Aspects of local conformal symmetry in 1+1 dimensions

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Aspects of local conformal symmetry in 1 + 1 dimensions

Abstract

In this dissertation we describe progress towards carving out the space of non-integrable conformal field theories (CFTs) in 1 + 1 dimensions using conformal bootstrap techniques. We develop and apply numerical and analytical techniques to explore the non-perturbative consequences of conformal symmetry, unitarity and locality on the spectrum and dynamics of 2D CFTs. Throughout we focus on compact irrational CFTs, which have a discrete spectrum consisting of an infinite number of primary operators and exhibit chaotic dynamics.

In the first part of the dissertation we report on numerical advances in charting the space of 2D CFTs. We use semidefinite programming to compute bounds on the gap in the spectrum of primary operators as a function of the central charge and on spectral functions that capture the spectral density of primary operators. We then develop recursion relations for arbitrary Virasoro conformal blocks, and explore the constraints of modular invariance of the genus-two partition function, leading to “critical surfaces” that bound the heavy structure constants of a CFT.

In the second part of the dissertation we develop the analytic bootstrap program for two dimensional conformal field theories, focusing on universal aspects of irrational unitary CFTs and holography. We show that the large-spin sector of irrational CFTs is universally governed by a “Virasoro mean field theory,” incorporating exact stress tensor dynamics into the analytic bootstrap and leading to non-perturbative insights about the bound-state dynamics of three-dimensional quantum gravity. We then leverage a generalized notion of modular invariance to derive a universal asymptotic formula for the average value of the structure constants whenever one or more of the operators has large conformal dimension. The large central charge limit makes contact with quantum gravity in three dimensions, where the averaging over heavy states corresponds to
coarse-graining over black hole microstates. Finally, we study the Euclidean gravitational path integral of pure $AdS_3$ quantum gravity, and use a modular bootstrap analysis to suggest new saddle point configurations that can be consistently included in the gravitational path integral and whose inclusion renders the dual torus partition function unitary.
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Citations to previously published research

Parts of this thesis contain research previously reported in the following articles:

Chapter 2 is a lightly edited version of the previously published paper

S. Collier, Y.-H. Lin, and X. Yin
*Modular Bootstrap Revisited*


Chapter 3 and appendix A is a lightly edited version of the previously published paper

S. Collier, Y.-H. Lin, P. Kravchuk, and X. Yin
*Bootstrapping the Spectral Function: On the Uniqueness of Liouville and the Universality of BTZ*


Chapter 4 and appendix B is a lightly edited version of the previously published paper

M. Cho, S. Collier, and X. Yin
*Recursive Representation of Arbitrary Virasoro Conformal Blocks*


Chapter 5 is a lightly edited version of the previously published paper

M. Cho, S. Collier, and X. Yin
*Genus Two Modular Bootstrap*


Chapter 6 and appendix C is a lightly edited version of the previously published paper

S. Collier, Y. Gobeil, H. Maxfield, and E. Perlmutter
*Quantum Regge Trajectories and the Virasoro Analytic Bootstrap*


Chapter 7 and appendix D is a lightly edited version of the paper that has previously appeared as

S. Collier, A. Maloney, H. Maxfield, and I. Tsiaraes
*Universal Dynamics of Heavy Operators in CFT*$_2$

Chapter 8 and appendix E is a lightly edited version of the paper that has previously appeared as

N. Benjamin, S. Collier, and A. Maloney
*Pure Gravity and Conical Defects*


Preprints can be accessed on [https://arXiv.org](https://arXiv.org).
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If everything has gone according to plan, by the time it comes time to compile one’s dissertation, the acknowledgements section should be the most challenging to write. Reader, I regret to inform you that this truism has been validated by my experience.

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For my parents, Carol and Bruce
and my love, Samantha.
Introduction and summary of results

Conformal field theories are universal structures in theoretical physics. The Wilsonian paradigm [1, 2] has been a powerful organizing principle for physical systems with many degrees of freedom, including statistical systems and relativistic quantum field theories. Such systems are described in terms of effective theories for the relevant degrees of freedom at the energy scale of interest, with physical parameters that vary as the energy scale is dialed according to the renormalization group equations. Upon renormalization group flow, in which high energy degrees of freedom are integrated out, quantum systems inherit scale invariance in the infrared. On quite general grounds, scale invariance in local, unitary, relativistic quantum field theories is enhanced to invariance under transformations that preserve angles, acting locally as simultaneous dilatations.
and rotations. This larger group of symmetries is known as the conformal group.\textsuperscript{1} Quantum field theories that enjoy this larger group of symmetries — conformal field theories — thus describe universality classes of quantum systems at low energies.\textsuperscript{2} For this reason conformal field theories also describe statistical or condensed matter systems at criticality where correlation lengths diverge, such as liquid-vapour critical points, Curie transitions in ferromagnets and quantum phase transitions.

Conformal field theory also provides an in principle ultraviolet-complete holographic definition of quantum gravity in higher-dimensional anti-de Sitter space \textsuperscript{[5–8]}. The reformulation in terms of a conformal field theory provides a window into non-perturbative quantum gravity and the emergence of spacetime — an opportunity to sharply pose and resolve puzzles in quantum gravity and black hole physics, including a version of the black hole information problem in a manifestly unitary setting \textsuperscript{[9]}.

In this dissertation we study conformal bootstrap constraints on unitary two dimensional conformal field theories, focusing on universal aspects of non-integrable CFTs and the corresponding lessons for $AdS_3$ quantum gravity.

\section*{1.1 Abstract conformal field theory, CFT data and conformal bootstrap}

Our tools for the quantitative study of strongly-coupled quantum field theories are regrettably rather limited. Conformal field theory provides a non-perturbative formulation of quantum field theory, without the need for any particular Lagrangian realization or weak-coupling description (which themselves are subject to the ambiguity of field redefinitions). The basic observables in conformal field theory are correlation functions of

\textsuperscript{1}In two dimensions a proof exists that scale invariance implies conformal invariance \textsuperscript{[3]}, however despite general expectations there is no such proof in higher dimensions. Even in two dimensions the issue is subtle however, and if one relaxes the principles of locality or unitarity there are counterexamples \textsuperscript{[4]}.

\textsuperscript{2}However, the long-distance dynamics need not necessarily be described by an interacting conformal field theory; the low-energy theory may be in a massive phase, with infrared dynamics governed by a topological quantum field theory.
local operators

\[ \langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle. \]  

(1.1.1)

Conformal symmetry provides an elegant organizing principle for the spectrum and dynamics of the conformal field theory. Each local operator \( \mathcal{O} \) is labelled by quantum numbers \( (\Delta, \ell) \), the scaling dimension and Lorentz representation (or spin), in addition to charges under any other global symmetries of the problem. In the case that the conformal field theory describes the critical point of a statistical system, these quantum numbers are related to critical exponents. As a result of the conformal symmetry (in combination with any other symmetry in the problem), the spectrum of local operators organizes into a privileged set of primary operators, which are in a sense the independent degrees of freedom of the theory, and descendant operators that are related to the primaries by the action of symmetry generators.

Arbitrary \( n \)-point correlation functions (in flat space in \( d > 2 \), and on arbitrary Riemann surfaces in \( d = 2 \)) are completely determined by a minimal set of CFT data. In conformal field theory, one computes correlation functions by making use of the operator product expansion (OPE), in which one expands a product of local operators

\[ \mathcal{O}_i(x_i)\mathcal{O}_j(x_j) = \sum_{\mathcal{O}_k \text{ primary}} C_{ijk} |x_i - x_j|^{\Delta_k - \Delta_i - \Delta_j} \mathcal{O}_k(x_j) (1 + \ldots), \]  

(1.1.2)

where the \( \ldots \) include the contributions from descendants of the primary operator \( \mathcal{O}_k \), which are completely fixed by conformal symmetry. The OPE is convergent when acting on the vacuum \([10]\), and (1.1.2) is to be regarded as an operator equation, valid inside correlation functions away from other operator insertions.

The OPE coefficients or structure constants \( C_{ijk} \) of primary operators \( \mathcal{O}_i, \mathcal{O}_j, \mathcal{O}_k \) represent the entirety of the dynamical content of the CFT. Together, the spectrum of local primary operators \( \mathcal{O}_i \), their quantum numbers \( \{(\Delta_i, \ell_i)\} \), and the set of structure constants of primary operators \( \{C_{ijk}\} \) form the complete set of CFT data from which one can determine any correlation function by (repeated) application of the OPE.\(^3\)

\(^3\)As stated this is at least true within the naive domain of convergence of the OPE; outside this domain, the correlators can be computed by analytic continuation.
To solve a conformal field theory thus amounts to determining the CFT data. An interesting open question that we will not be able to fully answer in this dissertation is the extent to which the low-lying CFT data specifies the CFT.

However, the CFT data are crucially far from arbitrary. Indeed, conformal symmetry when combined with foundational principles like unitarity and locality leads to infinitely many non-perturbative constraints on the allowed CFT data. The idea that such basic consistency conditions in fact highly constrain the space of allowed CFT data has come to be known as the conformal bootstrap [11–17]. In the conformal bootstrap program, one attempts to fully explore the consequences of the guiding principles of symmetries, quantum mechanics and locality, leading to precise predictions for observables at strong coupling in particular CFTs and a refined understanding of the space of CFTs.

This is in fact an old idea [11, 12]. Indeed, the bootstrap fantasy of exactly solving and classifying a class of integrable 2D CFTs was beautifully realized several decades ago [13, 18–20]. This research program has undergone a modern renaissance [15–17], instigated by a 2008 paper [14] that demonstrated that although individual bootstrap consistency conditions may not lead to an exact, analytic solution, they can still lead to powerful constraints on the allowed CFT data in higher dimensions. The prototypical example of a bootstrap constraint on CFT data is crossing symmetry of the sphere four-point function, reflecting associativity of the OPE, as illustrated in figure 1.1. This program has been has been extraordinarily exemplified in the numerical bootstrap study of four-point functions of operators in the three-dimensional CFT describing the critical point of the 3D Ising model, leading to world-class determinations of the critical exponents [21–24], a precise explanation of critical universality and a beautiful window into the structure of the spectrum of conformal field theory [25].

An alternative but complementary approach to the conformal bootstrap is to study the crossing equation in a kinematic regime such that the dynamics in one channel simplify drastically, enabling an analytical treatment and leading to constraints on the CFT data in the other channel that are universal to all CFTs. A classic result of this nature is the lightcone bootstrap [26, 27], which shows that the large-spin spectrum and
dynamics of any CFT in dimensions \( d > 2 \) approach that of mean field theory, leading to the formulation of a large-spin perturbation theory \([28-32]\). Another landmark result in the analytic bootstrap literature is the derivation of a Lorentzian OPE inversion formula \([33-35]\), an integral transform that acts on the four-point function of local operators in Lorentzian kinematics and extracts the intermediate spectral OPE data. Crucially, this formula is analytic in the spin of the intermediate operator,\(^4\) formalizing the idea that local operators in CFT are organized into rigid Regge trajectories, some of which asymptote at large spin to the flat trajectories of mean field theory. This goes beyond the large-spin expansion, implying a rigidity of the spectrum and OPE of any CFT.

In this dissertation we use both numerical and analytical techniques to explore the bootstrap constraints on unitary CFTs in \( 1 + 1 \) dimensions, which we presently introduce.

\subsection*{1.2 Conformal field theories in \( 1 + 1 \) dimensions}

In general spacetime dimensions \( d \), the conformal group is \( SO(2, d) \) in Lorentzian signature or \( SO(1, d + 1) \) in Euclidean signature. However, in two spacetime dimensions, the conformal algebra enjoys an infinite-dimensional enhancement from \( \mathfrak{sl}(2, \mathbb{C}) \) to two copies of the Virasoro algebra \([13]\). Although the vacuum spontaneously breaks the additional symmetries, one still has access to Virasoro conformal Ward identities that relate correlation functions by the action of the additional symmetry generators, and can thus use the

\(^4\)Technically, analyticity in spin only holds for spins greater than or equal to two, or more generally, above the “Regge intercept” of the theory, which is determined by the behaviour of the four-point function in a Lorentzian kinematic regime known as the Regge limit.
Figure 1.2: An example of a crossing equation for the two-point function of identical operators on a genus-two Riemann surface. As discussed in chapter 7, different OPE channels amount to different pair-of-pants decompositions of the Riemann surface. The role of conformal bootstrap is to constrain the structure constants by relating the different decompositions. The structure constants constrained by this crossing equation have been left implicit.

larger symmetry to organize the operator content of the theory.

Furthermore, conformal field theories in $1 + 1$ dimensions can be consistently formulated on surfaces of arbitrary topology in terms of the CFT data on the two-sphere [36–39]. The internal consistency of this construction is generally known as modular invariance, and leads to further constraints on the CFT data. Although crossing symmetry of the sphere four-point functions and modular covariance of the torus one-point functions constitute the complete set of independent consistency conditions on the CFT data [37], in practice it is difficult to work with a large set of crossing equations simultaneously, and so it is often convenient to study crossing equations on higher-genus Riemann surfaces, which package the same CFT data in different ways. See figure 1.2 for a cartoon of such a higher-genus crossing equation. Indeed in chapters 5 and 7 we will leverage this generalized notion of modular invariance to discover constraints on OPE data not accessible at the level of individual crossing equations on the sphere.

One might wonder what is to be gained by applying conformal bootstrap techniques to the study of $1 + 1$ dimensional conformal field theory given that 2D CFT is by now well-established textbook material [40–42]. Despite the fantastic successes of the exact analytic solutions and classifications of rational conformal field theory, in fact very little is known about generic compact irrational conformal field theories with central charge $c > 1$, which have a discrete spectrum of infinitely many primary operators with respect to the largest chiral algebra and exhibit non-integrable dynamics. In the absence of global symmetries beyond Virasoro conformal symmetry, there are even to the best of our knowledge no explicitly known examples. One might even expect such theories to be generic based on holographic reasoning. Much of the work in
this dissertation is devoted to closing this gap in our knowledge of 2D CFTs by combining the power of the infinite-dimensional conformal symmetry with the full constraints of modular invariance and unitarity.

Three-dimensional quantum gravity with negative cosmological constant is special for the same reasons that 2D CFT is special; the asymptotic symmetry algebra of three-dimensional general relativity with negative cosmological constant is two copies of the infinite-dimensional Virasoro algebra \([43]\), an observation that predates the \(AdS/CFT\) correspondence by over a decade. Furthermore, for kinematic reasons there are no propagating gravitational degrees of freedom in 3D gravity. However the quantum theory of gravity in \(AdS_3\) is nevertheless far from trivial — it has a rich structure including boundary gravitons (which one can think of as analogous to edge modes in a quantum Hall system), and black hole solutions \([44]\). The infinite-dimensional asymptotic symmetry algebra thus allows one the technology of 2D CFT to study foundational puzzles in quantum gravity and quantum black holes in a theoretical laboratory under robust analytic control. Some questions one might hope to address with bootstrap techniques in this arena include: does consistency of quantum gravity (equivalently, the CFT axioms of the dual) always require states below the Planck scale? What is the “simplest” theory of quantum gravity (and can we solve it)? Can one prove swampland constraints on UV completions of effective field theory with gravity \([45, 46]\), such as the weak gravity conjecture \([47]\)?

Finally, supersymmetric 2D CFTs also describe worldsheet dynamics in the Ramond-Neveu-Schwarz formulation of superstring perturbation theory \([42, 48-52]\). However the worldsheet CFT is peculiar, involving superconformal ghost systems, and needs not even be local in the usual sense in the presence of nontrivial Ramond-Ramond flux backgrounds \([53]\). Nevertheless, one may ambitiously hope to pose the search for weakly-coupled string theories as a worldsheet bootstrap problem.
1.3 Organization of this dissertation

This dissertation splits into two halves, roughly along parallel and complementary research directions that have emerged in the conformal bootstrap literature. The first part of the thesis (chapters 2–5) concerns numerical conformal bootstrap, computing bounds on low-lying operators, pinning down specific CFTs, and the computation of conformal blocks. The second part of the thesis (chapters 6–8) focuses on analytic bootstrap, universal aspects of irrational unitary compact 2D CFTs, and holography.

In chapter 2 we constrain the spectrum of two-dimensional unitary, compact conformal field theories with central charge $c > 1$ using invariance of the torus partition function under large diffeomorphisms of the torus. The systematic study of the resulting constraints is known as modular bootstrap. We use modular bootstrap to compute upper bounds on the gap in the dimension of primary operators of any spin, as well as in the dimension of scalar primaries as functions of the central charge using semi-definite programming. Our bounds refine those of Hellerman [54] and Friedan-Keller [55], and are in some cases saturated by known CFTs. In particular, we show that unitary CFTs with $c < 8$ must admit relevant deformations, and that a nontrivial bound on the gap of scalar primaries exists for $c < 25$. We also study bounds on the dimension gap in the presence of twist gaps, bounds on the degeneracy of operators, and demonstrate how “extremal spectra” which maximize the degeneracy at the gap can be determined numerically.

In chapter 3 we introduce spectral functions that capture the distribution of OPE coefficients and density of states in two-dimensional conformal field theories, and show that nontrivial upper and lower bounds on the spectral function can be obtained from semidefinite programming. We find substantial numerical evidence indicating that OPEs involving only scalar Virasoro primaries in a $c > 1$ CFT are necessarily governed by the structure constants of Liouville theory [56,57]. Combining this with analytic results in modular bootstrap, we conjecture that Liouville theory is the unique unitary $c > 1$ CFT whose primaries have bounded spins. We also use the spectral function method to study modular constraints on CFT spectra, and discuss some implications of our results on CFTs with large central charge and a large gap, in particular, the extent to
which the spectral density of BTZ black holes, as dictated by the Bekenstein-Hawking formula for black hole entropy \[58\], is universal.

The basic kinematic objects in 2D CFTs are Virasoro conformal blocks, which enumerate the contribution of the descendants of a given set of internal primary operators to a correlation function in a given OPE channel. They are completely fixed by Virasoro symmetry, but extremely computationally expensive to compute from their definitions. In chapter 4 we follow an old observation of Zamolodchikov \[59,60\] that the conformal blocks can be analytically continued as meromorphic functions of the internal weights or central charge with simple poles corresponding to degenerate representations, and derive recursive representations in the internal weights of \(N\)-point Virasoro conformal blocks on the sphere and on the torus in particular OPE channels. We also derive recursive representations in the central charge of arbitrary Virasoro conformal blocks on the sphere, the torus, and higher genus Riemann surfaces in the plumbing frame. This frame is particularly useful for numerical computations as the large central charge limit of the block in this frame is finite.

In chapter 5 we study the Virasoro conformal block decomposition of the genus two partition function of a two-dimensional CFT by expanding around a \(Z_3\)-invariant Riemann surface that is a three-fold cover of the Riemann sphere branched at four points, and explore constraints from genus two modular invariance and unitarity. In particular, we find “critical surfaces” that bound the heavy structure constants of a CFT in terms of the light structure constants — constraints beyond what is accessible via the crossing equation on the sphere.

The spectrum of every conformal field theory (CFT) above two dimensions contains an infinite set of Regge trajectories of local operators which, at large spin, asymptote to “double-twist” composites with vanishing anomalous dimension \[26,27\]. In two dimensions, due to the existence of local conformal symmetry, this and other central results of the conformal bootstrap do not apply. In chapter 6 we incorporate exact stress tensor dynamics into the CFT\(_2\) analytic bootstrap, and extract several implications for AdS\(_3\) quantum gravity. Our main tool is the Virasoro fusion kernel, which we newly analyze and interpret in the bootstrap context.
The contribution to double-twist data from the Virasoro vacuum module defines a “Virasoro Mean Field Theory” (VMFT); its spectrum includes a finite number of discrete Regge trajectories, whose dimensions obey a simple formula exact in the central charge $c$ and external operator dimensions. We then show that VMFT provides a baseline for large spin universality in two dimensions: in every unitary compact CFT$_2$ with $c > 1$ and a twist gap above the vacuum, the double-twist data approaches that of VMFT at large spin $\ell$. Corrections to the large spin spectrum from individual non-vacuum primaries are exponentially small in $\sqrt{\ell}$ for fixed $c$. We analyze our results in various large $c$ limits. Further applications include a derivation of the late-time behavior of Virasoro blocks at generic $c$; a refined understanding and new derivation of heavy-light blocks; and the determination of the cross-channel limit of generic Virasoro blocks. We deduce non-perturbative results about the spectrum and dynamics of bounds states in AdS$_3$ quantum gravity.

In chapter 7 we obtain an asymptotic formula for the average value of the operator product expansion coefficients of any unitary, compact two dimensional CFT with $c > 1$. This formula is valid when one or more of the operators has large dimension or — in the presence of a twist gap — has large spin. Our formula is universal in the sense that it depends only on the central charge and not on any other details of the theory. This result unifies all previous asymptotic formulas for 2D CFT structure constants, including those derived from crossing symmetry of four point functions, modular covariance of torus correlation functions, and higher genus modular invariance. We determine this formula at finite central charge by deriving crossing kernels for higher genus crossing equations, which give analytic control over the structure constants even in the absence of exact knowledge of the conformal blocks. The higher genus modular kernels are obtained by composing the elementary kernels for sphere four-point crossing and the modular S-transform of torus one-point functions, as in [37]. Our asymptotic formula is related to the DOZZ formula for the structure constants of Liouville theory [56,57], and makes precise the sense in which Liouville theory governs the universal dynamics of heavy operators in any CFT. We use technology developed along the way to prove the conjecture made in chapter 3 that Liouville theory is the unique solution to the crossing equations involving only scalar Virasoro primaries. The large central charge limit provides a link with 3D gravity, where the averaging over heavy
states corresponds to a coarse-graining over black hole microstates in holographic theories. Our formula also provides an improved understanding of the eigenstate thermalization hypothesis (ETH) in 2D CFT, and suggests that ETH can be generalized to other kinematic regimes in two dimensional CFTs.

Finally, in chapter 8 we revisit the spectrum of pure quantum gravity in AdS$_3$. The computation of the torus partition function — if computed from a Euclidean gravitational path integral that includes only smooth saddle points — leads to a density of states that is not physically sensible, as it has a negative degeneracy of states for some energies and spins. We consider a minimal cure for this non-unitarity of the pure gravity partition function, which involves the inclusion of additional states below the black hole threshold. We propose a geometric interpretation for these extra states: they are conical defects with deficit angle $2\pi(1 - 1/N)$, where $N$ is a positive integer. That only integer values of $N$ should be included can be seen from a modular bootstrap argument, and leads us to propose a modest extension of the set of saddle-point configurations that contribute to the gravitational path integral: one should sum over orbifolds in addition to smooth manifolds. The orbifold states are below the black hole threshold and are thus regarded as massive particles in AdS, but they are not perturbative states: they are too heavy to form multi-particle bound states. We compute the one-loop determinant for gravitons in these orbifold backgrounds, which confirms that the orbifold states are Virasoro primaries. We compute the gravitational partition function including the sum over these orbifolds and find a finite, modular invariant result; this finiteness involves a delicate cancellation between the infinite tower of orbifold states and an infinite number of instantons associated with $PSL(2, \mathbb{Z})$ images.
Modular bootstrap revisited

2.1 Introduction

It is commonly believed that unitary, compact conformal field theories in two dimensions with central charge $c > 1$ and no conserved Virasoro primary currents (of any spin) are ubiquitous. $^1$ In particular, the holographic duals of generic quantum theories of gravity in $AdS_3$ are expected to have such properties $[8, 63–68]$. Such theories are hard to construct using standard CFT techniques, however. For instance, rational CFTs always contain an extended chiral algebra and therefore extra conserved currents $[37, 69, 70]$. Product orbifolds and (GSO projected) superconformal theories also contain higher spin conserved currents $[71]$. Evidently there is

$^1$After all, spin-1 currents are always governed by current algebra in a unitary theory, and can be gauged away, while higher spin currents hint at some sort of integrability $[61, 62]$. 
a tremendous gap in our knowledge of two-dimensional CFTs: the vast majority of CFTs that are believed to exist are inaccessible with available analytic methods. In this chapter, we undertake initial steps towards closing this gap, by refining the modular bootstrap approach of [54,55,74–77]. A surprisingly rich set of new constraints on the CFT spectrum will be uncovered from unitarity and the modular invariance of the torus partition function alone.

To begin with, we consider the Virasoro character decomposition of the torus partition function of a CFT with no conserved current primaries (the latter assumption may be relaxed, as we will consider later), of the form

\[ Z(\tau, \bar{\tau}) = \chi_0(\tau) \bar{\chi}_0(\bar{\tau}) + \sum_{h, \bar{h} > 0} d(h, \bar{h}) \chi_h(\tau) \bar{\chi}_{\bar{h}}(\bar{\tau}), \]  

and impose positivity on the coefficients \( d(h, \bar{h}) \). The characters \( \chi_h \) with \( h > 0 \) are non-degenerate, and the spins \( h - \bar{h} \) are assumed to be integers. Modular invariance can be formulated as the “modular crossing equation”

\[ E_{0,0}(\tau, \bar{\tau}) + \sum_{h, \bar{h} > 0} d(h, \bar{h}) E_{h, \bar{h}}(\tau, \bar{\tau}) = 0, \]  

where

\[ E_{h, \bar{h}}(\tau, \bar{\tau}) = \chi_h(\tau) \bar{\chi}_{\bar{h}}(\bar{\tau}) - \chi_h(-1/\tau) \bar{\chi}_{\bar{h}}(-1/\bar{\tau}). \]

Constraints on the spectrum of Virasoro primaries, namely the set of \((h, \bar{h})\) in the sum as well as the coefficients \( d(h, \bar{h}) \), due to the positivity condition on \( d(h, \bar{h}) \), can be extracted using the powerful numerical method of semi-definite programming. One proceeds by assuming certain properties of the spectrum, say the presence of a gap in the dimensions, and seeks a linear functional \( \alpha \) that acts on functions of \((\tau, \bar{\tau})\), such that \( \alpha[E_{0,0}] \) takes the same sign as \( \alpha[E_{h, \bar{h}}] \) for all \((h, \bar{h})\) in the hypothetical spectrum. If such a linear functional is found, the positivity condition on \( d(h, \bar{h}) \) cannot be satisfied and the proposed spectrum would be ruled out.

\[ \text{See however [72,73] for families of candidate irrational CFTs that potentially admit no extra conserved currents.} \]

\[ \text{Typically, there are infinitely many primaries and we would only make assumptions on the properties of a finite subset of operators. We need } \alpha[E_{h, \bar{h}}] \text{ to have the same sign for sufficiently large } h \text{ or } \bar{h} \text{ in order to derive a useful constraint.} \]
In [54, 55, 76], the linear functional $\alpha$ was taken to be a linear combination of derivatives in $\tau_2$ up to a certain order, evaluated at the self-dual point $\tau = i$ (i.e., modular invariance was imposed only on the restriction of the partition function to the imaginary $\tau$ axis). Using such linear functionals, one finds an upper bound on the scaling dimension of the primaries that is insensitive to the spin. We denote this bound by $\Delta_{\text{HFK}}^{(N)}(c)$, where $N$ is the maximal derivative order of the linear functional.\(^4\) It was found in [55] that, for fixed $N$, $\Delta_{\text{HFK}}^{(N)}(c) = \frac{c}{6} + O(1)$ in the $c \to \infty$ limit.

In this chapter, we make two important refinements of the analysis of [55]. Firstly, we find substantial numerical evidence that the $N \to \infty$ limit and $c \to \infty$ limit of $\Delta_{\text{HFK}}^{(N)}(c)$ do not commute. The optimal gap $\Delta_{\text{HFK}}(c) \equiv \Delta_{\text{HFK}}^{(\infty)}(c)$ appears to have the property that its slope $d\Delta_{\text{HFK}}/dc$ decreases monotonically with $c$, and asymptotes to a value $b_{\text{HFK}}$ that lies between $\frac{1}{9}$ and $\frac{1}{12}$. Secondly, we find a stronger bound on the gap, $\Delta_{\text{mod}}^{(N)}(c)$, using linear functionals built out of derivatives in both $\tau$ and $\bar{\tau}$ up to total derivative order $N$. The optimal bound on the gap, $\Delta_{\text{mod}}(c) \equiv \Delta_{\text{mod}}^{(\infty)}(c)$, appears to have the same property that $d\Delta_{\text{HFK}}/dc$ decreases monotonically with $c$, and asymptotes to a value $b_{\text{mod}} \leq b_{\text{HFK}}$.\(^5\)

We can also obtain spin-dependent bounds, by applying the most general linear functional to the modular crossing equation. For instance, we have computed an upper bound $\Delta_{\text{mod}}^{s=0}(c)$ on the gap in the dimension of scalar primaries, regardless of the gaps in the spectra of higher-spin primaries. While initially we assume the absence of extra conserved currents, this assumption will later be relaxed, with very little difference in the resulting bounds.\(^6\) It is observed that $\Delta_{\text{mod}}^{s=0}(c) < 2$ for $c < 8$, that is, unitary CFTs with $c < 8$ must admit a relevant deformation. There is a kink on the bounding curve at $c = 8$, with scalar gap $\Delta_{\text{mod}}^{s=0} = 2$, which we believe is exact. The gap at this kink is saturated by the $E_8$ WZW model at level 1.\(^7\) Based on the numerical precision at large values of $c$ is insufficient in resolving the difference between $b_{\text{mod}}$ and $b_{\text{HFK}}$, if there is any.

More precisely, when conserved primary currents are allowed, we find no difference in the bound $\Delta_{\text{mod}}^{s=0}$ within the numerical error of the binary search for the optimal bound based on semi-definite programming for $1 \leq c \leq 8$ and a slightly weaker bound for $c > 8$ obtained using linear functionals up to a fixed derivative order. Our numerical extrapolation to infinite derivative order is not accurate enough to resolve the difference between the two bounds.

\(^4\) $N$ was taken to be 3 in [54] and up to 23 in [55].
\(^5\) More precisely, when conserved primary currents are allowed, we find no difference in the bound $\Delta_{\text{mod}}^{s=0}$ within the numerical error of the binary search for the optimal bound based on semi-definite programming for $1 \leq c \leq 8$ and a slightly weaker bound for $c > 8$ obtained using linear functionals up to a fixed derivative order. Our numerical extrapolation to infinite derivative order is not accurate enough to resolve the difference between the two bounds.

\(^6\) Note that while this CFT does contain extra conserved currents, its partition function happens to admit a formal decomposition in terms of non-degenerate Virasoro characters with non-negative coefficients, due to the contributions from twist-2 primaries. We refer to such a partition function as that of the \textbf{generic type}, which may be viewed as a limiting case of partition functions with no conserved current primaries.
this, we conclude that “perfect metals” \cite{78} do not exist for \( c \leq 8 \). Interestingly, \( \Delta_{\text{mod}}^{c=0}(c) \) diverges at \( c = 25 \), and in fact modular invariance is compatible with a spectrum that contains no scalar primaries for \( c > 25 \). We will give an explicit example of a spectrum that has such a property.

It is also possible to place upper bounds on the degeneracies of the lightest primaries, provided that their dimension lies in between \( \frac{c-1}{12} \) and \( \Delta_{\text{mod}} \) (the former is the upper bound on the twist gap). When this upper bound on the degeneracy is saturated, the entire spectrum of primaries is determined as zeroes of the optimal linear functional acting on the characters as a function of the weights.\(^8\) We refer to this spectrum as the extremal spectrum.\(^9\) We will demonstrate the extraction of the extremal spectrum in a number of examples. It is observed that when the upper bound on the (scalar) dimension gap is saturated for \( 1 \leq c \leq 4 \), the extremal spectrum always contains conserved spin-1 currents and marginal scalar primaries. Rather strikingly, we will uncover precisely the spectra of \( SU(2) \), \( SU(3) \), \( G_2 \) and \( SO(8) \) WZW models at level 1 from the extremal spectra with maximal gap at the respective values of central charge.

### 2.2 Constraining gaps in the operator spectrum

#### 2.2.1 Basic setup

The vacuum Virasoro character \( \chi_0 \) and the non-degenerate Virasoro character \( \chi_h \) are given by

\[
\chi_0(\tau) = q^{-\frac{c-1}{24}} \eta(\tau) (1 - q), \quad \chi_h(\tau) = \frac{q^{h-\frac{c-1}{24}}}{\eta(\tau)},
\]

\(^8\)A priori, the degeneracies of higher-dimension primaries in the extremal spectrum are not fixed by this procedure. However, in several examples of CFTs that realize the extremal spectrum, we find that the degeneracies of higher dimension operators agree with the respective upper bounds subject to the assumed gap.

\(^9\)Note that in our definition of the extremal spectrum, the degeneracies are only required to be positive, and not necessarily integers. Demanding the latter would slightly refine our bounds.
where \( q = e^{2\pi i \tau} \), \( \eta(\tau) \) is the Dedekind eta function, and \( c \) is the central charge. In general, the partition function of a compact, unitary CFT admits the character decomposition

\[
Z(\tau, \bar{\tau}) = \chi_0(\tau)\bar{\chi}_0(\bar{\tau}) + \sum_{j=1}^\infty \left[ d_j \chi_j(\tau)\bar{\chi}_0(\bar{\tau}) + \bar{d}_j \chi_0(\tau)\bar{\chi}_j(\bar{\tau}) \right] + \sum_{h, \bar{h}} d(h, \bar{h}) \chi_h(\tau)\bar{\chi}_{\bar{h}}(\bar{\tau}),
\]

(2.2.2)

where \( d_j \) and \( \bar{d}_j \) are the degeneracies of holomorphic and anti-holomorphic currents of spin \( j \), and \( d(h, \bar{h}) \) is the degeneracy of primary operators of weight \( (h, \bar{h}) \). For the rest of this chapter, we will assume that the spectrum is parity-invariant. The constraints we derive on the parity invariant-spectrum can be applied to parity non-invariant theories as well, if we simply consider the projection of the partition function \( Z(\tau, \bar{\tau}) \) onto its parity invariant part, \( \frac{1}{2} [Z(\tau, \bar{\tau}) + Z^*(\bar{\tau}, \tau)] \). For the parity invariant spectrum, we will label the primaries by their dimension \( \Delta = h + \bar{h} \) and spin \( s = |h - \bar{h}| \) \((\in \mathbb{Z}_{\geq 0}) \), and write the degeneracies as

\[
d_{\Delta, s} \equiv d\left( \frac{\Delta + s}{2}, \frac{\Delta - s}{2} \right) = d\left( \frac{\Delta - s}{2}, \frac{\Delta + s}{2} \right).
\]

(2.2.3)

As explained in the introduction, we will be primarily interested in genuinely irrational unitary CFTs with central charge \( c > 1 \), no conserved currents, i.e., \( d_j = \bar{d}_j = 0 \), and all nontrivial primaries obeying \( \Delta > s \). In deriving various numerical bounds, it will be convenient to allow for the limiting case \( \Delta \to s \) which corresponds to a contribution to the partition function of the form

\[
\lim_{\Delta \to s} \left( \chi_{\frac{s}{2}} \bar{\chi}_{\frac{s}{2}} + \chi_{\frac{s-1}{2}} \bar{\chi}_{\frac{s-1}{2}} \right) = \chi_s(\bar{\chi}_0 + \bar{\chi}_1) + (\chi_0 + \chi_1)\bar{\chi}_s.
\]

(2.2.4)

That is, the spectrum may contain conserved currents of spin \( s \), along with twist-2 primaries of dimension \( s + 1 \) and spin \( s - 1 \). We refer to such a partition function as that of the **generic type**. We will see in several instances that certain rational CFTs with partition functions of the generic type appear at kinks on the boundary of the domain of allowed spectra.
As in [55], it is convenient to work with the reduced partition function

\[ \hat{Z}(\tau, \bar{\tau}) \equiv |\tau|^{\frac{1}{2}} |\eta(\tau)|^2 Z(\tau, \bar{\tau}), \]  

(2.2.5)

decomposed into the reduced characters

\[ \hat{\chi}_0(\tau) \hat{\chi}_0(\bar{\tau}) = |\tau|^{\frac{1}{2}} |q^{-\frac{1}{24}} (1 - q)|^2, \quad \hat{\chi}_h(\tau) \hat{\chi}_h(\bar{\tau}) = |\tau|^{\frac{1}{2}} q^{h - \frac{1}{24}} \bar{q}^{\bar{h} - \frac{1}{24}}. \]  

(2.2.6)

Since \(|\tau|^{\frac{1}{2}} |\eta(\tau)|^2\) is invariant\(^{10}\) under \(\tau \rightarrow -1/\tau\), it suffices to consider the modular crossing equation for \(\hat{Z}\).

It is convenient to introduce the variable \(z\) defined by \(\tau = i \exp(z)\), so that the modular \(S\) transformation \(\tau \rightarrow -1/\tau\) takes \(z \rightarrow -z\). Modular invariance amounts to the statement that the partition function is an even function in \((z, \bar{z})\) (and that all spins are integers). To implement semi-definite programming on the modular crossing equation, we will apply to it the basis of linear functionals

\[ \partial_z^m \partial_{\bar{z}}^n |_{z=\bar{z}=0}, \quad m + n \text{ odd}. \]  

(2.2.7)

In other words, we consider linear functionals of the form

\[ \alpha \equiv \sum_{m+n \text{ odd}} \alpha_{m,n} (z, \bar{z}) \partial_z^m \partial_{\bar{z}}^n |_{z=\bar{z}=0}. \]  

(2.2.8)

and turn the modular crossing equation (2.1.2) into

\[ 0 = \alpha[\hat{Z}(\tau, \bar{\tau})] = \alpha[\hat{\chi}_0(\tau) \hat{\chi}_0(\bar{\tau})] + \sum_{s=0}^{\infty} \sum_{\Delta \in}\mathcal{I}_s d_{\Delta,s} \alpha \left[ \hat{\chi}_{\frac{\Delta-s}{2}}(\tau) \hat{\chi}_{\frac{\Delta+s}{2}}(\bar{\tau}) + \hat{\chi}_{\frac{\Delta+s}{2}}(\tau) \hat{\chi}_{\frac{\Delta-s}{2}}(\bar{\tau}) \right], \]  

(2.2.9)

where \(\mathcal{I}_s\) is a (typically infinite) discrete set that consists of the dimensions of primaries of spin \(s\).

One may proceed by hypothesizing, for instance, that there is a gap \(\Delta^*_s \geq s\) in the spectrum of spin-\(s\)

\(^{10}\)It is not invariant under the \(T\) transform \(\tau \rightarrow \tau + 1\), but we have already taken into account \(T\)-invariance of \(Z(\tau, \bar{\tau})\) by demanding that the spins are integers, so it suffices to examine the \(S\) transform.
primaries, i.e., \( \mathcal{I}_s \) consists of dimensions \( \Delta \geq \Delta^*_s \) only. If we can find a functional \( \alpha \), which amounts to a set of \( \alpha_{m,n} \in \mathbb{R} \) in (2.2.8), such that

\[
\alpha[\hat{\chi}_0(\tau)\hat{\bar{\chi}}(\bar{\tau})] > 0, \\
\alpha \left[ \hat{\chi}_{\Delta^-}(\tau)\hat{\bar{\chi}}_{\Delta^+}(\bar{\tau}) + \hat{\bar{\chi}}_{\Delta^-}(\tau)\hat{\chi}_{\Delta^+}(\bar{\tau}) \right] \geq 0, \quad \Delta \geq \Delta^*_s, \quad \forall s \in \mathbb{Z}_0,
\]

we would then arrive at a contradiction with the non-negativity of the degeneracies \( d_{\Delta,s} \) in the modular crossing equation, thereby ruling out the putative spectrum. In other words, we would have proven that the gap in the spin-\( s \) spectrum cannot exceed \( \Delta^*_s \) simultaneously for all \( s \).

As examples, we may take

- \( \Delta^*_s = \max(\Delta_{\text{mod}}, s) \), where \( \Delta_{\text{mod}} \) is the maximal gap in the scaling dimension spectrum of all primaries.
- \( \Delta^*_0 = \Delta^*_{\text{mod}}, \Delta^*_s = s \) (\( s \geq 1 \)), where \( \Delta^*_{s=0} \) is the maximal gap in the dimension of scalar primaries.
- \( \Delta^*_s = s + t_{\text{mod}} \), where \( t_{\text{mod}} \) is the maximal twist gap in the spectrum of all primaries.

To implement the above procedure numerically, we must restrict to a finite subset of the basis of linear functionals, say \( \partial^m \partial^p |_{z = \bar{z} = 0} \) with odd \( m + n \leq N \). We will refer to \( N \) as the “derivative order” of the linear functional. The upper bound on the gap \( \Delta_{\text{mod}} \) derived by exclusion using linear functionals up to derivative order \( N \), for instance, will be denoted \( \Delta^{(N)}_{\text{mod}} \). In other words, \( \Delta^{(N)}_{\text{mod}} \) is the smallest \( \Delta^* \) such that (2.2.10) can be satisfied with \( \Delta^*_s = \max(\Delta_{\text{mod}}, s) \), by a functional \( \alpha \) restricted to derivative order \( N \). While for each positive integer \( N \), \( \Delta^{(N)}_{\text{mod}} \) is a rigorous bound on the gap, the optimal bound would be obtained by taking the \( N \to \infty \) limit.

Our numerical analysis is performed using the SDPB package [79]. In practice, we also need to truncate the spectrum: while SDPB allows us to consider a spectrum that consists of operators of all dimensions (say, above a hypothetical gap), we will need to restrict the spins of operators to a (sufficiently large) finite range, \( s \leq s_{\text{max}} \). In seeking linear functionals for the purpose of excluding a given hypothetic spectrum, increasing \( s_{\text{max}} \) puts more constraints on the linear functional, and in principle we need to take the limit \( s_{\text{max}} \to \infty \).
In practice, however, the bound $\Delta_{\text{mod}}^{(N)}$ derived at a given derivative order $N$ stabilizes to within numerical precision once $s_{\text{max}}$ exceeds a certain value (typically of order $N$).

In optimizing the bounds with increasing derivative order $N$, we find that at larger values of the central charge $c$, one must also work to higher values of $N$ for the bound $\Delta_{\text{mod}}^{(N)}$ to stabilize. When such a stabilization is unattainable due to the computational complexity, we will need to numerically extrapolate $\Delta_{\text{mod}}^{(N)}$ to $N = \infty$, by fitting with a polynomial in $1/N$ (say, of linear or quadratic order).

