Hirzebruch's Proportionality Theorem in the Non-Compact Case

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Hirzebruch’s Proportionality Theorem in the Non-Compact Case

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Dedicated to Friedrich Hirzebruch

In the conference on Algebraic Topology [7] in 1956, F. Hirzebruch described a remarkable theorem relating the topology of a compact locally symmetric variety:

\[ X = \overline{D}/\Gamma, \]
\[ D = \text{bounded symmetric domain}, \]
\[ \Gamma = \text{discrete torsion-free co-compact group of automorphisms of } D \]

with the topology of the extremely simply rational variety \( \overline{D} \), the “compact dual” of \( D \). (See §3 for full definitions.) His main result is that the Chern numbers of \( X \) are proportional to the Chern numbers of \( \overline{D} \), the constant of proportionality being the volume of \( X \) (in a natural metric). This is a very useful tool for analyzing the structure of \( X \). Many of the most interesting locally symmetric varieties that arise however are not compact: they have “cusps”. It seems a priori very plausible that Hirzebruch’s line of reasoning should give some relation even in the non-compact case between the Chern numbers of \( X \) and of \( \overline{D} \), with some correction terms for the cusps. The purpose of this paper is to show that this is indeed the case. We hope that the generalization that we find will have applications.

The paper is organized as follows. In §1, we make a few general definitions and observations concerning Hermitian metrics on bundles with poles and describe an instance where such metrics still enable one to calculate the Chern classes of the bundle. This section is parallel to work of Cornalba and Griffiths [6]. In §2, which is the most technical, we prove a series of estimates for a class of functions on a convex self-adjoint cone. In §3, the results of §1 and §2 are brought together, and the Proportionality Theorem is proven. One consequence is that \( D/\Gamma \) has the property, defined by Iitaka [8], of being of logarithmic general type. Finally, in §4, we analyze the step from logarithmic general type to general type and reprove a Theorem of Tai that \( D/\Gamma \) is of general type if \( \Gamma \) is sufficiently small.
§ 1. Singular Hermitian Metrics on Bundles

In this section we will not be concerned specifically with the locally symmetric algebraic varieties $D/I$, but with general smooth quasi-projective algebraic varieties $X$. When $X$ is not compact, we want to study the order of poles of differential forms on $X$ at infinity, and when $E$ is moreover a vector bundle on $X$, we want to study Hermitian metrics on $E$ which also “have poles at infinity”. This situation has been studied by Cornalba-Griffiths [6]. The following idea of bounding various forms by local Poincaré metrics on punctured polycylinders at infinite is due to them. More precisely, we choose a smooth projective compactification $\bar{X}$:

$$X \subset \bar{X}$$

where $\bar{X} - X$ is a divisor on $\bar{X}$ with normal crossings.

Then we look at polycylinders:

$$A' \subset \bar{X} \quad \text{where} \quad A' = \text{unit disc} \quad r = \dim \bar{X}$$

where $A' \cap (\bar{X} - X) = \{ \text{union of coordinate hyperplanes} \}$

$$\{ z_1 = 0, z_2 = 0, \ldots, z_k = 0 \}$$

hence:

$$A' \cap X = (A^*)^k \times A'^{-k}.$$

In $A^*$ we have the Poincaré metric:

$$ds^2 = \frac{|dz|^2}{|z|^2 (\log |z|)^2}$$

and in $A$ we have the simple metric $|dz|^2$, giving us a product metric on $(A^*)^k \times A'^{-k}$ which we call $\omega^{(p)}$.

**Definition.** A complex-valued $C^\infty$ p-form $\eta$ on $X$ is said to have Poincaré growth on $\bar{X} - X$ if there is a set of polycylinders $U_\epsilon \subset \bar{X}$ covering $\bar{X} - X$ such that in
each $U_s$, an estimate of the following type holds:

$$|\eta(t_1, \ldots, t_p)|^2 \leq C_s \omega_{t_s}(t_1, t_1) \cdots \omega_{t_s}(t_p, t_p)$$

(all $t_1, \ldots, t_p$ tangent vectors to $\bar{X}$ at some point of $U_s \cap X$).

It is not hard to see that this property is independent of the covering $U_s$ of $\bar{X} - X$ (but unfortunately it does depend on the compactification $\bar{X}$). Moreover, if $\eta_1, \eta_2$ both have Poincaré growth on $\bar{X} - X$, then so does $\eta_1 \wedge \eta_2$. This leads to the basic property:

**Proposition 1.1.** A $p$-form $\eta$ with Poincaré growth on $\bar{X} - X$ has the property that for every $C^\infty(r-p)$-form $\zeta$ on $\bar{X}$,

$$\int_{\bar{X} - X} |\eta \wedge \zeta| < +\infty$$

hence $\eta$ defines a $p$-current $[\eta]$ on $\bar{X}$.

**Proof.** Since $\zeta$ has Poincaré growth, we are reduced to checking that if $\eta$ is an $r$-form with Poincaré growth, then

$$\int_{\bar{X}} |\eta| < +\infty.$$

In a polytube $U_s$, this amounts to the well-known fact that for all relatively compact $V \subset U_s$, the Poincaré metric volume of $V \cap (\partial^* \times \partial^* - b)$ is finite. QED

**Definition.** A complex-valued $C^\infty$ $p$-form $\eta$ on $X$ is good on $\bar{X}$ if both $\eta$ and $d\eta$ have Poincaré growth.

The set of all good forms $\eta$ is differential graded algebra for which we have the next basic property:

**Proposition 1.2.** If $\eta$ is a good $p$-form, then

$$d([\eta]) = [d\eta].$$

**Proof.** By definition of $d([\eta])$, this means that for all $C^\infty(r-p-1)$-forms $\zeta$ on $\bar{X}$,

$$\int_{\bar{X} - X} d\eta \wedge \zeta = - \int_{\bar{X} - X} \eta \wedge d\zeta.$$

This comes down to asserting that if $U_\varepsilon$ is a tube of radius $\varepsilon$ around $\bar{X} - X$, then

$$\lim_{\varepsilon \to 0} \int_{U_\varepsilon} (\eta \wedge \zeta) = 0.$$

If we take, for instance, $r=2$ and set up this integral in local coordinates $x$, $y$ on $\bar{X}$ near a point where $\bar{X} - X$ has 2 branches $x=0$ and $y=0$, then this comes

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1 Compare Cornalba-Griffiths [6], p. 25
down to the assertion
\[ \lim_{\varepsilon \to 0} \int_{\varepsilon}^{1} \frac{|dx|^2}{|x|^2 (\log |x|)^2} \cdot \frac{|dy|}{|y| - |(\log |y|)|} = 0 \]
which is easy to check. The general case is similar. QED

Next, let \( \tilde{E} \) be an analytic rank \( n \) vector bundle on \( \tilde{X} \), let \( E \) be the restriction of \( \tilde{E} \) to \( X \) and let \( h: E \to \mathbb{C} \) be a Hermitian metric on \( E \). For such \( h \) we define “good” as follows:

**Definition.** A Hermitian metric \( h \) on \( E \) is **good on** \( \tilde{X} \) if for all \( x \in \tilde{X} - X \) and all bases \( e_1, \ldots, e_n \) of \( \tilde{E} \) in a neighborhood \( \mathcal{U}' \) of \( x \), \( \tilde{X} - X \) is given as above by \( \prod_{i=1}^{k} z_i = 0 \), if \( h_{ij} = h(e_i, e_j) \), then

i) \( |h_{ij}| (\det h)^{-1} \leq C \left( \sum_{i=1}^{k} \log |z_i| \right)^{2n} \), for some \( C > 0, n \geq 1 \),

ii) the 1-forms \( (\partial h \cdot h^{-1})_{ij} \) are good on \( \tilde{X} \cap \mathcal{U} \).

The first point about good Hermitian metrics is that given \( (E, h) \), there is at most one extension \( \tilde{E} \) of \( E \) to \( \tilde{X} \) for which \( h \) is good. This follows from:

**Proposition 1.3.** If \( h \) is good, then for all polycylinders \( \mathcal{U}' \subset \tilde{X} \) in which \( \tilde{X} - X \) is given by \( \prod_{i=1}^{k} z_i = 0 \),

\[ \mathcal{I}(\mathcal{U}', \tilde{E}) = \{ s \in \mathcal{I}'; (x, E) \mid h(s, s) \leq C \cdot (\sum \log |z_i|)^{2n}, \text{for some } C, n \} \]

**Proof.** The inclusion “\( \subset \)” is immediate. As for “\( \supset \)”, if \( s = \sum_{i=1}^{n} a_i(z) e_i \) is a holomorphic section of \( E \) on \( \mathcal{U}' \cap X \), for which \( h(s, s) \) is bounded as above, then it follows that

\[ |a_i(z)| \leq C^' (\sum \log |z_i|)^{2m} \], for suitable \( C', m \).

Therefore \( \left( \prod_{i=1}^{k} z_i \right) \cdot a_i(z) \) is bounded on \( \mathcal{U}' \), hence is analytic, hence \( a_i(z) \) is meromorphic with simple poles on \( \tilde{X} - X \). But as no inequality

\[ \frac{1}{|z|^2} \leq C (\log |z|)^{2n} \]

holds, \( a_i(z) \) is in fact analytic. QED

The main result of this section is the following:

**Theorem 1.4.** If \( \tilde{E} \) is a vector bundle on \( \tilde{X} \) and \( h \) is a good Hermitian metric on \( E = \tilde{E}|_X \), then the Chern forms \( c_k(E, h) \) are good on \( \tilde{X} \) and the current \([c_k(E, h)]\) represents the cohomology class \( c_k(\tilde{E}) \in H^{2k}(\tilde{X}, \mathbb{C}) \).
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Proof. Let $h^*$ be a $C^\infty$ Hermitian metric on $\bar{E}$. Define

$$\theta = \overline{\partial}h \cdot h^{-1}, \quad \theta^* = \overline{\partial}h^* \cdot h^* \cdot h^{-1},$$

$$K = \overline{\partial}0, \quad K^* = \overline{\partial}\theta^*.$$

Intrinsically, $K$ and $K^*$ are Hom $(E,E)$-valued $(1,1)$-forms and $\theta - \theta^*$ is a Hom $(E,E)$-valued $(1,0)$-form. According to results in Bott-Chern [5], for each $k$ there is a universal polynomial $P_k$ with rational coefficients in the forms $K, K^*$ and $\theta - \theta^*$ such that on $X$:

$$c_k(E, h) - c_k(E, h^*) = d(\text{Tr} P_k(K, K^*, \theta - \theta^*)).$$

Now $K, K^*$ and $\theta - \theta^*$ are forms good on $\bar{X}$, hence the $(k, k)$-form $\text{Tr} P_k(K, K^*, \theta - \theta^*)$ is good on $\bar{X}$. It follows that $c_k(E, h)$ is good on $\bar{X}$ and that

$$[c_k(E, h)] = d[\text{Tr} P_k] + [c_k(E, h^*)]. \quad \text{QED}$$

§ 2. Estimates on Cones

The results of this section are purely preliminary. We have isolated all the inequalities needed for the general proportionality theorem which involve only the cone variables (cf. § 3, definition of Siegel Domain), and worked these out in this section.

The object of study then is a real vector space $V$ and

$$C \subset V,$$

$C$ an open, convex, non-degenerate (\(\bar{C} = \{(x,y) \in V \mid \langle x,y \rangle \geq 0, \forall y \in C\}\) cone. Most of our results relate only to those $C$ which are homogeneous and self-adjoint; for any $C$, we let $G \subset GL(V)$ be the group of linear maps which preserve $C$, and say $C$ is homogeneous if $G$ acts transitively. If, moreover, there is a positive-definite inner product $\langle , \rangle$ on $V$ for which

$$\bar{C} = \{x \in V \mid \langle x,y \rangle \geq 0, \forall y \in C\}\)$$

we say $C$ is self-adjoint. The classification of these is well known (see [1], p. 63), as in the fact that all such arise by considering formally real Jordan algebras $V$ and setting

$$C = \{x^2 \mid x \in V, x \text{ invertible}\}.$$

All convex non-degenerate cones $C$ carry several canonical metrics on them. First of all, there is a canonical Finsler metric on $C$, which is analogous to the Caratheodory metrics on complex manifolds ([10], p. 49):

$$\forall x \in C, t \in T_{x,C} \cong V, \text{ let:}$$

$$\rho_x(t) = \sup_{\bar{C}} \frac{|l(t)|}{l(x)}.$$
(Another canonical Finsler metric, analogous to Kobayashi’s metric in the complex case, can also be defined. First introduce on the positive quadrant $\mathbb{R}_+ \times \mathbb{R}_+$ the metric $\frac{dx^2}{x^2} + \frac{dy^2}{y^2}$; then we have on any cone the definition:

$$\rho_x(t) = \begin{cases} \text{canonical length in} \\ \text{the cone } C \cap (\mathbb{R}_+ x + \mathbb{R} t) \end{cases}$$

$$= \sqrt{\frac{1}{a_1^2} + \frac{1}{a_2^2}}$$

if $a_1 > 0$ and $a_2 < 0$ are determined by $x + a_1 t, x + a_2 t \in \overline{C}, \notin C$. We won’t need this second metric however.)

