2d String Theory and the Non-Perturbative $c = 1$ Matrix Model

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Abstract

A valuable testing ground for exploring new features of string theory and quantum gravity is the duality between $c = 1$ string theory and the $c = 1$ matrix quantum mechanics, which has been explored for over 30 years. In this thesis, we use efficient numerical techniques to evaluate Liouville correlation functions and to compute scattering amplitudes of closed strings in string perturbation theory, resolving several previous puzzles in the perturbative duality dictionary.

In addition to this, we present a worldsheet formalism for computing all non-perturbative corrections to these amplitudes, which require including disconnected worldsheet diagrams with ZZ-instanton boundary conditions. By matching these contributions against the dual matrix model, we propose the non-perturbative completion of $c = 1$ string theory.

We further extend the duality by introducing new degrees of freedom known as long strings, which from the worldsheet description are given by open strings on FZZT branes in a limit where the FZZT branes decouple and the open strings are infinitely stretched. The first few tree-level scattering amplitudes of these objects are computed, and show an impressive agreement with the corresponding amplitudes computed in the dual $U(N)$ matrix quantum mechanics, where long strings are given by states in non-singlet sectors of the $U(N)$ symmetry group.
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Chapter 2 is partly based on previous results in the literature, while the discussion of the non-perturbative matrix quantum mechanics is based on the paper

*ZZ Instantons and the Non-Perturbative Dual of c = 1 String Theory*
B. Balthazar, V. A. Rodriguez and X. Yin
arXiv:1907.07688 [hep-th]

Chapter 3 is an edited version of the paper

*The c = 1 string theory S-matrix revisited*
B. Balthazar, V. A. Rodriguez and X. Yin

Chapter 4 is an edited version of the two papers

*ZZ Instantons and the Non-Perturbative Dual of c = 1 String Theory*
B. Balthazar, V. A. Rodriguez and X. Yin
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*Multi-Instanton Calculus in c = 1 String Theory*
B. Balthazar, V. A. Rodriguez and X. Yin
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*Long String Scattering in c = 1 String Theory*
B. Balthazar, V. A. Rodriguez and X. Yin
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In our quest for finding a quantum theory of gravity, $c = 1$ string theory is one of the most valuable toy models at our disposal. This theory was studied extensively in the early 90s (for previous reviews, see [1, 2, 3, 4, 5]), when it was first understood that it is dual to a $U(N)$-gauged matrix quantum mechanics, called the $c = 1$ matrix quantum mechanics [6, 7, 8, 9, 10]. Over the years, it led to new insights into the structure of string theory and quantum gravity, such as the emergence of extra dimensions from a large-$N$ limit [11, 12], the existence of non-perturbative degrees of freedom in string theory [13, 14], and the subject of tachyon
condensation [15, 16, 17, 18].

\[ \phi = -\infty \]

**Figure 1.1**: The Liouville CFT describes a spatial background with an asymptotic region at \( \phi = -\infty \), where the effective string coupling vanishes. Interactions occur near the “Liouville wall”, where the effective string coupling is finite.

The worldsheet conformal field theory (CFT) of \( c = 1 \) string theory consists of the Liouville CFT, the timelike free boson \( X^0 \), and the usual \( b, c \) ghosts. The Liouville field \( \phi \) parameterizes a spatial direction in target space, with a non-trivial background. The effective string coupling is given by \( \sim g_s e^{\phi} \), with the region of large \( \phi \) being “cut-off” by the exponential tachyon potential, see Figure 1.1. Thus, this is a bosonic string theory in 1+1d, with an asymptotic region at \( \phi = -\infty \) and a continuum of massless closed string states.

The dual theory is given by a \( U(N) \)-gauged matrix quantum mechanics, which can be reformulated in terms of a system of \( N \) free fermions moving in an inverted quadratic potential, so that their single-particle Hamiltonian is \( H_i = \frac{p_i^2}{2} - \frac{\lambda_i^2}{2} \). Semiclassically, \( c = 1 \) string theory is equivalent to this matrix model in the double-scaling limit, where \( N \to \infty \) with the chemical potential \( -\mu < 0 \) held fixed. In this limit, the Fermi sea occupies the region \( E \leq -\mu \) and \( \lambda > 0 \), shown in Figure 1.2. Closed string excitations are dual to fluctuations of the Fermi surface, with the inverted quadratic potential playing the role of the “Liouville wall”.

One of the initial goals was to find the dual matrix quantum mechanics that completes \( c = 1 \)
string theory non-perturbatively. However, it was quickly realized that the description of the matrix quantum mechanics given so far is only valid semiclassically, and non perturbatively the fermions can tunnel to the “other side” of the inverted quadratic potential (the region $\lambda < 0$). Thus, while there are infinitely many matrix models that reproduce the perturbative scattering amplitudes of $c = 1$ string theory, they produce different non-perturbative corrections to these scattering amplitudes.

In addition to this, the Liouville CFT was not completely solved at the time, and several puzzles in the perturbative duality were not fully understood. For example, the tree-level $2 \to 2$ scattering amplitude of closed strings in the dual matrix model suffered from discontinuities in the physical regime of real energies [19, 20, 21]. On the other hand, the worldsheet scattering amplitudes of closed strings were computed by analytically continuing the closed string energies to be on “resonance”, in which case the relevant Liouville correlation function can be computed via Coulomb gas methods [22, 23]. This trick suggested that the tree-level $2 \to 2$ worldsheet scattering amplitude was exactly zero.

The structure constants of the Liouville CFT were found shortly after [24, 25], and also its
boundary structure constants [26, 27, 28, 29]. This led to a revival of the subject, where the matrix description was conjectured to arise from the low-energy degrees of freedom of a stack of rolling D0-branes [17, 18]. Furthermore, it was proposed that a single eigenvalue of the matrix quantum mechanics is (classically) dual to a (1,1) ZZ-brane in $c = 1$ string theory [18].

In this thesis, we revisit the duality between $c = 1$ string theory and its dual matrix model, focusing on the recent developments in [30, 31, 32, 33]. We begin by studying the perturbative dictionary of $c = 1$ string theory from the worldsheet formalism, motivated by the development of efficient numerical techniques to compute Liouville correlation functions [30]. In particular, we use the exact Liouville structure constants as well as efficient recursion relations for computing Virasoro conformal blocks numerically [34, 35, 36, 37], while the worldsheet moduli is integrated numerically too.

With these tools, we find striking agreement between the worldsheet and matrix model scattering amplitudes of closed strings at tree-level, for up to 4 closed strings. In [30], it is shown that the 1-loop $1 \to 1$ scattering also agrees between the two sides. Along the way, we resolve many of the confusions in the perturbative dictionary of $c = 1$ string theory. For example, the discontinuities of the scattering amplitudes are shown to be due to ambiguities in the definition of asymptotic states of massless particles in 1+1d, and they are calculated exactly using worldsheet unitarity cut methods.

In [14, 38], a worldsheet formalism for computing the leading non-perturbative contribution to closed string scattering amplitudes in critical string theory was proposed. More explicitly, they come from disconnected worldsheet diagrams with Dirichlet boundary conditions in all spacetime directions. We apply this formalism to the case of $c = 1$ string theory [32], where the
relevant boundary state has Dirichlet boundary condition along $X^0$ (with Euclidean Dirichlet label $x^E$) and $(m, 1)$ ZZ boundary condition in the Liouville CFT [26].

We further extend the formalism in $c = 1$ string theory to all non-perturbative orders [33]. In particular, we include multi-instanton configurations with $\ell \in \mathbb{Z}_{\geq 1}$ ZZ-instantons, given by ZZ-instantons of types $(m_i, 1)$ and at Euclidean times $x^E_i$, for $i = 1, ..., \ell$. Their contribution is taken into account by including disconnected worldsheet diagrams with multi-ZZ-instanton boundary conditions. There is a non-trivial moduli space measure coming from exponentiated empty diagrams with boundaries on different ZZ-instantons, which has poles at certain points in the moduli, and a choice of integration countour that avoids these poles is prescribed.

We find that the non-perturbative contributions to closed string scattering amplitudes computed this way are precisely reproduced\(^1\) by a unique matrix quantum mechanics, where the closed string vacuum is dual to the matrix model state with no incoming flux from the “other side”. Thus, by including non-perturbative effects mediated by ZZ-instantons, we propose the exact dual of $c = 1$ string theory.

Another driving force in the exploration of the duality between $c = 1$ string theory and the dual matrix model is the connection between $c = 1$ string theory and the two-dimensional black hole [39, 40]. In [41], it was argued that the $c = 1$ string theory background can be deformed to the two-dimensional black hole by condensing a large number of long strings. These are defined as open strings on FZZT branes of $c = 1$ string theory in a limit where the strings

---

\(^1\)There is a small amount of ambiguity in the worldsheet computation coming from empty diagrams with boundary conditions on the same ZZ-instanton, which suffer from IR divergences. Since we do not have a consistent way of regularizing these diagrams, we simply replace them by constants in the worldsheet calculation, and later fix them by matching against the corresponding scattering amplitudes computed from the dual matrix model.
are infinitely stretched. It was further proposed in [41] that long strings are dual to states in non-singlet sectors of the dual $U(N)$ ungauged matrix quantum mechanics, and therefore the two-dimensional black hole microstates are related to states in non-singlet sectors of the dual matrix model.

One of our results is the computation of tree-level scattering amplitudes of long strings in $c = 1$ string theory and from the dual matrix model description [31]. From the worldsheet formalism, Liouville correlation functions are calculated from the boundary structure constants of Liouville CFT with FZZT boundary conditions, together with recursion methods for computing Virasoro conformal blocks [42, 43]. We find an exact agreement between these scattering amplitudes and those of states in non-singlet sectors of the dual $U(N)$ gauged matrix quantum mechanics, thereby providing convincing evidence for the conjectured duality of [41]. This work is a stepping stone in obtaining a quantum mechanical description of black hole microstates.

The outline for this thesis is as follows. The $c = 1$ matrix quantum mechanics is described in chapter 2, where we review the formulation of the theory in terms of free-fermions and propose the non-perturbative dual of $c = 1$ string theory. From this proposal, we calculate closed string scattering amplitudes to all perturbative and non-perturbative orders. These scattering amplitudes are reproduced from the worldsheet side in the following two chapters, starting with the computation of perturbative scattering amplitudes in chapter 3, where we use numerical and analytical techniques to explicitly compute these amplitudes. In chapter 4, we first review ZZ-branes and ZZ-instantons in $c = 1$ string theory, before discussing the worldsheet formalism that computes non-perturbative corrections to closed string scattering amplitudes, via ZZ-instanton effects. Both the perturbative and non-perturbative worldsheet scattering amplitudes computed
this way are in exact agreement with the matrix model scattering amplitudes, thereby justifying the proposal for the non-perturbative dual of $c = 1$ string theory.

Finally, in chapter 5 we extend the duality dictionary to include long string degrees of freedom. In particular, we define FZZT branes and the long string limit, and compute the tree-level long→long+closed and long+long→long+long scattering amplitudes from the worldsheet formalism. After that, we describe the non-singlet sectors of the dual matrix model, and find an exact agreement with the tree-level scattering amplitudes of long strings calculated from the matrix model. Chapter 6 is a brief concluding section with some open questions.

Technical details and some further results are summarized in Appendices A-F in order to not distract from the main text. In Appendix A, we discuss close relatives to the Barnes G-function that appear in the structure constants of Liouville CFT. We briefly review Zamolodchikov’s recursion relation for the sphere 4-point Virasoro conformal blocks in Appendix B. Appendix C contains some of the technical and numerical details that enter into the calculation of worldsheet scattering amplitudes. In Appendix D we derive the reflection phase of the long string wavefunction from the worldsheet and matrix model descriptions [41, 44]. Appendix E contains a derivation of the $1 \to 2$ scattering amplitude of closed strings from the matrix model description. Finally, in Appendix F we describe the action of collective excitations of the Fermi sea using an alternative parametrization that does not require regularization.
The $c = 1$ Matrix Quantum Mechanics

In this chapter we describe the non-perturbative completion of $c = 1$ string theory, which we call the $c = 1$ matrix quantum mechanics. We begin by reformulating the Hamiltonian of this quantum mechanical system in terms of a system of free fermions. Then we provide a semiclassical description of their collective dynamics, using the formalism of collective field theory, from which we compute the tree-level $1 \rightarrow 2$ scattering amplitude of closed strings. Finally, we move beyond the semiclassical analysis and propose the quantum state that is dual to the closed string vacuum (2.36). Using this proposal, we compute scattering amplitudes of
closed strings to any perturbative and non-perturbative orders (2.50), by following the formalism of [20, 45].

2.1 The Hamiltonian

The matrix quantum mechanics is described in terms of a $N \times N$ Hermitian matrix $X$, with dynamics governed by the Hamiltonian

$$H = \text{Tr} \left( \frac{P^2}{2} + V(X) \right),$$

where $P_{ij} \equiv -i \partial / \partial X_{ji}$ is the canonical momenta conjugate to $X$, and the potential $V(X)$ is for now unspecified. The system has a $U(N)$ global symmetry under which $X$ transforms in the adjoint representation.

The matrix $X$ can be decomposed as

$$X = \Omega^{-1} \Lambda \Omega,$$

where $\Omega$ is a unitary matrix and $\Lambda = \text{diag} (\lambda_1, ..., \lambda_N)$. In terms of the variables $\Lambda, \Omega$, the wavefunction $\hat{\Psi}(\Omega, \Lambda) \equiv \Psi(X)$ has a $S_N \times U(1)^N$ gauge redundancy, which acts as

$$U(1)^N : \quad \Lambda \rightarrow \Lambda, \quad \Omega \rightarrow T^{-1} \Omega, \quad T = \text{diag} \left( e^{i \theta_1}, ..., e^{i \theta_N} \right),$$

$$S_N : \quad \Lambda \rightarrow W_{ij}^{-1} \Lambda W_{ij}, \quad \Omega \rightarrow W_{ij}^{-1} \Omega, \quad i \neq j \quad (2.3)$$
where $W_{ij}$ is defined by

\[
(W_{ij})_{k\ell} = \begin{cases} 
\delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk} & k = i \text{ or } k = j \\
\delta_{k\ell} & \text{otherwise}
\end{cases}
\]  

(2.4)

In particular, the action of $W_{ij}$ on $\Lambda$ exchanges $\lambda_i$ and $\lambda_j$.

The Hamiltonian $\hat{H}$ acting on $\hat{\Psi}(\Lambda, \Omega)$ is given by

\[
\hat{H} = \sum_{i=1}^{N} \left( -\frac{1}{2} \frac{\partial^2}{\partial \lambda_i^2} + V(\lambda_i) \right) + \frac{1}{2} \sum_{i\neq j} \left( -\frac{1}{\lambda_i - \lambda_j} \frac{\partial}{\partial \lambda_i} + \frac{R_{ij}R_{ji}}{(\lambda_i - \lambda_j)^2} \right),
\]  

(2.5)

where $R_{ij} \equiv \sum_{m=1}^{N} \Omega_{\ell m} \frac{\partial}{\partial \Omega_{k\ell m}}$ is the $U(N)$ symmetry generator. To obtain (2.5), note that it follows from (2.2) that the $i$-th row of $\Omega$ is a left-eigenvector of $X$, while the $k$-th column of $\Omega^\dagger$ is a right-eigenvector of $X$,

\[
\Omega_{ij}X_{jk} = \lambda_i \Omega_{ik}
\]  

(2.6)

\[
X_{ij}\Omega^\dagger_{jk} = \lambda_k \Omega^\dagger_{ik}.
\]

Varying both sides of the first equation, and acting on the resulting equation with $\Omega^\dagger_{k\ell}$ from the right, we find

\[
\frac{\partial \lambda_i}{\partial X_{mn}} = \Omega_{im}(\Omega^\dagger)_{ni}, \quad \frac{\partial \Omega_{ij}}{\partial X_{mn}} = \sum_{i\neq k} \frac{\Omega_{im}(\Omega^\dagger)_{nk}}{\lambda_i - \lambda_k} \Omega_{kj},
\]  

(2.7)

from which (2.5) follows.

The matrix model Hilbert space can be decomposed into irreducible representations $\mathcal{R}$ of the $U(N)$ symmetry group, subject to the $S_N \ltimes U(1)^N$ gauge redundancy in (2.3). The $U(1)^N$
restricts the states in $\mathcal{R}$ to the subspace of zero-weight states $V^0_\mathcal{R}$, while the $S_N$ further constrains the Hilbert space to the subsector invariant under its action. Thus, the Hilbert space in an irreducible representation $\mathcal{R}$ is written as

$$
\hat{\mathcal{H}}_\mathcal{R} = \left[ \tilde{L}^2(\mathbb{R}^N) \otimes V^0_\mathcal{R} \right]^{S_N},
$$

where $\tilde{L}^2(\mathbb{R}^N)$ is the space of square-integrable functions on $\mathbb{R}^N$ with respect to the measure

$$
d\mu = d\lambda_1...d\lambda_N\Delta^2,
$$

where

$$
\Delta \equiv \prod_{i<j}^N(\lambda_i - \lambda_j).
$$

This measure is inherited from the $U(N)$-invariant measure of $\Psi(X)$ after decomposing $X$ as in (2.2).

In fact, we can eliminate the term linear in $\lambda_i$ derivatives in (2.5) by the similarity transformation $\hat{H} = \Delta^{-1}H'\Delta$, so that the Hamiltonian $H'$ acts on the wavefunction $\Psi'(\Lambda, \Omega) \equiv \Delta \hat{\Psi}(\Lambda, \Omega)$ as

$$
H' = \sum_{i=1}^N \left( -\frac{1}{2} \frac{\partial^2}{\partial \lambda_i^2} + V(\lambda_i) \right) + \frac{1}{2} \sum_{i \neq j} R_{ij}R_{ji} \frac{2}{(\lambda_i - \lambda_j)^2}.
$$

The Hilbert space after the similarity transformation is given by

$$
\mathcal{H}'_\mathcal{R} = \left[ L^2(\mathbb{R}^N) \otimes V^0_\mathcal{R} \right]^{S'_N},
$$

where $L^2(\mathbb{R}^N)$ are square integrable functions on $\mathbb{R}^N$ with respect to the flat measure, and the action of $S'_N$ on the wavefunction $\Psi'(\Lambda, \Omega)$ is given by (2.3) together with an extra minus sign whenever two eigenvalues are exchanged.
2.2 The Singlet Sector and Collective Field Theory

We first discuss the gauged version of the duality in the semiclassical limit, in which case only the singlet sector of the \( U(N) \) symmetry contributes. Non-singlet sectors are discussed in chapter 5.

In the singlet sector, the wavefunction \( \Psi'(\Lambda, \Omega) \) is a function of \( \lambda_1, \ldots, \lambda_N \) only, and it is antisymmetric under swapping two eigenvalues. The eigenvalues describe a system of \( N \) non-relativistic non-interacting fermions, where the dynamics of a single fermion is governed by the Hamiltonian

\[
H_i = \frac{p_i^2}{2} + V(\lambda_i),
\]

where \( p_i \) is the canonical momentum conjugate to \( \lambda_i \).

The large-\( N \) limit in which this matrix model is dual to \( c = 1 \) string theory is described as follows. Consider the system at a fixed chemical potential \( -\mu \), with the potential\(^1\)

\[
V(\lambda) = -\frac{\lambda^2}{2} + g\frac{\lambda^4}{4!}.
\]

Due to Pauli’s exclusion principle, in the ground state fermions occupy energy levels up to the Fermi energy \( -\mu \). Note that the parameters \( N, g, \) and \( \mu \) are not mutually independent. The large-\( N \) limit of interest corresponds to \( N \to \infty \) with \( \mu \) held fixed, so that \( g \to 0 \). This “double-scaling limit” shown in Figure 2.1 essentially zooms near the \( \lambda = 0 \) region of \( V(\lambda) \),

\(^1\)The discussion is analogous for any other potential \( V(\lambda) \) provided that it is bounded either at positive or negative \( \lambda \), where we ignore for now instability issues due to tunneling effects since we are working in the semiclassical approximation.
where the inverted quadratic potential dominates. In the double-scaling limit, the potential is exactly given by \( V(\lambda) = -\lambda^2/2 \), and the semiclassical dual of the closed string vacuum consists of fermions filling the region \( \lambda > 0 \), up to the energy \( E = -\mu \).

![Semiclassical description of the free fermion Hamiltonian (2.12) before (left) and after (right) the double-scaling limit is taken.](image)

2.2.1 COLLECTIVE FIELD THEORY

The collective dynamics of the fermions is conveniently captured by a phase space description, where the Fermi sea consists of the region \( p^2 - \lambda^2 \leq -2\mu \). Closed strings are dual to low energy fluctuations of the Fermi surface, which come in from the asymptotic region of large \( \lambda \) by time-evolution under the Hamiltonian (2.12). The interacting region where fluctuations scatter corresponds to the tip of the Fermi sea at \( \lambda = \sqrt{2\mu} \), after which they move back to \( \lambda = \infty \), as shown in Figure 2.2.

To describe the collective dynamics of the system, it is convenient to use the formalism of collective field theory. Let \( p = p_\pm(\lambda) \) be the upper and lower branches of the Fermi surface as in Figure 2.2, which in the ground state are given by \( p_\pm^{(0)} = \pm \sqrt{\lambda^2 - 2\mu} \). Define the fermion
density by

\[ \rho(\lambda) \equiv \sum_{i=1}^{N} \delta(\lambda - \lambda_i) = \frac{1}{2\pi} (p_+ (\lambda) - p_- (\lambda)). \quad (2.14) \]

Fluctuations of the fermion density are characterized by the collective field \( \eta \),

\[ \rho(\lambda) = \rho^{(0)}(\lambda) + \frac{1}{\sqrt{\pi}} \partial_{\lambda} \eta(\lambda), \quad (2.15) \]

where \( \rho^{(0)}(\lambda) \equiv \frac{1}{\pi} \sqrt{\lambda^2 - 2\mu} \) is the ground state fermion density. Similarly, we define the fermion momentum density

\[ \Pi_p (\lambda) \equiv \sum_{i=1}^{N} p_i \delta(\lambda - \lambda_i) = \frac{p_+^2 (\lambda) - p_-^2 (\lambda)}{4\pi}, \quad (2.16) \]

which satisfies the (classical) commutation relation

\[ \{ \rho(\lambda), \Pi_p (\lambda') \} = \rho(\lambda') \frac{d}{d\lambda} \delta(\lambda - \lambda'). \quad (2.17) \]

Using (2.17) and (2.14), we find that the field \( \Pi_\lambda (\lambda) \) satisfying the (classical) canonical commu-
The Hamiltonian can be written in terms of the collective fields as follows:

\[
H = \sum_{i=1}^{N} \frac{P_i^2}{2} - \frac{\lambda_i^2}{2} + \mu N
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} dp \int_{\sqrt{2\mu} \lambda}^{\infty} d\lambda \left[ \mu \left( p_+ - p_- \right) + \frac{p_+^3}{6} - \frac{\lambda^2}{2} \right]
\]

\[
= \int_{\sqrt{2\mu} \lambda}^{\infty} d\lambda \left[ \frac{1}{2} \sqrt{\lambda^2 - 2\mu(\Pi_\lambda + (\partial_\lambda \eta)^2)} + \frac{\sqrt{\pi}}{2} (\Pi_\lambda)^2 \partial_\lambda \eta + \frac{\sqrt{\pi}}{6} (\partial_\lambda \eta)^3 \right]
\]

\[
= \int_{0}^{\infty} d\tau \left[ \frac{1}{2} (\Pi_\tau^2 + (\partial_\tau \eta)^2) + \frac{\sqrt{\pi}}{12\mu \sinh^2(\tau)} (3(\Pi_\tau)^2 \partial_\tau \eta + (\partial_\tau \eta)^3) \right],
\]

where in the last line we introduced the \( \tau \) coordinate, defined by the relation \( \lambda = \sqrt{2\mu} \cosh \tau \), and the field \( \Pi_\tau(\tau) \equiv \sqrt{2\mu} \sinh \tau \Pi_\lambda(\lambda(\tau)) \), which satisfies canonical commutation relations with \( \eta(\tau) \). At \( \tau = 0 \) we impose Dirichlet boundary conditions on the field \( \eta(\tau) \), which are consistent with the closed string scattering amplitudes described below.

The collective field Hamiltonian (2.19) describes a massless scalar field with a position dependent coupling. The asymptotic region corresponds to large \( \tau \), where the interaction coupling is exponentially suppressed. As we will see, this is very similar to \( c = 1 \) string theory if we identify the string coupling \( g_s \) with \( \mu^{-1} \), and the \( \tau \) coordinate with the Liouville field \( \phi \), which parameterizes the spatial direction, see the discussion in section 3.2. While the precise\(^2\) relation

\(^2\)In principle, the relation between \( \mu \) and \( g_s \) can be renormalized at higher orders. At the perturbative level, the lowest amplitude for which this would enter is the 1-loop 1 \( \rightarrow \) 2 amplitude, which has not been
between $\mu$ and $g_s$ is given by (3.38), the identification of $\tau$ and $\phi$ is only valid in the asymptotic region, and more generally the map between the two coordinates is non-local.

The $\tau = 0$ divergence of the last term in (2.19) is a consequence of the unnatural $\lambda$ parameterization of the Fermi sea, in which the tip of the Fermi sea is held fixed. Instead we can parameterize the Fermi surface in terms of the canonical momentum $p$, and introduce an IR cutoff at large $\lambda$. This is discussed in Appendix F, where it is shown that there are only right-moving massless modes with a finite cubic interaction in the $p$-parameterization. However, this makes the non-singlet sectors of the matrix model to be discussed in chapter 5 rather cumbersome, so we will stick to the $\lambda$ parameterization for now and regularize the divergent integrals as follows [46, 47]. We introduce a UV cutoff in $\lambda$ coordinates, so that we cutoff the $\lambda$ integral at $\lambda = \sqrt{2\mu} + \epsilon$. This translates to a UV cutoff in $\tau$ coordinates at $\tau = \delta$ for $\delta = \sqrt{\epsilon(\mu/2)^{-\frac{1}{4}}}$. Any divergences in $\epsilon$ that scale with $\mu$ differently from the finite term are dropped, as they are cancelled by an appropriate counter term.

From the point of view of string theory, the action obtained from the Hamiltonian (2.19) should be regarded as the string field action for $c = 1$ string theory with a suitable gauge choice for the string fields. This choice removes all the off-shell modes but the tachyon, and simplifies to a quantum field theory for a single field $\eta$. However, it is not known how to reproduce (2.19) starting from the string field theory formalism.

---

explicitly computed.
2.2.2 Tree-level $1 \rightarrow 2$ Scattering Amplitude

The collective fields can be decomposed in modes as follows,

$$\eta(\tau) = \int_0^\infty \frac{dp}{\sqrt{\pi}} \frac{1}{p} (a_p + a_p^\dagger) \sin(p\tau),$$
$$\Pi(\tau) = -i \int_0^\infty \frac{dp}{\sqrt{\pi}} (a_p - a_p^\dagger) \sin(p\tau),$$

where $a_p^\dagger, a_p$ are creation/annihilation operators for the fluctuations of the fermion density, satisfying $[a_p, a_{p'}^\dagger] = p\delta(p - p')$. The single-particle states are given by $|\omega\rangle = a^\dagger_\omega |0\rangle$, and the tree-level $1 \rightarrow 2$ scattering amplitude is obtained in the Born approximation as

$$S^{\text{pert},(0)}_{1\rightarrow2}(\omega; \omega_1, \omega_2) = \delta(\omega - \omega_1 - \omega_2) A^{\text{pert},(0)}_{1\rightarrow2}(\omega_1, \omega_2),$$
$$A^{\text{pert},(0)}_{1\rightarrow2}(\omega_1, \omega_2) = -2\pi i \langle \omega_1, \omega_2 | V_I | \omega \rangle$$

where $V_I$ is the cubic part of the Hamiltonian (2.19). The $\tau$ integral gives

$$\int_\delta^\infty \frac{d\tau}{\sinh^2 \tau} = -1 + \frac{1}{\delta} = -1 + \frac{\mu^\dagger}{2\sqrt{\epsilon}},$$

so that following the regularization described above, we find the scattering amplitude

$$A^{\text{pert},(0)}_{1\rightarrow2}(\omega_1, \omega_2) = i\omega_1 \omega_2,$$
where $\omega = \omega_1 + \omega_2$. In Appendix F, we reproduce this scattering amplitude using the $p$-parameterization of the Fermi sea, where no regularization is required.

The computation of scattering amplitudes in the collective field formalism can be carried out to higher orders, but for loop amplitudes a tadpole term has to be included [46, 47]. Another curious aspect noted in these works is that the characteristic large order behaviour of string perturbation theory, which grows as $(2L)!g_s^{2L}$ for large genus $L$, is reproduced from the quantum field theory (2.19) by the integrals over the (non-conserved) loop-momenta, which is of order $(2L)!$ for an amplitude with $L$ loops.

2.3 The Closed String Vacuum and Scattering Amplitudes from Second Quantization

The collective field theory formalism computes scattering amplitudes of closed strings in a perturbative expansion in $\mu^{-1}$ that quickly becomes practically difficult due to the large number of Feynman graphs. Furthermore, fermions in the inverted quadratic potential can tunnel to the “other” side, so that the wavefunctions are not localized in the region $\lambda > 0$. This means that the semiclassical description of the ground state given so far, in which only the region $\lambda > 0$ is occupied, is ill-defined non-perturbatively.

The goal of this section is to describe an alternative approach to calculating scattering amplitudes from the matrix model, which explores the exact solution of the single-particle Hamiltonian [48, 20]. In order to compute closed string scattering amplitudes, we will have to specify the state dual to the closed string vacuum. We will do so in (2.36), which will be justified in chapter 4 when we compute non-perturbative corrections to the scattering amplitudes of closed
strings from the worldsheet description, and match against those in the matrix model with the choice of closed string vacuum (2.36).

2.3.1 SINGLE-FERMION WAVEFUNCTIONS

The even and odd wavefunctions of the single-fermion Hamiltonian (2.12) at a given energy $E$ are given by

\[
\begin{align*}
\psi^+_E(\lambda) &= \frac{e^{-i\frac{\pi}{2}}}{2^{\frac{1}{4}}\pi} e^{\frac{\pi E}{4}} \left| \Gamma \left( \frac{1}{4} - i \frac{E}{2} \right) \right| \frac{1}{\sqrt{|\lambda|}} \left| M_{-i\frac{E}{2}, -\frac{1}{4}}(i\lambda^2) \right|, \\
\psi^-_E(\lambda) &= \frac{e^{-i\frac{3\pi}{4}}}{2^{\frac{1}{4}}\pi} e^{\frac{3\pi E}{4}} \left| \Gamma \left( \frac{3}{4} - i \frac{E}{2} \right) \right| \frac{\lambda}{|\lambda|^2} \left| M_{-i\frac{E}{2}, -\frac{1}{4}}(i\lambda^2) \right|,
\end{align*}
\]

(2.24)

where $M_{\kappa,\mu}(z)$ is a Whittaker function, which can be defined in terms of confluent Hypergeometric functions. The wavefunctions $\psi^\pm_E(\lambda)$ are normalized as

\[
\begin{align*}
\int_{-\infty}^{\infty} d\lambda \left( \psi^+_E(\lambda)\psi^+_E(\lambda) + \psi^-_E(\lambda)\psi^-_E(\lambda) \right) &= \delta(E_1 - E_2), \\
\int_{-\infty}^{\infty} d\lambda \psi^+_E(\lambda)\psi^-_E(\lambda) &= 0, \\
\int_{-\infty}^{\infty} dE \left( \psi^+_E(\lambda_1)\psi^+_E(\lambda_2) + \psi^-_E(\lambda_1)\psi^-_E(\lambda_2) \right) &= \delta(\lambda_1 - \lambda_2).
\end{align*}
\]

(2.25)
Alternatively, we can use the basis of in-eigenstates $|E\rangle_R^\text{in}$, $|E\rangle_L^\text{in}$, with wavefunctions $f^\text{in}_{E,R}(\lambda)$, $f^\text{in}_{E,L}(\lambda)$ given by

$$f^\text{in}_{E,L}(\lambda) \equiv \frac{\mu^E e^{i\Phi(E)}}{\sqrt{2} \left(1 + e^{-2\pi E}\right)^{\frac{1}{4}}} \left[ \left( \sqrt{k(E)} - i \frac{1}{\sqrt{k(E)}} \right) \psi^+_E(\lambda) + \left( \sqrt{k(E)} + i \frac{1}{\sqrt{k(E)}} \right) \psi^-_E(\lambda) \right],$$

$$f^\text{in}_{E,R}(\lambda) \equiv \frac{\mu^E e^{i\Phi(E)}}{\sqrt{2} \left(1 + e^{-2\pi E}\right)^{\frac{1}{4}}} \left[ \left( \sqrt{k(E)} - i \frac{1}{\sqrt{k(E)}} \right) \psi^+_E(\lambda) - \left( \sqrt{k(E)} + i \frac{1}{\sqrt{k(E)}} \right) \psi^-_E(\lambda) \right],$$

(2.26)

where

$$k(E) \equiv \sqrt{1 + e^{-2\pi E}} - e^{-\pi E},$$

$$\Phi(E) \equiv \frac{\pi}{4} + \frac{i}{4} \ln \left( \frac{\Gamma\left(\frac{1}{2} + iE\right)}{\Gamma\left(\frac{1}{2} - iE\right)} \right).$$

(2.27)

The in-basis is normalized so that

$$\langle E'|E\rangle_R^\text{in} = \langle E'|E\rangle_L^\text{in} = \delta(E - E'), \quad \langle E'|E'\rangle_L^\text{in} = 0.$$

(2.28)

**Figure 2.3**: In-basis wavefunctions.

The wavefunction $f^\text{in}_{E,R}(\lambda)$ has no incoming flux from the region $\lambda \ll 0$, while $f^\text{in}_{E,L}(\lambda)$ has
no incoming flux from the region $\lambda \gg 0$, as shown in Figure 2.3. In the asymptotic region, we have

\[
\begin{align*}
 f_{E,R}(\lambda) \overset{\lambda \gg 0}{=} & \frac{1}{\sqrt{2\pi |\lambda|}} \exp \left( -i \frac{\lambda^2}{2} - iE \log \left( \sqrt{2\mu} \right) \right) + R(E) \exp \left( i \frac{\lambda^2}{2} + iE \log \left( \sqrt{2\mu} \right) \right) , \\
 f_{E,R}(\lambda) \overset{\lambda \ll 0}{=} & \frac{1}{\sqrt{2\pi |\lambda|}} T(E) \exp \left( i \frac{\lambda^2}{2} + iE \log \left( \sqrt{2\mu} \right) \right) , \\
 f_{E,L}(\lambda) \overset{\lambda \gg 0}{=} & \frac{1}{\sqrt{2\pi |\lambda|}} T(E) \exp \left( i \frac{\lambda^2}{2} + iE \log \left( \sqrt{2\mu} \right) \right) , \\
 f_{E,L}(\lambda) \overset{\lambda \ll 0}{=} & \frac{1}{\sqrt{2\pi |\lambda|}} \exp \left( -i \frac{\lambda^2}{2} - iE \log \left( \sqrt{2\mu} \right) \right) + R(E) \exp \left( i \frac{\lambda^2}{2} + iE \log \left( \sqrt{2\mu} \right) \right) ,
\end{align*}
\]

(2.29)

where the reflection and transmission coefficients are

\[
\begin{align*}
 R(E) \equiv & \mu e^{i\Phi(E)} \frac{k(E) - \frac{1}{k(E)}}{k(E) + \frac{1}{k(E)}} e^{2i\Phi(E)} = i\mu e^{i\Phi(E)} \sqrt{\frac{1}{1 + e^{2\pi E}} \Gamma \left( \frac{1}{2} + iE \right) / \Gamma \left( \frac{1}{2} - iE \right) }, \\
 T(E) \equiv & \frac{-i\mu e^{i\Phi(E)}}{\sqrt{1 + e^{-2\pi E}}} e^{2i\Phi(E)} = \mu e^{i\Phi(E)} \sqrt{\frac{1}{1 + e^{-2\pi E}} \Gamma \left( \frac{1}{2} - iE \right) / \Gamma \left( \frac{1}{2} + iE \right) },
\end{align*}
\]

(2.30)

Note that in the complex-$E$ plane, $R(E)$ and $T(E)$ have simple poles at $E = -i \left( n + \frac{1}{2} \right)$ and no branch cuts, as expected for a non-relativistic quantum mechanical system with resonances and no bound states.

