PERIODS OF A MODULI SPACE OF BUNDLES ON CURVES.

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We will work over the complex numbers in this paper. For all curves $C$, and for all integers $(n, d)$, the problem arises of determining the structure of the "space" of all vector bundles $E$, with rank $n$ and degree $(\deg c_1(E))d$. The problem has been considerably clarified recently by the introduction of the concept of stable and semi-stable bundles: [4], [6], [10]. It has been proven, in particular, that for each $n$ and each line bundle $L$ on $C$ such that $n$ and $\deg L$ are relatively prime, then the set:

$$S_{n,L}(C) = \{ \text{set of all stable vector bundles } E \text{ on } C \text{ of rank } n \text{ such that } L^n E \cong L \}$$

has a natural structure of a non-singular projective variety of dimension $(n^2 - 1) \cdot (g - 1)$, where $g = \text{genus } (C)$. It is important to note that the map

$$E \mapsto E \otimes M$$

for a line bundle $M$ induces an isomorphism

$$S_{n,L}(C) \cong S_{n,L \otimes M^n}(C)$$

hence the variety $S_{n,L}(C)$ depends essentially only on the residue class of $\deg L \mod n$.

We wish to look at the case $g \geq 2$, $n = 2$, $\deg L$ odd. In this case, we may assume for simplicity that a base point $x_0 \in C$ has been chosen that $L$ is taken to be the line bundle whose sections form the sheaf $O_{-L}(x_0)$. We abbreviate $S_{2,L}(C)$ now to $S_{2,-}(C)$. The topology of these varieties has been described in [7] and when the genus of $C$ is 2, their complete structure is described in [8]. $S_{2,-}(C)$ has dimension $3g - 3$ and is known to be birationally equivalent to $P_{3g-3}$. In particular, it is simply connected and the invariants $h^{0,0} = h^{0,0}$ are all 0, ([9]). In [7], it is also proven that $B_2 = 1$, $B_3 = 2g$. Now for non-singular projective varieties $X$ with $h^{0,3} = h^{3,0} = 0$, a very interesting invariant is Weil's "intermediate jacobian" attached to $H^3(X)$. This is an abelian variety, which we shall denote $J^2(X)$, which is by definition:

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\[ J^2(X) \cong H^2(X, \mathbb{R}) / \text{Image}[H^3(X, \mathbb{Z})] \]

where \( H^2(X, \mathbb{R}) \) is given a complex structure via the decomposition

\[ H^2(X, \mathbb{R}) \otimes \mathbb{C} \cong H^{2,1} \oplus H^{1,2} \]

since this induces an isomorphism

\[ H^2(X, \mathbb{R}) \cong H^{1,2} = H^2(X, \Omega^1). \]

cf. [11], [1], [3]. Weil also showed that a polarization on \( X \) induces a polarization on \( J^2(X) \) in a canonical way.

If \( \text{Alb}(C) \) denotes the albanese, or jacobian, variety of \( C \), then our main result is:

**Theorem.** \( J^2[S_2^{-}(C)] \cong \text{Alb}(C). \)

Note that \( S_2^{-}(C) \) has a unique polarization since \( B_2 = 1 \), hence \( J^2(S_2^{-}(C)) \) has a canonical polarization, just as \( \text{Alb}(C) \) does. It is easy to check that our isomorphism is compatible with these canonical polarizations, hence by Torelli’s theorem, we conclude:

**Corollary.** If \( S_2^{-}(C_1) \cong S_2^{-}(C_2) \), then \( C_1 \cong C_2. \)

Before beginning the proof, we must recall Weil’s map relating \( J^2(X) \) to codimension 2 cycles on \( X \):

1. Let \( Y \) be an non-singular parameter space,
2. Let \( W \) be an algebraic cycle on \( X \times Y \) of codimension 2.

