f, and take the value at the origin. But $T^*_{-\alpha} = T^{*-1}_\alpha$, and $(T^*_\alpha f)(0) = f(\alpha)$, so (*) is proven. Thus the span of the $\Theta_{[\beta]}$'s is the same as the space of functions $\vartheta_{[s]} \circ \psi$, $s \in \Gamma(\mathscr{T})$.

As for the final assertion of the Proposition, on the one hand, Θ does satisfy the functional equation there; and, from the general theory of the space $\vartheta[\Gamma(\mathscr{T})]$ in § 8, we know that this functional equation has only a 1-dimensional set of solutions in $\vartheta[\Gamma(\mathscr{T})] \circ \psi$. *Q.E.D.*

Corollary 2. *All g-dimensional principally polarized abelian varieties X are isomorphic to* Proj $(\mathscr{S}(M_2))$, *where* M_2 *is the span of the* $\Theta_{[\beta]}$'s, $\beta \in \frac{1}{2}\Lambda$, *for some non-degenerate theta function* Θ *on V.*

Proof. Just take Θ to be the $\vartheta\begin{bmatrix}0\\0\end{bmatrix}$ attached to X as in § 9, and carried over to a function on V by a suitable isomorphism of V and $V(X)$. *Q.E.D.*

Corollary 3. *The open set* $M_\infty \subset \overline{M}_\infty$, *which in* § 9 *represents the moduli functor* $\mathscr{M}_\infty$, *is the open set whose geometric points represent non-degenerate theta functions, i.e.,*

$$E = \left\{\begin{matrix}\text{set of all systems of coset representatives}\\ r\colon \frac{1}{4}Z_2^{2g}/\frac{1}{2}Z_2^{2g} \to \frac{1}{4}Z_2^{2g}\end{matrix}\right\}.$$

For all $r \in E$, *let*

$$U_r = \left\{\begin{matrix}\text{open set in } \overline{M}_\infty \text{ defined by}\\ X_\alpha \neq 0, \text{ all } \alpha \in \text{Image}(r)\end{matrix}\right\}.$$

Then

$$M_\infty = \bigcup_{r \in E} U_r.$$

§ 11. Satake's Compactification

In this section, I want to analyze the degenerate theta functions Θ on V, in the sense of § 10. In particular, they all come from lower dimensional non-degenerate theta-functions via "cusps". This will show that the whole moduli scheme $\overline{M}_\infty$ is a disjoint union of copies of the M_∞'s for dimensions *g and lower* i.e., that $\overline{M}_\infty$ is the Satake compactification of M_∞[1].

[1] *Added in Proof.* A closer study has shown that $\overline{M}_\infty$ is *not normal* along $\overline{M}_\infty - M_\infty$. Its normalization is Satake's compactification.

Return to the discussion at the beginning of § 10: let V, Λ, e, e^* be given as before. First, I want to describe a way of forming degenerate theta functions on V out of theta functions on lower dimensional spaces.

Definition 1. A *cusp* is a subspace $W \subset V$ such that $W^\perp \subset W$, i.e., if $\alpha \in V$ has the property $e(\alpha, \beta) = 1$, all $\beta \in W$, then $\alpha \in W$.

Given a cusp W, let:

$$\tilde{V} = W/W^\perp$$
$$\tilde{\Lambda} = \Lambda \cap W / \Lambda \cap W^\perp$$
$$\tilde{e} = \text{induced skew-symmetric pairing, } \tilde{V} \times \tilde{V} \to k^*.$$

Lemma. *$\tilde{\Lambda}$ is a maximal isotropic lattice in $\tilde{V}$, (for $\tilde{e}$).*

Proof. Notice that $\Lambda/\Lambda \cap W$ is a free $\boldsymbol{Z}_2$-module. Therefore the sequence:

$$0 \to \Lambda \cap W \to \Lambda \to \Lambda/\Lambda \cap W \to 0$$

splits, and $\Lambda = \Lambda_1 \oplus (\Lambda \cap W)$ for some sub $\boldsymbol{Z}_2$-Module Λ_1. Let $V_1 = \boldsymbol{Q}_2 \cdot \Lambda_1$, so $V = V_1 \oplus W$. Now I claim:

$$(*) \qquad (\Lambda \cap W)^\perp = \Lambda + W^\perp.$$

[In fact, let $\alpha \in V$ satisfy $e(\alpha, \beta) = 1$, all $\beta \in \Lambda \cap W$. Since V_1 and W are dual vector spaces via e, there is a $\gamma \in W^\perp$ such that $e(\alpha, \beta) = e(\gamma, \beta)$ all $\beta \in V_1$. But then $\alpha - \gamma$ is orthogonal to both V_1 and $\Lambda \cap W$, hence orthogonal to Λ, hence $\alpha - \gamma \in \Lambda$. Thus $\alpha \in W^\perp + \Lambda$.]

Now to show $\tilde{\Lambda}$ is maximal isotropic, let $\alpha \in W$ have an image $\tilde{\alpha}$ in $\tilde{V}$ perpendicular to $\tilde{\Lambda}$, i.e., $\alpha \in (W \cap \Lambda)^\perp$. By $(*)$, $\alpha = \alpha_1 + \alpha_2$, where $\alpha_1 \in \Lambda$, $\alpha_2 \in W^\perp$. But then $\alpha_1 = \alpha - \alpha_2 \in W$. Therefore $\alpha_1 \in W \cap \Lambda$ so $\tilde{\alpha} = \tilde{\alpha}_1 \in \tilde{\Lambda}$. *Q.E.D.*

Definition 2. A cusp with origin is a cusp $W \subset V$, plus an element $\eta_0 \in \frac{1}{2}\Lambda$ such that

i) $e_*(\alpha) = e(\alpha, \eta_0)^2$, all $\alpha \in W^\perp \cap (\frac{1}{2}\Lambda)$.

ii) $e_*(\eta_0) = 1$.