The advantage of the Finsler metric is that (as in the complex case) it behaves in a monotone way when you replace $C$ by a smaller (or bigger) cone:

**Proposition 2.1.** i) If $C$ is an open convex non-degenerate cone in $V$, and $a \in \overline{C}$, $x \in C$, $t \in V$, then

$$\rho_{x+a}(t) \subseteq \rho_x(t) C.$$ 

ii) If $C_1 \subset C_2$ are 2 open convex non-degenerate cones in $V$, then for all $x \in C_1$, $t \in V$,

$$\rho_x(t) C_1 \subseteq \rho_x(t) C_2.$$ 

(The proofs are easy.)

Now suppose $C$ is homogeneous and self-adjoint. Then one can introduce a Riemannian metric on $C$ as follows. Chose a base point $e \in V$ which we take as the identity for the Jordan algebra and let $\langle \cdot, \cdot \rangle$ be an inner product on $V$ in terms of which $G = \mathfrak{g}$. Then ([1], p. 62), $C$ is self-adjoint with respect to $\langle \cdot, \cdot \rangle$. Moreover, $K = \text{Stab}(e)$ is a maximal compact subgroup of $G$ and if $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition with respect to $K$ and $*: G \to G$ the Cartan involution, then:

1) $\langle gx, g^* y \rangle = \langle x, y \rangle$, hence $K = \exp(\mathfrak{k})$ acts by orthogonal maps while $P = \exp(\mathfrak{p})$ acts by self-adjoint maps,

2) $(gx)^{-1} = g^*(x^{-1})$ (here $x^{-1}$ is the Jordan algebra inverse).

Now identifying $T_{e,C}$ with $V$, we use $\langle \cdot, \cdot \rangle$ to define a Riemannian metric on $C$ at $e$; since it is $K$-invariant, it globalizes to a unique $G$-invariant Riemannian metric on $C$, which we write $ds^2_C$. For later use, we need the following formula:

**Lemma.** Take $t_1, t_2 \in T_{e,C}$, and let $f(x) = \langle t_2, x^{-1} \rangle$ where $x^{-1}$ is the Jordan algebra inverse. Then

$$ds^2_C(t_1, t_2) = -D_{t_1}f$$

where $D_{t_1}$ is the derivative of functions on $V$ in the direction $t_1$.

**Proof.** Let $g \in \exp(\mathfrak{p})$ carry $e$ to $x$. Then by $G$-invariance:

$$ds^2_C(t_1, t_2) = ds^2_{C,e}(g^{-1}t_1, g^{-1}t_2) = \langle g^{-1}t_1, g^{-1}t_2 \rangle$$
and

\[ D_{t_1}(f \circ g)(e) = D_{g^{-1}t_1}(f \circ g)(e). \]

But

\[ f \circ g(x) = \langle t_2, (g x)^{-1} \rangle \]
\[ = \langle t_2, g^{-1} \cdot (x^{-1}) \rangle \]
\[ = \langle g^{-1} t_2, x^{-1} \rangle. \]

If \( \delta x \in V \) is small, then \((e + \delta x)^{-1} = e - \delta x + (\text{terms of lower order})\), hence

\[ D_{g^{-1}t_1}(f \circ g)(e) = -\langle g^{-1} t_2, g^{-1} t_1 \rangle. \quad \text{QED} \]

On a homogeneous cone \( C \), any \( 2G \)-invariant metrics are necessarily comparable, so we can deduce, from the monotone behavior of the Finsler metric \( \rho \), a weaker monotonicity for \( ds^2_C \):

**Proposition 2.2.** i) If \( C \) is a self-adjoint homogeneous cone in \( V \) and \( a \in C \), then there is a constant \( K > 0 \) such that

\[ ds^2_{C, a}(t, t) \leq K \cdot ds^2_C(t, t) \quad \text{all } t \in V, \ x \in C. \]

ii) If \( C_1 \subset C_2 \) are 2 self-adjoint homogeneous cones in \( V \), then there is a constant \( K > 0 \) such that

\[ ds^2_{C_2}(t, t) \leq K \cdot ds^2_{C_1}(t, t), \quad \text{all } t \in V, \ x \in C_1. \]

(The proofs are easy.)

The main estimates of this section deal with the following situation:

- \( C \) a self-adjoint homogeneous cone,
- \( G = \text{Aut}^o(V, C) \) (\( ^o \) means connected component),
- \( C_\bullet = \text{cone of positive definite } n \times n \text{ Hermitian matrices}, \)
- \( \rho : G \to GL(n, \mathbb{C}) \) a representation,
- \( H : C \to C_\bullet \text{ an equivariant, symmetric map with respect to } \rho. \)

Here \((\rho, H)\) **equivariant** means:

\[ H(g x) = \rho(g) H(x) \rho(g)^\dagger, \quad \text{all } x \in C, \ g \in G, \quad (1) \]

while \((\rho, H)\) **symmetric** means:

\[ \rho(g^\ast) = H(e) \cdot \rho(g)^{-1} \cdot H(e)^{-1}, \quad \text{all } g \in G. \quad (2) \]

Note that \( A \to H(e) \cdot \overline{A}^{-1} \cdot H(e)^{-1} \) is just the Cartan involution of \( GL(n, \mathbb{C}) \) with respect to the maximal compact subgroup given by the unitary group of \( H(e) \): calling this \( \mathfrak{h} \), we can rewrite the symmetry condition (2):

\[ \rho(g^\ast) = \rho(g)^\mathfrak{h}. \quad (2') \]
Condition (2) is actually independent of the choice of \( e \); if \( e' \in C \) is any other point, and \( g \mapsto g^{e'} \) is the corresponding Cartan involution then one can check that (1)+(2) imply:

\[
\rho(g^{e'}) = H(e')\rho(g)^{-1}H(e')^{-1}.
\]

(2')

We will also need for applications the slightly more general situation where \( \rho \) is fixed but \( H \) depends on some extra parameters \( t \in T \) with \( T \) compact. In this case, we ask that (1) and (2) hold for each \( H_t \). Note that we can change coordinates in \( \mathbb{C}^n \), to get a new pair:

\[
\rho'(g) = a \rho(g) a^{-1},
\]

\[
H_t'(g) = a H_t(g) a,
\]

satisfying the same identities. In this way, we can, for instance, normalize the situation so that

\[
H_u(e) = I_n
\]

hence:

\[
\rho(K) \subset U(n),
\]

\[
\rho(\exp p) = \{ \text{self-adjoint matrices, commuting} \}.
\]

\[
\{ \text{with } H_t(e), \text{ all } t \}
\]

This normalization will not affect our estimates. The first of these is:

**Proposition 2.3.** For all \( \lambda > 0 \), there is a constant \( K > 0 \) and an integer \( N \) such that

\[
\|H_t(x)\| \quad \text{and} \quad |\det H_t(x)|^{-1} \leq K \cdot \langle x, x \rangle^N, \quad \text{all } x \in (C + \lambda \cdot e).
\]

**Proof.** Let \( A \subset \exp(p) \) be a maximal \( \mathbb{R} \)-split torus. Then \( C = K \cdot A \cdot e \) and \( C + \lambda e = K \cdot (Ae + \lambda e) \). Write \( x = k(a(e)) \). Then

\[
\|H_t(x)\| = \|H_t(ae)\|
\]

\[
= \|\rho(a) \cdot H_t(e) \cdot \rho(a)\|
\]

\[
\leq \|\rho(a)\|^2 \cdot \|H_t(e)\|.
\]

We may change coordinates in \( \mathbb{C}^* \) by a unitary matrix so that all the matrices \( \rho(a) \) are diagonalized. Write then \( \rho(a) = \chi(a) \cdot \delta_{ij} \) where \( \chi_i: A \rightarrow \mathbb{R}^* \) are characters. Now we may coordinatize \( A \) by

\[
A \rightarrow A \cdot e \cong \mathbb{R}_e^r \subset V, \quad e \mapsto (1, \ldots, 1).
\]

Let \( a_1, \ldots, a_r \) be coordinates in \( \mathbb{R}_e^r \). Then

\[
\chi_i(a) = \prod_{j=1}^r a_j^{n_{ij}}, \quad s_{ij} \in \mathbb{R}.
\]
Note that if \( a \in C + \lambda e \), then \( a e \in \mathbb{R}^\ast_+ + (\lambda, \ldots, \lambda) \), i.e., \( a_i \geq \lambda \), all \( i \). Hence
\[
\|p(a)\|^2 \leq \max_{1 \leq i \leq n} (\chi_i(a))^2 \\
\leq K_1 \left( \sum_{i=1}^r |a_i|^2 \right)^{(\max s_i)r} \\
\leq K_1 \cdot K_2 \cdot (\langle ae, ae \rangle)^{(\max s_i)r}
\]
hence
\[
\|H_i(kae)\| \leq K_1 \cdot K_2 \cdot K_3 (\langle kae, kae \rangle)^{(\max s_i)r}.
\]
The same proof works for \( |\det|^{-1} \). QED

Next, let \( \xi \in V \), and let \( D_\xi \) be the derivative of \( V \) in the direction \( \xi \). We wish to estimate the matrix-valued functions
\[
(D_\xi H_i) \cdot H_i^{-1} : C \to M_d(\mathbb{C}).
\]
To do this, we prove first:

**Proposition 2.4.** For all \( 1 \leq \alpha, \beta \leq n \), let \( (D_\xi H_i) \cdot H_i^{-1} \) be the \( (\alpha, \beta) \)th entry in this matrix. There is a linear map
\[
C_{\alpha\beta, t} : V \to V
\]
depending continuously on \( t \) such that
\[
(D_\xi H_i) \cdot H_i^{-1})_{\alpha\beta}(x) = \langle C_{\alpha\beta, t}(\xi), x^{-1} \rangle.
\]
Moreover \( C_{\alpha\beta, t} \) has the property:
\[
\xi, \eta \in \mathbb{C} \\
\langle \xi, \eta \rangle = 0 \Rightarrow \langle C_{\alpha\beta, t}(\xi), \eta \rangle = 0.
\]

For some reason, I can't prove this by a direct calculation, but must resort to the following trick:

**Lemma.** Let \( C \subset V \) be a convex open cone, let \( e \in C \), and let \( f : C \to \mathbb{C} \) be a differentiable function. Suppose that for all \( W \subset V \), \( \dim W = 2 \), and \( e \in W \), \( f \) is linear on \( C \cap W \). Then \( f \) is linear.

**Proof.** The hypothesis means that
\[
f(ax + be) = af(x) + bf(e), \quad \text{all } x \in C, \ a, b \in \mathbb{R}_+.
\]
So we may extend \( f \) to all of \( V \) by the formula
\[
f(x) = f(x + ae) - af(e), \quad \text{provided } x + ae \in C.
\]
Note that
\[ f(tx) = f(tx + tae) - atf(e) = t(f(x + ae) - af(e)) = tf(x). \]
Thus for all \( x \in V, n \geq 1 \):
\[ f(x) = \frac{f(x/n) - f(0)}{n}. \]
\[ \therefore f(x) = D_x f(0) \]
and the right hand side is linear in \( x \).  QED

We prove \((D_x H_\cdot H^{-1})_{ab}(x^{-1})\) is bilinear in \( \xi, x \). Since it is linear in \( \xi \), it suffices to find a basis \( \{ \xi_k \} \) of \( V \) such that \((D_x H_\cdot H^{-1})_{ab}(x^{-1})\) is linear in \( x \). In fact, we will show that for every \( \xi \in C, (D_{\xi} H_\cdot H^{-1})_{ab}(x^{-1}) \) is linear in \( x \). To do this, by the lemma, it suffices to show that for every \( W \subset V \) with \( \dim W = 2, \xi \in W, (D_{\xi} H_\cdot H^{-1})_{ab}(x^{-1}) \) is linear on \( C \cap W \). We will do this for all \( a, b \) at once, so in verifying this we can change coordinates in \( \mathbb{C}^n \).

What we do is this: we let \( \xi \) be a new base point of \( C \) and we change coordinates so that \( H_\xi(\xi) = I_n \). This reduces us to verifying \((D_x H_\cdot H^{-1})_{ab}(x^{-1})\) is linear on \( C \cap W \) when \( \dim W = 2, e \in W \). Now any such \( W \) is part of a subspace of \( V \) of the form \( A \cdot e \), where \( A \subset \exp(p) \) is a maximal \( \mathbb{R} \)-split torus (of course, this \( p \) corresponds to the new choice of \( e \)). Moreover, as above, we can diagonalize \( \rho \) on \( A \):
\[ \rho(a, \ldots, a_r) = (\lambda_i(a) \delta_{ij}), \quad \lambda_i(a) = \prod_{j=1}^r a_j^{\gamma_i}. \]
Then
\[ H_i(ae) = \rho(a)\xi H_i(e), \quad \rho(a)H_i(e) = H_i(e)\rho(a). \]
Since \( e = (1, \ldots, 1) \), it follows:
\[ D_e(H_i(ae)) = \sum_{i=1}^r \frac{\partial}{\partial a_i} H_i(ae). \]
Using this, one calculates:
\[ (D_x H_\cdot H^{-1})_{ab}(ae) = \left( \sum_{j=1}^r \frac{2s_j}{a_j} \right) \delta_{ab} \]
and since on \( Ae, x^{-1} \) is given by \((a_1, \ldots, a_r) \mapsto (a_1^{-1}, \ldots, a_r^{-1})\), this proves that \((D_x H_\cdot H^{-1})_{ab}(x^{-1})\) is linear in \( x^{-1} \) on every \( W \subset Ae \).