Then we get

\[ w \in H^4(X \times Y, \mathbb{Z}), \text{ the fundamental class of } W \]

esp. \( w_{a,1} \in (H^3(X, \mathbb{Z})/\text{torsion}) \otimes H^1(Y, \mathbb{Z}), \text{ the } (3,1)\)-component of \( w. \)

Then \( w_{a,1} \) defines a map

\[ \phi_w : H^1(Y, \mathbb{R}) \xrightarrow{\text{linear maps which are integral on } H^1(Y, \mathbb{Z})} H^3(X, \mathbb{R}) / \text{Image } H^3(X, \mathbb{Z}) \]

\[ \text{Alb}(Y) \xrightarrow{\cong} J^2(X) \]
which is easily seen to be complex-analytic using the fact that \( w \) is of type \((2, 2)\) in the Hodge decomposition of \( H \). Note the obvious fact:

**Lemma 1.** \( \varphi_w \) is an isomorphism if and only if \( w_{a,1} \) is "unimodular," (i.e., written out as a matrix in terms of bases of \( H^a(X, \mathbb{Z})/\text{torsion}, H^1(Y, \mathbb{Z}) \), it is a square matrix with \( \det \equiv \pm 1 \)).

1. In the sequel, we abbreviate \( S_\infty(C) \) by \( S \). The first step in our proof is to construct a universal vector bundle \( E \) on \( S \times C \), i.e., one whose restriction to \( \{ t \} \times C \), for any \( t \in S \), is exactly the vector bundle \( E_t \) on \( C \) corresponding to the point \( t \in S \). This is a problem in descent theory. In fact, \( S \) can be described as a quotient \( R/PGL(\nu) \), where \( R \) is a non-singular quasi-projective variety, and \( PGL(\nu) \) acts freely on \( R \); and where there is a vector bundle \( F \) on \( R \times C \) whose restriction to \( \{ t \} \times C \), any \( t \in R \), is the vector bundle on \( C \) corresponding to the image of \( t \) in \( S \): cf. [10], p. 321. However, the difficulty is that the action of \( PGL(\nu) \) on \( R \) does not, a priori, lift to an action on \( F \). Instead, \( GL(\nu) \) acts on \( F \) satisfying

1) \( G_m = \text{center} (GL(\nu)) \) acts on \( F \) by homotheties

2) if \( \pi : GL(\nu) \to PGL(\nu) \) is the canonical map, and \( T_\nu \) represents the action of an element \( g \), then the diagram

\[
\begin{array}{ccc}
F & \xrightarrow{T_\nu} & F \\
\downarrow & & \downarrow \\
R \times C & \xrightarrow{T_{\pi(g)} \times 1_C} & R \times C
\end{array}
\]

commutes.

The way out of this type of impasse is to find a "functorial" way of associating to every vector bundle \( E \) on \( C \) (of the type being considered) a \( 1 \)-dimensional vector space \( \lambda(E) \) such that multiplication by \( \alpha \) in \( E \) induces multiplication by \( \alpha \) in \( \lambda(E) \). By functorial we mean that the procedure extends to families of such vector bundles: if \( E \) is a vector bundle on \( T \times C \) (for any algebraic scheme \( T \)) whose restriction to \( \{ t \} \times C \) is of the type under consideration, then we should get a line bundle \( \lambda(E) \) on \( T \). Moreover, for any diagram of vector bundles

\[
\begin{array}{ccc}
E_1 & \xrightarrow{g} & E_2 \\
\downarrow & & \downarrow \\
T_1 \times C & \xrightarrow{f \times 1_C} & T_2 \times C
\end{array}
\]
making $E_1$ into a fibre product of $E_2$ and $T_1 \times C$ over $T_2 \times C$, we should be given a definite isomorphism of $\lambda(E_1)$ with $f^*(\lambda(E_2))$. For example, if $T_1 = T_2 = \text{Spec}(C)$, $E_1 = E_2 = E$, and $g$ is multiplication by a scalar $\alpha \neq 0$, we are then given an induced automorphism of $\lambda(E)$: we want this automorphism to be multiplication by $\alpha$ too (it might turn out to be multiplication by $\alpha^n$ instead). All this data is subject to an obvious co-cycle condition: compare [5], p. 64. If we can find such data, we get as a consequence a line bundle $\lambda(F)$ on $R$, plus an action of $GL(v)$ on $\lambda(F)$ in which the center acts by homotheties. If we then define

$$F' = F \otimes p_1^*(\lambda(F)^{-1}),$$

we get a new vector bundle on $R \times C$ with the same restrictions to the fibres $\{t\} \times C$ as before; but where in the natural action of $GL(v)$ on $F'$, the action of the center $\mathbb{G}_m$ on $F$ and $p_1^*(\lambda(F)^{-1})$ cancel each other out, i.e., $PGL(v)$ acts on $F$. Then $F / PGL(v)$ is the sought-for universal vector bundle on $S \times C$.

