



A Note of Shimura's Paper "Discontinuous Groups and Abelian Varieties"

The Harvard community has made this article openly available. [Please share](#) how this access benefits you. Your story matters.

Citation	Mumford, David B. 1969. A note of Shimura's paper "Discontinuous groups and abelian varieties". <i>Mathematische Annalen</i> 181(4): 345-351.
Published Version	doi:10.1007/BF01350672
Accessed	October 23, 2017 5:48:36 AM EDT
Citable Link	http://nrs.harvard.edu/urn-3:HUL.InstRepos:3612771
Terms of Use	This article was downloaded from Harvard University's DASH repository, and is made available under the terms and conditions applicable to Other Posted Material, as set forth at http://nrs.harvard.edu/urn-3:HUL.InstRepos:dash.current.terms-of-use#LAA

(Article begins on next page)

A Note of Shimura's Paper "Discontinuous Groups and Abelian Varieties"

D. MUMFORD

Recently, there has been considerable discussion of families of abelian varieties parametrized by quotients of bounded symmetric domains by arithmetic subgroups. An exposition of this material can be found in the papers of Shimura, Kuga, Satake and myself in [1]. Subsequently in vol. 168, p. 171 of this journal, Shimura analyzed closely certain families of this type and showed that the abelian varieties in these families are characterized, surprisingly, by the existence of certain non-holomorphic endomorphisms. Furthermore, he says at the end of his paper (p. 199) that it does not seem that these families fall into the class which I constructed in [1c], which have the following property: they can be grouped into finite sets of families, i.e., families over a base space which is a finite union of varieties $V_i = \Gamma_i \backslash D_i$, such that the abelian varieties A_x in any one of these coarser families are exactly characterized by the rational (p, p) -forms on their powers $A_x \times \cdots \times A_x^*$.

The purpose of this note is to clarify the relationship of the different types of families of abelian varieties. First, we recall the most general type $\mathcal{X} = \{X(x) \mid x \in V\}$, as defined by Kuga. Second, we note which of these families are "of Hodge type", i.e., defined by my procedure. Thirdly, we prove that one of Kuga's families is of Hodge type if and only if it contains one abelian variety $X(x_0)$ of CM-type. As a corollary, it follows that Shimura's families in [2] are indeed of Hodge type, contrary to his guess. Lastly, we will give another example of a family of Hodge type not characterized by its ring of endomorphisms.

§ 1. Kuga's Families

Actually, we slightly generalize his definitions, for the sake of our application. We start with:

- (1) a rational vector space V of dimension $2g$,
- (2) a lattice $L \subset V$,
- (3) a non-degenerate skew-symmetric $A : V \times V \rightarrow \mathbb{Q}$, integral on $L \times L$,

* In the terminology of [1c], note that I showed that the *Hodge group* of an abelian variety was determined by these rational (p, p) -forms. But my families are given by the various different *Hodge types*, where a Hodge type is given by a Hodge group G , plus a conjugacy class of 1-dimensional tori $T \subset G$; where, however, the conditions on T imply that only a finite number of conjugacy classes of T 's can be used to form a Hodge type.

- (4) an algebraic group G , defined over \mathbb{Q} ,
 (5) a faithful symplectic representation

$$\varrho : G \rightarrow \mathrm{Sp}(V, A)$$

defined over \mathbb{Q} ,

- (6) an arithmetic subgroup $\Gamma \subset G$ such that $\varrho(\Gamma)$ preserves the lattice L .

We are interested in complex structures on $V \otimes \mathbb{R}$ so as to make $V \otimes \mathbb{R}/L$ into a complex torus. We find it convenient to adopt the following method of describing possible complex structures: let $T = \{z \in \mathbb{C} \mid |z| = 1\}$, regarded as a 1-dimensional algebraic group over the reals \mathbb{R} . Then, all complex structures on $V \otimes \mathbb{R}$ define a homomorphism of algebraic groups over \mathbb{R} :