Now say \( \xi, \eta \in \mathcal{C}, \langle \xi, \eta \rangle = 0 \). For a suitable choice of maximal torus \( A \), we have \( \xi, \eta \in A \cdot e \). In the coordinates \((a_1, \ldots, a_r)\) on \( A \cdot e \), let
\[ \xi = (\xi_1, \ldots, \xi_r), \]
\[ \eta = (\eta_1, \ldots, \eta_r). \]
Since \(\langle \cdot, \cdot \rangle\) on \(Ae\) makes \(A\) self-adjoint, it is a quadratic form of the type \(\langle ae, be \rangle = \sum \lambda_i a_i b_i\), so \(\langle \xi, \eta \rangle = 0\) means that for every \(i, \xi_i = 0\) or \(\eta_i = 0\). A calculation like that just made shows:

\[
(D_\xi H_i \cdot H_t^{-1})_{a\beta}(ae) = \left( \sum_{\gamma=1}^{r} 2s_{\xi_j} \xi_j \right) \cdot \delta_{\alpha \beta}
\]
i.e.,

\[
(D_\xi H_i \cdot H_t^{-1})(x^{-1})(a\cdot e) = \left( \sum_{\gamma=1}^{r} 2s_{\xi_j} \xi_j \cdot \delta_{\alpha \beta} \right).
\]

This is clearly zero if \(x = \eta\). QED

For the next result, suppose \(\delta\) is any vector field in the manifold of values of \(t\). Then

**Proposition 2.5.** For all vector fields \(\delta\) on \(T\),

\[
(\delta H_i \cdot H_t^{-1})(x)
\]
is independent of \(x\), i.e., depends on \(t\) alone.

**Proof.** By equivariance:

\[
\delta H_i(g \cdot x) = \rho(g) \cdot \delta H_i(x) \cdot \rho(g)^{-1}
\]
hence

\[
(\delta H_i \cdot H_t^{-1})(g \cdot e) = \rho(g) \cdot (\delta H_i \cdot H_t^{-1})(e) \cdot \rho(g)^{-1}.
\]

By symmetry:

\[
\rho(g^*) \cdot \delta H_i(e) = \delta H_i(e) \cdot \rho(g)^{-1}
\]
hence

\[
\rho(g^*) \cdot (\delta H_i \cdot H_t^{-1})(e) \cdot \rho(g^*)^{-1} = (\delta H_i \cdot H_t^{-1})(e).
\]

Together, these imply the Proposition. QED

Proposition 2.4 gives us estimates on \(\|D_\xi H_i \cdot H_t^{-1}\|\). To work these out, we fix a maximal flag of boundary components of \(C\). In the notation of [1], p.109, choosing this flag and the base point \(e \in C\) is equivalent to choosing in the Jordan algebra \(V\) a maximal set of orthogonal idempotents:

\[
e = \epsilon_1 + \cdots + \epsilon_r.
\]

Let

\[
C_i = \text{boundary component containing } \epsilon_{i+1} + \cdots + \epsilon_r.
\]

Then

\[
\tilde{C}_1 \supseteq \tilde{C}_2 \supseteq \cdots \supseteq \tilde{C}_r.
\]
is the flag. Also let
\[ \tilde{C} = C \cup C_1 \cup C_2 \cup \cdots \cup C_r \cup (0) \]
and let
\[ A = \sum_{i=1}^{r} \mathbb{R} \cdot e_i. \]

Let \( P \) be the parabolic group which stabilizes the flag \( \{ \tilde{C}_i \} \). Our estimates are based on:

**Proposition 2.6.** (1) Let \( \xi_1 \in \tilde{C} \) and let \( \xi'_1 \in V \) satisfy
\[ \left\{ \begin{array}{l}
\langle \xi_1, \eta \rangle = 0 \\
\eta \in \tilde{C}
\end{array} \right\} \Rightarrow \langle \xi'_1, \eta \rangle = 0. \]

Then for every compact set \( \omega \subset P \), there is a \( K > 0 \) such that:
\[ |\langle \xi'_1, x^{-1} \rangle| \leq K \sqrt{|d\tilde{s}_{\xi_1}(\xi'_1, \xi_1)|}, \quad \text{all } x \in \omega \cdot A \cdot e. \]

(2) Let \( \xi_1 \in \tilde{C}, \xi'_1 \in V \) be as above. Let \( \xi_2 \in \tilde{C} \). Then for every compact set \( \omega \subset P \), there is a \( K > 0 \) such that:
\[ |d\tilde{s}_{\xi_2}(\xi_1, \xi_2)| \leq K \sqrt{|d\tilde{s}_{\xi_1}(\xi_1, \xi_1)| \cdot |d\tilde{s}_{\xi_2}(\xi_2, \xi_2)|}. \]

**Proof.** We will use the Peirce decomposition of \( V \) defined by the idempotents \( e_i \):
\[ V = \bigoplus_{i \in J} V_{ij}, \]
where
\[ x \in V_{ij} \Rightarrow x = \sum a_i e_i \cdot x = \frac{a_i + a_j}{2} x. \]

This decomposition is orthogonal with respect to \( \langle \cdot, \cdot \rangle \) and \( C_k \) is an open cone in the subspace \( \bigoplus_{k \notin i \in J} V_{ij} \). If \( \xi_1 \in C_k \), then note that
\[ \xi_1 \perp \bigoplus_{i \notin j \in k} V_{ij} \]
and
\[ \bigoplus_{i \notin j \in k} V_{ij} \Rightarrow (\text{the boundary component of } C \text{ corresponding to } e_1 + \cdots + e_k). \]

This boundary component is open in \( \bigoplus_{i \notin j \in k} V_{ij} \). Thus
\[ \xi_1 \perp \bigoplus_{i \notin j \in k} V_{ij} \]
or
\[ \xi_1 \in \bigoplus_{i \notin j \in k} V_{ij}. \]
To prove (1), let \( x = gae, \ g \in \omega, \ ae = \sum a_i e_i \). Then
\[
\langle \xi', x^{-1} \rangle = \langle g^{-1} \xi', (ae)^{-1} \rangle = \sum_{i=1}^{r} \frac{1}{a_i} \langle (g^{-1} \xi')_{hi}, e_i \rangle
\]
where \((g^{-1} \xi')_{hi}\) is the component of \((g^{-1} \xi')\) in \(V_{ii}\). But \(\omega\) preserves the flag, so
\[
g^{-1} \xi' \in \bigoplus_{k<j} V_{ij}
\]
too. Thus
\[
\langle \xi', x^{-1} \rangle = \sum_{i=k+1}^{r} \frac{1}{a_i} \langle (g^{-1} \xi')_{hi}, e_i \rangle.
\]
As \(g\) varies in \(\omega\), \(\langle (g^{-1} \xi')_{hi}, e_i \rangle\) is bounded, hence
\[
|\langle \xi', x^{-1} \rangle| \leq K_1 \sum_{i=k+1}^{r} \frac{1}{a_i}.
\]
Let \(e^{(k)} = e_{k+1} + \cdots + e_r\) be the base point in \(C_k\) and note that, again by compactness of \(\omega\), there is a \(K_2 > 0\) such that
\[
g^{-1} \xi' \in C_k + K_2 e^{(k)}, \ \text{all} \ g \in \omega,
\]
hence
\[
ds^2_{C,ae}(K_2 e^{(k)}, K_2 e^{(k)}) \leq ds^2_{C,ae}(g^{-1} \xi', g^{-1} \xi'), \ \text{all} \ x \in C.
\]
Thus
\[
ds^2_{C,ae}(\xi', \xi_1) = ds^2_{C,ae}(g^{-1} \xi', g^{-1} \xi_1)
\geq K_2 ds^2_{C,ae}(g^{-1} \xi', g^{-1} \xi_1)
\geq K_2 ds^2_{C,ae}(a^{-1} e^{(k)}, a^{-1} e^{(k)})
\geq K_2 \sum_{i=k+1}^{r} \frac{1}{a_i} \langle e_i, e_i \rangle
\geq K_2 \left( \sum_{i=k+1}^{r} \frac{1}{a_i} \right)^2.
\]
The same procedure proves (2). If \(\xi_1, \xi_2 \in C_a, \ \xi_2 \in C_t\), the argument goes like this:
\[
ds^2_{C,ae}(\xi', \xi_2) = \langle a^{-1} (g^{-1} \xi')_h, a^{-1} (g^{-1} \xi_2)_h \rangle
\geq \sum_{i=k+1}^{r} \frac{1}{a_i} \langle (g^{-1} \xi')_{hi}, (g^{-1} \xi_2)_{hi} \rangle
\leq K_1 \sqrt{\sum_{i=k+1}^{r} \frac{1}{a_i^2}} \sqrt{\sum_{j=k+1}^{r} \frac{1}{a_j^2}}
\leq K_1 K_2 \sqrt{ds^2_{ae}(\xi_1, \xi_1) \cdot ds^2_{ae}(\xi_2, \xi_2)}.
\]
We can put everything we have said together as follows: Suppose \(N = \dim V\) and \(\xi_1, \ldots, \xi_N \in \mathcal{C}\) span \(V\). Define a simplicial cone \(\sigma \subset C\) by

\[
\sigma = \sum_{i=1}^{n} \mathbb{R}_+ \cdot \xi_i.
\]

Let \(l_i: V \to \mathbb{R}\) be a dual basis: \(l_i(\xi_j) = \delta_{ij}\). Then for all \(\rho, H\), as above, we have the estimates:

**Proposition 2.7.** For all vector fields \(\delta, \delta'\) to the \(T\)-space, and \(a \in \bar{C}\) there is a constant \(K > 0\) such that for all \(x \in \text{Int}(\sigma + a)\):

\[
\|D_{\xi_j} H \cdot H^{-1}(x)\| \leq \frac{K}{l_i(x) - l_i(a)},
\]

\[
\|\delta H \cdot H^{-1}(x)\| \leq K,
\]

\[
\|D_{\xi_j}(D_{\xi_j} H \cdot H^{-1})(x)\| \leq \frac{K}{(l_i(x) - l_i(a))(l_j(x) - l_j(a))},
\]

\[
\|\delta(D_{\xi_j} H \cdot H^{-1})(x)\| = 0,
\]

\[
\|\delta(D_{\xi_j} H \cdot H^{-1})(x)\| \leq \frac{K}{l_i(x) - l_i(a)},
\]

\[
\|\delta'(H \cdot H^{-1})(x)\| \leq K.
\]

**Proof.** Combine (2.4), (2.5) and (2.6) and the formula \(ds_t^2(t_1, t_2) = -D_{t_2}(\langle t_2, x^{-1} \rangle)\) to get estimates in terms of \(ds_t^2(\xi_i, \xi_j)\) on sets \(\omega \cdot A \cdot e\). Then apply (2.2)(i) and (ii) to the inclusion \(\text{Int}(\sigma + a) \to C\), plus Ash’s theorem that \(\text{Int}(\sigma + a) \subset \omega \cdot A \cdot e\) if \(\omega\) is large enough ([1], Ch. II, §4), plus the formula

\[
ds_t^2 = \sum_{i=1}^{n} \frac{d l_i^2}{l_i^2}
\]

for the canonical metric on the homogeneous convex cone \(\sigma\). QED

There is one final estimate that we need. For this, we first make a definition:

**Definition.** A linear map \(T: C^n \to C^n\) is called \(\rho\)-upper triangular if the following holds: For all maximal \(\mathbb{R}\)-split tori \(A \subset G\), let \(X(A) = \text{Hom}(A, \mathbb{G}_m)\) be the character group of \(A\). As is well known, there is a basis \(\gamma_1, \ldots, \gamma_r\) of \(X(A)\) such that the weights of \(A\) acting on \(V\) are contained in \(\gamma_i + \gamma_j, i \leq j\) (and contain \(2\gamma_i, 1 \leq i \leq r\)). Partially order the characters by defining

\[
\sum n_i \gamma_i \geq \sum m_i \gamma_i \quad \text{if} \quad n_i \geq m_i, \quad \text{all } i.
\]

Diagonalize \(\rho(A)\):

\[
C^n = \bigoplus_{\lambda \in X(A)} V_\lambda.
\]

Then \(T\) is \(\rho\)-upper triangular if for all \(A\), all \(\lambda_0 \in X(A)\),

\[
T(\bigoplus_{\lambda \neq \lambda_0} V_\lambda) \subseteq \bigoplus_{\lambda \neq \lambda_0} V_\lambda.
\]
The estimate we need is:

**Proposition 2.8.** If \( T \) is \( \rho \)-upper triangular, then for all \( a \in C(F) \), there is a constant \( K > 0 \) such that

\[
\| H_i(x) \cdot T \cdot H_i(x)^{-1} \| \leq K, \quad \text{all } x \in C(F) + a.
\]

**Proof.** Take \( a \) as a base point of \( C(F) \) and pick any maximal torus \( A \) so that \( A e = \mathbb{R}^n_+ \), \( a = (1, \ldots, 1) \), \( C(F) = K \cdot a \cdot e \). Therefore

\[
C(F) + a = \{ k a e : k \in K, a = (a_1, \ldots, a_n), a_i \geq 1, \text{ all } i \}.
\]