It is not hard to check that every cusp has at least one origin: we leave this to the reader. Given a cusp with origin, look at the map

$$\alpha \mapsto e_*(\alpha) \cdot e(\alpha, \eta_0)^2$$

where $\alpha \in \frac{1}{2}\Lambda \cap W$. If $\beta \in \frac{1}{2}\Lambda \cap W^\perp$, then

$$e_*(\alpha+\beta) \cdot e(\alpha+\beta, \eta_0)^2 = e_*(\alpha) \cdot e_*(\beta) \cdot e(\alpha, \beta)^2 \cdot e(\alpha, \eta_0)^2 \cdot e(\beta, \eta_0)^2$$
$$= e_*(\alpha) \cdot e(\alpha, \eta_0)^2.$$

Thus there is a quadratic form $\tilde{e}_*: \frac{1}{2}\tilde{\Lambda}/\tilde{\Lambda} \to \{\pm 1\}$ such that

$$(*) \qquad \tilde{e}_*(\tilde{\alpha}) = e_*(\alpha) \cdot e(\alpha, \eta_0)^2, \qquad \text{all } \alpha \in \tfrac{1}{2}\Lambda \cap W.$$

It is not hard to check that the new data $(\tilde{V}, \tilde{\Lambda}, \tilde{e}, \tilde{e}_*)$ has the standard form required in § 10 (i.e., that the associated Arf-invariant is 0). We leave this to the reader also.

Now let $\tilde{\Theta}$ be a theta-function on $\tilde{V}$.

Definition 3. For all $\alpha \in V$, let

$$T_{W,\eta_0}\Theta(\alpha) = \begin{cases} 0 & \text{if } \alpha \notin \eta_0 + W + \Lambda \\ e_*\left(\dfrac{\eta_1}{2}\right) e\left(\dfrac{\eta_1}{2}, \eta_0\right) e\left(\dfrac{\eta_0+\eta_1}{2}, \alpha\right) \tilde{\Theta}(\tilde{\alpha}_0) & \\ & \text{if } \alpha = \eta_0 + \eta_1 + \alpha_0,\ \eta_1 \in \Lambda,\ \alpha_0 \in W. \end{cases}$$

Proposition 1. *The above $T_{W,\eta_0}\tilde{\Theta}$ is well-defined (note that the $\alpha \in V$ may be decomposed in more than way as $\alpha = \eta_0 + \eta_1 + \alpha_0$), and is a theta-function on V.*

The proof of this Proposition is a ghastly but wholly straightforward set of computations. It took me several hours to do every bit and as I was no wiser at the end — except that I knew the definition was correct — I shall omit details here. Our main result is:

Theorem. *Let Θ be any theta-function on V, and let W be the subspace of V such that $S_\infty = W + \Lambda$* (cf. § 10). *Then W is a cusp, and if η_0 is any origin for W, Θ is equal to $T_{W,\eta_0}\tilde{\Theta}$ for some non-degenerate theta-function $\tilde{\Theta}$ on $\tilde{W}$. In particular, W is characterized by:*

$$\textit{coarse support}\ (\Theta) = W + \tfrac{1}{2}\Lambda.$$

The proof of this theorem will be based on the $\Theta \leftrightarrow \mu$ correspondence, given in Lemma 1, § 8. Before taking up the proof of the Theorem, we want to give this correspondence a more intrinsic formulation. Let $V = W_1 \oplus W_2$, where W_i are maximal isotropic subspaces, such that

i) $\Lambda = \Lambda_1 \oplus \Lambda_2$, $\Lambda_i = \Lambda \cap W_i$.

ii) $e_*(\alpha/2) = 1$, all α in Λ_1 or in Λ_2.

Then

a) Define a measure μ on W_1, from a theta function Θ on V via

$$\mu(\alpha_1 + 2^n\Lambda_1) = 2^{-ng} \sum_{\alpha_2 \in 2^{-n}\Lambda_2/\Lambda_2} e\left(\alpha_1, \frac{\alpha_2}{2}\right) \cdot \Theta(\alpha_1 + \alpha_2).$$

b) Define a theta function Θ on V, from a measure μ on W_1, via

$$\Theta(\alpha_1 + \alpha_2) = e\left(\alpha_1, \frac{\alpha_2}{2}\right) \int_{\alpha_1 + \Lambda_1} e(\alpha_2, \beta) \cdot d\mu(\beta).$$

Our proof will be based on the fact that any finitely additive measure μ (on the algebra of compact open subsets of W_1) has a *support*, i.e., a smallest closed set S such that:

$$\mu(U)=0, \quad \text{all compact open } U\text{'s in } W_1-S.$$

Proof. Say S_A and S_B are closed sets such that $\mu(U)=0$ if $U\subset W_1-S_A$ or $U\subset W_1-S_B$. Then let $U\subset W_1-(S_A\cap S_B)$ be a compact open set. We must decompose U into $U_A\cup U_B$, where $U_A\subset W_1-S_A$, and $U_B\subset W_1-S_B$, and U_A and U_B are compact and open. For all $x\in U\cap S_A$, note that $x\notin S_A$, so we can find a compact, open neighborhood U_x of x such that

$$U_x\subset U\cap(W_1-S_B).$$

Since $U\cap S_A$ is compact, it can be covered by a finite set of these U_x's: say

$$U\cap S_A\subset[U_{x_1}\cup\cdots\cup U_{x_n}].$$

Let $U_B=U_{x_1}\cup\cdots\cup U_{x_n}$. By construction $U_B\subset U\cap(W_1-S_B)$ and U_B is compact and open. Let $U_A=U-U_B$. Then U_A is also compact and open and since $U_B\supset U\cap S_B$, it follows that $U_A\subset U\cap(W_1-S_B)$. By assumption on S_A and S_B, we have $\mu(U_A)=0$ and $\mu(U_B)=0$. Therefore $\mu(U)=0$. This shows that the family of sets:

$$\mathscr{S}=\{S \text{ closed in } W_1 \mid \mu(U)=0 \text{ for all compact open sets } U\subset W_1-S\}$$

is closed under finite intersections. Now let

$$S^*=\bigcap_{S\in\mathscr{S}} S.$$

I claim $S^*\in\mathscr{S}$ too. Let $U\subset W_1-S^*$ be a compact open set. Since

$$W_1-S^*=\bigcup_{S\in\mathscr{S}}(W_1-S),$$

it follows that U is covered by the open sets $U\cap(W_1-S)$, where $S\in\mathscr{S}$. Since U is compact, it can be covered by a finite number of such open sets:

$$U\subset(W_1-S_1)\cup\cdots\cup(W_1-S_n)$$

where $S_1,\ldots,S_n\in\mathscr{S}$. Now let $T\in\mathscr{S}$ be a closed set contained in all these S_i. Then $U\subset W_1-T$. But $T\in\mathscr{S}$ means that this implies $\mu(U)=0$. So $\mu(U)=0$ whenever $U\subset W_1-S^*$, i.e., $S^*\in\mathscr{S}$ too. *Q.E.D.*