$$\varphi : T \rightarrow \mathrm{GL}(V)$$

via $\varphi(e^{i\theta}) = \text{mult. by } e^{i\theta}$ in $V \otimes \mathbb{R}$. Conversely, any such homomorphism puts a complex structure on $V \otimes \mathbb{R}$, hence makes $V \otimes \mathbb{R}/L$ into a complex torus, which we shall call X_φ . Furthermore, if φ satisfies the Riemann conditions (a) $\varphi(T) \subset \mathrm{Sp}(V, A)$, and (b) $A(x, \varphi(i) \cdot x) > 0$ all $x \in V$, $x \neq 0$, then X_φ is an abelian variety, with a canonical polarization induced by A . Now we want a family of abelian varieties $X(x)$ parametrized by points $x \in V = \Gamma \backslash G_{\mathbb{R}}^0 / K_{\mathbb{R}}^0$ for a certain maximal compact subgroup $K_{\mathbb{R}}^0$ of $G_{\mathbb{R}}^0$. In particular, the identity in $G_{\mathbb{R}}^0$ defines a double coset $\Gamma \cdot e \cdot K_{\mathbb{R}}^0$ which is a base point 0 of this base space V . Our family can be regarded as a family of perturbations of the "base abelian variety" $X(0)$ over 0. So as our next piece of data, we assume an arbitrary $X(0)$ is given:

- (7) a complex structure $\varphi_0 : T \rightarrow \mathrm{Sp}(V, A) \subset \mathrm{GL}(V)$ such that $J_0 = \varphi_0(i)$ satisfies:

$$A(x, J_0 x) > 0, \quad \text{all } x \neq 0 \text{ in } V.$$

Now, to every point $g \in G_{\mathbb{R}}^0$, let

$$X(g) = X_{\varrho(g)\varphi_0 \cdot \varrho(g)^{-1}} = \left\{ \begin{array}{l} \text{the polarized abelian variety } V \otimes \mathbb{R}/L, \text{ with complex} \\ \text{structure } \varrho(g) \cdot \varphi_0 \cdot \varrho(g)^{-1}, \text{ Riemann form } A \end{array} \right\}.$$

If $K_{\mathbb{R}}^0 = \{g \in G_{\mathbb{R}} \mid \varrho(g)\varphi_0 = \varphi_0\varrho(g)\}^0$, then $X(g)$ depends obviously only on the image of g in the coset space $G_{\mathbb{R}}^0 / K_{\mathbb{R}}^0$. Moreover, the automorphism $\varrho(\gamma)$ of V sets up isomorphisms of the abelian variety $X(g)$ with $X(\gamma g)$ for all $g \in G_{\mathbb{R}}^0$. Therefore $X(g)$ depends only on the image x of g in $\Gamma \backslash G_{\mathbb{R}}^0 / K_{\mathbb{R}}^0$ and we may write for simplicity $X(x)$, instead of $X(g)$.

We still want to ensure that $G_{\mathbb{R}}^0 / K_{\mathbb{R}}^0$ is a bounded symmetric domain and that $\{X(x)\}$ is a holomorphic family of abelian varieties. This is guaranteed by imposing on the data (1)—(7) the following condition (which is the integrated form of Satake's original condition (H_1)):

$$(H_1^*) \quad \varrho(G) \text{ is normalized by } \varphi_0(T).$$

It follows easily from (H_1^*) that G is reductive, that $K_{\mathbb{R}}^0$ is a maximal compact subgroup of $G_{\mathbb{R}}^0$, that $G_{\mathbb{R}}^0 / K_{\mathbb{R}}^0$ is a Hermitian symmetric space, and finally that

the map induced by τ :

$$G_{\mathbb{R}}^0/K_{\mathbb{R}}^0 \rightarrow \text{Sp}(V, A) / \left(\begin{array}{c} \text{centralizer} \\ \text{of } \varphi_0 \end{array} \right) \cong \text{Siegel's upper half-plane}$$

is holomorphic. The resulting family $\{X(x) \mid x \in \Gamma \backslash G_{\mathbb{R}}^0/K_{\mathbb{R}}^0\}$ glued together into a complex analytic fibre system of abelian varieties over the base space $\Gamma \backslash G_{\mathbb{R}}^0/K_{\mathbb{R}}^0$ will be denoted by $\mathcal{X}(G, \varrho, \varphi_0)$.