Change coordinates in \( \mathbb{C}^n \) so that

\[
\rho(a) = \begin{pmatrix}
\lambda_1(a) & 0 \\
\vdots & \\
0 & \lambda_n(a)
\end{pmatrix}.
\]

Now

\[
H_i(x) \cdot T \cdot H_i(x)^{-1} = \rho(k) H_i(e) \rho(a) \rho(k) \cdot T \cdot \rho(k)^{-1} \rho(a)^{-1} \rho(k)^{-1}
\]

so it suffices to bound \( \| \rho(a)^{2 \cdot \lambda_i} (\rho(k)^{-1} \cdot T \cdot \rho(k)) \cdot \rho(a)^{-2} \| \). This means we wish to bound:

\[
|\lambda_i(a)^2 \lambda_j(a)^{-2} (\rho(k)^{-1} \cdot T \cdot \rho(k))|_{ij}
\]

when \( k \) ranges over \( K \), and \( a = (a_1, \ldots, a_n) \) satisfies \( a_i \geq 1 \), all \( i \). This is equivalent to:

\[
(\rho(k)^{-1} \cdot T \cdot \rho(k))_{ij} \neq 0 \Rightarrow \lambda_j \geq \lambda_i
\]

(weights of \( A \) being partially ordered as in the definition). But for all \( i \) define

\[
W_i = \rho(k) \left( \bigoplus_{\lambda_j \geq \lambda_i} \mathbb{C} \cdot e_j \right)
\]

(where \( e_j \in \mathbb{C}^n \) is the \( j \)th unit vector). Note that \( k A k^{-1} \) maps \( W_i \) into itself and that \( W_i \) is one of the sums of weight spaces referred to in the definition. Therefore \( T(W_i) \subseteq W_i \), hence

\[
(\rho(k)^{-1} T (\rho(k)) e_j \in \bigoplus_{\lambda_j \geq \lambda_i} \mathbb{C} \cdot e_j
\]

which is precisely \((*)\). QED

**§ 3. The Proportionality Principle**

Let \( D \) be an \( r \)-dimensional bounded symmetric domain and let \( \Gamma \) be a neat\(^2\) arithmetic group acting on \( D \). Then \( X = D/\Gamma \) is a smooth quasi-projective

\(^2\) Recall that following a definition of Borel, a "neat" arithmetic subgroup \( \Gamma \) of an algebraic group \( \mathfrak{g} \subset GL(n, \mathbb{C}) \) is one such that for every \( x \in \Gamma, x \neq e \), the group generated by the eigenvalues of \( x \) is torsion-free. Every arithmetic group \( \Gamma \) has neat arithmetic subgroups of finite index.
variety, called a *locally symmetric variety*, or an *arithmetic variety*. In [1], Ash, Rapoport, Tai and I have introduced a family of smooth compactifications $\overline{X}$ of $X$ such that $\overline{X} - X$ has normal crossings. We must first recall how $\overline{X}$ is described locally. At the same time, we will need various details from the whole cumbersome apparatus used to manipulate $D$ so we will rapidly sketch these too. All results stated without proof can be found in [1].

By definition, $D \cong K \backslash G$, where $G$ is a semi-simple adjoint group and $K$ is a maximal compact subgroup. Inside the complexification $G_{\mathbb{C}}$ of $G$, there is a parabolic subgroup of the form $P_{\mathbb{C}} \cdot K_{\mathbb{C}}$ ($P_{\mathbb{C}}$ its unipotent radical which is, in fact, abelian and $K_{\mathbb{C}}$ the complexification of $K$) such that $K = G \cap (P_{\mathbb{C}} \cdot K_{\mathbb{C}})$ and $G \cdot (P_{\mathbb{C}} \cdot K_{\mathbb{C}})$ open in $G_{\mathbb{C}}$. This induces an open $G$-equivariant immersion

$$D \hookrightarrow \overline{D}$$

$$K \backslash G \quad P_{\mathbb{C}} \cdot K_{\mathbb{C}} \backslash G_{\mathbb{C}}.$$ 

Here $\overline{D}$ is a rational projective variety known as a flag space and $G_{\mathbb{C}}$ is an algebraic group acting algebraically on $\overline{D}$. Let $D$ be the closure of $D$ in $\overline{D}$. The maximal analytic submanifolds $F \subset D - D$ are called the boundary components of $D$. For each $F$, we set

- $N(F) = \{g \in G \mid g F = F\}$,
- $W(F) =$ unipotent radical of $N(F)$,
- $U(F) =$ center of $W(F)$, a real vector space of dimension $k$, say,
- $V(F) = W(F) / U(F)$: known to be abelian, centralizing $U(F)$. Via "exp", we get a section and write $W(F)$ set-theoretically as $V(F) \cdot U(F)$. Also $\dim V$ is even — let it be $2l$.

Next splitting $N(F)$ into a semi-direct product of a reductive part and its unipotent radical, we decompose $N(F)$ further:

$$N(F) = (G_{\mathbb{A}}(F) \cdot G_{\mathbb{I}}(F) \cdot M(F)) \cdot V(F) \cdot U(F),$$

where

- a) $G_{\mathbb{I}} \cdot M \cdot V \cdot U$ acts trivially on $F$, $G_{\mathbb{A}}$ mod a finite center being $\text{Aut}^0(F)$,
- b) $G_{\mathbb{I}} \cdot M \cdot V \cdot U$ commutes with $U(F)$, $G_{\mathbb{I}}$ mod a finite central group acts faithfully on $U(F)$ by inner automorphisms
- c) $M$ is compact.

Here $F$ is said to be rational if $\Gamma \cap N(F)$ is an arithmetic subgroup of $N(F)$. Mod $\Gamma$ there are only finitely many such $F$, and if $F_1, \ldots, F_k$ are representatives:

$$X \cup \bigcup_{i=1}^{k} \left( F_i / \Gamma \cap N(F_i) \right),$$

with suitable analytic structure is Satake-Baily-Borel's compactification of $X$. 
Next, for each \( F \), we define an open subset \( D_F \subset \overline{D} \) by
\[
D_F = \bigcup_{g \in U(F)_{\mathbb{C}}} g \cdot D.
\]
The embedding of \( D \) in \( D_F \) is Pjatetski-Shapiro’s realization of \( D \) as “Siegel Domain of 3rd kind”. In fact, there is an isomorphism:
\[
D_F \cong U(F)_{\mathbb{R}} \times \mathbb{C}^l \times F
\]
such that not only \( N(F) \) but even the bigger group \( G \cdot (M \cdot G)_{\mathbb{C}} \cdot V \cdot U_{\mathbb{C}} \) acts by “semi-linear transformations”:
\[
(x, y, t) \mapsto (Ax + a(y, t), B_t y + b(t), g(t))
\]
\((A, B_t) \text{ matrices}, a, b \text{ vectors}) \text{ and} \)
\[
D = \{(x, y, t) \mid \text{Im } x + l(y, y) \in C(F)\}
\]
where \( C(F) \subset U(F) \) is a self-adjoint convex cone homogeneous under the \( G_t \)-action on \( U(F) \) and \( l : \mathbb{C}^l \times \mathbb{C}^l \to U(F) \) is a symmetric \( \mathbb{R} \)-bilinear form.

Moreover
\[
U(F) \cong \text{group of automorphisms of } D : (x, y, t) \mapsto (x + a, y, t), a \in U(F),
\]
\[
U(F)_{\mathbb{R}} \cong \text{group of automorphisms of } D(F) : (x, y, t) \mapsto (x + a, y, t), a \in U(F)_{\mathbb{R}},
\]
\[
W(F) \cong \text{group of automorphisms of } D : (x, y, t) \mapsto (x + a(u, t), y + b(t), t) \text{ and}
\]
the group \( V(F) \) acts, for each \( t \), simply transitively on the space \( \mathbb{C}^l \) of possible \( y \)-values.

There is a technical lemma which we will need about this action:

**Lemma.** Let \( t_0 \in F, \ e_0 \in C(F) \), and let \( u_0 \in U(F)_{\mathbb{R}} \) be the map \((x, y, t) \mapsto (x + i e_0, y, t)\). Let \( e = (i e_0, 0, t_0) \) be a base point of \( D \), so that \( \text{Stab}_G(e) = K_i \), a maximal compact of \( G \) and \( \text{Stab}_{G_{\mathbb{C}}}(e) = K_{\mathbb{R}} \cdot P \). Moreover, \( \text{Stab}_{G_i}(e_0) = K_i \) is a maximal compact in \( G_i \). Since \( G_i \subset \text{Stab}(0, 0, t_0) \), \( u_0(G_i) u_0^{-1} \subset \text{Stab}(e) \) and we may look at
\[
\alpha : G_i \xrightarrow{\text{conj. by } u_0} \text{Stab}_{G_{\mathbb{C}}}(e) \xrightarrow{\text{mod } P} K_{\mathbb{C}}.
\]

If \( \ast \) is the Cartan involution of \( G_i \) with respect to \( K_i \), then:
\[
\alpha(g^\ast) = \overline{\alpha(g)}.
\]

**Proof.** This is a straightforward calculation, for instance, using the fundamental decomposition of \( g = \text{Lie } G \) via \( \text{sl}(2)_{\mathbb{C}} \subset g \) (cf. [1], p. 182) and the description of \( \text{Lie } (M \cdot G) \) in this decomposition for the standard boundary components \( F_\xi \) given in [1], p. 226.

We now describe local coordinates on \( \overline{X} \). Recall that \( \overline{X} \) is not unique but depends on the choice of certain auxiliary simplicial decompositions. We need not recall these in detail. The chief thing is that each \( \overline{X} \) is covered by a finite set of coordinate charts constructed as follows:

1) take a rational boundary component \( F \) of \( D \),
2) take \( \xi_1, \ldots, \xi_k \) a basis of \( \Gamma \cap U(F) \) such that \( \xi_i \in \overline{C(F)} \subset U(F) \) and in fact, \( \xi_i \in C(F) \cup C_1 \cup C_2 \cup \cdots \cup C_i = \overline{C} \), where \( \overline{C(F)} \supset \overline{C_1} \supset \overline{C_2} \supset \cdots \supset \overline{C}_i \) is a flag of
boundary components (cf. §2) and at least one $\xi_i$ is in $C(F)$: say

$$\xi_1, \ldots, \xi_m \in C(F), \quad \xi_{m+1}, \ldots, \xi_k \in \overline{C(F)} - C(F),$$

3) let $l_i: U(F)_{\xi_i} \to \mathbb{C}$ be dual to $\{\xi_i\}$, i.e., $l_i(\xi_j) = \delta_{ij}$,

4) consider the exponential:

$$\begin{array}{rcl}
D & \subset & (U(F)_{\xi} \times \mathbb{C}^l \times F) \\
\downarrow & & \downarrow \\
D/\Gamma \cap U(F) & \subset & (\mathbb{C}^{*k} \times \mathbb{C}^l \times F) \\
\downarrow & & \downarrow \\
\tilde{X} & & \\
\end{array}$$

5) Define $(D/\Gamma \cap U(F))^{-}$ to be the set of $P \in \mathbb{C}^k \times \mathbb{C}^l \times F$ which have a neighborhood $U$ such that

$$U \cap (\mathbb{C}^{*k} \times \mathbb{C}^l \times F) \subset (D/\Gamma \cap U(F)).$$

Note that

$$\begin{array}{c}
(D/\Gamma \cap U(F))^{-} \supset \bigcup_{i=1}^{m} \{(z, y, t) \mid z = (z_1, \ldots, z_k), z_i = 0\} = S(F, \{\xi_i\}).
\end{array}$$

6) The basic property of $\tilde{X}$ is that for suitable $F$, $\{\xi_i\}$, the covering map $p$ extends to a local homeomorphism

$$\tilde{p}: (D/\Gamma \cap U(F))^{-} \to \tilde{X}$$

and that every point of $\tilde{X}$ is equal to $\tilde{p}((z, y, t))$, where $z_i = 0$, some $1 \leq i \leq m$, for some such $F$, $\{\xi_i\}$.

