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AN ALGEBRAIC SURFACE WITH K AMPLE, $(K^2) = 9$, $p_g = q = 0$

By D. MUMFORD

Severi raised the question of whether there existed an algebraic surface X homeomorphic to $\mathbf{P}^2$ but not isomorphic to it (as a variety), and conjectured that such a surface did not exist. The essential problem in proving this is to eliminate the possibility that the canonical class K, as a member of the infinite cyclic group $H^2(X, \mathbf{Z})$ might be a *positive* multiple (in fact, 3) of the ample generator of $H^2(X, \mathbf{Z})$ instead of a negative multiple (in fact, -3) as it ought to be. That this is the problem is clear from Castelnuovo's criterion for rationality, and was analyzed and generalized to higher dimensions in the paper [3] of Hirzebruch and Kodaira where it was shown that in *odd* dimensions, $\mathbf{P}^n$ is the only variety in its homeomorphism class. Severi's question was finally answered by S. Yau [8] as a Corollary of his result that all varieties X on which K is ample carry a unique Kähler-Einstein metric. In fact, this result shows that when X is a surface on which K is ample, then the Chern numbers satisfy $c_1^2 \leq 3c_2$, with equality if and only if X is isomorphic to D_2/Γ (D_2 = unit ball in $\mathbf{C}^2$, $\Gamma \subset SU(2, 1)/(\text{center})$ a discrete torsion-free co-compact subgroup; Hirzebruch in [2] had much earlier shown that the surfaces D_2/Γ did satisfy $c_1^2 = 3c_2$). However, the question arises: how close can we come to a surface with K ample which mimics the topology of $\mathbf{P}^2$? In particular, does there exist such a surface with the *same Betti numbers as* $\mathbf{P}^2$? By the standard results on algebraic surfaces, this means that we seek a surface X such that:

$$p_g = q = 0, \quad \text{hence } \chi(\mathcal{O}_X) = 1$$

$$(c_1^2) = (K^2) = 9$$

$$B_0 = B_2 = B_4 = 1, \qquad B_1 = B_3 = 0, \qquad \text{hence } c_2 = 3$$

I shall exhibit one such surface. My method is not by complex uniformization, as used by Shavel [6] and Jenkins (unpublished) in the construction of a surface with K positive and the same Betti numbers as $\mathbf{P}^1 \times \mathbf{P}^1$, but by the p-adic uniformization introduced recently by Kurihara [1] and Mustafin [5]. After looking for such an example at some length, I would hazard the guess that there are, in fact, very few such surfaces (combining Yau's results with Weil's theorem [7] that discrete co-compact groups $\Gamma \subset SU(2, 1)$ are rigid, it follows that there are in any case only *finitely* many such surfaces). But it seems a difficult matter to find some way of enumerating all such surfaces.

1. p-adic uniformizations in general. In this section we wish to summarize and extend somewhat the results of Kurihara and Mustafin cited above, restricting ourselves however to the 2-dimensional case. Let R be a complete discrete valuation ring with fraction field K and residue field $k = R/\pi R$. We assume k is finite. The basis of the construction is a beautiful scheme $\mathfrak{X}$, locally of finite type over R, which may be described by charts as follows:

$$\mathfrak{X} = \bigcup_{A \in GL(3,K)} \operatorname{Spec} R\left[\frac{l_0}{l_1}, \frac{l_1}{l_2}, \pi\frac{l_2}{l_0}\right] - (C_0 \cup C_1 \cup C_2)$$

where $l_i = \sum_{j=0}^{2} A_{ij}x_j$, $A = (A_{ij})$
$C_0 =$ set of curves

$$\pi = \frac{l_0}{l_1} = 0, \qquad a\left(\frac{l_1}{l_2}\right)\cdot\left(\pi\frac{l_2}{l_0}\right) + b\left(\pi\frac{l_2}{l_0}\right) + c = 0$$

$a, b, c \in k$, $a \cdot c \neq 0$, plus the curves

$$\pi = \frac{l_0}{l_1} = 0, \qquad \left(\pi\frac{l_2}{l_0}\right) + c = 0$$

and $$\pi = \frac{l_0}{l_1} = 0, \qquad \frac{l_1}{l_2} + c = 0 \quad (c \in k^*).$$

$C_1, C_2 =$ similar sets of curves where the role of l_0/l_1, l_1/l_2, $\pi(l_2/l_0)$ are permuted cyclically.