Notice that the base point in Kuga's families is arbitrary. For any $g_0 \in G_{\mathbb{R}}^0$, we can replace φ_0 by $\varphi'_0 = \varrho(g_0) \cdot \varphi_0 \cdot \varrho(g_0)^{-1}$, and obtain the same family as before, but with a different base point $X(0)$.

§ 2. Hodge Groups and CM-Type

We continue to assume that the data (V, L, A) is given. Suppose

$$\varphi : T \rightarrow \text{Sp}(V, A)$$

is any complex structure on $V \otimes \mathbb{R}$ satisfying the Riemann positivity condition, so that the corresponding X_{φ} is an abelian variety.

Definition. The *Hodge group* $\text{Hg}(X_{\varphi})$ of X_{φ} is the smallest algebraic subgroup of $\text{Sp}(V, A)$ defined over \mathbb{Q} and containing $\varphi(T)$. Recall that $\text{Hg}(X_{\varphi})$ is always reductive, with compact center, and semi-simple part of Hermitian type.

Definition. If $\text{id.} : \text{Hg}(X_{\varphi}) \rightarrow \text{Sp}(V, A)$ is just the inclusion map, then the families

$$\mathcal{X}(\text{Hg}(X_{\varphi}), \text{id.}, \varphi)$$

are called the *families of Hodge type*.

These are exactly the families constructed in [1 c].

Definition. An abelian variety X over \mathbb{C} is of *CM-type* if X is isogenous to a product $X_1 \times \dots \times X_k$ of simple abelian varieties and there are fields $K_i \subset \text{Hom}(X_i, X_i) \otimes \mathbb{Q}$ such that $[K_i : \mathbb{Q}] \geq 2 \dim X_i$ (in which case, $[K_i : \mathbb{Q}] = 2 \dim X_i$ and $K_i = \text{Hom}(X_i, X_i) \otimes \mathbb{Q}$).

The following is well known ([3], § 5):

Proposition. X is of CM-type if and only if $\text{Hom}(X, X) \otimes \mathbb{Q}$ contains a commutative semi-simple \mathbb{Q} -algebra R such that $[R : \mathbb{Q}] \geq 2 \dim X$, and if R exists, $[R : \mathbb{Q}] = 2 \dim X$.

Yet another characterization of CM-type is:

Proposition. X_{φ} is of CM-type if and only if $\text{Hg}(X_{\varphi})$ is a torus algebraic group.

Proof. Any endomorphism of X_{φ} has a natural representation as an endomorphism of V over \mathbb{Q} , and it is easily seen that for any φ ,

$$\text{Hom}(X_{\varphi}, X_{\varphi}) \otimes \mathbb{Q} \cong \{g \in \text{Hom}(V, V)_{\mathbb{Q}} \mid g\varphi = \varphi g\}.$$

Since $\text{Hg}(X_\varphi)$ is generated by $\text{Im}(\varphi)$ and by its conjugates over \mathbb{Q} , it follows that:

$$\text{Hom}(X_\varphi, X_\varphi) \otimes \mathbb{Q} \cong \{g \in \text{Hom}(V, V)_{\mathbb{Q}} \mid gg' = g'g, \text{ all } g' \in \text{Hg}(X_\varphi)\}. \quad (*)$$

Now suppose that X_φ is of CM-type. Let R be the commutative semi-simple \mathbb{Q} -algebra given by the previous proposition. Then via the above isomorphism it follows that $\text{Hg}(X_\varphi)$ commutes with a maximal commutative semi-simple subalgebra R' of $\text{Hom}(V, V)$. But therefore $\text{Hg}(X_\varphi) \subset \text{units of } R'$, hence $\text{Hg}(X_\varphi)$ is itself commutative, hence it is a torus algebraic group. Conversely, if $\text{Hg}(X_\varphi)$ is a torus algebraic group, then as a subgroup of $GL(V)_{\mathbb{C}}$ it is diagonalizable. Therefore, the commutator of $\text{Hg}(X_\varphi)$ in $\text{Hom}(V, V)_{\mathbb{C}}$ and hence in $\text{Hom}(V, V)_{\mathbb{Q}}$ contains maximal commutative semi-simple subalgebras R' . Since for all such R' , $[R' : \mathbb{Q}] = \dim V = 2 \dim X_\varphi$, this implies by (*) that X_φ is of CM-type. QED.