### 2.2.2 The twist gap

In a unitary, compact two-dimensional CFT with central charge $c > 1$, the twist gap\footnote{In two-dimensional CFTs, the twist $t$ (here we do not use $\tau$ to avoid confusion with the modular parameter) is defined by $t = \Delta - s = 2 \min(h, \tilde{h})$.} $t_{\text{gap}}$ can be no larger than $\frac{c-1}{12}$. Furthermore, in the absence of conserved currents, there must be infinitely many high spin primaries whose twists accumulate to $\frac{c-1}{12}$. This can be seen as follows.\footnote{This argument is due to Tom Hartman.}

Consider the character decomposition of the partition function (2.1.1). In the limit $\bar{\tau} \to -i0^+$, $Z(\tau, \bar{\tau}) = Z(-1/\tau, -1/\bar{\tau})$ is dominated by the modular transformed vacuum character, which may be expressed as

$$
\lim_{\bar{\tau} \to -i0^+} \frac{Z(-1/\tau, -1/\bar{\tau})}{\chi_0(-1/\bar{\tau})} = \chi_0(-1/\tau) = \int_{\frac{c-1}{12}}^\infty dh' K(0, h') \chi_{h'}(\tau),
$$

where $K(h, h')$ is the modular kernel [64, 65]. On the other hand, we have

$$
\lim_{\tau \to -i0^+} \frac{Z(\tau, \bar{\tau})}{\chi_0(-1/\bar{\tau})} = \lim_{\tau \to -i0^+} \left[ \chi_0(\tau) \frac{\chi_0(\bar{\tau})}{\chi_0(-1/\bar{\tau})} + \sum_{h, \tilde{h} > 0} d(h, \tilde{h}) \chi_h(\tau) \frac{\bar{\chi}_{\tilde{h}}(\bar{\tau})}{\chi_0(-1/\bar{\tau})} \right].
$$

In the limit $\bar{\tau} \to -i0^+$, $\bar{\chi}_0(-1/\bar{\tau}) \sim [\bar{g}(-1/\bar{\tau})]^{-\frac{1}{12}}$ while $\bar{\chi}_{\tilde{h}}(\bar{\tau}) \sim (i\bar{\tau})^{\frac{1}{12}} [\bar{g}(-1/\bar{\tau})]^{-\frac{1}{12}}$, and thus

$$
\lim_{\tau \to -i0^+} \frac{\bar{\chi}_{\tilde{h}}(\bar{\tau})}{\chi_0(-1/\bar{\tau})} = 0, \text{ for } c > 1.
$$

(2.2.13)
Figure 2.1: **Top:** The colored curves are the upper bound on the twist gap as a function of the central charge for increasing derivative order (up to 43) from green to red. The results are consistent with the convergence to the bound on the twist gap $t_{\text{mod}} = \frac{c-1}{12}$ predicted by the analytic argument in the infinite-$N$ limit. **Bottom:** The bound on the twist gap $t_{\text{mod}}^{(N)}$ as a function of inverse derivative order for $c = 5$ and $c = 100$.

If we could exchange the limit and summation over primaries in (2.2.12), we would have concluded that the right-hand side of (2.2.12) vanished, which would contradict (2.2.11). In particular, in the $\tau \to i\infty$ limit (after taking $\bar{\tau} \to -i0^+$ first), the right-hand side of (2.2.11) is dominated by the character of weight $h' = \frac{c-1}{24}$. For (2.2.12) to be consistent, there must be infinitely many primaries with left conformal weight $h$ accumulating to $\frac{c-1}{24}$, or equivalently, their twists accumulating to $\frac{c-1}{12}$.

As a test of our numerical approach to modular bootstrap, we can indeed reproduce this twist gap bound,
by seeking linear functionals with the following positivity properties,

\[
\alpha \left[ \hat{\chi}_0(\tau)\hat{\chi}_0(\bar{\tau}) \right] > 0 \\
\alpha \left[ \hat{\chi}_{\frac{1}{2}+s}(\tau)\hat{\chi}_{\frac{1}{2}}(\bar{\tau}) + \hat{\chi}_{\frac{1}{2}}(\tau)\hat{\chi}_{\frac{1}{2}+s}(\bar{\tau}) \right] \geq 0, \quad t \geq t^*_\text{mod}, \quad \forall s \in \mathbb{Z}_{\geq 0}.
\]

(2.2.14)

The smallest \( t^*_\text{mod} \) such that there exists a linear functional \( \alpha \) satisfying the above equation then yields the strongest upper bound on the twist gap, which we denote \( t^*_\text{mod} \). Figure 2.1 shows \( t^{(N)}_\text{mod}(c) \) as a function of the central charge. Indeed it appears that the bounds are converging to the twist gap predicted by the above argument as the derivative order is taken to infinity. However, we observe that the convergence of the bound with \( N^{-1} \) is slower as the central charge is increased. This will turn out to be a generic feature of bounds we obtain from semi-definite programming, and will prevent accurate determinations of asymptotic bounds as \( c \to \infty \).

### 2.2.3 Refinement of Hellerman-Friedan-Keller bounds

Bounds on the gap in the dimension of all primaries, regardless of spin, were derived in \([54,55]\) by applying linear functionals of the form

\[
\alpha = \sum_{\text{odd } n \leq N} \alpha_n (\tau \partial_\tau + \bar{\tau} \partial_{\bar{\tau}})^n |_{\tau = i}
\]

(2.2.15)

to the modular crossing equation. We refer to (2.2.15) as HFK functionals, and the resulting bound on the dimension gap \( \Delta^{(N)}_{\text{HFK}} \).

The HFK functional amounts to restricting the partition function to the imaginary \( \tau \)-axis. Writing \( \tau = i\beta \), for real \( \beta \), the character decomposition is blind to the spins of operators in the spectrum. In particular, the reduced partition function takes the form

\[
\hat{Z}(\beta) = \hat{Z}_0(\beta) + \sum_\Delta d_\Delta \hat{Z}_\Delta(\beta),
\]

(2.2.16)
where $d_\Delta$ is the degeneracy including all primary operators with dimension $\Delta = h + \bar{h}$, and

\[
\tilde{Z}_0(\beta) = \beta^\frac{1}{2} e^{2\pi \beta \frac{c}{12}} (1 - e^{-2\pi \beta})^2, \quad \tilde{Z}_\Delta(\beta) = \beta^\frac{1}{2} e^{-2\pi \beta (\Delta - \frac{c}{12})}.
\]  

(2.2.17)

The HFK functional can be written as

\[
\alpha = \sum_{\text{odd } n \leq N} \alpha_n (\beta \partial_\beta)^n |_{\beta = 1}.
\]  

(2.2.18)

To place upper bounds on the dimension of the lightest operator in the spectrum, we search for linear functionals that satisfy the following positivity properties

\[
\alpha[\tilde{Z}_0(\beta)] > 0
\]  

\[
\alpha[\tilde{Z}_\Delta(\beta)] \geq 0, \quad \Delta \geq \Delta^*_\text{HFK}.
\]  

(2.2.19)

If such a functional can be found, then $\Delta^*_\text{HFK}$ is a rigorous upper bound on the gap. The bound $\Delta^{(N)}_{\text{HFK}}$ is obtained as the smallest $\Delta^*_\text{HFK}$ such that (2.2.19) holds.

In [55], the bound on the gap in the large $c$ limit was of primary interest. It was found that, for fixed $N$ (taken to be 3 in [54] and 23 in [55]), in the large $c$ limit,

\[
\Delta^{(N)}_{\text{HFK}} = \frac{c}{6} + O(1).
\]  

(2.2.20)

We employ the strategy of [55] but consider the extrapolation $\lim_{N \to \infty} \Delta^{(N)}_{\text{HFK}}(c)$ at each value of $c$, thereby exhausting the constraints from the HFK functionals (2.2.18). Examples of data points used to obtain such extrapolations are shown in figure 2.2. We observe numerically that as we increase $N$, the value of $N$ at which $\Delta^{(N)}_{\text{HFK}}(c)$ stabilizes grows with the central charge $c$. Thus, it becomes increasingly difficult to extrapolate to $\Delta^{(\infty)}_{\text{HFK}}(c)$ as we go to larger values of $c$, requiring that we work to larger derivative orders $N$ to obtain an
accurate extrapolation. Figure 2.3 shows the result of an extrapolation of $\Delta_{\text{HFK}}^{(\infty)}(c)/c$, as well as the slope $d\Delta_{\text{HFK}}^{(\infty)}(c)/dc$ of the bound as a function of the central charge, over a range of the central charge where the numerical extrapolation appears reliable (using results up to $N = 183$).

We observe that the slope of $\Delta_{\text{HFK}}(c)$ decreases monotonically, and we conjecture that this property holds for all values of $c$. Note that this slope falls well below $\frac{1}{5}$, the large $c$ asymptotic slope of $\Delta_{\text{HFK}}^{(N)}(c)$ for fixed $N$ (as was found in [55]). Since a reliable extrapolation requires going to derivative order $N$ that grows with $c$, so far we have not been able to perform a reliable extrapolation for $c \gtrsim O(10^2)$. Based on the numerical
results, together with the known constraint $\Delta_{\text{HFK}}(c) \geq \Delta_{\text{gap}}(c) = \frac{c-1}{12}$, we conjecture that in the large $c$ limit,

$$\lim_{c \to \infty} \frac{\Delta_{\text{HFK}}(c)}{c} = b_{\text{HFK}}, \quad \text{with} \quad \frac{1}{12} \leq b_{\text{HFK}} < \frac{1}{9}. \quad (2.2.21)$$

The need for computing the bound using HFK functionals to very large derivative order $N$ suggests that

$\{((\beta \partial_\beta)^n)_{|_{\beta=1}}\}_{\text{odd } n}$ is in fact a poor choice of basis for the optimal linear functional at large $c$. We will examine such optimal functionals in detail in section 2.2.5.

### 2.2.4 Bounds from full modular invariance

We now consider the stronger constraints obtained by imposing modular invariance on $\hat{Z}(\tau, \bar{\tau})$ on the entire upper half $\tau$-plane (as opposed to the imaginary $\tau$-axis), by applying to the modular crossing equation linear functionals of the form

$$\alpha = \sum_{\text{odd } m+n \leq N} \alpha_{m,n} \partial_z^m \partial_{\bar{z}}^n \big|_{z=\bar{z}=0} \quad (2.2.22)$$

that satisfy

$$\alpha \left[ \hat{\chi}_0(\tau) \hat{\chi}_0(\bar{\tau}) \right] > 0$$

$$\alpha \left[ \hat{\chi}_{\Delta-} (\tau) \hat{\chi}_{\Delta+} (\bar{\tau}) + \hat{\chi}_{\Delta+} (\tau) \hat{\chi}_{\Delta-} (\bar{\tau}) \right] \geq 0, \quad \Delta \geq \max(\Delta_{\text{mod}}^*, s), \quad (2.2.23)$$

for some $\Delta_{\text{mod}}^*$. If such a linear functional $\alpha$ is found, we learn that the gap in the dimension spectrum of all primaries is bounded from above by $\Delta_{\text{mod}}^*$. The bound that results from the smallest such $\Delta_{\text{mod}}^*$ will be denoted $\Delta_{\text{mod}}^{(N)}(c)$. The optimal bound would be obtained in the infinite $N$ limit, namely,

$$\Delta_{\text{mod}}(c) = \lim_{N \to \infty} \Delta_{\text{mod}}^{(N)}(c). \quad (2.2.24)$$

Note that, just as with the HFK bound, it is important that we take the $N \to \infty$ limit at fixed $c$. 

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Figure 2.4 shows the bound $\Delta_{\text{mod}}(c)$ obtained by a numerical extrapolation of $\Delta_{\text{mod}}^{(N)}$ to infinite derivative order $N$. We find with high numerical precision that

$$\Delta_{\text{mod}}(c) = \frac{c}{6} + \frac{1}{3}, \quad \text{for } c \in [1, 4].$$  \hspace{1cm} (2.2.25)

A kink appears at $c = 4$ and $\Delta_{\text{mod}} = 1$, where the slope of $\Delta_{\text{mod}}(c)$ jumps from $\frac{1}{6}$ to the left of the kink, to $\frac{1}{8}$ to the right of the kink. As the central charge is increased, the slope of $\Delta_{\text{mod}}(c)$ appears to decrease monotonically, just as $\Delta_{\text{HFK}}(c)$ seen in the previous subsection. For larger values of $c$, a numerical extrapolation to infinite derivative order $N$ is again needed. Based on the numerical results, we conclude that

$$\Delta_{\text{mod}}(c) < \frac{c}{8} + \frac{1}{2}, \quad c > 4.$$  \hspace{1cm} (2.2.26)

Furthermore, we conjecture that the slope $\Delta_{\text{mod}}(c)$ decreases monotonically, and asymptotic to a value

$$\lim_{c \to \infty} \frac{\Delta_{\text{mod}}(c)}{c} = b_{\text{mod}}, \quad \text{with } \frac{1}{12} \leq b_{\text{mod}} \leq b_{\text{HFK}} < \frac{1}{9}.$$  \hspace{1cm} (2.2.27)

Interestingly, $\Delta_{\text{mod}}(c)$ coincides with $\Delta_{\text{HFK}}(c)$ at $c = 4$ (where both are equal to 1), but the bounds do
not agree for $c$ above or below 4. The numerical evaluation of $\Delta^{(N)}_{\text{mod}}(c)$ is more time consuming than that of $\Delta^{(N)}_{\text{HFK}}(c)$, and we are unable to perform a reliable extrapolation of the large $c$ asymptotics of $\Delta_{\text{mod}}(c)$ directly. We can nonetheless analyze the difference between the two bounds, $\Delta_{\text{HFK}}(c) - \Delta^{(N)}_{\text{mod}}(c)$, for moderate values of $c$, as shown in figure 2.5. We observe that at a fixed derivative order $N$ of the linear functional, the difference between $\Delta_{\text{HFK}}(c)$ and $\Delta^{(N)}_{\text{mod}}(c)$ will initially grow with the central charge until it eventually begins to decrease and becomes negative. This is related to the observed phenomenon that as the central charge is increased, one must use linear functionals of larger and larger derivative order to obtain stabilized bounds. While it is possible that the asymptotic slope $b_{\text{mod}}$ is smaller than $b_{\text{HFK}}$, we have not been able to resolve their difference numerically.

2.2.5 The optimal linear functional

It is somewhat unexpected that such delicate numerical analysis is required to extract bounds from a very simple form of the modular crossing equation for the reduced partition function, and we still do not know the value of the asymptotic slopes, $b_{\text{HFK}}$ and $b_{\text{mod}}$. As an analytic derivation of the optimal bounds is not yet available, we may look for hints in the optimal linear functional $\alpha$. At a given derivative order $N$, by minimizing the gap bound $\Delta^*_{\text{HFK}}$ or $\Delta^*_{\text{mod}}$, we can determine the optimal linear functional, which we denote by $\alpha^{(N)}_{\text{HFK}}$ or $\alpha^{(N)}_{\text{mod}}$. Numerically, it appears that there are indeed well defined $N \to \infty$ limits on the optimal

![Figure 2.5: $\Delta_{\text{HFK}} - \Delta^{(N)}_{\text{mod}}$ as a function of central charge $c$, for $c \in [4, 22]$ and increasing $N$. The numerical values of the HFK bound are obtained by a linear extrapolation of $\Delta^{(N)}_{\text{HFK}}$ to $1/N \to 0$ using bounds for $137 \leq N \leq 175$.](image-url)
linear functionals,
\[ \alpha_{\text{HFK}} = \lim_{N \to \infty} \alpha_{\text{HFK}}^{(N)} = F_{\text{HFK}}(\beta \partial_{\beta})|_{\beta=1}, \]
\[ \alpha_{\text{mod}} = \lim_{N \to \infty} \alpha_{\text{mod}}^{(N)} = F_{\text{mod}}(\partial_z, \partial_{\bar{z}})|_{z=\bar{z}=0}. \] (2.2.28)

Here \( F_{\text{HFK}} \) is a power series in \( \beta \partial_{\beta} \) (to be evaluated at \( \beta = 1 \)), and \( F_{\text{mod}} \) is a power series in \( \partial_z, \partial_{\bar{z}} \) (to be evaluated at \( z = \bar{z} = 0 \)). Both \( F_{\text{HFK}} \) and \( F_{\text{mod}} \) can be computed numerically.

Figure 2.6 shows a few examples of the polynomials \( F_{\text{HFK}}^{(N)}(t) \) that represent the optimal HFK functional up to derivative order \( N \), that converge in the infinite \( N \) limit. For small values of \( c \), \( F_{\text{HFK}}(t) \) can be rather accurately fitted by a linear combination of \( \sin(at) \) and \( \sinh(bt) \), for some constants \( a, b \), suggesting that the optimal linear functional is well approximated by a linear combination of the modular crossing equation evaluated at two values of \( \beta \), one real and the other lying on the unit circle (analytically continued in \( \beta \)).

At large \( c \), however, the behavior of \( F_{\text{HFK}}(t) \) changes: it can be approximately fitted by \( te^{-at^2} \) over a large range of \( t \), suggesting that the optimal linear functional is more appropriately represented by an integral transform in \( \beta \) rather than taking derivatives at \( \beta = 1 \).\(^{13}\)

Similarly, we can obtain \( F_{\text{mod}}^{(N)}(w, \bar{w}) \) that represents the optimal functional obtained by imposing full modular invariance. It exhibits a nontrivial dependence on both the real and imaginary parts of \( w \), refining the HFK functional. An example is shown in figure 2.7.

\(^{13}\)It is tempting to suggest that the optimal functional in the large \( c \) limit has the form \( \partial_z e^{-a\partial_z^2}|_{z=0} \), for some \( c \)-dependent constant \( a \), but the latter by itself does not take the same sign when acting on arbitrarily high dimension characters as on the vacuum character, and thus cannot be used to derive an analytic bound on the gap.
Figure 2.7: A plot of $F_{\text{mod}}^{(N)}(u + iv, u - iv)$ with increasing derivative orders $N$ (as indicated by colors from green to red), at central charge $c = 5$.

### 2.3 Spin-dependent bounds

Using the complete basis of linear functionals (2.2.8), it is now possible to obtain more refined bounds on the spectrum that distinguish primaries of different spins. In this section, we present a number of results on the spin-dependent bounds.

#### 2.3.1 Gap in the spectrum of scalar primaries

By searching for linear functionals that satisfy the following positivity conditions

$$
\alpha \left[ \hat{\chi}_0(\tau) \hat{\chi}_0(\bar{\tau}) \right] > 0
$$

$$
\alpha \left[ \hat{\chi}_{-\Delta}(\tau) \hat{\chi}_{\Delta-}(\bar{\tau}) + \hat{\chi}_{\Delta+}(\tau) \hat{\chi}_{-\Delta}(\bar{\tau}) \right] \geq 0,
\begin{cases}
\Delta \geq \Delta_{s=0}, & s = 0 \\
\Delta \geq s, & s > 0
\end{cases}
$$

we can place upper bounds on the dimension of the lightest scalar primary operator, with no assumption on the spectra of other spins. Namely, if such a linear functional is found, we would learn that the gap in the
dimensions of scalar primaries must be bounded from above by $\Delta^*_s=0$. The smallest such $\Delta^*_s=0$, obtained using functionals up to derivative order $N$, will be denoted $\Delta^0_{s=0}(N)$. We can numerically extrapolate to infinite $N$, which results in the optimal bound $\Delta^0_{s=0}$ on the scalar gap. Our results are shown in figure 2.8.

Figure 2.8: Top-left: The upper bound on the gap in the dimension of scalar primaries obtained at increasing derivative order of the linear functional (from green to red, up to $N = 51$) as a function of the central charge. Top-right: The extrapolated bounds on the dimension of the lightest scalar operator $\Delta^0_{s=0}$ (red) as a function of the central charge, superposed with the bound on the overall gap $\Delta_{mod}$ (blue). For the bound on the scalar gap, the numerical extrapolation is performed by fitting $19 \leq N \leq 51$ bounds with a quadratic polynomial in $1/N$. Bottom: The extrapolated bound on the scalar gap (black) superposed with the bounds at fixed derivative orders (increasing from green to red) near the first two kinks.

By definition, $\Delta^0_{mod}(c) \geq \Delta_{mod}(c)$, and the two agree when $c \leq 4$ (where the bound is less than 1 and only scalars could lie below the bound). It appears that the discontinuity in the slope at $c = 4$ is absent in $\Delta^0_{mod}(c)$. However, there appear to be new kinks\(^\text{14}\) in the $\Delta^0_{mod}(c)$ curve when the bound attains integer

\(^{14}\)We would like to emphasize, however, that the extrapolation to infinite $N$ is not accurate enough to determine whether the apparent kinks at $\Delta^0_{mod} = 3, 4, \ldots$ represent genuine discontinuities in the derivative of the $\Delta^0_{mod}(c)$ curve.
values 2, 3, 4, \ldots. In particular, $\Delta^{s=0}_{\text{mod}}(c) < 2$ for $c < 8$, which implies that unitary CFTs with no conserved currents and $c < 8$ must admit relevant deformations. The kink at $c = 8$ and $\Delta^{s=0}_{\text{mod}} = 2$ is in fact realized by a Narain lattice CFT of 8 free compact bosons (even though this CFT contains conserved currents, its partition function is of generic type), as will be discussed, among other examples, in the next subsection.

We find no bound on the dimension of the lightest scalar operator for $c \geq 25$. At a given derivative order $N$, we denote by $c^{(N)}_{*}$ the central charge at which the scalar gap bound $\Delta^{s=0,N}_{\text{mod}}$ ceases to exist. As shown in figure 2.9, $c^{(N)}_{*}$ approaches 25 from below as $N \to \infty$.

The disappearance of an upper bound on the scalar gap for $c \geq 25$ has a very simple explanation. Our positivity criteria on the degeneracy of the primaries do not exclude the limit where the degeneracies of the primaries diverge, and the partition function becomes a divergent factor multiplied by the partition function of a noncompact CFT spectrum, namely that of a continuous spectrum with a finite density of states, with no $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ invariant vacuum. Indeed, consider the modular invariant function

$$Z(\tau, \bar{\tau}) = \frac{J(\tau) + \bar{J}(\bar{\tau})}{\tau^4 |\eta(\tau)|^2},$$

where $J(\tau)$ is related to the elliptic $j$-invariant $j(\tau)$ by $J(\tau) = j(\tau) - 744 = q^{-1} + 196884q + \mathcal{O}(q^2)$. When interpreted as the partition function of a noncompact CFT of central charge $c$, it admits a decomposition in
terms of Virasoro characters associated with primaries of nonzero spin and twist \( \geq \frac{c-25}{12} \). Thus, the absence of scalar primaries is consistent with modular invariance for \( c \geq 25 \).

### 2.3.2 Kinks and CFTs with generic type partition function

An obvious question is whether there are CFTs whose spectra saturate the dimension gap bounds. This turns out to be the case for a few values of the central charge, where the bound is saturated by a rational CFT whose partition function is of the generic type. These examples provide good consistency checks of our numerical methods.

- At \( c = 1 \), the bound \( \Delta_{\text{mod}} = \frac{1}{2} \) is saturated by the gap of the \( SU(2) \) WZW model at level 1, otherwise known as the free compact boson at the self-dual radius.
- At \( c = 2 \), the bound \( \Delta_{\text{mod}} = \frac{2}{3} \) is saturated by the \( SU(3) \) WZW model at level 1. This theory admits a description in terms of free bosons with target space \( T^2 \) at the \( \mathbb{Z}_3 \)-invariant point in both its Kähler and complex structure moduli spaces.
- At \( c = \frac{14}{5} \), the bound \( \Delta_{\text{mod}} = \frac{4}{5} \) is saturated by the \( G_2 \) WZW model at level 1.
- At \( c = 4 \), the bound \( \Delta_{\text{mod}} = 1 \) is saturated by the \( SO(8) \) WZW model at level 1. This theory also admits a description in terms of 8 free fermions with diagonal GSO projection. Note that this is the kink on the curve \( \Delta_{\text{mod}}(c) \).
- At \( c = 8 \), the bound \( \Delta_{\text{mod}}^{s=0} = 2 \) is saturated by the \( E_8 \) WZW model at level 1. This theory also admits a description in terms of 8 compact bosons at the holomorphically factorized point in its moduli space, where the holomorphic factor can be described as the Narain compactification on the root lattice of \( E_8 \). This is the first kink on the curve for the upper bound on the scalar dimension gap \( \Delta_{\text{mod}}^{s=0}(c) \).

In section 4, we will show that the spectra of these theories are in fact completely determined by the saturation of the gap bound.
2.3.3 Allowing for primary conserved currents

So far, we have focused on spectra with no conserved current primaries, together with the limiting case of generic type spectra, where conserved currents were allowed as long as their contributions to the partition function were combined with twist-2 primaries to give non-degenerate Virasoro characters. It is straightforward to relax this assumption by including degenerate characters of the form $\chi_j(\tau)\bar{\chi}_0(\bar{\tau})$ and $\chi_0(\tau)\bar{\chi}_j(\bar{\tau})$ in the partition function, as in (2.2.2), and try to rule out hypothetical spectra by seeking linear functionals that act non-negatively on the degenerate Virasoro characters as well as the non-degenerate characters present in the spectrum.

Before doing so, let us note that there are many constraints on the spectrum due to the associativity of the OPE that are not taken into account by modular invariance of the partition function alone. This is apparent in the presence of conserved current primaries$^{15}$: operators must form representations of an extended chiral algebra, and in particular, operators formed by taking product of left and right moving currents (e.g. of the form $J^a(z)\tilde{J}^b(\bar{z})$) are part of the spectrum.

In the discussion that follows, we will again assume that the spectrum is parity-invariant. If there are holomorphic and anti-holomorphic spin-1 currents in the CFT, then the spectrum consists of representations of a current algebra, whose characters always admit non-negative decompositions into non-degenerate Virasoro characters. In other words, if conserved spin-1 currents are present in a parity invariant CFT, then the partition function is necessarily of the generic type, except the case where there is a single $U(1)$ current algebra, which can easily be taken into account by replacing the vacuum Virasoro character by the $U(1)$ current algebra character.

We shall study the numerical bounds on the scalar dimension gap in the following three cases:

(I) Conserved current primaries of all spins are allowed. Due to the basic OPE constraints discussed above (under the assumption of a parity-invariant spectrum), it suffices to assume either there are no conserved

$^{15}$When the currents are not conserved, the constraints on their OPE are much more delicate.
spin-1 currents (i.e. case (II) below), or there is a $U(1)$ current (in both the left and right sector) while higher-spin primaries come with non-degenerate characters.

(II) Conserved current primaries of spins $j \geq 2$ are allowed.

(III) Conserved current primaries of spins $j \geq 3$ are allowed, i.e., we consider CFTs with a unique stress-energy tensor and no spin-1 currents.

The resulting upper bound on the scalar gap will be denoted $\Delta_{\text{mod}, j \geq 1}^s = 0 (N)$, $\Delta_{\text{mod}, j \geq 2}^s = 0 (N)$, and $\Delta_{\text{mod}, j \geq 3}^s = 0 (N)$ respectively.

In case (III), we find essentially no difference between $\Delta_{\text{mod}, j \geq 3}^s = 0 (N)$ and $\Delta_{\text{mod}}^s = 0 (N)$, up to the numerical error due to the finite resolution $\epsilon$ of the binary search implemented for determining the optimal bound,\textsuperscript{16} except for a tiny peak in their difference that is localized near $c \sim 12.5$ and narrows with increasing derivative truncation order $N$.

In cases (I) and (II), we find the same bounds $\Delta_{\text{mod}, j \geq 1}^s = 0 (N)$ and $\Delta_{\text{mod}, j \geq 2}^s = 0 (N)$ for the scalar gap, at sufficiently high derivative order $N$. For $N \geq 19$, we find no difference between $\Delta_{\text{mod}, j \geq 1}^s = 0 (N)$ and $\Delta_{\text{mod}}^s = 0 (N)$ (the latter obtained assuming the absence of conserved current primaries) for $c \leq 8$, up to our numerical resolution. A small difference $\Delta_{\text{mod}, j \geq 1}^s = 0 (N) - \Delta_{\text{mod}}^s = 0 (N)$ is found for $9 \lesssim c \lesssim 15$, as shown in figure 2.10, for $N$ up to 51. The numerics suggests that when conserved current primaries of all spins are allowed, the second kink near $c \sim 12.5$ for $\Delta_{\text{mod}}^s = 0 (c)$ may be shifted slightly to the left in the curve for $\Delta_{\text{mod}, j \geq 1}^s (c)$. Note that unlike the first kink in figure 2.8 at $c = 8$, we do not know of a candidate CFT that resides at this second kink.

In conclusion, apart from a small shift in the position of the second kink near $c \sim 12.5$, we do not find any significant weakening of the scalar gap bound $\Delta_{\text{mod}}^s = 0 (c)$ when conserved current primaries are allowed.

Unitary CFTs that admit only irrelevant deformations describe what are known as “perfect metals” [78]. It follows from our bound on the scalar dimension gap that perfect metals do not exist when the central

\textsuperscript{16}Here the resolution of our binary search for the optimal bound is taken to be $\epsilon = \frac{c-1}{48000}$.
### 2.3.4 Turning on a twist gap

As remarked in the introduction, while there is a nontrivial upper bound on the twist gap $t_{\text{mod}} = \frac{c-1}{12}$, we do not know any explicit construction of unitary, compact CFTs with $c > 1$ and nonzero twist gap.\(^{17}\) Obviously it would be of interest to exhibit such theories, if they are indeed as ubiquitous as one might expect (some candidates for irrational CFTs with no extra conserved currents have been considered in [72, 73]). Here we

\(^{17}\)Nor do we know one with zero twist gap but no conserved primary currents, i.e., a unitary, compact CFT with infinitely many non-conserved higher spin primaries whose twists accumulate to zero.
study the upper bound on the gap in the dimension of scalar primaries while imposing a nonzero gap in
the twist of all primary operators in the spectrum. This would in particular exclude theories that contain
conserved current primaries (regardless of whether their partition functions are of the generic type).

Figure 2.11 shows the bound on the scalar dimension gap $\Delta_{\text{mod}}(t_{\text{gap}})$ as a function of the twist gap $t_{\text{gap}}$, for various values of central charge $c$. As $t_{\text{gap}}$ is increased from 0 to its upper bound $t_{\text{mod}} = \frac{c-1}{12}$, $\Delta_{\text{mod}}(t_{\text{gap}})$ decreases smoothly (but interestingly, it never approaches $\frac{c-1}{12}$).

Figure 2.11: The gap in the dimension of scalar primaries as a function of the gap in the twist at increasing
derivative order (from green to red, up to $N = 47$) for $c = 4, 8, 12$.

If we further increase the twist gap for nonzero-spin primaries beyond $\frac{c-1}{12}$, the bound on scalar dimension
gap drops to $\frac{c-1}{12}$ as expected. An example of this is shown in figure 2.12.

Figure 2.12: The gap in the dimension of scalar primaries as a function of the gap in the twist of all nonzero-spin
primaries at increasing derivative order (from green to red, up to $N = 47$) for $c = 5$. The dotted lines denote
$t_{\text{gap}} = \frac{c-1}{12}$ and $\Delta_{\text{mod}} = \frac{c-1}{12}$.

As we will explain in sections 2.4.1 and 2.4.2, when the upper bound on the degeneracy of the lightest
operator is saturated, the rest of the spectrum (which we refer to as the “extremal spectrum”) is uniquely
determined by the zeroes of the optimal linear functional acting on the characters as a function of the
dimension. This provides a procedure to explicitly construct a spectrum for which the twist gap is nonzero.\(^\text{18}\)

As a proof of principle, in figure 2.13 we plot the optimal functional acting on spin-0 and spin-1 characters
for \(c = 4\) with the maximal twist gap and a value of the dimension gap close to the upper bound. The
resulting spectra with finite twist gap and maximal degeneracy of the lightest primary appear to be discrete
and are not obviously inconsistent.

![Figure 2.13: The optimal linear functional acting on spin-0 and spin-1 reduced characters with the maximal twist
gap imposed. The zeroes of this functional determine the dimensions of operators in the extremal spectrum.](image)

It is interesting to compare \(\Delta_{\text{mod}}^{s=0}(t_{\text{gap}})\) with the bound on scalar dimension gap when a nonzero twist gap
only for spin-1 primaries is introduced. We denote the latter bound by \(\tilde{\Delta}_{\text{mod}}^{s=0}(t_{\text{gap}})\). Obviously, by definition
\(\tilde{\Delta}_{\text{mod}}^{s=0}(t) \geq \Delta_{\text{mod}}^{s=0}(t)\). We find numerically that \(\tilde{\Delta}_{\text{mod}}^{s=0}(t)\) coincides with \(\Delta_{\text{mod}}^{s=0}(t)\) for \(t \leq \frac{c-1}{12}\). A transition
occurs at a larger value of \(t\), after which \(\tilde{\Delta}_{\text{mod}}^{s=0}(t)\) decays smoothly with \(t\) (potentially exponentially fast)
towards \(\frac{c-1}{12}\). We have not managed to obtain a reliable plot of the full curve of \(\tilde{\Delta}_{\text{mod}}^{s=0}(t_{\text{gap}})\) as the numerics
stabilize slowly with the truncation on derivative order \(N\), for an intermediate range of \(t_{\text{gap}}\).

\(^{18}\)However, this procedure does not on its own determine the degeneracies of the higher-dimension operators in the extremal spectrum.
2.4 Operator degeneracies and extremal spectra

2.4.1 Bounds on the degeneracy at the gap

If we impose a dimension gap $\Delta_{\text{gap}}$ on the spectrum (not to be confused with the upper bound on such a gap, which we denoted by $\Delta_{\text{mod}}$), we can use semi-definite programming to place universal bounds on the degeneracies of primary operators in such a CFT. In particular, if $\Delta_{\text{gap}}$ lies between the twist gap bound $\Delta_{\text{int}} = \frac{c-1}{12}$ and the upper bound on the dimension gap (that follows from modular invariance) $\Delta_{\text{mod}}$, one can place upper bounds on the degeneracies of primaries, as follows.

Fixing the dimension $\Delta_{\text{gap}}(\leq \Delta_{\text{mod}})$ of the lowest primary operator, consider all linear functionals $\rho$ of the form (2.2.8) such that

$$\rho \left[ \hat{\chi}_{\Delta-s}(\tau) \hat{\chi}_{\Delta+s}(\bar{\tau}) + \hat{\chi}_{\Delta+s}(\tau) \hat{\chi}_{\Delta-s}(\bar{\tau}) \right] \geq 0, \quad \Delta \geq \max \left( \Delta_{\text{gap}}, s \right).$$

(2.4.1)

Since the gap $\Delta_{\text{gap}}$ is allowed by the modular crossing equation, $\rho$ must be negative when acting on the vacuum character. We will normalize $\rho$ so that

$$\rho \left[ \hat{\chi}_0(\tau) \hat{\chi}_0(\bar{\tau}) \right] = -1.$$

(2.4.2)

The degeneracy\footnote{Of course, the coefficients $\{d_{\Delta,s}\}$ only have an interpretations as degeneracies of primary operators in the absence of conserved currents. In the presence of conserved currents, the $\{d_{\Delta,s}\}$ are simply the coefficients in the decomposition of the partition function into non-degenerate characters.} $d_{\Delta,s}$ of a primary of dimension $\Delta$ and spin $s$ is then subject to the following upper bound (this mirrors the upper bound on the squared OPE coefficients derived in the context of the four-point function bootstrap in $[80]$)

$$d_{\Delta,s} \leq \left( \rho \left[ \hat{\chi}_{\Delta-s}(\tau) \hat{\chi}_{\Delta+s}(\bar{\tau}) + \hat{\chi}_{\Delta+s}(\tau) \hat{\chi}_{\Delta-s}(\bar{\tau}) \right] \right)^{-1}.$$

(2.4.3)
Obviously, the optimal bound on the degeneracy would be obtained using the functional that maximizes 
\[ \rho \left[ \hat{\chi}_{\Delta_0} (\tau) \hat{\chi}_{\Delta_+} (\bar{\tau}) + \hat{\chi}_{\Delta_0} (\bar{\tau}) \hat{\chi}_{\Delta_+} (\tau) \right], \]
subject to the conditions (2.4.1) and (2.4.2).

We will illustrate this method starting with the special case of \( c = 1 \). Figure 2.14 shows the upper bound on the degeneracy of the lowest dimension scalar primaries above the vacuum as a function of the dimension gap \( \Delta_{\text{gap}} \) in the spectrum. As the derivative order \( N \) is increased, the degeneracy bound converges to \( \frac{3}{2} \) except for a sequence of peaks located at \( \Delta_{\text{gap}} = \frac{1}{2}, \frac{2}{5}, \frac{1}{8}, \ldots \) where the degeneracy bound is 4 or 2. This is in fact precisely consistent with what we know about the partition function of \( c = 1 \) CFTs, as we now explain.

\[ \text{c= 1} \]

\[ \begin{array}{c}
\text{degeneracy bound} \\
\hline
\text{N=15} \\
\text{N=31} \\
\text{N=47} \\
\text{\( \Delta_{\text{gap}} = \frac{2}{7}, N=47 \)} \\
\end{array} \]

**Figure 2.14:** The upper bound on the degeneracy of the lowest-lying operator for \( c = 1 \) as a function of the assumed gap in the spectrum for derivative orders \( N = 15, 31, 47 \).

To our knowledge, the only known unitary, compact \( c = 1 \) CFTs are the compact boson and its orbifolds [81]. The \( S^1/\mathbb{Z}_2 \) orbifold partition function is not of the generic type, due to the degenerate character of a conserved spin-4 current, except at the self-dual radius \( R = 1 \), where the CFT is equivalent to a compact boson at \( R = 2 \). The compact boson CFT at radius \( R \) has a reduced partition function of the form

\[ \hat{Z}_{\text{CB}}(R) = |\tau|^\frac{1}{2} \sum_{a,b \in \mathbb{Z}} q^{\frac{1}{4}(a+bR)^2} \bar{q}^{\frac{1}{4}(a-bR)^2}. \]

For \( R \geq 1 \), the gap in the spectrum is \( \Delta_{\text{gap}}(R) = \frac{1}{2\pi R} \). If we decompose the reduced partition function in terms of non-degenerate reduced characters, \( \hat{Z}_{\text{CB}}(R) \) generically has a single negative coefficient of \(-1\) at
weight \((1, 1)\). When the radius \(R\) is a half-integer \(R = \frac{n}{2}\), \(n \in \mathbb{Z}_{\geq 0}\) (or a \(T\)-dual equivalent thereof), however, the weight \((1, 1)\) coefficient becomes positive due to the appearance of extra marginal primaries, and thus the corresponding partition function is of the generic type to which our bounds apply despite the presence of conserved currents. When \(R = 1\), it is easy to see that the reduced partition function has degeneracy 4 at the gap:

\[
\hat{Z}_{\text{CB}}(1) = \hat{\chi}_0(\tau)\hat{\chi}_0(\bar{\tau}) + |\tau|^\frac{3}{2} \left[ 4(q\bar{q})^{\frac{1}{2}} + 3(q + \bar{q}) + 3qq + \ldots \right],
\]

(2.4.5)

which corresponds to the peak at \(\Delta_{\text{gap}} = \frac{1}{2}\) in figure 2.14. When the compactification radius takes another half-integer value, the degeneracy of the lowest-lying primary is 2:

\[
\hat{Z}_{\text{CB}}\left(\frac{n}{2}\right) = \hat{\chi}_0(\tau)\hat{\chi}_0(\bar{\tau}) + |\tau|^\frac{3}{2} \left[ 2(q\bar{q})^{\frac{1}{2}} + \ldots \right], \quad n \geq 3,
\]

(2.4.6)

which corresponds to peaks of the degeneracy bound at \(\Delta_{\text{gap}} = \frac{2n}{\pi^2}\) for \(n \geq 3\), as seen in figure 2.14.

At first sight, it might seem odd that the degeneracy bounds in figure 2.14 approach \(\frac{3}{2}\) at a generic value of \(\Delta_{\text{gap}}\). Recall that the compact boson reduced partition function will generically have a negative coefficient \(-1\) at weight \((1, 1)\) in its decomposition into non-degenerate reduced characters. On the other hand, from (2.4.5), the reduced partition function at the self-dual radius \(R = 1\) has a coefficient 3 at weight \((1, 1)\), and so the linear combination

\[
\frac{3}{4} \hat{Z}_{\text{CB}}(R) + \frac{1}{4} \hat{Z}_{\text{CB}}(1) = \hat{\chi}_0(\tau)\hat{\chi}_0(\bar{\tau}) + |\tau|^\frac{3}{2} \left[ \frac{3}{2}(q\bar{q})^{\frac{1}{2}} + \ldots \right]
\]

(2.4.7)

has non-negative coefficients; it is a partition function of the generic type. Furthermore, we see that the “degeneracy” at the gap \(\Delta_{\text{gap}} = \frac{1}{2R^2}\) is \(\frac{3}{2}\), as seen in figure 2.14.\(^{20}\) In this case, a more refined bound would be obtained if we demand that the degeneracies are integers.

\(^{20}\)One could also have considered the linear combinations \(\frac{3}{4} \hat{Z}_{\text{CB}}(R) + \frac{1}{4} \hat{Z}_{\text{CB}}(2)\) (for \(R > 2\)) or \(\frac{1}{2} \hat{Z}_{\text{CB}}(R) + \frac{1}{2} \hat{Z}_{\text{CB}}\left(\frac{R}{2}\right)\) for \(n = 3, 5, 6, 7, \ldots\) (for \(R > \frac{n}{2}\)), but the conclusion that \(\frac{3}{2}\) is the maximal leading coefficient is unchanged.
The bounding curves on the degeneracies of the lightest primary operators at larger values of the central charge are shown in figure 2.15. Note that the bound diverges as \( \Delta_{\text{gap}} \) approaches \( \frac{c-1}{12} \), as expected.\(^{21}\)

**Figure 2.15:** The upper bound on the degeneracy of the lowest-lying operators as a function of \( \Delta_{\text{gap}} \) for a few values of the central charge, and the growth of the degeneracy at the maximal gap \( \Delta_{\text{mod}} \) as a function of the central charge. In the bottom-right plot, the black points denote the special theories discussed in section 2.3.2.

### 2.4.2 Extremal spectrum from the optimal linear functional

When the degeneracy bound at the gap \( \Delta_{\text{gap}} \) is saturated (for \( \frac{c-1}{12} < \Delta_{\text{gap}} \leq \Delta_{\text{mod}} \)), the entire spectrum of the CFT is in fact determined \(^{22}\), for reasons we explain below. Such a spectrum will be called “extremal”, and the corresponding reduced partition function will be denoted \( \hat{Z}_{\text{ext}}(c, \Delta_{\text{gap}}) \).\(^{22}\) The spectrum of spin-\( s \) primaries will be denoted \( \mathcal{I}_{\text{s,ext}} \). The optimal functional \( \rho \) we use to determine the degeneracy bound,

\(^{21}\)This can be understood by noting that the partition function of Liouville theory, with gap \( \frac{c-1}{12} \), may be viewed as an infinite degeneracy limit of a compact CFT partition function.