For large negative $E$, the absolute value of $R(E)$ differs from 1 by a term of order $e^{2\pi E}$. This corresponds to the tunneling probability for a fermion at energy level $E$ in the single-fermion Hamiltonian (2.12), as computed in the WKB approximation:

\[
\exp \left( 2 \int_{-\lambda_0}^{\lambda_0} d\lambda \sqrt{2(V(\lambda) - E)} \right) = \exp(2\pi E),
\]

(2.31)
where $\pm \lambda_0$ are the turning points $V(\pm \lambda_0) = E$, and the factor of 2 is because this “bounce” trajectory starts at $\lambda_0$ and goes to $-\lambda_0$ and back. Similarly, $T(E)$ starts at order $e^{\pi E}$, since now the particle’s trajectory goes from $\lambda_0$ to $-\lambda_0$. Further corrections come from multiple bounce solutions, giving rise to an expansion in powers of $e^{2\pi E}$.

The out-basis consists of eigenstates $|E\rangle^\text{out}_R$, $|E\rangle^\text{out}_L$ with wavefunctions

\[
\begin{align*}
    f^\text{out}_{E,R} &\equiv (f^\text{in}_{E,R})^* = R^*(E)f^\text{in}_{E,R} + T^*(E)f^\text{in}_{E,L} \\
    f^\text{out}_{E,L} &\equiv (f^\text{in}_{E,L})^* = T^*(E)f^\text{in}_{E,R} + R^*(E)f^\text{in}_{E,L},
\end{align*}
\]  

(2.32)

The out-basis wavefunctions have no outgoing flux to the left or to the right, as depicted in Figure 2.4.

![Diagram](image)

**Figure 2.4:** Out-basis wavefunctions.

In the next subsection, the out-basis will be used to describe the wavefunctions of a hole in the Fermi surface. Recall that a hole is an unoccupation of a fermion eigenstate, and therefore moves in the opposite direction to the fermion. Thus, the out-basis for a fermionic particle corresponds to the in-basis for the hole, and vice-versa.
2.3.2 The Non-Perturbative Matrix Quantum Mechanics

Let $\Psi(\lambda), \Psi^\dagger(\lambda)$ be the second quantized fermion fields,

$$
\Psi(\lambda) \equiv \int_{-\infty}^{\infty} dE \left( f^{\text{in}}_{E,R}(\lambda)b^{\text{in}}_E + f^{\text{in}}_{E,L}(\lambda)c^{\text{in}}_E \right) = \int_{-\infty}^{\infty} dE \left( f^{\text{out}}_{E,R}(\lambda)b^{\text{out}}_E + f^{\text{out}}_{E,L}(\lambda)c^{\text{out}}_E \right),
$$

$$
\Psi^\dagger(\lambda) \equiv \int_{-\infty}^{\infty} dE \left( (f^{\text{in}}_{E,R}(\lambda))^\star (b^{\text{in}}_E)^\dagger + (f^{\text{in}}_{E,L}(\lambda))^\star (c^{\text{in}}_E)^\dagger \right) = \int_{-\infty}^{\infty} dE \left( (f^{\text{out}}_{E,R}(\lambda))^\star (b^{\text{out}}_E)^\dagger + (f^{\text{out}}_{E,L}(\lambda))^\star (c^{\text{out}}_E)^\dagger \right),
$$

where $b^{\text{in}}_E, (b^{\text{in}}_E)^\dagger$ are the annihilation and creation operators for the fermion eigenstates with wavefunctions $f^{\text{in}}_{E,R}$, and $c^{\text{in}}_E, (c^{\text{in}}_E)^\dagger$ are the annihilation and creation operators for the wavefunctions $f^{\text{in}}_{E,L}$. They satisfy the anti-commutation relations

$$
\{b^{\text{in}}_E, (b^{\text{in}}_E)^\dagger\} = \delta(E - E'), \quad \{c^{\text{in}}_E, (c^{\text{in}}_E)^\dagger\} = \delta(E - E'),
$$

where the other anti-commutators vanish, with an analogous story for the out creation/annihilation operators. The anti-commutation relations between in and out creation/annihilation operators can be obtained from the change of basis (2.32), and are given by

$$
\{b^{\text{in}}_E, (b^{\text{out}}_{E'})\} = R^\star(E)\delta(E - E'), \quad \{b^{\text{in}}_E, (c^{\text{out}}_{E'})\} = T^\star(E)\delta(E - E'),
$$

$$
\{c^{\text{in}}_E, (b^{\text{out}}_{E'})\} = T^\star(E)\delta(E - E'), \quad \{c^{\text{in}}_E, (c^{\text{out}}_{E'})\} = R^\star(E)\delta(E - E').
$$

We propose that the closed string vacuum in $c = 1$ string theory is dual to the matrix model
eigenstate $|\Omega\rangle$ where the fermions fill the energy eigenstates $|E\rangle^\text{in}_R$ for $E \leq -\mu$. Thus, it is defined by

$$
\begin{align*}
 b^\text{in}_E |\Omega\rangle &= 0, \quad E > -\mu \\
 (b^\text{in}_E)^\dagger |\Omega\rangle &= 0, \quad E < -\mu \\
 c^\text{in}_E |\Omega\rangle &= 0.
\end{align*}
$$

(2.36)

At the perturbative level, this matches with the description where the fermions fill the “right-side” of the potential, while the region to the left of the potential is empty. While not obvious at this state, we claim that the state (2.36) reproduces the non-perturbative corrections to closed string scattering amplitudes to be discussed in chapter 4.

To compute scattering amplitudes of closed strings, we consider excitations of the state $|\Omega\rangle$. At the perturbative level, closed strings are dual to fluctuations of the density operator $\rho(\lambda) \equiv \sum_{i=1}^{N} \delta(\lambda - \lambda_i)$, which in the second quantized formalism is given by the operator $\Psi^\dagger(\lambda)\Psi(\lambda)$. We will see shortly that at the non-perturbative level there are corrections to this mapping, but it is nevertheless useful to start by considering the state $\Psi^\dagger(\lambda, t)\Psi(\lambda, t) |\Omega\rangle$, where the $\Psi(\lambda, t)$ operators are in the Heisenberg picture.

An in-state is given by a massless left-moving particle with wavefunction $e^{i\omega t + i\omega \tau}$, where the $\tau$ coordinate is defined by $\lambda = \sqrt{2\mu} \cosh \tau$, as below (2.19). In $\lambda$-coordinates, the LSZ in-state of energy $\omega > 0$ is defined by the term that behaves as $e^{i\omega t + i\omega \log(\sqrt{2\mu}\lambda)}$ in the region $\lambda \gg 0$.

\footnote{Note that this choice of closed string vacuum breaks time-reversal symmetry. This will also be seen from the worldsheet formalism.}
Using (2.33) and (2.36), this contribution is given by

\[
\left[ \Psi^\dagger(\lambda, t) \Psi(\lambda, t) \right]_{\text{in}}(\Omega) = e^{i\omega t + i\omega \log(\sqrt{2} \lambda)} \int_{-\mu - \omega}^{-\mu} dE_1 R^*(E_1) \left( b_{E_1 + \omega}^\text{in} \right)^\dagger b_{E_1}^\text{out} \Omega ,
\]

\[
+ e^{i\omega t + i\omega \log(\sqrt{2} \lambda)} \int_{-\infty}^{-\mu} dE_1 T(E_1 - \omega)^* \left( c_{E_1 - \omega}^\text{in} \right)^\dagger b_{E_1}^\text{out} \Omega ,
\]

\[
+ e^{i\omega t + i\omega \log(\sqrt{2} \lambda)} \int_{-\mu - \omega}^{-\mu} dE_1 T^*(E_1) \left( b_{E_1 + \omega}^\text{in} \right) c_{E_1}^\text{out} \Omega .
\]

(2.37)

Note that the operators $\Psi^\dagger$ and $\Psi$ create a particle and a hole respectively, and for this reason we expanded these operators in their in-basis (as discussed below (2.32), the in-basis of the hole is the out-basis of the particle). The first term corresponds to a particle-hole pair coming from the asymptotic region $\lambda \gg 0$, while the last term corresponds to a particle coming from the region $\lambda \gg 0$ while a hole is coming from the region $x \ll 0$, by tunneling through the potential.

The second term corresponds to a particle coming from the region $x \ll 0$ while a hole comes from the region $x \gg 0$, but this term will not play a role in what follows.

Similarly, the LSZ out-state of energy $\omega' > 0$ is defined by the term that behaves as $e^{i\omega' t - i\omega' \log(\sqrt{2} \mu \lambda)}$ in the region $\lambda \gg 0$,

\[
\left[ \Psi^\dagger(\lambda, t) \Psi(\lambda, t) \right]_{\text{out}}(\Omega) = e^{i\omega' t - i\omega' \log(\sqrt{2} \mu \lambda)} \int_{-\mu - \omega'}^{-\mu} dE_1' R(E_1') \left( b_{E_1' + \omega'}^\text{out} \right)^\dagger b_{E_1'}^\text{in} \Omega ,
\]

(2.38)

Again, we expanded $\Psi^\dagger$ and $\Psi$ in the out-basis for the particle and hole, respectively. The out-state consists of an outgoing particle-hole pair in the $\lambda \gg 0$ region.
The overlap of an in-state with an out-state is given by

\[
\left[ \Omega | \Psi^{\dagger}(\lambda_2, t_2) \Psi(\lambda_2, t_2) \right]^\text{out}_{(\omega')} \left[ \Psi^{\dagger}(\lambda_1, t_1) \Psi(\lambda_1, t_1) | \Omega \right]^\text{in}_{(\omega)} = \frac{e^{\imath \omega t_1 + \imath \omega \log \left( \sqrt{2 \pi} \lambda_1 \right)} e^{-\imath \omega' t_2 + \imath \omega' \log \left( \sqrt{2 \pi} \lambda_2 \right)}}{2\pi \lambda_1} \frac{1}{2\pi \lambda_2} \int_{-\mu}^{-\mu-\omega} dE_1 \int_{-\mu-\omega'}^{-\mu} dE'_1 \left( R^*_{E_1} \right) \langle \Omega | \left( b_{E_1}^{\text{in}} \right)^\dagger b_{E'_1 + \omega'}^\text{out} \\
\times \left( b_{E_1 + \omega} \right)^\dagger \left( R^*_{E_1} b_{E_1}^\text{out} + T^*_{E_1} c_{E_1}^\text{out} \right) | \Omega \rangle \right)
\]

\[
= \frac{e^{\imath \omega t_1 + \imath \omega \log \left( \sqrt{2 \pi} \lambda_1 \right)} e^{-\imath \omega' t_2 + \imath \omega' \log \left( \sqrt{2 \pi} \lambda_2 \right)}}{2\pi \lambda_1} \frac{1}{2\pi \lambda_2} \int_0^\omega dx R(-\mu + \omega - x) R^*(-\mu - x),
\]

where (2.35) was used in the last line.

To obtain the scattering amplitude of closed strings from (2.39), we have to relate the density operator to closed string in/out states. The closed string states in the collective field formalism are given by \( |\omega \rangle = a_{\omega}^{\dagger} |0\rangle \), where \(|0\rangle\) is the Fock vacuum of the collective field theory that maps to the state \(|\Omega\rangle\) in the free fermion description. The density operator \( \rho(t, \lambda) \equiv \Psi^{\dagger}(t, \lambda) \Psi(t, \lambda) \) is related to \( \partial_{\tau} \eta(t, \tau) \) via (2.14). In the collective field language, the LSZ limit extracts

\[
\left[ \Psi^{\dagger}(\lambda, t) \Psi(\lambda, t) | \Omega \right]^\text{in}_{(\omega)} = \frac{1}{\sqrt{2\pi \mu} \sinh \tau} \left[ \partial_{\tau} \eta(t, \tau) |0\right]^\text{in}_{(\omega)}
\]

\[
= \frac{e^{\imath \omega t + \imath \omega \tau}}{2\pi \sqrt{2\mu} \sinh \tau} a_{\omega}^{\dagger} |0\rangle
\]

\[
= \frac{e^{\imath \omega t + \imath \omega \log \left( \sqrt{2 \pi} \lambda \right)}}{2\pi \lambda} |\omega\rangle,
\]

where in the last line we used the fact that \( \lambda = \sqrt{2\mu} e^\tau \) in the LSZ limit. Using this and a
similar relation for the out-state, we extract from (2.39) the $1 \to 1$ momentum-space $S$-matrix

$$S(\omega \to \omega') = \delta(\omega - \omega') A(\omega),$$

$$A(\omega) = \int_0^\omega dx R(-\mu + \omega - x) R^*(-\mu - x).$$

This formula has a simple interpretation, in which a particle-hole pair comes from the asymptotic region at large $\lambda$, reflects off the potential, and goes back to the region of large $\lambda$. The particle has energy $\omega - x$ above the Fermi sea and contributes the reflection coefficient $R(-\mu + \omega - x)$, while a hole of energy $-x$ below the Fermi sea contributes the hole reflection coefficient $R_h(-\mu - x) = R^*(-\mu - x)$, and in (2.41) we sum over all such particle-hole pairs.

Expanding (2.41) in $1/\mu$, this expression reproduces the $1 \to 1$ scattering amplitude of closed strings to any perturbative order, the first few of which are written in (2.52). Furthermore, since (2.41) is obtained from the exact solution of the free fermion Hamiltonian, it includes non-perturbative corrections in $e^{-2\pi\mu}$ coming from tunneling effects.

However, it turns out that (2.41) is in disagreement with the non-perturbative corrections to worldsheet scattering amplitudes to be calculated in chapter 4. The source of the disagreement are non-perturbative corrections to the asymptotic states. Instead of the in- and out-states (2.37) and (2.38), we propose that the closed string in-state and out-state are given in momentum
space by\(^4\)

\[
|\omega\rangle^{\text{in}} \equiv \int_{-\mu-\omega}^{-\mu} dE_1 \left( b_{E_1+\omega}^\dagger E_1 r^{\text{out}} + T^*(E_1) c_{E_1}^{\text{out}} \right) |\Omega\rangle,
\]

\[
|\omega\rangle^{\text{out}} \equiv \int_{-\mu-\omega}^{-\mu} dE_1 (1 + e^{2\pi E_1}) R(E_1) \left( b_{E_1+\omega}^\dagger b_{E_1}^{\text{in}} \right) |\Omega\rangle,
\]

which are normalized as

\[
in \langle \omega' | \omega \rangle^{\text{in}} = \omega \delta(\omega - \omega'),
\]

\[
out \langle \omega' | \omega \rangle^{\text{out}} = \omega \left[ 1 + e^{-2\pi \mu} \frac{e^{-\pi \omega}}{\pi \omega} \sinh(\pi \omega) \right] \delta(\omega - \omega').
\]

Note that while the in-state is the same as before, the out-state has a greater outgoing hole-flux.

This agrees with our expectation that a hole has leaked in from the "other side" of the inverted quadratic potential, which is unoccupied. This extra contribution is the fundamental reason why the closed string vacuum is given by the state $|\Omega\rangle$ in (2.36).

The overlap of the in- and out-states (2.42) gives our final result for the $1 \rightarrow 1$ scattering amplitude, which is

\[
A(\omega) = \int_0^\omega dx R(-\mu + \omega - x) R^{-1}(-\mu - x).
\]

Once again we interpret $R(-\mu + \omega - x)$ as the reflection of a fermion with energy $\omega - x$ above the Fermi sea, and $R^{-1}(-\mu - x)$ as the reflection of a hole of energy $-x$ below the Fermi sea.

From (2.44), we find that the scattering amplitude of closed strings is non-unitary at the non-perturbative level. The loss of unitarity is understood as being due to the existence of other asymptotic states beyond particle-hole pairs, such as fermions/holes tunneling to the "other

\(^4\)Note that we will only compute closed string scattering amplitudes, for which the second term in (2.37) does not contribute. Therefore, we drop this contribution when writing the asymptotic states.
side” of the inverted quadratic potential.

Let’s now discuss the convergence properties of the $1 \to 1$ scattering amplitude (2.44). From (2.30) it follows that the function $R(-\mu + \omega - x)R^{-1}(-\mu - x)$ can be written as

$$R(-\mu + \omega - x)R^{-1}(-\mu - x) = \left[ \frac{1 + e^{-2\pi \mu} e^{-2\pi x}}{1 + e^{-2\pi \mu} e^{2\pi(\omega-x)}} \right]^{\frac{1}{2}} K(\mu, \omega, x), \quad (2.45)$$

where

$$K(\mu, \omega, x) \equiv \mu^{\omega} \left[ \frac{\Gamma(1/2 - i(-\mu + \omega - x)) \Gamma(1/2 - i(-\mu - x))}{\Gamma(1/2 - i(-\mu - x)) \Gamma(1/2 + i(-\mu + \omega - x))} \right]^{\frac{1}{2}}. \quad (2.46)$$

The term in square brackets on the RHS of (2.45) gives a non-perturbative expansion of (2.45) in “instanton sectors”, that is

$$R(-\mu + \omega - x)R^{-1}(-\mu - x) = \sum_{n=0}^{\infty} e^{-2\pi \mu n} f_n(\omega, x) K(\mu, \omega, x), \quad (2.47)$$

where $f_n(\omega, x)$ is a $n$-order polynomial in $e^{-2\pi x}$ and $e^{2\pi(\omega-x)}$. As mentioned above, the interpretation of the non-perturbative corrections is that the fermions and holes can tunnel to the other side of the potential, and the expansion in non-perturbative orders exactly matches with the WKB approximation for these corrections, written in (2.31).

Consider first the $n = 0$ term of the sum in (2.47), and expand $K(\mu, \omega, x)$ in $1/\mu$ as an asymptotic series. After integrating in $x$, this expansion reproduces the perturbative $1 \to 1$ scattering amplitudes of closed strings, where the first two terms are given by (2.52). At non-perturbative orders $e^{-2\pi \mu n}$ for $n > 0$, the scattering amplitude also admits an expansion in $\mu^{-1}$, which is captured by the asymptotic expansion of $\int_0^\omega dx f_n(\omega, x) K(\mu, \omega, x)$. 

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In fact, these perturbative expansions are Borel summable [49, 50]. Thus, assuming the duality between $c = 1$ string theory and the matrix model, we find that the genus expansion of the $1 \to 1$ scattering amplitude computed in string perturbation theory is Borel summable to the function $\int_0^\infty dx K(\mu, \omega, x)$. Comparison against the matrix model result further shows that the Borel resummation is corrected by non-perturbative effects, which are computed from the worldsheet formalism in chapter 4.

Finally, let us point out that while there are infinitely many matrix quantum mechanics that reproduce the perturbative scattering amplitudes (2.52), the non-perturbative corrections in (2.53) uniquely pick\(^5\) the matrix model dual described so far. For example, if the closed string vacuum was dual to a state in which both $|E\rangle_R$ and $|E\rangle_L$ are filled\(^6\) for $E \leq -\mu$, the reflection phase of the hole $R_h(-\mu - x)$ would equal $(R(-\mu - x))^*$ as opposed to $(R(-\mu - x))^{-1}$ [48]. Furthermore, if we modify the inverted quadratic potential (for example by putting an infinite wall at $\lambda = 0$), then the particle reflection phase $R(-\mu + \omega - x)$ would also differ from $(R(-\mu - x))^{-1}$ non-perturbatively [48]. However, the non-perturbative effects computed from the worldsheet description that we discuss below are consistent with exactly the matrix model dual described above.

---

\(^5\)While it is a logical possibility that there are other matrix model duals that can reproduce the non-perturbative scattering amplitudes of closed strings to be discussed in chapter 4, we have not found another such dual. However, there is at least another possibility for the dual matrix model, where the closed string vacuum is dual to the state $T|\Omega\rangle$, where $T$ is the time-reversal symmetry operator and $|\Omega\rangle$ is the state given by (2.36), as discussed in footnote 3.

\(^6\)This is known as the type 0B matrix model [51, 52], and it is conjectured to be dual to a worldsheet-supersymmetric version of $c = 1$ string theory.
2.3.3 Diagrammatic Formalism

The calculation of $S$-matrix elements from the free fermion wavefunctions admits a diagrammatic representation that simplifies the evaluation of amplitudes with more closed strings \cite{20, 48}.

The $\ell \to k$ $S$-matrix is written in terms of the scattering amplitude $A_{\ell \to k}$ as

$$S_{\ell \to k}(\{\omega_i\};\{\omega_j\}) = \delta \left( \sum_i \omega_i - \sum_j \omega_j \right) A_{\ell \to k}(\{\omega_i\},\{\omega_j\}),$$

where the indices $i$ and $j$ run over the $\ell$ incoming and the $k$ outgoing closed strings, respectively. Each closed string introduces a vertex at the position $(t, \lambda)$, where $\lambda > 0$. We draw diagrams with $t$ pointing upwards and $\lambda$ pointing to the right. It is convenient at intermediate steps to fix the positions $\lambda_i$ for each closed string, while the final result is independent of this choice. The rules for constructing the diagrams are as follows

- Closed strings are denoted by a dashed line, and slope upward to the left (right) for incoming (outgoing) scattering states.
- Outgoing closed strings vertices are located at later times than incoming closed string vertices.
- Each closed string vertex bosonizes the composite boson into a particle-hole pair, with total energy equal to the closed string energy.

Figure 2.5: Closed string vertices for incoming (left) and outgoing (right) closed strings.
The particle/hole propagators are denoted by a thick line, and slope upwards to the left (right) if the particle/hole move away from (towards) the asymptotic region $\lambda \gg 0$.

- Arrows point in the direction of motion for a particle, and in the opposite direction for a hole.
- Each incoming particle/hole propagator ends in a “flip”, where the direction of motion is reversed. We include a factor $R(-\mu + q)$ or $R_h(-\mu - q)$ for the flip of a particle or hole of energy $E$, respectively.
- The final diagram looks like a 1-loop graph for the internal particle/hole. The momentum of the internal particle/hole is integrated over, but the range of integration is finite.
- Finally, there is a constant factor associated with each diagram (see [48] for details).

Figures 2.5, 2.6, 2.7 illustrate the ingredients of these diagrammatic rules. In the end, we sum over all diagrams with $\ell$ incoming and $k$ outgoing strings to reproduce the scattering amplitude $A_{\ell \rightarrow k}$.

Let’s now consider the simplest scattering amplitude, the $1 \rightarrow 1$ amplitude given by (2.44). In this case there is a single diagram that contributes, shown in Figure 2.8. The interpretation
Figure 2.8: Diagram that contributes to $1 \rightarrow 1$ scattering, where $0 < q < \omega$.

of the diagram is similar to the discussion below (2.41), with the hole reflection coefficient
$R_h(E) = (R(E))^{-1}$. In this case, the combinatoric factor is equal to 1, and integrating over the
loop momentum reproduces (2.44).

Figure 2.9: Diagrams contributing to the scattering $\{\omega\} \rightarrow \{\omega_1, \omega_2\}$, where $q$ lies in the range $0 < q < \omega_2$.

Next, let’s consider the $1 \rightarrow 2$ scattering amplitude. There are two diagrams that contribute,
as shown in Figure 2.9. Their total contribution is given by

$$ A_{1\rightarrow 2}(\omega_1,\omega_2) = - \int_{\omega_1}^{\omega} dx\ R(-\mu + \omega - x)(R(-\mu - x))^{-1} + \int_{0}^{\omega_2} dx\ R(-\mu + \omega - x)(R(-\mu - x))^{-1}, $$

(2.49)

where the factor of $-1$ in the first term on the RHS comes from the constant prefactor. Expanding this equation in $1/\mu$, at leading order we find the tree-level $1 \rightarrow 2$ amplitude (2.52). A rigorous derivation of this scattering amplitude is given in Appendix E.

More generally, the $1 \rightarrow k$ scattering amplitude has a single incoming boson, so every possible diagram will have a single flip for the particle and another for the hole. The different diagrams restrict the range of integration of the internal energy, and the constant factors are $\pm 1$. The full answer can be obtained from a generalization of the $1 \rightarrow 2$ scattering amplitude derived in Appendix E, and it is given by the expression [20, 48]

$$ A_{1\rightarrow k}(\omega_1, \cdots, \omega_k) = - \sum_{S_1 \sqcup S_2 = S} (-1)^{|S_2|} \int_0^{\omega(S_2)} dx\ R(-\mu + \omega - x)(R(-\mu - x))^{-1}, $$

(2.50)

where $S_1, S_2$ are disjoint subsets of $S = \{\omega_1, \cdots, \omega_k\}$ such that $S_1 \sqcup S_2 = S$. $|S_2|$ denotes the number of elements of $S_2$, and $\omega(S_2)$ is the sum of all elements of $S_2$. The $1 \rightarrow k$ scattering amplitude admits an expansion of the form

$$ A_{1\rightarrow k}(\omega_1, \cdots, \omega_k) = \sum_{g=0}^{\infty} \frac{1}{\mu^{k-1+2g}} A_{1\rightarrow k}^{\text{pert.}(g)}(\omega_1, \cdots, \omega_k) $$

$$ + \sum_{n=1}^{\infty} e^{-2\pi n \mu} \sum_{L=0}^{\infty} \frac{1}{\mu^L} A_{1\rightarrow k}^{\text{n-inst.}(L)}(\omega_1, \cdots, \omega_k). $$

(2.51)

Here, $A_{1\rightarrow k}^{\text{pert.}(g)}$ is the perturbative $1 \rightarrow k$ scattering amplitude of closed strings at genus $g$. 

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and $A_{1 \to k}^{\text{inst.}(L)}$ is the $n$-instanton contribution to the scattering amplitude, expanded at $L$-th order. As in the case of the $1 \to 1$ amplitude discussed below (2.45), the perturbative expansion about any instanton order is captured by the asymptotic expansion of $f_n(\omega, x)K(\mu, \omega, x)$ (after integrating over $x$), and it is Borel-summable.

Each term in (2.51) can be explicitly evaluated from the expressions (2.50) and (2.30). The simplest perturbative scattering amplitudes are given by

\begin{align}
A_{1 \to 1}^{\text{pert.}(0)}(\omega_1) &= \omega_1, \\
A_{1 \to 2}^{\text{pert.}(0)}(\omega_1, \omega_2) &= i\omega_1 \omega_2, \\
A_{1 \to 3}^{\text{pert.}(0)}(\omega_1, \omega_2, \omega_3) &= i\omega_1 \omega_2 \omega_3 (1 + i\omega), \\
A_{1 \to 1}^{\text{pert.}(1)}(\omega_1) &= \frac{1}{24} (i\omega^2 + 2i\omega^4 - \omega^5),
\end{align}

while the first few non-perturbative scattering amplitudes are

\begin{align}
A_{1 \to k}^{\text{inst.}(0)}(\omega_1, \ldots, \omega_k) &= -\frac{2^{k+1}}{4\pi} \sinh(\pi \omega) \prod_{i=1}^{k} \sinh(\pi \omega_i), \\
A_{1 \to 1}^{\text{inst.}(1)}(\omega_1) &= -\frac{i}{2\pi^2} \omega_1 \left( \frac{\pi \omega_1}{\tanh(\pi \omega_1)} - 1 \right) \sinh^2(\pi \omega_1). \\
A_{1 \to k}^{\text{inst.}(0)}(\omega_1, \ldots, \omega_k) &= \frac{(-1)^n}{2\pi^2} \frac{\Gamma(1/2 + n)}{\Gamma(1 + n)} e^{\pi n} \frac{2^{k+1}}{4\pi} \sinh^2(\pi \omega_1) \\
&= \frac{1}{2\pi^2} \frac{(-1)^n}{n} \frac{\Gamma(1/2 + n)}{\Gamma(1 + n)} e^{\pi n} \frac{2^{k+1}}{4\pi} \sinh^2(\pi \omega_1) \\
&= \frac{1}{2\pi^2} \frac{(-1)^n \Gamma(1/2 + n)}{n \Gamma(1 + n)} e^{\pi n} \frac{2^{k+1}}{4\pi} \sinh^2(\pi \omega_1).
\end{align}

In the next few chapters, we will reproduce these scattering amplitudes from the worldsheet formalism, starting with the perturbative scattering amplitudes in (2.52), and then computing the non-perturbative corrections (2.53).
It follows from similar methods that the tree-level $2 \to 2$ scattering amplitude is given by [19, 20, 21]

\[
\begin{align*}
\mathcal{S}^{(0)}_{2 \to 2}(\omega_1, \omega_2; \omega_3, \omega_4) &= \delta (\omega_1 + \omega_2 - \omega_3 - \omega_4) A^{(0)}_{2 \to 2}(\omega_1, \omega_2; \omega_3, \omega_4), \\
A^{(0)}_{2 \to 2}(\omega_1, \omega_2, \omega_3, \omega_4) &= i \omega_1 \omega_2 \omega_3 \omega_4 (1 + i \text{Imax}(\{\omega_j\})) ,
\end{align*}
\]

(2.54)

where $\text{Imax}(\{\omega_j\})$ picks out the $\omega_j$, $j = 1, 2, 3, 4$, with largest imaginary part. Thus, this amplitude suffers from discontinuities in the physical regime of real energies. It turns out that these discontinuities are exactly as expected from unitarity, and indeed the discontinuous jumps in the real part of (2.54), as well as the real part of $A^{\text{pert.(0)}}_{1 \to 3}$, can be calculated analytically using worldsheet unitarity cut methods, that we discuss in the next chapter.
In this chapter we describe the worldsheet formalism of $c = 1$ string theory, and compute scattering amplitudes of closed strings using string perturbation theory [30], thereby reproducing the matrix model results in (2.52) and (2.54). We start by reviewing the semiclassical quantization of Liouville conformal field theory (CFT), before discussing the computation of Liouville correlation functions from the CFT data. We then discuss the worldsheet formalism of the $1 + 1d$ $c = 1$ string theory, and compute the tree-level $1 \rightarrow 2$ scattering amplitude. Next, we analyze the tree-level $1 \rightarrow 3$ and $2 \rightarrow 2$ scattering amplitudes of closed strings. The real parts
of these amplitudes are computed exactly using unitarity cut methods, while the full amplitude
is evaluated using numerical techniques, and show striking agreement with the matrix model re-
sults (2.52) and (2.54). We further explain the discontinuities in the tree-level $2 \to 2$ scattering
amplitude (2.54) as a consequence of the subtle kinematics of massless particles in $1 + 1d$.

3.1 Liouville CFT

3.1.1 Semiclassical Quantization

On a Riemann surface $\Sigma$ with coordinates $z, \bar{z}$ and background metric $g_{mn}$, the Liouville CFT
is described semiclassically by the Euclidean action

$$S_L[\phi] = \frac{1}{4\pi} \int_{\Sigma} d^2 z \sqrt{g} \left( g^{mn} \partial_m \phi \partial_n \phi + Q R \phi + 4\pi \mu e^{2b\phi} \right), \quad (3.1)$$

where $\phi(z, \bar{z})$ is a scalar field and $R$ is the scalar curvature of $g_{mn}$. In the absence of the last
term in (3.1), the action is the same as that of the conformally invariant linear dilaton theory.
To preserve conformal invariance, we require the operator $e^{2b\phi}$ to be a marginal deformation
of the linear dilaton theory, with conformal weights $(h_b, \bar{h}_b) = (1, 1)$. This implies$^1$ that the
“background charge” $Q$ is related to the parameter $b$ by $Q = b + b^{-1}$, and the central charge of
the theory is the same as that of the linear dilaton theory, $c = \tilde{c} = 1 + 6Q^2$.

Note that if we shift $\phi$ by $-\frac{1}{2\mu} \ln \mu$, the dependence of (3.1) on the “cosmological constant”

$^1$Recall that in the linear dilaton theory, the scalar operator $e^{2\alpha \phi}$ has conformal weight $h_\alpha = \alpha(Q - \alpha).$
\( \mu \) comes only in the form

\[- \frac{Q \ln \mu}{8\pi b} \int_{\Sigma} d^2 z \sqrt{g} QR = - \frac{Q \ln \mu}{2b} \chi_{\Sigma}, \quad (3.2)\]

where \( \chi_{\Sigma} \) is the Euler characteristic of \( \Sigma \). Thus, the dependence of a Liouville correlation function on the parameter \( \mu \) is completely determined by the topology of the Riemann surface.

The action (3.1) is a good description of the dynamics in Liouville theory in the semiclassical limit \( b \to 0 \), where the exponential potential can be treated as a perturbation. It is useful to canonically quantize the semiclassical Liouville theory described by \( S_L[\phi] \) on the cylinder to gain some intuition. Consider the Lorentzian cylinder with coordinates \((\sigma, t)\), \( \sigma \sim \sigma + 2\pi \), \( t \in \mathbb{R} \) and line element \( ds^2 = -dt^2 + d\sigma^2 \). The scalar curvature of the cylinder vanishes, and after Wick rotation (3.1) simplifies to

\[ S^c_{cyl} L[\phi] = \frac{1}{4\pi} \int_0^{2\pi} d\sigma \int_{-\infty}^{\infty} dt \left( \partial_t \phi \partial_t \phi - \partial_\sigma \phi \partial_\sigma \phi - 4\pi \mu e^{2b\phi} \right). \quad (3.3) \]

In the semiclassical limit, we can treat the last term as a perturbation and canonically quantize \( \phi \) and its conjugate momentum \( \Pi(\sigma, t) \equiv \frac{\partial_\phi(\sigma, t)}{2\pi} \), so that \([\phi(\sigma, t), \Pi(\sigma', t)] = i\delta(\sigma - \sigma')\). Decomposing \( \phi, \Pi \) in Fourier modes, we find

\[ \phi(\sigma, t) = \phi_0 + \sum_{n \neq 0} \frac{1}{n} \left( a_n e^{i\sigma + int} + b_n e^{-i\sigma + int} \right), \quad (3.4) \]

\[ \Pi(\sigma, t) = \frac{p}{2\pi} + \frac{i}{2\pi} \sum_{n \neq 0} \left( a_n e^{i\sigma + int} - b_n e^{-i\sigma + int} \right), \quad (3.5) \]
where the modes $a_n, b_n$ satisfy

$$[\phi_0, p] = i, \quad (3.6)$$

$$[a_n, a_m] = \frac{n}{2} \delta_{n,-m}, \quad (3.7)$$

$$[b_n, b_m] = \frac{n}{2} \delta_{n,-m}, \quad (3.8)$$

and all other commutators vanish.

