Periods of a Moduli Space of Bundles on Curves

PERIODS OF A MODULI SPACE OF BUNDLES ON CURVES.

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We will work over the complex numbers in this paper. For all curves C, and for all integers (n, d), the problem arises of determining the structure of the "space" of all vector bundles E, with rank n and degree $(=\deg c_1(E))d$. The problem has been considerably clarified recently by the introduction of the concept of stable and semi-stable bundles: [4], [6], [10]. It has been proven, in particular, that for each n and each line bundle L on C such that n and $\deg L$ are relatively prime, then the set:

$$S_{n,L}(C) = \begin{cases} \text{set of all stable vector bundles } E \text{ on } C \text{ of} \\ \text{rank } n \text{ such that } \Lambda^n E \cong L \end{cases}$$

has a natural structure of a non-singular projective variety of dimension $(n^2-1)\cdot(g-1)$, where $g = \text{genus } (C)$. It is important to note that the map

$$E \mapsto E \otimes M$$

for a line bundle M induces an isomorphism

$$S_{n,L}(C) \xrightarrow{\approx} S_{n,L\otimes M^n}(C)$$

hence the variety $S_{n,L}(C)$ depends essentially only on the residue class of $\deg L \bmod n$.

We wish to look at the case $g \geqq 2$, $n = 2$, $\deg L$ odd. In this case, we may assume for simplicity that a base point $x_0 \in C$ has been chosen that L is taken to be the line bundle whose sections form the sheaf $O_{-C}(x_0)$. We abbreviate $S_{2,L}(C)$ now to $S_2^-(C)$. The topology of these varieties has been described in [7] and when the genus of C is 2, their complete structure is described in [8]. $S_2^-(C)$ has dimension $3g-3$ and is known to be birationally equivalent to $\boldsymbol{P}_{3g-3}$. In particular, it is simply connected and the invariants $h^{0,p} = h^{p,0}$ are all 0, ([9]). In [7], it is also proven that $B_2 = 1$, $B_3 = 2g$. Now for non-singular projective varieties X with $h^{0,3} = h^{3,0} = 0$, a very interesting invariant is Weil's "intermediate jacobian" attached to $H^3(X)$. This is an abelian variety, which we shall denote $J^2(X)$, which is by definition:

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$J^2(X) \cong H^3(X, \boldsymbol{R})/\mathrm{Image}[H^3(X, \boldsymbol{Z})]$

where $H^3(X, \boldsymbol{R})$ is given a complex structure via the decomposition

$$H^3(X, \boldsymbol{R}) \otimes \boldsymbol{C} \cong H^{2,1} \oplus H^{1,2}$$

since this induces an isomorphism

$$H^3(X, \boldsymbol{R}) \cong H^{1,2} = H^2(X, \Omega^1).$$

cf. [11], [1], [3]. Weil also showed that a polarization on X induces a polarization on $J^2(X)$ in a canonical way.

If $\mathrm{Alb}(C)$ denotes the albanese, or jacobian, variety of C, then our main result is:

THEOREM. $J^2[S_2^-(C)] \cong \mathrm{Alb}(C)$.

Note that $S_2^-(C)$ has a *unique* polarization since $B_2 = 1$, hence $J^2(S_2^-(C))$ has a *canonical* polarization, just as $\mathrm{Alb}(C)$ does. It is easy to check that our isomorphism is compatible with these canonical polarizations, hence by Torelli's theorem, we conclude:

COROLLARY. *If* $S_2^-(C_1) \cong S_2^-(C_2)$, *then* $C_1 \cong C_2$.

Before beginning the proof, we must recall Weil's map relating $J^2(X)$ to codimension 2 cycles on X:

let Y be an non-singular parameter space,
let W be an algebraic cycle on $X \times Y$ of codimension 2.

Then we get

$w \in H^4(X \times Y, \boldsymbol{Z})$, the fundamental class of W
esp. $w_{3,1} \in (H^3(X, \boldsymbol{Z})/\mathrm{torsion}) \otimes H^1(Y, \boldsymbol{Z})$, the $(3,1)$-component of w.