Here the glueing represented by the union sign is induced by the requirement that $\mathfrak{X}$ is irreducible and separated with function field

$$K\left(\frac{X_1}{X_0}, \frac{X_2}{X_0}\right),$$

which is the common fraction field of all affine rings.

The closed fibre $\mathfrak{X}_0$ of $\mathfrak{X}$ can be represented graphically by means of the Bruhat-Tits building Σ attached to PGL(3, K). In fact, the 3 sets:

Components E of $\mathfrak{X}_0$
Free rank 3 R-submodules $M \subset K\cdot X_0 \oplus K\cdot X_1 \oplus K\cdot X_2$, modulo $M \sim \pi^k\cdot M$
Vertices ν of Σ

are isomorphic. Moreover, the components of $\mathfrak{X}_0$ cross normally, and if E_i, M_i, ν_i, $i = 1, 2, 3$, correspond as above, then:

a) $E_1 \cap E_2$ is a curve $\Leftrightarrow M_1 \supsetneq \pi^k M_2 \supsetneq \pi M_1$, some $k \in \mathbf{Z}$

$\Leftrightarrow \nu_1, \nu_2$ are joined in Σ by a segment

b) $E_1 \cap E_2 \cap E_3$ is a triple point $\Leftrightarrow M_1 \supsetneq \pi^k M_2 \supsetneq \pi^l M_3 \supsetneq \pi M_1$, some $k, l \in \mathbf{Z}$ (or same with 2, 3 interchanged)

$\Leftrightarrow \nu_1, \nu_2, \nu_3$ are the vertices of 2-simplex in Σ.

To describe $\mathfrak{X}$ in a Zariski-open neighborhood of some component E of $\mathfrak{X}_0$, we can proceed geometrically as follows: let E correspond to M and let Y_0, Y_1, Y_2 be an R-basis of M. Start with $\mathbf{P}_R^2$ based on homogeneous coordinates Y_0, Y_1, Y_2 (hence with function field $K(X_1/X_0, X_2/X_0)$ still). First, blow up all k-rational points of the closed fibre $\mathbf{P}_k^2$ of $\mathbf{P}_R^2$ (if k has q elements, there are $q^2 + q + 1$ of these). Second, blow up the proper transforms on this scheme of all k-rational lines on the original closed fibre $\mathbf{P}_k^2$ (again there are $q^2 + q + 1$ of these). Call this $\mathfrak{X}_M$ and let $E_M \subset \mathfrak{X}_M$ be the proper transform of $\mathbf{P}_k^2$. Then a suitable Zariski-neighborhood of E_M in $\mathfrak{X}_M$ is isomorphic to a neighborhood of E in $\mathfrak{X}$. In particular, all surfaces E are rational surfaces gotten by blowing up $\mathbf{P}_k^2 (q^2 + q + 1)$ times and they meet the $2(q^2 + q + 1)$ adjacent components in rational curves. These rational curves are either exceptional curves C of the first kind, in which case $(C^2) = -1$, or proper transforms of lines along which $q + 1$ points have been blown up, in which case:

$$(C^2) = +1 - (q + 1) = -q.$$

Thus geometrically, if $C = E_1 \cap E_2$, then $(C^2)_{E_1} = -1$ and $(C^2)_{E_2} = -q$ or vice versa; this asymmetry corresponds in (a) above to whether

$$\dim_k(M_1/\pi^k M_2) = 1 \quad \text{or} \quad \dim_k(\pi^k M_2/\pi M_1) = 1$$

and in Σ to the orientation on the segment from v_1 to v_2.

Now if $\Gamma \subset \mathrm{PGL}(3, K)$ is a discrete torsion-free co-compact group, we define first a formal scheme $\mathfrak{X}/\Gamma$ over R by dividing the formal completion of $\mathfrak{X}$ along $\pi = 0$ by Γ (this is possible because Γ acts freely and discontinuously in the Zariski-topology on $\mathfrak{X}_0$). Secondly, one verifies that the dualizing sheaf $\omega_{\mathfrak{X}}$ is ample on each component of $\mathfrak{X}_0$, hence it descends to an invertible sheaf $\omega_{(\mathfrak{X}/\Gamma)}$ on $\mathfrak{X}/\Gamma$ with the same property: this allows one to conclude that $\mathfrak{X}/\Gamma$ can be algebraized to true projective scheme over R, which, for simplicity, we denote $\mathfrak{X}/\Gamma$. Since the generic fibre $\mathfrak{X}_\eta$ of $\mathfrak{X}$ is smooth over K, the generic fibre $(\mathfrak{X}/\Gamma)_\eta$ is also smooth over K, hence