We now come to the main results of this paper. Let $E_0$ be a $G$-equivariant analytic vector bundle of rank $n$ on $D$. $E_0$ is defined by the representation

$$\sigma: K \to GL(n, \mathbb{C})$$

of the stabilizer $K$ of the base point $e_+ \in D$ in the fibre $\mathbb{C}^n$ of $E_0$ over $e_+$. We complexify $\sigma$ and extend it to $P_+$ $K_\mathbb{C}$ by letting it kill $P_+$. Then $\sigma$ defines a $G_\mathbb{C}$-equivariant analytic vector bundle $\tilde{E}_0$ on $\tilde{D}$ also. In the other direction, we can divide $E_0$ by $\Gamma$ obtaining a vector bundle $E$ on $X$. Since $K$ is compact, $E_0$ carries a $G$-invariant Hermitian metric $h_0$, which induces a Hermitian metric $h$ on $E$. We claim:

**Main Theorem 3.1.** $E$ admits a unique extension $\tilde{E}$ to $\tilde{X}$ such that $h$ is a singular Hermitian metric good on $\tilde{X}$.
These various bundles are all linked as in this diagram:

\[ \begin{array}{ccc}
\tilde{E}_0 & \overset{\text{restriction}}{\longrightarrow} & E_0 \\
\downarrow & & \downarrow \\
D & \longrightarrow & X
\end{array} \quad E \quad \overset{\text{extension}}{\longrightarrow} \quad \tilde{E} \quad \overset{\text{extension}}{\longrightarrow} \quad \hat{X}.\]

**Proof.** We saw in §1 that \( \tilde{E} \), if it existed, has as its sections the sections of \( E \) with growth \( O \left( \frac{1}{\prod_{i=1}^m \log |z_i|} \right)^{2N} \) along \( \tilde{X} - X \). To see that the set of these sections defines an analytic vector bundle on \( \tilde{X} \), it suffices to check this locally, e.g., on \( (D/\Gamma \cap U(F))^{-} \). But now the bundle \( \tilde{E}_0 \) restricts to a bundle \( E_r \) on \( D_r \) with \( N(F) \cdot U(F)_k \) acting equivariantly. Now note that the subgroup \( U(F)_k \) acts simply transitively and holomorphically on the first factor \( U(F) \) of \( D_r \) in its Siegel Domain presentation. Since \( \mathbb{C}^l \times F \) is contractible and Stein (\( F \) is another bounded symmetric domain), it follows that \( E_r \) has a set of \( n \) holomorphic sections \( e_1, \ldots, e_n \) such that

i) \( e_i \) is \( U(F)_k \)-invariant,

ii) \( e_1(x), \ldots, e_n(x) \) are a basis of \( E_r(x) \), all \( x \in D_r \).

Dividing by \( \Gamma \cap U(F) \), \( E_r \) descends to a vector bundle \( E_r \) on \( \mathbb{C}^k \times \mathbb{H}^l \), which is also globally trivial via the same basic sections \( e_1, \ldots, e_n \). We can then extend \( E_r \) to \( \mathbb{C}^k \times \mathbb{H}^l \) so as to be trivial with these basic sections. We must show that along \( S(F, \{ \xi_i \}) \) the sheaf of sections of this extension is exactly the sheaf of sections of \( E_r \) on \( (D/\Gamma \cap U(F))^{-} \) with growth \( O \left( \frac{1}{\prod_{i=1}^k \log |z_i|} \right)^{2N} \) on the coordinate hyperplanes \( \bigwedge_{i=1}^k (z_i = 0) \). Equivalently, this means that \( h(e_i, e_j) \) and \( \det(h(e_i, e_j))^{-1} \) have this growth. To do this, it is convenient to use a 2nd basis of \( E_r \), which is \( C^\infty \) but not analytic. Note that \( V(F) \cdot U(F)_k \) acts simply transitively on the 1st 2 factors of \( D_r \). So we can find \( e_1', \ldots, e_n' \in \Gamma(D_r, E_r) \) such that

i') \( e_i' \) is \( V(F) \cdot U(F)_k \)-invariant,

ii') \( e_1'(x), \ldots, e_n'(x) \) are a basis of \( E_r(x) \), all \( x \in D_r \),

iii') On \( (0, 0) \times F \), \( e_i' = e_i \), hence are holomorphic sections.

\( \{e_i\} \) and \( \{e_i'\} \) are related by an invertible \( U(F)_k \)-invariant matrix \( S \); so that \( |S_{ij}| \) and \( |\det S|^{-1} \) are uniformly bounded on subsets 

\[ \left( \mathbb{C}^k \times \text{compact subset of } \mathbb{C}^k \times F \right) \Rightarrow \left( \mathbb{C}^k \times \mathbb{C}^l \times F \right). \]

Therefore it is enough to check that \( h(e_i', e_j), (\det h(e_i', e_j))^{-1} \) have the required growth.

Now if \( g \in G_i(F) \), note that because \( g \) normalizes \( V \cdot U_k \), \( g e_i' \) is another \( V \cdot U_k \) invariant section of \( E_r \). Therefore

\[ g \cdot (e_i') = \sum_{j=1}^n r_{ij}(t) \cdot e_j' \quad \text{(here } t \text{ is the coordinate on } F). \]
Since \( g(U(F)_{\mathcal{E}} \times \mathbb{C}^t \times \{t\}) = U(F)_{\mathcal{E}} \times \mathbb{C}^t \times \{t\} \), it follows that for each \( t \),
\[
g \mapsto \rho_t(g) = (r_{ij}(t))
\]
is an \( n \)-dimensional representation of \( G_t(F) \). In fact, as \( g(0, 0, t) = (0, 0, t) \), this is just the representation of the stabilizer of \( (0, 0, t) \) restricted to \( G_t(F) \). This shows that \( \rho_t \) is a holomorphic family of algebraic representations of \( G_t \) (this is not trivial because \( G_t \) has a positive dimensional center). Since \( G_t \) is reductive, we may change our basis \( \{e_i'\} \) so that \( \rho_t \) is in fact independent of \( t \).

Now consider the functions \( h_{ij} = h(e_i', e_j') \) on \( D \). Since \( h, e_i' \) and \( e_j' \) are \( U \cdot V \)-invariant, so is \( h_{ij} \), hence in Siegel domain notation, it is a function of \( u = \text{Im} x + \tilde{h}(y, y) \) and \( t \), i.e., is a function on \( C(F) \times F \). For each fixed \( t \in F \), and variable \( u \in C(F) \),
\[
H_t(u) = (h_{ij}(u, t))
\]
is a map
\[
C(F) \to C_n = \left\{ \text{cone of pos. def. } n \times n \right\} .
\]
I claim that \( (\rho, H_t) \) satisfy the hypothesis of §2. In fact,
\[
H_t(gu)_{ij} = h(g e_i', g e_j') = \sum_{k, l} r_{kl}(g) h(e_k', e_l') \bar{r}_{ij}(g)
\]
\[
= (\rho(g) \cdot H_t(u)) \cdot \overline{\rho(g)}_{ij}.
\]
Let \( e = (ie_0, 0, t_0) \) be a base point of \( D \). Since \( h \) is a \( K \)-invariant metric on \( E_0 \),
\[
h(k e_i', k e_j')(e) = h(e_i', e_j')(e), \quad \text{all } k \in K.
\]
Complexifying, we get too:
\[
h(k e_i', \bar{k} e_j')(e) = h(e_i', e_j')(e), \quad \text{all } k \in K_{\mathcal{E}}.
\]
Let \( u_0 \in U_{\mathcal{E}} \) be given by \( (x, y, z) \mapsto (x + i e_0, y, z) \). Then for all \( g \in G_t(F), u_0 g u_0^{-1}(e) = e \), so \( u_0 g u_0^{-1} = k \cdot p \), where \( k \in K_{\mathcal{E}}, p \in P_+ \). By the lemma above, \( u_0 g^* u_0^{-1} = \bar{k} \cdot p' \) for some other element \( p' \in P_+ \). Therefore:
\[
h(e_i', e_j')(e) = h(p e_i', p' e_j')(e) \quad \text{(since } P_+ \text{ acts trivially on } E_0(e))
\]
\[
= h(k p e_i', \bar{k} p' e_j')(e)
\]
\[
= h(u_0 g u_0^{-1}(e_i'), u_0 g^* u_0^{-1}(e_j'))(e)
\]
\[
= \sum r_{kl}(g) h(e_k', e_l')(e) \bar{r}_{ij}(g^*) \quad \text{(since } u_0 e_i' = e_i', \text{ all } i)\]
or
\[
H_{t_0}(e_0) = \rho(g) H_{t_0}(e_0) \cdot \overline{\rho(g^*)}.
\]
In fact, the same holds if \( t_0 \) is replaced by any \( t \in F \) as follows easily using the \( G_h(F) \)-invariance of \( h \) and the fact that \( G_t(F) \) and \( U(F)_{\mathcal{E}} \) commute with \( G_h(F) \).
Thus we have the full situation of §2. In particular, we have available all the bounds of §2.

As above, to describe local coordinates near boundary points of $\bar{X}$, choose a simplicial cone:

$$\sigma = \bigoplus_{i=1}^{k} \mathbb{R}^+ \cdot \xi_i \subset C(F); \quad \xi_i \in C(F) \iff 1 \leq i \leq m$$

and let $l_i$ be dual linear functionals on $U(F)_{\mathbb{R}}$. Then if $(x, y, t)$ are Siegel domain coordinates on $D(F)$, and $z_i = e^{2\pi i l_i(x)}$, then $(z, y, t)$ at points where at least one $z_i$ is $0 (1 \leq i \leq m)$ are local coordinates on $\bar{X}$. Moreover each point $P \in S(F, \{\xi_i\}) \subset \bar{X}$ has an open neighborhood whose intersection with $X$ is contained in the image of

$$\{(x, y, t) \mid u = \text{Im } x + l_i(y, y) \in \sigma + a, \ y \in Y', \ t \in F'\}$$

for some $\sigma$ as above, $a \in C(F)$, $Y'$ (resp. $F'$) a relatively compact subset of $Y$ (resp. $F$). Note that $\log |z_i| = -2\pi l_i(\text{Im } x)$. At all points $P \in S(F, \{\xi_i\})$, we have to estimate $h(s', s')^{\pm 1}$ in terms of

$$\left( \sum \log \left| \frac{z_i}{C} \right| \right)^{2N}$$

(choose $C$ large enough so that $\left| \frac{z_i}{C} \right| < 1$ in a neighborhood of $P$). This is the same as estimating $h(s', s')^{\pm 1}$ in terms of

$$(\sum l_i(u) + C)^{2N}$$

(choose $C$ large enough so that $l_i(u) + C > 0$ in a neighborhood of $P$). But if $l_i(u) + C > 0$, $(\sum l_i(u) + C)^{2N}$ is comparable with $(\sum l_i(u)^2)^N$, hence with $\langle u, u \rangle^N$. This is exactly the estimate that Proposition 2.3 gives us. Next we have to estimate the connection and the curvature. Now in terms of a holomorphic trivialization of $E_0$, the connection is given by $\partial h \cdot h^{-1}$. What we have is good control of $\delta_1 h \cdot h^{-1}$ and $\delta_2 (\delta_1 h \cdot h^{-1})$ for all vector fields $\delta_1$, $\delta_2$ in terms of the real analytic trivialization given by $\{e_i\}$. Write

$$e_i = \sum_{j=1}^{n} a_{ij} e'_j,$$

$$A = \text{matrix } (a_{ij}),$$

$$H^{an} = \text{matrix } h(e_i, e_j).$$

Then

$$H^{an} = A \cdot H \cdot A^{-1}.$$ 

From this you calculate:

$$\left( \text{Connexion in trivialization} \right)_{\{e_i\}} = \partial H^{an} \cdot (H^{an})^{-1}$$

$$= \partial A \cdot A^{-1} + A \cdot \partial H \cdot H^{-1} \cdot A^{-1} + A \cdot H \cdot (\partial \bar{A} \cdot A^{-1}) H^{-1} A^{-1}.$$
\[ d\left(\text{connexion in}\ \\{e_i\}\right) = d(\partial H^{\alpha} \cdot (H^{\alpha})^{-1}) \]
\[ = d(\partial A \cdot A^{-1}) + d(A \cdot \partial H \cdot H^{-1} \cdot A^{-1}) \]
\[ + d(A \cdot H \cdot (\partial \vec{A} \cdot \vec{A}^{-1}) \cdot H^{-1} \cdot A^{-1}) \]
\[ = d(\partial A \cdot A^{-1}) + dA \cdot (\partial H \cdot H^{-1}) \cdot A^{-1} \]
\[ + A \cdot d(\partial H \cdot H^{-1}) \cdot A^{-1} + A \cdot \partial H \cdot H^{-1} \cdot A^{-1} \cdot dA \cdot A^{-1} \]
\[ + A \cdot [H \cdot (\partial \vec{A} \cdot \vec{A}^{-1}) \cdot H^{-1}] \cdot A^{-1} \]
\[ + A \cdot [H \cdot d(\partial \vec{A} \cdot \vec{A}^{-1}) \cdot H^{-1}] \cdot A^{-1} \]
\[ + A \cdot [H \cdot (\partial \vec{A} \cdot \vec{A}^{-1}) \cdot H^{-1}] \cdot (dH \cdot H^{-1}) \cdot A^{-1} \]
\[ + A \cdot [H \cdot (\partial \vec{A} \cdot \vec{A}^{-1}) \cdot H^{-1}] \cdot A^{-1} \cdot dA \cdot A^{-1}. \]

Therefore, since \( A \) is a \( C^\infty \) metric on \( \vec{X} \), to show that the connexion and its differential have Poincaré growth on \( \vec{X} \) it suffices to prove that the 4 forms:

\[ \partial H \cdot H^{-1}, \]
\[ d(\partial H \cdot H^{-1}), \]
\[ H \cdot (\partial \vec{A} \cdot \vec{A}^{-1}) \cdot H^{-1}, \]
\[ H \cdot d(\partial \vec{A} \cdot \vec{A}^{-1}) \cdot H^{-1}. \]

have Poincaré growth on \( \vec{X} \). To check this for the first two, note that \( H \) is a function on \( C(F) \times F \), hence it suffices to bound \( \delta_1 H \cdot H^{-1}, \delta_2 (\partial_1 H \cdot H^{-1}) \) for all vector fields \( \delta_1, \delta_2 \) on \( C(F) \times F \). But the Poincaré metric is given in Siegel coordinates by:

\[ ds^2 = \sum \frac{|dz_j|^2}{|z_i|^2 \left( \log \frac{|z_i|}{C} \right)^2} + \sum |dy_i|^2 + \sum |dt_i|^2 \]
\[ = \sum \frac{|dz_j|^2}{(l_i(\text{Im } x) + C)^2} + \sum |dy_i|^2 + \sum |dt_i|^2 \]
(choose \( C \) large enough so that \( l_i(\text{Im } x) + C > 0 \) near the boundary point in question). Therefore the bounds of Proposition 2.7 imply that \( \partial H \cdot H^{-1} \) and \( d(\partial H \cdot H^{-1}) \) have Poincaré growth.