Then $w_{3,1}$ defines a map

$$\phi_W : \underbrace{\widehat{H^1(Y, \boldsymbol{R})} \Big/ \begin{array}{l}\text{linear maps}\\ \text{which are}\\ \text{integral on}\\ H^1(Y, \boldsymbol{Z})\end{array}}_{\textstyle \Vert \atop \textstyle \mathrm{Alb}(Y)} \longrightarrow \underbrace{H^3(X, \boldsymbol{R})/\mathrm{Image}\, H^3(X, \boldsymbol{Z})}_{\textstyle \Vert \atop \textstyle J^2(X)}$$

which is easily seen to be complex-analytic using the fact that w is of type $(2,2)$ in the Hodge decomposition of H. Note the obvious fact:

LEMMA 1. *ϕ_W is an isomorphism if and only if $w_{3,1}$ is "unimodular," (i.e., written out as a matrix in terms of bases of $H^3(X,\mathbf{Z})/torsion$, $H^1(Y,\mathbf{Z})$, it is a square matrix with* $\det = \pm 1$).

1. In the sequel, we abbreviate $S_2^-(C)$ by S. The first step in our proof is to construct a universal vector bundle E on $S \times C$, i.e., one whose restriction to $\{t\} \times C$, for any $t \in S$, is exactly the vector bundle E_t on C corresponding to the point $t \in S$. This is a problem in descent theory. In fact, S can be described as a quotient $R/PGL(\nu)$, where R is a non-singular quasi-projective variety, and $PGL(\nu)$ acts freely on R; and where there is a vector bundle F on $R \times C$ whose restriction to $\{t\} \times C$, any $t \in R$, is the vector bundle on C corresponding to the image of t in S: cf. [10], p. 321. However, the difficulty is that the action of $PGL(\nu)$ on R does not, a priori, lift to an action on F. Instead, $GL(\nu)$ acts on F satisfying

1) $\mathbf{G}_m =$ center $(GL(\nu))$ acts on F by homotheties
2) if $\pi\colon GL(\nu) \to PGL(\nu)$ is the canonical map, and T_g represents the action of an element g, then the diagram

$$\begin{tikzcd} F \arrow[r, "T_g"] \arrow[d] & F \arrow[d] \\ R \times C \arrow[r, "T_{\pi(g)} \times 1_C"'] & R \times C \end{tikzcd}$$

commutes.

The way out of this type of impasse is to find a "functorial" way of associating to every vector bundle E on C (of the type being considered) a 1-dimensional vector space $\lambda(E)$ such that multiplication by α in E induces multiplication by α in $\lambda(E)$. By functorial we mean that the procedure extends to families of such vector bundles: if E is a vector bundle on $T \times C$ (for any algebraic scheme T) whose restriction to $\{t\} \times C$ is of the type under consideration, then we should get a line bundle $\lambda(E)$ on T. Moreover, for any diagram of vector bundles

$$\begin{tikzcd} E_1 \arrow[r, "g"] \arrow[d] & E_2 \arrow[d] \\ T_1 \times C \arrow[r, "f \times 1_C"'] & T_2 \times C \end{tikzcd}$$

making E_1 into a fibre product of E_2 and $T_1 \times C$ over $T_2 \times C$, we should *be given* a definite isomorphism of $\lambda(E_1)$ with $f^*(\lambda(E_2))$. For example, if $T_1 = T_2 = \mathrm{Spec}(\boldsymbol{C})$, $E_1 = E_2 = E$, and g is multiplication by a scalar $\alpha \neq 0$, we are then given an induced automorphism of $\lambda(E)$: we want this automorphism to be multiplication by α too (it might turn out to be multiplication by α^n instead). All this data is subject to an obvious co-cycle condition: compare [5], p. 64. If we can find such data, we get as a consequence a line bundle $\lambda(F)$ on R, plus an action of $GL(\nu)$ on $\lambda(F)$ in which the center acts by homotheties. If we then define

$$F' = F \otimes p_1^*(\lambda(F)^{-1}),$$

we get a new vector bundle on $R \times C$ with the same restrictions to the fibres $\{t\} \times C$ as before; but where in the natural action of $GL(\nu)$ on F', the action of the center $\boldsymbol{G}_m$ on F and $p_1^*(\lambda(F)^{-1})$ cancel each other out, i.e., $PGL(\nu)$ acts on F. Then $F/PGL(\nu)$ is the sought-for universal vector bundle on $S \times C$.