Proposition. *Let μ be a non-zero even Gaussian measure on W_1* (i.e., *μ has the property* (A) *of Lemma* 1, § 8). *Then the support S of μ is a sub-vector space of W_1.*

Proof. Notice that if μ_1, μ_2 are 2 measures on W_1, and $\mu_1\times\mu_2$ is the induced measure on $W_1\times W_1$, then

$$\mathrm{Support}(\mu_1\times\mu_2)=\mathrm{Support}(\mu_1)\times\mathrm{Support}(\mu_2).$$

Let ξ: $W_1\times W_1\to W_1\times W_1$ be the map $\xi((x,y))=(x+y,x-y)$. By definition, a Gaussian measure μ is associated to a second measure ν such that

$$\xi_*(\mu\times\mu)=\nu\times\nu.$$

Therefore, if $S'=\mathrm{Support}\ (\nu)$, it follows that $\xi(S\times S)=S'\times S'$. In particular

$$\begin{aligned}\alpha\in S &\Leftrightarrow (\alpha,\alpha)\in S\times S\\ &\Leftrightarrow (2\alpha,0)=\xi((\alpha,\alpha))\in S'\times S'.\end{aligned}$$

Since S is non-empty, $0\in S'$, and $\alpha\in S\Leftrightarrow 2\alpha\in S'$, i.e., $S'=2S$. Therefore $0\in S$ too, and we find:

$$\begin{aligned}\alpha\in S &\Leftrightarrow (\alpha,0)\in S\times S\\ &\Leftrightarrow (\alpha,\alpha)=\xi((\alpha,0))\in S'\times S'\\ &\Leftrightarrow \alpha\in S'.\end{aligned}$$

Therefore $S=S'$ also. Finally,

$$\begin{aligned}\alpha,\beta\in S &\Rightarrow (\alpha,\beta)\in S\times S\\ &\Rightarrow (\alpha+\beta,\alpha-\beta)\in S'\times S'\\ &\Rightarrow \alpha+\beta,\alpha-\beta\in S'=S.\end{aligned}$$

Thus S is a closed subgroup of W_1, such that $S=2S$. Therefore S is a subvectorspace over $\boldsymbol{Q}_2$. *Q.E.D.*

Corollary. *For all* $\gamma_2\in W_2$, *all theta functions* Θ *on* V,

$$\mathit{Support}(\Theta)\subset\{\alpha\mid e(\alpha,\gamma_2)=1\}\ \Rightarrow\ \Theta(\alpha+\lambda\gamma_2)=e\left(\alpha,\frac{\lambda\gamma_2}{2}\right)\Theta(\alpha),$$
$$\text{all } \lambda\in\boldsymbol{Q}_2.$$

Proof. The assumption on the support of Θ implies (cf. (a) above) that $\mu(\alpha_1+2^n\Lambda_1)=0$ if $e(\alpha_1,\gamma_2)\neq 1$. Therefore,

$$\mathrm{Support}(\mu)\subset\{\alpha_1\in W_1\mid e(\alpha_1,\gamma_2)=1\}.$$

Since this support is a vector space,

$$\mathrm{Support}(\mu)\subset W_1\cap(\boldsymbol{Q}_2\cdot\gamma_2)^\perp.$$

Let H denote the hyperplane $W_1 \cap (\mathbf{Q}_2 \cdot \gamma_2)^{\perp}$. Then

$$\Theta(\alpha_1+\alpha_2)=e\left(\alpha_1, \frac{\alpha_2}{2}\right) \int_{(\alpha_1+\Lambda_1)\cap H} e(\alpha_2, \beta) \cdot d\mu(\beta).$$

Thus

$$\Theta(\alpha_1+\alpha_2+\lambda\gamma_2)=e\left(\alpha_1, \frac{\alpha_2+\lambda\gamma_2}{2}\right) \int_{(\alpha_1+\Lambda_1)\cap H} e(\alpha_2+\lambda\gamma_2, \beta) \cdot d\mu(\beta)$$

and since $e(\lambda\gamma_2, \beta)=1$ when $\beta \in H$, this comes out

$$=e\left(\alpha_1, \frac{\lambda\gamma_2}{2}\right) \cdot \left\{e\left(\alpha_1, \frac{\alpha_2}{2}\right) \int_{(\alpha_1+\Lambda_1)\cap H} e(\alpha_2, \beta) \cdot d\mu(\beta)\right\}$$

$$=e\left(\alpha_1, \frac{\lambda\gamma_2}{2}\right) \cdot \Theta(\alpha_1+\alpha_2). \qquad Q.E.D.$$

In fact, I claim that the same Corollary holds *for all* $\gamma \in V$, not just *for* $\gamma \in W_2$. This can be seen by noting that for any $\gamma \in V$, there is a symplectic automorphism T: $V \to V$ such that $T(\Lambda)=\Lambda$, i.e., $T \in \mathrm{Sp}(V, \Lambda)$, such that $T^{-1}(\gamma) \in W_2$. Going back to the action of the symplectic group introduced in § 9, we see that:

$$\begin{cases} \text{If } \Theta \text{ is a theta-function, then so is } \Theta', \text{ where} \\ \qquad \Theta'(\alpha)=e(\eta/2, \alpha)\, \Theta(T\alpha - T\eta) \\ \text{where } \eta \in \tfrac{1}{2}\Lambda \text{ satisfies} \\ \qquad e_*(\alpha/2) \cdot e_*(T\alpha/2)=e(\eta, \alpha), \quad \text{all } \alpha \in \Lambda. \end{cases}$$

Now assume $\mathrm{Supp}(\Theta) \subset \{\alpha \mid e(\alpha, \gamma)=1\}$. Then

$$\begin{aligned} \mathrm{Supp}(\Theta') &= \eta + T^{-1}(\mathrm{Supp}(\Theta)) \\ &\subset \eta + \{\alpha \mid e(\alpha, T^{-1}\gamma)=1\} \\ &\subset \{\alpha \mid e(\alpha, 2^n T^{-1}\gamma)=1\} \qquad (\text{if } n \geqslant 0). \end{aligned}$$