We work in the minisuperspace approximation where only the zero modes of $\phi, \Pi$ are kept. Interactions with other modes are suppressed in the $b \to 0$ limit, so this is a consistent truncation of the Hilbert space. The Hamiltonian in the minisuperspace approximation is given by

$$H = \frac{p^2}{2} + 2\pi \mu e^{2b\phi_0}, \quad (3.9)$$

up to an overall shift due to renormalization of the vacuum energy. In the asymptotic region $\phi_0 \to -\infty$, the theory is free and the solutions are plane waves. Incoming plane waves move towards the interacting region, where they reflect off the Liouville potential and move back towards the asymptotic region (see figure 3.1). Note that there is a single asymptotic region, and states are labelled by their momentum $p > 0$.

The wavefunctions in the $\phi_0$ representations can be solved exactly and are given by

$$\psi_{2p}(\phi_0) = \frac{2 \left( \frac{\pi \mu}{b^2} \right)^{-2i \frac{p}{\pi} \phi_0}}{\Gamma\left(-i \frac{p}{b} \phi_0\right)} K_{2i \frac{p}{b}} \left( 2\sqrt{\frac{\pi \mu}{b^2}} e^{b\phi_0} \right), \quad p \in \mathbb{R}_{\geq 0} \quad (3.10)$$
Figure 3.1: Solutions to the Liouville theory in the minisuperspace approximation describe plane waves reflecting off the Liouville wall, as in (3.12).

for a wavefunction of energy $2p^2$, which is normalized as

$$\int_{-\infty}^{\infty} d\phi_0 \psi_2p(\phi_0)\psi^{*}_{2p'}(\phi_0) = 2\pi\delta(2p - 2p').$$

In the asymptotic region, the wavefunction approaches

$$\psi_p(\phi_0) \xrightarrow{\phi_0 \to -\infty} e^{2ip\phi_0} + S_0(p)e^{-2ip\phi_0},$$

where the semiclassical reflection phase $S_0(p)$ is given by

$$S_0(p) = \frac{\Gamma\left(\frac{2ip}{b}\right)}{\Gamma\left(-\frac{2ip}{b}\right)} \left(\frac{\pi\mu}{b^2}\right)^{-\frac{2ip}{b^2}}.$$  \hfill (3.13)

3.1.2 Liouville CFT Data

In the context of $c = 1$ string theory, we are interested in $c = 25$ Liouville CFT, for which $b = 1$. The minisuperspace approximation is not valid in this case since the Liouville potential couples different Fourier modes of $\phi$. However, in the asymptotic region $\phi \to -\infty$ a subset of
the states are still described by (3.12), since the interaction between the zero modes and the other modes is suppressed in this region. The state-operator mapping maps these states to the vertex operators\(^2\)

\[
V_P \xrightarrow{\phi \to -\infty} e^{(Q+2iP)\phi} + S(P)e^{(Q-2iP)\phi}.
\]

Away from \(b \to 0\), the reflection phase \(S(P)\) is not equal to the semiclassical reflection phase \(S_0(p)\), but is given by [25]

\[
S(P) = - (\pi \mu \gamma(b^2) \frac{2iP}{b}) \frac{\Gamma \left( \frac{2iP}{b} \right)}{\Gamma \left( -\frac{2iP}{b} \right)} \frac{\Gamma \left( 2iPb \right)}{\Gamma \left( -2iPb \right)},
\]

where \(\gamma(x) \equiv \Gamma(x)/\Gamma(1-x)\). We normalize \(V_P\) so that the two-point function on the sphere is delta function normalized,

\[
\langle V_{P_1}(z_1, \bar{z}_1)V_{P_2}(z_2, \bar{z}_2) \rangle = \frac{\delta(P_1 - P_2)}{|z_1z_1|^{\frac{1}{b}P_1}}.
\]

With this normalization, in the asymptotic regime \(V_P\) becomes

\[
V_P \xrightarrow{\phi \to -\infty} S(P)^{-\frac{1}{2}}e^{(Q+2iP)\phi} + S(P)^{\frac{1}{2}}e^{(Q-2iP)\phi},
\]

where \(\phi\) is the usual field of the linear dilaton theory. The conformal weights of \(V_P\) are the same as those of the linear dilaton operators in the asymptotic region, namely \(h_P = \bar{h}_P = \frac{Q^2}{4} + P^2\).

Under the state-operator mapping, the states on the cylinder have energies \(h_P + \bar{h}_P - \frac{c+\hat{c}}{24} = \)

\(^2\)Confusingly, the notation \(V_P\) denotes a state with momentum \(2P\) on the cylinder. This is the standard notation in the literature that we will use throughout.
2P^2 - \frac{1}{12}, in agreement with the previous semiclassical analysis (up to the overall vacuum renormalization energy shift). In fact, \( V_{P \geq 0} \) parameterized by the “Liouville momentum” \( P \in \mathbb{R}_{\geq 0} \) form a complete basis\(^3\) for the (delta function) normalizable primary operators in the Hilbert space of Liouville CFT with \( c \geq 1 \). Under the state-operator correspondence, Virasoro descendants of \( V_P \) are mapped to states on the cylinder which in the asymptotic region have excited non-zero modes.

The sphere three-point function of primary operators in Liouville CFT takes the form

\[
\langle V_{P_1}(z_1, \bar{z}_1)V_{P_2}(z_2, \bar{z}_2)V_{P_3}(z_3, \bar{z}_3) \rangle = \frac{C(P_1, P_2, P_3)}{|z_{12}|^{2h_{P_1} + 2h_{P_2} - 2h_{P_3}}|z_{13}|^{2h_{P_1} + 2h_{P_3} - 2h_{P_2}}|z_{23}|^{2h_{P_2} + 2h_{P_3} - 2h_{P_1}}},
\]

where the structure constants \( C(P_1, P_2, P_3) \) are given by the “DOZZ” formula [24, 25]. For \( c = 25 \) Liouville CFT, the structure constants are given by

\[
C(P_1, P_2, P_3) = \frac{1}{\Upsilon_1(1 + i(P_1 + P_2 + P_3))} \left[ \frac{2P_1\Upsilon_1(1 + 2iP_1)}{\Upsilon_1(1 + i(P_2 + P_3 - P_1))} \times (2 \text{ permutations}) \right].
\]

The function \( \Upsilon_1(x) \), which is a special case of the Barnes double Gamma function, is defined in Appendix A.

The CFT data consists of the spectrum of Virasoro primaries and their structure constants, so that any correlation function in an arbitrary Riemann surface is completely fixed by this data and by Virasoro Ward identities. This leads to non-trivial consistency conditions on the

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\(^3\)Note that the identity operator and its descendants (including the stress-energy tensor) are not part of the spectrum of the Liouville CFT, since they are not a normalizable operators. While this may seem confusing, the situation is no different from other non-compact CFTs, such as the non-compact free boson.
CFT data, which are automatically satisfied provided crossing symmetry of the sphere 4-point function and modular covariance of the torus 1-point function are obeyed \[53\]. For the Liouville CFT, these conditions were explicitly checked in \[25, 54, 55\].

The correlation function of 4 Liouville primary operators \( V_{P_1} \) on the sphere is not known in closed form for generic \( P_i \in \mathbb{R}_{\geq 0} \), but it can be expanded in conformal blocks as follows. For simplicity, we fix three vertex operators\(^4\) at 0, 1, \( \infty \), so that the cross-ratio \( z \) is given by

\[
z \equiv \frac{z_{12} z_{34}}{z_{32} z_{14}} \rightarrow z_1. \tag{3.20}\]

The \( s \)-channel expansion of the sphere 4-point function is obtained by taking the OPE between operators \( V_P(z, \bar{z}) \) and \( V_{P_1}(0) \), and it is given by

\[
\langle V_P(z, \bar{z})V_{P_1}(0)V_{P_2}(1)V_{P_3}(\infty) \rangle_{\text{Liouville}} = \int_0^{\infty} \frac{dP'}{\pi} \mathcal{C}(P, P_1, P')\mathcal{C}(P_2, P_3, P') F^{(P')}_{P, P_1, P_2, P_3}(z) F^{(P')}_{P, P_1, P_2, P_3}(\bar{z}). \tag{3.22}\]

Here, \( F^{(P')}_{P, P_1, P_2, P_3}(z) \) is given by

\[
F^{(P')}_{P, P_1, P_2, P_3}(z) = \mathcal{F} \left( \frac{Q^2}{4} + P^2, \frac{Q^2}{4} + P_1^2, \frac{Q^2}{4} + P_2^2, \frac{Q^2}{4} + P_3^2; \frac{Q^2}{4} + P'^2 \right) \bigg| z \right), \tag{3.23}\]

where \( \mathcal{F}(h_1, h_2, h_3, h_4; h|z) \) is the holomorphic sphere 4-point Virasoro conformal block with \( c = 1 + 6Q^2 \), external weights \( h_1, \cdots, h_4 \), and internal weight \( h \). The Virasoro conformal block encodes the contribution of all the descendants of the internal primary operator of conformal

\(^4\)As usual, we define the operator at \( \infty \) by \( V_{P_1}(\infty) \equiv \lim_{z \to \infty} z^{2h} z^{2h_2} V_{P_3}(z, \bar{z}) \) and slightly abuse notation by dropping the prime.
weight $h$, and it is completely fixed by the Virasoro Ward identities. While it is not known in closed form in full generality, in Appendix B we discuss efficient recursion techniques for computing them numerically.

Similarly, the $t$-channel decomposition of the Liouville 4-point function is obtained from the OPE expansion between operators $V_P(z, \bar{z})$ and $V_{P_2}(1)$, and it is given by

$$\langle V_P(z, \bar{z})V_{P_1}(0)V_{P_2}(1)V_{P_3}(\infty)\rangle_{\text{Liouville}} = \int_0^{\infty} \frac{dP'}{\pi} C(P, P_2, P')C(P_1, P_3, P')F^{(P')}_{P', P_2, P_1, P_3}(1 - z)F^{(P')}_{P', P_2, P_1, P_3}(1 - \bar{z}).$$

Comparing (3.22) and (3.25), we find that the position dependence of the conformal block can be exchanged, $z \leftrightarrow 1 - z$, at the cost of exchanging the external weights in the conformal block – this is the condition of crossing invariance. The $u$-channel conformal block decomposition is obtained by taking the OPEs of $V_P(z, \bar{z})$ and $V_{P_3}(\infty)$,

$$\langle V_P(z, \bar{z})V_{P_1}(0)V_{P_2}(1)V_{P_3}(\infty)\rangle_{\text{Liouville}} = (z\bar{z})^{-2 - 2P} \int_0^{\infty} \frac{dP'}{\pi} C(P, P_3, P')C(P_1, P_2, P')F^{(P')}_{P, P_3, P_1, P_2}(1/z)F^{(P')}_{P, P_3, P_1, P_2}(1/\bar{z}).$$

Combining all crossing relations, we find that crossing symmetry of the sphere 4-point function is isomorphic to the group $S_3$, and relates sphere 4-point correlators with the cross-ratios

$$z, \quad 1 - z, \quad \frac{1}{z}, \quad \frac{1}{1 - z}, \quad \frac{z - 1}{z}, \quad \frac{z}{z - 1}.$$  

(3.28)

These relations map the region $D = \{|z - 1| < 1, \text{Re } z < 1/2\}$ to the full complex plane. We
make use of this below when computing scattering amplitudes of closed strings in $c = 1$ string theory.

### 3.2 The Worldsheet Theory

The worldsheet formulation of $c = 1$ string theory consists of a timelike free boson $X^0$, $c = 25$ Liouville CFT, and the usual $b, c$ ghost system. The total central charge of the theory is zero as required for Weyl anomaly cancellation on the worldsheet. The liouville field $\phi$ and the timelike free boson $X^0$ describe a $1 + 1d$ target spacetime, with a non-trivial background in the spatial direction due to the dynamics of the Liouville CFT, as shown in figure 3.2. Due to the linear dilaton term in (3.1), the effective string coupling grows as $g_s^{\text{eff}} \sim g_s e^\phi$, so that there is a single asymptotic region at $\phi \to -\infty$, while the region of arbitrarily strong coupling is cut-off by the Liouville potential. Note that, as a consequence of the discussion below (3.2), the parameter $\mu$ that appears in the Liouville action (3.1) can be absorbed in a renormalization of the string coupling, so it will be set to 1 from now on.

![Figure 3.2](image)

**Figure 3.2**: A pictorial representation of the $1 \to 2$ scattering amplitude in $c = 1$ string theory. An incoming string with energy $\omega$ approaches the Liouville wall from the weak coupling region. Near the Liouville wall, the effective string coupling is finite and the incoming closed string splits into two closed strings of energies $\omega_1, \omega_2$, which move back towards the asymptotic region at $\phi \to -\infty$. 

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Closed strings correspond to non-trivial representatives of worldsheet BRST cohomology classes. We take in (+) and out (−) closed strings to be represented by

$$V_{\omega}^{\pm} = g_s e^{\pm i\omega X^0} : V_{p=\frac{b}{2}} :$$

(3.29)

where $\omega > 0$. These vertex operators form a complete basis for the non-trivial normalizable BRST cohomology classes. Even though the closed strings (3.29) are massless, they are called “tachyons” due to an unfortunate resemblance with the tachyonic vertex operators in critical bosonic string theory. We write $|\omega\rangle^{\pm}$ for the corresponding closed string states in spacetime.

Perturbative scattering amplitudes of closed strings are computed in string perturbation theory following the usual worldsheet formalism. For example, the perturbative $1 \rightarrow k$ scattering amplitude admits an expansion in the string coupling given by

$$S^{\text{pert}}_{1\rightarrow k}(\omega; \omega_1, \ldots, \omega_k) = \sum_{g=0}^{\infty} g_s^{k-1+2g} S^{\text{pert},(g)}_{1\rightarrow k}(\omega; \omega_1, \ldots, \omega_k),$$

$$S^{\text{pert},(g)}_{1\rightarrow k}(\omega; \omega_1, \ldots, \omega_k) = \delta \left( \omega - \sum_{i=1}^{k} \omega_i \right) A^{\text{pert},(g)}_{1\rightarrow k}(\omega_1, \ldots, \omega_k),$$

(3.30)

where $S^{\text{pert},(g)}_{1\rightarrow k}(\omega; \omega_1, \ldots, \omega_k)$ is the genus-$g$ contribution to the $1 \rightarrow k$ scattering amplitude. These worldsheet scattering amplitudes will soon be matched against the corresponding matrix model amplitudes in (2.52) for the cases of tree-level scattering. In [30], the 1-loop $1 \rightarrow 1$ scattering amplitude has also been shown to agree with (2.52).

The perturbative scattering amplitudes are subject to the unitarity constraint of the $S$-matrix,

---

5To be precise, (3.29) should include the usual $c\bar{c}$ ghosts, but for ease of writing we will omit the ghosts in writing the vertex operators and include them separately when computing scattering amplitudes.
which for 1-closed string scattering states takes the form

\[ \sum_{n=1}^{\infty} \int_0^\infty d\omega_1 \ldots \int_0^\infty d\omega_n \delta \left( \omega - \sum_{i=1}^{n} \omega_i \right) \left| A_{1 \rightarrow n}(\omega_1, \ldots, \omega_n) \right|^2 = 1, \]  

(3.31)

for every \( \omega > 0 \). The denominator on the LHS is due to the normalization of the closed string asymptotic states \(|\omega \rangle^{\pm}\), which is given by

\[ \pm \langle \omega | \omega' \rangle^{\pm} = \omega \delta(\omega - \omega'), \]  

(3.32)

see\(^6\) (2.43).

3.3 Tree-level 1 \( \rightarrow \) 2 Amplitude

The simplest non-trivial scattering amplitude is the tree-level 1 \( \rightarrow \) 2 scattering amplitude of closed strings, shown in figure (3.2). In this case, there are no moduli to integrate over since the integration over the positions of the vertex operators is cancelled by the volume of the conformal Killing group. Thus, the amplitude is given by

\[ S_{1 \rightarrow 2}^{\text{pert,}(0)}(\omega, \omega_1, \omega_2) = \langle \bar{c} \bar{c} \mathcal{V}^+_\omega(z_1, \bar{z}_1) \bar{c} \bar{c} \mathcal{V}^-_{\omega_1}(z_2, \bar{z}_2) \bar{c} \bar{c} \mathcal{V}^-_{\omega_2}(z_3, \bar{z}_3) \rangle_S^2. \]  

(3.33)

The correlation function of the \( \mathcal{V}^{\pm}_{\omega} \) factorizes into a correlation function in the different CFTs.

\(^6\)Note that perturbative unitarity does not require including the non-perturbative corrections to the normalization of the asymptotic states given by (2.43).
The $X^0$ correlation function is

$$\langle e^{i\omega X^0}(z_1, \bar{z}_1) e^{-i\omega_1 X^0}(z_2, \bar{z}_2) e^{-i\omega_2 X^0}(z_3, \bar{z}_3) \rangle = i C_{S^2}^{X^0} |z_1 - z_2|^{\omega_1} |z_1 - z_3|^{\omega_2} |z_2 - z_3|^{-\omega_1 \omega_2},$$

while the $b, c$ correlation function is given by

$$\langle c\bar{c}(z_1, \bar{z}_1) c\bar{c}(z_2, \bar{z}_2) c\bar{c}(z_3, \bar{z}_3) \rangle = C_{S^2}^{b,c} |z_1 - z_2|^2 |z_1 - z_3|^2 |z_2 - z_3|^2.$$  \hfill (3.35)

Together with the three-point function in Liouville (3.19), we obtain

$$S_1^{(0)}(\omega; \omega_1, \omega_2) = ig_s^3 C_{S^2} \delta(\omega - \omega_1 - \omega_2) C \left( \frac{\omega}{2}, \frac{\omega_1}{2}, \frac{\omega_2}{2} \right)$$

$$= i \delta(\omega - \omega_1 - \omega_2) g_s^3 C_{S^2} \omega_1 \omega_2,$$

where $C_{S^2} \equiv C_{S^2}^{X^0} C_{S^2}^{b,c}$. This result exactly matches the matrix model scattering amplitude in the dual matrix model (2.52), provided we identify

$$g_s^3 C_{S^2} = \mu^{-1}. \hfill (3.37)$$

In the next section, we will explicitly compute the real part of the $1 \rightarrow 3$ scattering amplitude from the worldsheet formalism (see (3.57)), and comparing this result with the matrix model answer in (2.52) fixes

$$2\pi g_s = \mu^{-1}, \quad C_{S^2} = \frac{2\pi}{g_s}. \hfill (3.38)$$

Note that the complicated looking expression for $C \left( \frac{\omega}{2}, \frac{\omega_1}{2}, \frac{\omega_2}{2} \right)$ in (3.36) simplified dramati-
ally once the condition $\omega = \omega_1 + \omega_2$ was imposed. We will repeatedly encounter this seemingly mysterious feature in scattering amplitudes of $c = 1$ string theory, which is due to its the underlying dual description as a matrix quantum mechanics.

### 3.4 Tree-level Amplitude of 4 Closed Strings

Next we consider the tree-level $1 \rightarrow 3$ scattering amplitude of closed strings. In this case, the conformal killing group on the sphere fixes the position of the outgoing closed string vertex operators, while the position of the incoming closed string is to be integrated over. The amplitude is formally given by the expression

$$S^\text{pert.}(0)_{1 \rightarrow 3}(\omega; \omega_1, \omega_2, \omega_3) = \int d^2 z \left\langle V^+_{\omega}(z, \bar{z}) \bar{c} \bar{c} V^-_{\omega_1}(0) c \bar{c} V^-_{\omega_2}(1) c \bar{c} V^-_{\omega_3}(\infty) \right\rangle_{S^2},$$

$$A^\text{pert.}(0)_{1 \rightarrow 3}(\omega_1, \omega_2, \omega_3) = ig^4_s C_{S^2} \int d^2 z |z|^\omega_1 |1 - z|^\omega_2 \left\langle V^\omega_{\frac{1}{2}}(z, \bar{z}) V^\omega_{\omega_1}(0) V^\omega_{\omega_2}(1) V^\omega_{\omega_3}(\infty) \right\rangle_{\text{Liouville}},$$

(3.39)

where we used (3.35) for the ghost correlator, and the relevant $X^0$ correlation function is given by

$$\langle e^{i \omega X^0}(z, \bar{z}) e^{-i \omega_1 X^0}(0) e^{-i \omega_2 X^0}(1) e^{-i \omega_3 X^0}(\infty) \rangle = i C^X_{S^2} \delta \left( \omega - \sum_{i=1}^3 \omega_i \right) |z|^\omega_1 |z - 1|^\omega_2.$$
3.4.1 Regularization of the Moduli Integral

The expression (3.39) suffers from divergences when the vertex operators approach each other, i.e. from the vicinity of $z = 0, 1, \infty$. The $z$-integral of (3.39) near $z = 0$ is given by

\[
\int d^2z \int_0^\infty \frac{dP}{\pi} C\left(\frac{\omega}{2}, \frac{\omega_1}{2}, P\right) C\left(\frac{\omega_2}{2}, \frac{\omega_3}{2}, P\right) |z|^{-2-\frac{1}{2}(\omega-\omega_1)^2+2P^2} \sum_{n,m=0}^\infty a_{n,m} z^n \bar{z}^m, \tag{3.41}
\]

where we are dropping the prefactor $i\theta_4^4C_{s^2}$. The series expansion on the RHS is obtained by expanding the integrand of (3.39) in powers of $z, \bar{z}$, with the Liouville correlator decomposed in $s$-channel conformal blocks as in (3.22). Thus, the constants $a_{n,m}$ are rational functions of $\omega, \omega_1$, and $P$, with the leading term $a_{0,0}$ given by $a_{0,0} = 1$. In the physical domain where $\omega$ and $\omega_1$ are real, the $z$-integral is divergent for $P^2 < \frac{1}{4}\text{Re}((\omega - \omega_1)^2)$. Our regularization scheme is chosen to preserve analyticity in the external energies, and we motivate it as follows.

Consider instead the integral

\[
\int_0^\infty dx \frac{\theta(1-x)}{x^{1-a}} = \frac{1}{a}, \tag{3.42}
\]

for $\text{Re}(a) > 0$. To analytically continue this integral to $\text{Re}(a) < 0$, we add the “counter term”

\[- \int_0^\infty dx \frac{\theta(-\text{Re}(a))}{x^{1-a}}, \tag{3.43}\]

so that the full integral for $\text{Re}(a) < 0$ becomes

\[
\int_0^\infty dx \frac{\theta(1-x) - \theta(-\text{Re}(a))}{x^{1-a}} = - \int_1^\infty dx \frac{1}{x^{1-a}} = \frac{1}{a}, \tag{3.44}
\]
which is the desired analytic continuation in $a$.

To regularize the divergence in (3.41), we similarly introduce a counter term whose job is to remove the terms in the $n,m$ sum in (3.41) that give rise to a divergence. In particular, we subtract the $a_{n,m}$ satisfying

$$n = m < \frac{1}{4} \text{Re}((\omega - \omega_1)^2) - P^2. \quad (3.45)$$

Thus, the $s$-channel counter term is given by the $z$-integrand

$$R_s = \sum_{0 \leq n \leq \frac{1}{4} \text{Re}((\omega - \omega_1)^2)} \int_0^{\frac{1}{4} \text{Re}((\omega - \omega_1)^2) - n} \frac{dP}{\pi} a_{n,n} C\left(\frac{\omega}{2}, \frac{\omega_1}{2}, P\right)$$

$$\times C\left(\frac{\omega_2}{2}, \frac{\omega_3}{2}, P\right) |z|^{-2 - \frac{1}{2}(\omega - \omega_1)^2 + 2P^2 + 2n}. \quad (3.46)$$

We further need to include counter terms for the $t$- and $u$-channels (from the divergences near $z = 1$ and $z = \infty$, respectively), which are given by

$$R_t = R_s|_{z \to z - 1, \omega_1 + \omega_2}, \quad R_u = |z|^{-4} \left( R_s|_{z \to 1/z, \omega_1 + \omega_3} \right). \quad (3.48)$$

The fully regularized $1 \rightarrow 3$ amplitude is given by

$$A^{\text{pert.}(0)}_{1 \rightarrow 3}(\omega; \omega_1, \omega_2, \omega_3) = ig_s^4 C_s$$

$$\times \int d^2z \left[ |z|^{\omega_1} |1 - z|^{\omega_2} \left\langle V_{\frac{1}{2}}(z, \bar{z})V_{\frac{1}{2}}(0)V_{\omega_1}(1)V_{\omega_2}(\infty) \right\rangle_{\text{Liouville}} - R_s - R_t - R_u \right]. \quad (3.49)$$

In fact, in the physical regime of real energies this regularized $1 \rightarrow 3$ scattering amplitude still
suffers from a divergence in the moduli space integral, which we turn to next.

3.4.2 Subtleties of Massless Scattering in 1+1d

When the closed string energies are real, $\text{Im}(\omega - \omega_i) = 0$ and $\text{Re}((\omega - \omega_i)^2) > 0$, for $i = 1, 2, 3$.

Focusing on the divergence near $z = 0$ for now, the regulated $z$-integrand of (3.49) contains terms of the form

$$\int_{P_n}^{\infty} dP f(P)|z|^{-2+2(p^2-P_n^2)} \sim -\frac{f(P_n)}{4P_n|z|^2 \log |z|},$$

where $P_n$ is given by one of the following values

$$P_n = \frac{1}{2} \sqrt{\text{Re}((\omega - \omega_1)^2)} - n, \quad n = 0, \ldots, \left\lfloor \sqrt{\text{Re}((\omega - \omega_1)^2)} \right\rfloor.$$ \hspace{1cm} (3.51)

In fact, it turns out that $f(P_n) = 0$ vanishes for all values of $n$ except $n = 0$, due to zeroes\(^7\) of the coefficients $a_{n,n}$. For $n = 0$, there is a log log divergence in the $z$-integral, which can be understood as a consequence of the kinematics of massless particles in 1+1d, as we explain below. Note that there are further divergent contributions from the moduli integral near $z = 1$ and $z = \infty$.

![Figure 3.3: Wavepackets of massless particles move with constant separation in 1+1d, and have a natural notion of "ordering".](image)

\(^7\)While we have not found a rigorous proof of this, this can be numerically checked for $n$ up to $n \sim 10$ using the recursion relations for the Virasoro conformal block discussed in Appendix B.
Let’s consider the problem of massless scattering in a 1+1d relativistic quantum field theory of massless particles with cubic interactions\(^8\). In order to have well-defined asymptotic states, the LSZ prescription requires that wave packets are not interacting in the asymptotic region. This is usually accomplished by having far-separated wave-packets in the infinite past/future, which is impossible for the scattering of massless particles in 1+1d, as shown in Figure 3.3. Thus, multi-particle asymptotic states are continuously interacting, even in the asymptotic region.

\[ \omega \rightarrow \omega_1 \]

Figure 3.4: \( s \)-channel contribution to \( 1 \rightarrow 3 \) scattering at tree-level.

To see the problem more explicitly, consider the \( 1 \rightarrow 3 \) scattering amplitude in this quantum field theory. The momentum \((\omega, p)\) of incoming/outgoing on-shell massless particles satisfies the relation \( p = \pm \omega \), where \( \omega > 0 \) in the physical regime. In the \( s \)-channel, the intermediate particle has momentum

\[
(\omega_s, p_s) = (\omega - \omega_1, p - p_1) = (\omega - \omega_1, \omega - \omega_1),
\]

so this particle is also on-shell. Thus, unitarity of scattering amplitudes implies the \( s \)-channel scattering amplitude will be divergent due to a pole of the intermediate particle. Note that

\(^8\)In particular, this excludes integrable theories, which have no particle creation/annihilation.
analytically continuing the external energies keeps the intermediate state on-shell, so that this divergence is unavoidable.

$c = 1$ string theory avoids these problems because the effective string coupling is spatially varying. A continuum of off-shell tachyons with energy $\omega_s = \omega - \omega_1$ and momentum $P > 0$ are exchanged in the s-channel of the tree-level $1 \to 3$ amplitude in (3.49), since momentum is not conserved. As $\text{Im}(\omega - \omega_1) \to 0$, intermediate tachyons with momentum close to $P = \frac{1}{2} \sqrt{\text{Re} \left( (\omega - \omega_1)^2 \right)} \to \frac{1}{2} (\omega - \omega_1)$ satisfy the on-shell condition\(^9\), and give rise to the divergence in the scattering amplitude written in (3.50).

Our prescription is to give a positive imaginary part to the energies of the incoming/outgoing closed strings, while preserving energy conservation. The physical regime is recovered by taking the limit when the imaginary parts go to zero. This prescription renders the integral (3.50) finite, and it gives a real contribution to $A_{1 \to 3}^{\text{pert.(0)}}$. If it weren’t for this contribution, the $1 \to 3$ amplitude (3.49) would be manifestly imaginary, in contradiction with the matrix model result in (2.52).

Note that the exponential suppression of the string coupling as $\phi \to -\infty$ implies that the interaction between wavepackets dies off in this region, so that asymptotic states are well-defined. Furthermore, there is a notion of “ordering” in multi-particle asymptotic states, shown in Figure 3.3. The different orderings of asymptotic states correspond to an order of limits ambiguity in reaching the physical regime of real energies. This is the origin of the ambiguities of the tree-level $2 \to 2$ scattering amplitude (2.54) when the closed string energies are real.

\(^9\)Note that the absence of divergences in (3.50) for $P_n \neq 0$ is consistent with the fact that tachyons are the only on-shell states in $c = 1$ string theory.
which we reproduce analytically below\(^\text{10}\).

### 3.4.3 Real Part of the 1 → 3 Amplitude From Unitarity Cut Methods

The real part of the 1 → 3 scattering amplitude in the physical regime of real energies is obtained via “worldsheet unitarity cut methods” as follows. Let the closed strings energies have small imaginary parts, which by energy conservation satisfy \(\varepsilon_i \equiv \text{Im}(\omega - \omega_i) > 0\), for \(i = 1, 2, 3\). The contribution to the real part of the s-channel 1 → 3 amplitude comes from the vicinity of \(P = P_{n=0} \equiv P_0\) in the \(P\)-integral, see (3.50). Due to this, for \(P > P_0\) we can extend the range of the \(P\) integral to \((P_0, \infty)\). Since no regulators are needed in this range, it follows from (3.41) that the contribution from this range of the \(P\) integral is given by

\[
ig_s^4C_S^2 \int d^2z \int_{P_0}^\infty \frac{dP}{\pi} C(\frac{\omega}{2}, \frac{\omega_1}{2}, P)C(\frac{\omega_2}{2}, \frac{\omega_3}{2}, P)|z|^{-2-\frac{1}{2}(\omega-\omega_1)^2+2P^2} \sum_{m=0}^\infty a_{m,m}|z|^{2m} \tag{3.53}
\]

\[
= ig_s^4C_S^2 \int_{P_0}^\infty dP \frac{C(\frac{\omega}{2}, \frac{\omega_1}{2}, P)C(\frac{\omega_2}{2}, \frac{\omega_3}{2}, P)}{P^2 - P_0^2 - iP_0\varepsilon_1} \tag{3.54}
\]

where \(P_0 \gg \varepsilon_1 > 0\), and we have approximated the last term in the denominator \(\frac{1}{2}\text{Re}(\omega-\omega_1) = P_0\) up to terms of order \(\varepsilon_1\). In the second line we performed the \(z\)-integral\(^\text{11}\), and dropped the contribution from the terms in the sum with \(m > 0\), since in the limit \(\varepsilon_1 \rightarrow 0\) they do not contribute to the real part of \(A^{\text{pert.}(0)}_{1\rightarrow 3}\).

Note that we chose our regulator to preserve analyticity in \(P\) and \(\omega\), and since we are only interested in the contribution coming from the vicinity of \(P = P_0\), we can extend the range of the \(P\)-integral in (3.54) to \(P < P_0\). The real part of (3.54) comes from the imaginary part of

\(^{10}\)It is not obvious why the 1 → \(k\) scattering amplitudes do not suffer from these ambiguities.  
\(^{11}\)It can be numerically checked that the order of the integrals can be interchanged.
the propagator, which in the $\varepsilon_1 \to 0$ limit becomes

$$\text{Im} \frac{1}{P^2 - P_0^2 - iP_0\varepsilon_1} \to \frac{\pi}{2P_0^2} \delta(P - P_0).$$

(3.55)

Performing the integral in $P$, we find that the result is proportional to the on-shell contribution to $1 \to 3$ scattering in the $s$-channel, which is given by

$$\text{Re} A^{\text{pert.}(0)}_{1 \to 3} (\omega_1, \omega_2, \omega_3) = \frac{\pi}{2P_0 C_{S^2}} A^{\text{pert.}(0)}_{1 \to 2} (\omega \to \omega_1, 2P_0) A^{\text{pert.}(0)}_{1 \to 2} (2P_0 \to \omega_2, \omega_3)$$

$$= -\pi g_s^4 C_{S^2} (\omega - \omega_1)\omega_1\omega_2\omega_3$$

(3.56)

where we used $P_0 \to \frac{1}{2}(\omega - \omega_1)$ in the $\varepsilon_1 \to 0$ limit. The contributions from the $t$- and $u$-channels amount to exchanging $\omega_1 \leftrightarrow \omega_2$ and $\omega_1 \leftrightarrow \omega_3$ respectively, so in total we find

$$\text{Re} A^{\text{pert.}(0)}_{1 \to 3} (\omega_1, \omega_2, \omega_3) = -\pi g_s^4 C_{S^2} \omega_1\omega_2\omega_3 [(\omega - \omega_1) + (\omega - \omega_2) + (\omega - \omega_3)]$$

$$= -2\pi g_s^4 C_{S^2} \omega_1^2\omega_2\omega_3.$$

(3.57)

This result is in precise agreement with the matrix model amplitude (2.52) provided that $2\pi g_s^4 C_{S^2} = \mu^{-2}$. This relation and (3.37) fix $g_s$ and $C_{S^2}$ to be given by (3.38).

### 3.4.4 The 2 → 2 Amplitude

The tree level $2 \to 2$ amplitude is given by

$$S^{\text{pert.}(0)}_{2 \to 2} (\omega_1, \omega_2; \omega_3, \omega_4) = \int d^2 z \langle \psi^+_{\omega_1}(z, \bar{z})\psi^+_{\omega_2}(0)\psi^-_{\omega_3}(1)\psi^-_{\omega_4}(\infty) \rangle,$$

(3.58)
where the $z$ integral is regularized in an analogous manner to the $1 \to 3$ amplitude. The $2 \to 2$ amplitude can be formally obtained from the $1 \to 3$ amplitude by analytically continuing one of the outgoing energies $\omega_i$ to minus itself, together with an overall minus sign due to the fact that the DOZZ structure constants (3.19) are odd in the Liouville momenta. If we continue $\omega_2 \to -\omega_2$ while maintaining

$$\text{Im}(\omega_1 + \omega_2), \text{Im}(\omega_1 - \omega_3), \text{Im}(\omega_1 - \omega_4) > 0,$$

(3.59) then we find

$$S_{2 \to 2}^{\text{pert.}(0)}(\omega_1, \omega_2; \omega_3, \omega_4) = -S_{1 \to 3}^{\text{pert.}(0)}(\omega_1; -\omega_2, \omega_3, \omega_4)$$

(3.60)

When $\text{Imax}(\{\omega_j\}) = \omega_1$ for $j = 1, 2, 3, 4$ (recall the definition of $\text{Imax}$ below (2.54)), this analytic continuation is possible and agrees with the matrix model result (2.54).