Here's how to construct λ. We limit ourselves to the case $T = \mathrm{Spec}(\boldsymbol{C})$, E a vector bundle on C, since the generalization of λ to an arbitrary base will be clear. Recall E has rank 2, degree 1, and is stable:

a) $H^1(E \otimes (\Omega_C^1)^k) = (0)$, if $k \geqq 1$.

Proof. This group is dual to $H^0(\hat{E} \otimes (\Omega_C^1)^{1-k})$ and if this were non-zero, we would get a non-zero homomorphism

$$(\Omega_C^1)^{k-1} \to \hat{E}$$

hence a sub-line-bundle $G \subset \hat{E}$ of degree $\geqq 2(k-1)(g-1) \geqq 0$. This contradicts the stability of E.

b) If $V_k(E) = H^0(E \otimes (\Omega_C^1)^k)$, then

$$\mathrm{Dim}\, V_k(E) = (2g-2)(2k-1)+1.$$

Proof. Riemann-Roch.

c) Set $\lambda(E) = [\Lambda^{2g-1} V_1(E)]^{\otimes(3g-1)} \otimes [\Lambda^{6g-5} V_2(E)]^{\otimes(-g)}$.
Then multiplication by α in E induces the endomorphism, multiplication by α, in each $V_k(E)$, hence it induces multiplcation by α to the power

$$(3g-1)(2g-1)+(6g-5)(-g)$$

in $\lambda(E)$. This number happens to be 1!

We now know that E exists. Next consider the chern classes of E. We have

$$\begin{aligned} c_2(E) &\in H^4(S\times C,\boldsymbol{Z}) \cong (H^2(S,\boldsymbol{Z})\otimes H^2(C,\boldsymbol{Z})) \\ c_1(E) &\in H^2(S\times C,\boldsymbol{Z}) \cong H^2(C,\boldsymbol{Z})\otimes H^2(S,\boldsymbol{Z}) \\ &\qquad \otimes (H^3(S,\boldsymbol{Z})\otimes H^1(C,\boldsymbol{Z})) \\ &\qquad \otimes H^4(S,\boldsymbol{Z}). \end{aligned}$$

Note that any bundle $E\otimes p_1{}^*M$, M a line bundle on S, would have the same universal property that E does, so $c_1(E)$ is not very interesting. However, let

$$\alpha=(c_2(E))_{3,1}=[\text{component of } c_2(E) \text{ in } H^3(S,\boldsymbol{Z})\otimes H^1(C,\boldsymbol{Z})].$$

A simple computation of chern classes shows that α is independent of this modification of E. According to [7], $H^3(S,\boldsymbol{Z})$ and $H^1(C,\boldsymbol{Z})$ have the same rank. In fact:

PROPOSITION 1. *α is unimodular.*

This will be proven in § 2. Assuming this, it follows from Lemma 1 that if $W=$ the *algebraic* 2nd chern class of E, then Weil's map ϕ_W: $\mathrm{Alb}(C)\to J^2(S)$ is an isomorphism, as required. Although it is not essential, it will be convenient in § 2 to know that $H^3(S,\boldsymbol{Z})$ is torsion-free. In fact, the torsion subgroup of $H^3(X,\boldsymbol{Z})$—for any non-singular complete variety X over $\boldsymbol{C}$—is a *birational invariant* of X known as the "topological Brauer group" (cf. [12], Cor. (7.3) and equation (8.9), p. 59). And S is birationally equivalent to $\boldsymbol{P}_{3g-3}$ which has no H^3 at all!