Therefore, by the Corollary

$$\Theta'(\alpha+\lambda T^{-1}\gamma)=e\left(\alpha, \frac{\lambda T^{-1}\gamma}{2}\right) \Theta'(\alpha), \qquad \text{all } \lambda \in \mathbf{Q}_2,$$

from which

$$\Theta(\alpha+\lambda\gamma)=e\left(\alpha, \frac{\lambda\gamma}{2}\right) \cdot \Theta(\alpha)$$

follows immediately. We are now ready for the Proof itself:

Proof of Theorem. We know that the support of Θ meets $\frac{1}{2}\Lambda$ (cf. § 10): choose $\eta_0 \in \mathrm{Supp}(\Theta) \cap \frac{1}{2}\Lambda$. Then:

$$\mathrm{Supp}(\Theta)+\eta_0 \subseteq W+\Lambda$$

(§ 10, assertion (4.) at the beginning). Therefore, if $\gamma \in W^{\perp} \cap (2\Lambda)$ it follows that $e(\alpha, \gamma) = 1$, all $\alpha \in \operatorname{Supp}(\Theta)$. But then by Corollary above — as generalized —

$$\Theta(\alpha + \lambda \cdot \gamma) = e\left(\alpha, \frac{\lambda \gamma}{2}\right) \cdot \Theta(\alpha), \qquad \text{all } \lambda \in \boldsymbol{Q}_2 .$$

This shows that

$$(*) \qquad \Theta(\alpha + \gamma) = e\left(\alpha, \frac{\gamma}{2}\right) \cdot \Theta(\alpha), \qquad \text{all } \gamma \in W^{\perp}.$$

In particular, $\Theta(\eta_0 + \gamma) \neq 0$, all $\gamma \in W^{\perp}$, hence $W^{\perp} + \eta_0 \subseteq W + \Lambda + \eta_0$. Therefore $W^{\perp} \subseteq W$, i.e., W is a cusp.

Now suppose we take an arbitrary point α in the Support of Θ. We know that α can be written as:

$$\alpha = \eta_0 + \eta_1 + \alpha_0 , \qquad \eta_1 \in \Lambda, \ \alpha_0 \in W.$$

But then:

$$\begin{aligned}
\Theta(\alpha) &= e_*\left(\frac{\eta_1}{2}\right) \cdot e\left(\frac{\eta_1}{2}, \eta_0 + \alpha_0\right) \cdot \Theta(\eta_0 + \alpha_0) \\
&= e_*\left(\frac{\eta_1}{2}\right) \cdot e\left(\frac{\eta_1}{2}, \eta_0\right) \cdot e\left(\frac{\eta_0 + \eta_1}{2}, \alpha\right) \cdot \left[e\left(\alpha, \frac{\eta_0}{2}\right) \cdot \Theta(\eta_0 + \alpha)\right].
\end{aligned}$$

Define a function $\tilde{\Theta}$ on W by

$$\tilde{\Theta}(\alpha) = e\left(\alpha, \frac{\eta_0}{2}\right) \cdot \Theta(\alpha + \eta_0).$$

If $\gamma \in W^{\perp}$, we compute (using (*)):

$$\begin{aligned}
\tilde{\Theta}(\alpha + \gamma) &= e\left(\alpha + \gamma, \frac{\eta_0}{2}\right) \cdot \Theta(\alpha + \eta_0 + \gamma) \\
&= e\left(\gamma, \frac{\eta_0}{2}\right) \cdot e\left(\alpha + \eta_0, \frac{\gamma}{2}\right) \cdot e\left(\alpha, \frac{\eta_0}{2}\right) \cdot \Theta(\alpha + \eta_0) \\
&= \tilde{\Theta}(\alpha).
\end{aligned}$$

This shows that $\tilde{\Theta}$ is, in reality, a function on $\tilde{V} = W/W^{\perp}$, and that Θ is exactly the function $T_{W, \eta_0} \tilde{\Theta}$ obtained from $\tilde{\Theta}$ via Definition 3.

To check that η_0 is an origin for W, look at (*) when $\gamma^{\perp} \in W \cap \Lambda$. Then:

$$e\left(\alpha, \frac{\gamma}{2}\right) \cdot \Theta(\alpha) = \Theta(\alpha + \gamma) = e_*\left(\frac{\gamma}{2}\right) \cdot e\left(\frac{\gamma}{2}, \alpha\right) \cdot \Theta(\alpha)$$

hence

$$e_*\left(\frac{\gamma}{2}\right) = e(\alpha, \gamma) \qquad \text{if } \Theta(\alpha) \neq 0 .$$

So

$$e_*\left(\frac{\gamma}{2}\right)=e(\eta_0,\gamma), \qquad \text{all } \gamma\in W^\perp\cap\Lambda.$$

Moreover, using

$$\Theta(\eta_0)=\Theta(-\eta_0+2\eta_0)=e_*(\eta_0)\,\Theta(-\eta_0)$$

and

$$\Theta(-\eta_0)=\Theta(\eta_0)\neq 0,$$

we conclude that $e_*(\eta_0)=1$ too.

The fact that $\tilde{\Theta}$ is again a theta-function is simply a matter of applying the calculations of Prop. 1 in reverse and is quite straightforward. We omit this. The final point is that $\tilde{\Theta}$ is non-degenerate. But since $S_\infty\supseteq W$, we know that for all $\alpha\in W$, $\alpha=2^k\beta+\eta_1$, where $\Theta(\beta)\neq 0$, $\eta_1\in\Lambda$. Then $\beta=\eta_0+\eta_2+\beta_0$, $\eta_2\in\Lambda$, $\beta_0\in W$, and $\tilde{\Theta}(\beta_0)\neq 0$. Since

$$\alpha-2^k\beta_0=\eta_1+2^k\eta_0+2^k\eta_2\in W\cap\Lambda,$$

this shows that for all $\alpha\in W$, $\alpha=2^k\beta_0+\eta_3$, where $\tilde{\Theta}(\beta_0)\neq 0$, $\eta_3\in W\cap\Lambda$. This means exactly that the S_∞ for $\tilde{\Theta}$ is all of $\tilde{V}$, i.e., $\tilde{\Theta}$ is non-degenerate. *Q.E.D.*

The main Theorem can now be reformulated to give a Satake-like decomposition of $\overline{M}_\infty$. More precisely, for each integer $g\geqq 0$, let

$\overline{M}_\infty(g)=$ the Proj defined in § 9, Def. 3 with indices $\alpha\in\boldsymbol{Q}_2^{2g}$.