$$\omega_{(\mathfrak{X}/\Gamma)_\eta} = \Omega^2_{(\mathfrak{X}/\Gamma)_\eta}$$

hence $(\mathfrak{X}/\Gamma)_\eta$ is a surface of general type without smooth rational curves C with $(C^2) = -1$ or -2. It is not hard to compute the invariants of $(\mathfrak{X}/\Gamma)_\eta$: to do this, note that $(\mathfrak{X}/\Gamma)_0$ consists of finite set of rational surfaces, crossing each other (possibly crossing themselves) transversally in rational double curves and triple points. Let

E_α = normalizations of the components of $(\mathfrak{X}/\Gamma)_0$, $\quad 1 \leq \alpha \leq \nu_2$
C_β = normalizations of the double curves of $(\mathfrak{X}/\Gamma)_0$, $\quad 1 \leq \beta \leq \nu_1$
P_γ = triple points of $(\mathfrak{X}/\Gamma)_0$, $\quad 1 \leq \gamma \leq \nu_0$

We get an exact sequence:

$$0 \to \mathcal{O}_{(\mathfrak{X}/\Gamma)_0} \to \bigoplus_{\alpha=1}^{\nu_2} \mathcal{O}_{E_\alpha} \xrightarrow{\lambda} \bigoplus_{\beta=1}^{\nu_1} \mathcal{O}_{C_\beta} \xrightarrow{\mu} \bigoplus_{\gamma=1}^{\nu_0} \mathcal{O}_{P_\gamma} \to 0$$

hence

$$\begin{aligned}\chi(\mathcal{O}_{(\mathfrak{X}/\Gamma)_\eta}) &= \chi(\mathcal{O}_{(\mathfrak{X}/\Gamma)_0}) \\ &= \sum_\alpha \chi(\mathcal{O}_{E_\alpha}) - \sum_\beta \chi(\mathcal{O}_{C_\beta}) + \sum_\gamma \chi(\mathcal{O}_{P_\gamma}) \\ &= \nu_2 - \nu_1 + \nu_0.\end{aligned}$$

Let N be the number of orbits when Γ acts on the vertices of Σ. Clearly $N = \nu_2$. But each E_α contains $2(q^2 + q + 1)$ double curves, each on two E_α's, so

$$\nu_1 = N(q^2 + q + 1).$$

And each double curve passes through $(q + 1)$ triple points, each on three double curves, so

$$\nu_0 = N\frac{(q^2 + q + 1)(q + 1)}{3}.$$

Thus

$$\begin{aligned}\chi(\mathcal{O}_{(\mathfrak{X}/\Gamma)_\eta}) &= N\left[1 - (q^2 + q + 1) + \frac{(q^2 + q + 1)(q + 1)}{3}\right] \\ &= N\frac{(q - 1)^2(q + 1)}{3}.\end{aligned}$$

Next:

$$\begin{aligned}(c_{1,(\mathfrak{X}/\Gamma)_\eta}{}^2) &= (c_1(\omega_{(\mathfrak{X}/\Gamma)_0})^2) \\ &= \sum_\alpha (\mathrm{res}_{E_\alpha} c_1(\omega_{(\mathfrak{X}/\Gamma)_0})^2) \\ &= \sum_\alpha \Big((c_1(\omega_{E_\alpha}) + \sum_{\beta\neq\alpha} E_\alpha \cap E_\beta)^2\Big)\end{aligned}$$