\(^{22}\)This definition of the extremal spectrum does not guarantee that the degeneracies of operators are integers; in fact, the latter occurs only for a discrete set of values of \( c \) and \( \Delta_{\text{gap}} \), and it is obviously only in these cases that the extremal spectrum could potentially be realized by a physical CFT.
as in (2.4.3), satisfies

\[ 0 = \rho \left[ \hat{\chi}_0(\tau) \hat{\chi}_0(\bar{\tau}) + \sum_s \sum_{\Delta \in I^\text{ext}_s} d_{\Delta,s} \left( \hat{\chi}_{\Delta-s}^+(\tau) \hat{\chi}_{\Delta-s}^- (\bar{\tau}) + \hat{\chi}_{\Delta-s}^-(\tau) \hat{\chi}_{\Delta-s}^+ (\bar{\tau}) \right) \right] \]

\[ = \sum_s \sum_{\Delta \in I^\text{ext}_s, \Delta > \Delta_{\text{gap}}} d_{\Delta,s} \rho \left[ \hat{\chi}_{\Delta+s}^+(\tau) \hat{\chi}_{\Delta-s}^- (\bar{\tau}) + \hat{\chi}_{\Delta-s}^- (\tau) \hat{\chi}_{\Delta+s}^+ (\bar{\tau}) \right]. \tag{2.4.8} \]

Here, the contribution of the vacuum and that of the primaries at the gap cancel, due to the saturation of the degeneracy bound at the gap. Positivity of the coefficients \(d_{\Delta,s}\) and (2.4.1) then constrain the extremal spectrum to be such that the corresponding characters are annihilated by the linear functional \(\rho\), namely

\[ \rho \left[ \hat{\chi}_{\Delta+s}^+(\tau) \hat{\chi}_{\Delta-s}^- (\bar{\tau}) + \hat{\chi}_{\Delta-s}^- (\tau) \hat{\chi}_{\Delta+s}^+ (\bar{\tau}) \right] = 0, \quad \Delta \in I^\text{ext}_s, \quad \Delta > \Delta_{\text{gap}}. \tag{2.4.9} \]

Indeed, the extremal spectrum can be efficiently computed in this way. We will begin with the \(c = 1\) example. Figure 2.16 shows the value of the optimal functional acting on spin-0 reduced characters. For \(\Delta_{\text{gap}} = \frac{2}{n^2}\), the zeroes of the optimal functional correspond exactly to the scaling dimensions of scalar operators in the reduced compact boson partition function at radius \(R = \frac{n}{2}\) (2.4.6). On the other hand, at generic values of \(\Delta_{\text{gap}}\), the zeroes of the optimal functional precisely correspond to the scaling dimensions of scalar primaries in two different compact boson CFTs, whose partition functions combine to give one of generic type (but with non-integer coefficients) as in (2.4.7). Repeating this exercise with the optimal linear functional acting on nonzero spin characters also reveals zeroes at the locations predicted by the corresponding reduced compact boson partition functions (2.4.6,2.4.7). See figure 2.17 for some spin-1 examples.

One can also investigate the upper bound on the degeneracies of operators of higher dimensions that appear in the extremal spectrum. To do this, one fixes the gap and maximizes the action of the linear functional subject to (2.4.1,2.4.2) on the character corresponding to the higher-dimension operator of interest (rather than that of the operator whose dimension saturates the assumed gap). A priori, it need not be the case that a CFT that realizes the extremal spectrum also maximizes the degeneracies of the operators other than
Figure 2.16: The optimal linear functional that maximizes the degeneracy at the gap, acting on the spin-0 reduced characters. The zeroes of the optimal functional correspond to scalar operators in the extremal spectrum. The dotted lines correspond to the “dimensions of scalar operators” in the corresponding generic type compact boson reduced partition functions.

For the lowest-lying (subject to the assumption that the gap in the spectrum is $\Delta_{\text{gap}}$). However, for $c = 1$, fixing the gap and maximizing the degeneracies of other operators in the extremal spectra, we find upper bounds that agree with the corresponding coefficients in the reduced compact boson partition functions of generic type (2.4.6,2.4.7).
2.4.3 Extremal spectra with maximal gap

Let us now consider the extremal spectra at higher values of the central charge when the gap is maximized. For $1 \leq c \leq 4$, curiously, we find that conserved spin-1 currents and marginal scalar primaries generically occur in the extremal spectra when the gap is maximized — figure 2.18 shows the evidence for this for a few values of the central charge.

Given the generic presence of conserved spin-1 currents and marginal scalar primaries in the extremal spectra when the upper bound on the dimension gap is saturated, one may then ask what the maximal degeneracies of these operators are subject to the maximal gap. Figure 2.19 shows the growth of the upper bound of the degeneracy of spin-1 conserved currents subject to the maximal gap as a function of the central charge.

Given the presence of spin-1 conserved currents, the upper bound obtained from (2.4.3) with $(\Delta, s) = (2, 0)$ does not exactly bound the number of marginal scalar primaries in the extremal spectrum: rather, it is a bound on the coefficient of $\hat{\chi}_1 \hat{\bar{\chi}}_1$ in the decomposition of the reduced partition function into non-degenerate reduced characters. To obtain the bound on the number of marginal scalar primary operators, one must sum...
Figure 2.18: The action of the optimal functional on spin-0 and spin-1 reduced characters when the degeneracy of operators at the gap is maximized and the dimension gap bound is saturated for $c > 1$. The dotted lines highlight the gap and the presence of marginal scalar primaries in the spin-0 case, and conserved spin-1 currents in the spin-1 case.

Figure 2.19: Left: The extrapolated upper bound on the total (holomorphic and antiholomorphic) number of conserved spin-1 currents as a function of the central charge when the gap is maximized. Right: The extrapolated upper bound on the number of marginal scalar primaries as a function of the central charge. This is obtained by adding the naive bound obtained from (2.4.3) with $(\Delta, s) = (2, 0)$, which assumes a decomposition of the partition function into non-degenerate characters, to the bound on the total number of conserved spin-1 currents, cf., $N_1(q + \bar{q}) + N_0q\bar{q} = \frac{\Delta}{2\Delta} [q(1 - \bar{q}) + \bar{q}(1 - q)] + (N_0 + N_1)q\bar{q}$. In both plots, the black points denote the special theories discussed in section 2.3.2. In particular, the extremal spectra at central charge $c = 1, 2, \frac{14}{7}, 4$ are realized respectively by the $SU(2)$, $SU(3)$, $G_2$ and $SO(8)$ WZW models at level 1.

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up the bound obtained from (2.4.3) together with the bound on the degeneracy of spin-1 conserved currents. The resulting upper bound on the degeneracy of marginal scalar primaries is plotted as a function of the central charge in figure 2.19.

Of course, only when the upper bounds on the degeneracies at all weights converge to integers can the extremal spectrum with maximal gap be realized by a physical conformal field theory. Based on the locations of the zeroes of the optimal functional and the values of the maximal degeneracies, we may then attempt to guess the CFTs that realize the extremal spectra with maximal gap. Let’s consider in particular the case of \( c = 2, \Delta_{\text{gap}} = \frac{2}{3} \). The upper bound on the degeneracy of scalar primaries at the gap converges to 18, while the number of \( \Delta = 2 \) scalar primaries is bounded above (in the case that the gap is \( \Delta_{\text{gap}} = \frac{2}{3} \)) by 64. From figure 2.18, we see that the extremal spectrum also contains scalar primaries of dimension \( \frac{8}{3}, 4, \frac{14}{3}, \ldots \). The maximal degeneracies of these scalar primaries subject to \( \Delta_{\text{gap}} = \frac{2}{3} \) predicted by (2.4.3) are 72, 64, 450, ..., respectively. The extremal spectrum that saturates these bounds is that of the \( SU(3) \) WZW model at level 1. The partition function of this theory admits a decomposition into non-degenerate characters with precisely the operator dimensions of the \( \Delta_{\text{gap}} = \frac{2}{3} \) extremal spectrum, with integer coefficients equal to the predicted maximal degeneracies:

\[
(q \bar{q})^{\frac{1}{3}} \left\{ \hat{Z}_\text{ext} \left( 2, \frac{2}{3} \right) - \hat{\chi}_0(\tau) \hat{\bar{\chi}}_0(\bar{\tau}) \right\} = |\tau|^{\frac{1}{2}} \left[ 18(q \bar{q})^{\frac{1}{2}} + 8(q + \bar{q}) + 36(q^2 \bar{q}^2 + q^2 \bar{q}^2) + 8(q^2 + \bar{q}^2) + 48q\bar{q} + \ldots \right].
\] (2.4.10)

Let us now consider the theory that lives at the kink of the bounding curve in figure 2.4, with \( c = 4 \) and \( \Delta_{\text{gap}} = 1 \). The upper bound on the number of \( \Delta = 1 \) scalar primaries converges to 192, while the bound on \( \Delta = 2 \) scalar primaries converges to 784. Furthermore, the corresponding optimal functional has zeroes at every integer dimension when acting on spin-0 and spin-1 reduced characters. In fact, this theory is nothing
but the CFT of 8 free fermions (with diagonal GSO projection), with partition function

\[
Z_{\text{ext}}(4, 1) = \frac{1}{2} \left( \left| \frac{\Theta_2(\tau)}{\eta(\tau)} \right|^8 + \left| \frac{\Theta_3(\tau)}{\eta(\tau)} \right|^8 + \left| \frac{\Theta_4(\tau)}{\eta(\tau)} \right|^8 \right),
\]

(2.4.11)

where the \(\{\Theta_a(\tau)\}\) are the Jacobi theta functions. The expansion of this extremal partition function into non-degenerate characters yields precisely the operator spectrum and degeneracies predicted by (2.4.3) and the zeroes of the optimal functional:

\[
(q\bar{q})^{\frac{1}{2}} \left\{ \tilde{Z}_{\text{ext}}(4, 1) - \tilde{\chi}_0(\tau) \tilde{\chi}_0(\bar{\tau}) \right\} = |\tau|^\frac{1}{2} \left[ 28(q + \bar{q}) + 192(q\bar{q})^\frac{1}{2} + 105(q^2 + \bar{q}^2) + 728q\bar{q} + 1344(q^\frac{3}{2} \bar{q}^\frac{3}{2} + q^\frac{3}{2} \bar{q}^\frac{3}{2}) + \ldots \right].
\]

(2.4.12)

In particular, the 28 holomorphic conserved spin-1 currents correspond to the 28 fermion bilinears.

Another illustrative example is the case of the \(c = 8\) theory that saturates the upper bound on the gap for scalar primaries, \(\Delta_{\text{gap}}^{s=0} = 2\), populating the first kink in figure 2.8. Figure 2.20 shows the action of the optimal functional that maximizes the degeneracy of the dimension-two scalar operators on the spin-0 and spin-1 reduced characters. Notice that the functional has zeroes when acting on spin-0 characters of even integer dimensions and zeroes at odd integer dimensions when acting on spin-1 characters. Furthermore, maximizing the degeneracy of \(\Delta = 2\) scalar primaries subject to the scalar gap \(\Delta_{\text{gap}}^{s=0} = 2\) reveals an upper limit.
bound of 61504 marginal scalar primaries. In fact, the extremal spectrum is nothing but that of 8 compact bosons on the $\Gamma_8$ Narain lattice, with holomorphically factorized partition function\footnote{We use the notation \(Z_{\text{ext}, s=0}(c, \Delta_{\text{gap}}^{s=0})\) to refer to the partition function of the CFT with the degeneracy of scalar primaries saturating the scalar gap $\Delta_{\text{gap}}^{s=0}$ maximized.}

\[
Z_{\text{ext}, s=0}(8, 2) = (j(\tau)\bar{j}(\bar{\tau}))^{\frac{1}{2}}.
\]  

(2.4.13)

The partition function (2.4.13) admits a decomposition into non-degenerate characters with a spectrum of primary operators and non-negative integer coefficients predicted by the maximal degeneracies (2.4.3) and the zeroes of the optimal functional

\[
(q\bar{q})^{\frac{1}{24}} \left\{ Z_{\text{ext}, s=0}(8, 2) - \hat{\chi}_0(\tau)\hat{\chi}_0(\bar{\tau}) \right\} \\
= |\tau|^{\frac{1}{2}} \left[ 248(q + \bar{q}) + 3875(q^2 + \bar{q}^2) + 61008q\bar{q} + 30380(q^3 + \bar{q}^3) + 957125(q^2\bar{q} + q\bar{q}^2) + \ldots \right].
\]  

(2.4.14)

2.5 Discussion and open questions

By optimizing the linear functional acting on the modular crossing equation, we have uncovered a surprisingly rich set of constraints on the spectrum. However, the semi-definite programming approach becomes difficult at large values of the central charge: while we have concluded that the asymptotic slope of $\Delta_{\text{mod}}(c)$ at large $c$ lies between $\frac{1}{12}$ and $\frac{1}{9}$, we still do not know its accurate value (which amounts to an upper bound on the mass of the lightest massive particle in a theory of quantum gravity in $AdS_3$ in Planck units [54]). We have identified the shape of the optimal linear functional numerically, and hopefully this will eventually lead to an analytic derivation of the optimal bound on the dimension gap at large $c$.

It is nonetheless clear from our results that, at large $c$, the basis (2.2.7) is inefficient for representing the optimal linear functional. Presumably, the latter is more appropriately expressed as an integral transform, rather than derivatives taken at $\tau = -\bar{\tau} = i$. Our preliminary attempt at a multi-point bootstrap approach...
[83, 84] has yielded results consistent with $\Delta_{\text{mod}}$, but it did not improve the numerical efficiency due to the need for a polynomial approximation of functions of the conformal weight in implementing the semi-definite programming with SDPB.

In deriving most of our modular constraints, we have ignored the requirement that the degeneracies are integers. For instance, if the degeneracy at the maximal gap is not an integer, demanding that the degeneracy takes an integer value would slightly lower the upper bound on the gap. However, since the degeneracy bound grows exponentially with the central charge, the improvement of the bound by demanding integral degeneracy at the gap seems inconsequential.

One can also place bounds on the gap in the spectrum by considering the OPE of a pair of primaries (say at the gap), and using the crossing equation of the sphere 4-point function, by considering the decomposition of the 4-point function in Virasoro conformal blocks and imposing positivity of the coefficients. This is currently being investigated. Ultimately, one would like to combine the crossing equation for the sphere 4-point function with the modular covariance of the torus 1-point function. Perhaps the most efficient way to do this, instead of considering the crossing equations that involve many external operators, is to study the modular constraints from higher genus partition functions.
3 Bootstrapping the spectral function

3.1 Introduction

Enormous progress in the conformal bootstrap program has been made in recent years based on semidefinite programming \([14, 21, 22, 85–102]\). Typically, one aims to bound the scaling dimensions and OPE coefficients of the first few operators in the spectrum based on unitarity and crossing invariance of the 4-point function. Such methods are most powerful in constraining CFTs with simple low lying spectrum, but become less constraining when the spectrum becomes dense.

In this chapter we introduce the spectral function method, which allows for constraining not just the gap or the first few OPE coefficients but the distribution of OPE coefficients over a wide range of scaling dimensions.
While the method can be applied to CFTs in any spacetime dimension, we focus on a particular application to two-dimensional unitary CFTs. There the spectral functions are defined by truncating the Virasoro conformal block decomposition of a 4-point function in the scaling dimension of the internal primaries, evaluated at the self-dual cross ratio, or by truncating the Virasoro character decomposition of a partition function in the scaling dimension of the primaries, evaluated at the self-dual modulus. More precisely, consider a scalar 4-point function \[ g(z, \bar{z}) \equiv \langle \phi(0)\phi(1)\phi(z, \bar{z})\phi'(\infty) \rangle = \sum_{s, \Delta} C^2_{s, \Delta} F_{s, \Delta}(z, \bar{z}), \tag{3.1.1} \]

where \( F_{s, \Delta}(z, \bar{z}) \) are Virasoro conformal blocks for an internal primary of dimension \( \Delta \) and spin \( s \). The corresponding spectral function is defined by truncating the Virasoro conformal block decomposition of the four-point function \[ f(x) = \frac{1}{g(z = \bar{z} = \frac{1}{2})} \sum_{s, \Delta \leq x} C^2_{s, \Delta} F_{s, \Delta}(z = \bar{z} = \frac{1}{2}). \tag{3.1.2} \]

Of course, for a compact CFT with a discrete spectrum, \( f(x) \) will be composed of step functions. If the CFT is non-compact, then typically \( f(x) \) will be a monotonically increasing smooth function that takes value between 0 and 1. We will see that the crossing equation \[ \sum_{\Delta, s} C^2_{s, \Delta} [F_{s, \Delta}(z, \bar{z}) - F_{s, \Delta}(1 - z, 1 - \bar{z})] = 0 \tag{3.1.3} \]

combined with additional assumptions on the spectrum lead to upper and lower bounds on \( f(x) \). Numerically, the crossing equation can be utilized by applying to (3.1.3) linear functionals spanned by the basis \( \partial^n \partial^m_z |_{z=\bar{z}=1/2} \) for odd \( n + m \leq N \). The resulting upper and lower bounds on \( f(x) \) (which are rigorous bounds although not optimal) will be denoted \( f^+_N(x) \) and \( f^-_N(x) \).

If we make the assumption that the CFT contains only scalar Virasoro primaries, we find that \( f^+_N(x) \) and

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1In this chapter, our definition of a 2d CFT includes consistency on all Riemann surfaces, which follows from crossing symmetry and modular covariance of 1-point functions on the torus [37].

2The notation \( \phi'(\infty) \) stands for \( \lim_{z, \bar{z} \to \infty} z^{2h_0} \bar{z}^{2\bar{h}_0} \phi(z, \bar{z}) \).
\( f_N^- (x) \) become closer as \( N \) increases, for various values of the central charge \( c > 1 \). We conjecture that both converge to the spectral function of Liouville theory, which can be computed by integrating the square of the DOZZ structure constants \([56, 57]\) times the Virasoro conformal blocks. Note that this approach can be extended to the 4-point function involving a pair of different primaries, leading to spectral functions that encode the most general structure constants of the CFT.

Convergence of the upper and lower bounds \( f_N^\pm (x) \) to the same value \( f_\infty (x) \) is related to the completeness of the derivatives of scalar Virasoro blocks in a suitable space of functions. Conversely, this completeness statement implies the uniqueness of the solution to the crossing equations. We propose a numerical test for the completeness and find compelling evidence suggesting that it holds. We moreover obtain numerical approximations to the (conjecturally) unique solution to the crossing equations, which reproduce the DOZZ spectral function with high accuracy. In contrast to the semidefinite methods, this linear approach does not rely on the assumption that the OPE coefficients are real. The linear and semidefinite results above therefore lead us to conjecture that the DOZZ structure constants are the unique solution to the crossing equations for (not necessarily unitary) CFTs with only scalar primaries (of non-negative scaling dimensions) and \( c > 1 \).

Interestingly, we find that the bounds on the spectral function \( f_N^\pm (x) \) exist for external operator dimensions \( \Delta_\phi \geq \frac{c-1}{16} \left( \frac{3}{4} \right) \) of the Liouville threshold), and converge to a step function when \( \Delta_\phi \) is equal to \( \frac{c-1}{16} \). When \( \Delta_\phi < \frac{c-1}{16} \), the crossing equation cannot be satisfied with only scalar internal primaries, ruling out the possibility of such operators.\(^4\) We will see that all of these are in agreement with the analytic continuation of Liouville 4-point functions.

A caveat in the above uniqueness claim is that we have assumed a non-degenerate scalar spectrum. If degeneracies are allowed, then the operator algebra of Liouville CFT tensored with a topological quantum field theory (TQFT) (or equivalently, a finite dimensional commutative non-unital Frobenius algebra) would

\(^3\)However, it should be stressed that, unlike the semidefinite method, it is not entirely rigorous. Furthermore, it only ever works if we expect a unique solution. Therefore, it should be regarded as complementary to the much more general semidefinite method.

\(^4\)This was also observed in unpublished work of Balt van Rees.
also solve the crossing equation. In fact, such a TQFT can always be “diagonalized” by a basis change, and amounts to superselection sectors. We will give partial arguments suggesting that under our assumptions, “Liouville $\otimes$ TQFT” is the only possibility.

If we further invoke modular invariance, it will turn out that demanding that a unitary CFT contains only primaries of spins in a finite range ($s \leq s_{\text{max}}$ for some finite $s_{\text{max}}$) implies that the CFT must have a non-compact spectrum with only scalar primaries, and that the spectral density $\rho(\Delta)$ must be that of Liouville theory, namely

$$\rho(\Delta) \propto \frac{1}{\sqrt{\Delta - \frac{c-1}{24}}}.$$  

(3.1.4)

This leads us to conjecture that Liouville theory is the unique unitary CFT with $c > 1$ whose primaries have bounded spins.

The spectral function method also can be applied to modular bootstrap. In this context, we write the torus partition function as

$$Z(\tau, \bar{\tau}) \equiv \text{Tr} q^{L_0} \hat{\Delta} q^{ar{L}_0} \hat{\bar{\Delta}} = \sum_{\Delta, s} d_{\Delta, s} \chi_{\Delta, s}(\tau, \bar{\tau}),$$

(3.1.5)

where $\chi_{\Delta, s}$ is the Virasoro character associated with a primary of dimension $\Delta$ and spin $s$ and $d_{\Delta, s}$ is the degeneracy. The modular spectral function is defined by truncating the Virasoro character decomposition of the partition function

$$f_{\text{mod}}(x) = \frac{1}{Z(\tau = -\bar{\tau} = i)} \sum_{\Delta, s, \Delta \leq x} d_{\Delta, s} \chi_{\Delta, s}(\tau = -\bar{\tau} = i).$$

(3.1.6)

Once again, upper and lower bounds $f_{\text{mod}, N}(x)$ can be obtained by acting on the modular crossing equation

$$\sum_{\Delta, s} d_{\Delta, s} [\chi_{\Delta, s}(\tau, \bar{\tau}) - \chi_{\Delta, s}(-1/\tau, -1/\bar{\tau})] = 0$$

(3.1.7)

with linear functionals spanned by the basis $(\tau \partial \tau)^n (\bar{\tau} \partial \bar{\tau})^m |_{\tau = -\bar{\tau} = i}$, for odd $n + m \leq N$. In [101] (improving...
upon \([54, 55]\)), an upper bound \(\Delta_{\text{mod}}(c)\) on the gap in the scaling dimensions was computed numerically as a function of the central charge \(c\). When this bound is saturated, the entire spectrum is fixed by modular invariance, and is determined by the zeros of the optimal linear functional acting on the Virasoro characters. We will see in examples of small \(c\) (between 2 and 8) that under the assumption of maximal dimension gap, \(f^+_{\text{mod},N}(x)\) and \(f^-_{\text{mod},N}(x)\) converge with increasing \(N\) to step functions, corresponding to the spectral functions of known theories.

For larger values of \(c\), even when the dimension gap is maximized, the convergence of the bounds \(f^\pm_{\text{mod}}(x)\) to a sum of step functions is difficult to see numerically, because a good approximation of the optimal linear functional requires larger values of \(N\), and because the step function feature becomes invisible due to an exponentially large spectral density. Nonetheless, for \(50 \leq c \leq 300\), we find empirically that the horizontal average \(\overline{f}_{\text{mod},N}(x)\) of the upper and lower bounds converges rather quickly with \(N\), and the result is in good agreement with the total contribution from thermal \(AdS_3\) and BTZ black hole \([44]\) to the gravity partition function, which results in the modular spectral function

\[
\overline{f}_{\text{mod}}(x) = \frac{3}{4} + \frac{1}{4} \text{Erf} \left( \sqrt{\frac{6\pi}{c}} (x - \frac{c}{6}) \right) + \left( \frac{1}{c} \text{ corrections} \right).
\] (3.1.8)

Note that this asymptotic spectral function at large \(c\) is nontrivial when the dimension \(x\) lies in a window of width \(\sim \sqrt{c}\) around \(c/6\). The agreement with the numerical bounds confirms the validity of the effective field theory of pure gravity in \(AdS_3\) in the canonical ensemble, for temperatures of order 1 in AdS units.

Curiously, BTZ black holes corresponding to operators of scaling dimension \(\Delta\) in the range \(\frac{c}{12} < \Delta < \frac{c}{6}\) never dominate the canonical ensemble, and yet have macroscopic (AdS scale) horizon, provided that \(\Delta - \frac{c}{12}\) scales with \(c\). While the naive expectation from effective field theory is that the Bekenstein-Hawking entropy formula should be a valid counting of the microstates of such BTZ black holes, it is unclear to us whether this is a universal property of CFTs with sufficiently large gap.\(^5\) In principle, the modular spectral function

\[5\]Such a universality would in particular require the dimension gap bound \(\Delta_{\text{mod}}(c)\) to have asymptotic slope \(\frac{1}{12}\), namely \(\frac{d\Delta_{\text{mod}}(c)}{dc} \to \frac{1}{12}, \quad c \to \infty\), which is not ruled out by the result of \([101]\) but remains unproven (with no numerical evidence either).
bounds at large $c$ should either confirm or disprove such statements. To probe the density of states in the regime $\Delta = yc$ for $\frac{1}{12} < y < \frac{1}{6}$ and large $c$ would require exponential precision in determining the modular spectral function, which is beyond our current numerical capability.

This chapter is organized as follows. In section 2 we introduce the spectral function for the scalar 4-point function in a 2D CFT, and explain how to obtain upper and lower bounds $f_{N}^{\pm}(x)$ from semidefinite programming. We then specialize to the case where only scalar primaries are present, and demonstrate the convergence of the bounds toward the Liouville spectral function. In section 3, we examine the completeness of scalar Virasoro conformal blocks which would be implied by the aforementioned convergence, and we give numerical evidence that the completeness indeed holds. We then present analytic arguments based on modular invariance that a unitary CFT with $c > 1$ and Virasoro primaries of bounded spin must be a non-compact CFT with the same spectral density as that of Liouville. This together with the result of section 2 strongly supports the conjecture that Liouville theory is the only CFT with bounded spins. In section 4, we analyze the numerical bounds on the modular spectral function in a number of examples. We conclude with a discussion on the universality of the BTZ spectral density in large-$c$ CFTs with large gaps.

### 3.2 Spectral function bounds from semidefinite programming

#### 3.2.1 A sphere four-point spectral function

We begin by considering the conformal block decomposition of the sphere four-point function of a pair of scalar Virasoro primary operators $\phi_1, \phi_2$ of dimensions $\Delta_1, \Delta_2$,

$$ g_{12}(z, \bar{z}) = \langle \phi_1(z, \bar{z}) \phi_2(0) \phi_2(1) \phi_1'(\infty) \rangle $$

$$ = \sum_{s=0}^{\infty} \sum_{\Delta \in \mathbb{Z}_{12}, s} C_{12, s, \Delta}^{2} \mathcal{F}_{12, s, \Delta}(z, \bar{z}). \quad (3.2.1) $$
Here \( \mathcal{I}_{12,s} \) is the set of scaling dimensions of spin-s primary operators in the \( \phi_1 \phi_2 \) OPE and \( C_{12,s,\Delta} = C_{\phi_1 \phi_2 \mathcal{O}} \) is the OPE coefficient corresponding to the fusion of \( \phi_1 \) and \( \phi_2 \) into the primary \( \mathcal{O} \) with dimension \( \Delta \) and spin \( s \).\(^6\) The OPE coefficients are real in a unitary CFT. The conformal block \( F_{12;\Delta,s} \) takes the form

\[
F_{12;\Delta,s}(z, \bar{z}) = \frac{1}{1 + \delta_{s,0}} \left[ F_e^{\text{Vir}} \left( \frac{\Delta_1 + \Delta_2 - s + 8}{2}; z \right) F_e^{\text{Vir}} \left( \frac{\Delta_1 + \Delta_2 - s + 8}{2}; \bar{z} \right) \right. \\
+ \left. F_e^{\text{Vir}} \left( \frac{\Delta_1 + \Delta_2 - s + 8}{2}; \bar{z} \right) F_e^{\text{Vir}} \left( \frac{\Delta_1 + \Delta_2 - s + 8}{2}; z \right) \right]
\] (3.2.2)

where \( F_e^{\text{Vir}}(h_1, h_2, h_3, h_4; h; z) \) is the holomorphic Virasoro conformal block with external primaries of weight \( h_i \) and an internal primary of weight \( h \), in a CFT with central charge \( c \). Note that in writing the four-point function this way we have assumed a parity-invariant spectrum.\(^7\) An efficient method for computing Virasoro conformal blocks is Zamolodchikov’s recurrence relation \([57, 59]\), which we review in Appendix A.1. It computes the blocks as expansions in the “nome” \( q(z) \), defined as

\[
q(z) \equiv \exp(i\pi \tau(z)), \quad \tau(z) \equiv \frac{iF(1-z)}{F(z)}, \quad F(z) = 2F_1(1/2, 1/2, 1|z).
\] (3.2.3)

Note that as \( z \) ranges over the complex plane, \( q(z) \) takes value in an eye-shaped region on the unit disc, and the expansion of a conformal block in \( q \) converges on the entire unit disc. In the numerical approach, we apply Zamolodchikov’s recurrence relation up to a finite depth \( d_q \), which generates the correct \( q \)-series coefficients up to order \( q^{d_q} \). We then truncate the conformal block to this order as an approximation of the exact block.

It follows from the associativity of OPE that the four-point function is crossing symmetric, which amounts to the crossing equation

\[
\sum_{s=0}^{\infty} \sum_{\Delta \in \mathcal{I}_{12,s}} C_{12;s,\Delta}^2 [F_{12;\Delta,s}(z, \bar{z}) - F_{12;\Delta,s}(1 - z, 1 - \bar{z})] = 0.
\] (3.2.4)

\(^6\)When the operator spectrum is degenerate, \( C_{12;\Delta,s}^2 \) would be replaced by the sum of squares of OPE coefficients of all primaries of dimension \( \Delta \) and spin \( s \).

\(^7\)In what follows we specialize to the case where the spectrum only has scalar primary operators, so this distinction is trivial.
This relation puts highly nontrivial constraints on the spectrum and OPE coefficients of the CFT, some of which were analyzed in [68, 77, 100, 102, 103]. In previous works, one typically either focuses on a limit of the crossing equation in the cross ratio and extracts asymptotic properties of the spectrum, or numerically bounds the scaling dimension and OPE coefficients of the first few operators from the positivity assumption on $C_{12;\Delta,s}^2$.

We now introduce a “spectral function” that captures the distribution of OPE coefficients over a range of scaling dimensions of primaries in the $\phi_1\phi_2$ OPE, defined through the conformal block decomposition of the four-point function evaluated at the crossing-symmetric point $z = \bar{z} = \frac{1}{2}$, truncated on the dimension of internal primary operators:

$$f(\Delta_\ast) = \frac{1}{g_{12}(1/2, 1/2)} \sum_{s=0}^{|\Delta_\ast|} \sum_{\Delta \in L_{12,s}, \Delta \leq \Delta_\ast} C_{12;\Delta,s}^2 F_{12;\Delta}^{12;\Delta}(1/2, 1/2). \quad (3.2.5)$$

Note that due to the unitarity bound, $f(\Delta_\ast)$ receives no contribution from primary operators with spin $s > \Delta_\ast$. By definition, obviously, $f(\Delta_\ast)$ is a non-decreasing function that takes value between 0 and 1.

One can place bounds on the spectral function using semidefinite programming as follows. We would like to either maximize or minimize the spectral function subject to the crossing equation expanded around $z = \bar{z} = \frac{1}{2}$

$$0 = \sum_{s=0}^{\infty} \sum_{\Delta \in L_{12,s}} C_{12;\Delta,s}^2 \partial_z^m \partial_{\bar{z}}^n F_{12;\Delta}^{12;\Delta}(z, \bar{z})|_{z=\bar{z}=\frac{1}{2}}, \quad m + n \text{ odd}. \quad (3.2.6)$$

Note that $z = \frac{1}{2}$ corresponds to the nome $q = e^{-\pi}$, thus the $q$-expansion of conformal blocks converges rather quickly at this point. Consider a set of coefficients $y_{0,0}$ and $y_{m,n}$ ($m + n$ odd) such that

$$(y_{0,0} - \Theta(\Delta_\ast - \Delta)) F_{12;\Delta,s}^{12;\Delta}(1/2, 1/2) + \sum_{m+n \text{ odd}} y_{m,n} \partial_z^m \partial_{\bar{z}}^n F_{12;\Delta,s}^{12;\Delta}(z, \bar{z})|_{z=\bar{z}=\frac{1}{2}} \geq 0. \quad (3.2.7)$$

Here $\Theta(\Delta_\ast - \Delta)$ is the step function that takes value 1 for $\Delta \leq \Delta_\ast$ and 0 otherwise. $(\Delta, s)$ runs through all possibly allowed values of dimension and spin in the OPE. We could place additional assumptions on the
spectrum by restricting the range of \((\Delta, s)\) in (3.2.7). For instance, if we are to impose a dimension gap \(\Delta_{\text{gap}}\) or twist gap \(t_{\text{gap}}\), then we have respectively \(\Delta \geq \max(s, \Delta_{\text{gap}})\) or \(\Delta \geq s + t_{\text{gap}}\) for the spin-\(s\) (non-vacuum) primaries.\(^8\)

We shall seek the minimal \(y_{0,0}\) such that (3.2.7) holds, which we denote by \(y_{0,0}^{\text{min}}\). It follows that

\[
f(\Delta_\ast) = \frac{1}{g_{12}(1/2, 1/2)} \sum_{s=0}^{\infty} \sum_{\Delta \in \mathcal{I}_{12,s}} C^2_{12,s,\Delta} F_{12,s,\Delta}(1/2, 1/2) \Theta(\Delta_\ast - \Delta)
\]

\[
\leq \frac{1}{g_{12}(1/2, 1/2)} \sum_{s=0}^{\infty} \sum_{\Delta \in \mathcal{I}_{12,s}} C^2_{12,s,\Delta} \times \left[ y_{0,0}^{\text{min}} F_{12,s,\Delta}(1/2, 1/2) + \sum_{m+n \text{ odd}} y_{m,n} \partial^m_z \partial^n_{\bar{z}} F_{12,s,\Delta}(z, \bar{z}) \big|_{z=\bar{z}=1/2} \right]
\]

\[
= y_{0,0}^{\text{min}},
\]

where we have invoked unitarity by making use of the non-negativity of the squared structure constants, and applied the crossing equation. In other words, \(y_{0,0}^{\text{min}}\) is an upper bound on the value of the spectral function at \(\Delta_\ast\).

Likewise, if we minimize \(w_{0,0}\) subject to

\[
(w_{0,0} + \Theta(\Delta_\ast - \Delta)) F_{12,s,\Delta}(1/2, 1/2) + \sum_{m+n \text{ odd}} w_{m,n} \partial^m_z \partial^n_{\bar{z}} F_{12,s,\Delta}(z, \bar{z}) \big|_{z=\bar{z}=1/2} \geq 0,
\]

then

\[
f(\Delta_\ast) \geq - w_{0,0}^{\text{min}},
\]

i.e., \(-w_{0,0}^{\text{min}}\) is a lower bound on the value of the spectral function at \(\Delta_\ast\).

To obtain these bounds numerically we need to restrict to a finite subset of linear functionals acting on the crossing equation. We will do so by restricting the sums in (3.2.7) and (3.2.9) to odd \(m + n \leq N\); we refer to \(N\) as the “derivative order.” The upper and lower bounds on the spectral function derived from the above minimization procedure using linear functionals up to derivative order \(N\) will be denoted \(f^+_N(\Delta_\ast)\) and

\(^8\)In the case of a compact CFT, one must take care to additionally impose (3.2.7) and (3.2.9) on the vacuum block.
While these bounds at every $N$ are rigorous by themselves, the optimal bounds are obtained by extrapolating to the $N \to \infty$ limit.

The numerical implementation of the above procedure is performed using the SDPB package [79], with two practical modifications. Firstly, we will need to truncate the spectrum: while the application of SDPB does not require cutting off the dimension spectrum from above, a sufficiently large but finite truncation on the spin is necessary. In principle, the spin truncation means that we would not be taking into account all inequalities obeyed by the coefficients $y_{0,0}$ and $y_{m,n}$, resulting in stronger-than-correct bounds on the spectral function.\footnote{Indeed, the discussion in section 3.3.2 shows precisely why it is dangerous to truncate spectra of primaries on their spins.} Nonetheless, working at a fixed derivative order $N$, we generally find that the spectral function bounds stabilize to within numerical precision once the maximal spin $s_{\text{max}}$ is taken to be sufficiently large (empirically, $s_{\text{max}}$ at order $N$ is sufficient). For the application to theories with only scalar primaries in the next few subsections, of course, we do not need to worry about the spin truncation being sufficiently large. In this case, however, we must be especially careful in taking the truncation on the $q$-series of the conformal blocks to be sufficiently large, as the corrections to the approximate blocks would introduce nonzero spin primary contributions.

Secondly, since SDPB deals with the question of whether there exists a linear combination of polynomials $p_i(x)$ that is non-negative for all $x \geq 0$, the above minimization problem must be recast in the form of inequalities on polynomial functions of $\Delta$ on a semi-infinite line. For instance, suppose we impose a lower bound $\Delta^*_s$ on the dimension of spin-$s$ primaries as part of the a priori assumptions on the spectrum, then (3.2.7) is equivalently written as

$$y_{0,0} \mathcal{F}_{12,s,\Delta}(1/2,1/2) + \sum_{m+n \text{ odd}} y_{m,n} \partial_z \partial_{\bar{z}}^n \mathcal{F}_{12,s,\Delta}(z,\bar{z})|_{z=\bar{z}=1/2} \geq 0, \Delta \geq \Delta^*_s,$$

$$(y_{0,0} - 1) \mathcal{F}_{12,s,\Delta}(1/2,1/2) + \sum_{m+n \text{ odd}} y_{m,n} \partial_z \partial_{\bar{z}}^n \mathcal{F}_{12,s,\Delta}(z,\bar{z})|_{z=\bar{z}=1/2} \geq 0, \Delta^*_s \leq \Delta < \Delta^*_s. \quad \text{(3.2.11)}$$

By default, $\Delta^*_s$ can be set to the unitarity bound. While the first inequality in (3.2.11) can be implemented...
in SDPB by a simple shift in the variable $\Delta$, the second inequality which holds for $\Delta$ in an interval is more subtle. It is handled\(^{10}\) by converting the inequality to one on the semi-infinite line by a change of variable $\Delta = (\hat{\Delta} \Delta_s + \Delta^*_s)/(\hat{\Delta} + 1)$; now $\Delta^*_s \leq \Delta < \Delta_s$ amounts to $\hat{\Delta} \geq 0$.

### 3.2.2 Bounding the spectral function in a CFT with only scalar primaries

We now specialize to the case of CFT with only scalar primary operators. We do not specify the normalization of the primaries; as far as the spectral function is concerned, the external primaries are effectively normalized through the 4-point function (thus capturing relative OPE coefficients). This allows us to deal simultaneously with compact and non-compact CFTs. (By a non-compact CFT, we mean one with continuous spectrum and no $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$-invariant vacuum.) As alluded to in the introduction, there is only one known unitary CFT with $c > 1$ of this type, namely Liouville theory, and we will compare our bounds to the Liouville spectral function which can be obtained by numerically integrating the known OPE coefficients (given by DOZZ formula\([56, 57]\), as reviewed in appendix A.2) with the Virasoro conformal blocks.

We can write the four-point function involving a pair of primaries $\phi_1, \phi_2$ as

$$g_{12}(z, \bar{z}) = \int_0^\infty d\Delta \ C_{12;0,\Delta}^2 \mathcal{F}_{12;0,\Delta}(z, \bar{z}),$$

(3.2.12)

and the spectral function as

$$f(\Delta_s) = \frac{1}{g_{12}(1/2, 1/2)} \int_0^{\Delta_s} d\Delta \ C_{12;0,\Delta}^2 \mathcal{F}_{12;0,\Delta}(1/2, 1/2).$$

(3.2.13)

This accommodates both continuous and discrete spectra (in the latter case the integral will receive contributions from delta-functions). To place bounds on $f(\Delta_s)$, we simply solve the minimization problem (3.2.7), (3.2.9) for $s = 0$ only. This is implemented with SDPB with a given set of $c, \Delta_1$ and $\Delta_2$, while scanning over a range of $\Delta_s$, at increasing derivative orders $N$.

\(^{10}\)This trick is due to David Simmons-Duffin.
Figure 3.1: Upper and lower bounds on the spectral function from linear functionals of increasing derivative order (from green to red), assuming only scalar primaries for $c = 8$ with $\Delta_0 = 3/4\Delta_0$, $N = 5$ and $c = 8$, $\Delta_0 = 7/8\Delta_0$, $N = 9$. In all cases, the shaded regions are excluded and the black curve denotes the corresponding spectral function of (analytically continued) Liouville theory.

First, we consider the case where all external operators in the four-point function have the same scaling dimension (above or below the Liouville threshold, $\Delta_0 \equiv 2\xi$). Our results for $c = 8$ are summarized in figure 3.1. We observe that as the derivative order $N$ increases, the upper and lower bounds approach one another, narrowing the allowed range of the spectral function. Both bounds appear to be converging upon the spectral function of Liouville theory (whose background charge $Q$ is related to $c$ by $c = 1 + 6Q^2$), which sits in the middle of the allowed window.

There exist solutions to the scalar-only crossing equations when the external operator dimension drops...
below the Liouville threshold, so long as $\Delta_0^\phi \geq \frac{3}{4} \Delta_0$. For $\Delta_0^\phi < \frac{3}{4} \Delta_0$, solutions to the crossing equations with only scalar primaries in the OPE are excluded by our numerical analysis. When $\Delta_0^\phi = \frac{3}{4} \Delta_0$, we find that the upper and lower bounds on the spectral function converge quickly to a step function, i.e., $f^+_N(\Delta_*) \approx f^-_N(\Delta_*) \approx \Theta(\Delta_* - 2\xi)$, already at small derivative order $N$. This case and an example where $\Delta_0^\phi$ lies in between $\frac{3}{4} \Delta_0$ and the Liouville threshold are included in figure 3.1.

In fact, for $\Delta_0^\phi \in (\frac{c-1}{16}, \frac{c-1}{12})$, our bounds on the spectral function are entirely consistent with the analytic continuation of the Liouville spectral function to external operator dimensions below the Liouville threshold. Indeed, such analytically continued Liouville correlators arise in the study of certain normalizable BPS correlators in super-Liouville theory $[100]$ as a result of a relation due to Ribault and Teschner between $SL(2)$ WZW model correlators and Liouville correlators $[104]$. A priori, the crossing invariant Liouville 4-point function involves external primaries of scaling dimension $\Delta_i = 2\alpha_i(Q - \alpha_i)$, and an integration over internal primaries of scaling dimension $\Delta = 2\alpha(Q - \alpha)$, where both $\alpha_i$ and $\alpha$ lie on the half line $\frac{Q}{2} + i\mathbb{R}_{\geq 0}$. We can analytically continue $\alpha_i$ to the real axis, away from $\frac{Q}{2}$, provided that no pole in the structure constant $C(\alpha_1, \alpha_2, \alpha)$ as a function of $\alpha$ crosses the integration contour $\frac{Q}{2} + i\mathbb{R}$. This is possible for $\frac{Q}{2} < \alpha_1 + \alpha_2 < Q$, but fails for $\alpha_1 + \alpha_2 \leq \frac{Q}{2}$ when a pole in $\alpha$ crosses the contour and the 4-point function picks up a residue contribution that violates unitarity. Indeed, $\alpha_1 = \alpha_2 = \frac{Q}{2}$ corresponds to $\Delta_0^\phi = \frac{3}{4} \Delta_0$, and we find the step function behavior demonstrated in figure 3.1 whenever $\alpha_1 + \alpha_2 = \frac{Q}{2}$.\textsuperscript{11}

Next, we study the bounds on the spectral function for the 4-point function involving a pair of primaries $\phi_1$ and $\phi_2$ of different scaling dimensions, of the form (3.2.1). Note that for a non-compact CFT with only scalar primaries, such spectral functions capture the complete set of structure constants for three primaries of arbitrary weights. In figure 3.2 we plot the upper and lower bounds on the mixed correlator spectral function for $c = 8$ with external primaries of various dimensions $(\Delta_1, \Delta_2)$. Once again, the bounds narrow down the allowed window towards the spectral function of Liouville theory.

