Pathologies IV

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By David Mumford.

In this note I would like to use the beautifully simple method introduced by Tony Iarrobino [1]—when he proved that there are 0-dimensional subschemes of $\mathbf{P}^3$ which are not specializations of reduced subschemes—to prove here that there are also reduced and irreducible complete curves which are not specializations of non-singular curves. Since there are no global obstructions in deforming reduced curves, this also shows that there are complete reduced 1-dimensional local rings with no flat deformation which is generically smooth.

Start with a complete non-singular curve C of genus g with no automorphisms over an algebraically closed ground field k. Choose a point $x \in C$ and a large even integer ν. Note that if V is any k-vector space where

$$m_{x,C}^{2\nu} \subset V \subset m_{x,C}^{\nu}$$

then $k+V$ is a subring of $\mathcal{O}_{x,C}$. For each such V, define a new curve:

$$\pi : C \to C(V)$$

by:

(a) π is a bijection, and an isomorphism

$$\operatorname{res} \pi : C - \{x\} \xrightarrow{\approx} C(V) - \{\pi x\}.$$

(b) $\mathcal{O}_{\pi x, C(V)} = k + V$.

Note that if V_1, V_2 are two such vector spaces, then

$$C(V_1) \approx C(V_2) \Rightarrow V_1 = V_2.$$

(In fact, C is the normalization of each $C(V)$; hence any $o : C(V_1) \xrightarrow{\approx} C(V_2)$ lifts to $o' : C \to C$ which must be the identity, hence $k + V_1 = \mathcal{O}_{\pi(x), C(V_1)} = \mathcal{O}_{\pi(x), C(V_2)} = k + V_2$.) Moreover, the curves $C(V)$ can all be fitted together into

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a family if we fix the integer $\dim_k V/m_{x,C}^{2\nu}$: for all k, $0 \leqslant k \leqslant \nu$,

let G = Grassmanian of k-dimensional subspaces of $m_{x,C}^{\nu}/m_{x,C}^{2\nu}$,

let $\mathcal{V} \subset (m_{x,C}^{\nu}/m_{x,C}^{2\nu}) \otimes_k \mathcal{O}_G$ be the universal family,

let $C(\mathcal{V})$ be the scheme equal to $C \times G$ as topological space, with structure sheaf defined by:

$$\begin{array}{ccccc} \mathcal{O}_{C\times G} & \xrightarrow{\alpha} & (\mathcal{O}_{x,C}/m_{x,C}^{2\nu}) \otimes_k \mathcal{O}_G & \longrightarrow 0 \\ & & \cup & \\ \cup & & [k + (m_{x,C}^{\nu}/m_{x,C}^{2\nu})] \otimes_k \mathcal{O}_G & \\ & & \cup & \\ \alpha^{-1}(\mathcal{V} + \mathcal{O}_G) & - - - \rightarrow & \mathcal{V} + \mathcal{O}_G & \\ \|\text{def.} & & & \\ \mathcal{O}_{C(\mathcal{V})} & & & \end{array}$$

Since $[(\mathcal{O}_{x,C}/m_{x,C}^{2\nu}) \otimes_k \mathcal{O}_G]/\mathcal{V}$ is a locally free $\mathcal{O}_G$-sheaf, $\mathcal{O}_{C(\mathcal{V})}$ is flat over $\mathcal{O}_G$, i.e., $C(\mathcal{V})$ is flat over G.

Now choose $k = \nu/2$ and calculate:

(i) $\dim G = k(\nu - k) = \nu^2/4$

(ii) $p_a(C(V)) = g + \dim_k [\mathcal{O}_{x,C}/\mathcal{O}_{\pi(x),C(V)}]$

$$= g + \left(\frac{3\nu}{2} - 1\right)$$

Therefore if $\nu \gg 0$, $\dim G \geqslant 3p_a(C(V)) - 3$! I claim that this implies that almost all the curves $C(V)$ are not specializations of non-singular curves, because of:

LEMMA. *Let $p: \mathcal{C} \to S$ be a flat and proper family of reduced and irreducible singular curves $C_s = p^{-1}(s)$ such that*

(a) $\forall s \in S$, $\{s' \mid C_{s'} \approx C_s\}$ *is finite*

(b) $p_a(C_s) \geqslant 2$, *S is irreducible and* $\dim S \geqslant 3p_a(C_s) - 3$,

then almost all curves C_s are not specializations of non-singular curves.