The unitarity cut methods of the previous subsection extend straightforwardly to the $2 \to 2$ scattering amplitude. For example, when $\text{Imax}(\{\omega_j\}) = \omega_1$, the generalization of (3.57) to $2 \to 2$ scattering is

$$\text{Re} A_{2 \to 2}^{\text{pert.}(0)}(\omega_1, \omega_2; \omega_3, \omega_4) = -\pi g_4^2 C_{S^2} \omega_1 \omega_2 \omega_3 \omega_4 \left[ (\omega_1 + \omega_2) + (\omega_1 - \omega_3) + (\omega_1 - \omega_4) \right]$$

$$= -\omega_1^2 \omega_2 \omega_3 \omega_4,$$

(3.61) where in the last line we used (3.38). Note that for $\text{Imax}(\{\omega_j\}) = \omega_1$, the analytic continuation from the $1 \to 3$ scattering amplitude is given by (3.60), which is indeed what we find in (3.61).

If $\text{Imax}(\{\omega_j\}) = \omega_3$, it follows that $\text{Im}(\omega_1 - \omega_3) < 0$, and therefore the RHS of (3.55) in the
\( t \)-channel gets an overall minus sign. The net effect of this is that (3.61) is now given by

\[
\text{Re} A_{2 \to 2}^{\text{pert},(0)}(\omega_1, \omega_2; \omega_3, \omega_4) = -\pi g_s^4 C_{G2} \omega_1 \omega_2 \omega_3 \omega_4 \left[ (\omega_1 + \omega_2) - (\omega_1 - \omega_3) + (\omega_1 - \omega_4) \right] \\
= -\omega_1 \omega_2 \omega_3^2 \omega_4,
\]

which agrees precisely with the matrix model scattering amplitude (2.54) when \( \text{Im} \{\omega_j\} = \omega_3 \).

A similar argument holds when \( \text{Im} \{\omega_j\} = \omega_2, \omega_4 \).

Thus, we find that the discontinuities observed in the matrix model result in (2.54) are indeed reproduced from the worldsheet formalism, and are a consequence of unitarity. We interpret the ambiguity of (2.54) in the physical regime of real energies as ambiguities in the “ordering” of asymptotic states of massless particles in 1+1d, as discussed above.

It is interesting to explore in more detail what causes the “jump” in the 2 \( \to 2 \) scattering amplitude when \( \text{Im}(\omega_1 - \omega_3) \) crosses zero. As discussed in subsection 3.4.2, there is a continuum of exchanged off-shell particles in the \( t \)-channel labelled by their momentum \( P > 0 \). The particles with \( P \) close to \( \frac{1}{2}(\omega_1 - \omega_3) \) become on-shell in the limit when \( \text{Im}(\omega_1 - \omega_3) \to 0 \). It is interesting that this continuum in some sense tames the divergence, and gives only a finite jump in the 2 \( \to 2 \) scattering amplitude as \( (\omega_1 - \omega_3) \) crosses the real axis.

### 3.5 Numerical Results for 4-point Amplitude

To explicitly evaluate \( A_{1 \to 3}^{\text{pert},(0)} \) and \( A_{2 \to 2}^{\text{pert},(0)} \), we resort to numerical techniques. We focus on the physical regime of real energies, in which the imaginary parts of the external closed string energies are taken to be small.

The regularized 1 \( \to 3 \) scattering amplitude is given by (3.49), and it suffices to perform the

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moduli integral over the region $D = \{|z - 1| < 1, \text{Re } z < 1/2\}$ defined below (3.28), with the remaining contributions obtained by crossing symmetry. In this region, we use the $s$-channel conformal block expansion of the Liouville 4-point function written in (3.22). We sample over different values of intermediate momentum $P'$, and at each value we compute the Virasoro conformal block using Zamolodchikov’s recursion relation, which is reviewed in Appendix B, as well as the DOZZ structure constants given by (3.19). The Liouville 4-point function is obtained by numerically integrating over $P'$.

The coefficients $a_{n,m}$ in the regulator $R_s$ defined by (3.47) are extracted from the expansion in powers of $z, \bar{z}$ of the Virasoro conformal block of the Liouville correlator (together with the appropriate contribution from $X^0$ and the ghosts). After including the contributions from the crossed channels (this includes the regulators $R_{t,u}$), we numerically integrate over the worldsheet moduli $z$. Further details of this calculation are discussed in Appendix C. The $2 \to 2$ scattering amplitude is evaluated using analogous methods.

![Figure 3.5](image)

**Figure 3.5:** Numerical results for the real and imaginary parts of the $-iA_{1\to3}^{(0)}$ string amplitude (red dots) in comparison to the matrix model amplitude (blue dashed line), where $\omega \in \mathbb{R}_{>0} + i \epsilon$ and $\omega_1 = \omega_2 = \omega_3 = \omega/3$, and $\epsilon = 0.01$.

In Figure 3.5, we show the numerical results for the $1 \to 3$ scattering amplitude of closed
strings at tree level, for the special case of \( \omega \in \mathbb{R}_{>0} + i \epsilon \), and \( \omega_1 = \omega_2 = \omega_3 = \omega/3 \), where \( \epsilon > 0 \) is small. The result is in remarkable agreement with the \( 1 \rightarrow 3 \) matrix model amplitude (2.52) (the discrepancy is less than 0.2%).

\[ \omega \in \mathbb{R}_{>0} + i \epsilon \]

\[ \omega_1 = \omega_2 = \omega_3 = \omega/3 \]

\[ \epsilon > 0 \]

\[ \omega = \omega_4 \]

\[ \omega_1 = \omega + i \epsilon \]

\[ \omega_2 = \frac{\omega}{2} - i \frac{a}{3} \]

\[ \omega_3 = \frac{\omega}{4} + i \frac{a}{3} \]

\[ \omega_4 = \frac{5\omega}{4} + i \frac{a}{3} \]

**Figure 3.6:** Numerical results for the real and imaginary parts of the \( -iA^{(0)}_{2 \rightarrow 2} \) string amplitude (red dots) in comparison to the matrix model amplitude (2.54) (blue dashed line) for the momenta (3.63), with \( 0 \leq r \leq 0.9 \). The parameter \( a \) is taken to be \( a = 1.4 \) in (a), (b) and \( a = 0.2 \) in (c), (d). The green dashed line in (c), (d) corresponds to (2.54) with \( \text{Imax}\{\omega_j\} \) replaced by \( \text{Max}(\{\text{Re}\ \omega_j\}) = \omega_4 \) for the momenta (3.63), which clearly deviates from the string amplitude computed.

For the \( 2 \rightarrow 2 \) tree-level amplitude \( \{\omega_1, \omega_2\} \rightarrow \{\omega_3, \omega_4\} \), we consider the choice of momenta given by\(^{12}\)

\[ \omega_1 = r + i a, \quad \omega_2 = \frac{r}{2} - i \frac{a}{3}, \quad \omega_3 = \frac{r}{4} + i \frac{a}{3}, \quad \omega_4 = \frac{5r}{4} + i \frac{a}{3} \]

\(^{12}\)The assignment of a negative imaginary part to \( \omega_2 \) is against our prescription, but it does not change the expected result (2.54) provided \( |\text{Im}(\omega_2)| < \text{Imax}(\{\omega_j\}) \) for \( j = 1, 3, 4 \).
for $a = 1.4$ and $a = 0.2$, and varying real $r$. Note that for $r > \frac{4a}{9}$, the $z$-integral must be regularized as in (3.49), in order to preserve analyticity of the amplitude in the closed string energies. The numerical results for this choice of energies are shown in Figure 3.6, which shows excellent agreement with (2.54).

We find this convincing evidence for the duality between $c = 1$ string theory and the dual matrix quantum mechanics at tree-level. Furthermore, in [30] it was shown using similar methods that the 1-loop $1 \rightarrow 1$ scattering amplitude also agrees on the two sides. This provides further evidence that the perturbative duality extends beyond tree-level. In the next chapter, we go past perturbation theory to test the non-perturbative duality of $c = 1$ string theory.
In the worldsheet formalism, non-perturbative corrections to closed string scattering amplitudes are obtained by including disconnected worldsheet diagrams with boundary conditions that are localized in spacetime [14, 38]. In \( c = 1 \) string theory, this formalism was extended to all non-perturbative orders in [32, 33], where the relevant boundary conditions are called ZZ-instantons. In this chapter, we describe the prescription of [32, 33] to compute non-perturbative scattering amplitudes of closed strings, from which we obtain a precise agreement with the results in (2.53). These results justify our proposal for the dual of the closed string vacuum given in (2.36).
The outline of this chapter is as follows. We begin by reviewing the \( (m,n) \) ZZ boundary condition in \( c = 25 \) Liouville CFT. Next, we define ZZ-branes and ZZ-instantons, and review the duality between \((1,1)\) ZZ-branes and free fermions in the matrix model. Finally, we present the worldsheet formalism for computing non-perturbative corrections to closed string scattering amplitudes. We first describe the formalism at leading perturbative order in each instanton sector, and explicitly compute the contributions \( A_{1\to k}^{1-\text{inst.}(0)} \), \( A_{1\to k}^{2-\text{inst.}(0)} \), \( A_{1\to 1}^{3-\text{inst.}(0)} \), and \( A_{1\to 1}^{4-\text{inst.}(0)} \), and match them against the matrix model answer (2.53). After that, we discuss subleading perturbative contributions and the cancellation of open string divergences via the Fischler-Susskind-Polchinski mechanism [56, 57, 14].

4.1 ZZ Boundary Condition in Liouville CFT

Let's first discuss the ZZ boundary condition in the semiclassical limit. From the semiclassical Liouville action (3.1), we obtain the Liouville equation of motion

\[
\partial \bar{\partial} \phi(z, \bar{z}) = 4\pi \mu be^{2b\phi}, \tag{4.1}
\]

where \( z = \frac{\sigma_1 + i\sigma_2}{2} \), and \( \bar{z} = \frac{\sigma_1 - i\sigma_2}{2} \). On the disk, there is a solution that describes the geometry of the pseudosphere, given by

\[
e^{2b\phi(z, \bar{z})} = \frac{4R^2}{(1 - |z|^2)^2}, \tag{4.2}
\]

where \( R^{-2} = \pi \mu b^2 \), and \( z, \bar{z} \) are coordinates on the disk. In the limit \( |z| \to 1 \) where we approach the boundary, \( \phi \) goes to infinity. Thus, the semiclassical ZZ boundary condition is localized in the strong-coupling region.
Consider now the $c = 25$ Liouville CFT on the disk (for which the semiclassical limit is not valid). It is convenient to use the exponential mapping to interpret the boundary condition on the disk as a state in the bulk theory, called the boundary state $|B\rangle$, see Figure 4.1. Conformal invariance requires that the boundary state admits an expansion in terms of Ishibashi states \cite{58} as

$$|B\rangle = \int_0^{\infty} \frac{dP}{\pi} \Psi_B(P)|V_P\rangle,$$  \hfill (4.3)

where $|V_P\rangle$ denote the Ishibashi state constructed from the Liouville primary operator $V_P$.

A non-trivial constraint on the allowed boundary states comes from modular invariance. Consider the cylinder diagram with boundary conditions specified by boundary states $|B\rangle$, $|B'\rangle$. By inserting a complete basis of bulk states, we can compute this diagram in terms of the coefficients of the expansion in Ishibashi states, namely $\Psi_B(P)$ and $\Psi_{B'}(P)$. On the other hand, the cylinder diagram is given by the partition function of the theory on the strip, with boundary conditions specified by $|B\rangle$, $|B'\rangle$. This partition function admits an expansion in terms of states on the strip with integer coefficients. Modular invariance requires that these two interpretations are consistent, see Figure 4.2.
Figure 4.2: Modular invariance of the cylinder partition function. On the right, we have the partition function of the Hilbert space on the strip with boundary conditions $B$ and $B'$, evolved in Euclidean time by $2\pi/t$. On the left, we flipped the cylinder on its side and performed an overall rescaling. In this case, the cylinder is interpreted as the overlap $\langle B'|B \rangle$, in which states in the bulk propagate from Euclidean times $t_E = 0$ to $t_E = \pi t$.

The $(1,1)$ ZZ boundary condition is defined such that the only primary boundary operator in the strip Hilbert space is the identity operator. Define its boundary state by

$$|ZZ(1,1)\rangle = \int_0^\infty \frac{dP}{\pi} \Psi^{(1,1)}(P)|V_P\rangle. \quad (4.4)$$

Then modular invariance imposes the constraint

$$\tilde{\chi}_{(1,1)}(-1/\tau) = \int_0^\infty \frac{dP}{\pi} \left(\Psi^{(1,1)}(P)\right)^2 \chi_{1+P^2}(\tau), \quad (4.5)$$

where $\tau \equiv it$ parameterizes the modulus of the cylinder, see Figure 4.2. $\chi_h(\tau)$ is the $c = 25$ Virasoro character of weight $h$ and $\tilde{\chi}_{(m,n)}(\tau)$ is the $c = 25$ Virasoro character of the degenerate
primary of weight \(1 - \frac{(m+n)^2}{4}\), which are given by

\[
\chi_{\ell}(\tau) = \frac{q^{h-1}}{\eta(\tau)},
\]

\[
\hat{\chi}_{(m,n)}(\tau) = q^{-\frac{(m+n)^2}{4}} - q^{-\frac{(m-n)^2}{4}} \frac{\eta(\tau)}{\eta(\tau)},
\] (4.6)

where \(q = \exp(2\pi i \tau)\). The constraint (4.5) gives

\[
\Psi^{(1,1)}(P) = 2^{\frac{3}{4}} \sqrt{\pi} \sinh(2\pi P). \tag{4.7}
\]

More generally, the \((m,n)\) ZZ boundary state is given by

\[
|\text{ZZ}(m,n)\rangle = \int_0^\infty \frac{dP}{\pi} \Psi^{(m,n)}(P)|V_P\rangle, \tag{4.8}
\]

and it is defined so that the strip Hilbert space with \((1,1)\) ZZ boundary condition on one boundary and \((m,n)\) on the other consists of the Verma module of the \((m,n)\) degenerate primaries of the \(c = 25\) Virasoro algebra. Thus, we have the modular bootstrap constraint

\[
\hat{\chi}_{(m,n)}(-1/\tau) = \int_0^\infty \frac{dP}{\pi} \Psi^{(m,n)}(P)\Psi^{(1,1)}(P)\chi_{1+P^2}(\tau), \tag{4.9}
\]

from which it follows that

\[
\Psi^{(m,n)}(P) = 2^{\frac{3}{4}} \sqrt{\pi} \frac{\sinh(2\pi mP)\sinh(2\pi nP)}{\sinh(2\pi P)}. \tag{4.10}
\]

Note that (4.10) is symmetric under \(m \leftrightarrow n\), so we will henceforth restrict to \(m \geq n\).
The cylinder partition function with \((m, n)\) and \((m', n')\) ZZ boundary conditions is given by

\[
\int_0^\infty \frac{dP}{\pi} \Psi^{(m,n)}(P) \Psi^{(m',n')}(P) \chi_{1+P^2} \left(-\frac{1}{\tau}\right)
\]

\[
= \sum_{p=0}^{\min(m,m')-1} \sum_{q=0}^{\min(n,n')-1} \chi(m+m'-2p-1,n+n'-2l-1) \left(\tau\right). 
\]

(4.11)

In particular, the lowest weight state \(\varphi_{(m,n)}\) when the strip has the same \((m, n)\) ZZ boundary condition has conformal weight \(h = 1 - (n + m - 1)^2\). Thus, among the \((m, n)\) ZZ boundary conditions, only the \((1,1)\) does not have negative weight states and is therefore unitary. Nevertheless, other types of ZZ boundary conditions will also play a role in our discussion.

The one-point function of the Liouville primary operator \(V_P\) on the disk with \((m, n)\) ZZ-boundary condition is given by

\[
\langle V_P(z, \bar{z}) \rangle_{(m, n)}^{D_2} = \frac{\Psi^{(m,n)}(P)}{|z - \bar{z}|^{h_P}},
\]

(4.12)

where \(h_P\) is the conformal weight of \(V_P\). Other boundary structure constants can be bootstrapped from the crossing relations of correlation functions with boundary operators [42, 43, 26, 59], but they will not be needed below.

4.2 ZZ-BRANES AND ZZ-INSTANTONS IN \(c = 1\) STRING THEORY

4.2.1 ZZ-BRANE

A \((m, n)\) ZZ-brane in \(c = 1\) string theory is defined as a \((m, n)\) ZZ boundary condition in Liouville together with Neumann boundary conditions for \(X^0\) and \(b, c\) ghosts. Its boundary
state $|ZZ(m,n)\rangle_{c=1}$ is given by

$$|ZZ(m,n)\rangle_{c=1} \equiv |ZZ(m,n)\rangle \otimes |N\rangle_{X^0} \otimes |N\rangle_{b,c} \quad (4.13)$$

where $|ZZ(m,n)\rangle$ is given by (4.8), and $|N\rangle_{X^0}$ and $|N\rangle_{b,c}$ denote the Neumann boundary states in the $X^0$ and $b, c$ ghost CFTs. The $(m,n)$ ZZ-branes are heuristically like D0-branes, which extend in time but are localized in the spatial (Liouville) direction.

The lowest weight state of Liouville on the strip with $(m,n)$ ZZ boundary condition gives rise to an on-shell open-string tachyon of the $(m,n)$ ZZ-brane, represented by the boundary vertex operator\(^1\)

$$\Phi^{(m,n)} \equiv \# e^{i\omega X^0} \# \varphi_{(m,n)}, \quad M_{(m,n)}^2 = \omega^2 = -(m + n - 1)^2 \quad (4.14)$$

where $M_{(m,n)}$ is the mass of the open-string, and $\varphi_{(m,n)}$ is the lowest weight state of Liouville with $(m,n)$ ZZ boundary condition discussed below (4.11). In particular, this implies that all ZZ-branes suffer from open string tachyon instabilities.

From the matrix model description, the (1,1) ZZ-brane is dual to a classical configuration in which a fermion of energy $\mu$ comes in from the asymptotic region $\lambda = \infty$ in the asymptotic past, rolls up to the top of the potential $V(\lambda) = -\lambda^2/2$, and then rolls back towards $\lambda = \infty$ in the asymptotic future [17, 18, 60, 61]. At time $t = 0$, the fermion is momentarily resting at the top of the inverted quadratic potential, and the instability in this configuration is described from the string theory side by the open string tachyon $\Phi^{(1,1)}$ on the (1,1) ZZ-brane. As the

\(^1\)As in (3.29), we are omitting the $c$ ghosts.
open string tachyon condenses, the fermion rolls from the top of the potential, and the vacuum expectation value of \( \Phi^{(1,1)} \) is related to the position \( \lambda \) of the rolling fermion.

More generally, there is a family of exactly marginal deformations of the \( X^0 \) Neumann boundary condition, given by\(^2\) [62, 63]

\[
\lambda_0 \int d\xi \Phi^{(1,1)}(\xi) = \lambda_0 \int d\xi \cosh(X^0(\xi)).
\] (4.15)

The interpretation of these boundary states is that the fermion rolling from \( \lambda = \infty \) has energy \( \mu \cos^2(\pi \lambda_0) \), and reaches a turning point at \( \lambda = \sqrt{2\mu} \sin(\pi \lambda_0) \) [17, 18, 60, 61].

Consider now the disk bulk 1-point with the (1,1) ZZ-brane boundary condition deformed by (4.15). The closed string radiation produced as the tachyon rolls from the top of the potential is obtained by Fourier transforming in \( \omega \) and taking the late-time limit. For \( \lambda_0 < 0 \), the tachyon rolls to the “other side” indefinitely, and the classical rolling approximation breaks down.

\(^2\)We are writing this deformation in Minkowski signature, but to avoid dealing with non-normalizable operators, we should do this deformation in the Euclidean theory and then Wick rotate \( X^0 \rightarrow iX^0 \).
corresponds to the fact that the Fermi sea on the “left-side” of the inverted quadratic potential is empty, so the fermion keeps accelerating forever, as shown in Figure 4.3.

If $\lambda_0 > 0$, the late-time closed string radiation produced by the condensation of the (1,1) ZZ-brane is given by a “kink” in the fluctuation of the fermion density, which is dual to a fermion moving with constant speed on top of the Fermi surface [18, 61]. This result provides non-trivial evidence for the proposal that the (1,1) ZZ-brane is dual to a fermion on top of the inverted quadratic potential. One consequence of this is that the mass of the (1,1) ZZ-brane $M_{ZZ}$ is equal to the energy of the rolling fermion, that is $M_{ZZ} = \mu$.

Significantly less is understood about the matrix model interpretation of other $(m, n)$ ZZ-branes. These ZZ-branes admit other boundary deformations besides (4.15), such as

$$\lambda_0 \int d\xi \cos (M_{(m,n)}X^0(\xi)) \psi_{(m,n)}(\xi) = \lambda_0 \int d\xi \cosh \left( (m + n - 1)X^0(\xi) \right) \psi_{(m,n)}(\xi).$$

To our knowledge, it is not known if these are exactly marginal boundary deformations. It is an interesting question to find the late-time closed string radiation produced by the $(m, n)$ ZZ-branes.

### 4.2.2 ZZ-INSTANTON

The $(m, n)$ ZZ-instanton is defined as a $(m, n)$ ZZ boundary condition in Liouville CFT together with Dirichlet boundary condition for (Euclidean) $X^0$ and Neumann boundary conditions for

\[ ... \]
the $b,c$ ghosts. Its boundary state $|ZZ(m,n),x^E\rangle$ is defined as

$$|ZZ(m,n),x^E\rangle \equiv |ZZ(m,n)\rangle \otimes |D,x^E\rangle_{X^0} \otimes |N\rangle_{b,c},$$ (4.17)

where $|D(x^E)\rangle_{X^0}$ is the Dirichlet boundary state with Euclidean parameter $x^E$ (we will eventually Wick rotate to Lorentzian signature, as discussed in the next section). Besides a single ZZ-instanton, we also consider multi-instanton configurations, whose boundary states are given by direct sums of (4.17). We will see below that only ZZ-instantons of type $(m,1)$ are relevant for our discussion, and for this reason we write $\{m_1,...,m_\ell\}$ to label the direct sum of boundary states corresponding to ZZ-instantons of types $(m_i,1)$ and (Euclidean) Dirichlet labels $x^E_i$, for $i = 1,...,\ell$.

Consider the open-string spectrum of a $(m,1)$ ZZ-instanton. From (4.11), it follows that the boundary primary operators of Liouville CFT with $(m,1)$ ZZ boundary condition is given by the degenerate representations of the $c = 25$ Virasoro algebra, of conformal weights $-j^2 + 1$, for $j = 1,...,m$, which we call $V_j$ (for example, $V_0$ is proportional to the identity operator). For the $X^0$ CFT, the boundary primary operators with Dirichlet boundary conditions are labelled by $X_j$, and they are given by the “special states” of the non-compact free-boson CFT [64], with conformal weight $j^2$, for $j \in \mathbb{Z}_{\geq 0}$ (for example, $X_1 \equiv \partial X^0$). Thus, the on-shell open strings of a $(m,1)$ ZZ-instanton are given by $X_jV_j$ for $j = 1,...,m$. The vertex operator $X_1V_1$ is simply $\partial X^0$, which is responsible for shifting the collective coordinate $x^E$. The other vertex operators are not expected to be exactly marginal, and therefore we do not consider them further.

We now collect some results that will be needed in the next section, where we consider
worldsheet diagrams with \((m, n)\) ZZ-instanton boundary conditions. The action of a \((1, 1)\) ZZ-instanton, \(S_{(1,1)}\), is related to the mass of the \((1, 1)\) ZZ-brane \(M_{ZZ}\) by

\[
S_{(1,1)} = 2\pi M_{ZZ} = 2\pi \mu = \frac{1}{g_s},
\]

(4.18)

where in the last equality we used (3.38). The factor of \(2\pi\) is due to the fact that the ZZ-instanton and the ZZ-brane differ only by the Dirichlet/Neumann boundary condition for \(X^0\) [65]. Note that \(e^{-S_{(1,1)}}\) agrees exactly with the suppression of the leading non-perturbative correction of the matrix model scattering amplitudes, which is of order \(e^{-2\pi\mu}\) for a “bounce” of a fermion/hole at the Fermi surface, see (2.31).

More generally, the action of a \((m, n)\) ZZ-instanton is formally given by (minus) the empty disk diagram. Instead of computing this diagram from the worldsheet formalism, which requires some prescription for fixing the \(c\) ghost insertions, we compute it by inserting a bulk Liouville operator on the disk, and analytically continuing the Liouville momentum \(P \to i\). In this limit, the Liouville operator \(V_P\) has zero weight, and therefore it is a multiple of the identity operator. From this, we conclude that

\[
S_{(m,n)} = \lim_{P \to i} \frac{\Psi_{(m,n)}(P)}{\Psi_{(1,1)}(P)} S_{(1,1)} = mnS_{(1,1)},
\]

(4.19)

where we used the disk bulk 1-point functions (4.10). Note that the \((m, n)\) ZZ-instantons have actions that are consistent with the non-perturbative corrections due to multiple fermion/hole bounces.

The disk with a single closed string insertion \(\mathcal{V}^\pm\) and with \((m, n)\) ZZ-instanton boundary
condition is given by

\[
\{ (m,n), x \} = \langle c \bar{c} \mathcal{V}_\omega^{\pm} \rangle_{(m,n),x} = g_s \frac{C_{D_2}}{2\pi} \Psi^{(m,n)} \left( \frac{\omega}{2} \right) e^{\pm i\omega x} = 2e^{\pm i\omega x} \frac{\sinh(m\pi\omega) \sinh(n\pi\omega)}{\sinh(\pi\omega)}.
\]

where \( x \) is the (Lorentzian) collective coordinate of the ZZ-instanton and we used (4.10) in the last equation. The factor of \( \frac{1}{2\pi} \) comes from the volume of the residual conformal Killing group, which is not completely fixed with a single bulk insertion. The normalization \( C_{D_2} \) can be fixed from factorization of scattering amplitudes on the disk, but we will fix it shortly by comparison against the matrix model result.

Next, consider the empty cylinder diagram between two (1,1) ZZ-instantons at Euclidean times \( x_1^E \) and \( x_2^E \). The free-boson cylinder partition function with Dirichlet boundary conditions separated in Euclidean time by \( x_1^E \equiv x_1^E - x_2^E \) is given by \( e^{-t \frac{(x_1^E)^2}{2\pi}} / \eta(it) \), where \( t \) parameterizes the modulus of the cylinder, see Figure 4.2. The cylinder partition function for Liouville with (1,1) ZZ boundary conditions follows from (4.5), and it is given by \( \tilde{\chi}(it) = (e^{2\pi t} - 1) / \eta(it) \). The cylinder partition function for the \( b, c \), ghosts is given by \( \eta(it)^2 \) as usual, so they cancel the expansion in oscillator modes of the free boson and Liouville CFT. Integrating over the modulus \( t \), we find in total

\[
\int_0^\infty \frac{dt}{2t} \left( e^{2\pi t} - 1 \right) e^{-t \frac{(x_1^E)^2}{2\pi}} = \frac{1}{2} \ln \left[ \frac{(x_{12}^E)^2}{(x_{12}^E)^2 - 4\pi^2} \right],
\]

where we have divided by a factor of \( 2t \) which comes from the residual CKG of the cylinder,
and the labels 1 and 2 in the worldsheet diagram denote the ZZ-instantons at positions $x_1^E$ and $x_2^E$. Note that when $x_{12}^E < 2\pi$, there is a tachyonic open string stretched between the two (1,1) ZZ-instantons, and the moduli integral is formally ill-defined. Instead, we define the empty cylinder diagram for $x_{12}^E < 2\pi$ by analytic continuation in $x_{12}^E$ from the region $x_{12}^E > 2\pi$.

Finally, the empty cylinder diagram between a (1,1) and a (n,1) ZZ-instanton (with $n \neq 1$) is given by

$$\frac{1}{2} = \frac{1}{2} \int_0^\infty \frac{dt}{t} \left[ e^{2\pi t(n+1)^2/4} - e^{2\pi t(n-1)^2/4} \right] e^{-t(x_{12}^E)^2/4} = \frac{1}{2} \ln \left[ \frac{(x_{12}^E)^2 - ((n - 1)\pi)^2}{(x_{12}^E)^2 - ((n + 1)\pi)^2} \right], \quad (4.22)$$

where the dashed boundary condition in the worldsheet diagram indicates a ZZ-instanton of type different than (1,1). The steps are similar to (4.21), except that we use (4.9) for the cylinder partition function of Liouville. Lastly, we will need the cylinder diagram between two (n,1) ZZ-instantons, which is given by

$$\frac{1}{2} = \frac{1}{2} \int_0^\infty \frac{dt}{t} \left[ e^{2\pi t\nu^2} - 1 \right] e^{-t(x_{12}^E)^2/4} = \frac{1}{2} \ln \left[ \frac{(x_{12}^E)^2}{(x_{12}^E)^2 - (2n\pi)^2} \right]. \quad (4.23)$$

As above, we regularize the moduli integral by analytic continuation in $x_{12}^E$.

4.3 Non-Perturbative Contributions to Closed String Scattering Amplitudes

The scattering amplitudes of closed strings obtained from the $c = 1$ matrix model admits an expansion in instanton sectors, which for $1 \to k$ scattering is given by (2.51). In [14], a
worldsheet formalism for computing non-perturbative scattering amplitudes of closed strings was proposed, and applied at leading order in type IIB string theory in [38] (see also [66] for higher order corrections). The main idea is to include disconnected worldsheet diagrams with D-instanton boundary conditions, which are localized in spacetime.

Here we describe this prescription for the case of $c = 1$ string theory [32, 33], for which we include disconnected worldsheet diagrams with ZZ-instanton boundary conditions. Furthermore, we will extend the formalism to all non-perturbative orders, and therefore consider multi-ZZ-instanton configurations.

Below, we will explicitly evaluate the 4-instanton amplitude $A_{1\to1}^{4-\text{inst.}(0)}$, and find that the $(2, 2)$ ZZ-instanton does not contribute to this scattering amplitude. More generally, the scattering amplitude $A_{1\to1}^{n-\text{inst.}(0)}$ was computed for all $n$ in [33], and it was shown that all $(m, n)$ ZZ-instantons with $n \geq 2$ do not contribute to closed string scattering amplitudes.

The non-perturbative scattering amplitudes of closed strings computed in this section are non-unitary. For example, from the matrix model results (2.53) which we will reproduce in this section, we can explicitly verify that instead of (3.31), we now have

$$
\sum_{n=1}^{\infty} \frac{1}{\mathrm{in} \langle \omega | \omega \rangle} \int_{0}^{\infty} \prod_{i=1}^{k} \frac{d\omega_{i}}{\mathrm{out} \langle \omega_{i} | \omega_{i} \rangle} \delta \left( \omega - \sum_{i=1}^{n} \omega_{i} \right) |A_{1\to n}(\omega_{1}, ..., \omega_{n})|^{2} = 1 - P, \quad (4.24)
$$

where we slightly abused notation to write $\mathrm{in/out} \langle \omega' | \omega' \rangle^{\mathrm{in/out}}$ for the normalization of the asymptotic states with the delta function stripped-off, see (2.43). Here $P$ is a positive real number of order $e^{-\frac{1}{gs}}$. From the dual description, the interpretation is that we are missing the asymptotic states of the tunnelled particle and/or hole. We expect that from the worldsheet formalism
these asymptotic states are captured by $(1, 1)$ ZZ-branes with open string tachyon condensed to the “other side”. It would be interesting to verify this explicitly.

4.3.1 General Strategy

Let’s first discuss the contribution of a multi-instanton configuration $\{m_1, ..., m_\ell\}$, defined in section 4.2.2, to a scattering amplitude with $k + 1$ closed strings. The moduli of the multi-instanton is given by the collective coordinates $x_i^E$, for $i = 1, ..., \ell$, and we will eventually integrate over $x_i^E$ to restore time translation invariance.

We first sum over all disconnected worldsheet diagrams with $\{m_1, ..., m_\ell\}$ boundary conditions and $k + 1$ closed string insertions. In performing this sum, it is convenient to explicitly separate the worldsheet diagrams with closed string insertions from empty worldsheet diagrams. Once we include the symmetry factors, the infinite sum over empty diagrams exponentiates to a sum over connected empty worldsheet diagrams, as shown in Figure 4.4.

$$1 + \left( \begin{array}{c} \circ \ + 
\ldots \end{array} \right) + \frac{1}{2} \left( \begin{array}{c} \circ \ + 
\ldots \end{array} \right)^2 + \ldots = \exp \left( \begin{array}{c} \circ \ + 
\ldots \end{array} \right)$$

**Figure 4.4:** The formal sum over disconnected empty diagrams factorizes and exponentiates to a sum over connected empty diagrams. The disk diagram with $(m, 1)$ ZZ-instanton boundary condition gives the instanton action $S_{(m, 1)}$, while empty diagrams with more boundaries are suppressed by powers of $g_s$.

Let’s focus on the exponentiated empty diagrams. The leading contribution in the weak string coupling limit comes from the empty disk, of order $1/g_s$. For a $\{m_1, ..., m_\ell\}$ ZZ-instanton, whose boundary state is the direct sum of $|ZZ(m_i, 1), x_i^E\rangle$ for $i = 1, ..., \ell$, we simply sum over the empty
disks for each ZZ instanton. From (4.19), this gives

\[- \sum_{i=1}^{\ell} S_{(m_i,1)} = - \sum_{i=1}^{\ell} \frac{m_i}{g_s}, \]

which is the multi-instanton action. Thus, at order \( e^{-\frac{n^2}{g_s}} \), we have to sum over all multi-instanton configurations satisfying \( \sum_{i=1}^{\ell} m_i = n \). From now on, we simply drop the contributions of the empty disks and replace it by the exponential of the multi-instanton action.

The next contribution in the weak coupling limit comes from empty cylinders, where we can either have both boundaries ending on the same ZZ-instanton, or they can end on different ZZ-instantons. When the boundaries end on the same \((m, 1)\) ZZ-instanton, the cylinder diagram is formally divergent (it has a tachyonic and a massless IR divergence), and requires regularization. Instead, we simply replace its exponential by a constant \( \mathcal{N}_m \), that we will fix by comparing against the matrix model amplitude.

When the cylinder has boundaries on different ZZ-instantons, at Euclidean times \( x_i^E \) and \( x_j^E \) for \( i \neq j \), the cylinder diagram will depend explicitly on \( x_{i_j}^E \), as obtained in (4.21) for the case of two \((1, 1)\) ZZ-instantons. Thus, we find that this gives a measure on the moduli space of the ZZ-instantons, which we will discuss in more detail shortly.