\textsuperscript{11}This step function behavior is consistent with the fact that the 4-point conformal block with $\alpha_1 + \alpha_2 = \frac{Q}{2}$ and internal primary with $\alpha = \frac{Q}{2}$ is crossing invariant by itself. This conformal block is the same as the holomorphic part of the 4-point function $(e^{2\alpha_1 \phi(z)} e^{2\alpha_2 \phi(0)} e^{2\alpha_1 \phi(1)} e^{2\alpha_2 \phi(\infty)})$ in the linear dilaton CFT with background charge $Q$. Note that in the linear dilaton theory the closure of the OPE demands a non-unitary spectrum.
Figure 3.2: Upper and lower bounds on the mixed correlator spectral function for $c = 8$ and $(\Delta_1, \Delta_2) = (5/9\Delta_0, 8/9\Delta_0), (1, 1/2), (1, 7/8)$. The black curve denotes the (analytically continued) DOZZ spectral function. In (c), a small gap of $\Delta_{\text{gap}} = 0.01$ has been imposed to explicitly exclude the vacuum channel which would correspond to a singular conformal block for the mixed correlator.

Apart from the case of $\alpha_1 + \alpha_2 = Q/2$, our numerical upper and lower bounds have not quite converged convincingly to the (analytic continuation of) the Liouville spectral function, due to the computational complexity of computing bounds at high derivative order $N$. Our results nonetheless suggest such a convergence in the $N \to \infty$ limit, supporting our conjecture that the DOZZ structure constants $C(\alpha_1, \alpha_2, \alpha_3)$ are the unique solution to the crossing equations for unitary CFTs with $c > 1$ and only scalar primaries.

Note that the convergence of the bounds on the $\langle \phi \phi \phi \phi \rangle$ spectral function would determine the $\phi \phi$ OPE up to normalization; if this holds for all $\Delta_\phi$, it would then determine, assuming a non-degenerate spectrum,
the conformal block decomposition of $\langle \phi_1 \phi_1 \phi_2 \phi_2 \rangle$ as well. This then determines the most general $\phi_1 \phi_2$ OPE, up to normalization. Compatibility with all crossing equations fixes the normalizations of OPE coefficients to be DOZZ up to an overall scale factor which cannot be fixed for a non-compact CFT.\footnote{This is because we can always tensor with a non-unital Frobenius algebra $G_{\alpha}$ with a single generator $e$, $(e, e) = 1$, $e^2 = \alpha e$ for any $\alpha \in \mathbb{R}$.} Thus, in order to establish our conjecture for the uniqueness of the DOZZ solution for the scalar-only crossing equations in the non-degenerate case, it suffices to consider the OPE of pairs of identical primaries, and then the result for mixed correlators would follow.

One can notice that the bounds appear to change slowly with $N$ in certain regions of the plots. We also observed in the numerical studies of spectral functions in modular bootstrap that the convergence of upper and lower bounds is relatively slow for continuous spectra as compared to discrete spectra (see section 3.4.2) in the cases where we know that the solution to the modular crossing equation is unique. It appears to be quite difficult numerically to push these bounds to higher derivative orders $N$, due to the need to substantially increase the truncation order $d_q$ on the $q$-expansion of the Virasoro conformal blocks. This is discussed in appendix A.4.1. In the next subsection, we consider an alternative method of directly solving the linear system that determines the spectral function assuming that the optimal upper and lower bounds coincide. This method in fact does not rely on the assumption of reality of the OPE coefficients and appears to converge much faster to the DOZZ spectral function.

### 3.3 The linear method and the uniqueness of Liouville theory

#### 3.3.1 Solution of the linear constraints on the spectral function

For CFTs with only scalar primaries, if the upper and lower bounds on the spectral function indeed converge (thereby pinning down the Liouville spectral function as the only solution), namely $\gamma_{0,0}^{\min}$ in (3.2.7) agrees
with \(-w_{0,0}^{\min}\) in (3.2.9), we would have a solution to the linear equation

\[ \Theta(\Delta_\ast - \Delta) F_{12;0,\Delta}(1/2, 1/2) = y_{0,0} F_{12;0,\Delta}(1/2, 1/2) + \sum_{m+n \text{ odd}} y_{m,n} \partial_z^m \partial_{\bar{z}}^n F_{12;0,\Delta}(z, \bar{z})|_{z = \bar{z} = \frac{1}{2}}. \]  

That is to say, on a certain vector space of functions in \(\Delta\), the function \(\Theta(\Delta_\ast - \Delta) F_{12;0,\Delta}(1/2, 1/2)\) can be decomposed on the basis spanned by \(F_{12;0,\Delta}(1/2, 1/2)\) and \(\partial_z^m \partial_{\bar{z}}^n F_{12;0,\Delta}(z, \bar{z})|_{z = \bar{z} = \frac{1}{2}}\). Since the step functions are themselves complete, our conjecture of the DOZZ structure constants as the unique solution is related to the completeness of this basis on a suitably defined Hilbert space of functions in \(\Delta\). We note in passing that in [105], Teschner proved the existence of the braiding and fusion transformations (and derived the fusion coefficients) of the Virasoro conformal blocks using a method that suggests that the Virasoro blocks in question (namely, the chiral blocks with internal weight above the Liouville threshold) form a complete set for an appropriate space of functions of the internal weight. While we do not have a proof of this statement, here we will adopt the strategy of analyzing the linear problem directly in an attempt to solve for the coefficient \(y_{0,0}\) (for a truncated system). The stability of the solution and its convergence to the Liouville spectral function will provide strong evidence for the conjecture. Another way to arrive at (3.3.1) is the following. In a non-compact CFT with only scalar primaries the crossing symmetry equations, together with a normalization condition \(g_{12}(1/2, 1/2) = 1\), (3.3.1) can be written as

\[ \int_0^\infty d\Delta C_{12;0,\Delta}^2 F_{12;0,\Delta}(1/2, 1/2) = 1, \]

\[ \int_0^\infty d\Delta C_{12;0,\Delta}^2 \partial_z^m \partial_{\bar{z}}^m F_{12;0,\Delta}(1/2, 1/2) = 0, \quad n + m \text{ odd}. \]  

\(\text{It is not obvious that the Hilbert space structure is the fundamentally correct one; for example, it might be that the correct notion is denseness in some Banach space.}\)

\(\text{Note that the linear independence of } F_{12;0,\Delta}(1/2, 1/2) \text{ from } \partial_z^m \partial_{\bar{z}}^n F_{12;0,\Delta}(z, \bar{z})|_{z = \bar{z} = \frac{1}{2}} \text{ as functions of } \Delta \text{ is guaranteed by the existence of DOZZ structure constants as a solution to the crossing equation. Indeed, according to (3.3.2) below, DOZZ structure constants give a linear functional which is 1 on } F_{12;0,\Delta}(1/2, 1/2) \text{ but 0 on } \partial_z^m \partial_{\bar{z}}^n F_{12;0,\Delta}(z, \bar{z})|_{z = \bar{z} = \frac{1}{2}}.\)
We may equivalently express these equations as

\[ \langle v, p_{0,0} \rangle = 1, \]
\[ \langle v, p_{n,m} \rangle = 0, \quad n + m \text{ odd}, \quad (3.3.3) \]

where the vectors \( v, p_{n,m} \) represent the functions

\[ v(\Delta) = C_{12;0}^2 \Delta / f_v(\Delta), \]
\[ p_{n,m}(\Delta) = \partial^n_\Delta \partial^m_\Delta F_{12;0}(1/2, 1/2) / f_p(\Delta). \quad (3.3.4) \]

for some suitable choices of \( f_v(\Delta) \) and \( f_p(\Delta) \) (see appendix A.4.2 for details), while the inner product is defined by

\[ \langle x, y \rangle = \int_0^\infty x^*(\Delta) y(\Delta) d\mu(\Delta), \quad (3.3.6) \]

with the measure \( d\mu(\Delta) = f_v(\Delta)f_p(\Delta)d\Delta \).

We now hope for completeness of the set \( p_{n,m} \) (from hereon we only consider \( n = m = 0 \) or \( n + m \) odd), assuming that all functions in question have finite norm. We truncate by \( n + m \leq N \) and consider the approximation of \( v \) by its orthogonal projection \( v_N = P_N v \) onto \( P_N = \text{span}\{p_{n,m}\}_{n+m \leq N} \). Note that despite the notation \( v_N \), because of the equations (3.3.3), \( v_N \) is independent of a particular solution \( v \). It can be computed by evaluating the Gram matrix of \( p \) vectors and taking its inverse,

\[ v_N = \sum_{n,m}^{n+m \leq N} \sum_{n',m'}^{n'+m' \leq N} \langle v, p_{n,m} \rangle (G_N^{-1})_{n,m}^{n',m'} p_{n',m'}, \quad (3.3.7) \]

where

\[ (G_N)_{n',m'}^{n,m} = \langle p_{n,m}, p_{n',m'} \rangle, \quad n + m \leq N, \quad n' + m' \leq N. \quad (3.3.8) \]
The spectral function can be computed as the inner product

\[ f(\Delta) = \langle v, \theta_{\Delta} \rangle = \int_{0}^{\Delta^*} d\Delta C_{12,0,\Delta}^{2} F_{12,0,\Delta}(1/2, 1/2), \]  

(3.3.9)

where

\[ \theta_{\Delta}(\Delta) = \Theta(\Delta^* \Delta) p_{0,0}(\Delta). \]  

(3.3.10)

We have an estimate,

\[ \langle v, \theta_{\Delta} \rangle = \langle v, P_{\Delta} \theta_{\Delta} \rangle + \langle v, (1 - P_{\Delta}) \theta_{\Delta} \rangle = \langle v, \theta_{\Delta} \rangle + R_{\Delta}(\Delta), \]  

(3.3.11)

where

\[ |R_{\Delta}(\Delta)|^2 = |\langle v, (1 - P_{\Delta}) \theta_{\Delta} \rangle|^2 \leq |v|^2 |(1 - P_{\Delta}) \theta_{\Delta}|^2 = E_{\Delta} |v|^2 |\theta_{\Delta}|^2. \]  

(3.3.12)

Note that \( E_{\Delta} = |(1 - P_{\Delta}) \theta_{\Delta}|^2 / |\theta_{\Delta}|^2 \) is also independent of a particular solution \( v \) and is computable from (3.3.3).

If (3.3.1) holds in the norm induced from \( \langle \cdot, \cdot \rangle \), then \( E_{\Delta} \rightarrow 0 \). Conversely, if we show for all \( \Delta \) that \( \lim_{N \rightarrow \infty} E_{\Delta} = 0 \), it will imply that any normalizable solution to (3.3.3) and thus to (3.3.2) is equal to the limit \( \lim_{N \rightarrow \infty} v_{\Delta} \), which is unique if it exists. Our strategy would be therefore to evaluate \( v_{\Delta} \) and \( E_{\Delta} \) numerically and estimate their limits.

We first numerically evaluate \( f_{\Delta}(\Delta) = \langle v_{\Delta}, \theta_{\Delta} \rangle \) and find that it converges to the Liouville spectral density in the limit \( N \rightarrow \infty \). For example, in figure 3.3(a) the approximation \( f_{\Delta} \) is plotted at successive odd values of \( N \) up to \( N = 25 \) for \( c = 8 \) and \( \Delta_{\phi} = 7/12 \). We can see that the curves exhibit the expected convergence. Another example where the external operator dimension is far above the Liouville threshold is shown in figure 3.3(b), where we studied \( c = 2, \Delta_{\phi} = 55/12, \) up to \( N = 33 \) and \( d_{\phi} = 200 \). While \( f_{\Delta}(\Delta) \) oscillates wildly at smaller \( N \) (the case \( N = 27 \) is shown for comparison), the oscillation settles down substantially as \( N \) is increased.
In figure 3.3 we compare $f_N(\Delta_*)$ with the DOZZ spectral function for $c = 8$ and $c = 30$, with $\Delta_\phi$ at or above the Liouville threshold, as well as an example of a mixed correlator spectral function\footnote{The mixed correlator spectral function was obtained under a technical assumption of a lower dimension bound of $\frac{\Delta_0}{2}$ (note that this is below the Liouville threshold $\Delta_0$), see appendix A.4.2 for details.} with two different values of external operator dimensions. In all cases we find good agreement.

To further support the conjecture, we numerically compute the error estimate $E_N$ as a function of $N$. For example, in figure 3.5 we show $E_N$ as a function of $1/N$ for $\Delta_\phi = \frac{c-1}{12}$, $\Delta_* = \frac{c}{10}$, and $c = 8$. In the figure we also show a linear fit using $N \geq 11$. Empirically, we find that the result is consistent with $E_N \sim N^{-1}$. We study $E_N$ in more detail in appendix A.4.2.

The discussion above depends on the assumption that $v$ has finite norm. This assumption itself depends on the choice of measure. We describe our choice of measure and details of our implementation in appendix A.4.2. Here we simply note that with our choice, $v$ has finite norm if $C_{12;0,\Delta}^4$ is locally integrable on $[0, \infty)$, and the OPE expansion is convergent in the region $|z| < 1$. Discrete spectra have infinite norm since $C_{12;0,\Delta}^4$ involves squares of delta-functions, but such spectra are excluded by modular invariance.
Figure 3.4: Comparison of $f_N(\Delta_\ast)$ (solid, red) with the exact DOZZ spectral function (dashed, blue) for external operator dimension $\Delta_\phi$, and in the mixed correlator case, external operator dimensions $\Delta_1$ and $\Delta_2$ ($\Delta_0 \equiv \frac{1}{12}$ is the Liouville threshold as before).

Figure 3.5: Plot of $E_N$ for as a function of $1/N$, $c = 8$, $\Delta_\phi = \frac{7}{12}$, $\Delta_\ast = 0.8$, $N \leq 25$. The dashed curve is a linear fit for $N \geq 11$. 
### 3.3.2 Constraints from modular invariance

Strong constraints on the primary spectrum, especially in the scalar-only case, follow from modular invariance alone. In fact, there is a simple argument that shows any 2D CFT with $c > 1$ and primary operators with bounded spin must have a spectrum identical to that of Liouville theory: that is, the spectrum is non-compact, has scalar primaries only, and has a spectral density that is uniformly distributed in Liouville momentum $P = \sqrt{2(\Delta - \frac{c-1}{24})}$.

Suppose the primaries have spins no greater than $s_{\text{max}}$. We can write the reduced torus partition function in the following way:

$$
\tilde{Z}(\tau, \bar{\tau}) = \tau^\frac{1}{2} |\eta(\tau)|^2 Z(\tau, \bar{\tau})
$$

$$
= \tau^\frac{1}{2} \left( |q^{-\xi}(1-q)|^2 + \sum_{h,\tilde{h}>0} d(h,\tilde{h}) q^{h-\xi}\tilde{q}^{\tilde{h}-\xi} \right)
$$

$$
= \sum_{s,|s|\leq s_{\text{max}}} e^{2\pi is\tau_2} f_s(\tau_2),
$$

where $q = e^{2\pi i \tau}$, $\xi = \frac{c-1}{24}$, $d(h,\tilde{h})$ is the degeneracy of primary operators in the spectrum with conformal weights $(h,\tilde{h})$ and $f_s(x) = \sum_{\Delta \geq |s|} d(\frac{\Delta + s}{2}, \frac{\Delta - s}{2}) x^\Delta e^{-2\pi x(\Delta - 2\xi)}$. For now we assume that the CFT is compact, and the vacuum character is degenerate and so $s_{\text{max}} \geq 1$. The non-compact CFTs may be viewed as limiting cases, where the spectral density diverges and we divide the partition function by an infinite normalization factor which removes the vacuum contribution. Here we consider unitary CFTs; in particular, we assume that the degeneracies $d(h,\tilde{h})$ and the conformal weights $h,\tilde{h}$ are non-negative.

Now consider the following change of variables

$$
x = \tau_2, \ y = \frac{\tau_2}{|\tau|^2},
$$

chosen so that the modular $S$ transformation exchanges $x$ and $y$. We can then write the modular crossing
equation in terms of these variables as

$$\sum_s e^{2\pi is\sqrt{\frac{r^2-x^2}{2}}} f_s(x) = \sum_s e^{2\pi is\sqrt{\frac{r^2-y^2}} f_s(y)} \quad (3.3.15)$$

Of course, the functions above have branch cuts at $x = y^{-1}$, but since the sum over spins is finite by assumption, the analytic continuation around the branch is straightforward. Furthermore, $f_s(y)$ is an analytic function for $\text{Re}(y) > 0$. To proceed, we fix $x = re^{-i\alpha}$ with $r > 0$ and $0 < \alpha < \frac{\pi}{2}$ and $y = \epsilon$ with $\epsilon \to 0^+$, so that the modular crossing equation becomes

$$\sum_s e^{2\pi s\sqrt{\tau_1 (\sin\frac{\alpha}{2} + i \cos\frac{\alpha}{2})}} f_s(x) \approx \sum_s f_s(\epsilon)$$

$$= \sum_s f_s(\epsilon^{-1}), \quad (3.3.16)$$

where in the first line we dropped the phase factors $e^{2\pi i s \sqrt{\frac{r\epsilon^{-1} - \epsilon^2}{2}}}$ (which are close to 1 due to the boundedness of $s$) in front of $f_s(\epsilon)$; this is a valid approximation since $f_s(\epsilon)$ is positive for all $s$ (as follows from our unitarity assumption). In the second line we again invoked modular invariance (this particular equality is realized as the modular crossing equation with $\tau_1 = 0, \tau_2 = \epsilon$). In the case that the CFT is compact, the right-hand side is dominated by the contribution of the vacuum, in particular

$$\sum_s f_s(\epsilon^{-1}) \approx \epsilon^{-\frac{1}{2}} e^{4\pi\epsilon} \quad (3.3.17)$$

By comparing to the $\epsilon \to 0$ limit of the left-hand side, which is dominated by the term with maximal spin

$$\sum_s e^{2\pi s\sqrt{\tau_2 (\sin\frac{\alpha}{2} + i \cos\frac{\alpha}{2})}} f_s(\epsilon^{-i\alpha}) \approx e^{2\pi s_{\text{max}}\sqrt{\tau_2 (\sin\frac{\alpha}{2} + i \cos\frac{\alpha}{2})}} f_{s_{\text{max}}}(\epsilon^{-i\alpha}), \quad (3.3.18)$$

we arrive at a contradiction and deduce that unitary 2D CFTs with primary operators of bounded spin must have non-compact spectra: namely, there is no $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$-invariant vacuum and the dimension of the lowest-lying primary operator obeys $\Delta_{\text{min}} > 0$.  

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In fact, this same logic allows us to conclude that the dimension of the lowest-lying operator must obey
\[ \Delta_{\text{min}} \geq \frac{c-1}{12}. \]
In the \( \epsilon \to 0 \) limit, we have
\[
e^{2\pi s_{\text{max}} \sqrt{\tau} (\sin \frac{\varphi}{2} + i \cos \frac{\varphi}{2})} f_{s_{\text{max}}} (r e^{-i \alpha}) \approx \sum_s f_s (\epsilon^{-1})
\]
\[ = \tilde{Z} (\tau_1 = 0, \tau_2 = \epsilon^{-1}) \]
\[ = \epsilon^{-\frac{1}{2}} \int_0^{\infty} d\Delta \rho (\Delta) e^{-\frac{2\pi}{\sqrt{\tau}} (\Delta - 2\xi)}, \]
where \( \rho (\Delta) \) is the density of primary operators in the spectrum with dimension \( \Delta \) (of any spin). The two sides of the equation are clearly incompatible if the minimum scaling dimension for which \( \rho (\Delta) \) is nonzero is smaller than \( 2\xi \). Furthermore, by non-negativity of the spectral density, the right-hand side can grow no faster than \( \epsilon^{-\frac{1}{2}} \) as \( \epsilon \to 0^+ \). On the other hand, the absolute value of (3.3.18) grows like \( e^{2\pi s_{\text{max}} \sqrt{\tau} \sin \frac{\varphi}{2}} \) in this limit. Modular invariance thus demands that \( s_{\text{max}} = 0 \): that is, a unitary 2D CFT with primary operators of bounded spin must in fact have only scalar primary operators in addition to having a non-compact spectrum. Moreover in this case the modular crossing equation becomes
\[ f_0 (x) = f_0 (y), \]
which demands that \( f_0 (x) \) is a constant. Thus the required spectral density is nothing other than that of Liouville theory, namely
\[ \rho (\Delta) = \rho_{\text{Liouville}} (\Delta) \propto (\Delta - 2\xi)^{-\frac{1}{2}} \Theta (\Delta - 2\xi), \]
completing the argument. In particular, the dimension of the lowest-lying operator must be exactly \( \Delta_{\text{min}} = 2\xi \).

By this result, our conjecture that the DOZZ structure constants are the unique solution to the crossing equations for a unitary 2D CFT with central charge \( c > 1 \) and only scalar primaries, as supported by substantial numerical evidence in sections 3.2.2 and 3.3.1 leads us to conjecture that Liouville theory is the unique unitary \( c > 1 \) CFT with Virasoro primaries of bounded spin.\(^{17}\)

\(^{16}\)To normalize the reduced partition function of Liouville theory to 1, the constant of proportionality is \( \sqrt{Z} \).

\(^{17}\)Note that we have not made use of the torus 1-point function. A priori, the modular invariance of the torus 1-point function puts nontrivial constraints on the structure constants with a pair of primaries identified. For the purpose of establishing our conjecture regarding the uniqueness of Liouville, once the OPE coefficients are pinned
3.3.3 Degenerate spectrum and TQFT

In our analysis of the crossing equation so far, we have implicitly assumed that the scalar primaries are labeled by a continuous parameter, namely the scaling dimension $\Delta_\phi$, without further degeneracy. If this assumption is relaxed, one can construct more examples of (non-compact) $c > 1$ CFTs with only scalar primaries, by taking the tensor product of Liouville CFT with a topological quantum field theory (TQFT); the latter has a finite dimensional Hilbert space on the circle and its structure constants are governed by those of a commutative Frobenius algebra [107]. We conjecture that this is the only possibility.

Let us assume that the scalar primaries are labeled by their scaling dimension $\Delta$ and an extra index $i$, and denote the structure constants by

$$C_{ijk}(\Delta_1, \Delta_2, \Delta_3) = A_{ijk}(\Delta_1, \Delta_2, \Delta_3)C^{\text{DOZZ}}(\Delta_1, \Delta_2, \Delta_3),$$

where we have explicitly factored out the DOZZ structure constants. Our numerical results in the previous sections on the spectral function of mixed correlators of the form $\langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle$ indicate that for a CFT with degenerate scalar-only primary spectrum,

$$\sum_k (A_{ijk}(\Delta_1, \Delta_2, \Delta))^2 = B_{ij}(\Delta_1, \Delta_2)$$

is independent of $\Delta$. In fact, we can strengthen this result slightly. Let us consider a mixed correlator $\langle \phi_i \phi_j \phi_k \phi_\ell \rangle$ where $\phi_i, \phi_\ell$ have scaling dimension $\Delta_1$, $\phi_j, \phi_k$ have scaling dimension $\Delta_2$, and the crossing equation

$$\sum_m \sum_\Delta C_{ijm}(\Delta_1, \Delta_2, \Delta)C_{k\ell m}(\Delta_1, \Delta_2, \Delta)F_{12,0,\Delta}(z, \bar{z}) = \sum_m \sum_\Delta C_{i\ell m}(\Delta_1, \Delta_2, \Delta)C_{kjm}(\Delta_1, \Delta_2, \Delta)F_{12,0,\Delta}(1-z, 1-\bar{z}).$$

down to those of DOZZ by the crossing equation, the torus 1-point functions are already modular invariant [106].

\(^{18}\)To be precise, we do not need to require the TQFT to have a vacuum state (or the algebra to be unital).
By taking the part of (3.3.23) that is odd under $z \rightarrow 1 - z, \bar{z} \rightarrow 1 - \bar{z}$, our earlier claim of the uniqueness of scalar-only solution to the crossing equation implies that

$$
\sum_m A_{ijm}(\Delta_1, \Delta_2, \Delta)A_{k\ell m}(\Delta_1, \Delta_2, \Delta) + (j \leftrightarrow \ell)
$$

is independent of $\Delta$. On the other hand, for the even part of (3.3.23) under $z \rightarrow 1 - z, \bar{z} \rightarrow 1 - \bar{z}$, the numerical analysis described in appendix A.4.2 is consistent with the conjecture that $\{ \partial^n \partial^m F_{12,0,\Delta} | z = \bar{z} = \frac{1}{2}, n, m \in \mathbb{Z}_{\geq 0}, n + m \text{ even} \}$ form a complete basis on the space functions of $\Delta$ on the positive real axis defined by the same norm as in section 3.3.1, which implies that

$$
\sum_m A_{ijm}(\Delta_1, \Delta_2, \Delta)A_{k\ell m}(\Delta_1, \Delta_2, \Delta) = \sum_m A_{i\ell m}(\Delta_1, \Delta_2, \Delta)A_{jkm}(\Delta_1, \Delta_2, \Delta)
$$

(3.3.24)

for every $\Delta > 0$, and thus

$$
\sum_m A_{ijm}(\Delta_1, \Delta_2, \Delta)A_{k\ell m}(\Delta_1, \Delta_2, \Delta) = B_{ij\ell}(\Delta_1, \Delta_2)
$$

(3.3.25)

is independent of $\Delta$.

It is likely that by analyzing a system of crossing equations for multiple scalar correlators involving $\phi_i, \phi_j, \phi_k, \phi_\ell$ of generally different scaling dimensions, one could establish that the spectral function for $\langle \phi_i \phi_j \phi_k \phi_\ell \rangle$ (with only scalar Virasoro primaries in the OPEs) is proportional to that of Liouville CFT, which would be equivalent to the statement that

$$
\sum_m A_{ijm}(\Delta_1, \Delta_2, \Delta)A_{k\ell m}(\Delta_3, \Delta_4, \Delta) = \sum_m A_{i\ell m}(\Delta_1, \Delta_4, \Delta)A_{jkm}(\Delta_3, \Delta_2, \Delta)
$$

(3.3.26)

$$
= B_{ij\ell}(\Delta_1, \Delta_2, \Delta_3, \Delta_4)
$$

is independent of $\Delta$, extending (3.3.25). We leave the numerical bootstrap of the spectral function with four generic external weights to future work. We now argue that if (3.3.26) holds, then our conjecture follows.
To each pair-of-pants decomposition of a genus $g$ Riemann surface, represented by a trivalent graph, we may associate a sum of product of $A_{ijk}$'s, with indices contracted and scaling dimensions identified along each edge of the graph, which we denote by $\hat{Z}_g$. (3.3.26) implies the crossing relation between graphs with fixed weights on the edges, and by applying crossing one can always turn the trivalent graph into one that does not contain tadpole subgraphs.\(^{19}\) (3.3.26) further implies that $\hat{Z}_g$ is independent of the scaling dimension on every edge that connects a pair of distinct vertices, and thus the genus $g$ partition function of the CFT is equal to $\hat{Z}_g$ times the Liouville partition function. It then follows from modular invariance that $\hat{Z}_g$ is independent of the pair-of-pants decomposition, and depends on the genus $g$ only.

To proceed, pick a finite set of scaling dimensions $\Delta_a$, $a = 1, \ldots, M$ and let $N_\Delta$ be the number of degenerate primaries of dimension $\Delta$, which we will assume to be finite. Set $N = \max N_\Delta$ and extend the ranges of the discrete labels to run up to $N$ for all $\Delta_a$ by setting the previously undefined structure constants to zero. Then the totality of $A_{ijk}(\Delta_a, \Delta_b, \Delta_c)$ gives an element $\mathcal{A}$ in $\mathcal{C} = S^3(\oplus_a \mathbb{R}^N)$. The space $\mathcal{C}$ is equipped with an action of $\prod_a O(N)$, corresponding to changes of basis for the discrete labels. $\hat{Z}_g$ regarded as polynomials generate the algebra of $\prod_a O(N)$ invariants on $\mathcal{C}$. It follows that $\mathcal{A}$ is equivalent to any other $\mathcal{A}'$ with the same values of $\hat{Z}_g$ by a $\prod_a O(N)$ reparametrization. In particular, since $\mathcal{A}$ is such that values of $\hat{Z}_g$ on it are independent of the internal labels, $\mathcal{A}$ is equivalent to $\mathcal{A}_0$ in which all $A_{ijk}(\Delta_a, \Delta_b, \Delta_c)$ are replaced by $a_{ijk} = A_{ijk}(\Delta_1, \Delta_1, \Delta_1)$. It then follows that we can choose $N_{\Delta_a} = N$.

By taking various finite sets of scaling dimensions sharing the dimension $\Delta_1$, we find that $N_\Delta = N$ is independent of $\Delta$ and thus the density of states is given by $N$ copies of Liouville density. Furthermore, for any such finite set we have

$$A_{ijk}(\Delta_a, \Delta_b, \Delta_c) = a_{ijk}$$

(3.3.27)

up to a reparametrization of finite labels. Note that such reparametrizations depend on the choice of our finite set of $\Delta_a$'s, being defined only up to automorphisms of $a_{ijk}$. As we show below, these automorphisms

\(^{19}\)That is to say, modular constraints on the general torus 1-point function are not needed for the argument presented here.
are scarce. Compatibility between different $\Delta_a$ then completely fixes them after we fix the reparametrization for $\Delta_1$. Thus we can pass to the full continuous set of scaling dimensions, and conclude that the CFT in question is a tensor product of Liouville with a TQFT defined by the structure constants $a_{ijk}$ (or the partition functions $\tilde{Z}_g$).

In fact, we can always find a basis in which the structure constants $a_{ijk}$ are diagonalized. To see this, note that the crossing equation for $a_{ijk}$ implies that the matrices $M_i$ with entries $M_{ij} = a_{ijk}$ are mutually commuting $N \times N$ symmetric matrices, and thus can be simultaneously diagonalized by some $O(N)$ matrix $R$, namely $\Lambda'_{ijk} = \sum_m R_{jm} R_{kn} a_{imn}$ are diagonal in $jk$. Then $\Lambda_{ijk} = \sum_m R_{im} \Lambda'_m{}_{jk}$ is still diagonal in $jk$ and completely symmetric, and thus $\Lambda_{ijk} = \delta_{ijk} \lambda_k$. Multiplying by a diagonal matrix with $\pm 1$ entries if necessary, we can set $\lambda_m > 0$. (If some $\lambda_m = 0$ they do not contribute to the correlators and we can obviously add or remove such $\lambda$’s at will.) It is straightforward to check that the automorphisms of $\Lambda_{ijk}$ are just the permutations preserving the $\lambda$’s. The partition functions are $\tilde{Z}_g = \sum_n \lambda_n^{-1}$.

This diagonalization implies that the algebra defined by $a_{ijk}$ is given by $\oplus_n \mathcal{G}_n$. Here $\mathcal{G}_\lambda = \mathbb{R} e$ with $(e,e) = 1$ and $e^2 = \lambda e$. Forming the tensor product Liouville $\otimes \mathcal{G}_\lambda$ corresponds to rescaling all OPE coefficients by $\lambda$. The overall scale of OPE coefficients cannot be fixed in the absence of the vacuum, and thus we can regard all these theories as isomorphic to Liouville. Therefore, the TQFT structure amounts to superselection sectors.
3.4 The modular spectral function

3.4.1 The minimization problem

We now consider the decomposition of the reduced torus partition function of a compact, unitary CFT (assumed to be parity-invariant)\(^{20}\) with no conserved currents into non-degenerate Virasoro characters

\[
\hat{Z}(\tau, \bar{\tau}) = |\tau|^{\frac{1}{2}} |\eta(\tau)|^2 Z(\tau, \bar{\tau}) = \hat{\chi}_0(\tau) \hat{\chi}_0(\bar{\tau}) + \sum_{s \geq 0} \sum_{\Delta \in \mathcal{I}_s} \frac{1}{1 + \delta_s,0} d_{\Delta,s} \left( \hat{\chi}_{\Delta^{-s}}(\tau) \hat{\chi}_{\Delta^{-s}}(\bar{\tau}) + \hat{\chi}_{\Delta^{-s}}(\tau) \hat{\chi}_{\Delta^{-s}}(\bar{\tau}) \right),
\]

(3.4.1)

where \(\mathcal{I}_s\) is the discrete spectrum of dimensions of primary operators, \(d_{\Delta,s} = d(\frac{\Delta + s}{2}, \frac{\Delta - s}{2}) = d(\frac{\Delta - s}{2}, \frac{\Delta + s}{2})\), and the reduced characters are given by

\[
\hat{\chi}_0(\tau) = (-i \tau)^{\frac{1}{4}} q^{-\xi} (1 - q)\]

\[
\hat{\chi}_h(\tau) = (-i \tau)^{\frac{1}{4}} q^{h - \xi}.
\]

(3.4.2)

Analogously to the four-point spectral function introduced in section 3.2.1, we define a “modular spectral function” by truncating the Virasoro character decomposition of the reduced partition function up to a cutoff dimension \(\Delta_s\), evaluated at the self-dual modulus \(\tau = -\bar{\tau} = i\),

\[
f_{\text{mod}}(\Delta_s) = \frac{1}{Z(i, -i)} \left[ \hat{\chi}_0(i) \hat{\chi}_0(-i) + \sum_{s = 0}^{\lfloor \Delta_s \rfloor} \sum_{\Delta \in \mathcal{I}_s, \Delta \leq \Delta_s} \frac{d_{\Delta,s}}{1 + \delta_s,0} \left( \hat{\chi}_{\Delta^{-s}}(i) \hat{\chi}_{\Delta^{-s}}(-i) + \hat{\chi}_{\Delta^{-s}}(i) \hat{\chi}_{\Delta^{-s}}(-i) \right) \right].
\]

(3.4.3)

As with the four-point spectral function, it is straightforward to place bounds on \(f_{\text{mod}}(\Delta_s)\) due to modular

\(^{20}\)As in [101], the bounds we derive here assuming a parity-invariant spectrum can be applied to parity non-invariant theories as well by considering the parity-positive projection of the partition function.
invariance using semidefinite programming. Defining

\[ \hat{Z}_{\Delta, s}(\tau, \bar{\tau}) = \frac{1}{1 + \delta_{s,0}} \left[ \hat{\chi}_{\Delta, s}(\tau) \hat{\chi}_{\Delta, s}(\bar{\tau}) + \hat{\chi}_{\Delta, s}(\tau) \hat{\chi}_{\Delta, s}(\bar{\tau}) \right] \]  \hspace{1cm} (3.4.4)

and

\[ \hat{Z}_{0,0}(\tau, \bar{\tau}) = \hat{\chi}(\tau) \hat{\chi}(\bar{\tau}), \]  \hspace{1cm} (3.4.5)

the modular crossing equation demands that

\[ 0 = \partial_{\bar{\tau}} \partial_{\bar{\tau}}^m \left[ \hat{Z}_{0,0}(\tau, \bar{\tau}) + \sum_{s=0}^{\infty} \sum_{\Delta_s \in \mathcal{I}_s} d_{\Delta, s} \hat{Z}_{\Delta, s}(\tau, \bar{\tau}) \right] \]  \hspace{1cm} (3.4.6)

where we have redefined \( \tau = ie^{\frac{\pi}{L}}, \bar{\tau} = -ie^{-\frac{\pi}{L}} \). We then seek to minimize \( y_{0,0} \) subject to the inequalities

\[ (y_{0,0} - 1) \hat{Z}_{0,0}(i, -i) + \sum_{m+n \text{ odd}} y_{m,n} \partial_{\bar{\tau}}^m \partial_{\bar{\tau}}^n \hat{Z}_{0,0}(\tau, \bar{\tau}) \bigg|_{z = \bar{z} = 0} \geq 0 \]

\[ (y_{0,0} - \Theta(\Delta_s - \Delta)) \hat{Z}_{\Delta, s}(i, -i) + \sum_{m+n \text{ odd}} y_{m,n} \partial_{\bar{\tau}}^m \partial_{\bar{\tau}}^n \hat{Z}_{\Delta, s}(\tau, \bar{\tau}) \bigg|_{z = \bar{z} = 0} \geq 0, \]  \hspace{1cm} (3.4.7)

for arbitrary coefficients \( y_{m,n} \). In the first line we have singled out the inequality involving the vacuum primary. In the second line, we made the extra assumption of a gap \( \Delta^*_s \) in the spin-s sector of the spectrum, as will be useful in later applications. As before, the minimal such \( y_{0,0} \) gives an upper bound on the modular spectral function, since

\[ f_{\text{mod}}(\Delta_s) \leq \frac{1}{Z(i, -i)} \left[ \min_{y_{0,0}} \left( \hat{Z}_{0,0}(i, -i) + \sum_{s, \Delta} d_{\Delta, s} \hat{Z}_{\Delta, s}(i, -i) \right) \right. \]

\[ + \sum_{m+n \text{ odd}} y_{m,n} \partial_{\bar{\tau}}^m \partial_{\bar{\tau}}^n \left( \hat{Z}_{0,0}(\tau, \bar{\tau}) + \sum_{s, \Delta} d_{\Delta, s} \hat{Z}_{\Delta, s}(\tau, \bar{\tau}) \right) \bigg|_{z = \bar{z} = 0} \]

\[ = y_{0,0}^\text{min}. \]
Similarly, the minimal \( w_{0,0} \) subject to the constraints

\[
(w_{0,0} + 1) \hat{Z}_{0,0}(i, -i) + \sum_{m+n \text{ odd}} w_{m,n} \partial_z^m \partial_{\bar{z}}^n \hat{Z}_{0,0}((\tau, \bar{\tau})) \bigg|_{\bar{z} = 0} \geq 0
\]

\[
(w_{0,0} + \Theta(\Delta_s - \Delta)) \hat{Z}_{\Delta,s}(i, -i) + \sum_{m+n \text{ odd}} w_{m,n} \partial_z^m \partial_{\bar{z}}^n \hat{Z}_{\Delta,s}((\tau, \bar{\tau})) \bigg|_{\bar{z} = 0} \geq 0, \quad \Delta \geq \Delta_s^*, \quad s \geq 0,
\]

provides a nontrivial lower bound on the modular spectral function

\[
f_{\text{mod}}(\Delta_s) \geq -w_{0,0}^{\text{min}}.
\]

Working up to a finite derivative order \( m + n \leq N \), we denote the corresponding upper and lower bounds obtained in this way \( f_{\text{mod},N}^+(\Delta_s) \) and \( f_{\text{mod},N}^-(\Delta_s) \) respectively.

### 3.4.2 Some consistency checks

#### Extremal spectra with maximal gap

In [101], an upper bound \( \Delta_{\text{mod}}(c) \) on the gap in the scaling dimension of primary operators due to modular invariance of the torus partition function was computed numerically as a function of the central charge. Given a dimension gap \( \Delta_{\text{gap}}(\leq \Delta_{\text{mod}}(c)) \), an upper bound on the degeneracy of primaries at dimension \( \Delta_{\text{gap}} \) can be obtained provided \( \Delta_{\text{gap}} > \frac{c-1}{12} \). When this upper bound on the degeneracy at the gap is saturated, the entire modular invariant spectrum is determined by the locations of the zeros of the optimal linear functional (optimized with respect to the degeneracy bound) acting on the Virasoro characters. Such (candidate) CFT spectra were dubbed ‘extremal.’ Furthermore, it is expected that for each given \( c > 1 \), there is a unique modular invariant spectrum (imposing positivity but not the integral condition on the degeneracy of primaries) whose dimension gap saturates the upper bound \( \Delta_{\text{mod}}(c) \) [82].

In [101], a number of examples of CFTs with spectra that saturated the bound on the dimension gap were identified at small values of the central charge. Here, we study the bounds on the modular spectral function.
at these values of the central charge assuming the maximal dimension gap. We will find that the resulting bounds indeed pin down the extremal modular spectral functions. To compute the bounds on the modular spectral functions in these cases, we impose (3.4.7, 3.4.9) with \( \Delta_s^* = \max(s, \Delta_{\text{mod}}(c)) \).

For \( c = 2 \), the dimension gap bound of \( \Delta_{\text{mod}}(2) = \frac{2}{3} \) is realized by the spectrum of the \( SU(3) \) WZW model at level one. This theory admits a description in terms of free bosons with \( T^2 \) target space at the \( Z_3 \)-invariant point in its complex structure and Kähler moduli spaces, with partition function

\[
Z_{\text{ext}}(2, \frac{2}{3}) = \sum_{n, w \in \mathbb{Z}} q^{\frac{1}{2}k^2_L + \frac{1}{2}k^2_R} \frac{\eta(\tau)^4}{|\eta(\tau)|^4},
\]

where

\[
k^2_{L,R} = \frac{G^{mn}}{\alpha'} (n_m + B_{mk}w^k \pm G_{mk}w^k)(n_n + B_{nl}w^l \pm G_{nl}w^l)
\]

for \( G = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \). The bounds on the modular spectral function collapse precisely to this extremal modular spectral function when the maximal gap is imposed, as shown in figure 3.6.

For \( c = 4 \), the dimension gap bound of \( \Delta_{\text{mod}}(4) = 1 \) is realized by the spectrum of the \( SO(8) \) WZW model at level 1, which also admits a description in terms of 8 free fermions with diagonal GSO projection. This theory occupied the kink on the curve \( \Delta_{\text{mod}}(c) \). The partition function of this theory is given by

\[
Z_{\text{ext}}(4, 1) = \frac{1}{2} \left( \left| \frac{\Theta_2(\tau)}{\eta(\tau)} \right|^8 + \left| \frac{\Theta_3(\tau)}{\eta(\tau)} \right|^8 + \left| \frac{\Theta_4(\tau)}{\eta(\tau)} \right|^8 \right).
\]

Once again, in figure 3.6 we see that the bounds on the modular spectral function collapse to that of the extremal spectrum.

For \( c = 8 \), there is a nontrivial bound on the dimension gap in the spectrum of scalar primaries, \( \Delta_{\text{mod}}^8(8) = 2 \). This bound on the scalar gap is saturated by the spectrum of the \( E_8 \) WZW model at level one. This theory, which occupied the first kink on the bounding curve \( \Delta_{\text{mod}}^8(c) \), admits an equivalent description in

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terms of 8 compact bosons at the holomorphically factorized point in the moduli space; the holomorphic factor is described by the Narain compactification on $\Gamma_8$, the root lattice of $E_8$. The partition function is

$$Z_{\text{ext}, s=0}(8, 2) = |j(\tau)|^{\frac{3}{2}}, \quad (3.4.14)$$

where $j(\tau)$ is the elliptic $j$-invariant. Figure 3.6 shows that the bounds on the modular spectral function (derived using $\Delta_s^* = \Delta_{\text{mod}}^s + s$) collapse to that of the extremal spectrum.