All E_α's are just $B =$ (the blow-up of $\mathbf{P}_k{}^2$ at all $(q^2 + q + 1)$-rational points). Let $\pi: B \to \mathbf{P}_k{}^2$ be the blow-up map, let h be the divisor class of a line on $\mathbf{P}_k{}^2$, let $e_i \subset B$ be the exceptional divisors and let $l_j \subset B$ be the proper transforms of the lines. Then $c_1(\omega_{E_\alpha}) + \Sigma_{\beta\neq\alpha}(E_\alpha \cap E_\beta)$ corresponds on B to:

$$\begin{aligned} K_B + \sum_i e_i + \sum_j l_j &\equiv (\pi^{-1}(-3h) + \Sigma e_i) \\ &\quad + \left(\sum_i e_i\right) + \sum_j (\pi^{-1}(h) - \text{the } e_i \text{ meeting } l_j) \\ &\equiv \pi^{-1}((q^2+q-2)h) - (q-1)\left(\sum_i e_i\right) \end{aligned}$$

with self-intersection $3(q-1)^2(q+1)$. Thus

$$(c_{1,(\mathfrak{X}/\Gamma)_\eta}{}^2) = 3N(q-1)^2(q+1).$$

By Riemann-Roch,

$$c_{2,(\mathfrak{X}/\Gamma)_\eta} = N(q-1)^2(q+1).$$

To determine the irregularity of $(\mathfrak{X}/\Gamma)_\eta$, we can use the relative Picard scheme $Pic_{\mathfrak{X}/\Gamma}{}^0$: its closed fibre is $Pic_{(\mathfrak{X}/\Gamma)_0}{}^0$, and since $(\mathfrak{X}/\Gamma)_0$ has normal crossings and rational components, this is an algebraic torus. In particular points of finite order l, $p \nmid l$, are dense: these correspond to l-cyclic coverings of $(\mathfrak{X}/\Gamma)_0$ and such coverings lift to $(\mathfrak{X}/\Gamma)_\eta$. Therefore the points of finite order l, $p \nmid l$, of $(Pic_{\mathfrak{X}/\Gamma}{}^0)_0$ lift to points of $(Pic_{\mathfrak{X}/\Gamma}{}^0)_\eta$ and hence $Pic_{\mathfrak{X}/\Gamma}{}^0$ is flat over R. On the other hand, a line bundle on $(\mathfrak{X}/\Gamma)_0$ is a line bundle on $\mathfrak{X}_0$ with Γ action: if it is in Pic^0, it is the trivial line bundle on $\mathfrak{X}_0$ and a Γ-action is just a homomorphism from Γ to $\mathbf{G}_m$. Thus finally, using Kajdan's theorem [4] that $\Gamma/[\Gamma, \Gamma]$ is finite, we deduce

$$\begin{aligned} \text{irregularity of } (\mathfrak{X}/\Gamma)_\eta &= \dim\,(Pic_{\mathfrak{X}/\Gamma}{}^0)_\eta \\ &= \dim\,(Pic_{\mathfrak{X}/\Gamma}{}^0)_0 \\ &= \dim \operatorname{Hom}(\Gamma, \mathbf{G}_m) \\ &= rk_{\mathbf{Z}}\Gamma/[\Gamma, \Gamma] \\ &= 0. \end{aligned}$$

Thus the numbers $h^{p,q}$ of $(\mathfrak{X}/\Gamma)_\eta$ fit into the pattern:

$$\begin{array}{c|ccc} q & M-1 & 0 & 1 \\ & 0 & M & 0 \\ & 1 & 0 & M-1 \\ \hline & & & \to p \end{array} \qquad M = N\frac{(q-1)^2(q+1)}{3}$$

In particular, if $N = 1$, $q = 2$, then $M = 1$ and $(\mathfrak{X}/\Gamma)_\eta$ is a surface of the desired type. In this case, in fact $(\mathfrak{X}/\Gamma)_0$ is one rational surface, $\mathbf{P}^2$ blown up 7 times, crossing itself in 7 rational double curves, themselves crossing in 7 triple points. The confusion arising from trying to draw the result brings vividly to mind Lewis Carroll's comment on the sandy shore—"If seven maids with seven brooms were to sweep it for half a year, do you suppose, the Walrus said, that they could get it clear?"