Here's how to construct $\lambda$. We limit ourselves to the case $T = \text{Spec}(C)$, $E$ a vector bundle on $C$, since the generalization of $\lambda$ to an arbitrary base will be clear. Recall $E$ has rank 2, degree 1, and is stable:

a) $H^1(E \otimes (\Omega_C^1)^k) = (0)$, if $k \geq 1$.

**Proof.** This group is dual to $H^0(E \otimes (\Omega_C^1)^{1-k})$ and if this were non-zero, we would get a non-zero homomorphism

$$(\Omega_C^1)^{k-1} \to \mathbb{C}$$

hence a sub-line-bundle $G \subset E$ of degree $\geq 2(k-1)(g-1) \geq 0$. This contradicts the stability of $E$.

b) If $V_k(E) = H^0(E \otimes (\Omega_C^1)^k)$, then

$$\dim V_k(E) = (2g-2)(2k-1) + 1.$$

**Proof.** Riemann-Roch.

c) Set $\lambda(E) = [\Lambda^{2g-1}V_1(E)] \otimes (\lambda^0) \otimes [\Lambda^{8g-6}V_2(E)] \otimes \cdots$.

Then multiplication by $\alpha$ in $E$ induces the endomorphism, multiplication by $\alpha$, in each $V_k(E)$, hence it induces multiplication by $\alpha$ to the power
\[(3g - 1)(2g - 1) + (6g - 5)(-g)\]

in \(\lambda(E)\). This number happens to be 1!

We now know that \(E\) exists. Next consider the chern classes of \(E\). We have

\[
c_2(E) \in H^4(S \times C, \mathbb{Z}) \cong (H^2(S, \mathbb{Z}) \otimes H^2(C, \mathbb{Z}))
\]

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c_1(E) \in H^2(S \times C, \mathbb{Z}) \cong H^2(C, \mathbb{Z}) \otimes H^2(S, \mathbb{Z})
\]

\[
\otimes (H^3(S, \mathbb{Z}) \otimes H^1(C, \mathbb{Z}))
\]

\[
\otimes H^4(S, \mathbb{Z}).
\]

Note that any bundle \(E \otimes p_1^*M, M\) a line bundle on \(S\), would have the same universal property that \(E\) does, so \(c_1(E)\) is not very interesting. However, let

\[
\alpha = (c_2(E))_{2,1} = \text{[component of } c_2(E) \text{ in } H^3(S, \mathbb{Z}) \otimes H^1(C, \mathbb{Z})].
\]

A simple computation of chern classes shows that \(\alpha\) is independent of this modification of \(E\). According to [7], \(H^3(S, \mathbb{Z})\) and \(H^1(C, \mathbb{Z})\) have the same rank. In fact:

**Proposition 1.** \(\alpha\) is unimodular.

This will be proven in § 2. Assuming this, it follows from Lemma 1 that if \(W\) the algebraic 2nd chern class of \(E\), then Weil’s map \(\phi_W: \text{Alb}(C) \to J^2(S)\) is an isomorphism, as required. Although it is not essential, it will be convenient in § 2 to know that \(H^3(S, \mathbb{Z})\) is torsion-free. In fact, the torsion subgroup of \(H^3(X, \mathbb{Z})\)—for any non-singular complete variety \(X\) over \(C\)—is a birational invariant of \(X\) known as the “topological Brauer group” (cf. [12], Cor. (7.3) and equation (8.9), p. 59). And \(S\) is birationally equivalent to \(P_{3g-3}\) which has no \(H^3\) at all!