**Figure 3.6:** The upper (blue) and lower (red) bounds on the modular spectral function. **Top:** The bounds for $c = 2$ with an assumed dimension gap one-half of (left) and equal to (right) the maximal gap allowed by modular invariance. **Bottom:** The bounds on the modular spectral function for $c = 4$ with the maximal dimension gap (left) and for $c = 8$ with the maximal gap in the spectrum of scalar primaries (right). In all cases, the dotted lines denote the modular spectral function for the corresponding extremal spectrum.
Only scalar primaries

As an additional example to illustrate the convergence of the bounds on the modular spectral function, we revisit the case of a CFT with only scalar primary operators. In section 3.3.2, we showed that a unitary \( c > 1 \) CFT with primaries of bounded spins must have a non-compact spectrum with only scalar primaries and a density of states equal to that of Liouville theory. The modular spectral function of Liouville theory is given by

\[
    f_{\text{mod}}^{\text{Liouville}}(\Delta_s) = \text{Erf}(\sqrt{2\pi}(\Delta_s - 2\xi)).
\]

(3.4.15)

Assuming a scalar-only spectrum, and a dimension gap \( 2\xi \) (that is, we impose (3.4.7, 3.4.9) for \( s = 0 \) only with \( \Delta_0^* = 2\xi \)), the numerical results of upper and lower bounds on the modular spectral function are shown in figure 3.7. Note that while the bounds do appear convergent toward the Liouville modular spectral function (as they must), the rate of convergence is rather slow compared to the previous examples of discrete extremal spectra at small \( c \). On the other hand, such a slow convergence with \( N \) is qualitatively similar to our bounds on the 4-point spectral function in the scalar-only case, as analyzed in section 3.2.2, where we also expect a continuous spectrum, and also to the non-compact example in the next subsection.

We thus expect that the slow convergence is associated with continuity of the spectrum. This is natural
from the point of view of extremal functionals – the extremal spectrum should converge to the continuous one as $N \to \infty$, but at any finite $N$ the extremal spectrum has finitely many operators, which therefore should condense. On the other hand, in the discrete case we typically need only a small number of operators to accurately determine the partition function or the correlation function in the neighborhood of the crossing/modular symmetric point.

**No scalar primaries**

In [101], following the observation that the bound on the gap of the dimension of scalar primaries diverged as $c \to 25^-$, it was shown that for $c \geq 25$ there exist (non-compact) modular-invariant spectra with no scalar primary operators. This is due to the fact that the modular invariant function

$$Z^{\text{no-scalar}}(\tau, \bar{\tau}) = \frac{J(\tau) + \bar{J}(\bar{\tau})}{\tau_2^{\frac{1}{2}} |\eta(\tau)|^2},$$

(3.4.16)

where $J(\tau) = j(\tau) - 744$, may be interpreted as the partition function of a unitary non-compact CFT with no scalar primary operators, twist gap $\frac{c-25}{12}$ and dimension gap $\frac{c-13}{12}$ for $c \geq 25$. This spectrum turns out to saturate the bound on the dimension gap in the case that there are no scalar primaries in the spectrum.

Writing $J(\tau) = \sum_{s=-1}^{\infty} j_s q^s$, the modular spectral function takes the form

$$f^{\text{no-scalar}}_{\text{mod}}(\Delta_s) = \frac{1}{984} \sum_{s=-1}^{[\Delta_s - 2 \xi]} j_s \text{Erf}(\sqrt{2\pi}(\Delta_s - 2 \xi - s)).$$

(3.4.17)

To compute the bounds on the modular spectral function in this case, we impose (3.4.7,3.4.9) for $s > 0$ with $\Delta^*_s = \max(s, 2 \xi - 1)$. As shown in figure 3.8, the bounds on the modular spectral function do indeed appear to be converging to (3.4.17) as the derivative order of the linear functional is increased, suggesting that the no-scalar spectrum is unique for $c = 25$. Note that for $c = 25$ the dimension gap $\frac{c-13}{12}$ coincides with the unitarity bound (since we assume no scalars). We expect that for $c > 25$ the uniqueness holds only under the assumption of the dimension gap $\frac{c-13}{12}$. 

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Figure 3.8: Bounds on the modular spectral function with no scalar primaries in the spectrum and a dimension gap of $\Delta_{\text{gap}} = \frac{c-13}{12}$ for $c = 25$. The dotted black curve denotes the extremal modular spectral function (3.4.17).

### 3.4.3 CFTs at large $c$ with large gap

In [101], the upper bound on the dimension gap $\Delta_{\text{mod}}(c)$ due to modular invariance of the torus partition function was computed numerically for central charge up to $c \sim \mathcal{O}(10^2)$. As $c$ is increased, the convergence of the upper bound with increasing derivative order $N$ slows, and accurate determinations of the optimal bound on the gap require a careful extrapolation to the limit $N \to \infty$. Nonetheless, a conjecture on the monotonicity of the slope of the optimal bounding curve $\frac{d\Delta_{\text{mod}}(c)}{dc}$ leads one to conclude that the asymptotic slope is less than $\frac{1}{5}$. Potentially, the asymptotic slope could be as small as $\frac{1}{12}$, a possibility that is natural from the holographic perspective (see the discussion in the next section) but with no direct evidence from the analysis of the modular crossing equation.

Thus at large $c$ it has been difficult to determine $\Delta_{\text{mod}}(c)$ accurately, and furthermore the exponential growth of operator degeneracies (combined with the need to go to very large $N$ to get a good approximation of the optimal linear functional at large $c$) makes it practically impossible to resolve the discreteness of the spectrum by bounding the modular spectral function even when the bound $\Delta_{\text{mod}}(c)$ is saturated. Nonetheless, we can study the bounds on the modular spectral function assuming a gap $\Delta_{\text{gap}}$ close to $\Delta_{\text{mod}}(c)$, at values of $c$ where the value of $\Delta_{\text{mod}}(c)$ can be reliably computed by numerical extrapolation of $\Delta_{\text{mod}}^{(N)}(c)$ to $N = \infty$. Figure 3.9 shows plots of the bounds on the modular spectral function for $c = 50, 100, 300$ with assumed
dimension gap $\Delta_{\text{gap}}$ close to the bound $\Delta_{\text{mod}}(c)$.

\begin{figure}[h]
\centering
\subfloat[]{{\includegraphics[width=0.4\textwidth]{figure3a.png}}} \hfill
\subfloat[]{{\includegraphics[width=0.4\textwidth]{figure3b.png}}} \\
\subfloat[]{{\includegraphics[width=0.4\textwidth]{figure3c.png}}}
\caption{Upper and lower bounds on the modular spectral function in the case that the dimension gap is close to the maximal value allowed by modular invariance for $c = 50, 100, 300$. The dotted black curve shows the modular spectral function of perturbative pure gravity due to the thermal $AdS_3$ and Euclidean BTZ saddles in the gravitational path integral.}
\end{figure}

The plots reveal several interesting features that we believe are universal at large $c$ assuming a sufficiently large gap $\Delta_{\text{gap}}(> \frac{c-1}{12})$. Firstly, for $\Delta_* \ll \frac{c}{6}$, the upper and lower bounds on the modular spectral function converge to $f_{\text{mod}}(\Delta_*) = \frac{1}{2}$: that is, modular invariance demands that the vacuum character accounts for exactly half of the partition function at the self-dual temperature. Furthermore, the bounds appear to be convergent upon a smooth function that interpolates between $\frac{1}{2}$ and 1 in a window of size $\sim \sqrt{c}$ about $\Delta_* = \frac{c}{6}$.
It is generally expected that 2D CFTs at large $c$ with large gap should be holographically dual to a semiclassical theory of pure gravity in $AdS_3$. To the best of our knowledge, this statement has not been precisely formulated: how large does the gap need to be? If we merely demand that the dimension gap grows linearly in $c$, corresponding to a Planckian mass gap in the bulk theory, but with a coefficient less than $\frac{1}{12}$, then the entropy need not follow Bekenstein-Hawking in the entire range $\Delta \geq \frac{c}{6}$. This is the range of masses for which BTZ black holes dominate the canonical ensemble at its Hawking temperature. On the other hand, one might expect that CFTs with gap close to $\Delta_{mod}(c)$ (if they exist) are holographic duals to suitable non-perturbative completions of pure gravity in $AdS_3$, in the sense that observables such as the spectral density are correctly captured by the perturbative expansion around known saddle points of the gravitational path integral in the bulk up to $\exp(-c)$ corrections.

This suggests that we compare the bounds on the modular spectral function to that of pure gravity, which, up to a priori unknown non-perturbative corrections, is computed by the contributions from thermal $AdS_3$ and the Euclidean BTZ black hole saddle points, which are known to be perturbatively 1-loop exact. We derive this modular spectral function in appendix A.3, see in particular (A.3.3,A.3.5). The bounds shown in figure 3.9 indeed appear to be converging upon the pure gravity result (3.1.8) for dimensions above the assumed gap.

Note that for $\Delta_{*} \ll \frac{c}{6}$, that the bounds on the modular spectral function with large gap converge to $\frac{1}{2}$ can be explained by the fact that in the semiclassical limit, the gravitational path integral evaluated at the self-dual temperature is dominated by the contributions of two saddles mentioned above, which are exchanged by the modular $S$ transformation, and thus the vacuum contribution accounts for $\frac{1}{2}$. Interestingly, at large $c$, the upper and lower bounds on the vacuum contribution already converge to $\frac{1}{2}$ when the dimension gap is slightly above $\frac{c-1}{12}$, not necessarily close to $\Delta_{mod}(c)$. This is illustrated in figure 3.10, where we plot the bounds on the contribution of the vacuum to the modular spectral function as a function of the dimension gap for $c = 8, 50, 100$. Note that the vacuum contribution to the spectral function determines the partition.

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21In a non-compact CFT where the vacuum is absent, by gap we mean the dimension of the lightest primary.
function $Z(\tau, \bar{\tau})$ itself at the self-dual temperature ($\tau = -\bar{\tau} = i$). From the bulk perspective, that the vacuum accounts for $\frac{1}{2}$ of the modular spectral function amounts to the statement that the thermal $AdS_3$ and BTZ saddle points are the two dominant saddle points in the gravitational path integral, while all other saddle points are exponentially suppressed.\footnote{This property does not hold when the dimension gap is less than or equal to $c - \frac{1}{12}$, even if the former is of order $c$, at large $c$. A possible bulk interpretation is that there are singular saddle point contributions (such as the Euclidean continuation of the massless BTZ black hole) to the gravity path integral where the pure gravity perturbation theory breaks down.}

![Graphs showing bounds on the contribution of the vacuum to $f_{\text{mod}}$.](image)

**Figure 3.10**: Bounds on the contribution of the vacuum character to the modular spectral function as a function of the imposed gap in the dimensions of primary operators for $c = 8, 50, 100$.

In [64], it was shown that in theories where the light spectrum is appropriately sparse (a condition of which the maximal gap is an extreme case), the microcanonical entropy is given universally by the Cardy formula

\[ S_{\text{micro}} = \frac{c}{3} \log \left( \frac{c^2}{3\pi} \right) \]

at large $c$. A possible bulk interpretation is that there are singular saddle point contributions (such as the Euclidean continuation of the massless BTZ black hole) to the gravity path integral where the pure gravity perturbation theory breaks down.
to leading order for all operators with dimension $\Delta \geq \frac{c}{6}$. Note that the sparseness criterion is satisfied by our assumption on the gap, but our statement regarding the modular spectral function and thereby the spectral density extends to the regime of $\Delta$ slightly below $\frac{c}{6}$ (see further discussion in the next section).

We conjecture that in the large $c$ limit, assuming a gap sufficiently close to $\Delta_{\text{mod}}(c)$, the bounds $f_{\text{mod}}^{\pm}(c)$ converge onto the modular spectral function of pure gravity described above, up to order $\exp(-c)$ corrections. Indeed, we note that for $c \sim \mathcal{O}(10^2)$, the horizontal average of the bounds $\overline{f_{\text{mod},N}(\Delta_*)}$ is already well approximated by the pure gravity modular spectral function at moderate values of $N$, as shown in figure 3.11.

![Figure 3.11](image)

**Figure 3.11:** The horizontal average of the upper and lower bounds on the modular spectral function for $c = 100$ with dimension gap close to the upper bound imposed by modular invariance.

### 3.5 On the universality of the BTZ spectral density

The BTZ black hole in $AdS_3$ has a striking feature that is unlike black holes in other spacetime dimensions (in asymptotically either AdS or flat spacetime): a Planckian mass BTZ black hole has a macroscopic horizon radius (rather than, say, Planckian radius), provided that the mass is an order 1 fraction above the BTZ threshold in Planck units. The microstates of such a black hole would be dual to an operator in the CFT of dimension $\Delta > (1 + \epsilon) \frac{c}{12}$, where $\epsilon$ is an order 1 fraction that does not scale with $c$. When there is

As remarked earlier, it is likely that a gap not too far above $\frac{c}{12}$ will suffice, not necessarily close to $\Delta_{\text{mod}}(c)$.
a sufficiently large mass gap in the spectrum, standard effective field theory reasoning in the bulk would suggest that the the entropy of the BTZ black hole, which captures the degeneracy of microstates, should be computed from the Bekenstein-Hawking formula based on the Einstein-Hilbert action, as any local higher-derivative corrections to the Einstein-Hilbert action of pure gravity in three dimensions can be absorbed by field redefinition. This would predict a degeneracy or spectral density

\[ \rho(\Delta) \sim \exp \left[ 2\pi \sqrt{\frac{c}{3}(\Delta - \frac{c}{12})} \right] \]

(3.5.1)

to leading order in the large \( c \) limit, for \( \Delta/c > \frac{1}{12} \).

For \( \Delta > \frac{c}{6} \), i.e., above twice the BTZ threshold, this universal behavior of the spectral density was demonstrated in [64] to be a consequence of the sparseness of the spectrum and modular invariance. From the gravity perspective, this is also the regime in which the Euclidean BTZ black hole solution is the dominant saddle point of the Euclidean pure gravity path integral, i.e., the BTZ black hole dominates the canonical ensemble at its Hawking temperature, and therefore the spectral density must be (3.5.1) in order to produce the correct free energy above the self-dual temperature.

The regime \( \frac{c}{12} < \Delta < \frac{c}{6} \) is much more interesting. Here the BTZ black hole does not dominate the canonical ensemble. Its contribution to the gravitational free energy is non-perturbatively suppressed compared to the thermal \( AdS_3 \) contribution. A priori, since we do not know the most general non-perturbative contributions to the pure gravity path integral, we cannot draw any reliable conclusion on the spectral density in this regime. This also puts doubt on the validity of the Bekenstein-Hawking formula, despite the macroscopic size of the horizon. If the Bekenstein-Hawking formula is violated in this regime, it then indicates some sort of breakdown of the effective field theory reasoning based on locality.

What conclusion can we draw from our numerical bounds on the modular spectral function for \( c \sim O(10^2) \)? We saw that in a window of size \( \sqrt{c} \) around \( \frac{c}{6} \), the modular spectral function is constrained to well approximate the \( AdS_3 + BTZ \) answer, in agreement with the expectation from the known perturbative
contributions to the Euclidean gravity path integral. Unfortunately, our numerical results do not have sufficient resolution to allow for distilling a contribution of order \( \exp(-c) \), thus preventing us from concluding whether the Bekenstein-Hawking entropy of BTZ correctly accounts for the spectral density in the regime \( \Delta = yc \), for \( \frac{1}{12} < y < \frac{1}{6} \). In fact, if the latter is true, then the asymptotic slope of the modular bound on the dimension gap, \( \lim_{c \to \infty} d\Delta_{\text{mod}}(c)/dc \), must be equal to \( \frac{1}{12} \), but this has not been shown. Thus, the fate of the small-yet-large BTZ black holes below twice the BTZ threshold remains a mystery.

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4

Recursive representations of arbitrary Virasoro conformal blocks

4.1 Introduction

A two-dimensional conformal field theory is characterized by its spectrum of Virasoro primaries and their OPE coefficients. Given these data, all correlation functions of the CFT on any Riemann surface can be constructed, through the Virasoro conformal blocks [13, 37] which sum up all descendant contributions of the conformal families in consideration. Direct evaluation of the conformal blocks based on the definition by summing over Virasoro descendants is computationally costly and is practically intractable beyond the first
few levels even with computer algebra.

An efficient method for computing the sphere 4-point Virasoro conformal block was found by Zamolodchikov in [59], in the form of a recurrence relation in the central charge $c$. This is based on the observation that the conformal block can be analytically continued as a meromorphic function in $c$, whose poles are dictated by degenerate representations of the Virasoro algebra, together with a simplification in the large $c$ limit where the Virasoro block reduces to that of the global conformal group $SL(2)$. An analogous recurrence formula through the analytic continuation in the internal weight $h$ rather than the central charge was subsequently found in [60]. These recurrence formulae have played an essential role both in computing string amplitudes [108, 109] and in the numerical conformal bootstrap approach to 2D CFTs [100, 102, 110] (in [110], for instance, the explicit expansion of a Virasoro conformal block to level 200 was used).

The recursive representations have also been extended to super-Virasoro conformal blocks [111–113], and to torus 1-point conformal blocks [114, 115]. More general Virasoro conformal blocks (higher points, higher genera) are important to the computation of certain string amplitudes as well as for more sophisticated numerical conformal bootstrap analyses. Our aim is to provide a complete set of recurrence relations for efficient evaluation of Virasoro conformal blocks on a Riemann surface of any genus with any number of external primary vertex operator insertions.

The main results of this chapter are the following:

1. We extend the $c$-recursion relation to sphere and torus $N$-point Virasoro conformal blocks in all channels.

The first key observation, which is common to all recurrence relations discussed in this chapter, is that when we analytically continue in the central charge $c$ or the internal weights $h_i$, there is a pole whenever an internal Virasoro representation becomes degenerate and develops a null state at some level $r_s$ [13, 40]. The residue of this pole is proportional to the conformal block evaluated at the degenerate internal weight shifted by $r_s$, with a universal coefficient that is a known function of the internal and external weights.

With this understanding, the determination of the recurrence relation boils down to identifying the large
c or large internal weight limits. The large \( c \) limit of the sphere \( N \)-point Virasoro conformal block reduces to that of the global \( SL(2) \) block, which is relatively easy to compute. The large \( c \) limit of the torus \( N \)-point Virasoro conformal block turns out to reduce to the product of the torus vacuum character and a corresponding global \( SL(2) \) block. The factorization property of the large central charge limit of the ‘light’ block (with all weights held fixed) was originally observed in the case of the torus 1-point block in [115].

\[

c
\]

\[
\begin{align*}
&d_1 \\
&h_1 \\
&h_2 \\
&h_3 \\
&d_5 \\
&d_4 \\
&d_3
\end{align*}
\]

\[

d_1 \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet
\]

\[
\begin{align*}
&d_2 \\
&d_3 \\
&d_4 \\
&d_{N-1} \\
\end{align*}
\]

\[
\begin{align*}
&d_1 \\
&h_1 \\
&h_2 \\
&h_3 \\
&\ldots \\
&h_{N-3} \\
&d_N
\end{align*}
\]

\[
\begin{align*}
&h_N \quad 1 \quad 2 \quad 3 \\
&\vdots \\
&\vdots \\
&\vdots
\end{align*}
\]

\[
\begin{align*}
&N-1 \\
&h_{N-1} \\
&h_1 \\
&h_2 \\
&h_3 \\
&\ldots \\
&\ldots
\end{align*}
\]

**Figure 4.1:** The sphere six-point block in the trifundamental channel (left) and the torus two-point block in the OPE channel (right). Our \( c \)-recursion representation for arbitrary sphere and torus \( N \)-point blocks enables recursive evaluation of these blocks; we work these cases out explicitly in section 4.4.4.

(2) We find the \( h \)-recursion for the sphere \( N \)-point Virasoro blocks in the linear channel, and torus \( N \)-point Virasoro blocks in the necklace channel.

\[

c
\]

\[
\begin{align*}
&d_2 \\
&d_3 \\
&d_4 \\
&d_{N-1} \\
\end{align*}
\]

\[
\begin{align*}
&d_1 \\
&h_1 \\
&h_2 \\
&h_3 \\
&\ldots \\
&h_{N-3} \\
&d_N
\end{align*}
\]

\[
\begin{align*}
&h_N \quad 1 \quad 2 \quad 3 \\
&\vdots \\
&\vdots \\
&\vdots
\end{align*}
\]

\[
\begin{align*}
&N-1 \\
&h_{N-1} \\
&h_1 \\
&h_2 \\
&h_3 \\
&\ldots \\
&\ldots
\end{align*}
\]

**Figure 4.2:** The torus \( N \)-point block in the necklace channel (left) and the sphere \( N \)-point block in the linear channel (right).
To determine the $h$-recursion relations, we need to know the behavior of the Virasoro conformal block in suitable large internal weight limits, which turns out to be very subtle. In the case of the torus $N$-point block in the necklace channel, for instance, the simplification occurs when all internal weights $h_i$ along the necklace are taken to infinity simultaneously, with $h_i - h_j$ kept finite. In this limit, the necklace conformal block reduces to a non-degenerate torus Virasoro character. This observation is powerful enough to determine the recurrence relation for the necklace conformal block.

A degeneration limit of the torus $(N - 1)$-point necklace block gives the sphere $N$-point conformal block in the linear channel. In the latter case, our recurrence relation makes use of the limit where all the internal weights $h_i$ and a pair of external weights $d_1$ and $d_N$ are taken to infinity simultaneously along a line that begins on $d_1$ and ends on $d_N$, again with their differences $h_i - h_j, h_i - d_1$, and $h_i - d_N$ kept finite. Note that this is different from Zamolodchikov’s $h$-recurrence relation in the sphere 4-point case, where the recursion only applies to the internal weight. In particular, in our formulation of the sphere $N$-point recursion in the linear channel, it suffices to work with the standard cross ratios rather than Zamolodchikov’s elliptic nome.

(3) We give a complete set of recipes for the $c$-recursion relation for the most general $N$-point Virasoro conformal blocks on a genus $g$ Riemann surface, based on a plumbing construction through a given pair-of-pants decomposition.

In formulating the higher genus Virasoro conformal blocks, based on a particular pair-of-pants decomposition, one must choose a conformal frame defined by a choice of the fundamental domain and gluing maps along its boundaries. Differences in the choice of conformal frame not only lead to different parameterizations of the moduli, but also extra factors multiplying the conformal block due to the conformal anomaly. We choose to construct the (punctured) Riemann surface by gluing together 3-holed Riemann spheres, represented by 2-holed discs on the complex plane, with $SL(2, \mathbb{C})$ Möbius maps along their boundary components. Formally, since only $SL(2)$ maps are used in such a plumbing construction, it also makes sense to define a corresponding global $SL(2)$ block, by summing up $L_{-1}$ descendants at the holes.

We will show that in this frame, the genus $g$, $N$-point Virasoro conformal block remains finite in the
$c \to \infty$ limit. In particular, the same is true for the genus $g$ vacuum block, whose large $c$ limit is expected to exponentiate into the form $e^{-cF_0}$ to leading order, where $F_0$ is the holomorphic part of a suitably regularized Einstein-Hilbert action on a hyperbolic handlebody \cite{116, 117}. In our frame, $F_0$ is simply zero, and the $c \to \infty$ limit of the vacuum block is finite. Further, the finite part of the $c \to \infty$ vacuum block is given by the 1-loop partition function of 3D pure gravity on the hyperbolic handlebody, as computed in \cite{118}.

We will show that the $c \to \infty$ limit of the genus $g$ Virasoro conformal block factorizes into the product of the $c \to \infty$ vacuum block and the global $SL(2)$ block defined through the above mentioned plumbing construction. This is a generalization of the factorization property of the light block at large central charge first proven in the case of the torus 1-point block in \cite{115}.

The chapter is organized as follows. In section 4.2 we review Zamolodchikov’s recurrence relations for the sphere 4-point Virasoro conformal block. The $h$-recurrence relations for torus $N$-point necklace channel conformal block and the sphere $N$-point linear channel conformal block are derived in section 4.3. In section 4.4, we formulate and prove the $c$-recurrence relation for sphere and torus $N$-point Virasoro conformal blocks in arbitrary channels. The generalization to higher genus is presented in section 4.5. We conclude in section 4.6 with a discussion of potential applications of our results, and issues concerning the mapping of moduli parameters for the higher genus conformal blocks.
4.2 Review of the sphere 4-point Virasoro block

In this section, we review the recursive representations of the sphere 4-point Virasoro conformal blocks, originally derived in [59,60]. We follow the notations and derivations of [105,111,114,119] in detail, as we will generalize their features to higher-point cases in later sections.

4.2.1 Definition of Virasoro conformal block

Using the global $SL(2, \mathbb{C})$ invariance, the 4-point function of Virasoro primaries of weight $(d_i, \bar{d}_i), i = 1, ..., 4$, on the Riemann sphere can be brought to the form

$$\langle \phi_4'(\infty, \infty) \phi_3(1, 1) \phi_2(z, \bar{z}) \phi_1(0, 0) \rangle = \langle \nu_4 \times \bar{\nu}_4 | \phi_3(1, 1) \phi_2(z, \bar{z}) | \nu_1 \times \bar{\nu}_1 \rangle,$$

(4.2.1)

where $\phi'(\infty, \infty) = \lim_{w, \bar{w} \to \infty} w^{2d_4} \bar{w}^{2\bar{d}_4} \phi_4(w, \bar{w})$, $| \nu_1 \times \bar{\nu}_1 \rangle$ is the state corresponding to the primary operator $\phi_i$ inserted at the origin in radial quantization, and $\langle \nu_1 \times \bar{\nu}_1 |$ is the BPZ conjugate. Inserting a complete set of states in between $\phi_2$ and $\phi_3$, we can write

$$\langle \nu_4 \times \bar{\nu}_4 | \phi_3(1, 1) \phi_2(z, \bar{z}) | \nu_1 \times \bar{\nu}_1 \rangle =$$

$$\sum_{h, \bar{h}} \sum_{|M| = |N| = n} \langle \nu_4 \times \bar{\nu}_4 | \phi_3(1, 1) | L_{-N} \nu_h \otimes L_{-P} \bar{\nu}_h \rangle (G_{c,h}^n)^{NM} (G_{c,\bar{h}}^m)^{PQ} \langle L_{-M} \nu_h \otimes L_{-Q} \bar{\nu}_h | \phi_2(z, \bar{z}) | \nu_1 \times \bar{\nu}_1 \rangle.$$

(4.2.2)

Let us explain the notations here. The first sum is over the spectrum of Virasoro primaries of weights\(^{1}\) $(h, \bar{h})$ and the second sum is over descendants in the corresponding conformal family. $M, N, P, Q$ are integer partitions in descending order that label Virasoro descendants. $L_{-N}$ stands for a chain of Virasoro generators corresponding to a specific partition $N$ of the non-negative integer $n = |N|$. For example, $N = \{2, 1, 1\}$ with $|N| = 4$ gives rise to $L_{-N} = L_{-2} L_{-1} L_{-1}$. $G_{c,h}^n$ is the Gram matrix at level $n$ for a weight $h$ representation.

---

\(^{1}\)To avoid overly cluttered notation, we have omitted the labels of possibly degenerate primaries, which can be restored easily when necessary.
of the Virasoro algebra of central charge \( c \), and \( \left( G^{n}_{c,h} \right)^{NM} \) stands for the inverse Gram matrix element.

We will make extensive use of the 3-point function of general Virasoro descendants, which factorizes into its holomorphic and anti-holomorphic parts, of the form [105]

\[
\langle \xi_3 \otimes \bar{\xi}_3 | V_2(z, \bar{z}) | \xi_1 \otimes \bar{\xi}_1 \rangle = C_{321} \rho(\xi_3, \xi_2, \xi_1 | z) \rho(\bar{\xi}_3, \bar{\xi}_2, \bar{\xi}_1 | \bar{z}).
\]

(4.2.3)

Here \( V_i \) represents a general Virasoro descendant of the primary \( \phi_i \), while \( (\xi_i, \bar{\xi}_i) \) label the corresponding states in the Verma modules associated with the left and right Virasoro algebra. The structure constant \( C_{321} \) is the coefficient of the 3-point function of the primaries \( \phi_3, \phi_2, \phi_1 \). \( \rho(\xi_3, \xi_2, \xi_1 | z) \) is determined entirely by the Virasoro algebra in terms of the weights of the primaries, as we briefly review in appendix B.1; in particular, for primary states \( \nu_i \), we have \( \rho(\nu_3, \nu_2, \nu_1 | z = 1) = 1 \). Using this notation, the 4-point function can be written as

\[
\langle \nu_4 \otimes \bar{\nu}_4 | \phi_3(1, 1) \phi_2(z, \bar{z}) | \nu_1 \otimes \bar{\nu}_1 \rangle = \sum_{h, \bar{h}} C_{43(\bar{h}, h)} C_{(h, \bar{h})21} z^{h-d_2-d_1} F(h, z, d_1, d_2, d_3, d_4, c) \times \bar{z}^{\bar{h}-\bar{d}_2-\bar{d}_1} F(\bar{h}, \bar{z}, \bar{d}_1, \bar{d}_2, \bar{d}_3, \bar{d}_4, c),
\]

(4.2.4)

where \( F(h, z, d_1, d_2, d_3, d_4, c) \) is the holomorphic Virasoro conformal block of interest,

\[
F(h, z, d_1, d_2, d_3, d_4, c) = \sum_{\mu = |N| = 0} z^n \rho(\nu_4, \nu_3, L_{-N} \nu_1 | 1) \left( G^{n}_{c,h} \right)^{NM} \rho(L_{-M} \nu_2, \nu_2, \nu_1 | 1).
\]

(4.2.5)

Let us note a subtlety in our convention of \( \rho(\xi_3, \xi_2, \xi_1 | z) \) that will become particularly important later for the torus and higher genus conformal blocks. In the definition of this 3-point function, \( \xi_1 \) and \( \xi_2 \) are Virasoro descendants of the form \( L_{-N_1} | h_1 \rangle \) and \( L_{-N_2} | h_2 \rangle \) inserted on the complex plane at 0 and \( z \), while \( \xi_3 \) is the BPZ conjugate of a state of the form \( L_{-N_3} | h_3 \rangle \), inserted at \( \infty \). In constructing a more general conformal block, associated with a pair-of-pants decomposition of a punctured Riemann surface, we will be contracting such 3-point functions of descendants with inverse Gram matrices. This corresponds to a
plumbing construction where we cut out holes centered at 0, z, and \( \infty \) on the complex plane, resulting in 2-holed discs, and identify boundary components of pairs of 2-holed discs via \( SL(2, \mathbb{C}) \) Möbius maps. This amounts to a choice of conformal frame for the conformal block in question, which turns out to be particularly convenient for the \( c \)-recursive representation to be discussed later. A different choice of frame would generally lead to a conformal block that differs by a factor of the conformal anomaly.

One could consider a different 3-point function of descendants, \( \tilde{\rho}(\xi_3, \xi_2, \xi_1|w) \), defined as the matrix element of the Virasoro descendant \( \xi_2 \) inserted at position \( w \) on the cylinder \( w \sim w + 2\pi \), between the states \( |\xi_3\rangle \) and \( |\xi_1\rangle \) on the cylinder (say both defined at \( \text{Im} w = 0 \)). While the cylinder can be conformally mapped to the complex plane via \( z = e^{-iw} \), \( \xi_2 \) being a descendant does not transform covariantly. For instance, \( \tilde{\rho}(\xi_3, \xi_2, \xi_1|w = 0) \) coincides with \( \rho(\xi_3, \xi_2, \xi_1|z = 1) \) when \( \xi_2 \) is a primary, but not otherwise. For certain conformal blocks it may be convenient to use a plumbing construction based on gluing together 1-holed cylinders rather than 2-holed discs, which would amount to contracting 3-point functions like \( \tilde{\rho}(\xi_3, \xi_2, \xi_1|0) \) rather than \( \rho(\xi_3, \xi_2, \xi_1|1) \) with inverse Gram matrices. This would result in the block in a different conformal frame.

### 4.2.2 Simple pole structure and its residue

Let us now consider the analytic continuation of the Virasoro conformal block in \( h \) and in \( c \). The presence of the inverse Gram matrix in (4.2.5) introduces simple poles, corresponding to the values of \( h \) and \( c \) where the Virasoro representation admits a null state at the corresponding level. Therefore, one can write

\[
F(h, z, d_1, d_2, d_3, d_4, c) = f_h(h, z, d_1, d_2, d_3, d_4, c) + \sum_{r \geq 1, s \geq 1} \frac{V_{rs}(z, d_1, d_2, d_3, d_4, c)}{h - d_{rs}(c)}
\]

\[
= f_c(h, z, d_1, d_2, d_3, d_4, c) + \sum_{r \geq 2, s \geq 1} \frac{W_{rs}(z, h, d_1, d_2, d_3, d_4, c - c_{rs}(h))}{c - c_{rs}(h)},
\]

where \( f_h \) and \( f_c \) are entire holomorphic functions in \( h \) and in \( c \) respectively. In the first line, we have assumed a generic value of \( c \), whereas in the second we have assumed a generic value of \( h \). The pole positions \( d_{rs}(c) \)
and $c_{rs}(h)$ are $[13,40]^2$

$$d_{rs}(c) = \frac{(b + b^{-1})^2}{4} - \frac{(rb + sb^{-1})^2}{4} \quad \text{with} \quad c = 1 + 6(b + b^{-1})^2, \quad r = 1, 2, 3, ..., \quad s = 1, 2, 3, ...$$

$$c_{rs}(h) = 1 + 6(b_{rs}(h) + b_{rs}(h)^{-1})^2 \quad \text{with} \quad b_{rs}(h)^2 = \frac{rs - 1 + 2h + \sqrt{(r - s)^2 + 4(rs - 1)h + 4h^2}}{1 - r^2}, \quad (4.2.7)$$

$$r = 2, 3, 4, ..., \quad s = 1, 2, 3, ...$$

Note that the two types of residues $V_{rs}$ and $W_{rs}$ are related by

$$W_{rs}(z, h, d_i) = \frac{\partial c_{rs}(h)}{\partial h} V_{rs}(z, d_i, c = c_{rs}(h)). \quad (4.2.8)$$

The Verma module of the degenerate primary of weight $d_{rs}$ contains a null descendant at level $rs$. In the degeneration limit $h \rightarrow d_{rs}$, a new primary emerges at level $rs$ in place of the null state, which generates a sub-Verma module. The key observation in [59] was that the residue at $h = d_{rs}$ is proportional to the Virasoro block whose internal representation is given by this sub-Verma module, namely one with internal weight $d_{rs} + rs$. This can be seen from (4.2.5) as follows. Following [111,114,119], we write the null descendant at level $rs$ corresponding to $d_{rs}$ as

$$\chi_{rs} = \sum_{|M| = rs} \chi_{rs}^M L_{-M} \nu_{d_{rs}}, \quad (4.2.9)$$

where the normalization convention is such that the coefficient $\chi_{rs}^{1,1,\cdots,1}$ of $L_{-1}^r$ is equal to 1. For any Verma module associated to a primary of weight $h$, one can choose a basis for the level $rs$ and higher descendants that includes the states

$$L_{-N} \chi_{rs}^h \quad \text{with} \quad \chi_{rs}^h = \sum_M \chi_{rs}^M L_{-M} \nu_k. \quad (4.2.10)$$

Here, $\chi_{rs}^M$ is the coefficient that appears in (4.2.9), whereas $\chi_{rs}^h$ denotes a state (at level $rs$, which is not null for generic $h$). Other basis states are chosen generically. By definition, $\lim_{h \rightarrow d_{rs}} \chi_{rs}^h = \chi_{rs}$. The residue $V_{rs}$ in (4.2.6) receives contributions only from descendants of the form $L_{-N} \chi_{rs}^h$ (whose level is $rs + |N|$), and is

2 As a subscript, $rs$ stands for separate labels $r$ and $s$, not to be confused with the product $rs$. 

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given by

\[ V_{rs}(z, d, c) = \lim_{h \to d_{rs}} (h - d_{rs}) F(h, z, d, c) \]

\[ = A_{rs}^c z^{rs} \sum_{|N| = |M| = n \geq 0} z^n \rho(\nu_4, \nu_3, L_{-N} \chi_{rs}|1) \left( G_{c,d_{rs}+rs}^n \right)^{NM} \rho(L_{-M} \chi_{rs}, \nu_2, \nu_1|1), \]  

(4.2.11)

where

\[ A_{rs}^c = \lim_{h \to d_{rs}} \left( \frac{\langle \chi_{rs}^h \rangle}{h - d_{rs}} \right)^{-1} = \frac{1}{2} \prod_{m=1-r}^{r} \prod_{n=1-s}^{s} (mb + nb^{-1})^{-1}, \quad (m, n) \neq (0, 0), (r, s) \]  

(4.2.12)

is guessed in [59] and checked in [120]. A key property that will be used repeatedly later is the factorization

\[ \rho(L_{-M} \chi_{rs}, \nu_2, \nu_1|1) = \rho(L_{-M} \chi_{d_{rs}+rs}, \nu_2, \nu_1|1) \rho(\chi_{rs}, \nu_2, \nu_1|1). \]  

(4.2.13)

Here, \( \nu_{d_{rs}+rs} \) stands for a primary of weight \( d_{rs}+rs \). The second factor on the RHS is the fusion polynomial

\[ \rho(\chi_{rs}, \nu_2, \nu_1|1) = P_{c}^{\nu_{rs} \nu_{d_{rs}+rs}} \left[ \begin{array}{c} d_1 \\ d_2 \end{array} \right] \]

\[ = \prod_{p=1-r}^{r} \prod_{q=1-s}^{s} \frac{\lambda_1 + \lambda_2 + pb + qb^{-1}}{2} \frac{\lambda_1 - \lambda_2 + pb + qb^{-1}}{2}, \]  

(4.2.14)

where the products are taken over \( p+r = 1 \) mod 2, \( q+s = 1 \) mod 2, and \( \lambda_i \) are defined by \( d_i = \frac{1}{4}(b + b^{-1})^2 - \frac{1}{4} \lambda_i^2 \).

By plugging (4.2.13) into (4.2.11) and comparing with (4.2.5), we determine the residue

\[ V_{rs}(z, d, c) = z^{rs} A_{rs}^c P_{c}^{\nu_{rs} \nu_{d_{rs}+rs}} \left[ \begin{array}{c} d_1 \\ d_2 \end{array} \right] \left[ \begin{array}{c} d_4 \\ d_3 \end{array} \right] F(h \to d_{rs} + rs, z, d, c). \]  

(4.2.15)

Indeed, the residue is proportional to the Virasoro conformal block with internal weight evaluated at the null descendant value \( d_{rs} + rs \). This sets a recursive representation of the Virasoro block, once the regular term \( f_h \) or \( f_c \) in (4.2.6) is known. In particular, the presence of the factor \( z^{rs} \) in (4.2.15) allows for the determination
of the coefficient at any given order in the power series expansion in $z$ by finitely many iterations of (4.2.6).

### 4.2.3 Determining the regular part

First, let us determine the regular part $f_c(h, z, d_i, c)$ in (4.2.6) by studying the conformal block in the large-$c$ limit. The latter is computable by inspecting the definition (4.2.5). It follows from Ward identities that the 3-point function of the form $\rho(\nu_4, \nu_3, L_{-N} \nu_0|1)$ is independent of $c$, simply because there are no non-$L_{-1}$ Virasoro generators acting on $\nu_4$ and $\nu_3$. Meanwhile, the inverse Gram matrix elements are suppressed in the large $c$ limit, except for one matrix element that corresponds to the inner product of a pair of $L_{n-1}$ descendants,

$$\lim_{c \to \infty} (G_{c.h})^{L_{n-1} L_{n-1}} = \frac{1}{n!(2h)_n}, \quad (4.2.16)$$

where $(a)_n \equiv a(a+1)(a+2)...(a+n-1)$ is the Pochhammer symbol. This gives the only level $n$ term in (4.2.5) that survives at $c \to \infty$. Using $\rho(L_{n-1} \nu_h, \nu_2, \nu_1) = (h + d_2 - d_1)_n$, we obtain the result

$$f_c(h, z, d_i, c) = \sum_{n=0}^{\infty} z^n \frac{(h + d_2 - d_1)_n (h + d_3 - d_4)_n}{n!(2h)_n} = _2F_1(h + d_2 - d_1, h + d_3 - d_4, 2h, z). \quad (4.2.17)$$

In particular, $f_c$ is independent of $c$. This feature will make a reappearance in other cases to be considered later. It is often asserted that “the large-$c$ limit of the Virasoro block is the global $SL(2)$ block”, referring to the fact that only the contributions of the $L_{n-1}$ descendants survive in the large $c$ limit here. We will see later that this is not true for the large $c$ limit of torus and higher genus Virasoro conformal blocks, but suitable modifications of the statement do hold.

Together with $W_{rs}$ acquired by (4.2.8) and (4.2.15), we have a complete $c$-recursive representation of the
sphere 4-point Virasoro conformal block

\[ F(h, z, d_i, c) = {}_2F_1(h + d_2 - d_1, h + d_3 - d_4, 2h, z) \]

\[ + \sum_{r \geq 2, s \geq 1} \frac{\partial c_{rs}(h)}{\partial h} \frac{z^{rs} A_{rs}^c}{c - c_{rs}(h)} P_{rs}^{c_{rs}} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} F(h \to h + rs, z, d_i, c \to c_{rs}). \]  

(4.2.18)

The story for the \( h \)-regular part \( f_h \) is more complicated. In [60], Zamolodchikov considered a semiclassical limit of large \( c \) with ratios \( c/h, c/d_i \) kept finite, where the conformal block is expected to be the exponential of a “classical block” of order \( c \). Through the monodromy equation related to the classical block, the large-\( h \) behavior was determined as a function of the elliptic nome \( q \), related to the cross ratio \( z \) by \( q = \exp\left(\frac{i\pi K'(z)}{K(z)}\right) \), where \( K(z) \) is the complete elliptic integral of the first kind. The final answer is

\[ z^{h-d_1-d_2} F(h, z, d_i, c) \]

\[ = (16q(z))^{h - \frac{(c-1)}{24}} \frac{\theta_3(q(z))^{\frac{(c-1)}{2}}}{\theta_3(q(z))^{\frac{(c-1)}{2}}} - 4(d_1 + d_2 + d_3 + d_4) H(c, h, d_i, q(z)), \]  

(4.2.19)

where \( H(c, h, d_i, q) \) is determined recursively,

\[ H(c, h, d_i, q) = 1 + \sum_{rs \geq 1} \frac{(16q)^{rs} h - d_{rs}}{r s c} P_{rs}^{c_{rs}} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} H(c, h \to d_{rs} + rs, d_i, q). \]  

(4.2.20)

An alternative viewpoint on the \( q \)-expansion was provided in [121]. There, the 4-punctured sphere was mapped to the “pillow” geometry \( T^2/\mathbb{Z}_2 \) with four corners. There is an external vertex operator insertion at each corner. The \( q \)-expansion has the natural interpretation in terms of matrix elements of the propagator along the pillow, between states created by pairs of vertex operators at the corners. The \( q \)-expansion of the Virasoro conformal block converges uniformly on the unit \( q \)-disc \( |q| < 1 \), which extends beyond the complex \( z \)-plane; for this reason, it is typically preferred in evaluations at high precision such as in numerical bootstrap, as well as for analytic continuation to Lorentzian signature. The solution to the recursion relations was studied in [122].
At the moment, it is unclear whether there is a useful analog of the $q$-expansion for more general Virasoro conformal blocks (higher points, higher genus). In the next section, we will instead work with an expansion in the cross ratio $z$ for the sphere $N$-point block in the linear channel, and derive a recursion relation that involves simultaneous shifts of the internal weights and a pair of external weights. In particular, we will not derive the analog of $f_h$ in (4.2.6), but rather a different kind of large-weight limit of the conformal block. The specialization of our $h$-recursion formula to the sphere 4-point block case differs from Zamolodchikov’s $h$-recursion in several ways: we do not make use of the elliptic nome, the regular (non-polar) part is very simple, but the recursion involves shifting both internal and external weights.