To check the result for the last two, we need to know what sort of a function \( A \) is. Firstly, since \( e_i \) and \( e'_i \) are both \( U_e \)-invariant, \( A \) is \( U_e \)-invariant, i.e., is a function of \( y \) and \( t \) alone. Therefore it has derivatives only in the \( y \) and \( t \) directions, so to say these have Poincaré growth is just to say they are bounded along \( \vec{X} \). Next, for all \( t_0 \in F \), the action of \( V \) on vector space of points \( (y, t_0) \) puts a complex structure on \( V \). Thus it defines a splitting \( V \equiv V_0^+ \oplus V_0^- \) where \( V_0^\pm \) are complex subspaces and \( V_0^- \) acts trivially on the points \( (y, t_0) \), while \( V_0^+ \) acts simply transitively. If we fix one \( t_0 \), then \( V_0^+ \) still acts simply transitively and holomorphically on the vector spaces of points \( (y, t) \) for \( t \) near \( t_0 \). Thus it is natural to choose our holomorphic basis \( e_i \) to be in fact \( V_0^+ \cdot U_e \)-invariant.
Moreover, it is easy to see that \( G_t \cdot V \) normalizes \( V_{t_0}^+ \cdot U_c \) for all \( t_0 \), hence that the action of \( G_t \cdot V \) in terms of the holomorphic basis is given by

\[
g(e_j) = \sum_{j=1}^{n} \tilde{\eta}_j(t) \cdot e_j
\]

where

\[
g \mapsto \tilde{\rho}_t(g) = (\text{matrix } \tilde{\eta}_t(t))
\]

is a representation of \( G_t \cdot V \). Comparing \( \rho \) and \( \tilde{\rho} \), we get:

\[
\rho(g) = g^* A^{-1} \cdot \tilde{\rho}_t(g) \cdot A.
\]

Since \( e_i = e'_i \) on \((0, 0) \times F\), \( \rho(g) = \tilde{\rho}_t(g) \) for all \( g \in G_t \) and \( A(0, t) = I_n \). Thus if \( \nu(0, t) = (y, t) \),

\[
I_n = \rho(\nu) = A(y, t)^{-1} \cdot \tilde{\rho}_t(\nu) \cdot I_n,
\]

or

\[
A(y, t) = \tilde{\rho}_t(\nu).
\]

Now we use the simple:

**Lemma.** Let \( \sigma \) be an algebraic representation of \( G_t \cdot V \), and let \( \sigma_0 \) be the restriction of \( \sigma \) to \( G_t \). Then for all \( \nu \in V \), \( \sigma(\nu) \) is \( \sigma_0 \)-upper triangular.

**Proof.** Let \( A \subset G_t \) be a maximal \( \mathbb{R} \)-split torus. As in §2, there is a basis \( \gamma_1, \ldots, \gamma_r \) of the character group of \( A \) such that \( A \) acts on the vector space \( U(F) \) containing the cone \( C(F) \) through the weights \( \gamma_i + \gamma_j \). Then its action on \( V(F) \) is through the weights \( \gamma_i \) (cf. [1], p. 224). Now if \( V_i(F) \subset V(F) \) is the root space corresponding to \( \gamma_i \), and if we diagonalize \( \sigma(A) \):

\[
\mathbb{C}^n = \bigoplus_{\lambda \in X(A)} W_{\lambda},
\]

then

\[
V_i(F)W_j \subset W_{\lambda + \gamma_i}
\]

hence \( V(F) \) acts in a \( \sigma_0 \)-upper triangular fashion. QED

Thus \( A(y, t) \) is \( \rho \)-upper triangular for all \( y, t \), hence so are

\[
A^{-1} \cdot \partial A \quad \text{and} \quad d(A^{-1} \cdot \partial A).
\]

Applying Proposition 2.8, it follows that

\[
H \cdot (A^{-1} \cdot \partial A) \cdot H^{-1} \quad \text{and} \quad H \cdot (d(A^{-1} \cdot \partial A)) \cdot H^{-1}
\]

are bounded in a neighborhood of every point of \( \bar{X} \), as required. This completes the proof of the Main Theorem.

A natural question is whether these vector bundles \( \tilde{E} \) are in fact pull-backs of vector bundles on less blown up compactifications of \( D/T \). Thus Baily and Borel
defined in [2] a “minimal” but usually highly singular compactification \((D/Γ)^*\) of \(D/Γ\). Unfortunately \(\bar{E}\) is only rarely a vector bundle on \((D/Γ)^*\) (we will see below one case where it is however). However, in [1], Ash, Rapoport, Tai and I defined not only smooth compactifications of \(D/Γ\) but also a bigger class of compactifications with toroidal singularities (cf. [9]). These are important because when you try to resolve \((D/Γ)^*\), often there is a \(\bar{D}/Γ\) with relatively simple structure on the boundary but still with some toroidal singularities. It is easy to see that the construction above of \(\bar{E}\) goes through equally well on all of these compactifications: it gives vector bundles on all of them such that whenever compactification \(a\) dominates compactification \(b\), then extension \(a\) is the pull-back of extension \(b\).

The Main Theorem, plus Hirzebruch's original proof of his proportionality theorem for compact locally symmetric varieties \(X\), gives us easily the proportionality theorem in the general case:

**Proportionality Theorem 3.2.** As above, fix:

\[ X = \text{an arithmetic variety } D/Γ, \ D = K \backslash G, \]
\[ \bar{X} = \text{a smooth compactification as in [1]}, \]
\[ \bar{D} = \text{compact dual of } D. \]

Then there is a constant \(K\), which in terms of a natural choice of metric on \(D\) is the volume of \(X\), such that for all:

\[ \bar{E}_0 = G_{e}\text{-equivariant analytic rank } n \text{ vector bundle on } \bar{D} \]
\[ \text{defined by a representation of } \text{Stab}_{G_{e}}(e) \text{ trivial on } P_+, \]
\[ \bar{E} = \text{corresponding vector bundle on } \bar{X}, \]

the following formula holds:

\[ c^\alpha(\bar{E}) = (-1)^{\dim X} \cdot K \cdot c^\alpha(\bar{E}_0), \quad \text{all } \alpha = (\alpha_1, \ldots, \alpha_n), \sum \alpha_i = \dim X. \]

**Proof.** As above, choose a \(G\)-invariant Hermitian metric \(h_0\) on \(E_0\). By the Main Theorem, \(h_0\) defined a “good” Hermitian metric \(h\) on \(E\), hence its Chern forms \(c_\lambda(E, h)\) represent the Chern classes of \(\bar{E}\). But on \(D\), \(c_\lambda(E, h)\) are \(G\)-invariant forms, so:

\[ c^\alpha(\bar{E}) = \int_X c^\alpha(E, h) \]
\[ = \int_{\text{al Domain}} c^\alpha(E_0, h_0) \]
\[ = \text{vol}(F) \cdot c^\alpha(E_0, h_0)(e). \]

Now if \(G^c\) is a compact form of \(G\):

\[ \text{Lie } G = \mathfrak{t} \oplus \mathfrak{p}, \]
\[ \text{Lie } G^c = \mathfrak{t} \oplus \mathfrak{i} \mathfrak{p} \]
then $\tilde{E}_o$ has a unique $G*$-invariant Hermitian metric $\tilde{h}$ equal to $h_o$ at $e$. So

$$c^s(\tilde{E}_o) = \int c^s(\tilde{E}_o, \tilde{h})$$

$$= \text{vol}(\tilde{D}) \cdot c^s(\tilde{E}_o, \tilde{h})(e).$$

Then—and this is the essence of Hirzebruch's remarkable proof—a simple local calculation shows (cf. [7]):

$$c_1(E, h)(e) = (-1)^k c_k(\tilde{E}_o, \tilde{h}).$$

This proves the result.

To apply this result, it is important to describe the bundles $\tilde{E}$ as closely as possible. Firstly, we can characterize their sections, precisely as a special case of the general definition of automorphic forms given by Borel [3]. Let $\rho: K \to GL(n, \mathbb{C})$ be a representation of $K$, and let

$$E_o = G \times \mathbb{C}^n$$

be the associated $G$-equivariant vector bundle over $D = K \backslash G$ (i.e., $E_o$ = set of pairs $(g, a) \ mod(g, a) \sim (kg, \rho(k)a)$). $E_o$ has a complex structure as follows: complexify $\rho$ and extend it to $K_{k*} : P_{k*}$ to be trivial on $P_{k*}$. Then $E_o$ is the restriction to $D$ to the bundle

$$\tilde{E}_o = G_{k*} \times (k_{k*} : P_{k*}) \mathbb{C}^n$$

on $\tilde{D}$, and in the definition of $\tilde{E}_o$, everything is analytic. Borel introduces a measure of size on $G$ by:

$$\|g\|_G = \text{tr}(Ad g^{*^{-1}} \cdot g)$$

$*$ = Cartan involution on $G$ w.r.t. $K$,

and defines holomorphic $\rho$-automorphic form $f$ to be a function

$$f: G \to \mathbb{C}^n$$

such that

1. $f(kg\gamma) = \rho(k)f(g), \text{ all } k \in K, \gamma \in \Gamma$,
2. $f$ induces a holomorphic section of $E_o$,
3. $|f(g)| \leq C \cdot \|g\|_G^n, \text{ some } n \geq 1, C > 0.$

Then one can show:

**Proposition 3.3.** In the above notation:

$$\Gamma(\tilde{X}, \tilde{E}) \cong (\text{vector space of holomorphic } \rho\text{-automorphic forms}).$$

**Sketch of Proof.** The problem is to check that the bound (3) is equivalent to requiring that the corresponding section of $\tilde{E}$ over $X$ has growth $O((\sum \log |z_i|)^{2n})$.
along $\bar{X}$. But $\|g\|_G$ defines a measure of size on $D$ and on $X$ by:

$$\forall x \in D: \|x\|_D = \|g\|_G \quad \text{if } x \text{ corresponds to the coset } K \cdot g,$$

$$\forall x \in X, \text{ image of } x: \|x\|_X = \min_{y \in \Gamma} \|y(x)\|_D = \min_{y \in \Gamma} \|g\|_G.$$  

Then holomorphic $\rho$-automorphic forms are clearly holomorphic sections $s$ of $\bar{E}$ over $X$, such that

$$h(s, s)(x) \leq C_1 \|x\|_X^n, \quad \text{some } n \geq 1, \ C_1 > 0.$$  

But if $d_\rho$ is a $G$-invariant distance function on $D$, then it is easy to see (using $G = K \cdot A = K$) that $d_\rho(x, e)$ and $\log \|x\|$ are bounded with respect to each other. In another paper [4], Borel has proven that if $x$ is restricted to a Siegel set $\mathcal{S} = \omega \cdot A, e \subset D$, then

$$\min_{\gamma \in \Gamma} d_\rho(x\gamma, e) \approx d_\rho(x, e) \approx d_\rho(a(x), e)$$

(here $x = \omega(x) a(x) \cdot e$ and $\approx$ means the differences are bounded). Applying this to a subset of a Siegel Domain of 3rd kind of the type $\{(x, y, t)| y \in V', t \in F'$, $\Re x \in U', \Im x - l_i(y, y) \in \sigma + a\}$ where $U' \subset U$, $V' \subset V$, $F' \subset F$ are compact subsets and $\sigma \subset C(F)$ is a simplicial cone and $a \in C(F)$, we see that

$$\min_{\gamma \in \Gamma} d_\rho((x, y, t)\gamma, e)$$

can be bounded above and below by expressions

$$C_2 \log(\langle \Im x - l_i(y, y), e \rangle), \quad e \in C(F), \ C_2 > 0$$

hence $\|x\|_X$ can be bounded above and below by expressions

$$\langle \Im x - l_i(y, y), e \rangle^n, \quad e \in C(F), \ n \geq 1.$$  

Describing $\sigma$ as $l_i \geq 0$ as above ($l_i$ linear functionals on $U$), this is of the same size as

$$\left(\sum l_i(\Im x) + C_3\right)^n$$

and as $z_i = e^{2\pi i l_i(x)}$, this is equal to

$$(-\sum \log(|z_i|/C_4))^n. \ \text{QED}.$$

Next, there are 2 particular equivariant bundles where we can describe $\bar{E}$ more completely:

**Proposition 3.4.** a) If $E_0 = \Omega_{p\sigma}$ the cotangent bundle, with canonical $G$-action, then $\bar{E} = \Omega_{p\sigma} \log$, the bundle on $\bar{X}$ whose sections in a polycylinder $A^n \subset \bar{X}$ such that

$$A^n \cap (\bar{X} - X) = \bigcup_{i=1}^{k} (\text{coordinate hyperplanes } z_i = 0).$$
are given by
\[ \sum_{i=1}^{k} a_i(z) \frac{dz_i}{z_i} + \sum_{i=k+1}^{*} b_i(z)dz_i. \]

b) If \( E_0 \) is the canonical line bundle \( \Omega^1_0 \), then \( \tilde{E} \) is the pull-back of an ample line bundle \( \mathcal{O}(1) \) on the Baily-Borel compactification \( X^* \) of \( X \). The sections of \( \mathcal{O}(n) \) are the modular forms with respect to the \( n \)th power of the canonical automorphism factor given by the Jacobian, hence \( \mathcal{O}(n), n \geq 0 \), is the very ample bundle used by Baily and Borel to embed in \( X^* \) in \( \mathbb{P}^k \).