2. The Example. The example is based on the 7th roots of unity: fix the notation:

$$\zeta = e^{2\pi i/7}$$

$$\lambda = \zeta + \zeta^2 + \zeta^4 = \left(\frac{-1+\sqrt{-7}}{2}\right)$$

$$\bar{\lambda} = \zeta^3 + \zeta^5 + \zeta^6 = \left(\frac{-1-\sqrt{-7}}{2}\right)$$

We have the fields:

$$\begin{array}{ll} \mathbf{Q}(\zeta) & \\ \Big|\ \text{deg } 3 & \Big\} \quad \text{Galois group } \mathbf{Z}/3\mathbf{Z}, \text{ generator } \sigma,\ \sigma(\zeta) = \zeta^2 \\ \mathbf{Q}(\lambda) & \\ \Big|\ \text{deg } 2 & \Big\} \quad \text{Galois group } \mathbf{Z}/2\mathbf{Z}, \text{ generator } z \to \bar{z} \end{array}$$

Note $\mathbf{Q}(\lambda)$ is a UFD, $2 = \lambda \cdot \bar{\lambda}$ is the prime factorization of 2 and $7 =$

$-(\sqrt{-7})^2$ is the prime factorization of 7. We set $\mathbf{Q}(\zeta) = V$ and think of it only as a 3-dimensional vector space over $\mathbf{Q}(\lambda)$. We put the Hermitian form

$$h(x, y) = \mathrm{tr}_{\mathbf{Q}(\zeta)/\mathbf{Q}(\lambda)}(x\bar{y}) = [x\bar{y} + \sigma(x\bar{y}) + \sigma^2(x\bar{y})]$$

on V. Taking $1, \zeta, \zeta^2$ as a basis of V, we find that h has the matrix

$$H = \begin{pmatrix} 3 & \bar{\lambda} & \bar{\lambda} \\ \lambda & 3 & \bar{\lambda} \\ \lambda & \lambda & 3 \end{pmatrix}$$

so that h is positive definite with determinant 7. Note that V contains the lattice $L = \mathbf{Z}[\zeta]$, with basis $1, \zeta, \zeta^2$ over $\mathbf{Z}[\lambda]$. Define

$$\begin{aligned}\Gamma_1 = {} & \mathbf{Q}(\lambda)\text{-linear maps } \gamma: V \to V \text{ which preserve the form } h \\ & \text{and map } L[1/2] \text{ to } L[1/2]\end{aligned}$$

Since 2 splits in $\mathbf{Q}(\lambda)$, the λ-adic completion of $\mathbf{Q}(\lambda)$ is isomorphic to the 2-adic completion $\mathbf{Q}_2$ of $\mathbf{Q}$ (in fact, in $\mathbf{Q}_2$, we may take $\lambda = (\text{unit})\cdot 2$, $\bar{\lambda} = \text{unit}$). So we have a canonical map $V \to (\lambda\text{-adic completion of } V) \cong \mathbf{Q}_2\cdot 1 \oplus \mathbf{Q}_2\cdot\zeta \oplus \mathbf{Q}_2\cdot\zeta^2$ and a canonical homomorphism

$$\Gamma_1 \to \mathrm{GL}(3, \mathbf{Q}_2) \to \mathrm{PGL}(3, \mathbf{Q}_2).$$

From standard results on the theory of arithmetic groups*, the image $\overline{\Gamma_1}$ of Γ_1 is discrete and co-compact. We introduce 3 elements of Γ_1: the first is σ itself; the second is

$$\tau(x) = \zeta\cdot x.$$

Note that $\sigma^3 = e$, $\tau^7 = e$ and $\sigma\tau\sigma^{-1} = \tau^2$, so together σ and τ generate

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*Consider $U(V, h)$ as an algebraic group over $\mathbf{Q}$. Γ_1 is its $\mathbf{Z}[1/2]$-rational points. Since U is compact at the infinite place, Γ_1 is discrete and co-compact in $U(V, h)(\mathbf{Q}_2)$. But over $\mathbf{Q}_2$, $U(V, h) \cong \mathrm{GL}(3)$, so mod scalars Γ_1 defines a discrete co-compact subgroup of $\mathrm{PGL}(3, K)$.

a subgroup $\Gamma_2 \subset \Gamma_1$ of order 21. The third is a map ρ given by

$$\rho(1) = 1$$
$$\rho(\zeta) = \zeta$$
$$\rho(\zeta^2) = \lambda - \frac{\lambda^2}{\bar{\lambda}}\zeta + \frac{\lambda}{\bar{\lambda}}\zeta^2$$