4.3 $h$-recursion for torus $N$-point Virasoro conformal blocks in the necklace channel (and sphere $N$-point blocks in the linear channel)

\[ w_1 = 2\pi \tau \]
\[ w_2 = 2\pi \tau_1 \]
\[ w_3 = 0 \]
\[ w_4 = 2\pi \]

**Figure 4.4:** $N$-cylinder decomposition of necklace channel for $N = 3$ case.

In this section, we derive recursion relations in the internal weights $h_i$ for torus $N$-point Virasoro blocks in the necklace channel for $N > 1$. The $N = 1$ case is studied in [114] and we will discuss this case in the
next section. The necklace channel is where complete sets of states are inserted in between every consecutive pair of external operators as shown in figure 4.2. This channel can also be viewed as the $N$-cylinder channel, where the torus is decomposed into $N$ cylinders, each of which contains exactly one external operator of weight $d_i$ at its origin. The $N = 3$ case is illustrated in figure 4.4. The result for the torus necklace channel reduces in a limit to the sphere block in the linear channel, thereby giving a recursion formula for the latter as well.

4.3.1 Definition of the Virasoro block in the necklace channel

Consider a torus of modulus $\tau$, parameterized by a complex coordinate $z$, with the identification $z \sim z + 2\pi \sim z + 2\pi \tau$. We consider $N$ primary operators $\phi_i$ of weights $(d_i, \bar{d}_i)$ inserted at positions $w_i$ for $i = 1, \cdots, N$. We set $w_N = 0$ by convention, and write $w_i = 2\pi(\tau - \sum_{k=1}^{i-1} \tau_k)$. In the necklace channel, the torus is decomposed into $N$ cylinders, of moduli $\tau_1, \tau_2, \cdots, \tau_N$, with $\sum_{k=1}^{N} \tau_k = \tau$. We will also write $q_i = e^{2\pi i \tau_i}$.

The torus $N$-point function is decomposed in terms of Virasoro conformal blocks in this channel as

$$
(O_1(w_1)O_2(w_2)\cdots O_{N-1}(w_{N-1})O_N(0))_{\tau^2} = \sum_{(h_1, \bar{h}_1) \cdots (h_N, \bar{h}_N)} \left(\prod_{i=1}^{N} C_{h_i, d_i, \bar{h}_{i+1}} q_i^{h_i-c/24} \bar{q}_i^{\bar{h}_i-c/24}\right) \times F(q_1, h_1, d_1, \cdots, q_N, h_N, d_N, c) F(\bar{q}_1, \bar{h}_1, \bar{d}_1, \cdots, \bar{q}_N, \bar{h}_N, \bar{d}_N, c) .
$$

(4.3.1)

Here $h_i, \bar{h}_i$ are the weights of the internal primaries. $F(q_1, h_1, d_1, \cdots, q_N, h_N, d_N, c)$ is the holomorphic torus $N$-point necklace conformal block,

$$
F(q_1, h_1, d_1, \cdots, q_N, h_N, d_N, c) = \sum_{n_1, \cdots, n_N=0}^{\infty} \left(\prod_{i=1}^{N} q_i^{n_i}\right) \sum_{|A_1|=|B_1|=n_1} \cdots \sum_{|A_N|=|B_N|=n_N} \left[\prod_{i=1}^{N} (G_{h_i}^{n_i})^{A_i, B_i} \rho(L_{-B_i}, d_i, L_{-A_{i+1}}, h_{i+1})\right] .
$$

(4.3.2)

The index $i$ ranges from 1 to $N$ cyclic, i.e. $i = N + 1$ is identified with $i = 1$. Here we have made use of an exponential mapping from each of the cylinders to the annulus, relating the matrix element of the primary $\phi_i$ at $w_i$ between a pair of descendant states to the 3-point function on the $z$-plane with the primary $\phi_i$. 

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inserted at $z = 1$. By a slight abuse of notation, in the sphere 3-point function of descendants $\rho$ we have labeled the primaries $\nu_i$ simply by their weights $h_i$, and have set $z = 1$.

### 4.3.2 Polar part

Again due to the presence of the inverse Gram matrix, (4.3.2) has simple poles in $h_i$ or $c$ at values corresponding to degenerate Virasoro representations. Focusing on a single internal weight $h_i$, we have a simple pole expansion

$$F = U_i + \sum_{1 \leq r, s_i} \frac{V_{r,s_i}}{h_i - d_{r,s_i}}, \quad (4.3.3)$$

where $U_i$ is the $h_i$-regular part of the conformal block. The factor in (4.3.2) responsible for the pole at $h_i = d_{r,s_i}$ is

$$\rho(L_{-B_i} h_{i-1}, d_{i-1}, L_{-A_i} h_i) \left( G_{h_i}^{n_i} \right)^{A_i B_i} \rho(L_{-B_i} h_i, d_i, L_{-A_{i+1}} h_{i+1}). \quad (4.3.4)$$

In the limit $h_i \to d_{r,s_i}$, we can repeat the arguments in section 4.2.2, now making use of a factorization property of the 3-point function involving null states that slightly generalizes (4.2.13)

$$\rho(L_{-B} \chi_{rs}, \nu_2, L_{-A} \nu_1 | 1) = \rho(L_{-B} \nu_{d_{r,s} + rs}, \nu_2, L_{-A} \nu_1 | 1) \rho(\chi_{rs}, \nu_2, \nu_1 | 1). \quad (4.3.5)$$

The derivation of this relation is discussed in appendix B.2. Therefore, the residue coefficient is captured by

$$\rho(L_{-B_i} h_{i-1}, d_{i-1}, L_{-A_i} \chi_{r,s_i}) \left( G_{d_{r,s_i} + r,s_i}^{n_i} \right)^{A_i B_i} \rho(L_{-B_i} \chi_{r,s_i}, d_{i}, L_{-A_{i+1}} h_{i+1})$$

$$= \rho(h_{i-1}, d_{i-1}, \chi_{r,s_i}) \rho(\chi_{r,s_i}, d_{i}, h_{i+1}) \quad (4.3.6)$$

$$\times \rho(L_{-B_i} h_{i-1}, d_{i-1}, L_{-A_i} \nu_{d_{r,s_i} + r,s_i}) \left( G_{d_{r,s_i} + r,s_i}^{n_i} \right)^{A_i B_i} \rho(L_{-B_i} \nu_{d_{r,s_i} + r,s_i}, d_{i}, L_{-A_{i+1}} h_{i+1}).$$
Following section 4.2.2, and using the definition of the fusion polynomial, the residue in (4.3.3) is determined to be

\[ V_{r_i, s_i} = q_i^{r_i, s_i} R_{r_i, s_i}(h_{i-1}, h_{i+1}, d_{i-1}, d_i, c) F(h_i \to d_{r_i, s_i} + r_i s_i), \]  

(4.3.7)

with

\[ R_{r_i, s_i}(h_{i-1}, h_{i+1}, d_{i-1}, d_i, c) = A_{r_i, s_i} c^{r_i, s_i} \left[ p_{i-1} \right] P_r c^{s_i} \left[ h_{i+1} \right] \]  

(4.3.8)

### 4.3.3 Regular part

The \( h_i \)-regular part \( U_i \) in (4.3.3) is in fact quite complicated; fortunately, we do not need to compute \( U_i \) directly. Let us define \( a_i = h_i - h_1 \), for \( i = 2, \ldots, N \), and consider the limit \( h_i \to \infty \) with all \( a_i \)'s held fixed. In other words, we take the simultaneous large \( h_i \) limit, with the differences \( h_i - h_j \) kept finite. We will see that a drastic simplification of the conformal block occurs in this limit, giving rise to the regular part

\[ F(\mathbf{q}, h_1, d_1, \ldots, q_N, h_N, d_N, c) \to \prod_{n=1}^{\infty} \frac{1}{1 - (q_1 q_2 \ldots q_N)^n}, \]  

(4.3.9)

which takes the form of a (non-degenerate) torus character.

Let us begin with a basis of level \( n \) descendants of a primary \( |h_i \rangle \), of the form \( L_{-A}|h_i \rangle \), where \( A \) is a partition of the integer \( n \) in descending order. We will write \( |A| = n \), and \( |A| \) for the number of Virasoro generators in \( L_{-A} \) (the length of the partition). Note that in the large \( h \), fixed \( c \) limit, the inner product \( \langle h|L_{-A} \rangle L_{-B}|h \rangle \) scales like \( h^{[A]} \) for \( A = B \), no faster than \( h^{[A]-1} \) for \( |A| = |B| \), \( A \neq B \), and no faster than \( h^{-\min(|A|,|B|)} \) for \( |A| \neq |B| \). We can thus construct via the Gram-Schmidt process an orthogonal basis of the form

\[ e_{-A}|h \rangle = L_{-A}|h \rangle + \sum_{|B| = n, \ |B| \leq |A|, \ B \neq A} f_{A}^{B}(c, h) L_{-B}|h \rangle, \]  

(4.3.10)
such that

\[ f_B^A(c, h) \sim \mathcal{O}(h^{-1}), \quad [B] = [A], \quad B \neq A; \]
\[ f_B^A(c, h) \sim \mathcal{O}(h^0), \quad [B] < [A], \]

in the large \( h \), fixed \( c \) limit. The norm of the basis state \( \ell_{-A}|h\rangle \) scales like

\[ \langle h|\ell^\ell_{-A}|\ell_{-A}|h\rangle \sim h^{[A]} . \tag{4.3.12} \]

In the large \( h_1 \) limit with \( a_i = h_i - h_1 \) fixed (\( i = 2, \cdots, N \)), the torus \( N \)-point block in the necklace channel (4.3.2) becomes

\[ F \to \sum_{n_1, \ldots, n_N=0}^{\infty} \left( \prod_{i=1}^{N} q_i^{n_i} \right) \sum_{|A_1|=n_1} \cdots \sum_{|A_N|=n_N} \left[ \prod_{i=1}^{N} \frac{\rho(\ell_{-A_1} h_1, d_i, \ell_{-A_{i+1}} h_1)}{\langle h_1|\ell^\ell_{-A_1}|\ell_{-A_i}|h_1\rangle} \right] . \tag{4.3.13} \]

Here we have traded every internal weight \( h_i \) with \( h_1 \), which is valid to leading order. Let us investigate the large \( h_1 \) behavior of the numerator,

\[ \rho(\ell_{-A} h_1, d_i, \ell_{-A_{i+1}} h_1) = \sum_{|C|=|A_i|, \ |B|=|A_{i+1}|} f_C^A f_B^{A+1} \rho(L_{-C} h_1, d_i, L_{-B} h_1) , \tag{4.3.14} \]

where we have extended the definition of \( f_B^A \) in (4.3.10) by setting \( f_A^A = 1 \) (no summation over \( A \)) and \( f_B^A = 0 \) for \( [B] > [A] \). We can now evaluate the 3-point functions on the RHS using the Ward identities discussed in appendix B.1. Moving \( L_{-C} \) to the right past \( d_i \), one picks up commutator terms involving \([L_m, \nu_{d_i}]\), but the latter does not scale with \( h_1 \). Thus, to leading order in the large \( h_1 \) limit, we may freely move \( L_{-C} \) through \( d_i \) to obtain

\[ \rho(L_{-C} h_1, d_i, L_{-B} h_1) \sim \rho(h_1, d_i, L_{-C} L_{-B} h_1) \sim \mathcal{O}(h_1^{\min([B],[C])}) . \tag{4.3.15} \]

It then follows from (4.3.11) that the terms in (4.3.13) that survive in the large \( h_1 \) limit have \( A_1 = A_2 = \ldots \)
\[\cdots = A_N, \text{ with} \]
\[\rho(\ell_A h_1, d_i, \ell_A h_1) \to 1. \quad (4.3.16)\]
Thus, the sum in (4.3.13) collapses to (4.3.9).

### 4.3.4 h-recursion representation

We can now combine the above results on the polar part and the large \(h_1\), fixed \(a_i\) asymptotics to obtain a complete recursive representation of torus \(N\)-point Virasoro conformal blocks in the necklace channel. First, we fix \(a_i\) for \(i = 2, 3, \ldots, N\) and view the necklace block as a meromorphic function of \(h_1\). Its simple pole expansion takes the form
\[
F(q_1, h_1, d_1, q_2, a_2, d_1, \ldots, q_N, a_N, d_N, c) = \prod_{n=1}^{\infty} \frac{1}{1 - (q_1 q_2 \ldots q_N)^n} + \sum_{i=1}^{N} \sum_{r_i s_i \geq 1} B_{r_i s_i} \frac{1}{h_1 + a_i - d_{r_i s_i}}, \quad (4.3.17)
\]
where we have extended the definition of \(a_i\) by including \(a_1 = 0\). The residues \(B_{r_i s_i}\) are determined using (4.3.7),
\[
B_{r_1 s_1} = q_1^{r_1 s_1} R_{r_1 s_1} (d_{r_1 s_1} + a_N, d_{r_1 s_1} + a_2, d_N, d_1, c) \\
\times F(h_1 \to d_{r_1 s_1} + r_1 s_1, a_1 \to a_1 - r_1 s_1 \text{ for } i = 2, \ldots, N),
\]
\[
B_{r_2 s_2} = q_2^{r_2 s_2} R_{r_2 s_2} (d_{r_2 s_2} - a_2, d_{r_2 s_2} - a_2 + a_3, d_1, d_2, c) F(h_1 \to d_{r_2 s_2} - a_2, a_2 \to a_2 + r_2 s_2),
\]
\[
B_{r_N s_N} = q_N^{r_N s_N} R_{r_N s_N} (d_{r_N s_N} - a_N + a_{N-1}, d_{r_N s_N} - a_N, d_{N-1}, d_N, c) \\
\times F(h_1 \to d_{r_N s_N} - a_N, a_N \to a_N + r_N s_N),
\]
\[
B_{r_i s_i} = q_i^{r_i s_i} R_{r_i s_i} (d_{r_i s_i} - a_i + a_{i-1}, d_{r_i s_i} - a_i + a_{i+1}, d_{i-1}, d_i, c) F(h_1 \to d_{r_i s_i} - a_i, a_i \to a_i + r_i s_i) \quad \text{for } i = 3, \ldots, N - 1.
\]
We caution the reader that the shifted conformal blocks on the RHS still depend on the original \(a_i = h_i - h_1\). While they are independent of \(h_1\) as functions of \(a_i\), they would still contain \(h_1\) dependence when viewed as
functions of the $h_i$'s.

Defining a reduced conformal block $f$ by factoring out the torus character,

$$F(q_1, h_1, d_1, q_2, a_2, d_1, ..., q_N, a_N, d_N, c) = \prod_{n=1}^{\infty} \frac{1}{1 - (q_1 q_2 ... q_N)^n} f(q_1, h_1, d_1, q_2, a_2, d_1, ..., q_N, a_N, d_N, c),$$

we can express the recursion relation as

$$f(q_1, h_1, d_1, q_2, a_2, d_1, ..., q_N, a_N, d_N, c) = 1 + \sum_{r, s_1 \geq 1} q_1^{r_1} R_{r_1 s_1} (d_{r_1 s_1} + a_N, d_{r_1 s_1} + a_2, d_N, d_1, c) \frac{h_1 - d_{r_1 s_1}}{s_1} f(h_1 \to d_{r_1 s_1} + r_1 s_1, a_i \to a_i - r_1 s_1 \text{ for } i = 2, ..., N)$$

$$+ \sum_{i=2}^{N} \sum_{r, s_i \geq 1} q_i^{r_i} R_{r_i s_i} (d_{r_i s_i} - a_i + a_i-1, d_{r_i s_i} - a_i + a_{i+1}, d_{i-1}, d_i, c) \frac{h_1 + a_i - d_{r_i s_i}}{s_i} f(h_1 \to d_{r_i s_i} - a_i, a_i \to a_i + r_i s_i).$$

This is a complete $h$-recursion representation of the torus $N$-point block in the necklace channel.

### 4.3.5 Sphere $N$-point block in the linear channel

The sphere $N$-point Virasoro conformal block in the linear channel can be obtained as a limit of the torus $(N-1)$-point necklace block, by sending $q_{N-2}, q_{N-1} \to 0$. The weights $h_{N-2}$ and $h_{N-1}$ will now be viewed as weights of a pair of external primary operators. This makes it clear that our $h$-recursion relation will involve simultaneous shift of internal weights together with a pair of external weights, which is rather different from the procedure of [60].

It is nonetheless useful to write the recurrence relation in the sphere linear channel in a set of notations adapted to the Riemann sphere as below. The linear channel conformal block amounts to inserting complete bases of states between successive pairs of external operators, except for the two pairs at the ends, as shown in figure 4.2. This conformal block has been studied in [123] from the perspective of the AGT relation. Mapping the torus to the annulus by exponentiation, the expansion parameters $q_i$ used in the previous
The sphere $N$-point function admits the Virasoro conformal block decomposition

\[
\langle O_N(\infty)O_{N-1}(1)O_{N-2}(z_{N-2})...O_2(z_2)O_1(0)\rangle_{S^2} = \sum_{(h_1,\tilde{h}_1),...,(h_{N-3},\tilde{h}_{N-3})} C_{h_1,d_1} C_{h_{N-1},d_{N-1}} C_{h_{N-2},d_{N-2}} \left( \prod_{i=1}^{N-4} C_{h_{i+1},d_{i+2}} \right)
\]

\[
\times \frac{h_1-d_2-d_1 \tilde{h}_1-d_2-d_i}{z_i} \left( \prod_{i=3}^{N-2} \frac{h_{i-1}-d_i-h_{i-2} \tilde{h}_{i-1}-d_i-h_{i-2}}{z_i} \right)
\]

\[
\times F(q_1, h_1, ..., q_{N-3}, h_{N-3}, d_1, ..., d_N, c) F(\tilde{q}_1, \tilde{h}_1, ..., \tilde{q}_{N-3}, \tilde{h}_{N-3}, \tilde{d}_1, ..., \tilde{d}_N, c),
\]

where $F(q_i, h_i, d_j, c)$ is the linear channel block

\[
F(q_i, h_i, d_j, c) = \sum_{n_1, ..., n_{N-3}=0}^{\infty} \left( \prod_{i=1}^{N-3} q_i^{n_i} \right) \left( \prod_{i=1}^{N-4} G_{h_i}^{n_i} \right) \rho(L-B_{i+1}, h_{i+2}, d_{i+2}, L-A, h_i)
\]

\[
\times \frac{A-B}{d_N, d_{N-1}, L-A, h_{N-3}} \rho(L-B_1, h_1, d_2, d_1).
\]

For any $i$ between 1 and $N-3$, we could analytically continue the conformal block in $h_i$, and write a simple pole expansion analogously to (4.3.3), (4.3.7),

\[
F(q_i, h_i, d_j, c) = U_i + \sum_{1 \leq r \leq n_i} \frac{V_{r,s_i}}{h_i - d_{r,s_i}},
\]
where the residues are given by

\[
V_{r, s_1} = q_1^{r s_1} R_{r s_1} (d_1, h_2, d_2, d_3, c) F(h_1 \to d_{r s_1} + r_1 s_1),
\]

\[
V_{r, s_i} = q_i^{r s_i} R_{r s_i} (h_{i-1}, h_{i+1}, d_{i+1}, d_{i+2}, c) F(h_i \to d_{r s_i} + r_i s_i), \quad 2 \leq i \leq N - 4,
\]

\[
V_{N-3s_{N-3}} = q_{N-3}^{r s_{N-3}} R_{r s_{N-3}} (h_{N-4}, d_N, d_{N-2}, d_{N-1}, c) F(h_{N-3} \to d_{r s_{N-3}} + r_{N-3} s_{N-3}).
\]

To determine the regular part and thereby the full recurrence relation via the large weight limit, it is important to specify how this limit is taken. As in the torus case, we will consider the simultaneous large \(d_1, h_1, \ldots, h_{N-3}, d_N\) limit. In other words, we will define \(a_i = h_i - h_1\) for \(i = 2, \ldots, N - 3\) and \(e_1 = d_1 - h_1\), \(e_N = d_N - h_1\), and consider the limit \(h_1 \to \infty\) with \(a_i, e_1, e_N\) held fixed. Following the same arguments as in section 4.3.3, in (4.3.23) only the terms with the equal internal levels \(n_1 = \ldots = n_{N-3}\) may survive. However, due to the extra inverse Gram matrix element \(G^{A_{N-3}}_{B_{N-3}}\), only the internal level zero contribution survives (this can also be understood as effectively sending \(q_{N-3} R_{N-3} \) to zero). Therefore, in this limit, we have simply \(F(q_i, h_i, d_j, c) \to 1\).

Combining these results, we obtain the following recursive representation of the sphere \(N\)-point Virasoro block in the linear channel

\[
F(q_i, h_1, a_2, \ldots, a_{N-3}, e_1, e_N, c) = 1
\]

\[
+ \sum_{r_1 s_1 \geq 1} q_1^{r s_1} R_{r s_1} (d_{r s_1} + e_1, d_{r s_1} + a_2, d_2, d_3, c) \frac{F(h_1 \to d_{r s_1} + r_1 s_1, a_i \to a_i - r_1 s_1, e_j \to e_j - r_1 s_1)}{h_1 - d_{r s_1}}
\]

\[
+ \sum_{i=2}^{N-4} \sum_{r_i s_i \geq 1} q_i^{r s_i} R_{r s_i} (d_{r s_i} - a_i + a_{i-1}, d_{r s_i} - a_i + a_{i+1}, d_{i+1}, d_{i+2}, c) \frac{F(h_1 \to d_{r s_i} - a_i, a_i \to a_i + r_i s_i)}{h_1 + a_i - d_{r s_i}}
\]

\[
+ \sum_{r s \geq 1} q_{N-3}^{r s} R_{r s} (d_N - a_{N-3} + e_N, d_N - a_{N-3} + a_{N-4}, d_{N-1}, d_{N-2}, c) \frac{F(h_1 \to d_N - a_{N-3}, a_{N-3} \to a_{N-3} + r s)}{h_1 + a_{N-3} - d_N}
\]

\[
	imes F(h_1 \to d_N - a_{N-3}, a_{N-3} \to a_{N-3} + r s).
\]

(4.3.26)

Again, it is important to keep in mind that the shifted blocks on the RHS are functions of \(a_i = h_i - h_1\), and thus when viewed as functions of the \(h_i\)’s, they still contain \(h_1\) dependence.
Let us comment that there is another expression for the sphere $N$-point linear channel block in terms of a $q_i$ expansion (which also easily extends to the torus necklace channel) obtained from the AGT relation [123,124]. In the language of the latter, such channels include only fundamental, anti-fundamental, or bi-fundamental hypermultiplets, whose Nekrasov instanton partition functions have simplified expressions. The instanton partition function gives a combinatorial formula for the Virasoro conformal blocks in these channels. Of course, these expressions should agree with (4.3.20) and (4.3.26). This can be verified by showing that the residues and large weight asymptotics agree. It is not hard to check that the simultaneous large weight limit of the combinatorial formula of [123] is finite. The residues were checked in [125,126] for a small number of external operators.

4.4 $c$-recursion for all sphere and torus Virasoro conformal blocks

In this section, we derive recursive representation in the central charge $c$ for sphere and torus $N$-point Virasoro conformal blocks in arbitrary channels. The pole structure of the blocks in $c$ is similar to the analytic property in $h$ considered in the previous section: the poles are associated with degenerate Virasoro representations, while the residues are given by appropriate fusion polynomials multiplying the blocks with shifted weights, as will follow from a generic factorization property of 3-point functions of Virasoro descendants.

The key feature that will allow for the determination of $c$-recursion relations in all channels (in contrast to just the linear and necklace channels in our $h$-recursion relation) will be a drastic simplification in the large $c$ limit. In this limit, the block reduces to the product of the Virasoro vacuum block (i.e. all primaries, both internal and external, are replaced by the identity operator) and a global $SL(2)$ block that captures the contributions of $L_{-1}$ descendants of the primaries only. In the sphere case, the vacuum block is just 1, while for the torus, the vacuum block is the Virasoro vacuum character. The global block will be relatively simple to compute.

Throughout this chapter we construct Virasoro conformal blocks in terms of $\rho(\xi_3, \xi_2, \xi_1)$, the 3-point
function of descendants on the plane. As remarked in section 4.2.1, this is natural in the conformal frame where the Riemann surface in question is formed by plumbing together 2-holed discs with \( SL(2) \) maps. In describing torus and higher-genus conformal blocks, we could alternatively have made use of \( \tilde{\rho}(\xi_3, \xi_2, \xi_1) \), the matrix element of the descendant \( \xi_2 \) between \( \langle \xi_3 \rangle \) and \( \langle \xi_1 \rangle \) on the cylinder, which would be natural in an alternative conformal frame in which the Riemann surface is formed by plumbing together 1-holed cylinders. While \( \tilde{\rho}(\xi_3, \xi_2, \xi_1) \) can in principle be put in the form \( \rho(\xi_3', \xi_2', \xi_1') \) via the exponential map from the cylinder to the plane, the conformally transformed descendant \( \xi_2' \) generally differs from \( \xi_2 \). Different conformal frames not only lead to different parameterizations of the moduli, but also conformal blocks that differ by a conformal anomaly factor (a simple example is the Casimir energy on the cylinder). The simplification at large \( c \) mentioned above only holds in the conformal frame defined by the plumbing construction based on 2-holed discs; for this purpose, \( \rho(\xi_3, \xi_2, \xi_1) \) rather than \( \tilde{\rho}(\xi_3, \xi_2, \xi_1) \) is the appropriate 3-point function building block.

### 4.4.1 Factorization of 3-point functions with degenerate representations and the poles of conformal blocks

Previously, in our derivation of the \( h \)-recursive representation of the necklace and linear channel blocks, a key ingredient that allowed for the determination of the polar part of the block was the factorization property of 3-point functions that involve descendants of degenerate primaries (4.2.13) and (4.3.5). Here we will need a slightly more general set of identities,

\[
\rho(L_{-N\chi_{rs}}, L_{-M\nu_2}, L_{-P\nu_3}|1) = \rho(L_{-N\nu_{d_{rs}+rs}}, L_{-M\nu_2}, L_{-P\nu_3}|1)\rho(\chi_{rs}, \nu_2, \nu_3|1) \\
\rho(L_{-N\nu_1}, L_{-M\chi_{rs}}, L_{-P\nu_3}|1) = \rho(L_{-N\nu_1}, L_{-M\nu_{d_{rs}+rs}}, L_{-P\nu_3}|1)\rho(\nu_1, \chi_{rs}, \nu_3|1) \\
\rho(L_{-N\nu_1}, L_{-M\nu_2}, L_{-P\chi_{rs}}|1) = \rho(L_{-N\nu_1}, L_{-M\nu_2}, L_{-P\nu_{d_{rs}+rs}}|1)\rho(\nu_1, \nu_2, \chi_{rs}|1) \\
\rho(L_{-N\chi_{rs}}, L_{-M\nu_2}, L_{-P\chi_{rs}}|1) = \rho(L_{-N\nu_{d_{rs}+rs}}, L_{-M\nu_2}, L_{-P\nu_{d_{rs}+rs}}|1)\rho(\chi_{rs}, \nu_2, \chi_{rs}|1). \tag{4.4.1}
\]

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We remind the reader that $\chi_{rs}$ is the level $rs$ null descendant of a primary of weight $d_{rs}$, of the form (4.2.9). (4.4.1) follows from Ward identities and the property that $\chi_{rs}$ behaves like a Virasoro primary, as explained in more detail in appendix B.2.

On the RHS of (4.4.1), the first factors will lead to the recursive representation, as they contribute to new conformal blocks with shifted internal weight $d_{rs} + rs$. The second factors are fusion polynomials $P_{rs}^c$ (B.2.2). Together, (4.4.1) will determine the residue of a Virasoro conformal block on its poles either at a degenerate value of an internal weight, $h_i \rightarrow d_{rs}$, or at a value of the central charge $c \rightarrow c_{rs}(h_i)$ such that an internal weight $h_i$ becomes that of a degenerate Virasoro representation. This statement applies to any $N$-point, genus $g$ Virasoro block in any given channel, as will become clear in the next section.

Consider for example the sphere 6-point block shown in figure 4.1, which we refer to as the “trifundamental” channel block.\textsuperscript{3} We may build the 6-punctured sphere by connecting three 2-punctured discs and a single two-holed disc through the following plumbing construction. Consider the 2-punctured and 2-holed discs

$$D_i = \{ z_i \in \mathbb{C} : |z_i| < r_i, \ z_i \neq 0, 1 \}, \ i = 1, 2, 3$$
$$D_4 = \{ z_4 \in \mathbb{C} : \tilde{r}_1 < |z_4| < \tilde{r}_3, \ |z_4 - 1| > \tilde{r}_2 \}. \quad (4.4.2)$$

We glue each boundary component of $D_4$ with the boundary of $D_i, i = 1, 2, 3$, via the $SL(2)$ maps

$$|z_4| = \tilde{r}_1 : \ z_4 = q_1 z_1, \ |q_1| = \frac{\tilde{r}_1}{q_1},$$

$$|z_4 - 1| = \tilde{r}_2 : \ z_4 - 1 = q_2 z_2, \ |q_2| = \frac{\tilde{r}_2}{r_2}, \quad (4.4.3)$$

$$|z_4| = \tilde{r}_3 : \ z_4 = \frac{1}{q_3 z_3}, \ |q_3| = \frac{1}{\tilde{r}_3 r_3}.$$ 

The result of the plumbing construction is a Riemann sphere with 6 punctures at

$$0, \ q_1, \ 1, \ 1 + q_2, \ \infty, \ \frac{1}{q_3}. \quad (4.4.4)$$

\textsuperscript{3}The terminology comes from the corresponding quiver theory in the context of the AGT relation, which involves a trifundamental hypermultiplet [127].
Figure 4.5: Plumbing construction for sphere 6-point conformal block in the trifundamental channel.

The 6 external vertex operators will be inserted at these 6 points, parameterized by the plumbing parameters $q_1, q_2, q_3$. Note that $q_1, q_2, q_3$ are not on equal footing. In such a parameterization, the Virasoro block is given by

$$F(q_i, h_i, d_j, c) = \sum_{N,M,P,Q,A,B} q_i^{[N]} q_2^{[P]} q_3^{[A]} \rho(L_N h_1, d_2, d_1) \rho(L_P h_2, d_4, d_3) \rho(L_A h_3, d_6, d_5)$$

$$\times \rho(L_M h_1, L_Q h_2, L_B h_3) \left( G_{h_1}^{[N]} \right)^{NM} \left( G_{h_2}^{[P]} \right)^{PQ} \left( G_{h_3}^{[A]} \right)^{AB}. \tag{4.4.5}$$

Here the summation is over integer partitions $N, M, P, Q, A, B$, with $|N| = |M|, |P| = |Q|, |A| = |B|$, that label Virasoro descendants.
A simple pole expansion of this conformal block in one of the weights, say $h_1$, takes the form

$$F(q_i, h_i, d_j, c) = U_1 + \sum_{rs \geq 1} \frac{q^{r_s}_1 A^r_{rs} P^{r_s}_c \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \begin{bmatrix} h_3 \\ h_2 \end{bmatrix}}{h_1 - d_{rs}} F(h_1 \rightarrow d_{rs} + rs),$$

(4.4.6)

where the residue is readily determined using the factorization property (4.4.1) as before. The $h_1$-regular part $U_1$ is more complicated. Instead of trying to determine $U_1$ directly, we can inspect similar polar terms in $h_2$ and $h_3$, and write a simple pole expansion in the central charge $c$ using (4.2.8), of the form

$$F(q_i, h_i, d_j, c) = U_c + \sum_{r \geq 2, s \geq 1} \left[-\frac{\partial c_{rs}(h_1)}{\partial h_1}\right] \frac{q^{r_s}_1 A^r_{rs} P^{r_s}_c \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \begin{bmatrix} h_3 \\ h_2 \end{bmatrix}}{c - c_{rs}(h_1)} F(h_1 \rightarrow h_1 + rs, c \rightarrow c_{rs}(h_1))$$

$$+ \sum_{r \geq 2, s \geq 1} \left[-\frac{\partial c_{rs}(h_2)}{\partial h_2}\right] \frac{q^{r_s}_2 A^r_{rs} P^{r_s}_c \begin{bmatrix} d_3 \\ d_4 \end{bmatrix} \begin{bmatrix} h_1 \\ h_3 \end{bmatrix}}{c - c_{rs}(h_2)} F(h_2 \rightarrow h_2 + rs, c \rightarrow c_{rs}(h_2))$$

$$+ \sum_{r \geq 2, s \geq 1} \left[-\frac{\partial c_{rs}(h_3)}{\partial h_3}\right] \frac{q^{r_s}_3 A^r_{rs} P^{r_s}_c \begin{bmatrix} d_5 \\ d_6 \end{bmatrix} \begin{bmatrix} h_3 \\ h_2 \end{bmatrix}}{c - c_{rs}(h_3)} F(h_3 \rightarrow h_3 + rs, c \rightarrow c_{rs}(h_3)).$$

(4.4.7)

Now the $c$-regular part $U_c$ is the only term that survives in the large-$c$ limit. This will be analyzed next.

### 4.4.2 Large $c$, fixed $h_i$ limit of Virasoro conformal blocks

In the previous subsection, we have seen that the factorization property (4.4.1) fixes the polar part of the recursive representation of an arbitrary Virasoro conformal block, and the problem reduces to determining the large $c$ limit of the block, such as $U_c$ in the case of the trifundamental block (4.4.7). We now show that a gen-
eral Virasoro conformal block built out of descendant 3-point functions of the form \( \rho(L_{-A}h_1, L_{-B}h_j, L_{-C}h_k) \) \((A, B, C \text{ stand for integer partitions})\) and inverse Gram matrices remains finite in the \( c \to \infty \) limit (rather than growing with \( c \)). Furthermore, it will turn out that the large \( c \) limit of a Virasoro conformal block factorizes into the product of the large \( c \) limit of the vacuum block (defined by setting all internal and external representations to the vacuum) and the global \( SL(2) \) conformal block.

Note that the construction of the Virasoro block using descendant 3-point functions and inverse Gram matrices amounts to a plumbing construction based on gluing together 2-holed discs via the inversion map, which specifies the coordinate charts for the punctured Riemann surface as well as the conformal frame of the conformal block. We have already seen such an example in (4.4.5). In the case of higher genus conformal blocks, this choice of conformal frame fixes the conformal anomaly in such a way that the blocks are \( c \)-independent to leading order.

We illustrate the large \( c \) factorization property by considering the genus two Virasoro conformal block in the channel where the genus two Riemann surface is formed by plumbing together a pair of 2-holed discs. This conformal block takes the form\(^4\)

\[
F(h_1, h_2, h_3, c) = \sum_{|A|=|B|, |C|=|D|, |E|=|F|} q_1^{[A]} q_2^{[C]} q_3^{[E]} G_{h_1}^{AB} G_{h_2}^{CD} G_{h_3}^{EF} \rho(L_{-A}h_1, L_{-C}h_2, L_{-E}h_3) \rho(L_{-B}h_1, L_{-D}h_2, L_{-F}h_3).
\]

The strategy here closely parallels that of section 4.3.3, with slight modifications. We begin with a basis of level \( n \) descendants of a primary \( |h\rangle \) of the form \( L_{-A}|h\rangle \), where \( A \) is a partition of the integer \( n = |A| \) in descending order. We will denote by \( \langle A \rangle \) the number of non-\( L_{-1} \) Virasoro generators in \( L_{-A} \). Note that in the large \( c \) limit with \( h \) fixed, the inner product \( \langle h|L_{-A}^\dagger L_{-B}|h\rangle \) scales like \( c^{\langle A \rangle} \) for \( A = B \), no faster than \( c^{\langle A \rangle-1} \) for \( \langle A \rangle = \langle B \rangle \), \( A \neq B \), and no faster than \( c^{\min(\langle A \rangle, \langle B \rangle)} \) for \( \langle A \rangle \neq \langle B \rangle \). We can thus construct via the

\[^4\text{This object was referred to as the "naive" conformal block in [127].}\]
Gram-Schmidt process an orthogonal basis of the form

\[ L_{-A}|h\rangle = L_{-A}|h\rangle + \sum_{|B|=n, \langle B\rangle \leq \langle A\rangle, B \neq A} g^A_B(c, h) L_{-B}|h\rangle, \quad (4.4.9) \]

such that

\[ g^A_B(c, h) \sim O(c^{-1}), \quad \langle B\rangle = \langle A\rangle, \quad B \neq A; \]
\[ g^A_A(c, h) \sim O(c^0), \quad \langle B\rangle < \langle A\rangle, \]

in the large \( c \), fixed \( h \) limit. The norm of the basis state \( L_{-A}|h\rangle \) scales like

\[ \langle h| L_{-A} L_{-A}|h\rangle \sim c^{\langle A\rangle}. \quad (4.4.11) \]

(4.4.8) can now be written as

\[ F(h_1, h_2, h_3, c) = \sum_{A, B, C} q^{|A|}_1 q^{|B|}_2 q^{|C|}_3 \frac{\rho(L_{-A} h_1, L_{-B} h_2, L_{-C} h_3)^2}{\langle h_1| L_{-A} L_{-A}|h_1\rangle \langle h_2| L_{-B} L_{-B}|h_2\rangle \langle h_3| L_{-C} L_{-C}|h_3\rangle}. \quad (4.4.12) \]

By construction of (4.4.9), the three-point function \( \rho(L_{-A} h_1, L_{-B} h_2, L_{-C} h_3) \) scales with \( c \) no faster than \( c^{\langle A\rangle + \langle B\rangle + \langle C\rangle} \). Therefore, (4.4.8) is finite in the \( c \to \infty \) limit. Moreover, in this limit the only surviving contribution to \( \rho(L_{-A} h_1, L_{-B} h_2, L_{-C} h_3) \) comes from the \( L_{-A} \) term in \( L_{-A} \) (4.4.9), i.e.

\[ \rho(L_{-A} h_1, L_{-B} h_2, L_{-C} h_3) \rightarrow \rho(L_{-A} h_1, L_{-B} h_2, L_{-C} h_3). \quad (4.4.13) \]

To prove the large \( c \) factorization into the vacuum Virasoro block and the global \( SL(2) \) block, we write Virasoro chains as \( L_{-A} = L_{-A'} L_{-1}^{k_A} \), where \( L_{-A'} \) does not include any \( L_{-1} \) generators (by convention, \( A' \) and \( A \) are both integer partitions in descending order). The RHS of (4.4.13) is now written as

\[ \rho(L_{-A'} L_{-1}^{k_A} h_1, L_{-B'} L_{-1}^{k_B} h_2, L_{-C'} L_{-1}^{k_C} h_3). \quad (4.4.14) \]
To leading order in the large $c$ limit, the (non-$L_{-1}$) Virasoro generators in $L_{-A'}, L_{-B'}, L_{-C'}$ must be contracted pairwise via the Ward identities. In particular, the dependence on the weights $h_i$ is suppressed by $h_i/c$ relative to the leading order scaling $c^{(A) + (B) + (C)}$ (when $(A) + (B) + (C)$ is even and the pairwise contraction is available). What remains is the 3-point function of $L_{-1}$ descendants. Thus, in the large $c$ limit we can replace

$$
\rho(\mathcal{L}_{-A}h_1, \mathcal{L}_{-B}h_2, \mathcal{L}_{-C}h_3) \rightarrow \rho(\mathcal{L}_{-A}\nu_0, \mathcal{L}_{-B}\nu_0, \mathcal{L}_{-C}\nu_0)\rho(L_{-1}^{k_A}h_1, L_{-1}^{k_B}h_2, L_{-1}^{k_C}h_3),
$$

(4.4.15)

where $\nu_0$ is the vacuum primary. Note that if the 3-point function of vacuum descendants on the RHS of (4.4.15) vanishes, the contribution to the conformal block also vanishes in the $c \rightarrow \infty$ limit, due to the factorization property of the 2-point function

$$
(h|L_{-A}^\dagger L_{-A}|h) \rightarrow \langle \nu_0|L_{-A}^\dagger L_{-A}|\nu_0\rangle \langle h|L_{-1}^{k_A}L_{-1}^{k_A}|h\rangle.
$$

(4.4.16)

Rewriting the summation over partitions $A, B, C$ in (4.4.12) in terms of $(A', k_A)$, $(B', k_B)$, $(C', k_C)$, where $A', B', C'$ involve only non-$L_{-1}$ generators, and $k_A, k_B, k_C$ counts the length of the $L_{-1}$ chains, we arrive at the large $c$ limit

$$
\lim_{c \rightarrow \infty} F(h_1, h_2, h_3) = \lim_{c \rightarrow \infty} \sum_{A', B', C'} q_1^{A'} q_2^{B'} q_3^{C'} \frac{\rho(L_{-A'}\nu_0, L_{-B'}\nu_0, L_{-C'}\nu_0)^2}{\langle \nu_0|L_{-A'}^\dagger L_{-A'}|\nu_0\rangle \langle \nu_0|L_{-B'}^\dagger L_{-B'}|\nu_0\rangle \langle \nu_0|L_{-C'}^\dagger L_{-C'}|\nu_0\rangle}
\times \sum_{k_1, k_2, k_3 \geq 0} q_1^{k_1} q_2^{k_2} q_3^{k_3} \frac{\rho(L_{-1}^{k_1}h_1, L_{-1}^{k_2}h_2, L_{-1}^{k_3}h_3)^2}{\langle h_1|L_{-1}^{k_1}L_{-1}^{k_1}|h_1\rangle \langle h_2|L_{-1}^{k_2}L_{-1}^{k_2}|h_2\rangle \langle h_3|L_{-1}^{k_3}L_{-1}^{k_3}|h_3\rangle}.
$$

(4.4.17)

The first factor on the RHS is the large $c$ limit of the vacuum block (note that the vacuum is annihilated by $L_{-1}$), while the second factor is the global $SL(2)$ conformal block which by definition is independent of the central charge.

Clearly, the above proof can be straightforwardly extended to any Virasoro conformal blocks built from contracting 3-point functions of descendants with inverse Gram matrices, as the argument was simply based
on power counting in the large $c$ limit. Note that the vacuum Virasoro block on the sphere is equal to 1, and vacuum Virasoro block on the torus (in any channel) is equal to the vacuum Virasoro character. Thus, the large $c$ limit for any $N$-point sphere or torus Virasoro conformal block in any channel $C$ (corresponding to a pair-of-pants decomposition of the $N$-punctured Riemann surface) is given by

$$\lim_{c \to \infty} \text{(sphere Virasoro block in channel } C\text{)} = \text{(sphere global } SL(2)\text{ block in channel } C\text{)}$$

$$\lim_{c \to \infty} \text{(torus Virasoro block in channel } C\text{)} = \text{(vacuum Virasoro character) } \times \text{(torus global } SL(2)\text{ block in channel } C\text{)}.$$  

(4.4.18)

Together with the residue structure of the $c$-polar part discussed in the previous section, we obtain a complete $c$-recursive representation for any $N$-point sphere and torus Virasoro conformal block. In the next two subsections, we will give the explicit formulae in several examples.