**Proof.** Using Siegel Domain coordinates \((x, y, t)\) on \( D(F) \), a \( U(F)_{\mathbb{R}} \)-invariant basis of \( \Omega^1_{F(F)} \) is given by \( \{dx_i, dy_j, dt_k\} \). Therefore these span the corresponding bundle on \( X \) near the boundary \( F \). But here
\[ \{z_i = e^{2\pi i t(x)}, y_j, t_k\} \]
are coordinates and these differentials are \( \{\frac{dz_i}{z_i}, dy_j, dt_k\} \). This proves (a).

To prove (b), recall that \( X^* \) is set-theoretically the union of \( X \) and of \( F/\Gamma \cap N(F) \) for all rational boundary components \( F \). Moreover, if \( P \in F/\Gamma \cap N(F) \subset X^* \), then there exists a neighborhood \( U \subset X^* \) and an open set \( V \subset D \) such that \( V \) maps to \( U \cap X \) and \( V \) is a \( (G_i(F) \cdot V(F) \cdot U(F)) \cap \Gamma \)-bundle over \( U \cap X \). Now say \( \{s_i\} \) is the \( U(F)_{\mathbb{R}} \)-invariant holomorphic basis of \( E_0 \) on \( D(F) \) used to extend \( \tilde{E} \) over the \( F \)-boundary points of \( X \). If we verify that each \( s_i \) is \( (G_i(F) \cdot V(F) \cdot U(F)) \cap \Gamma \)-invariant it follows that \( \tilde{E} \) is trivial on \( U \cap X \) and moreover, if \( \pi: \tilde{X} \rightarrow X^* \) is the canonical birational map, then \( \{s_i\} \) are a basis of \( \pi_* \tilde{E} \) over \( U \). Thus \( \pi_* \tilde{E} \) is a vector bundle which pulls-back to \( \tilde{E} \) on \( \tilde{X} \). Now in the case in question, \( E \) is a line bundle. \( s_1 \) can be identified with the differential form
\[ (\bigwedge^i dx_i) \wedge (\bigwedge^j dy_j) \wedge (\bigwedge^k dt_k) \]
on \( D(F) \), and \( G_i \cdot V \cdot U \) acts on it by multiplication by the Jacobian determinant in the Siegel Domain coordinates. But Baily-Borel ([2], Prop. 3.14) showed that the Jacobian on \( (G_i \cdot V \cdot U) \cap \Gamma \) was a root of unity. Since \( \Gamma \) is neat, it is one and \( (G_i \cdot V \cdot U) \cap \Gamma \) indeed fixes \( s_1 \). The last assertion is just a restatement of Proposition 3.3 for this special case. QED

The following consequence of the proportionality principle seems to be more or less well known to experts, but does not seem to be contained in any published articles:

**Corollary 3.5.** Let \( L = (\Omega^*_0)^{-1} \) be the ample line bundle on \( \tilde{D} \), and let
\[ P(l) = \chi(\mathcal{L}^\otimes l) \]
be the Hilbert polynomial of \( \tilde{D} \). Let \( \pi: \tilde{X} \rightarrow X^* \) map a smooth compactification of \( X \) onto Baily-Borel's compactification. Let \( n_1 = \dim(X^* - X) \). Then there exists a
polynomial $P_l(l)$ of degree at most $n_1$ such that for all $l \geq 2$:

$$\dim [\text{cusp forms on } D \text{ w.r.t. } \Gamma \text{ of weight } l] = \text{vol}(X) \cdot P(l-1) + P_1(l).$$

**Proof.** The Riemann-Roch theorem gives us a “universal polynomial” $Q$ such that if $L$ is any line bundle on a smooth projective variety $W$, then

$$\chi(L) = Q(c_1(L); c_1(\Omega^1_W), \ldots, c_n(\Omega^n_W)).$$

Therefore if $n = \dim D$,

$$(-1)^n \text{vol}(X) \cdot P(-l) = (-1)^n \text{vol}(X) \cdot \chi((\Omega^n_D)^\otimes l) = (-1)^n \text{vol}(X) \cdot Q(lc_1(\Omega^1_D); c_1(\Omega^1_D), \ldots, c_n(\Omega^n_D)) = Q(lc_1(\Omega^1_\mathcal{X}(\log)); c_1(\Omega^1_\mathcal{X}(\log)), \ldots, c_n(\Omega^n_\mathcal{X}(\log)))$$

by Proportionality Theorem 3.2.

Consider a typical term

$$\left[l c_1(\Omega^1_\mathcal{X}(\log))\right]^k \cdot c^k(\Omega^k_\mathcal{X}(\log)), \quad |x| + k = n.$$  

Now by Proposition 3.4.b:

$$c_1(\Omega^1_\mathcal{X}(\log)) = \pi^* H,$$

$H$ an ample divisor on $X^*$. Let $n_1 = \dim(X^* - X)$. If $k > n_1$, the cycle class $H^k$ on $X^*$ is represented by a cycle supported on $X$ alone, hence so is $\pi^* H$. Thus if $k > n_1$,

$$(l \cdot \pi^* H)^k \cdot c^k(\Omega^k_\mathcal{X}(\log)) = (l \cdot \pi^* H)^k \cdot c^k(\Omega^k_\mathcal{X}).$$

Therefore

$$Q(lc_1(\Omega^1_\mathcal{X}(\log)); c_1(\Omega^1_\mathcal{X}(\log)), \ldots, c_n(\Omega^n_\mathcal{X}(\log))) = Q(lc_1(\Omega^1_\mathcal{X}(\log)); c_1(\Omega^1_\mathcal{X}), \ldots, c_n(\Omega^n_\mathcal{X})) + (\text{poly. of degree } \leq n_1) = \chi(\Omega^1_\mathcal{X}(\log)^\otimes l) + (\text{poly. of degree } \leq n_1) = (-1)^n \chi(\Omega^1_\mathcal{X}(\log)^\otimes(l-1) \otimes \Omega^k_\mathcal{X}) + (\text{poly. of degree } \leq n_1)$$

by Serre duality.

Thus for suitable $P_1$ of degree at most $n_1$:

$$\text{vol}(X) \cdot P(l-1) = \chi(\Omega^n_\mathcal{X}(\log)^\otimes(l-1) \otimes \Omega^k_\mathcal{X}) - P_1(l).$$

But since $(\Omega^k_\mathcal{X}(\log)^\otimes N)$ is generated by its sections and maps $\bar{X}$ to $X^*$ of the same dimension for $N \gg 0$, Kodaira Vanishing (cf. [13]) applies if $l \geq 2$ and we have

$$h^0(\Omega^k_\mathcal{X}(\log)^\otimes l-1 \otimes \Omega^k_\mathcal{X}) = \text{vol}(X) \cdot P(l-1) + P_1(l).$$
The left-hand side is exactly the space of sections of \( \Omega_X^1(\log)^l \) which vanish on the boundary. By Proposition 3.3, these are exactly the cusp forms of weight \( l \). QED

§4. Applications: General Type and Log General Type

The purpose of this section is to consider the application of the preceding theory to the question of when \( D/\Gamma \) is of general type, and to reprove as a consequence of our theory the following theorem of Y.-S. Tai ([1], Ch.IV, §1).

Tai's Theorem 4.1. If \( \Gamma \) is any arithmetic variety acting on a bounded symmetric domain \( D \), then there is a subgroup \( \Gamma_0 \subset \Gamma \) of finite index such that for all \( \Gamma_i \subset \Gamma_0 \) of finite index, the variety \( D/\Gamma_i \) is of general type.

We recall that if \( X \) is any variety of dimension \( n \), we say that \( X \) is of general type, if for one (and hence all) smooth complete varieties \( \tilde{X} \) birational to \( X \), the transcendence degree of the ring

\[
\bigoplus_{N=0}^{\infty} \Gamma(\tilde{X}, (\Omega_\tilde{X})^N) \]

is \((n+1)\). More generally, the transcendence degree of this ring minus one is called the Kodaira dimension of \( X \).

Recall that Iitaka [8] has recently introduced a complementary theory of "logarithmic Kodaira dimension" for arbitrary varieties \( Y \). In fact, he first chooses a smooth blow-up \( Y' \) of \( Y \) and then a smooth compactification \( \bar{Y} \) of \( Y' \) such that \( \bar{Y} - Y' \) has normal crossings and defines \( \Omega_Y^1(\log) \) as the complex of 1-forms

\[
\sum_{i=1}^{k} a_i(z) \frac{dz}{z_i} + \sum_{i=k+1}^{n} a_i(z)dz_i
\]

if, locally, \( \bar{Y} - Y' \) is given by \( \prod_{i=1}^{k} z_i = 0 \). By definition \( \Omega_Y^1(\log) = \Lambda^* (\Omega_Y^1(\log)) \). He then looks at the "logarithmic canonical ring":

\[
R = \bigoplus_{N=0}^{\infty} \Gamma(\bar{Y}, \Omega_Y^1(\log)^N).
\]

He shows that this ring, as well as all other vector spaces of global forms with logarithmic poles (obtained from decomposing \( \Omega_Y^1(\log) \otimes \cdots \otimes \Omega_Y^1(\log) \) under the symmetric group and taking global sections) are independent of the choice of \( Y' \) and \( \bar{Y} \). He then defines the logarithmic Kodaira dimension of \( Y \) to be the transcendence degree of \( R \) minus 1. We may restate Proposition 3.4(b) in this language as follows:

**Proposition 4.2.** If \( \Gamma \) is a neat\(^3\) arithmetic group, then \( D/\Gamma \) is a variety of logarithmic general type, i.e., its logarithmic Kodaira dimension equals its dimension.

\(^3\) Some hypothesis on elements of finite order is needed because \( H/SL_2(\mathbb{Z}) \cong \mathbb{A}^1 \) which is not of log general type!
Proof. In fact, by Proposition 3.4, \( R \) is just the homogeneous coordinate ring of the Baily-Borel compactification of \( D/\Gamma \).

Note that \( D/\Gamma \) of logarithmic general type is weaker than saying \( D/\Gamma \) is general type.

I would like to add one comment to his theory which, in some cases, makes it easier to apply: one does not need smooth compactifications, but merely a toroidal compactification \( \bar{Y} \) of \( Y' \) (cf. [9], p. 54). This means that locally \( Y' \subset \bar{Y} \) is isomorphic to \( (\mathbb{C}^*)^r \subset X_g \), where \( X_g \) is an affine torus embedding (i.e., \( \mathbb{C}^* \)). \( X_g \) is normal affine and translations by \( \mathbb{C}^* \) extend to an action of \( \mathbb{C}^* \) on \( X_g \). On \( X_g \), define \( \Omega^1_X(\log) \) to be the sheaf generated by the \( (\mathbb{C}^*) \)-invariant 1-forms. Carrying these over, we define \( \Omega^1(\log) \) to be the coherent sheaf of 1-forms on \( \bar{Y} \), regular on \( Y' \), isomorphic locally to \( \Omega^1_X(\log) \). If \( \bar{Y} \to \bar{Y} \) is an “allowable” modification of toroidal embedding of \( Y ('9'), p. 87, then \( p^*(\Omega^1_Y(\log)) \cong \Omega^1(\log) \). In particular, there is always a smooth allowable modification \( \bar{Y} \) (‘9), p. 94). So Itaka’s spaces of forms with log poles can be calculated equally well on a smooth \( \bar{Y} \) or a toroidal \( \bar{Y} \).

This extension is helpful in checking the analog of the above Proposition for the moduli space of curves:

**Proposition 4.3.** Let \( \mathfrak{M}^{(n)}_g \) be the moduli space of smooth curves of genus \( g \) with level \( n \) structure. If \( n \geq 3 \), then \( \mathfrak{M}^{(n)}_g \) is of log general type.