Further empty diagrams are of order \( g_s \) or higher, and give subleading contributions. We interpret the contributions from these diagrams with all boundaries on the same \((m, 1)\) ZZ-instanton as renormalizing the action \( S_{(m,1)} \), while empty diagrams with boundaries on different ZZ-instantons renormalize the moduli space measure \( \mu(x_1, \ldots, x_{\ell}) \). Thus, integrating over the
moduli $x_i^E$, the contribution from the $\{m_1, \ldots, m_\ell\}$ multi-instanton is schematically given by\(^3\)

\[
\prod_{i=1}^\ell \left( N_{m_i} e^{-S(m_i)} \right) \int \prod_{i=1}^\ell dx_i^E \mu(x_1^E, \ldots, x_\ell^E) \times \text{(worldsheet diagram)}_{k+1 \text{ closed strings}}
\]

\[
\mu(x_1^E, \ldots, x_\ell^E) = \frac{1}{S} \exp \left[ \sum_{1 \leq i, j \leq \ell} j + O(g_s) \right].
\]  

(4.26)

In the first line, we have explicitly included the constant $N_{m_i}$ which come from the cylinder diagram ending on the same ZZ-instanton\(^4\). The expression $(\text{worldsheet diagram})_{k+1 \text{ closed strings}}$ denotes disconnected, non-empty diagrams with $k + 1$ closed string insertions in total. Finally, the measure factor $\mu(x_1^E, \ldots, x_\ell^E)$ includes contributions from empty diagrams with boundaries on different ZZ-instantons, the leading contribution coming from the empty cylinder diagrams.

There is also a symmetry factor of $1/\prod_a \ell_a!$, where $\ell_a$ is the number of $m_i$‘s that are equal to $a$.

Let’s consider the measure on the moduli space in more detail, focusing on the case of two $(1, 1)$ ZZ-instantons for simplicity. From (4.21), we find

\[
\mu(x_1^E, x_2^E) = \frac{1}{2} \frac{(x_{12}^E)^2}{(x_{12}^E)^2 - 4\pi^2},
\]  

(4.27)

where on the RHS we included the symmetry factor $\frac{1}{2}$. In the limit $x_{12}^E \to 0$, the measure is proportional to $(x_{12}^E)^2$, which is interpreted as the Vandermonde determinant obtained from gauge fixing to diagonal form the non-Abelian collective coordinates of coincident ZZ-instantons.

\(^3\)Here, the thick line for the cylinder diagram can denote a ZZ-instanton of any type, as opposed to the convention followed in (4.22) and in the discussion below.

\(^4\)This constant also absorbs a possible constant factor in the integration measure for the collective coordinate $x_i^E$. 

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When $x_j^E = 2\pi$, there is a pole in the moduli space measure, and we need to specify an integration contour that avoids the pole. Our prescription is to perform all moduli integrals in Lorentzian coordinates\(^5\), which amounts to Wick rotating $x_j^E \rightarrow ix_j$, and integrating over $x_j$ (the factor of $i$ from this Wick rotation is absorbed in $\mathcal{N}_{m_j}$), for $j = 1, ..., \ell$. This prescription regularizes all the divergences in the moduli integration, so that we can finally integrate over the (Lorentzian) collective coordinates $x_i, i = 1, ..., \ell$.

We now consider the disconnected, non-empty diagrams. At leading order in the $g_s$ expansion, the diagram that dominates has $k + 1$ disconnected disks, with one closed string insertion each, as shown in Figure 4.5. These are the diagrams that contribute to the scattering amplitudes $\mathcal{A}^{n-\text{inst.}(0)}$ in (2.53), which we now compute for $1 \leq n \leq 4$. After that, we discuss contributions from subleading non-empty diagrams, where a delicate cancellation in divergences occurs via the Fischler-Susskind-Polchinski mechanism [56, 57, 67].

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{disconnected_diagram.png}
\caption{At order $e^{-\frac{\pi}{g_s}}(g_s)^0$, the non-empty disconnected diagrams contributing to the $1 \rightarrow k$ scattering consist of $k + 1$ disconnected disks with 1 closed string insertion each, and with multi-instanton boundary condition \{m_1, ..., m_\ell\}.}
\end{figure}

\(^5\)This choice of integration contour is equivalent to a prescription for going around the pole in the Euclidean integral. The choice where the Euclidean contour goes around the pole in the opposite way reproduces, for different constants $\mathcal{N}_{m_j}$, the time-reversed scattering amplitudes. Thus, the breaking of time-reversal symmetry by the state (2.36) is also manifest from the worldsheet formalism.
4.3.2 1-Instanton Correction to the $1 \to k$ Amplitude

At this order, the moduli space consists of a single collective coordinate and hence the instanton measure is constant. Thus, we find the following contribution to $1 \to k$ scattering at order $e^{-\frac{1}{g_s}}$,

$$S_{1\rightarrow n}^{1\text{-inst.},(0)}(\omega_1,\ldots,\omega_n) = N_1 e^{-S(1,1)} \int_{-\infty}^{\infty} dx \langle c\bar{c}V^+(z,\bar{z}) \rangle^{D_2}_{1,1,x_0} \prod_{i=1}^{n} \langle c\bar{c}V^-_{\omega_i}(z,\bar{z}) \rangle^{D_2}_{1,1,x_0}$$

$$= 2\pi N_1 e^{-\frac{1}{g_s}} \left( \frac{2^{\frac{3}{4}}}{\sqrt{\pi}} g_s C_{D_2} \right)^{n+1} \delta \left( \omega - \sum_{i=1}^{n} \omega_i \right) \sinh(\pi \omega) \prod_{i=1}^{n} \sinh(\pi \omega_i), \quad (4.28)$$

where in the last line we used (4.20).

Comparison against the matrix model result (2.53) fixes

$$N_1 = -\frac{1}{8\pi^2}, \quad C_{D_2} = \frac{2^{\frac{3}{4}} \sqrt{\pi}}{g_s}. \quad (4.29)$$

4.3.3 2-Instanton Correction to the $1 \to k$ Amplitude

Let’s first discuss the 2-instanton contribution to the $1 \to 1$ scattering of closed strings. There are two contributions to be taken into account, a single $(2,1)$ ZZ-instanton, and a $\{1,1\}$ multi-instanton configuration.

Consider first the contribution from the $(2,1)$ ZZ-instanton. In this case, the (non-empty) diagrams that contribute are two disconnected disks with 1 closed insertion each, with the same
(2, 1) ZZ-instanton boundary condition. This diagram is evaluated in (4.20), so in total we find
\[
\begin{aligned}
e^{-\frac{2}{g_s}N_2} \int dx_1 \left( \left( \begin{array}{c}
1 \\
\end{array} \right) \left( \begin{array}{c}
1 \\
\end{array} \right) \right) & = 4N_2 e^{-\frac{2}{g_s}} \int_{-\infty}^{\infty} dx \sinh(2\pi \omega) \sinh(2\pi \omega_1) e^{i(x-\omega_1)} \\
& = 8\pi \delta(\omega - \omega_1) N_2 e^{-\frac{2}{g_s} \sinh(2\pi \omega_1)^2}.
\end{aligned}
\]
\hspace{1cm} (4.30)

For the contribution from the \{1, 1\} ZZ-instantons, the (non-empty) diagrams that contribute are two disconnected disks with 1 closed string insertion each. However, each disk can have either ZZ-instanton boundary condition, with Lorentzian collective coordinates \(x_1\) or \(x_2\). Furthermore, we have to include the measure factor for two \(1, 1\) ZZ-instantons given by (4.27). The contribution where the two disks 1-point have the same ZZ-instanton boundary condition, say at \(x_1\), is given by
\[
\begin{aligned}
e^{-\frac{2}{g_s}N_1^2} \frac{1}{2} \int dx_1 dx_2 \exp \left( 2 \left( \begin{array}{c}
1 \\
\end{array} \right) \left( \begin{array}{c}
1 \\
\end{array} \right) \right) & = e^{-\frac{2}{g_s}N_1^2} \frac{1}{2} \frac{2}{2} \sinh(\pi \omega_1) \sinh(\pi \omega_2) \int dx_1 dx_2 \frac{(x_{12})^2}{(x_{12})^2 + (2\pi)^2} e^{i(\omega_1 - \omega_2)x_1} \\
& = e^{-\frac{2}{g_s}N_1^2} 4\pi \delta(\omega_1 - \omega_2) \sinh^2(\pi \omega_1) \int d(x_{12}) \frac{(x_{12})^2}{(x_{12})^2 + (2\pi)^2}.
\end{aligned}
\]
\hspace{1cm} (4.31)

Note that there is an IR divergence from the limit \(x_{12} \to \infty\). This divergence is due to the fact that we are including a “disconnected” diagram in our calculation. In computing the scattering amplitudes, we should normalize by the vacuum amplitude, which also gets instanton corrections. In particular, expanding the vacuum amplitude at one-instanton level, we find that we have to subtract the disconnected diagram given by the \(1 \to 1\) scattering amplitude at one-instanton level, times a single \(1, 1\) ZZ-instanton coming from the vacuum amplitude that does not contribute to the scattering, as shown in Figure 4.6. The net result is that, for the diagrams
with disks ending on the same boundary condition, we make the replacement
\[
\frac{(x_{12})^2}{(x_{12})^2 + 4\pi^2} \to \frac{(x_{12})^2}{(x_{12})^2 + 4\pi^2} - 1.
\]

(4.32)

\[\text{Figure 4.6: Summary of 2-instanton contributions to the } 1 \to 1 \text{ closed string amplitude. The first diagram represents worldsheets with boundary on the direct sum of two } (1, 1) \text{ ZZ-instantons. The second diagram represents subtraction of the disconnected instanton amplitude in which one of the instantons contributes to the vacuum amplitude. The third diagram represents the } (2, 1) \text{ ZZ-instanton contribution.}\]

When the two disks 1-point have different ZZ-instanton boundary conditions, the contribution is given by

\[
e^{-\frac{2}{g_s} N_1^2 \frac{1}{2}} \int dx_1 dx_2 \exp \left(2 \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 1 & 2 \end{bmatrix} \right) \begin{bmatrix} 1 \star \star + \star \star \star \\ 1 \star \star + \star \star \star \end{bmatrix} \]

\[
= e^{-\frac{2}{g_s} N_1^2 \frac{1}{2}} \frac{1}{2} \sinh(\pi \omega_1) \sinh(\pi \omega_2) \int dx_1 dx_2 \frac{(x_{12})^2}{(x_{12})^2 + (2\pi)^2} \left( e^{i\omega_1 x_{12} - i\omega_2 x_2} + e^{i\omega_1 x_2 - i\omega_2 x_{12}} \right)
\]

\[
= e^{-\frac{2}{g_s} N_1^2 \frac{1}{4} \pi \delta(\omega_1 - \omega_2) \sinh^2(\pi \omega_1)} \int dx_{12} \frac{(x_{12})^2}{(x_{12})^2 + (2\pi)^2} \left( e^{i\omega_1 x_{12}} + e^{-i\omega_1 x_{12}} \right)
\]

(4.33)

Combining the contributions (4.30), (4.31), and (4.33), we find the result (note that (4.31)
comes with a factor of 2 since the two disks 1-point can have collective coordinate $x_1$ or $x_2$)

$$e^{-\frac{2}{gs}N_1^2}\frac{1}{2} \int dx_1 dx_2 \left[ 2 \left( \exp \left( 2 \frac{1}{(x_{12})^2 + (2\pi)^2} \right) - 1 \right) + \left( \frac{1}{(x_{12})^2 + (2\pi)^2} \right) \exp \left( 2 \frac{1}{(x_{12})^2 + (2\pi)^2} \right) \right]$$

$$+ e^{-\frac{2}{gs}N_2} \int dx_1 \left( \frac{1}{(x_{12})^2 + (2\pi)^2} \right)$$

$$= e^{-\frac{2}{gs}2\pi\delta(\omega_1 - \omega_2)} \left\{ N_1^2 2 \sinh^2(\pi\omega_1) \left[ 2 \int d(x_{12}) \left( \frac{(x_{12})^2}{((x_{12})^2 + (2\pi)^2} - 1 \right) \right.ight.$$  

$$+ \int d(x_{12}) \frac{(x_{12})^2}{((x_{12})^2 + (2\pi)^2} \left( e^{i\omega_1(x_{12})} + e^{-i\omega_1(x_{12})} \right) \] + N_2^4 \sinh^2(2\pi\omega_1) \right\}$$

$$= e^{-\frac{2}{gs}2\pi\delta(\omega_1 - \omega_2)} \left[ - N_1^2 8\pi^2 \sinh^2(\pi\omega_1) \left( 1 + e^{-2\pi\omega_1} \right) + N_2^4 \sinh^2(2\pi\omega_1) \right].$$

(4.34)

Using (4.29), comparison against the matrix model amplitudes (2.53) fixes

$$N_2 = \frac{3}{64\pi^2}.$$  

(4.35)

The generalization of the two-instanton amplitude to $1 \rightarrow k$ scattering is straightforward. Let the incoming closed string have energy $\omega$, while the outgoing closed strings have energies $\omega_1, ..., \omega_k$. For the contribution coming from the $\{1,1\}$ multi-instanton configuration, the disconnected diagrams of interest correspond again to disk bulk 1-point functions, with boundary condition in either ZZ-instanton. We also have to subtract the contribution from the vacuum amplitude, so that in total we find
\[
e^{-\frac{2}{\pi^2} N_1^2 \frac{1}{2}} \int dx_1 dx_2 \left[ \left( \frac{1}{\omega + \omega} \right) \times \ldots \times \left( \frac{1}{\omega + \omega} \right) \exp \left( 2 \frac{1}{\omega} \right) - 2 \frac{1}{\omega} \times \ldots \times \frac{1}{\omega} \right]
\]

\[
e^{-\frac{2}{\pi^2} \omega \omega_N^2} \left( \omega - \sum_{i=1}^{k} \omega_{i} \right) N_1^2 2^k \sinh(\pi \omega_l) \prod_{i=1}^{k} \sinh(\pi \omega_i)
\]

\[
\times \int dx_{12} \left[ \frac{(x_{12})^2}{((x_{12}))^2 + 4\pi^2} \left( 1 + e^{-i\omega x_{12}} \right) k \prod_{i=1}^{k} \left( 1 + e^{i\omega_i x_{12}} \right) - 2 \right].
\]

(4.36)

The integral in the last line can be evaluated as

\[
\int dx_{12} \left[ \frac{(x_{12})^2}{((x_{12}))^2 + 4\pi^2} \sum_{S_1 \cup S_2 = S} 2 \cos \frac{(\omega + \omega(S_1) - \omega(S_2)) x_{12}}{2} - 2 \right]
\]

\[
= -4\pi^2 \sum_{S_1 \cup S_2 = S} e^{-\pi(\omega(S_1) - \omega(S_2))} = -4\pi^2 e^{-\pi \omega} \prod_{i=1}^{k} 2 \cosh(\pi \omega_i),
\]

(4.37)

where we have defined \( S \equiv \{\omega_1, \ldots, \omega_n\} \), and \( S_1, S_2 \) are disjoint subsets of \( S \) such that \( S_1 \cup S_2 = S \), and \( \omega(S_i) = \sum_{\omega \in S_1} \omega \).

For the (2, 1) ZZ-instanton, the only disconnected diagram of interest involves disk bulk 1-point functions with ends on the same ZZ-instanton. The steps are similar to (4.28), and we find

\[
e^{-\frac{2}{\pi^2} N_2} \int dx_1 \times \ldots \times dx_1
\]

\[
e^{-\frac{2}{\pi^2} \omega \omega_N^2} \left( \omega - \sum_{i=1}^{k} \omega_{i} \right) N_2^2 2^{k+1} \sinh(2\pi \omega) \prod_{i=1}^{k} \sinh(2\pi \omega_i).
\]

(4.38)

Combining (4.36) and (4.38), we precisely reproduce (2.53) once we use the value of \( N_2 \) given by (4.35).
4.3.4 3-Instanton Correction to the $1 \to k$ Amplitude

At order $e^{-3/g_s}$ there are several types of contributions: three $(1,1)$ ZZ-instantons, a $(1,1)$ together with a $(2,1)$ ZZ-instanton, and a single $(3,1)$ ZZ-instanton. We label these instanton configurations by $\{1,1,1\}$, $\{2,1\}$, and $\{3\}$, respectively. In the following subsections we evaluate each contribution separately. The relevant worldsheet correlators, namely the disk 1-point function and the cylinder partition function, have been explicitly evaluated in (4.20) and (4.22).

$\{1,1,1\}$ ZZ-instantons

We begin with the case of three $(1,1)$ ZZ-instantons, located at time coordinates $x_1$, $x_2$, $x_3$. The worldsheet diagram at order $e^{-3/g_s}$ is again given by a pair of discs, each containing one closed string vertex operator, such that the boundaries of the disks lie on one or two out of the three instantons. Extra care must be taken in subtracting off the disconnected instanton diagrams so as to normalize the vacuum amplitude, shown schematically in Figure 4.7.

\[\text{Figure 4.7: The } \{1,1,1\} \text{ ZZ-instanton contribution with subtraction of disconnected instanton diagrams.}\]
The contribution from a pair of disks ending on the same \((1,1)\) ZZ-instanton is

\[
e^{-\frac{a}{g_s} \mathcal{N}_1^3 \frac{1}{3!}} \int dx_1 dx_2 dx_3 \left[ \frac{1}{\mathcal{N}_1^3} \frac{1}{3!} \exp \left( \begin{array}{ccc} 1 & 1 \\ \star & \star & \star \\ \end{array} \right) \right] \exp \left( \begin{array}{ccc} 2 & 1 \\ \star & \star & \star \\ \end{array} \right) + \exp \left( \begin{array}{ccc} 1 & 2 \\ \star & \star & \star \\ \end{array} \right) \right] \] \[
- \exp \left( \begin{array}{ccc} 2 \\ \star \\ \end{array} \right) - \exp \left( \begin{array}{ccc} 2 \\ \star \\ \end{array} \right) - \exp \left( \begin{array}{ccc} 2 \\ \star \\ \end{array} \right) + 2 \right].
\]

(4.39)

The first two subtractions are due to the diagram with a disconnected instanton of type \(\{1\}\), whereas the third subtraction is due to a disconnected instanton of type \(\{1,1\}\). The last term in the bracket takes care of the over-subtraction of diagrams with two disconnected instantons of type \(\{1\}\).

The contribution from a pair of disks ending on two separate \((1,1)\) ZZ-instantons is computed by

\[
e^{-\frac{a}{g_s} \mathcal{N}_1^3 \frac{1}{3!}} \int dx_1 dx_2 dx_3 \left[ \frac{1}{\mathcal{N}_1^3} \frac{1}{3!} \exp \left( \begin{array}{ccc} 1 & 2 \\ \star & \star & \star \\ \end{array} \right) + \exp \left( \begin{array}{ccc} 2 & 1 \\ \star & \star & \star \\ \end{array} \right) \right] \exp \left( \begin{array}{ccc} 2 \\ \star \\ \end{array} \right) \right] \] \[
- \exp \left( \begin{array}{ccc} 2 \\ \star \\ \end{array} \right) - \exp \left( \begin{array}{ccc} 2 \\ \star \\ \end{array} \right) - \exp \left( \begin{array}{ccc} 2 \\ \star \\ \end{array} \right) + 1 \right].
\]

(4.40)

Here the subtraction involves only a single disconnected \(\{1\}\) instanton.

After evaluating the integrals (along the Lorentzian time contour), (4.39) and (4.40) together give

\[
e^{-\frac{a}{g_s} 2\pi \delta(\omega_1 - \omega_2) \mathcal{N}_1^3 \frac{64\pi^4}{3} \sinh^2(\pi \omega_1)}(1 + e^{-2\pi \omega_1} + e^{-4\pi \omega_1}).
\]

(4.41)

\(\{2,1\}\) ZZ-INstantons

Next, we consider a \((2,1)\) ZZ-instanton at time \(x_1\) and a \((1,1)\) ZZ-instanton at time \(x_2\). The measure factor is computed by the cylinder diagram between these two boundary conditions, as
in (4.22). We should also subtract off diagrams with a disconnected instanton, either of (2, 1) or (1, 1) type, as shown in Figure 4.8.

![Diagram of ZZ-instantons with subtraction of disconnected instanton diagrams](image)

**Figure 4.8:** The \( \{2, 1\} \) ZZ-instanton contribution with subtraction of disconnected instanton diagrams.

The contribution from a pair of disks ending on the same ZZ-instanton is given by

\[
e^{-\frac{3}{4\pi}N_1N_2} \int dx_1 dx_2 \left[ 1 \cdot 1 \left( \exp \left( 2 \cdot \frac{2}{\omega} \right) - 1 \right) + 2 \cdot 2 \left( \exp \left( 2 \cdot \frac{1}{\omega} \right) - 1 \right) \right] = e^{-\frac{3}{4\pi}8\pi\delta(\omega_1 - \omega_2)N_1N_2} \int d\Delta x \left( \sinh^2(\pi\omega_1) + \sinh^2(2\pi\omega_1) \right) \left[ \frac{(\Delta x)^2 + \pi^2}{(\Delta x)^2 + 9\pi^2} - 1 \right],
\]

where the contribution from a pair of disks ending on the two different ZZ-instantons is

\[
e^{-\frac{3}{4\pi}N_1N_2} \int dx_1 dx_2 \left( 1 \cdot 2 + 2 \cdot 1 \right) \exp \left( 2 \cdot \frac{1}{\omega} \right)
= e^{-\frac{3}{4\pi}8\pi\delta(\omega_1 - \omega_2)N_1N_2} \int d\Delta x \frac{(\Delta x)^2 + \pi^2}{(\Delta x)^2 + 9\pi^2} 2 \cos(\omega_1 \Delta x) \sinh(\pi\omega_1) \sinh(2\pi\omega_1).
\]

After evaluating the \( \Delta x \)-integral, we find the total contribution from \( \{2, 1\} \) ZZ-instanton configuration to be

\[
- e^{-\frac{3}{4\pi}2\pi\delta(\omega_1 - \omega_2)N_1N_2} \frac{32\pi^2}{3} \sinh^2(\pi\omega_1) \left( e^{2\pi\omega_1} + 3 + 3e^{-2\pi\omega_1} + 2e^{-4\pi\omega_1} \right).
\]
\{3\} ZZ-INSTANTON

Finally, the contribution from a single (3, 1) ZZ-instanton at time $x$ is given by

$$e^{-\frac{3}{gs}N_3 \int dx \left( \frac{1}{x^3} \right) \left( \frac{1}{x^3} \right)} = e^{-\frac{3}{gs}8\pi \delta(\omega_1 - \omega_2)N_3 \sinh^2(3\pi\omega_1)}, \quad (4.45)$$

where the normalization factor $N_{3,1}$ is so far undetermined.

Combining (4.41), (4.44), and (4.45), the order $e^{-3/gs}$ contribution to the $1 \to 1$ closed string amplitude remarkably agrees with the matrix model result (2.53) provided that we make the identification

$$N_3 = -\frac{5}{192\pi^2}. \quad (4.46)$$

4.3.5 4-INSTANTON CORRECTIONS TO THE CLOSED STRING $1 \to 1$ AMPLITUDE

The final example we consider is the order $e^{-4/gs}$ correction to the $1 \to 1$ closed string amplitude. There are various ZZ-instanton configurations that could contribute: \{1, 1, 1, 1\}, \{2, 1, 1\}, \{3, 1\}, \{2, 2\}, \{4\}, all of which involve ZZ-instantons of type $(m, 1)$. In view of (4.19), one may further suspect that a single ZZ-instanton of type $(2, 2)$ could contribute at this order (not to be confused with \{2, 2\}, which means two ZZ-instantons of type $(2, 1)$). We will find a remarkable agreement of the total result with the matrix model, provided a suitable choice of the measure normalization factor $N_4$ for the $(4, 1)$ ZZ-instanton, and surprisingly, if we assume that the $(2, 2)$ ZZ-instanton does not contribute, i.e. $N_{(2,2)} = 0$. 

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\{1, 1, 1, 1\} ZZ-instantons

We begin with four \( (1, 1) \) ZZ-instantons, located at times \( x_1, x_2, x_3, x_4 \). The worldsheet diagrams again involve a pair of discs, with boundaries ending on either one or two out of the four instantons. The subtraction of disconnected diagrams is summarized schematically in Figure 4.9.

\[ e^{-\frac{4}{g_s} N_1^{4\pi}} \frac{1}{47} \int \prod_{i=1}^{4} dx_i \left\{ \exp \left( 2 \sum_{1 \leq i < j \leq 4} \right) \right\} \]

\[ \begin{array}{c}
\{1, 1, 1, 1\} \\
ZZ\text{-instantons}
\end{array}
\quad
\begin{array}{c}
\text{Disconnected} \\
\{1\} ZZ\text{-instanton}
\end{array}
\quad
\begin{array}{c}
\text{Disconnected} \\
\{1, 1\} ZZ\text{-instantons}
\end{array}
\quad
\begin{array}{c}
\text{Disconnected} \\
\{1, 1\} ZZ\text{-instantons}
\end{array}
\]

\textbf{Figure 4.9:} The \{1, 1, 1, 1\} ZZ-instanton contribution with subtraction of disconnected diagrams.

The contribution to the \( 1 \to 1 \) amplitude is computed as

\[
e^{-\frac{4}{g_s} N_1^{4\pi}} \frac{1}{47} \int \prod_{i=1}^{4} dx_i \left\{ \exp \left( 2 \sum_{1 \leq i < j \leq 4} \right) \right\} \]

\[ -3 \left( \exp \left( 2 \sum_{1 \leq i < j \leq 4} \right) \right) \]

\[ +6 \left( \sum_{1 \leq i < j \leq 4} \right) \]

\[ = -e^{-\frac{4}{g_s} 2\pi \delta(\omega_1 - \omega_2)} N_1^{4\pi} 64 \pi^6 \sinh^2(\pi \omega_1) \left( 1 + e^{-2\pi \omega_1} + e^{-4\pi \omega_1} + e^{-6\pi \omega_1} \right). \]

(4.47)
\{2,1,1\} ZZ-INSTANTONS

Next we turn to the case of one \((2,1)\) ZZ-instanton and a pair of \((1,1)\) ZZ-instantons, located at times \(x_1, x_2, x_3\) respectively. The subtraction of disconnected diagrams is summarized in Figure 4.10.

**Figure 4.10:** The \(\{2,1,1\}\) ZZ-instanton contribution and subtraction of disconnected diagrams.

The contribution to the amplitude is evaluated as

\[
e^{-\frac{4}{\pi} N_1^2 N_2 \frac{1}{2}} \int dx_1 dx_2 dx_3 \left[ \left( \begin{array}{ccc} 1 & 1 & 2 \\ 2 & 2 & 3 \\ 3 & 3 & \end{array} \right) \right]
\times \left( \exp \left( 2 \begin{array}{c} 1 \\ 2 \\ \end{array} \right) + 2 \begin{array}{c} 3 \\ 3 \\ \end{array} \right) - \exp \left( 2 \begin{array}{c} 1 \\ \end{array} \right) - \exp \left( 2 \begin{array}{c} 3 \\ \end{array} \right) + 2
+ \left( \begin{array}{ccc} 1 & 2 \\ 2 & 1 \\ \end{array} \right) \exp \left( 2 \begin{array}{c} 1 \\ \end{array} \right) - \exp \left( 2 \begin{array}{c} 3 \\ \end{array} \right) - 1
+ 2 \left( \begin{array}{ccc} 3 & 1 \\ 1 & 3 \\ \end{array} \right) \exp \left( 2 \begin{array}{c} 3 \\ \end{array} \right) - \exp \left( 2 \begin{array}{c} 1 \\ \end{array} \right) - 1 \right]
= e^{-\frac{4}{\pi\pi} 2\pi\delta(\omega_1 - \omega_2)N_1^2 N_2 32\pi^4}
\times \left[ \sinh^2(2\pi\omega_1) + \sinh^2(\pi\omega_1) \left( 2 + e^{-2\pi\omega_1} + e^{-6\pi\omega_1} \right) + 2 \sinh(\pi\omega_1) \sinh(2\pi\omega_1) \left( e^{-3\pi\omega_1} + e^{-5\pi\omega_1} \right) \right].
\]

(4.48)
\{3,1\} ZZ-INSTANTONS

The contribution from the configuration of a \{3,1\} together with \{1,1\} ZZ-instanton is evaluated similarly to the case of section 4.3.4 as

\[
e^{-\frac{4}{s_k}N_1N_3} \int dx_1 dx_2 \left[ 1 \begin{array}{c} 1 \end{array} \right] \left( \exp \left( \begin{array}{c} 2 \end{array} \right) - 1 \right) + \begin{array}{c} 2 \end{array} \left( \exp \left( \begin{array}{c} 2 \end{array} \right) - 1 \right)
\]

\[
+ \left( \begin{array}{c} 2 \end{array} + \begin{array}{c} 1 \end{array} \right) \exp \left( \begin{array}{c} 2 \end{array} \right)
\]

\[
= -e^{-\frac{4}{s_k}2\pi\delta(\omega_1 - \omega_2)N_1N_3} 12\pi^2 \left[ \sinh^2(\pi\omega_1) + \sinh^2(3\pi\omega_1) + 2 \sinh(\pi\omega_1) \sinh(3\pi\omega_1)e^{-4\pi\omega_1} \right].
\]

(4.49)

\{2,2\} ZZ-INSTANTONS

The contribution from a pair of \{2,1\} ZZ-instantons is evaluated similarly to the case of section 4.3.3 as

\[
e^{-\frac{4}{s_k}N_2^2 \frac{1}{2}} \int dx_1 dx_2 \left[ 2 \begin{array}{c} 1 \end{array} \right] \left( \exp \left( \begin{array}{c} 2 \end{array} \right) - 1 \right)
\]

\[
+ \left( \begin{array}{c} 1 \end{array} + \begin{array}{c} 2 \end{array} \right) \exp \left( \begin{array}{c} 1 \end{array} \right)
\]

\[
= -e^{-\frac{4}{s_k}2\pi\delta(\omega_1 - \omega_2)N_2^2} 16\pi^2 \sinh^2(2\pi\omega_1) \left( 1 + e^{-4\pi\omega_1} \right).
\]

(4.50)

\{4\} ZZ-INSTANTON AND THE PUTATIVE \{2,2\} ZZ-INSTANTON

Now we turn to new types of instantons that emerge at order \[e^{-4/s^k}\]. The contribution to the 1 \rightarrow 1 closed string amplitude from the ZZ-instanton configuration \{4\}, i.e. a single \{4,1\}
ZZ-instanton, is

\[ e^{-\frac{4}{gs} N_4} \int dx_1 \quad \begin{array}{c} 1 \end{array} \quad \begin{array}{c} 1 \end{array} = e^{-\frac{4}{gs} 2\pi \delta(\omega_1 - \omega_2)} N_{(4,1)} 4 \sinh^2(4\pi \omega_1), \quad (4.51) \]

where the normalization constant \( N_{(4,1)} \) is to be determined. A single \((2,2)\) ZZ-instanton, on the other hand, would give a contribution of the form

\[ e^{-\frac{4}{gs} N_{(2,2)}} \int dx_1 \quad \begin{array}{c} 1 \end{array} \quad \begin{array}{c} 1 \end{array} = e^{-\frac{4}{gs} 2\pi \delta(\omega_1 - \omega_2)} N_{(2,2)} \frac{4 \sinh^4(2\pi \omega_1)}{\sinh^2(\pi \omega_1)}. \quad (4.52) \]

Combining the results (4.47), (4.48), (4.49), (4.50), (4.51), (4.52), we find perfect agreement with the matrix model amplitude \( A_{1 \rightarrow 1}^{4-\text{inst.}(0)} \) in (2.53) provided

\[ N_4 = \frac{35}{2048\pi^2}, \quad N_{(2,2)} = 0. \quad (4.53) \]

The absence of \((2,2)\)-type ZZ-instanton contribution leads us to suspect that in fact \( N_{(k,\ell)} = 0 \) whenever \( k, \ell \geq 2 \) (recall that \((k,\ell)\) and \((\ell,k)\) ZZ-boundary conditions are equivalent in \( c = 25 \) Liouville theory), i.e. only the ZZ-instantons of type \((m,1)\) can contribute to closed string amplitudes in \( c = 1 \) string theory. In [33], this was confirmed by the computation of the closed string \( 1 \rightarrow 1 \) amplitude to order \( e^{-n/g_s} \), for all \( n \).

4.3.6 Perturbative Expansion of 1-Instanton Correction

We now discuss contributions to the scattering amplitude of the form \( A_{1 \rightarrow k}^{n-\text{inst.}(L)} \), where \( L > 0 \), see (2.51). We focus in the case \( A_{1 \rightarrow 1}^{1-\text{inst.}(1)}(\omega) \) for simplicity, but a similar story is expected to
hold at higher orders.

![Diagram](image)

**Figure 4.11:** Three of the non-empty worldsheet diagrams that contribute to the next-to-leading order non-perturbative correction to the closed string $1 \rightarrow 1$ amplitude, as explained in the text

There are five different contributions, three of which are shown in Figure 4.11: a disk with two bulk insertions, an annulus bulk 1-point times a disk bulk 1-point (and the same diagram with the closed string vertex operators exchanged), an empty disk with two holes times two disconnected disks with 1 closed string insertion each, and an empty one-holed torus times two disconnected disks with 1 closed string insertion each. Focusing on the first three for now, they give the contribution

$$ e^{-S_{(1,1)}} \int d \tau_0 \left( \langle V^+_{\omega_1} V^-_{\omega_2} \rangle D_2 \tau_0 + \langle V^+_{\omega_1} \rangle A_2 \langle V^-_{\omega_2} \rangle D_2 \tau_0 + \langle V^+_{\omega_1} \rangle D_2 \tau_0 + \langle V^-_{\omega_1} \rangle A_2 \langle V^+_{\omega_2} \rangle D_2 \tau_0 \right). \quad (4.54) $$

However, it turns out that each of these diagrams suffers from divergences in the moduli space integration. The divergence comes from the open string channel, in the limit where the diagrams look like disks with a long thin strip attached, see Figure 4.12. For the disk bulk 2-point, the strip is attached to two disk bulk 1-point diagrams, while for the annulus bulk 1-point the strip is attached to the same disk bulk 1-point.

There are two sources of divergences. One comes from an off-shell tachyonic mode, and it gives rise to a power divergence that can be subtracted as usual. The other divergence comes from the “on-shell” open string mode $\partial X^0$, and it gives rise to a logarithmic divergence. In
practice, we introduce an IR cutoff in the moduli space integral of these two moduli spaces to cut-off the divergent limit of integration. Note that a $\partial X^0$ insertion implements a shift of the collective coordinate $x$, so that the logarithmic divergence in the diagrams in (4.55) is of the form

$$e^{-S_{(1,1)}} \int dx_0 \left( 2\partial_{x_0} (V^+_{\omega_1}) \frac{D_2}{ZZ} \partial_{x_0} (V^-_{\omega_1}) \frac{D_2}{ZZ} \partial_{x_0} (V^+_{\omega_2}) \frac{D_2}{ZZ} \partial_{x_0} (V^-_{\omega_2}) \right) = e^{-S_{(1,1)}} \int dx_0 \partial_{x_0}^2 \left( (V^+_{\omega_1}) \frac{D_2}{ZZ} (V^-_{\omega_1}) \frac{D_2}{ZZ} \right),$$

(4.55)

which vanish upon integration over the collective coordinate! This is the Fischler-Susskind-Polchinski mechanism [56, 57, 14], and it ensures the consistency of the perturbative string expansion about each instanton saddle.
Note that while the logarithmic divergences have canceled out, there is a finite term left over, of the form
\[
e^{-S(1,1)} \int dx_0 \partial x_0 \langle \mathcal{V}_{\omega}^+ \rangle_{Z_{\infty},x_0} \partial x_0 \langle \mathcal{V}_{\omega_1}^- \rangle_{Z_{\infty},x_0},
\]
(4.56)
This finite ambiguity is a consequence of the fact that, in the worldsheet formalism, we have no way of identifying the IR cutoffs in the annulus diagram and the disk diagram moduli spaces. In [68], the finite term was explicitly calculated using string field theory methods.