An analogous large $c$ factorization property holds for higher genus Virasoro conformal blocks as well, provided that we define the latter in the appropriate conformal frame, based on plumbing together 2-holed discs. This will be discussed in section 4.5.

### 4.4.3 Global $SL(2)$ blocks

Here we briefly describe the evaluation of global $SL(2)$ blocks. Consider as an example the sphere 6-point block in the trifundamental channel (4.4.5), defined in terms of the plumbing parameters $q_1, q_2, q_3$. Its corresponding global block reads

$$G(q_i, h_i, d_j, c) = \sum_{i,j,k=0}^{\infty} q_i^j q_j^k \rho(L_{-1}^i h_1, d_2, d_4) \rho(L_{-1}^j h_2, d_4, d_3) \rho(L_{-1}^k h_3, d_6, d_5)$$

$$\times \frac{\rho(L_{-1}^i h_1, L_{-1}^j h_2, L_{-1}^k h_3)}{\langle h_1 | L_{-1}^i L_{-1}^j | h_1 \rangle \langle h_2 | L_{-1}^j L_{-1}^k | h_2 \rangle \langle h_3 | L_{-1}^k L_{-1}^i | h_3 \rangle}.$$  

(4.4.19)
The global block is generally simple enough to evaluate in closed form. For instance,

$$\rho(L_{-1}^ih_1, d_2, d_1) = (h_1 + d_2 - d_1)_i,$$

(4.4.20)

where \((a)_i\) is the Pochhammer symbol. The most general 3-point function of \(L_{-1}\) descendants is

$$\rho(L_{-1}^i h_1, L_{-1}^j h_2, L_{-1}^k h_3) = (h_1 + i - h_2 - j + 1 - h_3 - k)_i s_{ik}(h_1, h_2, h_3),$$

(4.4.21)

where we have defined \(s_{ik}(h_1, h_2, h_3) = \rho(L_{-1}^i h_1, h_2, L_{-1}^k h_3)\). It has a known closed form expression \([128]\)

$$s_{km}(h_1, h_2, h_3) = \sum_{p=0}^{\min(m, k)} \frac{k!}{p!(k-p)!} (2h_3 + m - 1)^{(p)} m^{(p)}$$

$$\times (h_3 + h_2 - h_1)_{m-p}(h_1 + h_2 - h_3 + p - m)_{k-p},$$

(4.4.22)

where \((a)^{(p)} = a(a-1)...(a-p+1)\) is the descending Pochhammer symbol. The inverse norms in (4.4.19) are given by

$$\frac{1}{\langle h| L_n^i L_{-1}^m |h \rangle} = \frac{1}{n!(2h)^n}.$$  

(4.4.23)

Combining these, we arrive at the following closed form expression for the trifundamental global block

$$G = \sum_{i,j,k=0}^{\infty} q_1^i q_2^j q_3^k (h_1 + d_2 - d_1)_i (h_2 + d_4 - d_3)_j (h_3 + d_6 - d_5)_k$$

$$\times \frac{(h_1 + i - h_2 - j + 1 - h_3 - k)_i s_{ik}(h_1, h_2, h_3)}{i!j!k!(2h_1)_i(2h_2)_j(2h_3)_k}.$$  

(4.4.24)

The extension of such results to any global block is evident. Let us note that for a given channel of an \(N\)-point, genus \(g\) conformal block, based on a pair-of-pants decomposition, the global \(SL(2)\) block is only defined in the plumbing construction based on 2-holed discs glued together via \(SL(2)\) maps.
4.4.4 Examples of \( c \)-recursive representations

**Sphere 6-point block in the trifundamental channel**

Our first nontrivial example is the sphere 6-point block in the trifundamental channel (4.4.5). (Note that the \( h \)-recursive representation given in the previous section is not available in this channel.) Combining the large \( c \) limit and the polar structure determined earlier, we have the following \( c \)-recursion formula

\[
F(q_i, h_i, d_j, c) = U_c + \sum_{r \geq 2, s \geq 1} \frac{-\partial c_{rs} (h_1)}{\partial h_1} \left\{ \begin{array}{c} q_1^{rs} A_{rs}^{c_{rs} (h_1)} \quad \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \quad \begin{bmatrix} h_3 \\ h_2 \end{bmatrix} \\ c - c_{rs} (h_1) \end{array} \right\} F(h_1 \to h_1 + rs, c \to c_{rs} (h_1)) \\
+ \sum_{r \geq 2, s \geq 1} \frac{-\partial c_{rs} (h_2)}{\partial h_2} \left\{ \begin{array}{c} q_2^{rs} A_{rs}^{c_{rs} (h_2)} \quad \begin{bmatrix} d_3 \\ d_4 \end{bmatrix} \quad \begin{bmatrix} h_3 \\ h_1 \end{bmatrix} \\ c - c_{rs} (h_2) \end{array} \right\} F(h_2 \to h_2 + rs, c \to c_{rs} (h_2)) \\
+ \sum_{r \geq 2, s \geq 1} \frac{-\partial c_{rs} (h_3)}{\partial h_3} \left\{ \begin{array}{c} q_3^{rs} A_{rs}^{c_{rs} (h_3)} \quad \begin{bmatrix} d_5 \\ d_6 \end{bmatrix} \quad \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \\ c - c_{rs} (h_3) \end{array} \right\} F(h_3 \to h_3 + rs, c \to c_{rs} (h_3)),
\]

with

\[
U_c = \sum_{i,j,k=0}^{\infty} q_i^1 q_j^2 q_k^3 (h_1 + d_2 - d_1)i(h_2 + d_4 - d_3)j(h_3 + d_6 - d_5)k \times \frac{(h_1 + i - h_2 - j + 1 - h_3 - k)js_{ik}(h_1, h_2, h_3)}{i!j!k!(2h_1)_i(2h_2)_j(2h_3)_k}.
\]

**Torus 1-point block**

Our next example is the torus 1-point block, which was already considered in [114,115]. Properties of this block were used to derive an asymptotic formula for the average value of heavy-heavy-light OPE coefficients...
from modular covariance of the torus 1-point function in [129]. The block is given by

\[ F(q, h, d, c) = \sum_{|N| = |M| = n} q^n \rho(L_{-N} h, d, L_{-M} h) (G^n h)^{NM}, \]  

(4.4.27)

where \( q = e^{2\pi i \tau} \), \( \tau \) being the modulus of the torus. Our conformal frame is defined by identifying the inner and outer boundaries of the annulus via the rescaling \( z \mapsto q^{-1} z \) on the complex plane, and thus the Casimir energy factor \( q^{-\frac{c}{24}} \) is absent. This distinction is rather minor in the present example, but will be important in more complicated examples to be discussed later.

The recursive representation of the torus 1-point block in the internal weight \( h \) reads [114]

\[ F(q, h, d, c) = \prod_{n=1}^{\infty} \frac{1}{1 - q^n} f(q, h, d, c), \]

\[ f(q, h, d, c) = 1 + \sum_{rs \geq 1} \frac{A_c^c \rho_{rs} \left( \begin{array}{c} d \\ d_{rs} + rs \\ h - d_{rs} \end{array} \right) \rho_{rs} \left( \begin{array}{c} d \\ d_{rs} \end{array} \right) }{h - d_{rs}} f(h \rightarrow d_{rs} + rs). \]  

(4.4.28)

Note that here we encounter a 3-point function involving a pair of null states \( \chi_{rs} \), resulting in the product of two fusion polynomials that involve the weight \( d_{rs} \) and \( d_{rs} + rs \) respectively. The corresponding global \( SL(2) \) block is [114, 115]

\[ g(q, h, d) = \frac{1}{1 - q} \, _2F_1 \left( d, 1 - d; 2h; \frac{q}{q - 1} \right) \]  

(4.4.29)

As originally observed in [115], the large \( c \) limit of the torus 1-point block reduces to the product of the vacuum Virasoro character with the global block,

\[ \lim_{c \rightarrow \infty} F(q, h, d, c) = \prod_{n=2}^{\infty} \frac{1}{1 - q^n} g(q, h, d) = \prod_{n=1}^{\infty} \frac{1}{1 - q^n} \, _2F_1 \left( d, 1 - d; 2h; \frac{q}{q - 1} \right). \]  

(4.4.30)
We arrive at the following $c$-recursive representation, in agreement with [115]

\[
F(q, h, d, c) = \left[ \prod_{n=1}^{\infty} \frac{1}{1-q^n} \right] {}_2F_1 \left( d, 1-d; 2h; \frac{q}{q-1} \right)
\]

\[
+ \sum_{r \geq 2, s \geq 1} \left[ -\frac{\partial c_{rs}(h)}{\partial h} \right] \frac{q^{rs} A_{rs}^{c_{rs}(h)} P^{rs}_{rs}(h) \left[ \begin{array}{c} d \\ h + rs \\
\end{array} \right] P^{rs}_{c_{rs}(h)} \left[ \begin{array}{c} d \\ h \\
\end{array} \right]}{c - c_{rs}(h)}
\]

\[\times F(h \to h + rs, c \to c_{rs}(h)),\]

where we have used that $d_{rs}(c_{rs}(h)) = h$.

**Torus 2-point block in the OPE channel**

The last example is the torus 2-point conformal block in the OPE channel, that is, two external vertex operators fusing into one that is inserted on the torus. Our conformal frame is defined by the plumbing construction illustrated in figure 4.6.

![Figure 4.6: The plumbing construction of the torus 2-point function in the OPE channel.](image)
We begin with a 2-punctured disc and a 2-holed disc,

\[ D_1 = \{ w \in \mathbb{C} : |w| < r_1, \ w \neq 0, 1 \}, \]

\[ D_2 = \{ u \in \mathbb{C} : \epsilon < |u| < \frac{\epsilon}{|\eta|}, \ |u-1| > r_2 \}. \]  

The $SL(2)$ gluing maps identify

\[ |u - 1| = r_2 : \ u - 1 = rv, \ |v| = \frac{r_2}{r_1}, \ \text{and} \]

\[ u \sim qu. \]  

The result of the plumbing construction is the annulus on the $u$ plane with the identification $u \sim qu$ and two vertex operators inserted at $u = 1$ and $u = 1 + v$. In terms of the parameters $q_1, q_2$ previously used for the necklace channel, we have

\[ q = q_1 q_2, \ \ v = \frac{1 - q_2}{q_2}. \]  

The Virasoro conformal block in this frame is given by

\[ F(q, h_1, v, h_2, d_1, d_2, c) = \sum_{N, M, P, Q} q^{N|\nu|P} \left( G_{h_1}^{N|\nu|P} \right)^{NM} \rho(L_{-N} h_1, L_{-P} h_2, L_{-M} h_1) \left( G_{h_2}^{P} \right)^{PQ} \rho(L_{-Q} h_2, d_1, d_2). \]  

It is important that $\rho$ is defined as the 3-point function of descendants on the plane (as opposed to on the
cylinder), as is clear from the above plumbing construction. The $c$-recursive representation takes the form

$$F = U_c + \sum_{r \geq 2, s \geq 1} \left[ -\frac{\partial c_{rs}(h_1)}{\partial h_1} \right] \frac{q^{r s} A_{rs}^{crs(h_1)} P_{crs(h_1)}^{rs}}{c - c_{rs}(h_1)} \left[ \begin{array}{c} h_2 \\ h_1 + rs \\ \end{array} \right] P_{crs(h_1)}^{rs} \left[ \begin{array}{c} h_2 \\ h_1 \\ \end{array} \right]$$

$$\times F(h_1 \to h_1 + rs, c \to c_{rs}(h_1))$$

$$+ \sum_{r \geq 2, s \geq 1} \left[ -\frac{\partial c_{rs}(h_2)}{\partial h_2} \right] \frac{v^{rs} A_{rs}^{crs(h_2)} P_{crs(h_2)}^{rs}}{c - c_{rs}(h_2)} \left[ \begin{array}{c} h_1 \\ h_1 \\ d_2 \\ d_1 \\ \end{array} \right] F(h_2 \to h_2 + rs, c \to c_{rs}(h_2)).$$

The $c$-regular part $U_c$ is again $c$-independent, and is given by the product of the torus vacuum character and the global block,

$$U_c = \prod_{n=2}^{\infty} \frac{1}{1 - q^n} \sum_{i, j \geq 0} q^i v^j (1 - h_2 - j) s_i(h_1, h_2, h_1)(h_2 + d_1 - d_2)_j.$$

4.5 Generalization to higher genus

![Diagram of building blocks for the plumbing construction](4.7)

Figure 4.7: Building blocks for the plumbing construction. Discs with 2 punctures (left), 1 hole and 1 puncture (middle), and 2 holes (right). Here we describe them using spheres with at least one hole around $\infty$, which are $SL(2)$-equivalent to discs with the appropriate number of holes and punctures.

We now describe the extension of $c$-recursive representation to $N$-point Virasoro conformal blocks on
arbitrary higher genus Riemann surfaces in an arbitrary channel. The $N$-punctured genus $g$ Riemann surface will be constructed by plumbing together $2g - 2 + N$ discs with either 2 holes, 1 hole and 1 puncture, or 2 punctures as illustrated in figure 4.7. For instance, a 2-holed disc is the domain

$$D = \{ z \in \mathbb{C} : |z| > r_1, |z - 1| > r_2, |z| < r_3 \}.$$  

(4.5.1)

Boundary components of the holed/punctured discs will be identified pairwise using $3g - 3 + N \text{ } SL(2, \mathbb{C})$ Möbius maps. For instance, we may glue the inner boundary $|z| = r_1$ of a 2-holed disc $D$ with the outer boundary $|\tilde{z}| = \tilde{r}_3$ of another 2-holed disc $\tilde{D}$ via $\tilde{z} = z/q$. The moduli of the $N$-punctured genus $g$ Riemann surface will be parameterized by $3g - 3 + N$ plumbing parameters $q_i$.

Figure 4.8: The plumbing construction for a genus two Riemann surface in two channels.

The plumbing construction not only gives a parameterization of the moduli, but also specifies the conformal frame in which the Virasoro conformal block is defined. As already mentioned, this is a particularly convenient frame for the $c$-recursive representation, because (1) the Virasoro conformal block remains finite in the $c \to \infty$ limit in this frame, and (2) the global $SL(2)$ block is naturally defined in this frame since only $SL(2)$ gluing maps are involved.

To build the Virasoro block, we begin with 3-point functions $\rho$ of Virasoro descendants inserted at $z = 0, 1, \infty$ on the plane, associated with each holed/punctured disc. A puncture corresponds to an external primary, while a hole corresponds to an internal descendant of the form $L_{-A\nu_h}$. Each gluing map in the plumbing construction amounts to contracting a pair of descendants from two $\rho$’s, say of primary weight
and level $N$, with the inverse Gram matrix, multiplied by a power of the plumbing parameter, $q^N$ (by convention, we have separated $q^h$ as an overall prefactor that multiplies the conformal block). We have already seen this through a number of examples: for instance, the sphere 6-point block in the trifundamental channel (4.4.5) corresponds to figure 4.5; the genus two conformal block corresponding to the left figure of figure 4.8 was considered in (4.4.8).

As described in section 4.4.1, the factorization property of descendant 3-point functions $\rho$ involving null states leads to the determination of the residues of the conformal block at its poles either in one of the internal weights or in the central charge. For instance, the genus two block (4.4.8) has the simple pole expansion in one of its internal weights $h_1$,

$$F = U_1 + \sum_{rs \geq 1} q^{r^2} A_{rs} \left( \begin{array}{c} h_3 \\ h_2 \\ h_1 \end{array} \right)^2 \left( \begin{array}{c} \rho^{rs} \\ h_2 \\ h_1 \end{array} \right) F(h_1 \to d_{rs} + rs),$$

(4.5.2)

where $U_1$ is regular in $h_1$. Similar results of course hold for the simple pole expansion in $h_2$ and in $h_3$, with regular parts $U_2$ and $U_3$ respectively. The $U_i$'s are a priori complicated. Instead, we now pass to the simple pole expansion in $c$, which is readily read off from the polar terms in $h_1, h_2, h_3$ (this is very similar to the (4.4.25) for the sphere 6-point trifundamental block). It then remains to determine the regular part of the conformal block in $c$, which is equivalent to knowing the large $c$ limit.

As we showed in section 4.4.2, the Virasoro conformal block in the plumbing frame built out of of 3-point functions of descendants contracted with inverse Gram matrices has a very simple large $c$ limit: it reduces to the product of the $c \to \infty$ limit of the vacuum Virasoro block and the global $SL(2)$ block (both defined in the plumbing frame). That is,

$$\lim_{c \to \infty} (\text{genus } g \text{ Virasoro block in channel } C) = \lim_{c \to \infty} (\text{genus } g \text{ vacuum block in channel } C) \times (\text{genus } g \text{ global block in channel } C).$$

(4.5.3)
A genus two example of this was shown in (4.4.17).\(^5\)

As already pointed out, the global $SL(2)$ block is easy to compute explicitly in any channel. It is less obvious how to determine the vacuum Virasoro block in the $c \to \infty$ limit on a genus $g$ Riemann surface (since all external and internal primaries are set to identity, there are no more punctures) in a general channel in the plumbing frame, as it receives contributions from all 3-point functions of descendants of the vacuum Verma module. The answer, in fact, is already known, as the holomorphic part of the 1-loop partition function of 3D pure gravity on the corresponding genus $g$ hyperbolic handlebody [118].

Firstly, note that the vacuum block has the special property that it depends only on the choice of a genus $g$ handlebody that “fills in” the Riemann surface, i.e. different channels corresponding to the same handlebody (related by crossing moves at the level of sphere 4-point functions) lead to the same answer. In the Schottky parameterization of the moduli, the Riemann surface is realized as a quotient of the form

$$(\mathbb{C} \cup \{\infty\} - \Lambda)/\langle \alpha_1, \cdots, \alpha_g \rangle,$$ (4.5.4)

where $\alpha_i$’s are loxodromic elements of $PSL(2, \mathbb{C})$ that act on the Riemann sphere $\mathbb{C} \cup \{\infty\}$ via Möbius transformation, and $\langle \alpha_1, \cdots, \alpha_g \rangle$ is the free group generated by $\alpha_1, \cdots, \alpha_g$, known as the Schottky group. $\Lambda$ is the limit set of the Schottky group action. Now given any element $\gamma$ of the Schottky group, as an element of $PSL(2, \mathbb{C})$ it is conjugate to

$$\begin{pmatrix} q^{1/2}_\gamma & 0 \\ 0 & q^{-1/2}_\gamma \end{pmatrix},$$

with $|q_\gamma| < 1$. Obviously, $q_\gamma$ depends only on the conjugacy class of $\gamma$.

Now the $c = \infty$ vacuum block in the plumbing frame is given by the product formula [118]

$$\prod_{\gamma \in \mathcal{P}} \prod_{n=2}^{\infty} (1 - q^n_\gamma)^{-\frac{1}{2}},$$ (4.5.5)

where $\mathcal{P}$ is the set of primitive conjugacy classes of the Schottky group. The relation between the Schottky

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\(^5\)We have also confirmed (4.5.3) for the genus two block (4.4.8) up to total level 12 in $q_1, q_2, q_3$ by scanning over a set of numerical values of the internal weights $h_i$ with Mathematica.
parameters and the plumbing parameters is straightforward. Let us consider as a simple example the genus two partition function. We construct a Virasoro block for the genus two partition function in the plumbing frame by gluing together two 2-holed discs. Gluing one disc into a hole of the other disc leads to a 3-holed disc, where a pair of inner holes are glued together and the remaining inner hole is glued to the outer boundary. The two $PSL(2, \mathbb{C})$ maps used in sewing up the boundaries of the 3-holed disc are precisely generators of the Schottky group. The generalization of this procedure to higher genus (involving the gluing of $2(g - 1)$ 2-holed discs) is entirely straightforward. Thus, the result of [118] combined with the global $SL(2)$ block provide the required $c$-regular part in the plumbing frame, thereby completing the $c$-recursive representation of a general genus $g$ conformal block.

Note that if we move to a different conformal frame, the vacuum block would pick up a conformal anomaly factor, of the form $\exp(-c\mathcal{F}_0)$, where $\mathcal{F}_0$ is a function of the moduli. From the holographic perspective, $\mathcal{F}_0$ is the holomorphic part of the regularized Einstein-Hilbert action evaluated on a genus $g$ hyperbolic handlebody [116,117], and the choice of conformal frame is tied to a choice of the cutoff surface along the conformal boundary. The logarithm of the vacuum conformal block is expected to have a $1/c$ asymptotic expansion of the form $-\sum_{k=0}^{\infty} c^{1-k} \mathcal{F}_k$, where $\mathcal{F}_k$ is the holomorphic part of the $k$-loop free energy of the 3D pure gravity at the handlebody saddle point of the gravitational path integral [118]. For our purposes here, $\exp(-\mathcal{F}_1)$ is what survives in the large $c$ limit in the plumbing frame, and serves as the seed that determines the $c$-recursion relation.

To go from the plumbing parameters $q_i$ or the Schottky parameterization of the moduli to the period matrix of the genus $g$ Riemann surface is rather nontrivial (see [130] for the construction of such a mapping in the genus two case). This is now the main technical obstacle before our recursive representation can be applied to, say, higher genus modular bootstrap.
4.6 Discussion

In the first part of this work, we derived the $h$-recursion representation of Virasoro conformal blocks for the sphere linear channel and torus necklace channel. The key to this derivation was the determination of the $h$-regular part by taking a simultaneous large weight limit, such that every 3-point function of descendants that appears in the conformal block involves a finite weight primary and a pair of large weights (either primary or descendants), which leads to a drastic simplification of the Virasoro block. Such a limit is not available however for more general conformal blocks, such as the sphere 6-point block in the trifundamental channel.\footnote{In this case, sending 3 of the external weights together with 3 internal weights to infinity while holding their differences fixed indeed still gives a finite limit, but we have not been able to find a closed form expression for the result.} For practical computations, while our $h$-recursive representation does compute order-by-order the expansion of Virasoro block in the plumbing parameters, it is not quite as efficient as that of \cite{60}: even in the sphere 4-point case, the residues of the recursive formula involve shifted blocks with a pair of new external weights that now depend on the original internal weight.

The $c$-recursion representations appear to be much more powerful, both in that they apply to arbitrary channel Virasoro conformal blocks on any Riemann surface (provided that we work in the plumbing frame), and they are more efficient for practical evaluation of the $q_i$-expansion.\footnote{For instance, using Mathematica on a laptop, symbolic evaluation of the torus 2-point function in the necklace channel for both $q_1$ and $q_2$ up to level 10 takes typically $\sim O(10)$ minutes using $c$-recursion, while the same evaluation using $h$-recursion at level 7 takes $\sim O(10)$ minutes.} It is now possible to compute efficiently the torus 2-point Virasoro blocks in both the necklace channel and the OPE channel, making it possible to analyze the torus 2-point conformal bootstrap for unitary CFTs with semidefinite programming. Note that unlike the conformal bootstrap where crossing symmetry of the sphere four-point function is imposed, here (and generically in higher genus bootstrap) there are multiple internal weights over which the positivity properties must be imposed. This is currently under investigation.

Even though a complete set of consistency constraints on a 2D CFT is captured by the crossing relation of the sphere 4-point function and the modular covariance of the torus 1-point function, the numerical approach
to bootstrapping unitary CFTs can hardly incorporate more than a few external operators simultaneously. For this reason it has been rather difficult to combine modular bootstrap and the sphere crossing equation in a useful way. The higher genus conformal bootstrap based on the modular crossing equation would effectively take into account the OPEs of all primaries in the spectrum, without having to work with them individually. This could be very useful especially for theories with large degeneracy/density in the operators.

The remaining complication in implementing higher genus modular bootstrap is to efficiently go between the plumbing parameterization of the moduli and the period matrix, since the latter has a simple modular transformation property while the former transforms in a complicated manner under the modular group. These have been studied in the genus two case in [117,130], but a more efficient computational algorithm will be needed for applications to bootstrap.

Finally, let us mention that our recursive formula allows for the evaluation of torus (and potentially higher genus) correlation functions in Liouville CFT and the $SL(2)$ WZW model, based on integrating a continuous family of conformal blocks with known structure constants. This makes it possible to perform direct numerical evaluation of string loop amplitudes in $c = 1$ string theory [131,132], doubled scaled little string theory [100,102,133–135], and string theory in $AdS_3$ [136].

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Genus two modular bootstrap

5.1 Introduction

The conformal bootstrap program in two dimensions aims to classify and solve two-dimensional conformal field theories (CFTs) based on the associativity of the operator product expansion (OPE) and modular invariance \([11, 13, 137]\). A complete set of consistency conditions is given by the crossing equations for sphere 4-point functions and modular covariance of the torus 1-point function for all Virasoro primaries in the CFT \([36, 37]\). In practice, while one may obtain nontrivial constraints on a specific OPE by analyzing a specific sphere 4-point function \([100, 102]\), or on the entire operator spectrum of the CFT by analyzing the torus partition function \([54, 55, 75, 101]\), it has been generally difficult to implement these constraints
simultaneously.

In this chapter, we analyze modular constraints on the genus two partition function of a general unitary CFT. The modular crossing equation for the Virasoro conformal block decomposition of the genus two partition function encodes both the modular covariance of torus 1-point functions for all primaries and the crossing equation for sphere 4-point functions of pairs of identical primaries. It in principle allows us to constrain the structure constants across the entire spectrum of the CFT.

A technical obstacle in carrying out the genus two modular bootstrap has been the difficulty in computing the genus two conformal blocks. Recently in [138] we found a computationally efficient recursive representation of arbitrary Virasoro conformal blocks in the plumbing frame, where the Riemann surface is constructed by gluing two-holed discs with $SL(2,\mathbb{C})$ maps. For a general genus two Riemann surface, however, it is rather cumbersome to map the plumbing parameters explicitly to the period matrix elements on which the modular group $Sp(4,\mathbb{Z})$ acts naturally [130].

To circumvent this difficulty, let us recall a well-known reformulation of the modular invariance of the genus one partition function. A torus of complex modulus $\tau$ can be represented as the 2-fold cover of the Riemann sphere, branched over four points at 0, 1, $z$, and $\infty$. $\tau$ and $z$ are related by

$$\tau = i \frac{K(1-z)}{K(z)}, \quad K(z) = 2F_1\left(\frac{1}{2}, \frac{1}{2}, 1 \mid z\right). \quad (5.1.1)$$

The torus partition function $Z(\tau, \bar{\tau})$ is equal, up to a conformal anomaly factor [139], to the sphere 4-point function of $Z_2$ twist fields of the 2-fold symmetric product CFT, $\langle \sigma_2(0)\sigma_2(z, \bar{z})\sigma_2(1)\sigma_2(\infty) \rangle$. The modular transformation $\tau \to -1/\tau$ corresponds to the crossing transformation $z \to 1 - z$. In this way, the modular invariance of the torus partition function takes a similar form as the crossing equation of the sphere 4-point function, except that the sphere 4-point conformal block is replaced by the torus Virasoro character.

Usually in the numerical implementation, the crossing equation is rewritten in terms of its $(z, \bar{z})$-derivatives evaluated at $z = \bar{z} = \frac{1}{2}$. While a priori this requires computing the conformal block (the torus character in
this example) at generic \( z \), one could equivalently compute instead the conformal block at \( z = \frac{1}{2} \) with extra insertions of the stress-energy tensor, or more generally Virasoro descendants of the identity operator at a generic position (on either sheet of the 2-fold cover).

Of course, the above reformulation is unnecessary for analyzing the modular invariance of the genus one partition function, as the torus Virasoro character itself is quite simple. However, it becomes very useful for analyzing genus two modular invariance. Let us begin by considering a 1-complex parameter family of \( \mathbb{Z}_3 \)-invariant genus two Riemann surfaces that are 3-fold covers of the Riemann sphere, branched at 0, 1, \( z \), and \( \infty \). Following [140], we will refer to them as “Renyi surfaces”; such surfaces have been studied in the context of entanglement entropy [141, 142]. For instance, the period matrix of the surface is given by

\[
\Omega = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \frac{\text{2F1}(\frac{2}{3}, \frac{1}{3}; 1 | 1 - z)}{\sqrt{3} \text{2F1}(\frac{2}{3}, \frac{1}{3}; 1 | z)}. \tag{5.1.2}
\]

The genus two partition function of the CFT in question on the Renyi surface is given, up to a conformal anomaly factor, by the sphere 4-point function of \( \mathbb{Z}_3 \) twist fields in the 3-fold symmetric product CFT, whose conformal block decomposition takes the form

\[
\langle \sigma_3(0) \bar{\sigma}_3(z) \sigma_3(1) \bar{\sigma}_3(\infty) \rangle = \sum_{i,j,k \in \mathcal{I}} C^2_{ijk} F_c(h_i, h_j, h_k | z) F_c(\tilde{h}_i, \tilde{h}_j, \tilde{h}_k | \bar{z}). \tag{5.1.3}
\]

Here \( \mathcal{I} \) is the index set that labels all Virasoro primaries of the CFT, \( C_{ijk} \) are the structure constants, and \( F_c(h_1, h_2, h_3 | z) \) is the holomorphic genus two Virasoro conformal block in a particular conformal frame, with central charge \( c \) and three internal conformal weights \( h_1, h_2, h_3 \). We will see that \( F_c \) can be put in the form

\[
F_c(h_1, h_2, h_3 | z) = \exp \left[ c F^{cl}(z) \right] G_c(h_1, h_2, h_3 | z), \tag{5.1.4}
\]

where the factor \( \exp \left[ c F^{cl}(z) \right] \) captures the large \( c \) behavior of the conformal block, essentially due to the conformal anomaly. \( G_c \) is the genus two conformal block in the plumbing frame of [138] (with a different
parameterization of the moduli) whose $c \to \infty$ limit is finite. It admits a recursive representation\footnote{In contrast to the form of the recursion formulae presented in [138], here we include the factor $z^{h_1+h_2+h_3}$ in the definition of the blocks, so that the residue coefficients do not depend on $z$.}

$$
G_c(h_1, h_2, h_3|z) = G_\infty(h_1, h_2, h_3|z) + \sum_{i=1}^{3} \sum_{r,s \geq 1, r \neq s} A_{rs}^{\infty}(h_i, h_j, h_k) \frac{c - c_{rs}(h_i)}{c} G_{c_{rs}(h_i)}(h_i \to h_i + rs|z),
$$

where $c_{rs}(h)$ is a value of the central charge at which a primary of weight $h$ has a null descendant at level $rs$, and $A_{rs}^{\infty}$ are explicitly known functions of the weights.

The $Z_3$ cyclic permutations of the three sheets are themselves elements of the $Sp(4, \mathbb{Z})$ modular group. A nontrivial $Sp(4, \mathbb{Z})$ involution that commutes with the $Z_3$ is the transformation $z \to 1 - z$. This gives rise to a genus two modular crossing equation,

$$
\sum_{i,j,k \in \mathbb{Z}} C_{ijk}^2 \left[ F_c(h_i, h_j, h_k|z) F_c(h_i, h_j, h_k|z) - F_c(h_i, h_j, h_k|1 - z) F_c(h_i, h_j, h_k|1 - z) \right] = 0.
$$

Together with the non-negativity of $C_{ijk}^2$ for unitary theories, this crossing equation now puts nontrivial constraints on the possible sets of structure constants. For instance, we will find examples of critical surfaces $S$ that bound a (typically compact) domain $D$ in the space of triples of conformal weights $(h_1, h_2, h_3; \tilde{h}_1, \tilde{h}_2, \tilde{h}_3)$, such that the structure constants $C_{ijk}$ with $(h_i, h_j, h_k; \tilde{h}_i, \tilde{h}_j, \tilde{h}_k)$ outside the domain $D$ are bounded by those within the domain $D$. In particular, applying this to noncompact unitarity CFTs, one concludes that there must be triples of primaries in the domain $D$ whose structure constants are nonzero. We emphasize that the existence of a compact critical surface for the structure constants is a genuinely nontrivial consequence of genus two modular invariance, which does not follow simply from a combination of bounds on spectral gaps in the OPEs (from analyzing the crossing equation of individual sphere 4-point functions) and modular invariance of the torus partition function (which does not know about the structure constants).

The crossing equation for (5.1.3) does not capture the entirety of genus two modular invariance, since the Renyi surfaces lie on a 1 complex dimensional locus (5.1.2) in the 3 complex dimensional moduli space of genus two Riemann surfaces. Instead of considering general deformations of the geometry, equivalently we
can again insert stress-energy tensors on the Renyi surface, or more generally insert Virasoro descendants of
the identity operator in the twist field correlator (5.1.3) (on any of the three sheets). This will allow us to
access the complete set of genus two modular crossing equations, through the conformal block decomposition
of (5.1.3) with extra stress-energy tensor insertions, which is computable explicitly as an expansion in \( z \) (or
better, in terms of the elliptic nome \( q = e^{\pi i \tau} \), where \( \tau \) is related to \( z \) by (5.1.1)).

Explicit computation of the genus two Virasoro conformal block of the Renyi surface in the twist-field
frame will be given in section 5.2. The genus two modular crossing equation will be analyzed in section 5.3.
In particular, we will find critical surfaces for structure constants simply by taking first order derivatives of
the modular crossing equation with respect to the moduli around the crossing invariant point. In section 5.4,
we formulate the crossing equation beyond the \( Z_3 \)-invariant locus in the moduli space of genus two Riemann
surfaces. We conclude with some future prospectives in section 5.5.

5.2 The genus two conformal block

In this section, we will study the genus two Virasoro conformal block with no external operators, focusing
on the \( Z_3 \)-invariant Renyi surface that is a 3-fold branched cover of the Riemann sphere with four branch
points. The latter can be represented as the curve

\[
y^3 = \frac{(x - x_1^+)(x - x_2^+)}{(x - x_1^-)(x - x_2^-)}
\]

in \( \mathbb{P}^1 \times \mathbb{P}^1 \). The genus two partition function of the CFT on the covering surface can be viewed as a correlation
function of the 3-fold symmetric product CFT on the sphere: up to a conformal anomaly factor (dependent
on the conformal frame), it is given by the 4-point function of \( Z_3 \) twist fields \( \sigma_3 \) and anti-twist fields \( \bar{\sigma}_3 \),

\[
\langle \sigma_3(x_1^+) \bar{\sigma}_3(x_1^-) \sigma_3(x_2^+) \sigma_3(x_2^-) \rangle.
\]
5.2.1 OPE of $Z_3$-twist fields in $\text{Sym}^3(\text{CFT})$

We will begin by analyzing the OPE of the $Z_3$ twist field $\sigma_3$ and the anti-twist field $\bar{\sigma}_3$. The 3-fold symmetric product CFT on the sphere with the insertion of $\sigma_3(z_1)$ and $\bar{\sigma}_3(z_2)$ can be lifted to a single copy of the CFT on the covering space $\Sigma$, which is also a Riemann sphere. Let $t$ be the affine coordinate on the covering sphere. It suffices to consider the special case $z_1 = 0$, $z_2 = 1$, where the covering map can be written as

$$z = \frac{(t + \omega)^3}{3\omega(1 - \omega)t(t - 1)},$$

(5.2.2)

where $\omega = e^{2\pi i/3}$. The branch points $z_1 = 0$, $z_2 = 1$ correspond to $t = -\omega$ and $t = 1 + \omega$ respectively. We have chosen this covering map (up to $SL(2, \mathbb{C})$ action on $\Sigma$) such that the three points $t_1 = 0$, $t_2 = 1$, and $t_3 = \infty$ on $\Sigma$ are mapped to $z = \infty$.

Now let us compute the 3-point function of the pair of twist fields $\sigma_3(0)$, $\bar{\sigma}_3(1)$, and a general Virasoro...
descendant operator in the 3-fold tensor product CFT of the form

\[ \Phi = \bigotimes_{i=1}^{3} L_{-N_i} \phi_i \]  

(5.2.3)

inserted at \( z = \infty \) (as a BPZ conjugate operator). Here we will keep track of the holomorphic \( z \)-dependence only, and omit the anti-holomorphic sector. For each \( i = 1, 2, 3 \), \( \phi_i \) is a primary of weight \( h_i \) in a single copy of the CFT, \( N_i = \{ n_{i1}^{(i)}, \ldots, n_{ik}^{(i)} \} \) is a partition of the integer \( |N_i| \) in descending order, and \( L_{-N_i} \) is the Virasoro chain \( L_{-n_i^{(i)}} \cdots L_{-n_k^{(i)}} \). Following [139], we can write

\[ \frac{\langle \sigma_3(0) \bar{\sigma}_3(1) \Phi(\infty) \rangle}{\langle \sigma_3(0) \bar{\sigma}_3(1) \rangle} = \langle O'_1(0) O'_2(1) O'_3(\infty) \rangle . \]  

(5.2.4)

Here \( O'_i(t_i) \) is the conformally transformed operator of \( L_{-N_i} \phi_i \) on the \( i^{th} \) covering sheet,\(^2\)

\[ O'_i(t_i) = (z'(t_i))^{-h_i} L_{-N_i}^t \phi'_i(t_i) = (z(t_i))^{-2h_i} [3\omega(1-\omega)]^{-h_i} L_{-N_i}^t \phi'_i(t_i), \]  

(5.2.5)

where \( \phi'_i(t_i) \) is the corresponding primary in the \( t \)-frame. \( L_{-N}^t = L_{-n_1}^t \cdots L_{-n_k}^t \) is the lift of \( L_{-N} \) (acting on an operator at \( z = \infty \)) to the \( t \)-plane. When acting on an operator at \( t = t_i, \) \( L_{-n}^t \) is given by

\[ L_{-n}^t = -\oint_{C_t} \frac{du}{2\pi i} \frac{(u-z(u))^{1+n}}{z'(u)} \left[ T_{uu}(u) - \frac{c}{12} \{ z(u), u \} \right] \]

\[ = -[3\omega(1-\omega)]^{-n} \text{Res}_{u\to t} u^{1-n}(u-1)^{-n}(u+\omega)^{1+3n}(u+\omega^2)^{-2} \left[ T_{uu}(u) - \frac{c}{(u+\omega)^2(u+\omega^2)^2} \right] , \]  

(5.2.6)

where we used the Schwarzian derivative

\[ \{ z,t \} = \frac{12}{(t+\omega)^2(t+\omega^2)^2} . \]  

(5.2.7)

The contour integral in (5.2.6) is taken on the \( t \)-plane, parameterized by the variable \( u \). \( C_t \) is a small

\(^2\)The factor \( z(t_i)^{-2h_i} \) drops out of the correlator (5.2.4) due to the normalization convention of \( \Phi(\infty) \).
counterclockwise circular contour around $t_i$ for $t_1 = 0$ and $t_2 = 1$. For $t_3 = \infty$, $C_\infty$ is taken to be a large clockwise circular contour on the $t$-plane. Note that the sign convention for the residue at infinity is such that \( \text{Res}_{u \to \infty} \frac{1}{u} = -1 \). The overall minus sign on the RHS of (5.2.6) is due to the orientation of the original $z$-contour (where we replace $L_{-n}$ acting on an operator at $z = \infty$ by $L_n$ acting on the product operator $\sigma_3(0)\tilde{\sigma}_3(1)$).

We proceed by putting (5.2.6) into the explicit form

\[
\mathcal{L}_{-n}^t = \sum_{m \geq -n} a_{-n,m}^t L_m + c b_n^t, \tag{5.2.8}
\]

where

\[
a_{-n,m}^0 = -[3\omega(1 - \omega)]^{-n} \text{Res}_{u \to 0} u^{-n-m-1}(u-1)^{1-n}(u+\omega)^{1+3n}(u+\omega^2)^{-2},
\]

\[
a_{-n,m}^1 = -[3\omega(1 - \omega)]^{-n} \text{Res}_{u \to 1} u^{-1-n}(u-1)^{-n-m-1}(u+\omega)^{1+3n}(u+\omega^2)^{-2}, \tag{5.2.9}
\]

\[
a_{-n,m}^\infty = -[3\omega(1 - \omega)]^{-n} \text{Res}_{u \to \infty} u^{-n+m-1}(u-1)^{1-n}(u+\omega)^{1+3n}(u+\omega^2)^{-2},
\]

and

\[
b_n^t = [3\omega(1 - \omega)]^{-n} \text{Res}_{u \to t} u^{-1-n}(u-1)^{1-n}(u+\omega)^{-1+3n}(u+\omega^2)^{-4}, \tag{5.2.10}
\]

for $t = 0, 1, \infty$. On the RHS of (5.2.8), $L_m$ is understood to be acting on an operator inserted at $t = 0, 1, \text{or } \infty$.