**Sketch of Proof.** The proof follows the ideas of [12], §5 very closely. Let \( H_0 \) be the Hilbert scheme of \( \epsilon \)-canonically embedded smooth curves of genus \( g \). Let \( H_0 \to \bar{H} \) be the covering defined by the set of level \( n \) structures on these curves. Let \( \bar{H}, \mathfrak{M}_0 \) be the compactified spaces allowing stable singular curves as well. Let \( \bar{H}_0, \mathfrak{M}_0 \) be the normalization of \( \bar{H}, \mathfrak{M}_0 \) in the coverings \( H_0, \mathfrak{M}_0 \). The group \( G = \text{PGL}(\nu) (\nu = (2e-1)(g-1)) \) acts on \( H_0 \) and on \( H_0^{(n)} \), freely on the latter, so that \( \mathfrak{M}_0 \cong H_0/G, \mathfrak{M}_0^{(n)} \cong H_0^{(n)}/G \). We have the diagram:

\[
\begin{array}{ccc}
\bar{H}_0^{(n)} & \to & \mathfrak{M}_0^{(n)} \\
\downarrow & & \downarrow \\
\bar{H}_0 & \to & \mathfrak{M}_0 \\
\downarrow & & \downarrow \\
H_0 & \to & \mathfrak{M}_0 \\

\end{array}
\]

where \( D \) and \( \mathcal{C} \) are the universal curves. Recall from [12] the notation: whenever \( \mathcal{A} \to S \) is a flat family of stable curves,

\[
\lambda = \mathcal{A}^* p_*(\omega_{C/S}), \\
\omega_{C/S} = \text{relative dualizing sheaf}
\]

and if \( p \) is smooth over all points of \( S \) of depth zero, then \( \Delta \subset S \) is the divisor of singular curves and

\[
\delta = \mathcal{O}_S(\Delta).
\]
Now on all 3 families above, we wish to show $\lambda^{13} \otimes \delta^{-1}$ and the sheaf of top logarithmic forms $\Omega^2(\log)$ are isomorphic line bundles. Firstly, for $p: \tilde{D}_x \to \tilde{H}_x$, we proceed like this: a) a simple modification of the proof of Theorem 5.10 [12] shows:

$$\lambda^{13} \otimes \delta^{-2} \cong A^{3x-3} (p_* (\Omega^1_{\tilde{D}_x} \otimes \omega_{\tilde{D}_x})).$$

b) Since $\tilde{H}_x$ represents the functor of $e$-canonical stable curves, $T_{\tilde{H}_x(C)}$ is canonically isomorphic to the vector space of deformations of $C$. This has a subspace consisting of deformations of the $e$-canonical embedding where $C$ doesn't change, and a quotient space of the deformations of $C$ alone:

$$0 \to \text{Lie } G \to T_{\tilde{H}_x(C)} \to \text{Ext}^1 (\Omega^1_{\tilde{C}}, \mathcal{O}_C) \to 0$$

or dually:

$$0 \to H^0 (\Omega^1_C \otimes \omega_C) \to \Omega^1_{\tilde{H}_x} \otimes \mathcal{K}(C) \to (\text{Lie } G)' \to 0.$$  

Therefore globally, we get

$$0 \to p_* (\Omega^1_{\tilde{D}_x} \otimes \omega_{\tilde{D}_x}) \to \Omega^1_{\tilde{H}_x} \to (\text{Lie } G)' \otimes \mathcal{O}_{\tilde{H}_x} \to 0$$

hence if $m = \dim \tilde{H}_x$,

$$\Omega^m_{\tilde{H}_x} \cong A^{3x-3} p_* (\Omega^1_{\tilde{D}_x} \otimes \omega_{\tilde{D}_x}) \cong \lambda^{13} \otimes \delta^{-2}$$

$$\therefore \Omega^m_{\tilde{H}_x} (\log) \cong \lambda^{13} \otimes \delta^{-1}.$$  

Secondly, for $p: \tilde{D}^\diamond \to \tilde{H}_x$, $\lambda^{13} \otimes \delta^{-1}$ pulls back to the analogous sheaf on $\tilde{H}_x^\diamond$. Moreover, because $\tilde{H}_x^\diamond \to \tilde{H}_x$ is ramified only along $A$ which has normal crossings, $\tilde{H}_x^\diamond$ has toroidal singularities and $\Omega^m_{\tilde{H}_x} (\log)$ pulls back to $\Omega^m_{\tilde{H}_x^\diamond} (\log)$. Finally both bundles descend to $\tilde{\mathbb{H}}_x^\diamond$ and by Proposition 1.4 [11], are still isomorphic. Finally, it is proven in [12] (Th.5.18 and 5.20: cf. diagram in §5) that $\lambda^{13} \otimes \delta^{-1}$ is sample on $\tilde{\mathbb{H}}_x^\diamond$. QED

In certain cases, there is a way of deducing that coverings of a variety of log general type are actually of general type. To explain this, suppose we are given a smooth quasi-projective variety $Y$, and a tower of connected étale Galois coverings:

$$\pi_\gamma: Y_\gamma \to Y, \quad \text{group } \Gamma_\gamma.$$

We assume that any 2 covers $\pi_\gamma, \pi_\delta$ are dominated by a third one $\pi_\eta$:

$$Y_\gamma \xrightarrow{\pi_\gamma} Y \xrightarrow{\pi_\delta} Y_\delta \xrightarrow{\pi_\delta} Y_\eta \xrightarrow{\pi_\eta}. \quad \Gamma_\gamma \xrightarrow{\pi_\gamma} \Gamma_\delta \xrightarrow{\pi_\delta} \Gamma_\delta \xrightarrow{\pi_\delta} \Gamma_\eta.$$
Let \( \tilde{Y} \) be a smooth compactification of \( Y \) with normal crossings at infinity. Extend the covering \( Y \) to a finite covering

\[
\pi_\alpha : \tilde{Y} \rightarrow \tilde{Y}
\]

be defining \( \tilde{Y} \) to be the normalizations of \( \tilde{Y} \) in the function field of \( Y \). We now make the definition:

**Definition.** The tower \( \{ \pi_\alpha \} \) is locally universally ramified over \( \tilde{Y} - Y \) if for all \( x \in \tilde{Y} - Y \), we take a nice neighborhood of \( x \):

\[
\begin{align*}
\Delta^\alpha & \subset \tilde{Y}, \\
\Delta^\alpha \cap (\tilde{Y} - Y) & = \text{(union of coordinate hyperplanes)} \\
 & = \left( z_1 = 0, \ldots, z_k = 0 \right)
\end{align*}
\]

then for all \( m \), there is an \( \alpha \) and a commutative diagram:

\[
\begin{array}{ccc}
\pi_\alpha^{-1}(\Delta^\alpha) & \xrightarrow{\text{res}_\Delta} & \Delta^\alpha \\
\downarrow & & \downarrow \\
\pi_\alpha^{-1}(\tilde{Y} - Y) & \xrightarrow{\text{res}_\Delta} & \Delta^\alpha \cap (\tilde{Y} - Y)
\end{array}
\]

In other words, \( \pi_\alpha^{-1}(\Delta^\alpha \cap (\tilde{Y} - Y)) \) is cofinal in the set of all unramified coverings of \( \Delta^\alpha \cap (\tilde{Y} - Y) \).

Then we assert:

**Proposition 4.4.** Let \( Y \subset \tilde{Y} \) be as above and let \( \pi_\alpha : \tilde{Y}_\alpha \rightarrow \tilde{Y} \) be a tower of coverings unramified over \( Y \) and locally universally ramified over \( \tilde{Y} - Y \). Then if \( Y \) is logarithmically of general type, there is an \( \alpha_0 \) such that for all \( \alpha_1 \) such that the covering \( \pi_{\alpha_1} \) dominates \( \pi_{\alpha_0} \), \( Y_{\alpha_1} \) is of general type.

**Proof.** Let \( \Delta = \tilde{Y} - Y, n = \dim Y \) and let \( \omega = \Omega_\alpha^Y (\log) \). Then we know that

1. for some \( N \), there are differentials \( \eta_0, \ldots, \eta_n \in \Gamma(\tilde{Y}, \omega^{\otimes N}) \) such that \( \eta_1/\eta_0, \ldots, \eta_n/\eta_0 \) are a transcendence base of the function field \( \mathbb{C}(Y) \),
2. \( h^N(\tilde{Y}, \omega^{\otimes N}) \geq C_1 N^{n+1} \) if \( N \geq N_0 \).

From (b) it follows that

\[
\Gamma(\tilde{Y}, \omega^{\otimes N}) \rightarrow \Gamma(\Delta, \omega^{\otimes N} \otimes \mathcal{O}_\Delta)
\]

has a non-zero kernel for some \( N \); let \( \zeta \) be in the kernel. Replacing \( \eta_i \) by \( \eta_i \otimes \zeta \), we may assume that all \( \eta_i \) are zero on \( \Delta \). Now let’s examine locally what happens to \( \eta_i \) when lifted to a covering of \( \tilde{Y} \): let \( \Delta^\alpha \subset \tilde{Y} \) be a polycylinder such that \( \Delta^\alpha \cap (\tilde{Y} - Y) = V \left( \prod_{i=1}^k z_i \right) \). Write out \( \eta_i \):

\[
\eta_1 = a_i(z) \cdot \prod_{i=1}^k z_i : \left( \frac{dz_1 \wedge \cdots \wedge dz_k}{z_1 \cdots z_k} \right)^N,
\]
Let \( w_i = z_i^{1/m_i}, 1 \leq i \leq k, w_i = z_i, k + 1 \leq i \leq n \) and let

\[
\pi_m : \Delta^n \to \Delta^n
\]

be the covering of the \( z \)-polycylinder by the \( w \)-polycylinder. Then

\[
\frac{dw_i}{w_i} = \frac{dz_i}{z_i}, \quad 1 \leq i \leq k, \\
\frac{dw_i}{w_i} = dz_i, \quad k + 1 \leq i \leq n
\]

hence

\[
\pi_m^\#(\eta) = m^N a_i \cdot \prod_{i=1}^k w_i^{m_i} \cdot \left( \frac{dw_1 \wedge \cdots \wedge dw_k}{w_1 \cdots w_k} \right)^N.
\]

So if \( m \geq N \), \( \pi_m^\#(\eta) \) is a holomorphic differential form on \( \Delta^n \). Now for each \( x \in \bar{Y} - Y \), fix a neighborhood \( U_x \subset \bar{Y} \) of this type and a covering

\[
\pi_{s(x)} : \bar{Y}_{s(x)} \to \bar{Y}
\]

which, over \( U_x \), dominates \( \pi_{s_x} \). \( \bar{Y} - Y \) is covered by a finite number \( \{U_{s_x}\} \) of \( U_{s_x} \)'s, so we can find one cover \( \pi_{s_x} \) which dominates all the covers \( \pi_{s(x)} \). I claim that if \( \pi_{s_x} \) dominates \( \pi_{s_{x_0}} \), hence \( \pi_{s(x)} \), then \( \pi_{s_{x_0}}^\#(\eta) \) has no poles on a desingularization \( \bar{Y}_{s_x} \) over \( \bar{Y}_{s_x} \). This is clear because it has an open covering by open sets \( V_i \) sitting in a diagram

\[
\begin{array}{c}
V_i \subset \bar{Y}_{s_x} \\
\end{array}
\]

so \( \pi_{s_{x_0}}^\#(\eta) \) regular \( \Rightarrow \pi_{s_{x_0}}^\#(\eta) \) regular on \( V_i \). But now \( \pi_{s_{x_0}}^\#(\eta/\eta_0) \) are a transcendence base of the function field of \( \bar{Y}_{s_x} \), so \( \bar{Y}_{s_x} \) is of general type. QED

Let's consider the case \( Y = D/I \) again. If \( I \) is an arithmetic group, then for every positive integer \( n \), we have its level \( n \) subgroup \( \Gamma(n) \), i.e., if

\[
\Gamma = \mathcal{G}(\mathbb{Z}), \quad \mathcal{G} \text{ algebraic group over Spec } \mathbb{Z}
\]

then

\[
\Gamma(n) = \text{Ker}[\mathcal{G}(\mathbb{Z}) \to \mathcal{G}(\mathbb{Z}/n\mathbb{Z})].
\]

It's easy to see that

\[
\pi_n : D/\Gamma(n) \to D/\Gamma
\]
is locally universally ramified over $D/\Gamma - D/\Gamma$. In fact, let $F$ be a rational boundary component. Then near boundary points associated to $F$, the pair $D/\Gamma \subset D/\Gamma$ is isomorphic to

$$
D(F)/\varGamma \subset (D(F)/\varGamma \cap \mathbb{Z})_a
\quad \mathbb{C}^* \times \mathbb{C}^m \times F
\quad \mathbb{C}^n \times \mathbb{C}^m \times F.
$$

Thus if $U \subset D/\varGamma$ is a small neighborhood of a point corresponding in the above chart to $(0, y, t)$, $\pi_1(U \cap (D/\Gamma))$ is isomorphic to $U(F) \cap \Gamma$. Thus we must check that for all $n$, there is an $m$ such that

$$U(F) \cap \Gamma(m) \subset n: U(F) \cap \Gamma.$$

But if $F$ is rational, $U(F)$ is an algebraic subgroup of the full group $\varGamma$ which is defined over $\mathbb{C}$, so this is clear. So now Proposition 4.2, Proposition 4.4 and this remark altogether imply Tai’s theorem.

It is now known that this same method will show that at least some high non-abelian levels of $\mathbb{H}_g$ are varieties of general type too. It is not simple however to check that the Teichmüller tower is locally universally ramified at infinite. This has recently been proven by a ingenious use of dihedral level by T.-L. Brylinski.

References

8. Iitaka, S.: On logarithmic Kodaira dimension of algebraic varieties. (To appear)

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