§ 3. The Theorem

Theorem. a) *Every family $\mathcal{X}(\text{Hg}(X_\varphi), \text{id.}, \varphi)$ of Hodge type contains abelian varieties of CM-type.*

b) *If $\mathcal{X}(G, \varrho, \varphi_0)$ contains a member of CM-type, then $\mathcal{X}(G, \varrho, \varphi_0)$ is isomorphic to a family of Hodge type.*

Proof. It is well known that for any algebraic group G over \mathbb{Q} , $G_{\mathbb{Q}}$ is dense in $G_{\mathbb{R}}^0$. This implies that for such groups G , every $G_{\mathbb{R}}$ -conjugacy class of maximal algebraic tori in G , defined over \mathbb{R} , contains tori defined over \mathbb{Q} . Namely, let $T_1 \subset G$ be a maximal torus defined over \mathbb{R} . If $a \in (T_1)_{\mathbb{R}}$ is a regular element, then T_1 is the centralizer of a , and a has an open neighborhood $U \subset G_{\mathbb{R}}^0$ such that the centralizer of any $a' \in U$ is a conjugate of T_1 . If $a' \in U \cap G_{\mathbb{Q}}$, then the centralizer of a' is a conjugate of T_1 defined over \mathbb{Q} , as required. Now apply these results to $\text{Hg}(X_\varphi)$. Let $K = \text{centralizer of } \varphi(T)$, regarded as an algebraic subgroup of $\text{Hg}(X_\varphi)$ defined over \mathbb{R} . Then K has at least one maximal algebraic torus $T_1 \subset K$ defined over \mathbb{R} ([1 e], p. 26). Since $\varphi(T) \subset \text{center}(K)$, $\varphi(T) \subset T_1$. Moreover, if T'_1 is any torus in $\text{Hg}(X_\varphi)$ containing T_1 , then T'_1 will centralize $\varphi(T)$, hence $T'_1 \subset K$, hence $T'_1 = T_1$: i.e., T_1 is a maximal algebraic torus of $\text{Hg}(X_\varphi)$ too. By our first remark, there are elements $g \in \text{Hg}(X_\varphi)_{\mathbb{R}}$ such that $T_2 = g T_1 g^{-1}$ is defined over \mathbb{Q} . But then $T_2 \supset g \varphi(T) g^{-1}$. Therefore $T_2 \supset \text{Hg}(X_{g\varphi g^{-1}})$, and so $\text{Hg}(X_{g\varphi g^{-1}})$ must be an algebraic torus. Therefore $X_{g\varphi g^{-1}}$ is of CM-type and this proves (a).

To prove (b), first replace φ_0 by $\varrho(g)\varphi_0\varrho(g)^{-1}$ for a suitable $g \in G_{\mathbb{R}}^0$ so that the abelian variety $X(0)$ is of CM-type. This does not alter the family $\mathcal{X}(G, \varrho, \varphi_0)$. Let $T_1 = \text{Hg}(X_{\varphi_0})$. This is an algebraic torus, defined over \mathbb{Q} . Since T_1 is generated by $\varphi_0(T)$ and its \mathbb{Q} -conjugates, all of which normalize $\varrho(G)$, it follows that T_1 normalizes $\varrho(G)$. So $G^* = \varrho(G)$. T_1 is an algebraic subgroup of $\text{Sp}(V, A)$ defined over \mathbb{Q} , with $\varrho(G)$ as a normal subgroup. Note that G^* is still reductive since $\varrho(G)$ and T_1 are reductive. In particular, $G^* = (\text{semi-simple part}) \cdot (\text{central torus})$, hence $G^* = \varrho(G) \cdot (\text{central torus})$. Therefore, the two collections of complex