$M_\infty(g)=$ the open set in $\overline{M}_\infty(g)$ whose geometric points are the non-degenerate theta functions.

If $h<g$, we define a vast number of closed immersions

$$i_W\colon\ \overline{M}_\infty(h)\to\overline{M}_\infty(g)$$

as follows: let $W\subseteq\boldsymbol{Q}_2^{2g}$ be a cusp such that $2h=\dim(W/W^\perp)$. For each such W, choose an origin $\eta_0\in\frac{1}{2}\boldsymbol{Z}_2^{2g}$, and a symplectic isomorphism:

$$\phi\colon\ \boldsymbol{Q}_2^{2h}\xrightarrow{\approx} W/W^\perp$$

such that

$$\phi(\boldsymbol{Z}_2^{2h})=W\cap\Lambda/W^\perp\cap\Lambda,$$

$$\chi(\tfrac{1}{2}{}^t a_1\cdot a_2)=\tilde{e}_*(\tfrac{1}{2}\phi(a)), \qquad \text{all } a\in\boldsymbol{Z}_2^{2h}.$$

Then i_W is defined by the homomorphism of the homogeneous coordinate ring:

$$i_W^*(X_\alpha^{(g)})=\begin{cases}0 & \text{if } \alpha\notin\eta_0+W+\boldsymbol{Z}_2^{2g}\\ e_*\left(\frac{\eta_1}{2}\right)e\left(\frac{\eta_1}{2},\eta_0\right)e\left(\frac{\eta_0+\eta_1}{2},\alpha\right)\cdot X^{(h)}_{\phi^{-1}(\alpha_0)} & \\ \qquad\qquad \text{if } \alpha=\eta_0+\alpha_0+\eta_1,\ \alpha_0\in W,\ \eta_1\in\boldsymbol{Z}_2^{2g}.\end{cases}$$

(Here $X_\alpha^{(g)}, X_\alpha^{(h)}$ are the coordinates used to define $\overline{M}_\infty(g)$, $\overline{M}_\infty(h)$ respectively). Then we get the restatement:

Main Theorem.

$$\overline{M}_\infty(g) = \left\{\begin{matrix}\textit{disjoint union of the locally}\\ \textit{closed subschemes } i_W(M_\infty(h))\end{matrix}\right\},$$

the union being taken over all cusps $W \subseteq Q_2^{2g}$.

§ 12. Analytic Theta Functions

In this section, we work over the field $\boldsymbol{C}$ of complex numbers. We have 2 purposes: (a) to sketch an approach to the classical theory of Θ-functions, analogous to our theory of algebraic Θ-functions, and (b) to use this to compute our algebraic Θ-functions via the classical ones, when $k = \boldsymbol{C}$.

We will make use of the following lemma:

Lemma 1. *Let X be a compact Kähler manifold. Then the operator*

$$\frac{1}{2\pi i}\partial\bar\partial$$

defines a surjection:

$$\left\{\begin{matrix}C^\infty \textit{ real}\\ \textit{functions on } X\end{matrix}\right\} \to \left\{\begin{matrix}\textit{real closed } C^\infty\ (1,1)\textit{-forms } \Omega \textit{ on } X,\\ \textit{with 0 cohomology class}\end{matrix}\right\}$$

with kernel consisting only of constants.

Corollary. *Let* $\boldsymbol{L}$ *be an analytic line bundle on X. Let* $c_1(\boldsymbol{L}) \in H^2(X, \boldsymbol{C})$ *be its first chern class. Then for all real closed* $C^\infty(1, 1)$*-forms* Ω *whose cohomology class equals* $c_1(\boldsymbol{L})$, *there is one and (up to a constant) only one Hermitian structure* $\| \; \|$ *on* $\boldsymbol{L}$ *whose associated curvature form is* Ω.

The lemma is standard and we omit the proof. The Corollary can be proven by choosing one Hermitian structure $\| \; \|_0$ on $\boldsymbol{L}$: let Ω_0 be its curvature form. Then any other Hermitian structure on $\boldsymbol{L}$ is given by $\rho \cdot \| \; \|_0$, where ρ is a positive real C^∞ function on X: and its curvature form Ω is

$$\Omega = \frac{1}{2\pi i}\partial\bar\partial \log \rho + \Omega_0.$$

Now use the Lemma and everything comes out. *Q.E.D.*

In particular, when X is an abelian variety, an analytic line bundle $\boldsymbol{L}$ on X has one and (up to a constant) only one Hermitian structure $\| \; \|$ whose curvature form Ω is a translation-invariant (1, 1)-form. In what follows, we will always put this Hermitian structure on line bundles on abelian varieties. In this case, Ω is determined by its value at the origin.

Now let $\hat{X}$ be the universal covering space of X. $\hat{X}$ is a complex vector space, and if

$$p\colon \hat{X} \to X$$

is the canonical homomorphism, dp induces a canonical identification between $\hat{X}$ and the tangent space of X at the origin (or at any other point). Therefore, any translation-invariant real 2-form Ω on X defines and is defined by a real-linear skew-symmetric form:

$$E\colon \hat{X} \times \hat{X} \to \boldsymbol{R}.$$

E is a (1, 1)-form if and only if $E(ix, iy) = E(x, y)$, all $x, y \in X$. Moreover, let $\Lambda = \text{kernel}\,(p)$. Λ is a lattice in X, canonically isomorphic to $H_1(X, \boldsymbol{Z})$. Since the first chern class of a line bundle is integral, if E represents $c_1(\boldsymbol{L})$, then E must take integral values on $\Lambda \times \Lambda$:

$$E(\Lambda \times \Lambda) \subseteq \boldsymbol{Z}.$$

If we lift $\boldsymbol{L}$ to $\hat{X}$, we have a situation in which the following lemma applies:

Lemma 2. *Let Y be a complex vector space, and let $\boldsymbol{L}_1, \boldsymbol{L}_2$ be 2 analytic-Hermitian line bundles on Y. Then a holomorphic-unitary isomorphism $\phi\colon \boldsymbol{L}_1 \xrightarrow{\sim} \boldsymbol{L}_2$ exists if and only if the curvature forms of $\boldsymbol{L}_1, \boldsymbol{L}_2$ are equal; if so, ϕ is unique up to a scalar of absolute value 1.*

Proof. Standard methods.