2. We start by recalling the results of [6]. In fact, let \(S_0\) be the subset of \(SU(2)^{2g}\) consisting of points \((A_1, \cdots, A_{2g})\) such that

\[
\prod_{i=1}^{g} A_{2i-1} A_{2i} A_{2i-1}^{-1} A_{2i}^{-1} = -I.
\]

Then \(S_0\) is an orientable submanifold of \(SU(2)^{2g}\) and there is a natural map

\[
p: S_0 \to S,
\]

which is a principal fibration with group \(PU(2)\). The map \(p\) may be determined as follows. Let \(\tilde{C}\) be the simply-connected covering of \(C\) which is
ramified over $x_0$ with ramification index 2. The group $\pi$ of this covering is generated by elements $a_1, \cdots, a_{2g}$ subject to the single relation

$$\prod_{i=1}^{g} a_{2i-1}a_{2i}a_{2i-1}^{-1}a_{2i}^{-1} = e.$$ 

Thus a point of $S_0$ may be regarded as a representation of $\pi$, and this representation defines a stable bundle $E$ over $C$ of rank 2, with $\Lambda^2E \cong L$, and hence a point of $S$. So we get a map $p: S_0 \to S$. Notice that the $a_i$ determine elements of $H_1(C; \mathbb{Z})$ and hence of $H_1(C; \mathbb{Z})$, and that these elements form a basis for $H_1(C; \mathbb{Z})$; let $\{\alpha_i\}$ be the dual basis of $H^1(C; \mathbb{Z})$.

**Lemma 2.** $p^*: H^2(S; \mathbb{Z}) \to H^2(S_0; \mathbb{Z})$ is an isomorphism.

**Proof.** Since $H^*(PU(2); \mathbb{Z}) = 0$ and $H^1(S; \mathbb{Z}) = 0$ ($S$ is simply-connected), the spectral sequence of the fibration $p$ gives rise to an exact sequence

$$H^0(S; H^2(PU(2); \mathbb{Z})) \to H^2(S; \mathbb{Z}) \xrightarrow{p^*} H^2(S_0; \mathbb{Z}) \to H^0(S; H^2(PU(2); \mathbb{Z})).$$

Now the first group in this sequence is $\mathbb{Z}$ and the last is $\mathbb{Z}$. Moreover $H^2(S; \mathbb{Z})$ is torsion-free (see §1) and has the same rank as $H^2(S_0; \mathbb{Z})$ by the results of [7]. The lemma now follows.

**Lemma 3.** The homomorphism $H^3(SU(2)^{2g}; \mathbb{Z}) \to H^3(S_0; \mathbb{Z})$ induced by the inclusion of $S_0$ in $SU(2)^{2g}$ is an isomorphism.

**Proof.** Lemma 3 of [7] shows that the homomorphism

$$H_3(S_0; \mathbb{Z}) \to H_3(SU(2)^{2g}; \mathbb{Z})$$

is surjective, except possibly for some 2-primary torsion. However, in this simple case, the same argument can be used to prove that the homomorphism is really surjective. It follows at once that $H^3(SU(2)^{2g}; \mathbb{Z})$ is contained in $H^3(S_0; \mathbb{Z})$ as a direct summand. The lemma now follows from the fact that the ranks of these two groups are equal (see [7]) and that $H^3(S_0; \mathbb{Z})$ is torsion-free by Lemma 2.

Now let $p_i: S_0 \to SU(2)$ denote the projection on the $i$-th factor and let

$$\beta_i = p_i^* [\text{generator of } H^3(SU(2); \mathbb{Z})].$$

Then by Lemma 3 the $\beta_i$ form a basis for $H^3(S_0; \mathbb{Z})$. In view of Lemma 2, it is now sufficient to prove:

**Proposition 2.** $c_2[(p \times 1_C)^* E]_{3,1} = \sum_{i=1}^{2g} \beta_i \otimes \alpha_i$. 
Now choose embedding \( s_i : S^1 \to C \to x_0 \) which represent the generators \( a_i \) of \( \pi \). Then
\[
 s_i^*(a_j) = 0 \quad \text{if} \quad i \neq j
\]
= generator of \( H^1(S^1; \mathbb{Z}) \) \( i = j \).

Hence Proposition 2 will follow at once from

**Proposition 3.**

\[
c_2[(1_{S_0} \times s_i)^*(p \times 1_C)^*E]_{a,1} = \beta_i \otimes \left[ \text{generator of } H^1(S^1; \mathbb{Z}) \right].
\]

We now need to recall a few more details from [6]. Let \( E_\rho \) be the bundle over \( C \) corresponding to the representation \( \rho \in S_0 \). Then ([6] Remark 6.2) we can write down coordinate transformations for \( E_\rho \) as follows.