As mentioned above, there are other diagrams contributing to this scattering amplitude, namely the empty disk with two holes times two disconnected disks with one closed string insertion each, and the empty one-holed torus times two disconnected disks with one closed string insertion each. The empty disk with two holes and one-holed torus are formally divergent and require regularization. Instead, we simply replace their contribution by a constant to be fixed.

In [32], \( A_{1 \to 1}^{1-\text{inst},(1)}(\omega) \) was explicitly computed numerically, and precisely reproduced the matrix model answer (2.53). The contribution from the empty disk with two holes + one-holed torus, as well as the finite ambiguity (4.56), were taken into account by leaving two overall constants unfixed, which were fixed by matching against the matrix model result. The result for the constant coefficient of (4.56) was also consistent with the result in [68].
Long Strings

c = 1 string theory admits semi-infinite D-branes called FZZT branes, which extend into the asymptotic region. In a certain limit, FZZT branes recede into the asymptotic region where the string coupling is suppressed, but highly energetic open strings can still stretch into the region of finite string coupling, where they can scatter with other long or closed strings.

In [41], long strings were conjectured to be dual to states in non-singlet representations of the dual c = 1 matrix model, and their explicit wavefunctions were found in [44]. In this section, we find striking agreement in the computation of the tree-level long→long+closed and
long+long→long+long scattering amplitudes from the two descriptions, which provides non-trivial evidence for the conjecture.

5.1 FZZT Branes and Long Strings

5.1.1 Semiclassical description

Besides the ZZ boundary condition discussed in the previous chapter, the Liouville CFT also admits the FZZT conformally invariant boundary condition. In the semiclassical limit of small $b$, the FZZT boundary condition is described by a boundary contribution to the Liouville action (3.1),

$$S_L[\phi] = \frac{1}{4\pi} \int d^2 z \sqrt{g} \left( g^{mn} \partial_m \phi \partial_n \phi + QR \phi + 4\pi \mu e^{2\phi} \right) + \int_{\partial \Sigma} d\xi \ g^{\frac{1}{2}} \left( \frac{Qk}{2\pi} \phi + \mu_B e^{\phi} \right), \quad (5.1)$$

where $d\xi \ g^{\frac{1}{2}}$ is the boundary line element and $k$ is the extrinsic curvature of the boundary.

Note that while the parameter $\mu$ depends only on the Euler characteristic of $\Sigma$ (see (3.2)), the dependence on the ratio $\mu_B/\sqrt{\mu}$ is not topological, and it parameterizes different FZZT boundary conditions.

In the asymptotic region $\phi \to -\infty$, both the bulk and the boundary Liouville potentials are negligible, and asymptotic states are described by incoming and outgoing plane waves. As $\phi$ increases, at $b\phi \sim -\ln(\mu_B/\sqrt{\mu})$ the boundary Liouville potential becomes large, and incoming waves reflect back to the asymptotic region. Thus, the FZZT describes a semi-infinite boundary condition which “ends” at $b\phi \sim -\ln(\mu_B/\sqrt{\mu})$, with a continuum spectrum of states.

Consider the semiclassical Euclidean action (5.1) on the infinite strip of width $2L$, parame-
tered by the coordinates $(\sigma, \tau)$ and with line element $ds^2 = d\sigma^2 + d\tau^2$. Defining $\varphi(\tau, \sigma) \equiv \phi(\tau, \sigma) + \frac{1}{2b} \ln \mu$, the equations of motion are given by

$$(\partial^2_\tau + \partial^2_\sigma)\varphi = 4\pi be^{2b\varphi}, \quad (5.2)$$

and the boundary conditions are

$$\partial_\sigma \varphi + \frac{2\pi \mu_B}{\sqrt{\mu}} e^{b\varphi} = 0, \quad \sigma = 2L, \quad (5.3)$$

$$\partial_\sigma \varphi - \frac{2\pi \mu_B}{\sqrt{\mu}} e^{b\varphi} = 0, \quad \sigma = 0.$$

We solve these equations on the Lorentzian strip, where we continue $\tau \to it$. A set of Lorentzian solutions is given by [41]

$$4\pi e^{2b\varphi} = \left[\sqrt{1 + \lambda^2 \cosh(bt)} + \lambda \cosh(b(\sigma - L))\right]^{-2}, \quad \lambda \equiv \frac{\mu_B}{\sqrt{\mu}} \frac{\sqrt{\pi}}{\sinh(bL)}. \quad (5.4)$$

This solution describes an open string coming from the asymptotic region $\phi = -\infty$ where it moves relativistically, reflecting off the bulk/boundary Liouville potentials, and going back to $\phi = -\infty$ in the asymptotic future. Note that the dynamics of the endpoints of the open string $\sigma = 0, 2L$ is different from the dynamics of the tip $\sigma = L$. The former get “stuck” on the boundary Liouville wall for times $|t| \lesssim L$, while the tip of the open string keeps moving relativistically until $|t| \sim 1$, after which the string is maximally stretched and the motion is reversed.

At the classical level, the “long string limit” is defined by taking $L \to \infty$ while keeping $\lambda$
finite. In this limit, the boundary Liouville potential becomes large, so that open strings of finite length are localized at $\phi = -\infty$, where they don’t interact with closed strings since the string coupling is exponentially suppressed, see Figure 5.1. However, long strings of length $\sim 2L$ are still able to stretch into the bulk where the string coupling is finite. The limit of large $L$ implies that the ends of the long string immediately get stuck at the boundary Liouville wall, while the tip moves freely for a long time, with a constant force due to the string tension. The energy $E$ of the long string is due to the string tension $1/2\pi$, and since it has total length $2L$ we find $E = L/\pi$. Note that there are corrections to this result from both the bulk/boundary Liouville potentials, which are suppressed in the $b \to 0$ limit.

Figure 5.1: A long string in $c = 1$ string theory is the high energy limit of an open string ending on an FZZT brane receding to the weak coupling region.

It is useful to think of the tip of the long string as a relativistic particle in a linear potential $V(\phi) = (\phi - \phi_c)/\pi$. The force is $1/\pi$ because it is twice the string tension since the string is folded, and we have introduced an IR cutoff at $\phi = \phi_c$ where the ends of the long string are stuck. Energy conservation implies that this particle has momentum $p$ given by

$$|p| = E - \frac{\phi - \phi_c}{\pi} = \frac{\phi_m - \phi}{\pi},$$

(5.5)
where $\phi_m$ is the turning point of the trajectory, $E = (\phi_m - \phi_c)/\pi$, and we assumed that the particle moves relativistically. The phase factor of ingoing/outgoing waves is given via the WKB approximation by

$$\pm i \int_{\phi_m}^\phi |p| \, d\phi = \pm \frac{i}{2\pi} (\phi - \phi_m)^2.$$  \hfill (5.6)

Thus, the wavefunction for the tip of the long string is given semiclassically by

$$\psi(\phi) \sim e^{-i\frac{1}{2\pi} (\phi - \pi\epsilon)^2} - e^{i\delta} \sqrt{\frac{i}{2\pi}} (\phi - \pi\epsilon)^2,$$  \hfill (5.7)

where we define the renormalized energy $\epsilon \equiv E + \frac{\phi_c}{\pi} = \frac{\phi_m}{\pi}$ by subtracting off the divergent part of $E$, and $\delta(\epsilon)$ is a reflection phase.

This wavefunction will be obtained from the matrix model description in section 5.3, while the reflection phase $\delta(\epsilon)$ is computed in Appendix D.

### 5.1.2 FZZT Boundary Conformal Field Theory

In terms of the boundary CFT, the FZZT boundary conditions are parameterized by a label $s$, which is related to the parameter $\mu_B$ by

$$\cosh(2\pi bs) = \frac{\mu_B}{\sqrt{\mu}} \sqrt{\sin(\pi b^2)},$$  \hfill (5.8)

although we emphasize that this relation is only true in the limit $b \to 0$. From the semiclassical analysis above it follows that the FZZT($s$) boundary condition “ends” at the position $\phi \sim -2\pi s$. 

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The FZZT\((s)\) boundary state is given by

\[
|\text{FZZT}(s)\rangle = \int_0^\infty \frac{dP}{\pi} \Psi_s(P)|V_P\rangle, \tag{5.9}
\]

where we expanded the boundary state as a sum of Ishibashi states, see (4.3). The Hilbert space between a \((1,1)\) ZZ and a FZZT\((s)\) boundary condition consists solely of the boundary Virasoro primary of weight \(1 + s^2\) and its Virasoro descendants. Thus, modular invariance of the cylinder partition function with these boundary conditions requires that

\[
\chi_{1+s^2}(-1/\tau) = \int_0^\infty \frac{dP}{\pi} \Psi^{(1,1)}(P)\Psi^{\text{FZZT}}_s(P)\chi_{1+P^2}(\tau), \tag{5.10}
\]

where we use the same conventions as in section 4.1. This condition completely fixes \(\Psi^{\text{FZZT}}_s(P)\), which is given by

\[
\Psi^{\text{FZZT}}_s(P) = 2^{1/2} \sqrt{\frac{\cos(4\pi sP)}{\sinh(2\pi P)}}. \tag{5.11}
\]

The FZZT boundary condition is unitary for \(s \in \mathbb{R}_{\geq 0}\) and also when \(s\) is a purely imaginary number with \(0 < \text{Im} \ s < 1\). When \(s\) is purely imaginary and \(1/2 < \text{Im} \ s < 1\), there is an extra Virasoro primary of conformal weight \(h = 1 + s^2\), which is normalizable [69].

We focus on FZZT\((s)\) boundary conditions for \(s \in \mathbb{R}_{\geq 0}\). In this case, the boundary primary operators between two FZZT boundary conditions form a continuum. We label the boundary primary operators between FZZT boundary conditions of types \(s_1\) and \(s_2\) by \(\psi^{s_1,s_2}_P(x_1)\) for \(P \geq 0\), which are normalized so that the boundary 2-point function on the upper half-plane is
given by

$$
\langle \psi^{s_1,s_2}_{P_1}(x_1)\psi^{s_2,s_1}_{P_2}(x_2) \rangle = \frac{\pi}{|x_1 - x_2|^2 h_P^4} \delta(P_1 - P_2),
$$

(5.12)

where $x_1, x_2 \in \mathbb{R}$. With this choice of normalization, $\psi^{s_1,s_2}_P$ admits a free-field representation in the asymptotic region $\phi \to -\infty$, given by

$$
\psi^{s_1,s_2}_P \to (d^{s_1,s_2}(P))^{-\frac{1}{2}} e^{(1+i\phi)} + (d^{s_1,s_2}(P))\frac{1}{2} e^{(1-i\phi)},
$$

(5.13)

where $d^{s_1,s_2}(P)$ is the reflection phase of a boundary primary operator on the strip [27],

$$
d^{s_1,s_2}(P) = \frac{\Gamma_1(2iP) S_1(1+i(s_1+s_2-P))S_1(1-i(P+s_1+s_2))}{\Gamma_1(-2iP)S_1(1+i(P+s_1-s_2))S_1(1+i(P+s_2-s_1))}.
$$

(5.14)

The functions $S_1(x)$ are special functions related to the Barnes $G$-function, and it is defined in Appendix A.

Correlation functions on arbitrary Riemann surfaces with boundaries are computed from the boundary structure constants, namely the disk bulk 1-point given by (5.11), the disk boundary 3-point, and the disk bulk-boundary 2-point. These structure constants are completely fixed by crossing and modular invariance of correlation functions with boundaries. The disk with 3 boundary primary insertions is given by [28]

$$
\langle \psi^{s_1,s_3}_{P_3}(x_3)\psi^{s_2,s_2}_{P_2}(x_2)\psi^{s_2,s_1}_{P_1}(x_1) \rangle = \frac{C^{s_1,s_2,s_3}(P_1,P_2,P_3)}{|x_3 - x_2|^{-h_{P_1} + h_{P_2} + h_{P_3}}|x_1 - x_3|^{-h_{P_2} + h_{P_1} + h_{P_3}}|x_2 - x_1|^{-h_{P_3} + h_{P_1} + h_{P_2}}}.
$$

(5.15)
where \( C^{s_1, s_2, s_3}(P_1, P_2, P_3) \) is the boundary structure constant, given by the formula

\[
C^{s_1, s_2, s_3}(P_1, P_2, P_3) = 2^{3} \pi^{\frac{5}{2}} \frac{(d^{s_1, s_3}(P_3))^\frac{1}{2}}{(d^{s_2, s_1}(P_1))^\frac{1}{2} (d^{s_3, s_2}(P_2))^\frac{1}{2}} \Gamma_1(1 - i(P_1 + P_2 + P_3)) \\
\times \frac{\Gamma_1(1 + i(P_2 + P_3 - P_1)) \Gamma_1(1 + i(P_2 - P_3 - P_1)) \Gamma_1(1 + i(P_3 - P_2 - P_1))}{\Gamma_1(2) \Gamma_1(2iP_3) \Gamma_1(-2iP_2) \Gamma_1(-2iP_1)} \\
\times \frac{S_1(1 + i(P_3 + s_1 - s_3)) S_1(1 + i(P_3 - s_1 - s_3))}{S_1(1 + i(P_2 + s_2 - s_3)) S_1(1 + i(P_2 - s_2 - s_3))} \int_{\mathbb{R} + \mathbf{i} \mathbb{R}^{+}} dt \prod_{k=1}^{4} \frac{S_1(U_k + it)}{S_1(V_k + it)}.
\]

The variables \( U_k \) and \( V_k \) for \( k = 1, \ldots, 4 \) are defined as

\[
\begin{align*}
U_1 &= 1 + i(s_1 + s_2 - P_1), & U_2 &= 1 + i(s_2 - s_1 - P_1), \\
U_3 &= 1 + i(s_2 - s_3 + P_2), & U_4 &= 1 + i(s_2 - s_3 - P_2), \\
V_1 &= 2 + i(s_2 - s_3 + P_3 - P_1), & V_2 &= 2 + i(s_2 - s_3 - P_1 - P_3), \\
V_3 &= 2 + 2is_2, & V_4 &= 2.
\end{align*}
\]

Though not immediately evident from (5.15), \( C^{s_1, s_2, s_3}(P_1, P_2, P_3) \) is real when \( P_1, P_2, P_3 \in \mathbb{R} \), and it is invariant with respect to exchanging the Liouville momenta.

The disk bulk-boundary 2-point function is given by [29]

\[
\langle V_{P_1}(z, \bar{z}) \psi_{P_2}^{s_1, s_2}(x) \rangle = \frac{\mathcal{R}^{s}(P_1; P_2)}{|z - \bar{z}|^{2h_{P_1} - h_{P_2}} |z - x|^{2h_{P_2}}}
\]

(5.18)
where $\mathcal{R}^s(P_1; P_2)$ is the bulk-boundary structure constant, given by

$$
\mathcal{R}^s(P_1; P_2) = 2\pi \frac{3}{2} (s(P_1))^{-\frac{1}{2}} (d^s(P_2))^{-\frac{1}{2}} \frac{\Gamma_1(1 - iP_2)^3 \Gamma_1(1 - i(2P_1 + P_2)) \Gamma_1(1 + i(2P_1 - P_2))}{\Gamma_1(2) \Gamma_1(1 + iP_2) \Gamma_1(-2iP_2) \Gamma_1(2 + 2iP_1) \Gamma_1(-2iP_1)} \times \int_{-\infty}^{\infty} dt e^{4\pi i ts} \frac{S_1((1/2)(1 + i(2P_1 + P_2)) + it) S_1((1/2)(1 + i(2P_1 + P_2) - it))}{S_1((1/2)(3 + i(2P_1 - P_2)) + it) S_1((1/2)(3 + i(2P_1 - P_2) - it)).}

(5.19)

Although it is not immediately obvious, $\mathcal{R}^s(P_1; P_2)$ is real when $P_1, P_2 \in \mathbb{R}$.

5.1.3 FZZT BRANES AND LONG STRINGS

A FZZT(s) brane in $c = 1$ string theory is defined as a boundary state with FZZT(s) boundary condition in the Liouville CFT, and Neumann boundary conditions in the $X^0$ and ghosts CFT.

In (+) and out (−) open string states states with ends between FZZT branes of types $s_1$ and $s_2$ are represented by the boundary vertex operators$^1$

$$
\Psi_{\omega}^{s_1, s_2, \pm} \equiv g_o \# e^{\pm i\omega X^0} \# \psi_{\omega}^{s_1, s^2},

(5.20)
$$

where $\# \ldots \#$ denotes boundary normal ordering and $g_o$ is the open string coupling. The precise relation between $g_o$ and $g_s$ is determined below (see (5.25)), and it is of the form $g_o \sim \sqrt{g_s}$.

In section 4.2 we argued that the ZZ brane is heuristically like a D0-brane that is localized in the strong coupling region. Similarly, the FZZT(s) brane describes a D1-brane that starts at a position $\phi_c(s)$ and extends to the asymptotic region $\phi = -\infty$. From (5.10) it follows that the open string stretched between a ZZ and FZZT(s) branes has mass squared equal to $s^2$. For

$^1$We omit writing the $c$ ghosts for simplicity, and explicitly include them in the worldsheet scattering amplitudes below.
large values of $s$, the energy of this open string diverges, since the energy due to the string tension increases as we move the FZZT($s$) brane away from the bulk. Thus, $\phi_c(s)$ agrees with the semiclassical result $\phi_c(s) = -2\pi s$ for large $s$.

At the level of the boundary CFT, the “long string limit” of an open string vertex operator of energy $\omega$ is defined by [41]

$$\omega \to \infty, \quad s \to \infty, \quad \epsilon \equiv \omega - 2s \text{ fixed.} \quad (5.21)$$

This is the precise analogue of the long string limit described for the classical solution (5.4), after using the mapping (5.8). $\epsilon$ is called the “renormalized energy” of the long string, or simply energy when there is no room for confusion.

The interpretation is the same as for the classical long string, shown in Figure 5.1. A very energetic open string moves from the asymptotic region at $\phi = -\infty$ where the bulk/boundary Liouville potentials vanish. At the position $\phi \sim -2\pi s$, the ends of the open string get stuck on the FZZT($s$) brane, while the tip of the long string continues moving since it has enough energy to reach deep into the bulk. The effective string coupling is finite in the bulk, which leads to non-trivial scattering amplitudes with other long/closed strings. Note that interactions with the ends of the long string or other open strings of finite energy are exponentially suppressed, since the effective string coupling on the FZZT($s$) brane is of order $g_s e^{-2\pi s} \to 0$ as $s \to \infty$.

Finally, we should comment that the matrix model description of a FZZT($s$) brane is unknown (see [70, 71] for some conjectures). Nevertheless, their long string limit was conjectured to be dual to non-singlet sectors of the matrix quantum mechanics [41]. We now turn to the
computation of non-trivial scattering amplitudes between long strings and closed strings from the worldsheet formalism, which will be exactly reproduced from the dual matrix model description in section 5.3. In Appendix D, we provide further evidence for this conjecture by matching the reflection phase of the tip of the long string (5.7).

5.2 LONG STRING SCATTERING FROM THE WORLDSHEET

In this section we compute worldsheet scattering amplitudes between long strings and closed strings. We restrict the discussion to open strings with ends on a single FZZT(s) brane.

The scattering amplitude $A_{L_i C_j \rightarrow L_k C_l}$ of long strings $L$ and closed strings $C$ is obtained from the scattering amplitudes of open strings and closed strings $A_{\Psi^+_i \psi^+_j \rightarrow \Psi^-_k \psi^-_l}$ as

$$
A_{L_i C_j \rightarrow L_k C_l} (\{\epsilon_i, \omega_j\} \rightarrow \{\epsilon_k, \omega_l\}) \equiv 
\lim_{s \rightarrow \infty} \left( \prod_i \frac{1}{\sqrt{\epsilon_i + 2s}} \right) \left( \prod_k \frac{1}{\sqrt{\epsilon_k + 2s}} \right) A_{\Psi^+_i \psi^+_j \rightarrow \Psi^-_k \psi^-_l} (\{\epsilon_i + 2s, \omega_j\} \rightarrow \{\epsilon_k + 2s, \omega_l\}),
$$

(5.22)

where the long string $L_i$ has renormalized energy $\epsilon_i$, the closed string $C_j$ has energy $\omega_j$, and $\Psi^+_i$ and $\psi^+_j$ are the corresponding open and closed string vertex operators. The closed string vertex operators are still normalized as in (2.52). A consequence of energy conservation is that the number of long strings in the in- and out-states are the same.

Although not immediately obvious, the open string vertex operator in (5.20) corresponding to an open string asymptotic state $|\omega\rangle_o$, is normalized as $_o\langle \omega | \omega' \rangle_o = \omega \delta(\omega - \omega')$. Thus, before taking the long string limit of the scattering amplitude of open+closed strings on the RHS of (5.22), we have to multiply by $1/\sqrt{\epsilon_i + 2s}$ for each long string with renormalized energy $\epsilon_i$, in
order to obtain finite scattering amplitudes.

5.2.1 Long→Long+Closed Scattering Amplitude

According to (5.22), the tree-level long→long+closed scattering amplitude is obtained from the scattering amplitude between two open strings and a closed string on the disk. We work on the upper half-plane, and fix the closed string at position $z = i/2$ and one of the open strings at position $z = 0$, while the other open string is integrated over the real line parametrized by $x \in \mathbb{R}$. Thus we find

$$ A_{\Psi_{\omega_1}^{s,+} \rightarrow \Psi_{\omega_2}^{s,-} V_{\omega_3}^{-}} = 2 \int_{0}^{\infty} dx \left\langle \Psi_{\omega_1}^{s,+}(0) \Psi_{\omega_2}^{s,-}(x) V_{\omega_3}^{-}(i/2) \right\rangle, $$

$$ \left\langle \Psi_{\omega_1}^{s,+}(0) \Psi_{\omega_2}^{s,-}(x) V_{\omega_3}^{-}(i/2) \right\rangle = i g_s^2 C_{D^2} \delta(\omega_1 - \omega_2 - \omega_3) $$

$$ \times 2^{-2\omega_1 \omega_3} |x|^{2\omega_1 \omega_2} |x - i/2|^{-2\omega_2 \omega_3} \left\langle \psi_{\omega_1}^{s,+}(0) \psi_{\omega_2}^{s,-}(x) V_{\omega_3/2}(i/2) \right\rangle_{\text{Liouville}}. $$

On the RHS of the second equation, we have explicitly evaluated the free-boson and ghost correlators via the doubling trick. The overall factor of 2 comes from restricting the moduli integral to $x > 0$, since it is symmetric under $x \leftrightarrow -x$.

$C_{D^2}$ is a constant that is independent of the boundary condition for the Liouville CFT, since it comes from the normalization in the $X^0$ and ghosts correlation functions. Thus, it can be related to the constant $C_{D^2}$ in (4.29) obtained for the ZZ-instanton boundary condition, and it is given by\(^2\) (4.29).

\(^2\) First note that the ghosts have the same boundary condition in both cases, so their contribution is the same. $X^0$ has Dirichlet boundary condition for the ZZ instanton and Neumann boundary conditions for FZZT brane. There is a factor of $1/2\pi$ from relating the Dirichlet and Neumann boundary conditions, but this is cancelled by a factor of $2\pi$ in the integration over the zero mode for the Neumann boundary condition.
To compute \( \langle \psi_{\omega_1}^{s,s}(0)\psi_{\omega_2}^{s,s}(x)V_{\omega_3/2}(i/2) \rangle_\text{Liouville} \), we use the bulk-boundary OPE to reduce it to a sum over disk 3-boundary correlation functions. This sum can be organized into a sum over boundary primaries, together with an appropriate conformal block that includes the contributions of the Virasoro descendants. More explicitly, we have

\[
\langle \psi_{\omega_1}^{s,s}(0)\psi_{\omega_2}^{s,s}(x)V_{\omega_3/2}(i/2) \rangle_\text{Liouville} = 2^{2h_1}|x - i/2|^{-2h_2} \left( \frac{x - i/2}{x + i/2} \right)^{-h_1} \times \int_0^\infty \frac{dP}{\pi} R^s(\omega_3/2; P) C^{s,s,s}(P, \omega_1, \omega_2) i^{-h_1+h_2-h} F(h_1, h_2, h_3, h_3; h |\eta). \tag{5.24}
\]

Here \( R^s \) is the bulk-boundary structure constant and \( C^{s,s,s} \) is the 3-boundary structure constant, discussed in section 5.1.2. Furthermore, we use the doubling trick on the disk to write the disk bulk-boundary-boundary conformal block as the holomorphic sphere 4-point Virasoro block with \( c = 25 \), written in (3.23). The arguments in the conformal block are the external weights \( h_1 = 1 + \omega_1^2, h_2 = 1 + \omega_2^2, h_3 = 1 + \omega_3^2/4, \) the internal weight \( h = 1 + P^2, \) and the cross-ratio \( \eta = 2x/(x - i/2). \) The phase factor in (5.24) is fixed by demanding that the correlator is real.

The integration over the moduli \( x \) in (5.23) is a priori divergent for intermediate Liouville momenta \( P \lesssim \omega_3 \), and requires regularization. We regularize it by preserving analyticity in the external energies, in a similar way to the regularization of closed string scattering amplitudes (3.49). However, it turns out that in the \( s \to \infty \) limit the contribution from the \( P \)-integral in (5.24) is dominated by values of \( P \) of the order \( P \sim \omega_1 \). Since \( \omega_1 \sim 2s \gg \omega_3 \) in this limit, we can simply perform the \( P \)-integral in (5.24) for \( P > \omega_3 \), as the remaining contribution is negligible. The exact regularization of (5.24) is discussed in Appendix C, where we also present numerical evidence for dropping, in the long string limit, the region of the \( P \)-integral that needs
regularization.

**Numerical Results**

To compute the scattering amplitude (5.23), we resort to numerical methods. The Liouville correlator is evaluated as in (5.24) (with the intermediate momentum $P$ restricted to the range $P > \omega_3$). The Virasoro conformal block appearing in (5.24) can be evaluated numerically using Zamolodchikov’s recursion relation, which is discussed in Appendix B. We sample over different values of the intermediate momentum, and at each value we combine the conformal block with the structure constants. We then numerically integrate over the intermediate Liouville-momentum to obtain the Liouville correlator (5.24), and further numerically integrate over the worldsheet modulus $x$ as in (5.23). The technical and numerical details are discussed in Appendix C.

![Figure 5.2](image.png)

**Figure 5.2:** The blue dots are a sample of the amplitude $A_{\Psi_{\omega_1} \rightarrow \Psi_{\omega_2} - \nu_{\omega_3}} / \sqrt{\omega_1 \omega_2}$, dropping the prefactors in the first line of (5.23), for increasing values of $s$. Here we use $\epsilon_2 = 0.7$ and $\omega_3 = 0.5$. The dashed grey line is an exponential fit of the amplitudes at large $s$, whereas the solid red line marks the long string limit of the amplitude.
To take the long string limit, we evaluate $\mathcal{A}_{\Psi^{s,s^+}_{\omega_1} \rightarrow \Psi^{s,s^-}_{\omega_2}}$ numerically at fixed renormalized long string energies $\epsilon_1, \epsilon_2$, for increasing values of $s$. We further include the prefactors from the normalization of the long string energies in (5.22), and we find an exponential convergence of the amplitude $\frac{1}{\sqrt{\omega_1 \omega_2}} \mathcal{A}_{\Psi^{s,s^+}_{\omega_1} \rightarrow \Psi^{s,s^-}_{\omega_2}}$ as we take $s \rightarrow \infty$, shown in Figure 5.2. The exponential convergence is expected since the effective string coupling on the FZZT brane vanishes exponentially with $s$.

![Image of a 3D plot with axes labeled $\epsilon_2$, $\omega_3$, and $\mathcal{A}_{\Psi^{s,s^+}_{\omega_1} \rightarrow \Psi^{s,s^-}_{\omega_2}}$](image)

**Figure 5.3:** The long $\rightarrow$ long + closed string amplitude $\mathcal{A}_{L \rightarrow L+C}$ (rescaled by $\mu$) as a function of the outgoing long string energy $\epsilon_2$ and the closed string energy $\omega_3$.

The long $\rightarrow$ long + closed amplitude $\mathcal{A}_{L \rightarrow L+C}$, defined as the limit in (5.22), is then obtained from an exponential numerical fit. We perform this calculation over a range of renormalized energy $\epsilon_2$ of the outgoing long string, and over a range of the closed string energy $\omega_3$, and plot the results in Figures 5.3, 5.4 and 5.5.

Figure 5.4 shows the amplitude $\mathcal{A}_{L \rightarrow L+C}$ at a fixed value of the closed string energy $\omega_3$, as a function of the renormalized energy of the outgoing long string $\epsilon_2$. Its qualitative features can
Figure 5.4: Long string amplitude $A_{L \to L+1}$ evaluated at fixed $\omega_3$ as a function of $\epsilon_2$. Here we have rescaled the amplitude by $\mu = \frac{1}{\sqrt{g_0}}$. In (b) we plot the logarithm of the absolute value of the amplitude over a range of sufficiently large negative $\epsilon_2$. The red line represents a linear fit of slope 6.25, which is in reasonable agreement with our expectation that the amplitude is modulated by a $e^{2\pi \epsilon_2}$ profile in this regime.

be partially understood as follows. For $\epsilon_2 \ll -1$, the tip of the long string always remains in the weak coupling region, and reaches at most $\phi \sim \epsilon_2/(2T)$, where $T = \frac{1}{2\pi}$ is the tension of the string. At the latter location, the effective string coupling is $g_s e^{2\phi} \sim g_s e^{2\pi \epsilon_2}$, indicating that the amplitude of emitting a closed string is exponentially suppressed. This is consistent with our numerical results, in that the latter exhibits an oscillatory behavior at sufficiently large negative $\epsilon_2$ modulated by an exponential suppression profile that fits with $\sim e^{2\pi \epsilon_2}$, as shown in Figure 5.4b.

On the other hand, for $\epsilon_2 \gg 1$, we expect the tip of the long string to reach deep into the Liouville barrier, up to the location $\phi \sim \frac{1}{2} \ln \epsilon_2$ where the renormalized energy of the long string is dominated by the Liouville potential energy $\sim e^{2\phi}$. Now the effective string coupling at the tip of the long string is $g_s e^{2\phi} \sim g_s \epsilon_2$, giving rise to the linear behavior of the amplitude in $\epsilon_2$, as seen in Figure 5.4a for sufficiently large energy $\epsilon_2$.  

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Figure 5.5: Long string amplitude $A_{L \to L+C}$ (rescaled by $\mu$) evaluated at fixed outgoing long string energy $\epsilon_2$ as a function of closed string energy $\omega_3$.

We will see that the worldsheet results agree precisely with the matrix model amplitude (5.48), and furthermore allow us to fix

$$C_{D^2} = \frac{1}{2g_o^2}, \quad g_o^2 = \frac{1}{2\pi^2} g_s = \frac{1}{2\pi^2 \frac{\mu}{\pi^2}},$$

(5.25)

where for the second equation we used (4.29) and (3.38).

Figure 5.5 shows $A_{L \to L+C}$ as a function of the closed string energy $\omega_3$ (which is positive by definition), at a fixed value of the outgoing long string renormalized energy $\epsilon_2$. The result is approximately linear for sufficiently large $\omega_3$, and oscillatory for small $\omega_3$. We do not know a
semi-classical explanation of this behavior.

5.2.2 Long+Long→Long+Long Scattering Amplitude

![Liouville disk diagrams](image)

Figure 5.6: Three of the Liouville disk diagrams that contribute to long + long → long + long string scattering. The other three diagrams (not shown) are related by exchanging $P_3 \leftrightarrow P_4$. The solid and dashed lines represent incoming and outgoing long strings respectively. Diagrams (a) and (c) are suppressed in the long string limit, as the interaction between the incoming strings occurs on the boundary.

Consider now the tree-level $2 \to 2$ scattering of a pair of long strings. This scattering amplitude can be obtained from the long string limit of the disk 4-point amplitude of open strings, which is given by

$$A_{\psi_{\omega_1}^{s,s} + \psi_{\omega_2}^{s,s} \to \psi_{\omega_3}^{s,s} - \psi_{\omega_4}^{s,s}} = ig_4 C_{D2} \delta(\omega_1 + \omega_2 - \omega_3 - \omega_4)$$

$$\times \sum_{j=1}^{3} \int_{I_j} dx \left| x \right|^{-2\omega_1 \omega_2} \left| x - 1 \right|^{2\omega_2 \omega_3} \langle \psi_{\omega_1}^{s,s}(0) \psi_{\omega_2}^{s,s}(x) \psi_{\omega_3}^{s,s}(1) \psi_{\omega_4}^{s,s}(\infty) \rangle_{\text{Liouville}} + (3 \leftrightarrow 4),$$

(5.26)

where $C_{D2}$ is given by (4.29), as in the case of long→long+closed scattering. The $I_j$ ($j = 1, 2, 3$) on the RHS denote integration domains corresponding to the three different orderings of the boundary vertex operator insertions on the disc, as shown in Figure 5.6.

In the long string limit, the diagram corresponding to Figure 5.6b and the one related by
exchanging $P_3 \leftrightarrow P_4$ are the only ones that contribute to the scattering amplitude, whereas the four diagrams corresponding to Figures 5.6a,c and the ones related by $P_3 \leftrightarrow P_4$ are exponentially suppressed. This is because the process depicted in Figure 5.6b involves a pair of long strings recombining in the region of spacetime where the effective string coupling is finite, whereas those of Figure 5.6a,c involves a pair of long strings joining their ends on the FZZT brane, where the effective string coupling is suppressed in the long string limit. Thus, in evaluating the Liouville disk 4-point we will henceforth keep only the contribution from the diagram in Figure 5.6b, as well as the diagram related to it by exchanging $P_3 \leftrightarrow P_4$. In Appendix C, we provide numerical evidence for dropping the diagrams in Figures 5.6a,c.