structures:

$$(I) \{ \varrho(g) \varphi_0 \varrho(g)^{-1} \mid g \in G_{\mathbb{R}}^0 \},$$

$$(II) \{ g \varphi_0 g^{-1} \mid g \in G_{\mathbb{R}}^{*,0} \}$$

are exactly the same. Moreover, $G_{\mathbb{R}}^0 / (\text{centr. of } \varphi_0)$ is canonically isomorphic to $G_{\mathbb{R}}^{*,0} / (\text{centr. of } \varphi_0)$. Therefore the 2 families $\mathcal{X}(G, \varrho, \varphi_0)$ and $\mathcal{X}(G^*, \text{id.}, \varphi_0)$ (where $\text{id.}: G^* \rightarrow \text{Sp}(V, A)$ is the inclusion map) are isomorphic. Finally, let $\varphi = g \varphi_0 g^{-1}$ where $g \in G_{\mathbb{R}}^{*,0}$ is a generic point of G^* over a field of definition of $\varphi_0(T)$. Then $\text{Hg}(X_\varphi)$ is the smallest \mathbb{Q} -rational subgroup of G^* containing all the tori $g \varphi_0^{\sigma} g^{-1}$, $g \in G^*$, $\sigma \in \text{Aut}(\mathbb{C})$. Hence $\text{Hg}(X_\varphi)$ is just the smallest subgroup of G^* (defined over any field) containing all the tori $g T_1 g^{-1}$, $g \in G^*$. In particular $\text{Hg}(X_\varphi)$ is a normal subgroup of G^* , defined over \mathbb{Q} and containing $\varphi(T)$ and $\varphi_0(T)$. Therefore $G^* = \text{Hg}(X_\varphi) \cdot G_2$ where G_2 commutes with $\text{Hg}(X_\varphi)$ and hence with $\varphi(T)$ and with $\varphi_0(T)$. It follows that the sets (I) and (II) of complex structures are equal to the set:

$$(III) \{ g \varphi g^{-1} \mid g \in \text{Hg}(X_\varphi)_{\mathbb{R}}^0 \}.$$

It also follows that $G_{\mathbb{R}}^{*,0} / (\text{centr. of } \varphi_0)$ is isomorphic to $\text{Hg}(X_\varphi)_{\mathbb{R}}^0 / (\text{centr. of } \varphi)$, hence the 2 families $\mathcal{X}(G^*, \text{id.}, \varphi_0)$ and $\mathcal{X}(\text{Hg}(X_\varphi), \text{id.}, \varphi)$ are isomorphic. QED.

§ 4. An Example

Since families of abelian varieties which are *not* characterized by their endomorphism rings are fairly mysterious, it seems worth-while to present as an example what seems to be the *only* family of this type of 4-dimensional abelian varieties. (In dimensions 1, 2 and 3, all families are characterized by endomorphisms.)

To define this family, we need an apparently "well-known" construction for central simple algebras: if $L \supset K$ is a finite separable extension of degree n , and D is a central simple algebra over L , with $[D : L] = e^2$, then there is canonical central simple algebra $\text{Cor}_{L/K}(D)$ over K , with $[\text{Cor}(D) : K] = e^{2n}$, and with a homomorphism

$$\text{Nm} : D^* \rightarrow \text{Cor}_{L/K}(D)^*$$

of units. It is simply the corestriction map in the cohomology theory of groups applied to the Brauer group. To construct it, let Ω be a separable closure of K and let $\sigma_1, \dots, \sigma_n : L \rightarrow \Omega$ be the distinct K -isomorphisms from L to Ω . Let $D^{(i)} = D \otimes_L (\Omega, \sigma_i)$ be the central simple Ω -algebra obtained by base change with respect to σ_i . Then $\text{Gal}(\Omega/K)$ acts on

$$E = D^{(1)} \otimes \dots \otimes D^{(n)}$$

in a natural way, i.e. if $\tau : \Omega \rightarrow \Omega$ is a K -isomorphism, then $\tau \circ \sigma_i = \sigma_{\pi(i)}$ for some permutation π , and the maps