2. We start by recalling the results of [6]. In fact, let S_0 be the subset of $SU(2)^{2g}$ consisting of points $(A_1,\cdots,A_{2g})$ such that

$$\prod_{i=1}^{g} A_{2i-1}A_{2i}A_{2i-1}{}^{-1}A_{2i}{}^{-1}=-I.$$

Then S_0 is an orientable submanifold of $SU(2)^{2g}$ and there is a natural map

$$p\colon S_0\to S,$$

which is a principal fibration with group $PU(2)$. The map p may be determined as follows. Let $\tilde{C}$ be the simply-connected covering of C which is ramified over x_0 with ramification index 2. The group π of this covering is generated by elements $a_1, \cdots, a_{2g}$ subject to the single relation

$$[\prod_{i=1}^{g} a_{2i-1}a_{2i}a_{2i-1}^{-1}a_{2i}^{-1}]^2 = e.$$

Thus a point of S_0 may be regarded as a representation of π, and this representation defines a stable bundle E over C of rank 2, with $\Lambda^2 E \cong L$, and hence a point of S. So we get a map $p\colon S_0 \to S$. Notice that the a_i determine elements of $\pi_1(C)$ and hence of $H_1(C;\mathbf{Z})$, and that these elements form a basis for $H_1(C;\mathbf{Z})$; let $\{\alpha_i\}$ be the dual basis of $H^1(C;\mathbf{Z})$.

LEMMA 2. *$p^*\colon H^3(S;\mathbf{Z}) \to H^3(S_0;\mathbf{Z})$ is an isomorphism.*

Proof. Since $H^1(PU(2);\mathbf{Z}) = 0$ and $H^1(S;\mathbf{Z}) = 0$ (S is simply-connected), the spectral sequence of the fibration p gives rise to an exact sequence

$$H^0(S;H^2(PU(2);\mathbf{Z})) \to H^3(S;\mathbf{Z}) \xrightarrow{p^*} H^3(S_0;\mathbf{Z}) \to H^0(S;H^3(PU(2);\mathbf{Z})).$$

Now the first group in this sequence is $\mathbf{Z}_2$ and the last is $\mathbf{Z}$. Moreover $H^3(S;\mathbf{Z})$ is torsion-free (see § 1) and has the same rank as $H^3(S_0;\mathbf{Z})$ by the results of [7]. The lemma now follows.

LEMMA 3. *The homomorphism $H^3(SU(2)^{2g},\mathbf{Z}) \to H^3(S_0;\mathbf{Z})$ induced by the inclusion of S_0 in $SU(2)^{2g}$ is an isomorphism.*

Proof. Lemma 3 of [7] shows that the homomorphism

$$H_3(S_0;\mathbf{Z}) \to H_3(SU(2)^{2g},\mathbf{Z})$$

is surjective, except possibly for some 2-primary torsion. However, in this simple case, the same argument can be used to prove that the homomorphism is really surjective. It follows at once that $H^3(SU(2)^{2g};\mathbf{Z})$ is contained in $H^3(S_0;\mathbf{Z})$ as a direct summand. The lemma now follows from the fact that the ranks of these two groups are equal (see [7]) and that $H^3(S_0;\mathbf{Z})$ is torsion-free by Lemma 2.

Now let $p_i\colon S_0 \to SU(2)$ denote the projection on the i-th factor and let

$$\beta_i = p_i^*[\text{generator of } H^3(SU(2);\mathbf{Z})].$$

Then by Lemma 3 the β_i form a basis for $H^3(S_0;\mathbf{Z})$. In view of Lemma 2, it is now sufficient to prove:

PROPOSITION 2. $c_2[(p \times 1_C)^*E]_{3,1} = \sum_{i=1}^{2g} \beta_i \otimes \alpha_i.$

Now choose embedding $s_i\colon S^1 \to C - x_0$ which represent the generators a_i of π. Then

$$\begin{aligned} s_i^*(\alpha_j) &= 0 & i \neq j \\ &= \text{generator of } H^1(S^1;\mathbf{Z}) & i = j. \end{aligned}$$

Hence Proposition 2 will follow at once from

PROPOSITION 3.

$$c_2[(1_{S_0} \times s_i)^*(p \times 1_C)^*E]_{3,1} = \beta_i \otimes [\textit{generator of } H^1(S^1;\mathbf{Z})].$$