In particular, let $Y = \hat{X}$, and let $\boldsymbol{M} = p^*(\boldsymbol{L})$ be induced from an abelian variety. Give $\boldsymbol{L}$ and hence $\boldsymbol{M}$ the Hermitian structure with constant curvature form E. The above lemma has 2 applications:

(I) Construction of a nilpotent group $\mathscr{G}$: If $x \in X$, and T_x denotes translation by x, then the lemma shows that $\boldsymbol{M}$ and $T_x^* \boldsymbol{M}$ are holomorphic-unitary isomorphic. If

$$\mathscr{G}(\boldsymbol{M}) = \{(x, \Phi) \mid \Phi \text{ a holo.-unit. isom. of } \boldsymbol{M} \text{ with } T_x^* \boldsymbol{M}\},$$

then $\mathscr{G}(\boldsymbol{M})$ is, as before, a group lying in an exact sequence:

$$1 \to \boldsymbol{C}_1^* \to \mathscr{G}(\boldsymbol{M}) \to X \to 0$$

($\boldsymbol{C}_1^*$ = complex numbers of absolute value 1).

(II) Construction of canonical "trivialization" of $\boldsymbol{M}$: Let $\mathbf{1}$ denote the trivial analytic line bundle over X with canonical section 1. To put a Hermitian structure on $\mathbf{1}$, we may set $\|1\| =$ any positive real C^∞-function. For example, let

$$\|1\|(x) = e^{-\pi/2H(x,x)}$$

where H is a Hermitian form on X. The corresponding curvature form $E: \hat{X} \times \hat{X} \to \boldsymbol{R}$ is easily checked to equal Im (H). But

$$H \mapsto E = \operatorname{Im}(H)$$

sets up an isomorphism:

$$\begin{Bmatrix}\text{hermitian} \\ \text{forms on } X\end{Bmatrix} \xrightarrow{\sim} \begin{Bmatrix}\text{real skew-symmetric forms } E \text{ on } X \\ \text{such that } E(i\,x, i\,y) = E(x, y)\end{Bmatrix},$$

so for each $\boldsymbol{L}$ on X with translation-invariant curvature form, we have a unique Hermitian structure on $\mathbf{1}$ of the above type so that $\mathbf{1} \cong \boldsymbol{L}$. In particular, we get a canonical

$$\mathbf{1} \cong \boldsymbol{M}.$$

We can now develop a theory along similar lines to our algebraic theory. For example, if H is positive definite, then let:

$\mathscr{H}$ = Hilbert space of L^2-holomorphic sections of $\boldsymbol{M}$ over $\hat{X}$.

Then $\mathscr{G}(\boldsymbol{M})$ has a natural unitary representation on $\mathscr{H}$, it is irreducible, and it turns out to be the only irreducible unitary representation of $\mathscr{G}(\boldsymbol{M})$ in which $\boldsymbol{C}_1^* \subset \mathscr{G}(\boldsymbol{M})$ acts by its natural character. This is the situation described by CARTIER [*2*], and studied by CARTIER and many others, e.g., MACKEY, FOCK, WEIL etc. Exactly as in § 1, $\mathscr{G}(\boldsymbol{M})$ governs the "descent" of the Hermitian bundle $\boldsymbol{M}$ to the abelian variety X, (or to other ones $X' = [\hat{X}/\text{another lattice}]$), and the "descent" of holomorphic sections of $\boldsymbol{M}$ to holomorphic sections of its descended form. Thus we get:

Proposition 1. *There is a* $1-1$ *correspondence between*

1. *Hermitian-analytic line bundles* $\boldsymbol{L}'$ *on* X *such that* $p^* \boldsymbol{L}' \cong \boldsymbol{M}$,

2. *subgroups* $K \subset \mathscr{G}(\boldsymbol{M})$, *such that* $K \cap \boldsymbol{C}_1^* = \{1\}$ *whose image in* $\hat{X}$ *is* $\Lambda = \ker (p: \hat{X} \to X)$.

Moreover, the holomorphic sections of $\mathbf{M}$ *of the form* $p^*(s')$, $s' \in \Gamma(X, \boldsymbol{L}')$, *are exactly those sections* s *which are invariant under* K, i.e.,

$$s = T_{-x}^*(\phi(s)), \qquad \text{all } (x, \phi) \in K.$$

Proof. Straightforward.

Finally, via the canonical trivialization of $\boldsymbol{M}$, holomorphic sections of $\boldsymbol{M}$ correspond to holomorphic functions on $\hat{X}$: thus each section $s \in \Gamma(X, \boldsymbol{L})$ defines a holomorphic function on $\hat{X}$. These are the classical theta-functions.

As far as moduli are concerned, the simplest and most basic result is the following: we set out to classify triples consisting of –

1. a complex vector space Y, of dimension 2;

2. an analytic, Hermitian line bundle $\boldsymbol{M}$ on Y, with curvature form $E = \operatorname{Im} H$, H positive definite.

3. Parametrized lattices in Y, i.e., monomorphisms

$$\alpha\colon \boldsymbol{Z}^{2g} \to Y$$

such that

$$E(\alpha x, \alpha y) = {}^t x_1 \cdot y_2 - {}^t x_2 \cdot y_1$$

if

$$x = (x_1, x_2), \qquad y = (y_1, y_2).$$

Such triples arise if we start with a principally polarized abelian variety $(X, \boldsymbol{L})$, together with a symplectic isomorphism:

$$\beta\colon \boldsymbol{Z}^{2g} \xrightarrow{\sim} H_1(X, \boldsymbol{Z}).$$

Namely, let $Y = \hat{X}$, $\boldsymbol{M} = p^* \boldsymbol{L}$ with canonical Hermitian structure, and let β define α via the natural maps $H_1(X, \boldsymbol{Z}) \cong \operatorname{Ker}(p\colon \hat{X} \to X) \subset \hat{X}$. Conversely, the triple $(Y, \boldsymbol{M}, \alpha)$ determines X and β, and $\boldsymbol{L}$ up to replacing $\boldsymbol{L}$ by $T_x^* \boldsymbol{L}$, some $x \in X$.

Let $\mathfrak{H}$ = SIEGEL's $g \times g$ upper half-plane. Then the moduli result is:

Proposition 2. *There is a natural bijection between the set of isomorphism classes of triples $(Y, \boldsymbol{M}, \alpha)$ and $\mathfrak{H}$. In this bijection, $\tau \in \mathfrak{H}$ corresponds to*

$$Y = \boldsymbol{C}^g,$$

$$\boldsymbol{M} = \boldsymbol{1} \quad \text{with hermitian structure} \quad \|1\|(x) = e^{-\frac{\pi}{2} {}^t x \cdot B \cdot \bar{x}},$$

$$\alpha((x_1, x_2)) = x_1 + \tau \cdot x_2$$

where $B = (\operatorname{Im} \tau)^{-1}$.