It can be checked easily that $\rho \in \Gamma_1$. It can also be checked that

$$(\rho \cdot \tau)^3 = \text{multiplication by } \lambda/\bar{\lambda}.$$

Note that the scalar matrices in Γ_1 are exactly

$$\pm(\lambda/\bar{\lambda})^k \cdot I_3$$

PROPOSITION. *ρ, σ, τ and $-I$ generate Γ_1. All torsion elements in $\overline{\Gamma_1}$ are conjugate to either $\sigma^i \cdot \tau^j$ or to $(\rho \cdot \tau)^i$ (some $0 \leq i \leq 2$, $0 \leq j \leq 6$).*

Proof. Consider the action of Γ_1 on $\Sigma_0' =$ [the set of free rank $3\mathbf{Z}_2$-submodules of $\mathbf{Q}_2{}^3$]. Let M_0 be the submodule $\mathbf{Z}_2 \cdot 1 \oplus \mathbf{Z}_2 \cdot \zeta \oplus \mathbf{Z}_2 \cdot \zeta^2$ or $\mathbf{Z}_2{}^3$ for short. If an element $\alpha \in \Gamma_1$ maps M_0 to itself, then back in V, α is given by a 3×3 matrix with coefficients in $\mathbf{Z}[\lambda][1/\bar{\lambda}]$. Since α is H-unitary, its coefficients are also in $\mathbf{Z}[\lambda][1/\lambda]$, so α in fact has coefficients in $\mathbf{Z}[\lambda]$ and maps L to itself. But in L, it is easy to see that $\{\pm\zeta^i\}$ are the only elements $x \in L$ with $h(x, x) = 3$. So α permutes them. Then $\pm\tau^i \circ \alpha$ also carries the element $1 \in L$ to itself. Now the equations

$$h(x, x) = 3$$
$$h(1, x) = \bar{\lambda}$$

have only 3 solutions in L: $x = \zeta$, ζ^2 or ζ^4. So $(\pm\tau^i \cdot \alpha)$ carries ζ to ζ, ζ^2 or ζ^4. Then $(\pm\sigma^j \circ \tau^i \circ \alpha)$ fixes 1 *and* ζ and it is easy to check that such a map must be the identity. Thus $\pm\Gamma_2$ is the stabilizer of M_0.

As in the Bruhat-Tits building, call M, $M' \in \Sigma_0'$ adjacent if $M \supset M'$ and $\dim_{\mathbf{Z}/2\mathbf{Z}} M/M' = 2$ or vice versa. Then $\rho(M_0) \subset M_0$ and is adjacent to M_0. Because $M/2M \cong (\mathbf{Z}/2\mathbf{Z})^3$, there are only 7 modules $M' \subset M_0$ adjacent to M_0. One checks easily that these are the modules $\tau^i\rho(M_0)$, $0 \leq i \leq 6$. Thus $(\tau^i\rho)^{\pm 1}(M_0)$ is the set of all $M \in \Sigma_0'$ adjacent to M_0. Since Σ_0' is connected under adjacency, this shows that *all* elements of Σ_0' can be expressed as:

$$(\tau^{i_1}\rho)^{\epsilon_1} \cdot \ldots \cdot (\tau^{i_k}\rho)^{\epsilon_k}(M_0), \qquad 0 \le i_l \le 5, \qquad \epsilon_l = \pm 1.$$

Thus the subgroup of Γ_1 generated by ρ, σ, τ and $-I$ acts as transitively as Γ_1 on Σ_0' and M_0 has the same stabilizer in both groups, so they are equal.

Now let $\alpha \in \Gamma_1$ be torsion in $\overline{\Gamma_1}$. If α is torsion in Γ_1, then α fixes e.g. the module

$$M_1 = \sum_{i=1}^{\mathrm{order}(\alpha)} \alpha^i(M_0)$$

and, if $M_1 = \beta(M_0)$, then $\beta^{-1}\alpha\beta$ fixes M_0. Thus $\beta^{-1}\alpha\beta \in \pm\Gamma_2$ and $\bar{\alpha}$ is conjugate to $\sigma^j \circ \tau^i$, some i, j. In general, we consider $\det \alpha$. Then $|\det \alpha|^2 = 1$ and $\det \alpha \in \mathbf{Z}[\lambda][1/2]$, hence

$$\det \alpha = \pm(\lambda/\bar{\lambda})^i.$$

Replacing α by $\pm(\lambda/\bar{\lambda})^i\alpha^{\pm 1}$, we may assume $\det \alpha = \lambda/\bar{\lambda}$. Then