$$d \otimes a \mapsto d \otimes \tau(a)$$

induce semi-linear-isomorphisms $D^{(i)} \simeq D^{(\pi(i))}$, hence a semi-linear automorphism of E . Let $\text{Cor}_{L/K}(D)$ be the subalgebra of E left fixed by this action: this will be a central simple K -algebra such that $\text{Cor}_{L/K}(D) \otimes_K \Omega \cong E$. Finally define $\text{Nm}(d)$ to be the element

$$(d \otimes 1) \otimes \cdots \otimes (d \otimes 1)$$

of E , which is clearly in $\text{Cor}_{L/K}(D)$.

Now let K be a totally real cubic number field, and let D be a quaternion division algebra over K such that

$$\text{Cor}_{K/\mathbb{Q}}(D) \text{ splits, i.e. } \cong M_8(\mathbb{Q}), \quad (1)$$

$$D \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{K} + \mathbb{K} + M_2(\mathbb{R}). \quad (2)$$

Then we get a natural homomorphism:

$$\text{Nm} : D^* \rightarrow GL(8, \mathbb{Q}).$$

Let $\bar{}$ be the standard involution of D , and let

$$G = \{x \in D^* \mid x \cdot \bar{x} = 1\}.$$

Then G is an algebraic group over \mathbb{Q} , which is \mathbb{Q} -simple, but which by (2), is a \mathbb{Q} -form of the \mathbb{R} -algebraic group $SU(2) \times SU(2) \times SL(2, \mathbb{R})$. Moreover, let $V = \mathbb{Q}^8$; then, via Nm , G has an algebraic representation in V defined over \mathbb{Q} , which is a \mathbb{Q} -form of \mathbb{R} -representation:

$$\begin{aligned} SU(2) \times SU(2) \times SL(2, \mathbb{R}) &\rightarrow \underbrace{SO(4) \times SL(2, \mathbb{R})}_{\text{acting on } \mathbb{R}^4 \otimes \mathbb{R}^2} \end{aligned}$$

Over \mathbb{R} , this representation leaves invariant a unique symplectic form (up to scalars), so there is a unique symplectic form $A : V \times V \rightarrow \mathbb{Q}$ left fixed by our \mathbb{Q} -representation. Let $L \subset V$ be any lattice, and $\Gamma \subset G$ any arithmetic subgroup preserving L . Finally, let

$$\varphi_0 : T \rightarrow \text{Sp}(V, A)_{\mathbb{R}}$$

be the homomorphism:

$$\varphi_0 : T \rightarrow SO(4) \times SL(2, \mathbb{R}) \subset \text{Sp}(V, A)_{\mathbb{R}},$$

$$e^{i\theta} \rightarrow I_4 \otimes \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}.$$

This gives data (1)–(7) as in § 1, and (H_1^*) is obviously satisfied. Moreover, since G is \mathbb{Q} -simple, $\varrho(G)$ is exactly the Hodge group of a generic conjugate $\varrho(a) \cdot \varphi_0 \cdot \varrho(a)^{-1}$, so we have a family of Hodge type. Finally, V is an absolutely irreducible representation of G , so whenever $\text{Hg}(X_\varphi) = \varrho(G)$, X_φ has no non-trivial endomorphisms.

References

1. Algebraic groups and discontinuous subgroups. Proc. Symp. in Pure Math., vol. 9, Am. Math. Soc., 1966.
 - 1a. Article of Shimura, p. 312.
 - 1b. Article of Kuga, p. 338.
 - 1c. Article of Mumford, p. 347.
 - 1d. Article of Satake, p. 352.
 - 1e. Article of Borel and Springer, p. 26.
2. Shimura, G.: Discontinuous groups and abelian varieties. Math. Ann. **168**, 171—199 (1967).
3. —, and Y. Taniyama: Complex multiplication of abelian varieties. Publ. of Math. Soc. Japan, 1961.

Professor D. Mumford
Department of Mathematics
Harvard University
Cambridge, Mass. 02138, USA

(Received November 4, 1968)