We now need to recall a few more details from [6]. Let E_ρ be the bundle over C corresponding to the representation $\rho \in S_0$. Then ([6] Remark 6.2) we can write down coordinate transformations for E_ρ as follows. Choose a finite open covering $\{U_i\}$ $(i = 0, 1, \cdots, m)$ of C such that every non-empty intersection of the sets U_i is contractible. Assume $x_0 \in U_0$, $x_0 \notin U_i$ for $i \neq 0$. Assume moreover that there exist discs D_i in $\tilde{C}$ such that U_0 is the quotient of D_0 by $\mathbf{Z}_2$ and that for $i \neq 0$, D_i maps homeomorphically onto U_i. For every i, j, k, where $k = i$ or j, let $W_{ij,k}$ be a connected component of $v^{-1}(U_i \cap U_j) \cap D_k$ (where $v\colon \tilde{C} \to C$ is the covering map). If $U_i \cap U_j = \emptyset$, $i \neq j$, $W_{ij,k}$ maps homeomorphically onto $U_i \cap U_j$; let γ_{ij} be the element of π such that $\gamma_{ij} W_{ij,j} = W_{ji,i}$. Then a set of coordinate transformations g_{ij} for E_ρ is given by

$$\begin{aligned} g_{ij} &= \rho(\gamma_{ij}) \text{ on } U_i \cap U_j,\ i \neq 0,\ j \neq 0 \\ &= f_i \cdot \rho(\gamma_{0i}) \text{ on } U_0 \cap U_i,\ i \neq 0, \end{aligned}$$

where f_i is an analytic scalar function on $U_0 \cap U_i$ which is independent of ρ. Note that the coordinate transformations depend differentially on ρ, so that the same g_{ij} (now regarded as functions on $S_0 \times U_i \cap S_0 \times U_j$) define a differentiable bundle E' over $S_0 \times C$ which is a differentiable family of analytic bundles over C.

Now $E' \mid \{\rho\} \times C \cong E_\rho \cong (p \times 1_C)^*E \mid \{\rho\} \times C$ for all $\rho \in S_0$. Since E is stable, it follows that

$$\dim H^0(C; \underset{\text{Anal.}}{\operatorname{Hom}}(E', (p \times 1_C)^*E) \mid \{\rho\} \times C) = 1$$

for all ρ. So by Proposition 2.7 of [2],

$$\bigcup_{\rho \in S_0} H^0(C; \underset{\text{Anal.}}{\operatorname{Hom}}(E', (p \times 1_C)^*E) \mid \{\rho\} \times C)$$

has a natural structure of differentiable line bundle over S_0. Let L be the induced line bundle over $S_0 \times C$. There is then an obvious isomorphism

$$E' \otimes L \cong (p \times 1_C)^* E.$$

So

$$c_2[(p \times 1_C)^* E]_{3,1} = c_2(E')_{3,1}.$$

Using the above explicit description of the bundle E', we see that for any continuous map $s\colon S^1 \to C - x_0$, $(1_{S_0} \times s)^* E'$ can be described as follows: take a trivial bundle of rank 2 over $S_0 \times [0,1]$ and glue its ends together by means of the map

$$S_0 \to SU(2),$$

defined by

$$\rho \mapsto \rho(a), \text{ where } a \in \pi \text{ corresponds to } s.$$

Apply this when $s = s_i$, $a = a_i$, and $\rho(a_i) = p_i(\rho)$; so Proposition 3 will follow at once from

LEMMA 4. *Let W be a space and let F be the bundle over $W \times S^1$ obtained by glueing together the two ends of the trivial bundle of rank 2 over $W \times [0,1]$ by means of the map $f\colon W \to SU(2)$. Then*

$$c_2(F) = f^*[\textit{generator of } H^3(SU(2);\mathbf{Z})] \otimes [\textit{generator of } H^1(S^1;\mathbf{Z})].$$

Proof. F is the bundle induced by f from the bundle obtained by taking $W = SU(2)$, $f = 1_{SU(2)}$ in the construction. Hence it is sufficient to prove the lemma for this special case. But then it follows from the fact that $H^*(BSU(2))$ is generated by c_2.

This completes the proof of Proposition 3 and hence of our theorem.

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