The final topic I want to discuss is the relation between the classical and algebraic theories. Let's start with:

X = abelian variety;

L = symmetric, ample, degree 1 sheaf on X. [Assume for simplicity that L is so chosen among its translates $T_x^* L$, $x \in X_2$, that its unique section is *even*; equivalently, that the Arf invariant of Q, where $e_*^L(x) = (-1)^{Q(x)}$, is 0.]

Let

$\boldsymbol{L}$ = line bundle on X whose holomorphic sections are L;

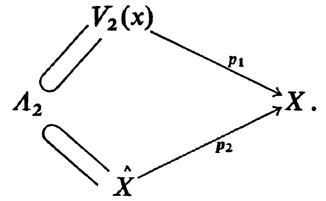
$\hat{X}$ = universal covering space of X;

$V_2(X)$ = 2-Tate group of X.

Also, let Λ_2 = inverse image in $\hat{X}$ of $\mathrm{tor}_2(X)$, i.e.,

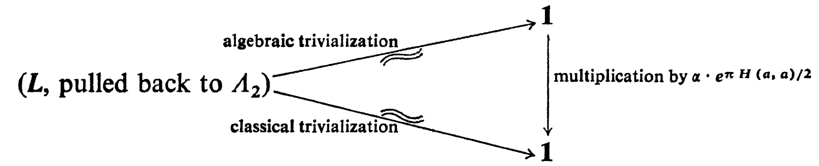
$$\bigcup_n 2^{-n}\cdot\Lambda, \quad \text{if} \quad \Lambda=\mathrm{Ker}(p\colon \ddot{X}\to X).$$

Then we have canonical maps:



Note that Λ_2 is dense in both $V_2(X)$ and X. We have "trivialized" $\boldsymbol{L}$ when it is pulled up to $V_2(X)$ or to X, in § 8 and just above. Thus we have 2 distinct trivializations of $\boldsymbol{L}$ on Λ_2. The main result is that these differ by an elementary factor:

Theorem 3. *Let* **1** *denote the trivial complex line bundle on* Λ_2. *Then the following diagram commutes*:



where $\alpha\in\boldsymbol{C}^*$ *and* $E=\mathrm{Im}\,(H)$ *is the curvature form of* $\boldsymbol{L}$.

Proof. Let $\boldsymbol{M}_i=p_i^*\boldsymbol{L}$ = induced line bundle on $V_2(X)$ or $\hat{X}$. Let ψ: $\boldsymbol{M}_2\xrightarrow{\approx}\boldsymbol{1}$ be the classical trivialization. The algebraic trivialization of $\boldsymbol{M}_1$ is based on finding a distinguished collection of isomorphisms

$$\varphi_a\colon \boldsymbol{M}_1\to T_a^*\boldsymbol{M}_1,$$

all $a\in V_2(X)$. In fact, let ι = inverse map in all our groups, and let ρ: $\boldsymbol{M}_i\xrightarrow{\sim}\iota^*\boldsymbol{M}_i$ be the isomorphism induced by the symmetry of $\boldsymbol{L}$. Then, for all elements $2a\in V_2(X)$, φ_{2a} is characterized by the existence of φ_a satisfying:

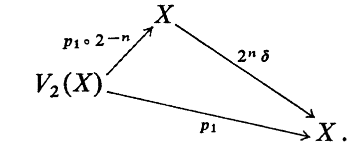
i) $\varphi_{2a}=T_a^*\varphi_a\circ\varphi_a$,

ii) $\iota^*\varphi_a\circ\rho=T_{-a}^*[\rho\circ\varphi_a^{-1}]$,

iii) φ_a is induced by an algebraic isomorphism

$$\varphi'_a\colon (2^n\delta)^*\boldsymbol{L}\xrightarrow{\sim}(2^n\delta)^*(T_{p_1(a)}^*\boldsymbol{L})$$

for some n, i.e., via the factorization:



But introduce, for all $a \in X$, isomorphisms ψ_a from $\boldsymbol{M}_2$ to $T_a^* \boldsymbol{M}_2$ via:

$$\boldsymbol{M}_2 \xrightarrow[\psi]{\approx} \mathbf{1} \xrightarrow[\text{mult. by } f_a(x)]{\approx} T_a^* \mathbf{1} \xleftarrow[T_a^* \psi]{\approx} T_a^* \boldsymbol{M}$$

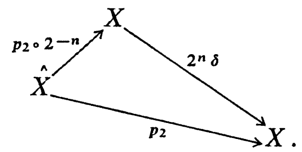
where

$$f_a(x) = e^{\pi[H(x,a) + H(a,a)/2]}.$$

Also introduce

$$\rho' \colon \boldsymbol{M}_2 \xrightarrow[\psi]{\approx} \mathbf{1} \xrightarrow[\text{canonical identification}]{} \iota^* \mathbf{1} \xleftarrow[\iota^* \psi]{\approx} \iota^* \boldsymbol{M}.$$

One checks easily that ψ_a and ρ' are holomorphic and unitary isomorphisms. Therefore ρ and ρ' can differ only by a constant: and since both are the identity at $0 \in X$, $\rho = \rho'$. Moreover, if $a \in 2^{-n} \Lambda$, then the algebraic isomorphism φ_a': $(2^n \delta)^* \boldsymbol{L} \xrightarrow{\sim} (2^n \delta)^* T_{p_2(a)}^* \boldsymbol{L}$, referred to in (iii) above, induces an isomorphism $\varphi_a'' \colon \boldsymbol{M}_2 \to T_a^* \boldsymbol{M}_2$ via the factorization



Since φ_a'' is also holomorphic and unitary, it differs from ψ_a only by a constant. Next, note that $\{f_a\}$ satisfy the identities:

i') $f_{2a}(x) = f_a(x+a) \cdot f_a(x)$,

ii') $f_a(-x) = f_a(x-a)^{-1}$.

These translate readily into the identities on the $\{\psi_a\}$:

i'') $\psi_{2a} = T_a^* \psi_a \circ \psi_a$.

ii'') $\iota^* \psi_a \circ \rho = T_{-a}^* [\rho \circ \psi_a^{-1}]$.