Proof. If the conclusion is false, then after replacing S by a Zariski open subset we can extend the family $\mathcal{C}/S$ like this:

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{C}^* \\ \downarrow & & \downarrow \\ S & \longrightarrow & S^* \end{array} \quad ; \quad \begin{array}{l} S^* \text{ irreducible,} \\ \dim S^* = \dim S + 1 \end{array}$$

so that $\mathcal{C}^*$ is generically smooth over S^*. In fact $\mathcal{C}$ will carry a relatively ample L, so we may use $p_* L^{\otimes n}$ $(n \gg 0)$ to embed $\mathcal{C}$ in some $\mathbf{P}^N$-bundle $\mathcal{P}$ over S. Moreover, if a C_S $(s \in S_0)$ is abstractly a specialization of a non-singular curve, so is the embedded curve $C_S \subset \mathbf{P}^N$. So take S^* to be a suitable subvariety of the Hilbert scheme of $\mathcal{P}$ over S. Once we have $\mathcal{C}^*/S^*$, consider the two induced families:

$$\mathcal{C}_i^* = \mathcal{C}^* \times_{S^*} (S^* \times S^*) \qquad \text{(formed via } p_i : S^* \times S^* \to S^*, i = 1,2)$$

and the scheme

$$I = \mathbf{Isom}_{S^* \times S^*}(\mathcal{C}_1^*, \mathcal{C}_2^*)$$

whose points over $(s_1, s_2) \in S^* \times S^*$ are isomorphisms $o : C_{s_1} \to C_{s_2}$. Look at the morphisms:

$$\begin{array}{c} I \\ q \;\downarrow\!\!\uparrow\; \delta \\ S^* \end{array} \qquad \begin{array}{l} q(o : C_{s_1} \to C_{s_2}) = s_1 \\ \delta(s) = [\text{id.} : C_s \to C_s] \end{array}$$

Since $\dim S^* = \dim S + 1 > 3p_a(C_s) - 3$, whenever C_s is non-singular, the same non-singular curve must occur in the family $\mathcal{C}^*/S^*$ infinitely often; thus when C_s is non-singular, some component of $q^{-1}(s)$ through $\delta(s)$ is positive-dimensional. Now by upper semi-continuity of dimensions of fibres of a morphism, it follows that for every s, $q^{-1}(s)$ has a positive-dimensional component through $\delta(s)$. Now let $D_1 = \mathrm{Im}(S \to S^*)$, $D_2 = \{s \mid C_s \text{ is singular}\}$; then $\bar{D}_1$ is a component of D_2 and let $D_1^0 = D_1$-(closure of $D_2 - D_1$). Choose $s \in D_1^0$ and consider how $q^{-1}(s)$ can have a positive-dimensional component γ through $\delta(s)$. By (b), $\mathrm{Aut}(C_s)$ is finite; by (a), there are only finitely many $s' \in D_1$ with $C_s \approx C_{s'}$; certainly $C_s \not\approx C_{s'}$ if $s' \in S^* - D_2$ because C_s is singular while $C_{s'}$ is non-singular; and since $s \not\in$ (closure $D_2 - D_1$), γ cannot lie over $(s) \times \overline{D_2 - D_1}$ in $S^* \times S^*$. Thus there is nowhere for γ to go! Contradiction.

REFERENCES.

[1] A. Iarrobino, "Reducibility of the families of 0-dimensional schemes on a variety," *Inv. Math.* **15** (1972), pp. 72–77.