The integration domain $I_2$ corresponding to the diagram in Figure 5.6b is given by $I_2 = \{1 < x < \infty\}$. Let’s consider first the region $1 < x < 2$. In this case, we compute the Liouville correlator by doing the boundary OPE between $\psi_{\omega_3}^{s,s}(1)$ and $\psi_{\omega_2}^{s,s}(x)$ (the $t$-channel), which admits the conformal block decomposition

$$
\langle \psi_{\omega_1}^{s,s}(0) \psi_{\omega_2}^{s,s}(x) \psi_{\omega_3}^{s,s}(1) \psi_{\omega_4}^{s,s}(\infty) \rangle_{\text{Liouville}} = \int_0^\infty \frac{dP}{\pi} C_{s,s,s}(\omega_2, \omega_3, P) C_{s,s,s}(P, \omega_1, \omega_4) F(h_2, h_3, h_1, h_4; h \mid x - 1).
$$

(5.27)

When $2 < x$, we expand the Liouville correlator in the $u$-channel,

$$
\langle \psi_{\omega_1}^{s,s}(0) \psi_{\omega_2}^{s,s}(x) \psi_{\omega_3}^{s,s}(1) \psi_{\omega_4}^{s,s}(\infty) \rangle_{\text{Liouville}} = x^{-2-2\omega_2} \int_0^\infty \frac{dP}{\pi} C_{s,s,s}(\omega_2, \omega_4, P) C_{s,s,s}(P, \omega_1, \omega_3) F(h_2, h_4, h_1, h_3; h \mid 1/x).
$$

(5.28)

When combined with the $X^0$ and ghost correlators as in (5.26), there will be divergences in the
moduli space integral from the vicinity of $x = 1$ and $x = \infty$. As usual, we regularize by adding a counter term to the moduli integrand which preserves analyticity in the external energies. Similarly to the case of the long→long+closed amplitude, it turns out that the region where the regulator is needed gives a negligible contribution in the long string limit, and therefore we will simply restrict the $P$-integral in (5.27) and (5.28) to the range where no regularization is necessary, namely $P > \omega_2 - \omega_3$ and $P > \omega_2 - \omega_4$ for the $t$ and $u$-channels respectively. The careful regularization of the moduli space integrand is done in Appendix C, where it is explicitly observed that restricting to these ranges does not affect the long string amplitude.

To integrate over the worldsheet moduli, we combine the $X^0$ and ghosts correlators with the Liouville disk correlator expanded in the $t$-channel for the range $1 < x < 2$, and in the $u$-channel for the range $2 < x$. We further do the change of variables $x \to 1 - 1/x$ for the former and $x \to 1/x$ for the latter, from which we find that the contribution to (5.26) that survives in the long string limit is given by

$$A_{\psi_{\omega_1}^{s,s}}^{\text{large } s} \Psi_{\omega_2}^{s,s} + \Psi_{\omega_3}^{s,s} \rightarrow \Psi_{\omega_4}^{s,s} = \int_0^{1/2} dx \left[ x^{2\omega_1 \omega_3} (1 - x)^{2\omega_1 \omega_4} \times \int_{\Lambda}^{\infty} \frac{dP}{\pi} C^{s,s,s} (P, \omega_1, \omega_3) C^{s,s,s} (P, \omega_4, \omega_2) F(h_1, h_3, h_4, h_2; h | x) + (1 \leftrightarrow 2) \right],$$

where $\Lambda \equiv \max(\omega_2 - \omega_3, \omega_2 - \omega_4)$, and the answer is independent of $\Lambda$ in the long string limit. The conformal weights appearing in the conformal blocks are $h_i \equiv 1 + \omega_i^2$ and $h \equiv 1 + P^2$.

**Numerical Results**

To compute the scattering amplitude of long strings $A_{L+L \rightarrow L+L}$, we have to evaluate the contribution to the scattering amplitude of 4 open-strings on the disk given by (5.29), combine it
Figure 5.7: Long string amplitude $A_{L+L \to L+L}$ as a function of incoming long string renormalized energies $\epsilon_1, \epsilon_2$, evaluated at a fixed outgoing long string renormalized energy $\epsilon_3 = 0.5$.

Figure 5.8: Long string amplitude $A_{L+L \to L+L}$ as a function of incoming long string renormalized energy $\epsilon_2$, evaluated at a fixed outgoing long string renormalized energies $\epsilon_1 = 3.0$ and $\epsilon_3 = 0.5$. In (b) we plot the logarithm of the absolute value of the amplitude over a range of sufficiently large negative $\epsilon_2$. The red line represents a linear fit of slope 0.21, which is in reasonable agreement with our expectation that the amplitude is modulated by a $e^{2\pi \text{min}(|\epsilon_1|)}$ profile in this regime.

with the prefactors in (5.22), and finally take the long string limit $s \to \infty$. The steps are very similar to the long→long+closed scattering amplitude, and the technical and numerical details are discussed in Appendix C. A sample of the results as a function of $\epsilon_1, \epsilon_2$, at a generic fixed
value of $\epsilon_3$, is shown in Figure 5.7. These results are in perfect agreement with the matrix model amplitude computed in (5.50), provided $g_0$ is related to $C_D$ as in (5.25), which is consistent with the long→long+closed scattering amplitude.

Some qualitative features of the results shown in Figures 5.7 and 5.8 can be understood semiclassically. For fixed $\epsilon_3$ and large $\epsilon_1, \epsilon_2$, the pair of long strings interact at the tip of the outgoing long string of least energy (namely $\epsilon_3$), where the effective string coupling is independent of $\epsilon_1, \epsilon_2$, which explains the plateau in Figure 5.7. For sufficiently large negative $\epsilon_i$, the amplitude is exponentially suppressed due to the suppression of the string coupling at the tip of the $i$-th long string, as shown in Figure 5.8.

5.3 Long String Scattering Amplitudes in the Dual Matrix Model

Long strings in $c = 1$ string theory are conjectured to be dual to states in the $c = 1$ matrix quantum mechanics Hilbert space (2.11) that transform in non-singlet representations of the $U(N)$ symmetry group [41]. In this section, we describe the long string Hamiltonian and compute the long→long+closed and long+long→long+long tree-level scattering amplitudes.

5.3.1 Non-singlet Sector and Long String Hamiltonian

The sector of the Hilbert space of the $c = 1$ matrix quantum mechanics (2.11) that describes $n$ long strings transforms in the $U(N)$ representation $\mathcal{R} = \text{adj}^\otimes n$. Note that, due to the zero-weight constraint (2.3), the matrix model Hilbert space can be completely decomposed into such sectors. It is convenient to think of the long strings as having their ends attached to different FZZT branes, and therefore states with multiple long strings are interpreted as
distinct particles.

The Hilbert space $\mathcal{H}_R'$ describing $n$ long strings is denoted by $\mathcal{H}_n$, and it is spanned by the states

$$\psi_{i_1 j_1, \ldots, i_n j_n}(\lambda_1, \ldots, \lambda_N)|i_1 j_1, \ldots, i_n j_n\rangle,$$

where due to the zero-weight condition $\{i_1, \ldots, i_n\}$ is a permutation of $\{j_1, \ldots, j_n\}$, while the $S'_N$ symmetry in (2.11) requires that $\psi$ is odd under swapping $\lambda_i$ with $\lambda_j$ and at the same time swapping $i$ and $j$ among all of the indices of $\psi$.

Let’s start by discussing a single long string state of the form

$$\psi_{ij}(\lambda_1, \ldots, \lambda_N)|ij\rangle = \sum_{i=1}^{N} w(\lambda_i)\psi(\lambda_1, \ldots, \lambda_N)|ii\rangle \equiv |w\rangle,$$

where $\psi(\lambda_1, \ldots, \lambda_N)$ is the ground state wave function of the singlet sector of the matrix model, namely the wavefunction of the state$^3$ (2.36). Note that $w(\lambda_i)$ is viewed as a function $w(\lambda)$ evaluated at the eigenvalue $\lambda_i$, so that $\psi_{i_1 j_1, \ldots, i_n j_n}$ obeys the required $S'_N$ symmetry.

The Hamiltonian for a single long string is given by the expression (2.10), namely

$$H = \sum_{i=1}^{N} \left( -\frac{1}{2} \frac{\partial^2}{\partial \lambda_i^2} + V(\lambda_i) \right) + \frac{1}{2} \sum_{i \neq j} R_{ij} R_{ji} (\lambda_i - \lambda_j)^2,$$

where the $U(N)$ generators $R_{ij}$ act on the adjoint representation. In particular, their action on

$^3$In this section we only consider perturbative scattering amplitudes of long strings, which are insensitive to exactly how the “other side” of the matrix model inverted quadratic potential is filled.

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the states (5.30) is induced from their action on the adjoint representation, which is given by

$$R_{ij}|k\ell\rangle = \delta_{jk}|i\ell\rangle - \delta_{i\ell}|kj\rangle.$$  

(5.33)

The long string Hamiltonian (5.32) acts on (5.31) as [72]

$$H|w\rangle = \sum_i \left[ -\frac{1}{2}w''(\lambda_i) - w'(\lambda_i) \frac{\partial}{\partial \lambda_i} + E_0 w(\lambda_i) + \sum_{j \neq i} \frac{w(\lambda_i) - w(\lambda_j)}{(\lambda_i - \lambda_j)^2} \right] \psi_0|ii\rangle,$n

(5.34)

where $E_0$ is the ground state energy, which will henceforth be dropped. Even though it is not immediately obvious, we will see in section 5.3.2 that in the weak string coupling limit $\mu \to \infty$ the last term on the RHS dominates. Using the eigenvalue density $\rho$ defined in (2.14), we rewrite this term in the continuum description as

$$\int d\lambda \rho(\lambda) \frac{w(\lambda_i) - w(\lambda)}{(\lambda_i - \lambda)^2} \psi_0|ii\rangle. \ 

(5.35)$$

The long string state of energy $E$ is given by (5.31) with $w(\lambda) = w_E(\lambda)$, and satisfies the eigen-energy equation [41]

$$\int_{\sqrt{2\pi}}^{\infty} d\lambda' \rho_0(\lambda') \frac{w_E(\lambda) - w_E(\lambda')}{(\lambda - \lambda')^2} = E w(\lambda), \ 

(5.36)$$

as well as the normalization condition

$$\int_{\sqrt{2\pi}}^{\infty} d\lambda \rho_0(\lambda) w_E(\lambda) w^{*}_{E'}(\lambda) = \delta(E - E'), \ 

(5.37)$$

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where \( \rho^{(0)}(\lambda) = \frac{1}{\pi} \sqrt{\lambda^2 - 2\mu} \) is the ground state fermion density in the singlet sector.

In order to define the \( \lambda' \)-integral on the LHS of (5.36), we use a principal value prescription to regularize the divergence at \( \lambda' = \lambda \). This is the continuum version of the sum over distinct eigenvalues \( \lambda_j \neq \lambda \) in the last term in (5.34). Furthermore, (5.36) suffers from an IR divergence from the region of large \( \lambda' \). We introduce an IR cutoff in the \( \tau \) coordinates at \( \tau = L \), or in \( \lambda \) coordinates at \( \lambda = \sqrt{2\mu} \cosh L \), where \( \lambda = \sqrt{2\mu} \cosh \tau \), from which we find

\[
\int_{\sqrt{2\mu}}^{\sqrt{2\mu}\cosh L} d\lambda' \rho_0(\lambda') \frac{w_E(\lambda)}{(\lambda - \lambda')^2} \sim \frac{L}{\pi} \quad (5.38)
\]

for large \( L \). This linear divergence with \( L \) is interpreted as the energy of the long string of length \( 2L \), which is divergent due to the string tension. Indeed, the renormalized long string energy defined by (5.21) in the worldsheet formalism is identified with

\[
\epsilon \equiv E - \frac{L - 1}{\pi} \quad (5.39)
\]

in the matrix model description. The finite shift on the RHS of (5.39) is in principle ambiguous, but we have fixed (5.39) by explicit comparison of the scattering amplitudes on the worldsheet and matrix model descriptions.

The Hamiltonian (5.36) and the normalization (5.37) can be rewritten in \( \tau \) coordinates as

\[
\frac{1}{4\pi} \int_0^{\infty} d\tau' \left[ \frac{1}{(\sinh \frac{\tau + \tau'}{2})^2} - \frac{1}{(\sinh \frac{\tau - \tau'}{2})^2} \right] h_\epsilon(\tau') - \frac{1}{\pi \tanh \tau} h_\epsilon(\tau) = \epsilon h_\epsilon(\tau),
\]

\[
\int_0^{\infty} d\tau h_\epsilon(\tau) h_{\epsilon'}(\tau) = \delta(\epsilon - \epsilon'), \quad (5.40)
\]

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where we have defined the long string wavefunction as

\[ h_\epsilon(\tau) \equiv \sqrt{\pi} \rho_0(\lambda) w_E(\lambda). \]  (5.41)

We impose\(^4\) the boundary condition \( h_\epsilon(0) = 0 \) on the wavefunction \( h_\epsilon(\tau) \). The exact wavefunction to (5.40) is given by [44]

\[
h_\epsilon(\tau) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{\pi}} \frac{\sin(k\tau) \sinh(\pi k)}{\sinh^2(\pi k) + e^{2\pi \epsilon}} \sin \left[ \pi \int_{k_0}^{k} \frac{dk'}{\sqrt{\sinh^2(\pi k') + e^{2\pi \epsilon}}} \left( \epsilon - \frac{k'}{\tanh(\pi k')} \right) \right],
\]  (5.42)

where \( k_0 = \frac{i}{\pi} \arcsin e^{\pi \epsilon} \).

In Appendix D, we show that \( h_\epsilon(\tau) \) exactly matches the wavefunction for the tip of the long string (5.7), and also that the reflection phases on the two descriptions agree [41, 44]. This provides further evidence for the ambiguous constant term in (5.39). We now turn to the matrix model computation of the long→long+closed and long+long→long+long scattering amplitudes.

### 5.3.2 Long→Long+Closed Scattering Amplitude

The long→long+closed scattering amplitude at tree-level is obtained, via the Born approximation, from the interaction part of the long string Hamiltonian. The long string in-state of energy \( E_1 \) is given by \( |w_{E_1}\rangle \). The out-state describing a closed string of energy \( \omega_3 \) and a long string of energy \( E_2 \) is given by \( a_\omega^\dagger |w_{E_2}\rangle \), where \( a_\omega^\dagger, a_\omega \) are the creation/annihilation modes of the fermion collective excitations, as discussed in (2.20).

\(^4\)This boundary condition gives non-trivial scattering amplitudes with closed strings satisfying Dirichlet boundary condition at \( \tau = 0 \), described below (2.19). Nevertheless, it should be possible to solve for the long string wavefunction using a \( p \)-parametrization of the Fermi sea, described in Appendix F.
Consider now the interaction part of the long string Hamiltonian, which is given by (5.34) with the free long string Hamiltonian (5.36) subtracted off, namely

$$H_1^{\text{int}}|_{w_{E_1}} = \sum_i \left[ -\frac{1}{2} w''_{E_1}(\lambda_i) - w'_{E_1}(\lambda_i) \frac{\partial}{\partial \lambda_i} + \frac{1}{\sqrt{\pi}} \int d\lambda \partial \lambda \eta(\lambda) \frac{w_{E_1}(\lambda_i) - w_{E_1}(\lambda)}{(\lambda_i - \lambda)^2} \right] \psi_0|_{ii},$$

(5.43)

where we used (2.15) to evaluate the last term. At the level of the Born approximation, the long→long+closed scattering amplitude is given by the matrix element

$$\langle w_{E_2} | b_{\omega_3} H_1^{\text{int}} | w_{E_1} \rangle,$$

(5.44)

where $E_1 = E_2 + \omega_3$. The contribution to this matrix element from the first term on the RHS of (5.43) is given by

$$-\frac{1}{2} \langle w_{E_2} | b_{\omega_3} \sum_i w''_{E_1}(\lambda_i) \psi_0|_{ii} \rangle = -\frac{1}{2 \sqrt{\pi}} \int d\lambda \langle \psi_0 | b_{\omega_3} \partial \lambda \eta(\lambda) | \psi_0 \rangle w'_{E_2}(\lambda) \partial^2 \lambda w_{E_1}(\lambda).$$

(5.45)

This contribution scales as $\mu^{-2}$ in the weak coupling limit, and therefore does not contribute to the tree-level long→long+closed scattering amplitude.

Now consider the contribution from the second term on the RHS of (5.43) to the matrix element (5.44). This can be written in terms of the fermion momentum density (2.16) as

$$-\langle w_{E_2} | b_{\omega_3} \sum_i w'_{E_1}(\lambda_i) \frac{\partial}{\partial \lambda_i} \psi_0|_{ii} \rangle = -i \int d\lambda \langle \psi_0 | b_{\omega_3} \Pi_p(\lambda) | \psi_0 \rangle w^*_{E_2}(\lambda) \partial \lambda w_{E_1}(\lambda)$$

$$= -\frac{\omega_3}{\sqrt{2\pi \mu}} \int_0^\infty d\tau \sin(\omega_3 \tau) \frac{h^*_{e_2}(\tau)}{\sinh \tau} \partial \tau \left( \frac{h_{e_1}(\tau)}{\sinh \tau} \right),$$

(5.46)

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where in the last line we rewrote the result in $\tau$ coordinates and used the results for the commutation relations of the closed string modes defined in section 2.2. Furthermore, in the last line we kept only the terms that scale as $\mu^{-1}$ in the weak coupling limit, which will contribute to the scattering amplitude at tree-level.

Finally, the last term on the RHS of (5.43) gives the contribution

$$\frac{1}{\sqrt{\pi}} \langle w_{E_2} | \omega_3 \sum_i \int d\lambda \partial_\lambda \eta(\lambda) \frac{w_{E_1}(\lambda_i) - w_{E_1}(\lambda)}{(\lambda_i - \lambda)^2} \psi_0 | \psi \rangle \quad (5.47)$$

where subleading terms in $1/\mu$ have been dropped.

Combining the contributions at order $1/\mu$ from the matrix element (5.44), the tree-level long→long+closed scattering amplitude is given by

$$A_{L\rightarrow L+C}^{\text{tree}} = -2\pi i \langle w_{E_2} | \omega_3 \mathcal{H}_{\text{int}}' | w_{E_1} \rangle \quad (5.48)$$

This scattering amplitude can also be obtained by a collective field formalism for the adjoint sector of the matrix quantum mechanics [73, 74].

To evaluate the scattering amplitude (5.48), we first compute the wavefunctions $h_\epsilon(\tau)$ by numerically performing the $k$-integral in (5.42). With these in hand, we numerically integrate over $\tau, \tau'$ in (5.48). The results are shown in Figure 5.9, and they are in striking agreement.
Figure 5.9: (a): Numerical results for the matrix model scattering of the tree level long → long + closed amplitude (5.48) as a function of the outgoing long string renormalized energy $\epsilon_1$ and outgoing closed string energy $\omega_3$. (b) and (c): Ratio of numerical results computed from the worldsheet to those computed from the matrix model for tree-level long → long + closed string amplitude.

with the worldsheet amplitude computed in section 5.2.1. By comparing the worldsheet and matrix model answers, we fix $C_{D^2}$ in terms of $g_0$, with the result being given to high numerical accuracy by (5.25). Assuming the relation (5.25), the error between the worldsheet and matrix model results is shown in Figure 5.9b,c.
5.3.3 Long+Long→Long+Long Scattering Amplitude

We now turn to the calculation of the $2 \rightarrow 2$ scattering amplitude of long strings at tree-level. The state describing a pair of long strings transforms in the bi-adjoint representation of the $U(N)$ symmetry group, of which there are two types satisfying the zero weight constraint, namely

\[
\sum_{i,j=1}^{N} w_{E_1}(\lambda_i)w_{E_2}(\lambda_j)\psi_0(\lambda_1, \cdots, \lambda_N)|ii, jj\rangle \equiv |w_{E_1}, w_{E_2}\rangle_{11,22},
\]

(5.49)

\[
\sum_{i,j=1}^{N} w_{E_1}(\lambda_i)w_{E_2}(\lambda_j)\psi_0(\lambda_1, \cdots, \lambda_N)|ij, ji\rangle \equiv |w_{E_1}, w_{E_2}\rangle_{12,21}.
\]

It is convenient to let the long strings end on different FZZT branes, labelled by their FZZT parameters $s_i$ for $i = 1, \ldots, 4$. The first state in (5.49) is interpreted as having a long string of energy $E_1$ with ends on $s_1 \rightarrow s_2$, and another long string of energy $E_2$ with ends on $s_3 \rightarrow s_4$, where the arrow denotes the orientation of the long string. On the other hand, the second state in (5.49) is interpreted as having a long string of energy $E_1$ with ends on $s_1 \rightarrow s_4$, and a long string of energy $E_2$ with ends on $s_3 \rightarrow s_2$, as shown in Figure 5.10.

![Figure 5.10: The 11,22 → 12,21 long string scattering by reconnecting.](#)

We are interested in the tree-level scattering amplitude between a pair of long strings that
interact by reconnecting in the bulk, thereby turning a state of type $11,22$ to a state of type $12,21$. The scattering amplitude is computed in the Born approximation from the matrix element of the interaction potential, which is given by (5.32) with the free Hamiltonian subtracted off. Thus, we find the long+long$\rightarrow$long+long scattering amplitude

$$A_{L+L\rightarrow L+L}^{\text{tree}} = -2\pi i_{12,21} \left[ w_{E_3, E_4} \left( \frac{1}{2} \sum_{i \neq j} \frac{R_{ij} R_{ji}}{(\lambda_i - \lambda_j)^2} - (E_1 + E_2) \right) \right] \left| w_{E_1, E_2} \right|_{11,22}$$

$$= 2\pi i \int d\lambda d\lambda' \rho_0(\lambda) \rho_0(\lambda') \frac{w_{E_3}^*(\lambda) w_{E_4}^*(\lambda') (w_{E_1}(\lambda) - w_{E_1}(\lambda')) (w_{E_2}(\lambda) - w_{E_2}(\lambda'))}{(\lambda - \lambda')^2}$$

$$= \frac{\pi i}{\mu} \int_0^{\infty} d\tau d\tau' \sinh \tau \sinh \tau' \left( \frac{h_{e_{1}^*}(\tau)}{\sinh \tau} - \frac{h_{e_{1}}(\tau')}{\sinh \tau'} \right) \left( \frac{h_{e_{2}^*}(\tau)}{\sinh \tau} - \frac{h_{e_{2}}(\tau')}{\sinh \tau'} \right),$$

where in the last line we rewrote the expression in $\tau$ coordinates. At this order in the string coupling, the scattering amplitudes $11,22 \rightarrow 11,22$ or $12,21 \rightarrow 12,21$ between a pair of long strings of the same type vanishes.

**Figure 5.11:** (a) Numerical results for the matrix model scattering of the tree level long $+$ long $\rightarrow$ long $+$ long amplitude (5.50) as a function of incoming long string renormalized energies $\epsilon_1, \epsilon_2$ at fixed outgoing long string renormalized energy $\epsilon_3 = 0.5$. (b) Ratio of numerical results computed from the worldsheet to those computed from the matrix model for tree-level long $+$ long $\rightarrow$ long $+$ long string amplitude.
We evaluate the scattering amplitude (5.50) numerically. The results are shown in Figure 5.11a, and are in excellent agreement with the worldsheet result discussed in section 5.2.2. Furthermore, matching the worldsheet and matrix model results fixes the constant $C_{D^2}$ in terms of $g_o$, which is consistent with the relation (5.25). Assuming this relation, we plot in Figure 5.11b the ratio between the worldsheet and matrix model results.
In this thesis, we revisited the perturbative dictionary of \( c = 1 \) string theory and explored new features that go beyond string perturbation theory. We started by exploiting numerical techniques for computing Liouville correlation functions to explicitly compute worldsheet scattering amplitudes of closed strings. This approach helped us resolve several previous puzzles in the perturbative dictionary.

We also proposed a worldsheet formalism for computing non-perturbative contributions to closed string scattering amplitudes, by including disconnected worldsheet diagrams with \( ZZ \)-
instanton boundary conditions. This formalism can be applied to all non-perturbative orders, and by matching non-perturbative scattering amplitudes we were led to a natural proposal for the non-perturbative dual of $c = 1$ string theory.

Furthermore, we argued that there are long string degrees of freedom in $c = 1$ string theory, which are dual to states in nonsinglet sectors of the dual matrix model. Evidence for these proposals was based on a detailed matching of scattering amplitudes computed on the two sides.

There are several open questions remaining. One unanswered question is how to restore non-perturbative unitarity of scattering amplitudes in $c = 1$ string theory. As discussed before, we expect that the missing degrees of freedom are ZZ-branes with open string tachyons condensed to the “other side”. While this is a natural suggestion, such a calculation is currently beyond the realm of the worldsheet formalism, and possibly requires the use of open+closed string field theory.

The non-perturbative methods developed in $c = 1$ string theory can be extended to the string theories dual to types 0A and 0B matrix models [52, 51]. In the case of type 0B, the closed string vacuum describes a state in the dual matrix model where the fermions fill both sides of the inverted quadratic potential. Unlike $c = 1$ string theory, the exclusive scattering amplitudes of closed strings are expected to be unitary non-perturbatively, without having to include other types of non-perturbative degrees of freedom. Similarly, the matrix model dual of type 0A is known to be a quiver theory, which is also expected to have non-perturbatively unitary closed string scattering amplitudes. It would be interesting to verify if the scattering amplitude mediated by instantons indeed satisfy these unitarity conditions.
It has been proposed that the Euclidean $c = 1$ string theory at finite temperature can be deformed to the 2d black hole background, by condensing a large number of FZZT branes in a suitable ’t Hooft-like limit [41]. In this limit, the black hole background is described by a thermal bath of long strings, and we can explore (Lorentzian) scattering amplitudes on this thermal bath. From the worldsheet description, the analytic continuation from Euclidean to Lorentzian signature is not obvious since we need to specify a contour in field space. However, Lorentzian scattering amplitudes on a thermal bath are expected to be well-defined from the matrix model description, and it would be interesting to explore what these scattering amplitudes can teach us about the black hole information paradox. This goal would benefit enormously from an exact solution of the $c = 1$ matrix quantum mechanics in non-singlet sectors, where the long strings live.

It is also interesting to extend the methods developed in this thesis to theories in higher dimensions. One such example is the calculation of non-perturbative effects in type IIB string theory. While such contributions to the scattering amplitudes have been computed in the past [38] at order $e^{-\frac{1}{\alpha}}$, subleading perturbative and non-perturbative corrections remain to be understood. It is particularly interesting to explore the higher non-perturbative corrections, which will involve new rules compared to the case of $c = 1$ string theory (see [66] for some progress in this direction).

Another possible application is to the near-horizon theory of NS5 branes known as “little string theories” (see [75, 76, 77, 78] and references therein), which also have a linear dilaton background. When the NS5 branes are separated, there is a Liouville wall, and it would be interesting to explore if there is another asymptotic region beyond the Liouville wall, as is
the case of $c = 1$ string theory. Furthermore, the Liouville wall disappears if the NS5 branes coincide, and we can ask how to generalize our methods to this more challenging setup.

To conclude, there are many questions left to explore, and the fun part has only just began!
In this Appendix we discuss some of the properties of the special functions that appear in the structure constants in Liouville CFT.

The function $\Gamma_1(x)$ is defined in terms of the Barnes $G$-function $G(x)$ by

$$\Gamma_1(x) \equiv \frac{(2\pi)^{\frac{x-1}{2}}}{G(x)}.$$  \hspace{1cm} (A.1)

It is a meromorphic function on the complex plane, with (not necessarily simple) poles at
where \( x \in \mathbb{Z}_{\leq 0} \). Furthermore, it satisfies the recursion relation

\[
\Gamma_1(x + 1) = \frac{\sqrt{2\pi}}{\Gamma(x)} \Gamma_1(x).
\]  

(A.2)

Alternatively, \( \Gamma_1(x) \) admits an integral representation in the range \( 0 < \text{Re} \, x < 2 \), which is given by

\[
\ln \Gamma_1(x) = \int_0^\infty \frac{dt}{t} \left[ e^{-xt} - e^{-t} - \frac{(1 - x)^2}{2e^t} - 2\frac{1 - x}{t} \right],
\]

(A.3)

and it can be defined on the whole complex plane by analytic continuation using (A.2).

The function \( \Upsilon_1(x) \) is the “Barnes double Gamma function”, defined by

\[
\Upsilon_1(x) \equiv \frac{1}{\Gamma_1(x) \Gamma_1(2 - x)}.
\]

(A.4)

It is an entire analytic function, with zeroes at \( x \in \mathbb{Z}\setminus\{1\} \). It satisfies the following recursive and reflection properties

\[
\Upsilon_1(x + 1) = \gamma(x) \Upsilon_1(x), \quad \gamma(x) \equiv \frac{\Gamma(x)}{\Gamma(1 - x)},
\]

(A.5)

\[
\Upsilon_1(2 - x) = \Upsilon(x).
\]

Furthermore, it admits an integral representation in the range \( 0 < \text{Re} \, x < 2 \) which is given by

\[
\ln \Upsilon_1(x) = \int_0^\infty \frac{dt}{t} \left[ (1 - x)^2 e^{-t} - \frac{\sinh^2 \left[ \frac{(1 - x)t}{2} \right]}{\sinh^2 \frac{t}{2}} \right],
\]

(A.6)

and it is extended outside this range by analytic continuation using the recursive property
above.

Finally, the function \( S_1(x) \) is defined as

\[
S_1(x) \equiv \frac{\Gamma_1(x)}{\Gamma_1(2 - x)},
\]

and it has poles at \( x \in \mathbb{Z}_{\leq 0} \) and zeros at \( x \in \mathbb{Z}_{\geq 2} \). It satisfies a recursive and a reflection property, given by

\[
S_1(x + 1) = 2\sin(\pi x)S_1(x),
\]

\[
S_1(2 - x)S_1(x) = 1.
\]

Furthermore, \( S_1(x) \) is given for large values of \( |\text{Im } x| \) by

\[
S_1(x) \sim e^{\mp i \frac{\pi}{2}(x(x-2)+\frac{5}{6})}, \quad \text{Im}(x) \to \pm \infty.
\]

The integral representation of \( S_1(x) \) for the range \( 0 < \text{Re } x < 2 \) is

\[
\ln S_1(x) = \int_0^\infty \frac{dt}{t} \left[ \frac{\sinh(2t(1 - x))}{2\sinh^2 t} - \frac{1 - x}{t} \right].
\]
It is commonly stated that correlation functions of 2d CFTs are completely fixed by the spectrum of primary operators and their three-point functions via the Virasoro algebra. To compute for example the sphere 4-point function, we have to include the contributions from the Virasoro descendants of a given primary, which is encoded by the Virasoro conformal block. Even though the sphere 4-point Virasoro conformal block is not known in closed form, there are efficient recursive techniques for computing it numerically, which we will review in this Appendix (see [36, 37] for more details).
Consider a generic 2d CFT of central charge $c$ on the Euclidean plane, which is parameterized by complex coordinates $z, \bar{z}$ and line element $ds^2 = dzd\bar{z}$. The sphere 4-point function of scalar primary operators $\phi_i(z_i, \bar{z}_i)$ of conformal weights $h_i = \tilde{h}_i, \ i = 1, 2, 3, 4$, can be expressed in terms of three-point functions and conformal blocks as

$$
\langle \phi_1(z_1, \bar{z}_1)\phi_2(z_2, \bar{z}_2)\phi_3(z_3, \bar{z}_3)\phi_4(z_4, \bar{z}_4) \rangle \\
= |z_{14}|^{-4h_1}|z_{24}|^{2(h_1-h_2+h_3-h_4)}|z_{34}|^{2(h_1+h_2-h_3-h_4)}|z_{23}|^{2(h_4-h_1-h_2-h_3)} \\
\times \sum_j C_{12j}C_{34j} F(h_1, h_2, h_3, h_4; h_j|z) F(\tilde{h}_1, \tilde{h}_2, \tilde{h}_3, \tilde{h}_4; \tilde{h}_j|\bar{z}),
$$

where $F(h_1, h_2, h_3, h_4; h_j|z)$ is the sphere 4-point Virasoro conformal block of external weights $h_i, \ i = 1, 2, 3, 4$, intermediate weight $h_j$ and cross-ratio $z$, which is defined as

$$
z \equiv \frac{z_{12}z_{34}}{z_{14}z_{32}}.
$$

(B.1)

The sphere 4-point Virasoro conformal block, which we will henceforth refer to as “conformal block” unless explicitly stated otherwise, is completely fixed by the Virasoro Ward identities, and there is a known algorithm for computing it order by order in an expansion in powers of $z$. Nevertheless, the calculation quickly becomes too difficult even for numerical methods.

Luckily, there are efficient recurrence relations to compute conformal blocks numerically. In particular, we will frequently use Zamolodchikov’s recursion relation [34, 35], which takes the
form

\[
\mathcal{F}(h_1, h_2, h_3, h_4; h_j | z) = [16q(z)]^{\frac{w-1}{24}} z^{\frac{w-1}{24}-h_2(1-z)\frac{w-1}{24}-h_1-h_3} \\
\times [\theta_3(q(z))]^{\frac{w-1}{8}-d(h_1+h_2+h_3+h_4)} H(\lambda^2, h|q(z)),
\]

where the nome \(q(z)\) is defined as

\[
q(z) \equiv e^{i\pi \tau(z)}, \quad \tau(z) \equiv i \frac{K(1-z)}{K(z)}, \quad K(z) \equiv 2 F_1 \left( \frac{1}{2}, \frac{1}{2}, 1 \middle| z \right),
\]

and \(\theta_3(\tau) = \sum_{n \in \mathbb{Z}} q^{n^2}\) is one of the Jacobi theta functions. The function \(H(\lambda^2, h|q)\) satisfies a recursion relation in the intermediate conformal weight \(h\), given by

\[
H(\lambda^2, h|q) = 1 + \sum_{m \geq 1, n \geq 1} q^{mn} R_{m,n}(\lambda) \frac{h - h_{m,n}}{H(\lambda^2, h_{m,n} + mn|q)},
\]

where \(h_{m,n}\) are the conformal weights of the degenerate representations of the Virasoro algebra of central charge \(c\). In terms of the parameters \(Q\) and \(b\) defined as

\[
c = 1 + 6Q^2, \quad Q = b + 1/b,
\]

\(h_{m,n}\) is given by

\[
h_{m,n} = \frac{Q^2}{4} - \lambda_{m,n}^2, \quad \lambda_{m,n} = \frac{1}{2} \left( \frac{m}{b} + nb \right).
\]
Furthermore, \( \lambda_i^2 \equiv \frac{Q_i^2}{4} - h_i \), for \( i = 1, 2, 3, 4 \). Finally, the residue \( R_{m,n}(\{\lambda_i\}) \) is given by

\[
R_{m,n}(\{\lambda_i\}) = \frac{2 \prod_{r,s} (\lambda_1 + \lambda_2 - \lambda_{r,s}) (\lambda_1 - \lambda_2 - \lambda_{r,s}) (\lambda_3 + \lambda_4 - \lambda_{r,s}) (\lambda_3 - \lambda_4 - \lambda_{r,s})}{\prod'_{k,\ell} \lambda_{k,\ell}},
\]

(B.8)

where the product \((r,s)\) in the numerator is over the range

\[
r = -m + 1, -m + 3, ..., m - 1,
\]

\[
s = -n + 1, -n + 3, ..., n - 1,
\]

while the product of \((k,\ell)\) in the denominator is over the range

\[
k = -m + 1, -m + 3, ..., m,
\]

\[
\ell = -n + 1, -n + 3, ..., n,
\]

but the latter excludes \((k,\ell) = (0,0)\) and \((k,\ell) = (m,n)\).