Finally, i'', ii'', plus the fact that φ_a' induces ψ_a, shows that ψ_a and φ_a induce the *same isomorphism* of $\boldsymbol{L}$ on Λ_2, with $T_a^*(\boldsymbol{L}$ on $\Lambda_2)$, all $a \in \Lambda_2$.

Finally, to compare the 2 trivializations, start with the unit section 1 of $\mathbf{1}$ on Λ_2. This goes over, via the algebraic trivialization, to *a* section s of $\boldsymbol{L}$ on Λ_2 such that, for all $a \in \Lambda_2$,

$$s(a) = \phi_a(0)\,[s(0)]$$

(i.e., $\phi_a(0)$ is the induced isomorphism from the fibre L_0 or $(M_1)_0$ to the fibre $L_{p_1(a)}$ or $(M_1)_a$) But under the classical trivialization ψ, $\psi_a(0)$ corresponds to the isomorphism of fibres:

$$\begin{array}{ccc} \mathbf{1}_0 & \xrightarrow{\text{mult. by } e^{\pi/2H(a,a)}} & \mathbf{1}_0 \\ \| & & \| \\ C & & C. \end{array}$$

Therefore, the section s goes over, under the classical trivialization, to a section of $\mathbf{1}$ which, if it has value α at 0, has value

$$\alpha \cdot e^{\pi/2\, H(a,a)}$$

at a. All in all, the section 1 of $\mathbf{1}$ has gone into the section

$$g(a) = \alpha \cdot e^{\pi/2\, H(a,a)}$$

of $\mathbf{1}$. *Q.E.D.*

Corollary. *If the unique section s of $\mathbf{L}$ (up to scalars) defines*

a) *the holomorphic function Θ_h on $\hat{X}$ via the classical trivialization,*

b) *the* 2-adic *theta-function Θ_a on $V_2(X)$ via the algebraic trivialization, then*

$$\Theta_h(x) = \alpha \cdot e^{\frac{\pi}{2} H(x,x)} \cdot \Theta_a(x)$$

all $x \in \Lambda_2$.

To calculate Θ_h and hence Θ_a by analytic means, we must know the "descent data"

$$K \subset \mathcal{G}(M_2)$$

that defines L on X. Let $e_*: \frac{1}{2}\Lambda/\Lambda \to \{\pm 1\}$ be the quadratic character defined by L. Then, as we saw in § 8, the descent data for the pull-back M_1 of L is the group:

$$\{(x, \phi) \mid x \in \Lambda \cdot Z_2,\ \phi = e_*(\tfrac{1}{2}x) \cdot \phi_x\}.$$

In view of the proof of the theorem, this implies that

$$K = \{(x, \psi) \mid x \in \Lambda,\ \psi = e_*(\tfrac{1}{2}x) \cdot \psi_x\}.$$

(Notation as in proof of Theorem). Now a K-invariant section s of M_2 is one which satisfies $T_a^*(s) = \phi(s)$, all $(a, \phi) \in K$. Going back to the definition of ψ_a, one sees that if $f = \psi(s)$ is the function on $\hat{X}$ corresponding to s, then f is K-invariant if and only if

$$(*) \qquad f(x+a) = e_*(\tfrac{1}{2}a)\, f_a(x) \cdot f(x)$$

all $x \in \hat{X}$, $a \in \Lambda$. From this it follows that Θ_h must be the unique holomorphic function satisfying $(*)$.

To go further and write down this Θ_h as an infinite series, it is convenient to introduce coordinates. Let

$$i\colon Z^{2g} \xrightarrow{\approx} \Lambda \qquad \text{be a symplectic isomorphism.}$$

Coordinatize $\hat{X}$ via

$$\hat{X} \cong C^g$$

so that $i((n_1, 0)) = n_1$, and let τ be the $g \times g$ matrix defined by

$$i((0, n_2)) = \tau \cdot n_2 .$$

Because of our assumption on e_*^L, hence on e_*, if we choose coordinates correctly, we can assume that

$$e_*[\tfrac{1}{2}\, i(n_1, n_2)] = (-1)^{{}^t n_1 \cdot n_2}.$$

As we saw in Prop. 2, if we now express:

$$H(z, z) = {}^t z \cdot B \cdot \bar{z}$$

then $B = (\operatorname{Im} \tau)^{-1}$. Finally, set

$$\Theta_h(z) = e^{\frac{\pi}{2} {}^t z \cdot B \cdot z} \cdot \sum_{n \in Z^g} e^{2\pi i [\frac{1}{2} {}^t n \cdot \tau \cdot n + {}^t n \cdot z]}.$$

It is easy to check that this is a holomorphic function satisfying (*). Therefore, this is the sought-for theta-function. Combining this with the Corollary, we find

$$\Theta_a(z) = e^{\frac{\pi}{2} {}^t z \cdot B \cdot (z - \bar{z})} \cdot \sum_{n \in Z^g} e^{2\pi i [\frac{1}{2} {}^t n \cdot \tau \cdot n + {}^t n \cdot z]} \qquad \text{all } z \in \bigcup_k 2^{-k} \Lambda .$$

If

$$z = i((\alpha_1, \alpha_2)), \qquad \alpha_i \in \bigcup_k 2^{-k} \cdot (Z^g),$$

then after rearranging, one finds

$$\Theta_a(\alpha_1, \alpha_2) = e^{-\pi i {}^t \alpha_1 \cdot \alpha_2} \cdot \sum_{n \in \alpha_2 + Z^g} e^{2\pi i [\frac{1}{2} {}^t n \cdot \tau \cdot n + {}^t n \cdot \alpha_1]}.$$

The function so defined clearly extends to a locally constant function defined for all $\alpha_1, \alpha_2 \in Q^{2g}$: it is the sought-for algebraic theta function defined in § 8. Comparing this with the formula in Lemma 1, § 8, expressing Θ_a in terms of the finitely additive measure μ on Q_2^g, we also get an analytic description for μ:

$$\begin{cases} \mu \text{ is countably additive,} \\ \mu = \sum_{x \in D} e^{\pi i {}^t x \cdot \tau \cdot x} \cdot \delta_x , \\ \delta_x = \text{delta measure at } x, \\ D = \bigcup_k 2^{-k} Z^g . \end{cases}$$

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(Received February 20, 1967)