Choose a finite open covering \( \{U_i\} \) (\( i = 0, 1, \ldots, m \)) of \( C \) such that every non-empty intersection of the sets \( U_i \) is contractible. Assume \( x_0 \in U_o, x_0 \notin U_i \) for \( i \neq 0 \). Assume moreover that there exist discs \( D_i \) in \( \hat{C} \) such that \( U_o \) is the quotient of \( D_0 \) by \( \mathbb{Z}_2 \) and that for \( i \neq 0 \), \( D_i \) maps homeomorphically onto \( U_i \).

For every \( i, j, k \), where \( k = i \) or \( j \), let \( W_{ij,k} \) be a connected component of \( v^{-1}(U_i \cap U_j) \cap D_k \) (where \( v : \hat{C} \to C \) is the covering map). If \( U_i \cap U_j = \emptyset \), \( i \neq j \), \( W_{ij,k} \) maps homeomorphically onto \( U_i \cap U_j \); let \( \gamma_{ij} \) be the element of \( \pi \) such that \( \gamma_{ij} W_{ij,k} = W_{ij,k} \).

Then a set of coordinate transformations \( g_{ij} \) for \( E_\rho \) is given by
\[
g_{ij} = \rho(\gamma_{ij}) \text{ on } U_i \cap U_j, \quad i \neq 0, \quad j \neq 0
\]
\[
= f_i^* \rho(\gamma_{0i}) \text{ on } U_0 \cap U_i, \quad i \neq 0,
\]
where \( f_i \) is an analytic scalar function on \( U_0 \cap U_i \) which is independent of \( \rho \).

Note that the coordinate transformations depend differentially on \( \rho \), so that the same \( g_{ij} \) (now regarded as functions on \( S_0 \times U_i \cap S_0 \times U_j \)) define a differentiable bundle \( E' \) over \( S_0 \times C \) which is a differentiable family of analytic bundles over \( C \).

Now \( E' \mid \{\rho\} \times C \cong E_\rho \cong (p \times 1_C)^*E \mid \{\rho\} \times C \) for all \( \rho \in S_0 \). Since \( E \) is stable, it follows that
\[
\dim H^0(C; \text{Hom}(E', (p \times 1_C)^*E) \mid \{\rho\} \times C) = 1
\]
for all \( \rho \). So by Proposition 2.7 of [2],
\[
\bigcup_{\rho \in S_0} H^0(C; \text{Hom}(E', (p \times 1_C)^*E) \mid \{\rho\} \times C)_{\text{Anal.}} = 0
\]
has a natural structure of differentiable line bundle over $S_0$. Let $L$ be the induced line bundle over $S_0 \times C$. There is then an obvious isomorphism

$$E' \otimes L \cong (p \times 1_C)^* E.$$ 

So

$$c_2[(p \times 1_C)^* E]_{3,1} = c_2(E')_{3,1}.$$ 

Using the above explicit description of the bundle $E'$, we see that for any continuous map $s : S^1 \to C \rightarrow x_0$, $(1_{S_0} \times s)^* E'$ can be described as follows: take a trivial bundle of rank 2 over $S_0 \times [0, 1]$ and glue its ends together by means of the map

$$S_0 \to SU(2),$$ 

defined by

$$\rho \mapsto \rho(a),$$ 

where $a \in \pi$ corresponds to $s$.

Apply this when $s = s_0$, $a = a_0$, and $\rho(a_0) = p_0(\rho)$; so Proposition 3 will follow at once from

**Lemma 4.** Let $W$ be a space and let $F$ be the bundle over $W \times S^1$ obtained by gluing together the two ends of the trivial bundle of rank 2 over $W \times [0, 1]$ by means of the map $f : W \to SU(2)$. Then

$$c_2(F) = f^*[\text{generator of } H^0(SU(2); \mathbb{Z})] \otimes [\text{generator of } H^1(S^1; \mathbb{Z})].$$

**Proof.** $F$ is the bundle induced by $f$ from the bundle obtained by taking $W = SU(2)$, $f = 1_{SU(2)}$ in the construction. Hence it is sufficient to prove the lemma for this special case. But then it follows from the fact that $H^*(BSU(2))$ is generated by $c_2$.

This completes the proof of Proposition 3 and hence of our theorem.

**References.**