In practice, we can efficiently implement this recursion formula by first solving for \( H(\lambda_i^2, h_{m,n} + mn|q) \) for a finite set of integers \( m,n \) subject to \( mn \leq N \). We need only keep terms of order up to \( q^N \) in \( H(\lambda_i^2, h_{m,n} + mn|q) \), and with these in hand we can use them to find \( H(\lambda_i^2, h|q) \) for generic internal weight \( h \) up to order \( q^N \). As an example, consider the case \( N = 2 \) where the set of \((m,n)\) that contribute are \((1,1), (1,2), (2,1)\). Then we solve for \( H(\lambda_i^2, h_{m,n} + mn|q) \) via
the matrix equation

\[
\begin{pmatrix}
H(\lambda_i^2, 1|q) \\
H(\lambda_i^2, h_{1,2} + 2|q) \\
H(\lambda_i^2, h_{2,1} + 2|q)
\end{pmatrix}
= 
\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}
+ 
\begin{pmatrix}
0 & q^2R_{1,2} & q^2R_{1,2} \\
qR_{1,1} & 0 & q^2R_{2,1} \\
qR_{1,2} & q^2R_{2,1} & 0
\end{pmatrix}
\begin{pmatrix}
H(\lambda_i^2, 1|q) \\
H(\lambda_i^2, h_{1,2} + 2|q) \\
H(\lambda_i^2, h_{2,1} + 2|q)
\end{pmatrix},
\]

(B.11)

After solving this equation, we can easily obtain from (B.5) a series expansion for \(H(\lambda_i^2, h|q)\) that is valid up to order \(q^2\).

In \(c = 1\) string theory, we will frequently use this technique to compute conformal blocks for the \(c = 25\) Liouville CFT. One subtle issue in the application of Zamolodchikov’s recursion relation to this value of central charge is that there are poles in the coefficient multiplying \(H(\lambda_i^2, h_{m,n} + mn|q)\) in the recursion relation. To circumvent this problem, we take \(c = 25 + \epsilon\) where \(\epsilon\) is a small number, and take \(\epsilon\) smaller and smaller until the result converges, via multiplication by a zero that appears in the coefficient at a given \(q\) expansion order. Note that this cancellation is guaranteed by the fact that there are no degenerate representations for a \(c > 1\) CFT.
Details of Worldsheet Scattering Amplitudes Computations

In this Appendix, we provide several of the technical details that enter into the numerical evaluation of worldsheet scattering amplitudes, which were omitted in the main text.
C.1 Tree-level Scattering Amplitude of 4 Closed Strings

We focus this discussion on the $1 \to 3$ scattering amplitude for simplicity, the details of the $2 \to 2$ scattering amplitude are identical upon replacing $\omega_2 \to -\omega_2$ in the free-boson correlator.

The $1 \to 3$ scattering amplitude of closed strings at tree-level is obtained from the regularized expression in (3.49). There are two integrations involved in this expression, one over the moduli $z \in \mathbb{C}$, and the other over the intermediate Liouville momentum $P$. Let’s first discuss how to compute the integrand at a fixed value of $z$ and $P$.

Consider the contribution coming from the first term on the RHS of (3.49), in a region of moduli $z$ close to the origin. In this case, the Liouville 4-point function admits an expansion in $s$-channel conformal blocks expansion, as in (3.22). The Virasoro conformal blocks are computed numerically using Zamolodchikov’s recursion relation, discussed in detail in Appendix B, which gives a convergent expansion in the nome parameter $q(z)$ on the punctured complex plane $z \in \mathbb{C}\{1, \infty\}$.

Recall that the definition of the regulator $R_s$ given by (3.47) requires knowing the coefficients $a_{n,n}$ of the $s$-channel OPE expansion. The Virasoro conformal block expansion in $q(z)$ obtained from Zamolodchikov’s recursion relation, truncated at order $q(z)^n$, can be re-expanded in a series in $z$ that is accurate up to order $z^n$. After combining with the expression for the antiholomorphic Virasoro conformal block, and the correlation functions of free boson and ghosts, we extract the coefficients $a_{n,n}$ from the coefficient of the $z^n \bar{z}^n$ term.

The contributions near the points $z = 1$ and $z = \infty$ can be similarly obtained, by doing the $t$ and $u$ channel expansion in conformal blocks, which are given by (3.25) and (3.27) respectively.
Figure C.1: Contributions to the real (left) and imaginary (right) parts of the amplitude $-iA_{1\rightarrow 3}^{(0)}$ from a range of Liouville momentum $P$, after having already performed the $z$-integral in the domain $D = \{|z - 1| < 1, 0 < \text{Re}(z) < 1/2\}$. In this example we take $\omega = 1.4 + i\epsilon, \omega_1 = \omega_2 = \omega_3 = \omega/3$, with $\epsilon = 0.01$, and the Virasoro conformal block was computed up to order 12 in its $q$ expansion.

Using these expansions in different regions of moduli space gives numerical expressions for the Liouville correlator as well as the regulators for every value of $z$. It can be checked that the numerical Liouville conformal blocks quickly converge everywhere on the complex $z$ plane as the truncation order is increased, but convergence is significantly faster if the appropriate channel is used. For example, near $z = 1$ the $t$-channel conformal block expansion converges much faster than the $s$-channel as the truncation order increases.

To integrate over $z$ and $P$, we sample over a finite range of values of $P$, and for each value of $P$ we perform the $z$ integral. Due to the identifications in (3.28), crossing symmetry allows us to map the integration domain $z \in \mathbb{C}$ to the region $D = \{|z - 1| < 1, \text{Re} z < 1/2\}$, at the price of introducing new correlation functions in the moduli integrand related by exchanging the external operators dimensions. In the region $D$, Zamolodchikov’s recursion relation quickly converges with the truncation order. Similarly, we do a change of coordinates in $R_{s,t,u}$ to map the integration domain to the same region $D$.

To perform the moduli integral over the region $D$, for fixed values of the Liouville momentum $P$, we proceed as follows. We sample over the domain $D'_{\delta} = \{|z - 1| < 1, \delta < \text{Re} z < 1/2\}$ for
some small $\delta$, interpolate over these values, and numerically integrate over $D_\delta'$. In the region $D_\delta = \{|z - 1| < 1, 0 < \text{Re } z < \delta\}$, the integral over $z$ converges slowly, and therefore in this region we expand the integrand near $z = 0$ in a series in powers of $z$, and perform the $z$-integral of the first few leading terms analytically.

Finally, the integration over $P$ is performed by sampling over different values of $P$, interpolating over these values, and numerically integrating. Near $P = \frac{1}{2}\sqrt{\text{Re} \left((\omega - \omega_i)^2\right)}$, the $P$-integral diverges in the strict physical regime, see (3.50). Instead, we must assign a small imaginary part to the closed string energies. As we take this small imaginary part to zero, the real contribution to the $1 \to 3$ scattering amplitude becomes sharply peaked near $P = \frac{1}{2}\sqrt{\text{Re} \left((\omega - \omega_i)^2\right)}$ (see the discussion in section 3.4.3). Thus, we take a very fine sampling in $P$ in the vicinity of this region. A generic result of the $P$-integrand is shown in Figure C.1.

### C.2 Tree-level Long→Long+Closed Scattering Amplitude

The formal expression (5.23) that enters in the computation of the long→long+closed scattering amplitude suffers from a divergence from the moduli integration near $x = 0$. In this limit, the boundary operators collide, and to properly define this scattering amplitude we have to regularize the integrand by adding an appropriate counter term. The philosophy is the same as for the case of sphere scattering amplitudes of closed strings discussed in section 3.4.1. Near $x = 0$, the moduli integrand admits an expansion in powers of $x$ given by

$$\int \frac{dP}{\pi} R^s(\omega_3/2; P) C^{s,s,s}(P, \omega_1, \omega_2) 2^{2p_2 - 2\omega_3^2} x^{-1 + p_2 - \omega_3^2} \sum_{n=0} a_n x^n, \quad (C.1)$$
where we have dropped the prefactors of the RHS in the first line of (5.23). The coefficients \( a_n \) are functions of the external energies \( \omega_i \) and intermediate Liouville momenta \( P \). This expression shows that the \( x \)-integral diverges for \( P \leq \sqrt{\text{Re}(\omega_3^2)} \), which we regularize by adding a counter term to the integrand that cancels the divergences near \( x = 0 \), while preserving analyticity in the energies. The regulator is given by

\[
R(x) = \sum_{0 \leq n \leq \omega_3} a_n \int_0^{\sqrt{\omega_3^2 - n}} \frac{dP}{\pi} R^s(\omega_3/2; P)C^{s,s,s}(P, \omega_1, \omega_2)2^{2P^2-2\omega_3^2}x^{-1+P^2-\omega_3^2+n}, \tag{C.2}
\]

so that the regularized expression that computes \( A_{\psi^{s,s}_{\omega_1} \rightarrow \psi^{s,s}_{\omega_2} - \bar{\psi}^{s}_{\omega_3}} \) is

\[
\int_0^\infty dx \left[ 2^{-2-2\omega_1\omega_2}x^{2\omega_1\omega_2}|x-i/2|^{-2\omega_2\omega_3}\langle \psi^{s,s}_{\omega_1}(0)\psi^{s,s}_{\omega_2}(x)V_{\omega_3/2}(i/2) \rangle_{D^2, \text{Liouville}} - R(x) \right], \tag{C.3}
\]

where again we are omitting the prefactors on the first line of the RHS of (5.23). Furthermore, there is a log log divergence in the moduli integral coming from the vicinity of \( P = \sqrt{\text{Re}(\omega_3^2)} \), similar to that found in the scattering of 4 closed strings discussed in section 3.4.2. This was interpreted as due to ambiguities of the asymptotic states of massless particles in \( 1+1d \), and to obtain well-defined, finite amplitudes we assign a small imaginary part to the energies of the open and closed strings \( \omega_i \rightarrow \omega_i + \delta_i \), where \( \delta_i > 0 \) for \( i = 1, 2, 3 \) (and preserving energy conservation).

The regularized expression (C.3) can be evaluated as follows. To compute the Liouville correlator, we expand in Virasoro conformal blocks as in (5.24), which we evaluate numerically. The structure constants have been written explicitly in section 5.1.2, and we evaluate them by doing the integrals numerically. The Virasoro conformal blocks are evaluated as an expansion
in the nome \( q(x) \) using Zamolodchikov’s recursion relation, as discussed in Appendix B. We expand to sufficient high order so that the results have converged for sufficiently large values of the modulus \( x \). Furthermore, the coefficients \( a_n \) that enter in the counter term (C.3) can be obtained from the Virasoro conformal block after re-expanding the nome expansion in powers of \( x \).

We are then left with two integrals to compute, one over the intermediate Liouville momentum \( P \) and the other over the modulus \( x \). We will first do the integral over \( x \), and then over \( P \). For the \( P \)-integral, we sample over finitely many points in a finite integration range, as contributions coming from large values of \( P \) are negligible. For a fixed value of \( P \), we evaluate the \( x \)-integral by splitting the integration domain \( x \in (0, \infty) \) in three regions, \( D_1 = (0, \delta), D_2 = (\delta, \Lambda), D_3 = (\Lambda, \infty) \), where \( \delta \ll 1 \) and \( \Lambda \gg 2s \). The integration over \( D_1 \) is done analytically by keeping the leading terms that contribute at small \( x \). In \( D_2 \), we sample over finitely many values of \( x \), interpolate over these values and numerically integrate. Finally, for the region \( D_3 \) we note that the integrand (C.3) decays as \( x^{-2} \) for large \( x \). Thus, we fit a finite set of points to a monomial of the form \( c_2 x^{-2} \), and integrate this monomial analytically in the region \( D_3 \). Altogether, we have the contribution at a fixed value of intermediate Liouville momentum \( P \), and finally we interpolate between these to numerically integrate in \( P \).

Near \( P = \sqrt{\text{Re}(\omega_3^2)} \), we must choose our \( P \)-sampling very finely as there is a contribution that becomes sharply peaked in the limit when \( \text{Im} \omega_3 \to 0 \). However, we find in practice that the dominant contribution in the \( P \)-integral comes from a region that scales as \( P \sim 2s \) in the long string limit, when \( s \to \infty \). Since \( \omega_3 \) is held fixed in this limit, the contribution coming from the region of the \( P \)-integral that needed regularization and fine sampling is suppressed.
A generic illustration of this is shown in Figure C.2. Thus, a posteriori we find that we can restrict the integration in $P$ to the range $(\omega_3 + a, \infty)$, where $a \ll 1$, so that no regularization is needed.

Finally, we perform this computation for increasing values of the FZZT parameter $s$, multiply by the prefactors in (5.22), and perform an exponential fit in $s$, since the result is expected to converge exponentially with $s$. From this we extract the $s = \infty$ limit, which gives the long→long+closed scattering amplitude shown in section 5.2.1.
The long+long→long+long scattering amplitude is obtained from the long string limit of the scattering amplitude between 4 open strings on an FZZT\((s)\) brane. The latter is formally given by the expression (5.26), however there are divergences when the open string vertex operator \(\psi_{\omega_2}^{s,s}(x)\) collides with the other open string insertions, at \(x = 0, 1, \infty\). The regularization by introducing counter terms to the \(x\)-integrand is similar to the cases above, and the fully regularized expression is given by

\[
A_{\psi_1^{s,s} + \psi_2^{s,s} \to \psi_3^{s,s} - \psi_4^{s,s}} = i g_0^4 C_D 2 \delta(\omega_1 + \omega_2 - \omega_3 - \omega_4) \\
\times \left\{ \sum_{j=1}^3 \int_{I_j} dx \left| x - \frac{2\omega_1\omega_2}{x} - 1 \right| \langle \psi_{\omega_1}^{s,s}(0) \psi_{\omega_2}^{s,s}(x) \psi_{\omega_3}^{s,s}(1) \psi_{\omega_4}^{s,s}(\infty) \rangle_{\text{Liouville}} \right\} + (3 \leftrightarrow 4),
\]

where the regions \(I_j, j = 1, 2, 3\) correspond to each of the diagrams in Figure 5.6, and the regulators \(R_{s,t,u}(x)\) are given by

\[
R_s(x) = \sum_{0 \leq n \leq (\omega_1 + \omega_2)^2} a_n \int_0^{\sqrt{(\omega_1 + \omega_2)^2 - n}} \frac{dP}{\pi} C_s^{s,s}(P, \omega_1, \omega_2) C_s^{s,s}(P, \omega_3, \omega_4) |x|^{-1 + P^2 - (\omega_1 + \omega_2)^2 + n},
\]

\[
R_t(x) = \sum_{0 \leq n \leq (\omega_1 - \omega_3)^2} b_n \int_0^{\sqrt{(\omega_1 - \omega_3)^2 - n}} \frac{dP}{\pi} C_s^{s,s}(P, \omega_1, \omega_3) C_s^{s,s}(P, \omega_2, \omega_4) |1 - x|^{-1 + P^2 - (\omega_1 - \omega_3)^2 + n},
\]

\[
R_u(x) = x^{-2} \sum_{0 \leq n \leq (\omega_1 - \omega_4)^2} c_n \int_0^{\sqrt{(\omega_1 - \omega_4)^2 - n}} \frac{dP}{\pi} C_s^{s,s}(P, \omega_1, \omega_4) C_s^{s,s}(P, \omega_2, \omega_3) |1/x|^{-1 + P^2 - (\omega_1 - \omega_4)^2 + n}.
\]
The Liouville correlator can be computed by doing an expansion in Virasoro conformal blocks. In each of the regions $I_j$, only two of the conformal block expansions converges. For example, for $I_1$, we can use the $s$-channel expansion (near $x = 0$) or the $t$-channel expansion (near $x = 1$), but not the $u$-channel expansion near $x = \infty$.

In practice, we can use crossing symmetry to map the $x$ integration domain over the regions $I_j$, $j = 1, 2, 3$ to a single integral over $x \in (0, 1/2)$. For example, the integration domain $I_2$ corresponding to the diagram in Figure 5.6b is given by $I_2 = \{1 < x < \infty\}$. In the region \{1 < x < 2\}, we compute the Liouville correlator by taking the OPE between $\psi^{s,s}_{\omega_3}(1)$ and $\psi^{s,s}_{\omega_2}(x)$ (the $t$-channel conformal block expansion), while for $x > 2$ we take the OPE between the operators $\psi^{s,s}_{\omega_2}(x)$ and $\psi^{s,s}_{\omega_4}(\infty)$ (the $u$-channel conformal block expansion). Further doing a change of variables $x \rightarrow 1 - \frac{1}{x}$ for the former region and $x \rightarrow 1/x$ for the latter, the total contribution to (C.4) from the diagram in Figure 5.6b (including the counter term) is given by

\[
\mathcal{A}^{(I_2)}_{\psi^{s,s}_{\omega_3} + \psi^{s,s}_{\omega_2} \rightarrow \psi^{s,s}_{\omega_3} - \psi^{s,s}_{\omega_4}} = \int_0^{1/2} dx \left[ x^{2\omega_1 \omega_3} (1 - x)^{2\omega_1 \omega_4} \int_0^\infty \frac{dP}{\pi} C^{s,s,s}(P, \omega_1, \omega_3) C^{s,s,s}(P, \omega_4, \omega_2) F(h_1, h_3, h_4, h_2; h | x) \right. \\
+ (1 \leftrightarrow 2) - \frac{1}{x^2} R_t \left( 1 - \frac{1}{x} \right) - \frac{1}{x^2} R_u \left( \frac{1}{x} \right) \right] - \int_1^{\infty} dx \left[ R_t \left( 1 - \frac{1}{x} \right) + R_u \left( \frac{1}{x} \right) \right],
\]

(C.6)

where $h_i = 1 + \omega_i^2$, $h = 1 + P^2$, and the prime on the LHS denotes that we have excluded the contributions from the diagrams in Figures 5.6a,c. There are similar expressions for the contributions from these diagrams.

The strategy to evaluate the contribution (C.6) and also the contributions from the other
diagrams is the same as in the previous cases. We want to perform the integral over the modulus $x$ and the intermediate Liouville momentum $P$, and we assign small imaginary parts for the open string energies $\omega_i$, $i = 1, 2, 3, 4$. For a fixed value of $x$ and $P$, we can evaluate the integrand in (C.6) from the boundary structure constants and using Zamolodchikov’s recursion relation to compute the Virasoro conformal blocks. We sample over finitely many values of $P$, and do the $x$ integral over the region $x \in (0, 1/2)$ for each $P$ value. Near $x = 0$, the $x$-integral converges slowly and we integrate analytically, while away from this point we perform the integral numerically by sampling the integrand (C.6) over finitely many points. Finally, we interpolate over the range of $P$ values and numerically integrate over $P$.

Special care is needed in the vicinity of $P$ that becomes sharply peaked when the imaginary part of the energies approach zero. For the contribution (C.6), the peak occurs for $P = \sqrt{\text{Re}((\omega_1 - \omega_3)^2)}$, which is finite in the long string limit. Furthermore, we numerically verified that the $P$-integral grows with $s$, so that in the long string limit the contribution from the range $P \leq \sqrt{\text{Re}((\omega_1 - \omega_3)^2)}$ can be dropped and we only integrate over $P > \omega_1 - \omega_3 + a$, where $a \ll 1$ and we can take $\omega_i \in \mathbb{R}$, for $i = 1, 2, 3, 4$.

We evaluate the scattering amplitude of open strings for increasing values of $s$, multiply by the prefactors in (5.22), and perform an exponential fit in $s$. From this limit we extract the long+long→long+long scattering amplitude, as shown in Figure C.3.

The story for the other diagrams in Figures 5.6a,c is almost analogous, except that these contributions vanish in the long string limit, as discussed in section 5.2.2. We checked this explicitly in Figure C.3. Furthermore, in these cases there are regulators that need to be included up to values $P < \sqrt{\text{Re}((\omega_1 + \omega_2)^2)}$, therefore requiring a precise calculation of the
Figure C.3: Contributions to the open string scattering amplitudes on FZZT brane from (a) the diagrams in Figures 5.6a,c and two other diagrams that are exponentially suppressed in the long string limit, and (b) the diagram in Figure 5.6b and another diagram that converge to $A_{L+L+L+L}$ exponentially in the long string limit, evaluated numerically as a function of $s$, at fixed renormalized long string energies $\epsilon_1 = 0.2$, $\epsilon_2 = 0.4$, $\epsilon_3 = 0.35$. In comparison to (C.4), we have included a factor of $1/\Pi_{1=1}^{1} \sqrt{\omega_i}$ due to the normalization of the long string asymptotic state. Exponential fits are shown as dashed gray curve, whereas the long string limit amplitude is marked in red.

regulators, and fine sampling in the $P$ integration over many different peaks. At the end of the day, these contributions vanish in the long string limit as was expected.
Long String Reflection Phase

D.1 Long string reflection phase from the worldsheet

A long string is defined as an open string of energy $\omega = P$ in the long string limit (5.21). Thus, we can obtain the reflection phase of the long string from the appropriate limit of the open string reflection phase on a FZZT(s) brane (5.14). Using the properties of the special functions
\[ \frac{\Gamma_1(2iP)}{\Gamma_1(-2iP)} = S_1(1 + 2iP) \frac{\Gamma(2iP)}{\Gamma(-2iP)}. \]  

(D.1)

Using \( P = 2s + \epsilon \) and the asymptotic formula (A.9) for the function \( S_1(x) \), we find that the reflection phase \( \delta_\phi(\epsilon) \) is given by

\[
e^{i\delta_\phi(\epsilon)} = \lim_{s \to \infty} d^{s,s}(2s + \epsilon) = S_1(1 - i\epsilon) e^{\frac{i\pi}{2} \epsilon^2} e^{i\delta_{\text{div}}(\epsilon,s)},
\]

\[
\delta_{\text{div}}(\epsilon, s) = -4\pi s^2 + 8s (\ln(s) + 2 \ln(2) - 1) + 4\epsilon (2 \ln(2) + \ln(s)) - \frac{2\pi}{3}.
\]

The contribution \( \delta_{\text{div}}(\epsilon, s) \) to the reflection phase is ill-defined in the limit \( s \to \infty \). To understand where this contribution comes from, note that we introduced the FZZT\((s)\) brane and the open strings only as a trick to describe an infinitely long string. In particular, \( d^{s,s}(P) \) gives the wavefunction (5.13) and the reflection phase for the open string in the asymptotic region \( \phi = -\infty \), which includes a contribution coming from the reflection of the ends of the open string off the boundary Liouville potential. However, we are only interested in the reflection phase of the process where the tip of the open string is moving while the ends are stuck on the FZZT\((s)\) brane, at position \( \sim \phi = -2\pi s \), which is captured by the wavefunction (5.7). Thus, we should subtract the contribution \( \delta_{\text{div}}(\epsilon, s) \) from \( \delta_\phi(\epsilon) \), since the former is interpreted as due to the reflection of the ends of the long string. Note that subtracting \( \delta_{\text{div}}(\epsilon, s) \) leaves a constant ambiguity in \( \delta_\phi(\epsilon) \) as well as an ambiguity linear in \( \epsilon \), but this will not be important for us in what follows.

The function \( S_1(1 - i\epsilon) \) can be evaluated using the integral representation of \( S_1(x) \) discussed
in Appendix A, which gives

\[ \ln S_1(1 - i\epsilon) = i \int_0^\infty \frac{dt}{t} \left[ \frac{\sin(2\epsilon t)}{2 \sinh^2 t} + \frac{\epsilon}{t} \right] = -i\pi \int_0^\epsilon d\epsilon' \epsilon' \coth(\pi \epsilon'). \]  

(D.3)

Finally, the wavefunction for the tip of the long string (5.7) includes a somewhat unconventional factor in the definition of the reflection phase \( \delta(\epsilon) \). It is related to (the regularized) \( \delta_o(\epsilon) \) by

\[ \delta(\epsilon) = \delta_o(\epsilon) - \frac{\phi_m^2}{\pi} = \delta_o(\epsilon) - \pi \epsilon^2 \]  

(D.4)

where we have used the relation \( \phi_m = \pi \epsilon \) which is true semiclassically, and corrections are only expected to be constants which would only affect the linear and constant parts of \( \delta(\epsilon) \), which are ambiguous anyway. Thus in total, we find the reflection phase for the tip of the long string to be

\[ \delta(\epsilon) = -\pi \int_0^\epsilon d\epsilon' \epsilon' \coth(\pi \epsilon') - \pi \epsilon^2 = -\pi \int_0^\epsilon d\epsilon' \epsilon' \left[ 1 + \coth(\pi \epsilon') \right], \]  

(D.5)

where the last equation holds modulo terms that are constant or linear in \( \epsilon \). In the next section, we will show how the long string wavefunction (5.42) exactly reproduces this reflection phase.

D.2 Long string reflection phase in the matrix model description

It is interesting to study the wavefunction (5.42) in more detail [44]. First let’s examine the phase factor \( F(k, \epsilon) \), where

\[ F(k, \epsilon) \equiv \pi \int_{k_0}^k \frac{dk'}{\sqrt{\sinh^2(\pi k') + e^{2\pi \epsilon}} \left( \epsilon - \frac{k'}{\tanh(\pi k')} \right)}. \]  

(D.6)
We take the branch cut of the square-root to lie between $\pm k_0(\epsilon)$. The integral in (D.6) is not known explicitly, but its derivative with respect to $\epsilon$ gives

$$
\partial_\epsilon F(k, \epsilon) = \left[ \frac{\pi \sinh(\pi k')}{\sqrt{\sinh^2(\pi k') + e^{2\pi \epsilon}}} \left( k' + \frac{\epsilon e^{2\pi \epsilon}}{1 - e^{2\pi \epsilon} \tanh(\pi k')} \right) \right]_{k_0(\epsilon)}^k.
$$

(D.7)

In obtaining this result, we used the fact that the derivative with respect to $\epsilon$ acting on the lower limit of the integration range in (D.6) vanishes, so the only non-zero contribution comes from $\partial_\epsilon$ acting on the integrand. The integral can be done by noting that the integration domain in (D.6) is half the integration from $-k$ to $k$, with the contour going around the branch cut from above. The result is

$$
\partial_\epsilon F(k, \epsilon) = \frac{\pi \sinh(\pi k)}{\sqrt{\sinh^2(\pi k) + e^{2\pi \epsilon}}} \left( k + \frac{\epsilon e^{2\pi \epsilon}}{1 - e^{2\pi \epsilon} \tanh(\pi k)} \right).
$$

(D.8)

For large $k$, this becomes

$$
\partial_\epsilon F(k, \epsilon) = \pi \left( k + \frac{\epsilon e^{2\pi \epsilon}}{1 - e^{2\pi \epsilon}} \right).
$$

(D.9)

On the other hand, in the limit $\epsilon \ll 0$ the phase $F(k, \epsilon)$ can be obtained explicitly, and it is given by

$$
F(k, \epsilon) = \pi \left( \epsilon k - \frac{k^2}{2} - \frac{1}{12} \right),
$$

(D.10)
so that for large $k$, we find

$$
F(k, \epsilon) = \pi \left( \epsilon k - \frac{k^2}{2} - \frac{1}{12} \right) + \frac{1}{2} \delta(\epsilon),
$$

$$
\delta(\epsilon) \equiv -\pi \int_{-\infty}^{\epsilon} d\epsilon' \epsilon' (1 + \coth(\pi \epsilon')).
$$

(\text{D.11})

$\delta(\epsilon)$ is precisely the reflection phase of the tip of the long string (D.5), which we will now show is also the reflection phase of the wavefunction $h_\epsilon(\tau)$ in the adjoint sector of the dual matrix model (5.42).

Consider the wavefunction $h_\epsilon(\tau)$ given by (5.42) in the asymptotic region of large $\tau$. The $k$-integral can be evaluated using a stationary phase approximation, where the stationary point is located at large $|k|$. For large positive $k$, we can approximate the $k$-integrand by (D.11) and perform the $k$-integral in a Gaussian approximation. The contribution coming from the stationary point of large negative $k$ gives the same contribution. In total, we find

$$
h_\epsilon(\tau) \sim -\frac{1}{\sqrt{2\pi}} \left( e^{\frac{(\tau + \pi \epsilon)^2}{2\pi}} + i \frac{\pi}{2} \delta(\epsilon) - i \frac{\pi}{4} e^{-i\frac{(\tau + \pi \epsilon)^2}{2\pi}} - i \frac{\pi}{2} \delta(\epsilon) + i \frac{\pi}{4} \right).
$$

(\text{D.12})

This wavefunction is precisely the wavefunction of the tip of the long string (5.7), with asymptotic region at $\tau = \infty$. Furthermore, the reflection phase $\delta(\epsilon)$ in (D.11) precisely matches the result obtained in the worldsheet formalism (D.5).
Matrix Model Scattering Amplitudes

In this Appendix we discuss the $1 \rightarrow 2$ scattering amplitude of closed strings using the second quantization formalism of the $c = 1$ matrix quantum mechanics.

The in-state picked out in the LSZ limit was written in (2.42). For the out-state, there are
two closed strings and therefore two particles and hole excitations. Consider the state

\[ |\omega_1, \omega_2\rangle_{\text{out}} = \int dE_1 (1 + e^{2\pi E_1}) R(E_1) (b^\text{out}_{E_1 + \omega_1})^\dagger b^\text{in}_{E_1} \]

\[ \times \int_{-\mu - \omega_2}^{-\mu} dE_2 (1 + e^{2\pi E_2}) R(E_2) (b^\text{out}_{E_2 + \omega_2})^\dagger b^\text{in}_{E_2} |\Omega\rangle, \] (E.1)

where the integration in \( E_1 \) is over the real line. The terms that contribute to a non-vanishing overlap with \( |\omega\rangle_{\text{in}} \) are given by

\[ |\omega_1, \omega_2\rangle_{\text{out}} = \int_{-\mu - \omega_2}^{-\mu} dE (1 + e^{2\pi E}) R(E) (b^\text{out}_{E + \omega})^\dagger b^\text{in}_{E} |\Omega\rangle \]

\[ - \int_{-\mu - \omega_1}^{-\mu} dE (1 + e^{2\pi E}) R(E) (b^\text{out}_{E + \omega})^\dagger b^\text{in}_{E} |\Omega\rangle, \] (E.2)

where we used the anticommutation relations in section 2.3.2. From this, we find the \( 1 \rightarrow 2 \) scattering amplitude

\[ A_{1\rightarrow2}(\omega_1, \omega_2) = - \int_{\omega_1}^{\omega} dx R(-\mu + \omega - x)(R(-\mu - x))^{-1} + \int_{0}^{\omega_2} dx R(-\mu + \omega - x)(R(-\mu - x))^{-1}, \]

(E.3)

which justifies the prefactors for the diagrams in Figure 2.9, as written in (2.49). It is straightforward to generalize this approach to \( 1 \rightarrow k \) scattering, from which the general formula (2.50) follows.
An alternative parameterization of collective excitations of the Fermi surface

In this Appendix we describe an alternative parameterization of the collective excitations of the Fermi sea in the singlet sector of the $c = 1$ matrix quantum mechanics. That is, we view the collective fields as functions of the momentum $p$ rather than position $\lambda$ of the single-Fermion/eigenvalue phase space. While not strictly needed in our calculations of scattering amplitudes so far, the $p$-parameterization has the advantage that it avoids dealing with a
collective field Hamiltonian that is singular at the “tip” $\tau = 0$ of the Fermi surface, thereby eliminating the need for the regularization described in section 2.2.

To derive the collective field Hamiltonian in the $p$-parameterization, we need to introduce an IR regulator at large $\lambda$ to make the Fermi surface compact. We do so by adding a quartic term to the single Fermion Hamiltonian, now written as

$$H_i = \frac{p^2}{2} - \frac{\lambda^2}{2} + \alpha \lambda_i^4,$$  \hspace{1cm} (F.1)

where $\alpha$ is a positive parameter that will be taken to zero in the end.

The total Hamiltonian of the system can be written as

$$H = \int \frac{d\lambda dp}{2\pi} \epsilon \theta(\epsilon_F - \epsilon) + \mu N$$
$$= \int \frac{dp}{2\pi} \left[ \alpha \left( \frac{\lambda_+(p)^5}{5} - \frac{\lambda_-(p)^5}{5} \right) - \left( \frac{\lambda_+(p)^3}{6} - \frac{\lambda_-(p)^3}{6} \right) + \left( \mu + \frac{p^2}{2} \right) (\lambda_+(p) - \lambda_-(p)) \right],$$  \hspace{1cm} (F.2)

where $\lambda_{\pm}(p)$ are the maximal/minimal values of the eigenvalue $\lambda$ of given $p$ in the Fermi sea, related to the momentum density $\Pi(p)$ and eigenvalue density $\phi(p)$ by

$$\Pi(p) \equiv \sum_{i=1}^{N} \delta(p - p_i) = \frac{1}{2\pi} \int_{\lambda_-(p)}^{\lambda_+(p)} d\lambda = \frac{\lambda_+(p) - \lambda_-(p)}{2\pi},$$

$$\phi(p) \equiv \sum_{i=1}^{N} \lambda_i \delta(p - p_i) = \frac{1}{2\pi} \int_{\lambda_-(p)}^{\lambda_+(p)} \lambda d\lambda = \frac{\lambda_+(p)^2 - \lambda_-(p)^2}{4\pi}.$$  \hspace{1cm} (F.3)

$\Pi(p)$ and $\phi(p)$ obey the Poisson bracket \( \{ \Pi(p), \phi(p') \} = \partial_p (p - p') \Pi(p) \).
The ground state of the Fermi sea corresponds to the profile

\[
\lambda^0_\pm (p) = \frac{1}{2} \sqrt{\frac{1}{\alpha} \pm \sqrt{1 - 8p^2 / \alpha - 16(\alpha \mu)}} , \tag{F.4}
\]

whereas fluctuations of the Fermi surface can be parameterized as

\[
\lambda_\pm (p) = \lambda^0_\pm (p) + \sqrt{\pi} (\Pi_\eta \pm \partial_\eta) . \tag{F.5}
\]

Here \( \eta(p) \) is the collective field in \( p \)-parameterization, and \( \Pi_\eta \) its conjugate canonical momentum density. One can substitute this into (F.2), pass to the \( \tau \) coordinate defined by \( p = \sqrt{2\mu} \sinh \tau \) (which now ranges over the entire real line), and derive the action \( S[\eta] \). It is then straightforward to take \( \alpha \to 0 \) at the level of the action, resulting in

\[
S[\eta] = \int_{-\infty}^\infty d\tau \left( -i \partial_\tau \eta - (\partial_\tau \eta)^2 + \frac{\sqrt{\pi}}{3\mu \cosh^2 \tau} (\partial_\tau \eta)^3 \right) \tag{F.6}
\]

describing a relativistic right-moving massless field with cubic interaction. Notice that the interaction term is perfectly regular at \( \tau = 0 \), and we can, for instance, recover from it the same tree-level \( 1 \to 2 \) amplitude of collective-field/closed-strings \( A^{\text{pert.}}_{1\to2} = \frac{1}{p} i \omega \omega_1 \omega_2 \), given by (2.23).
References


[44] Lukasz Fidkowski. Solving the eigenvalue problem arising from the adjoint sector of the c=1 matrix model. 6 2005.


[70] Davide Gaiotto. Long strings condensation and FZZT branes. 3 2005.