$$(\lambda/\bar{\lambda})^{-1}\alpha^3$$

has determinant 1 and is torsion in PGL(3), so it is torsion in GL(3). Now consider α and α^3/λ acting on $\mathbf{Q}_2^3$. Since $\mathbf{Z}_2[\alpha, \alpha^3/\lambda]$ is a finite ring over $\mathbf{Z}_2$, there is a free rank $3\mathbf{Z}_2$-module $M \subset \Phi_2{}^3$ such that $\alpha(M) \subset M$, $\alpha^3/\lambda(M) \subset M$. Since $\alpha^3/(\lambda/\bar{\lambda})$ is torsion and $\bar{\lambda}$ is a λ-adic unit, it follows that

$$M \supset \alpha(M) \supset \alpha^2(M) \supset \alpha^3(M) = 2M.$$

As before, replacing α by a conjugate, we can assume $M = M_0$. But now it is easily checked that the 21 maps $\sigma^i \circ \tau^j$ act simply transitively on the flags

$$M_0/2M_0 \supset (2\text{-dim}^l \text{ subspace}) \supset (1 \text{ dim}^l \text{ subspace}).$$

So conjugating α by $\sigma^i \circ \tau^j$, we can assume

$$\alpha(M_0) = (\rho \circ \tau)(M_0)$$

$$\alpha^2(M_0) = (\rho \circ \tau)^2(M_0).$$

Then $(\rho \circ \tau)^{-1} \circ \alpha$ carries M_0 into itself and fixes a flag in $M_0/2M_0$. The former implies that $(\rho \circ \tau)^{-1} \circ \alpha = \pm \sigma^i \circ \tau^j$, some i, j, and the latter implies that $i = j = 0$. Thus in $\overline{\Gamma_1}$, $\alpha = \rho \circ \tau$. Q.E.D.

It remains to choose a suitable subgroup $\Gamma \subset \Gamma_1$ of finite index such that:

a) Γ/scalars is torsion-free
b) Γ acts transitively on Σ_0', (hence Γ/scalars acts transitively on Σ_0, the vertices of the Bruhat-Tits building).

It will then follow from the results of Section 1 that the corresponding surface $(\mathfrak{X}/\Gamma)_\eta$ is a surface of the desired type. To find Γ, it is convenient to use a congruence subgroup for the prime 7. In fact, consider the maps:

$$\begin{aligned} \mathbf{Z}[\lambda, \tfrac{1}{2}] &\to \mathbf{Z}[\lambda, \tfrac{1}{2}]/(\sqrt{-7}) \cong \mathbf{Z}/7\mathbf{Z} \\ \lambda, \bar{\lambda} &\mapsto 3 \\ L[\tfrac{1}{2}] &\to \underbrace{L[\tfrac{1}{2}]/(\sqrt{-7})L[\tfrac{1}{2}]}_{\text{call this } L_0} \cong (\mathbf{Z}/7\mathbf{Z})^3 \end{aligned}$$

The induced form h_0 on L_0 is of rank 1 and has null-space $L_1 \subset L_0$ spanned by $\zeta - 1$, $\zeta^2 - 1$. Taking 1, $\zeta - 1$, $(\zeta - 1)^2$ as a basis of L_0, it is easy to check that mod 7:

$$\sigma \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 4 \end{pmatrix}, \qquad \tau \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\rho \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \rho \circ \tau \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 1 & 5 & 4 \\ 0 & 1 & 1 \end{pmatrix}$$

In particular, considering the action of Γ_1 on L_1, we get a homomorphism

$$\pi:\Gamma_1 \to GL(2, Z/7\mathbf{Z}) \cap \{X \mid \det X = \pm 1\} \underset{\text{def}}{=} G.$$

The group G on the right has order $2^5 \cdot 3 \cdot 7$. Let H be a 2-Sylow subgroup and define $\Gamma = \pi^{-1}(H)$. Since all 21 elements $\sigma^i \tau^j$ and all 3 elements $(\rho \circ \tau)^i$ except e have non-zero images in G of orders 3 or 7, Γ is torsion-free. As the full group Γ_1 is set-theoretically $\Gamma \times \Gamma_2$, Γ acts transitively on $\Sigma_0{}'$. This completes the construction.

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