Putting these together, the 3-point function of interest is

\[
\frac{\langle \sigma_3(0)\tilde{\sigma}_3(1)\Phi(\infty) \rangle}{\langle \sigma_3(0)\tilde{\sigma}_3(1) \rangle} = C_{123} [3\omega(1 - \omega)]^{-h_1-h_2-h_3} \rho(L_{-N_1}h_1, L_{-N_2}^1h_2, L_{-N_3}^0h_3), \tag{5.2.11}
\]

where $C_{123} = \langle \phi_1(0)\phi_2(1)\phi_3(\infty) \rangle$ is the structure constant of the primaries, and $\rho(\xi_3, \xi_2, \xi_1)$ is the 3-point function of Virasoro descendants at $\infty, 1, 0$ on the plane, defined as in [105,138]. We remind the reader that so far we have only taken into account the holomorphic part of the correlator, for the purpose of deriving the holomorphic Virasoro conformal block in the next section.
5.2.2 The conformal block decomposition of \( \langle \sigma_3(0) \tilde{\sigma}_3(z) \sigma_3(1) \tilde{\sigma}_3(\infty) \rangle \)

Now we turn to the 4-point function of twist fields, \( \langle \sigma_3(0) \tilde{\sigma}_3(z) \sigma_3(1) \tilde{\sigma}_3(\infty) \rangle \), and compute the contribution from general untwisted sector descendants of the form \( \Phi = \bigotimes_{i=1}^{3} L_{-N_i} \phi_i \) in the \( \sigma_3(0) \tilde{\sigma}_3(z) \) OPE, for a given triple of primaries \( \phi_1, \phi_2, \phi_3 \). Again, we focus only on the holomorphic sector. This is given by

\[
\sum_{\{N_i\}, \{M_i\}} \prod_{k=1}^{3} G_{h_k}^{N_k M_k} \langle \sigma_3(0) \tilde{\sigma}_3(z) \rangle \left[ \bigotimes_{i=1}^{3} L_{-N_i} |\phi_i\rangle \right] \left[ \bigotimes_{i=1}^{3} \langle \phi_i | L_{-M_i}^{\dagger} \right] \sigma_3(1) \tilde{\sigma}_3(\infty) = \\
= \sum_{\{N_i\}, \{M_i\}} \sum_{k=1}^{3} z^{-2h_\sigma + \sum_{i=1}^{3} (h_i + |N_i|)} \prod_{k=1}^{3} G_{h_k}^{N_k M_k} \langle \sigma_3(0) \tilde{\sigma}_3(1) \rangle \bigotimes_{i=1}^{3} L_{-N_i} |\phi_i(\infty)\rangle \langle \sigma_3(1) \tilde{\sigma}_3(\infty) \rangle \langle \sigma_3(0) \tilde{\sigma}_3(1) \rangle \bigotimes_{j=1}^{3} L_{-M_j} |\phi_j(\infty)\rangle \\
= \sum_{\{N_i\}, \{M_i\}} \sum_{k=1}^{3} z^{-2h_\sigma + \sum_{i=1}^{3} (h_i + |N_i|)} \prod_{k=1}^{3} G_{h_k}^{N_k M_k} \left| z^\prime (t_k) \right| \left( \prod_{i=1}^{3} L_{-N_i} |\phi_i(t_i)\rangle \right) \left( \prod_{j=1}^{3} L_{-M_j}^{\dagger} |\phi_j(t_j)\rangle \right). 
\]

(5.2.12)

Here the summation is over integer partitions in descending order \( N_i \) and \( M_i \), for \( i = 1, 2, 3 \), and \( G_{h_k}^{N_k M_k} \) are the inverse Gram matrix elements for a weight \( h \) Verma module (nontrivial only for \( |N| = |M| \)). \( L_{-M}^{\dagger} \) is defined as the complex conjugation of \( L_{-M} \) (not to be confused with the adjoint operator), which simply amounts to replacing \( \omega \) by \( \omega^2 \) in (5.2.8)-(5.2.10). The appearance of the complex conjugate factors is due to the exchange of \( \sigma_3 \) with \( \tilde{\sigma}_3 \) in the last two factors in the third line of (5.2.12). \( h_\sigma \) is the holomorphic conformal weight of the \( Z_3 \) twist field, given by

\[
h_\sigma = \left( 3 - \frac{1}{3} \right) \frac{c}{24} = \frac{c}{9}.
\]

(5.2.13)

Using the covering map in the previous section, we arrive at the genus two conformal block for the Renyi surface in the twist field frame

\[
\mathcal{F}_c(h_1, h_2, h_3 |z) = 3^{-3} \sum_{\{N_i\}, \{M_i\}} \sum_{k=1}^{3} \prod_{k=1}^{3} G_{h_k}^{N_k M_k} \left[ \bigotimes_{i=1}^{3} L_{-N_i} h_3, L_{-N_i}^{\dagger} h_2, L_{-N_i}^{0} h_1 \right] \rho(L_{-M_1}^{\infty} h_3, L_{-M_2}^{\dagger} h_2, L_{-M_1}^{0} h_1) \rho(L_{-M_2}^{\infty} h_3, L_{-M_1}^{\dagger} h_2, L_{-M_1}^{0} h_1),
\]

(5.2.14)
where $L_{-N}^{0,1,\infty}$ are given by (5.2.8)-(5.2.10).

Let us comment on the $h_i \to 0$ limit, which is rather delicate. If one of the $h_i$ vanishes, say $h_1 = 0$, corresponding to the vacuum channel in one of the three handles of the genus two surface, then the only conformal blocks that appear in the genus two partition function involve $h_2 = h_3$. For $h_2 = h_3 > 0$, the $h_1 = 0$ block is given by the $h_1 \to 0$ limit of (5.2.14). This is not the case however for the vacuum block, where all three weights $h_i$ vanish: in fact the vacuum block differs from the simultaneous $h_i \to 0$ limit of (5.2.14). This is because the latter contains nonvanishing contributions from null descendants of the identity operator that are absent in the vacuum block.

### 5.2.3 Recursive representation

As already mentioned in the introduction, the genus two conformal block (5.2.14) admits a recursive representation in the central charge of the form (5.1.4), (5.1.5). The recursion formula is useful in computing the $z$-expansion to high orders efficiently, and can be derived by essentially the same procedure as in [138].

The only new feature is that the twist field frame considered here is different from the plumbing frame of [138], which leads to the conformal anomaly factor $\exp [c F_{cl}(z)]$ in (5.1.4). While in principle $F_{cl}(z)$ can be determined by evaluating a suitable classical Liouville action as in [139], we find it more convenient to compute $F_{cl}(z)$ by directly inspecting the large $c$ limit of $\log F_c(h_1, h_2, h_3 | z)$. Indeed, the latter is linear in $c$ in the large $c$ limit (with a leading coefficient that is independent of the internal weights), with the following series expansion in $z$.

\[
F_{cl}(z) = -\frac{2}{9} \log(z) + 6 \left( \frac{z}{27} \right)^2 + 162 \left( \frac{z}{27} \right)^3 + 3975 \left( \frac{z}{27} \right)^4 + 96552 \left( \frac{z}{27} \right)^5 + 2356039 \left( \frac{z}{27} \right)^6 \\
+ 57919860 \left( \frac{z}{27} \right)^7 + \frac{2869046823}{2} \left( \frac{z}{27} \right)^8 + 35771031918 \left( \frac{z}{27} \right)^9 + \frac{4486697950566}{5} \left( \frac{z}{27} \right)^{10} \\
+ \mathcal{O}(z^{11}).
\]

(5.2.15)
Note that $\xi F^{cl}(z)$ agrees with the semiclassical Virasoro sphere 4-point conformal block of central charge $c$ in the vacuum channel with four external primaries of weight $\frac{h_i}{2\pi} = \frac{c}{2\pi}$ [103, 143]. For numerical computations, we can pass to the elliptic nome parameter $q = e^{\pi i \tau}$, where $\tau$ is related to $z$ via (5.1.1). The $q$-expansion converges much faster than the $z$-expansion\(^3\) evaluated at the crossing symmetric point $z = \frac{1}{2}$, which corresponds to $q = e^{-\pi}$.

After factoring out $\exp[c F^{cl}(z)]$, the remaining part of the conformal block $G_c(h_1, h_2, h_3|z)$ as a function of the central charge $c$ has poles at

$$c_{rs}(h) = 1 + 6(b_{rs}(h) + b_{rs}(h)^{-1})^2, \quad \text{with} \quad b_{rs}(h)^2 = \frac{r s - 1 + 2h + \sqrt{(r - s)^2 + 4(rs - 1)h + 4h^2}}{1 - r^2},$$

(5.2.16)

where $r \geq 2$ and $s \geq 1$, for $h = h_i$, $i = 1, 2, 3$. The residue at a pole $c = c_{rs}(h_i)$ is proportional to the conformal block with central charge $c_{rs}(h_i)$ and the weight $h_i$ shifted to $h_i + rs$. The precise recursion formula is

$$G_c(h_1, h_2, h_3|z) = G_\infty(h_1, h_2, h_3|z)$$

$$\quad + \sum_{r \geq 2, s \geq 1} \left[ - \frac{\partial c_{rs}(h_1)}{\partial h_1} \right] \frac{A^c_{rs}(h_1)}{c - c_{rs}(h_1)} G^{c_{rs}(h_i)}(h_1 \rightarrow h_1 + rs|z)$$

$$\quad + (2 \text{ cyclic permutations on } h_1, h_2, h_3),$$

(5.2.17)

where $A^c_{rs}$ is the constant

$$A^c_{rs} = \frac{1}{2} \prod_{m=1-r}^r \prod_{n=1-s}^s (mb + nb^{-1})^{-1}, \quad (m, n) \neq (0, 0), (r, s),$$

(5.2.18)

\(^3\)As explained in [103], the $q$-expansion of $F^{cl}$ in general need not have unit radius of convergence, due to possible zeroes of the conformal block. In the present example, the radius of convergence nonetheless appears to be 1. We thank Y.-H. Lin for pointing out this subtlety and providing numerical verifications.
for \( c = 1 + 6(b + b^{-1})^2 \), and \( P^c_{rs} \) is the fusion polynomial

\[
P^c_{rs} \left[ \begin{array}{c} d_1 \\ d_2 \\ \end{array} \right] = \prod_{p=1-r}^{r-1} \prod_{q=1-s}^{s-1} \frac{\lambda_1 + \lambda_2 + pb + qb^{-1}}{2} \frac{\lambda_1 - \lambda_2 + pb + qb^{-1}}{2}, \tag{5.2.19}
\]

where \( \lambda_i \) are related to the weights \( d_i \) by

\[
d_i = \frac{1}{4} (b + b^{-1})^2 - \frac{1}{4} \lambda_i^2.
\]

The remaining undetermined piece in the formula (5.2.17) is the \( c \to \infty \) limit \( G_\infty(h_1, h_2, h_3|z) \). It was shown in [138] that \( G_\infty(h_1, h_2, h_3|z) \) is the product of the vacuum block and \( SL(2, \mathbb{C}) \) global block in the plumbing frame. The vacuum block is given by the holomorphic part of the gravitational 1-loop free energy of the genus two hyperbolic handlebody, computed in [118]. To translate the result of [118] into the vacuum part of our \( G_\infty \) requires expressing the Schottky parameters of the Renyi surface in terms of \( z \); this can be achieved through the map between Schottky parameters and the period matrix (5.1.2). Furthermore, the global block of [138] is naturally expressed in terms of the plumbing parameters, whose map to \( z \) is nontrivial.

The implementation of an efficient recursive computational algorithm for the genus two conformal blocks in the twist field frame will require knowing \( G_\infty \), which is in principle computable given the above ingredients, based on the map from \( z \) to the Schottky parameters and the plumbing parameters of the Renyi surface. Here we simply evaluate the \( z \)-expansion (5.2.14) directly, strip off the conformal anomaly factor and then take the \( c \to \infty \) limit, giving the result

\[
G_\infty(h_1, h_2, h_3|z) = \left( \frac{z}{27} \right)^{h_1 + h_2 + h_3} \left( 1 + \frac{h_1 + h_2 + h_3}{2} + \frac{(h_2 - h_3)^2}{54h_1} + \frac{(h_3 - h_1)^2}{54h_2} + \frac{(h_1 - h_2)^2}{54h_3} \right) + O(z^2). \tag{5.2.20}
\]

As already noted, the analog of \( G_\infty \) for the vacuum block, \( G^0_\infty(z) \), is not the same as the simultaneous \( h_i \to 0 \) limit of (5.2.20). The first few terms in the \( z \)-expansion of \( G^0_\infty(z) \) is given explicitly by

\[
G^0_\infty(z) = 1 + 3 \left( \frac{z}{27} \right)^4 + 168 \left( \frac{z}{27} \right)^5 + 6567 \left( \frac{z}{27} \right)^6 + 222012 \left( \frac{z}{27} \right)^7 + 6960036 \left( \frac{z}{27} \right)^8 + O(z^9). \tag{5.2.21}
\]
5.2.4 Mapping to the 3-fold-pillow

In this section we consider the Renyi surface in the 3-fold-pillow frame, which makes obvious certain positivity properties of the genus two conformal block. Following [121], the map from the plane (parameterized by $w$) to the pillow (parameterized by $v$) is given by

$$v = \frac{1}{(\theta_3(\tau))^2} \int_0^w \frac{dx}{\sqrt{x(1-x)(z-x)}}. \quad (5.2.22)$$

The four branch points on the plane at 0, $z$, 1, $\infty$, where the $Z_3$ twist fields and anti-twist fields are inserted, are mapped to $v = 0, \pi, \pi(\tau + 1), \pi\tau$ respectively, where $\tau$ is given by (5.1.1). The covering surface is turned into a 3-fold cover of the pillow, with the twist fields inserted at the four corners.

\[ \text{Figure 5.2: Left: The pillow geometry is the quotient } T^2/Z_2. \text{ The four branch points on the plane 0, } z, 1, \infty \text{ are mapped to the } Z_2 \text{ fixed points } v = 0, \pi, \pi(\tau + 1), \pi\tau \text{ respectively. Right: The pillow with the } Z_3 \text{ twist fields inserted at the corners. In section 5.4 we will obtain the full set of genus two modular crossing equations by inserting the stress-energy tensor or more generally arbitrary Virasoro descendants of the identity at the front center on each sheet of the 3-fold-pillow.} \]
The Renyi surface conformal block in the twist field frame can be mapped to the pillow frame as

\[ F_c(h_1, h_2, h_3 | z) = (z(1-z))^{\bar{z}} e^{-2h_\sigma \theta_3(\tau)} \sum_{n=0}^{\infty} A_n(h_1, h_2, h_3) q^n, \tag{5.2.23} \]

where \( q = e^{\pi i \tau}, \ h_\sigma = c_9 \). For instance, the first coefficients \( A_0 \) and \( A_1 \) are given by

\[
A_0 = 2^{-\bar{z}} \left( \frac{16}{27} \right)^{h_1 + h_2 + h_3}, \\
A_1 = 2^{-\bar{z}-1} \left( \frac{16}{27} \right)^{h_1 + h_2 + h_3 + 1} \left[ \left( \frac{h_1 - h_2}{h_3} \right)^2 + \left( \frac{h_2 - h_3}{h_1} \right)^2 + \left( \frac{h_3 - h_1}{h_2} \right)^2 \right]. \tag{5.2.24} 
\]

We also record here the first few coefficients \( A_n^0 \) in the \( q \)-expansion of the vacuum block in the pillow frame analogous to (5.2.23), which, as already emphasized, differ from the \( h_i \to 0 \) limit of (5.2.24),

\[
A_0^0 = 2^{-\bar{z}}, \quad A_1^0 = 0, \quad A_2^0 = 2^{-\bar{z}-1} \frac{25c}{243}, \quad A_3^0 = 0, \\
A_4^0 = 2^{-\bar{z}-3} \frac{1875c^2 + 83110c + 524288}{177147}, \quad A_5^0 = \frac{2^{-\bar{z}+21}}{4782969}, \tag{5.2.25} \\
A_6^0 = 2^{-\bar{z}-4} \frac{140625c^3 + 1869750c^2 + 349131040c + 2969567232 + 2147483648c^{-1}}{387420489}.
\]

Importantly, all of the coefficients \( A_n(h_1, h_2, h_3) \) are non-negative, as they can be interpreted as inner products of level \( n \) descendant states created by pairs of twist-anti-twist fields on two corners of the pillow, similarly to the sphere 4-point block analyzed in [121]. Indeed, we have explicitly verified the positivity of \( A_n(h_1, h_2, h_3) \) with \( c > 1 \) and \( h_i > 0 \), for \( n \leq 5 \).

### 5.3 The genus two modular crossing equation

#### 5.3.1 Some preliminary analysis

Now we consider the genus two modular crossing equation restricted to the Renyi surface, as given by (5.1.6). Some crude but rigorous constraints on the structure constants in unitary CFTs can be deduced even without
appealing to the details of the $z$-expansion of the genus two conformal block. First, let us write the twist field 4-point function \((5.1.3)\) in the pillow coordinates,

$$
\langle \sigma_3(0)\sigma_3(z, \bar{z})\sigma_3(1)\sigma_3(\infty) \rangle = \left| (z(1-z))^{\frac{c}{2}-2h+\theta_3(\tau)^{\frac{3c}{2}-16h+q}q^{-\frac{c}{2}}} \right|^2 \times \sum_{i,j,k}^{\infty} \sum_{n,m=0}^{\infty} C_{ijk}^2 A_n(h_i, h_j, h_k) A_m(\tilde{h}_i, \tilde{h}_j, \tilde{h}_k) q^{h_i+h_j+h_k+n\tilde{q}_i+\tilde{q}_j+\tilde{q}_k+m} \tag{5.3.1}
$$

In the last line, we simply grouped terms of the same powers of $q$ and $\tilde{q}$ together in the sum. The index set $J$ is by construction the union of \(\sum_{i=1}^{3} h_i + Z_{\geq 0}, \sum_{i=1}^{3} \tilde{h}_i + Z_{\geq 0}\) for all triples of conformal weights \(\{(h_i, \tilde{h}_i), i = 1, 2, 3\}\) that appear in nonzero structure constants, including the case where one of the primaries is the identity and the structure constant reduces to the two-point function coefficient. It follows from the non-negativity of the coefficients $A_n$ that $\tilde{C}_{h, \tilde{h}}^2$ are non-negative quantities in a unitary CFT.

Let us now apply \((5.3.1)\) to a unitary noncompact CFT, where the $SL(2)$-invariant vacuum is absent and the identity is not included in the spectrum of $(\delta$-function) normalizable operators. $\tilde{C}_{h, \tilde{h}}^2$ now only receives contributions from the structure constants of nontrivial primaries. Applying first order derivatives in $z$ and $\bar{z}$ to the crossing equation, and evaluating at $z = \bar{z} = \frac{1}{2}$, we have

$$
\sum_{(h, \tilde{h}) \in J'} \tilde{C}_{h, \tilde{h}}^2 \partial_{z|z=\frac{1}{2}} \left[ (z(1-z))^{\frac{c}{2}-4h+\theta_3(\tau)^{\frac{3c}{2}-32h+q}q^{h+\tilde{h}}-\frac{c}{4}} \right] = 0. \tag{5.3.2}
$$

In the above equation, the factor multiplying $\tilde{C}_{h, \tilde{h}}^2$ is negative for $\Delta \equiv h + \tilde{h}$ below a certain “critical dimension” $\Delta_{crit}$ and positive for $\Delta > \Delta_{crit}$. It follows immediately that there must be a nonzero $\tilde{C}_{h, \tilde{h}}^2$ for $\Delta < \Delta_{crit}$, i.e. there must be a triple of primaries with nonzero structure constant, whose total scaling dimension is less than $\Delta_{crit}$, in any unitary noncompact CFT of central charge $c$. The value of (or rather, an upper bound on) the critical dimension is easily computed from \((5.3.2)\) to be

$$
\Delta_{crit} = \left( \frac{1}{4} - \frac{3}{4\pi} \right) c + \frac{8}{\pi} h_\sigma + \frac{9\pi + 5}{36\pi} c \approx 0.29421c. \tag{5.3.3}
$$

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As a consistency check, the Liouville CFT of central charge $c$ has nonzero structure constants for triples of primaries of total scaling dimension above the threshold $\frac{c-1}{4}$, which is indeed less than (5.3.3).

Although rigorous, the bound (5.3.3) is quite crude. To deduce similar results in compact CFTs, it will be important to distinguish the contributions of Virasoro descendants from those of the primaries in (5.3.1). We will refine our analysis in the next subsection by computing the $z$ or $q$-expansion of the genus two conformal block to higher orders.

### 5.3.2 Critical surfaces

As is standard in the numerical bootstrap \cite{14,16,85}, we can turn the genus two modular crossing equation (5.1.6) into linear equations for $C_{ijk}^2$ by acting on it with the linear functional

$$\alpha = \sum_{n+m=\text{odd}} a_{n,m} \partial^n z \partial^m \bar{z} \bigg|_{z=\bar{z}=\frac{1}{2}},$$

(5.3.4)

where $a_{n,m}$ are a set of real coefficients, and obtain constraints on the structure constants of the general form

$$\sum_{i,j,k \in \mathcal{I}} C_{ijk}^2 F_c^\alpha (h_i, h_j, h_k; \tilde{h}_i, \tilde{h}_j, \tilde{h}_k) = 0,$$

(5.3.5)

where $F_c^\alpha$ is a function of a triple of left and right conformal weights. For typical choices of the linear functional $\alpha$, $F_c^\alpha$ will be negative on a domain $D$ in the space of triples of conformal weights, and positive on the complement of the closure of $D$. A critical surface $S$ is defined to be the boundary of $D$ where $F_c^\alpha$ vanishes. With an appropriate choice of sign in $\alpha$, the domain $D$ consists of triples of low lying weights (we will see that the critical surface is often compact), and the equation (5.3.5) implies that the structure constants outside of $D$ are bounded by those that lie within $D$.

Clearly, the critical surface $S$ depends on the choice of $\alpha$. It is of interest to find critical surfaces that bound a domain $D$ that is as “small” as possible, so that we can bound as many structure constants as possible based on the knowledge of a small set of structure constants of low dimension operators in any
Figure 5.3: Top: Three-dimensional plots of the domain $D_h^{(3)}$ for $c = 1, 4, 25$. Bottom: Plots of the cross-sections of these domains for various values of $h_1$. The structure constants of primaries with twists $(\tau_1, \tau_2, \tau_3) = (2h_1, 2h_2, 2h_3)$ outside these critical domains are bounded by those whose twists lie within the domains.

unitary CFT. Here we will consider the simplest nontrivial linear functional $\alpha$ which involves only first order derivatives in $z$ or in $\bar{z}$. In this case, the critical surface is the locus

$$a_{1,0}W_c(h_1, h_2, h_3) + a_{0,1}W_c(\tilde{h}_1, \tilde{h}_2, \tilde{h}_3) = 0,$$  \hfill (5.3.6)  

where $W_c(h_1, h_2, h_3) \equiv \partial_z \log F_c(h_1, h_2, h_3|z)|_{z=\frac{1}{2}}$. For instance, we can choose $a_{0,1} = 0$, and the critical surface $W_c(h_1, h_2, h_3) = 0$ bounds a compact domain $D_h$ in $\mathbb{R}^3_{\geq 0}$ parameterized by the holomorphic weights $(h_1, h_2, h_3)$, and bound structure constants of triples of primaries of higher twists by those of lower twists.
From (5.2.23), we have

$$W_c(h_1, h_2, h_3) = \frac{\pi^2}{K(\frac{1}{2})^2} \left[ h_1 + h_2 + h_3 - \left( \frac{1}{8} + \frac{5}{72\pi} \right) c + \sum_{n=0}^{\infty} \frac{nA_n(h_1, h_2, h_3)e^{-n\pi}}{\sum_{n=0}^{\infty} A_n(h_1, h_2, h_3)e^{-n\pi}} \right].$$  (5.3.7)

The last term in the bracket is always positive (assuming $c > 1$ and $h_i > 0$), thus the domain $W_c < 0$ lies within the region $h_1 + h_2 + h_3 < \left( \frac{1}{8} + \frac{5}{72\pi} \right) c$ and is compact. This is what we have seen in the previous subsection.

For numerical evaluation we may work with the truncated version

$$W_c^{(N)}(h_1, h_2, h_3) = \frac{\pi^2}{K(\frac{1}{2})^2} \left[ h_1 + h_2 + h_3 - \left( \frac{1}{8} + \frac{5}{72\pi} \right) c + \sum_{n=0}^{N} \frac{nA_n(h_1, h_2, h_3)e^{-n\pi}}{\sum_{n=0}^{N} A_n(h_1, h_2, h_3)e^{-n\pi}} \right].$$  (5.3.8)

The domain $D^{(N)}_h = \{(h_1, h_2, h_3) \in \mathbb{R}_{\geq 0}^3 : W_c^{(N)}(h_1, h_2, h_3) < 0\}$ becomes smaller with increasing $N$ (and of course, converges to $D_h$ in the $N \to \infty$ limit). In figure 5.3 we plot some examples of the critical domain $D^{(N)}_h$ with $N = 3$, for central charges $c = 1$, 4, and 25. The location of the critical surface converges rather quickly with the $q$-expansion order: an example of a slice of the critical domain in the $c = 4$ case is shown in figure 5.4.

In particular, in the limit $h_1 \to 0$, with $h_2, h_3$ fixed at generic positive values, the coefficients $A_n$ diverge like $h_1^{-1}P_n(h_2, h_3)$, where $P_n$ is a rational function of $h_2, h_3$ that vanishes quadratically along $h_2 = h_3 (> 0)$. 

Figure 5.4: A slice of the $c = 4$ critical domain $D^{(N)}_h$, which converges quickly with the truncation order $N$ of the $q$-expansion.
Figure 5.5: Slices of the critical domain $D_h^{(3)}$ for $c = 8$ and $c = 10$ at small values of $h_1$. For $c = 8$, the critical domain $D_h$ intersects the $h_1 = 0$ plane only along a segment of the diagonal line $h_2 = h_3$. This is not the case for $c = 10$.

For $h_2 \neq h_3 > 0$, for instance, we have $\lim_{h_i \to 0} W_c^{(1)}(h_1, h_2, h_3) = \frac{\pi^2}{K(\frac{1}{2})} [h_2 + h_3 - \left(\frac{1}{8} + \frac{5}{72\pi}\right) c + 1]$, which is always positive for $c < 6.79787$. A slightly more intricate analysis of $\lim_{h_i \to 0} W_c^{(2)}$ shows that it is positive for $c < 9.31751$. Consequently, for this range of the central charge $c$, the domain $D_h^{(2)}$ (and thereby $D_h$) meets the $h_1 = 0$ plane along a segment of the line $h_2 = h_3$ only. This is demonstrated in figure 5.5.

For $c > 1$, we observe that $W_c(h_1, h_2, h_3)$ is minimized in the limit $h_1 = h_2 = h_3 \to 0$, where it approaches a negative value $-r_c$ (note that in the simultaneous $h_i \to 0$ limit $W_c$ depends on the ratios of the $h_i$'s). For $a_{0,1}$ and $a_{0,1}$ both positive, the domain $D$ bounded by the critical surface $S$ then lies strictly within the domain

$$W_c(h_1, h_2, h_3) < \frac{a_{0,1}}{a_{1,0}} r_c,$$

$$W_c(\tilde{h}_1, \tilde{h}_2, \tilde{h}_3) < \frac{a_{1,0}}{a_{0,1}} r_c. \quad (5.3.9)$$

Let us choose $a_{0,1} = a_{1,0}$, and define $\bar{D}$ as the domain $W_c(h_1, h_2, h_3) < r_c$ in $\mathbb{R}^3_{\geq 0}$. Now the compact domain $D_\Delta = \bar{D} + \bar{D}$ (the set of sums of vectors from each set) in $\mathbb{R}^3_{\geq 0}$ may be viewed as a critical domain in the triple of scaling dimensions $(\Delta_1, \Delta_2, \Delta_3)$, with $\Delta_i = h_i + \tilde{h}_i$; in the sense that structure constants of triples of
primaries of dimensions $(\Delta_1, \Delta_2, \Delta_3)$ outside $D_\Delta$ are bounded by those that lie within $D_\Delta$. The figures show some examples of $D_\Delta$ are shown in figure 5.6.

A subtlety pointed out at the end of section 5.2.2 is that the simultaneous $h_i \to 0$ limit of the genus two conformal block with three positive internal weights is distinct from the vacuum block. If we define $W_{c,0}$ to be (5.3.7) computed using the vacuum block, we would find a result that is slightly below

$$\lim_{h_1 = h_2 = h_3 \to 0^+} W_c(h_1, h_2, h_3) = -r_c.$$  
Since we seek critical surfaces such that the structure constants of “heavy primaries” outside are bounded by those of the “light primaries” that lie inside the surface, the vacuum block which enters the genus two partition function with coefficient 1 is not relevant, and thus the result

\footnote{Note that if $\tilde{D}$ is convex, then $D_\Delta$ is simply $\tilde{D}$ rescaled by a factor of 2, but in fact $\tilde{D}$ is generally not convex in the region where one of the weights is small.}
suffices.

5.4 Beyond the $\mathbb{Z}_3$-invariant surface

In order to write the modular crossing equation for the partition functions on genus two Riemann surfaces of general moduli in a computationally useful manner, we will still work at the $\mathbb{Z}_3$-invariant Renyi surface and expand around the crossing-invariant point $z = \frac{1}{2}$, but with extra insertions of stress-energy tensors $T(z_j)$ and $\tilde{T}(\bar{z}_j)$ on any of the three sheets.

Under the crossing $z \to 1 - z$, the transformation of the stress-energy tensors is simple. For instance, it suffices to work with the insertion of $V = L_{-N}\bar{L}_{-\bar{N}} \cdot 1$ on one of the sheets at the point $w$. Here $L_{-N} = L_{-n_1} \cdots L_{-n_k}$ is a Virasoro chain, and $\bar{L}_{-\bar{N}}$ is defined similarly. The crossing transformation sends $V$ to the operator $(-)^{|N|+|\bar{N}|}L_{-N}\bar{L}_{-\bar{N}} \cdot 1$ inserted at the position $1 - w$. The point $w$ is mapped to the pillow coordinate via (5.2.22). In particular, with $z = \frac{1}{2}$, $\tau = i$, the points $w = \frac{1}{2} \pm \frac{i}{2} \pi$ are mapped to $v = \frac{1+i}{2} \pi$ (up to monodromies), i.e. the center on the front and back of the pillow.

We can now define the modified conformal blocks with $\tilde{L}_{-R_i}(x)$ insertion on the $i$-th sheet,

$$F(h_1, h_2, h_3; R_1, R_2, R_3; w|z) = 3^{-3} \sum_{i=1}^{3} h_i \sum_{\{N_i\}, \{M_i\}} \sum_{|N_i|, |M_i|} z^{-2h_\sigma + \sum_{i=1}^{3} (h_i + |N_i|) \tau \sum_{k=1}^{3} (|M_k| - |N_k| - |R_k|) \rho(L_{-N_3}h_3, L_{-N_2}h_2, L_{-N_1}h_1) \rho(L_{-M_3}^*, h_3, L_{-M_2}^*, h_2, L_{-M_1}^*, h_1)$$

$$\times \times \sum_{|P_i| = |N_i|, |Q_i| = |M_i|} \prod_{k=1}^{3} G_{h_k}^{N_k} P_k G_{h_k}^{M_k} Q_k \rho(L_{-Q_k} h_k, L_{-R_k} \text{id}, L_{-P_k} h_k).$$

Here the level sum takes the form of a series expansion in $w$ and $z/w$. For numerical evaluation, it is far more efficient to reorganize the sum as an expansion in $q_1 \equiv e^{i(\tau - v)}$ and $q_2 \equiv e^{iv}$ instead, where $\tau$ and $v$ are given by (5.1.1) and (5.2.22). As is evident from the pillow frame, evaluating at $z = \frac{1}{2}$ and $w = \frac{1+i}{2} \pi$, the
effective expansion parameters are \(|q_1| = |q_2| = e^{-\pi/2}\), with unit radius of convergence. Explicitly, we have

\[
\frac{z}{w} = 4q_2 - 8q_2^2 + 8q_1q_2 + 12q_2^3 - 32q_1q_2^2 + 4q_1^2q_2 - 16q_2^4 + 64q_1q_2^3 - 48q_1^2q_2^2 + \ldots,
\]

and \(w\) is given by the same series expansion with \(q_1\) and \(q_2\) exchanged.

For example, the conformal block with a single stress-energy tensor inserted in the first sheet, up to total level 2 in \(q_1\) and \(q_2\), is given by

\[
\mathcal{F}(h_1, h_2, h_3; R_1 = \{2\}, R_2 = \emptyset, R_3 = \emptyset; q_1, q_2)
= \left(\frac{16q_1q_2}{27}\right)^{h_1 + h_2 + h_3} \left(\frac{16q_1q_2}{q_1^2}\right)^{\frac{1}{\sqrt{3}}} \left(\frac{h_1}{16} + \frac{1}{16} \left(18h_1q_1 - i\sqrt{3}(h_2 - h_3)(q_1 - q_2)\right)\right)
+ \frac{1}{216h_1h_2h_3} \left[ h_1h_2h_3q_1^2 \left(8c + 413h_1 - 4(h_2 + h_3) - 48i\sqrt{3}(h_2 - h_3)\right) \right.
+ 4q_2q_1 \left(3(c + 9)h_3h_2 + h_2^3 + h_3^3 + h_1^4(h_2 + h_3) - 2h_1^2(h_2^2 + h_3^2) \right.
+ h_1h_2h_3(h_2 - h_3) \left(h_2 - h_3 + 12i\sqrt{3}\right) + h_2h_3(h_2 - h_3)^2 \right] + h_1h_2h_3q_2^2(8c + 35h_1 - 4h_2 - 4h_3) \right]
+ \ldots
\]

(5.4.3)

If we symmetrize (5.4.1) with respect to \(R_1, R_2, R_3\), we recover the conformal block of the \(\mathbb{Z}_3\)-invariant Renyi surface considered in the previous section, differentiated with respect to \(z\), up to a conformal anomaly factor. In particular, summing over insertions of a single stress-energy tensor on one of the three sheets, we find

\[
\mathcal{F}(h_1, h_2, h_3; R_1 = \{2\}, R_2 = \emptyset, R_3 = \emptyset; q_1, q_2) + (2\text{ cyclic permutations on } R_1, R_2, R_3)
= C(q_1, q_2)\partial_q \mathcal{F}(h_1, h_2, h_3; q)|_{q=q_1q_2} + cB(q_1, q_2)\mathcal{F}(h_1, h_2, h_3; q = q_1q_2),
\]

(5.4.4)

where the first term on the RHS is due to deformation of the modulus \(z\) or \(q\) and the second term is due to the conformal anomaly (from a Weyl transformation that flattens out the pillow geometry after the insertion of the stress-energy tensor). The functions \(C\) and \(B\) are independent of \(h_i\) and \(c\); they admit series expansions.
in \(q_1\) and \(q_2\) of the form

\[
C(q_1, q_2) = \frac{q_2}{16q_1} + \frac{q_2}{2} + \frac{q_2}{8} \left(15q_1 + 8q_2 + \frac{2q_2^2}{q_1}\right) + \frac{1}{2}q_2 \left(9q_1^2 + 16q_1q_2 + 3q_2^2\right) + \ldots,
\]

\[
B(q_1, q_2) = \frac{1}{72q_1^2} + \frac{1}{9q_1} + \frac{19q_1^2 + 4q_1q_2 + 5q_2^2}{36q_1^2} + \frac{17q_1^2 + 8q_1q_2 + 11q_2^2}{9q_1} + \ldots.
\]

(5.4.5)

A complete set of genus two modular crossing equations can now be written as

\[
(-)^{j_3}(\{R_j\} + \{\tilde{R}_j\}) \sum_{(h, \tilde{h})} C_{h_1, h_2, h_3; h_1, \tilde{h}_2, \tilde{h}_3} \mathcal{F}(h_1, h_2, h_3; R_1, R_2, R_3; w|z) \mathcal{F}(\tilde{h}_1, \tilde{h}_2, \tilde{h}_3; \tilde{R}_1, \tilde{R}_2, \tilde{R}_3; \tilde{w}|\tilde{z})
\]

\[
= \sum_{(h, \tilde{h})} C_{h_1, h_2, h_3; h_1, \tilde{h}_2, \tilde{h}_3} \mathcal{F}(h_1, h_2, h_3; R_1, R_2, R_3; 1 - w|1 - z) \mathcal{F}(\tilde{h}_1, \tilde{h}_2, \tilde{h}_3; \tilde{R}_1, \tilde{R}_2, \tilde{R}_3; 1 - \tilde{w}|1 - \tilde{z}).
\]

(5.4.6)

If we take into account all possible choices of integer partitions \(R_j\) and \(\tilde{R}_j\), it suffices to evaluate this equation at the crossing-invariant point \(z = \tilde{z} = \frac{1}{2}\), with the choice \(w = \frac{1+i}{2}\), \(\tilde{w} = \frac{1-i}{2}\). The consequence of (5.4.6) in constraining structure constants in unitary CFTs is currently under investigation.

### 5.5 Discussion

The main results of this chapter are the formulation of genus two modular crossing equations in an explicitly computable manner, by working on the Renyi surface as well as expanding around it. As an application, we found compact critical surfaces that bound domains \(D \subset \mathbb{R}^3_{\geq 0}\) such that structure constants \(C_{ijk}\) involving a triple of primaries whose dimensions \((\Delta_i, \Delta_j, \Delta_k)\) or twists \((\tau_i, \tau_j, \tau_k)\) are outside of \(D\) are bounded by those that lie within \(D\). The existence of the compact critical surface is a nontrivial consequence of genus two modular invariance that does not follow easily from the analysis of individual OPEs: roughly speaking, the crossing equation for the sphere 4-point function bounds light-light-heavy structure constants in terms of light-light-light ones, but the genus two modular crossing equation also bounds light-heavy-heavy and heavy-heavy-heavy structure constants in terms of light-light-light ones.

In deriving the critical surface, we have used merely a tiny part of the genus two crossing equation, namely
the first order $z$ and $\bar{z}$ derivatives of the Renyi surface crossing equation evaluated at the crossing invariant point $z = \bar{z} = \frac{1}{2}$. Clearly, stronger results for the critical surfaces (that bound smaller domains) should be obtained by taking into account higher order $z$ and $\bar{z}$ derivatives of the crossing equation. This is rather tricky to implement numerically through semidefinite programming, simply due to the fact the genus two conformal block decomposition involves 3 continuously varying scaling dimensions and 3 spins. To implement the crossing equation through \[79\], for instance, one may attempt to vary the sum of the 3 scaling dimensions, and sample over their differences as well as truncating on the spins, but such a sampling would involve a huge set of conformal blocks that is hard to handle numerically. At the moment this appears to be the main technical obstacle in optimizing the genus two modular bootstrap bounds.\(^5\)

Many more constraints on the structure constants $C_{ijk}$ can in principle be obtained by consideration of higher order derivatives of the genus two crossing equation. For instance, combining first and third order derivatives, analogously to \[14,54\], one can deduce the existence of structure constants $C_{ijk}$ with say the dimensions $(\Delta_i, \Delta_j, \Delta_k)$ lying within a small domain (typically, such a domain is strictly larger than one that is bounded by a critical surface). The genus two modular invariance potentially has the power to constrain CFTs with approximately conserved currents (i.e. primaries with very small twist): if such a current operator propagates through one of the three handles of the genus two surface, modular invariance should constrain the pairs of operators propagating through the other two handles according to representations of an approximate current algebra or $W$-algebra. Typically, when OPE bounds or (genus one) modular spectral bounds are close to being saturated \[101\], one finds that there are necessarily low twist operators in the spectrum. For instance, this strategy may be used to severely constrain (and possibly rule out) unitary compact CFTs with central charge $c$ slightly bigger than 1.

There is another genus two conformal block channel (the “dumbbell channel”) that we have not discussed so far, namely one in which the genus two surface is built by plumbing together a pair of 1-holed tori. The conformal block decomposition of the genus two partition function in this channel involves the torus 1-point

\(^5\)A potentially more efficient numerical approach would be based on sum-of-squares optimization, as is explained to us by D. Simmons-Duffin.
functions, or the structure constants $C_{ijj}$ where a pair of primaries are identified. The modular covariance of the torus 1-point function cannot be used by itself to constrain $C_{ijj}$ in a unitary CFT, since $C_{ijj}$ does not have any positivity property in general. In the dumbbell channel decomposition of the genus two partition function, the structure constants appear in the combination $C_{ijj}C_{ikk}$, allowing for the implementation of semidefinite programming. In our present approach via expansion around the Renyi surface, it appears rather difficult to perform the conformal block decomposition in the dumbbell channel explicitly. How to incorporate this channel in the genus two modular bootstrap is a question left for future work.

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6.1 Introduction and summary

Recent years have seen enormous progress in understanding generic conformal field theories (CFTs), largely through conformal bootstrap methods. Some of the most powerful results exist at large $N$ or in an expansion in large spin, where analytic methods reveal features of the operator product expansion (OPE) that are universal to all CFTs. In two dimensions, the enhancement of spacetime symmetry to the infinite-dimensional Virasoro algebra would seem to aid efforts to analytically explore the space of CFTs and, via holography, the
properties of AdS\(_3\) quantum gravity. While this has proven true in certain kinematic and parametric limits, Virasoro symmetry has been frustratingly difficult to harness for theories deep in the irrational regime at finite central charge, without any small parameters.

In this chapter, we will combine some of the maxims of recent analytic bootstrap studies in higher dimensions with the power of Virasoro symmetry to uncover universal properties of irrational CFTs at finite central charge, and their implications for AdS\(_3\) quantum gravity. These results represent the complete, exact summation of the stress tensor contributions to certain OPE data. To do this, we will leverage the power of an underexploited tool, the Virasoro fusion kernel.

### 6.1.1 Motivation by inversion

A key conceptual and technical tool in the modern conformal bootstrap is the Lorentzian inversion formula for four-point functions [33] (see also [34,35]). This is a transform which acts on a four-point correlation function to extract the spectral data of intermediate states, providing an inverse to the conformal block expansion. The Lorentzian inversion formula (as opposed to a Euclidean inversion formula [145]) makes manifest the analyticity in spin of CFT OPE data: in particular, it shows that CFT operators live in analytic families – “Regge trajectories” – which asymptote at large spin to the double-twist (or higher multi-twist) operators whose existence was first discovered by the lightcone bootstrap [26,27,66] (see [25,28,32,146,147] for subsequent developments prior to [33]). In this way, the Lorentzian inversion formula goes beyond the large spin expansion, and implies a rigid structure of the OPE in any CFT.

Inversion may be carried out block-by-block, inverting T-channel conformal blocks to find the dual spectrum and OPE coefficients in the S-channel. These data are encoded in the poles and residues of the 6\(j\) symbol for the conformal group \(SO(d + 1,1)\), also known as the crossing kernel, as it decomposes blocks in one channel in the cross-channel [148]. Inverting the contribution of the unit operator gives the OPE data of double-twist operators of Mean Field Theory (MFT), schematically of the form \(\mathcal{O}_i \square^{n} \partial^{l} \mathcal{O}_j\) with traces subtracted. In many cases of interest – most notably, at large \(N\) or in the lightcone limit – the exchange of
the unit operator parametrically dominates the correlation function in a particular channel. Including other T-channel operators gives corrections and additional contributions to this. Analysis in the lightcone limit implies that the spectrum of any $d > 2$ (unitary, compact) CFT approaches that of MFT at large spin, with anomalous dimensions suppressed by inverse powers of spin $\ell^{-2h_t}$, where $h_t$ is the twist ($2h_t = \Delta_t - \ell_t$) of some T-channel operator. These are the double-twist Regge trajectories present in every CFT in $d > 2$, dual to towers of two-particle states in AdS, which are well separated, and hence non-interacting, at large spin.

What about two dimensions? From the above perspective, it is well-appreciated that two-dimensional CFTs are exceptional. Unitarity does not impose a gap in twist above the vacuum, and in particular, every operator in the Virasoro vacuum module – namely, the stress tensor and its composites – has zero twist. Since these operators all contribute to the large spin expansion at leading order, the analysis of higher dimensions is not valid. So the question remains, in a two-dimensional CFT at finite central charge, what do the double-twist Regge trajectories look like? Phrasing the same question in dual quantum gravitational language: in AdS$_3$ quantum gravity coupled to matter, what is the spectrum of two-particle states? Because the gravitational potential does not fall off at large distance in three dimensions, the interactions do not become weak, even at large spin. By solving this problem, one might also hope to infer lessons from the two-dimensional case for summing stress tensor effects on Regge trajectories in higher dimensions. In $d > 2$, stress tensor dynamics are not universal, as the $TT$ OPE can contain arbitrary primaries consistent with the symmetries. Even when there is Einstein gravity in the IR thanks to a large higher spin gap in the CFT [149], summing stress tensor dynamics to access Planckian processes in gravity is out of reach.

It is not possible to solve this problem by global inversion of a Virasoro block because no simple expression is known for the block. Moreover, the Lorentzian inversion formula does not incorporate Virasoro symmetry, as its output is the OPE data for the primaries under the finite-dimensional global conformal algebra (quasiprimaries), rather than under the Virasoro algebra.\footnote{Here and throughout we take “primary” to mean Virasoro primary, and “quasiprimary” to mean $\mathfrak{sl}(2)$ primary.} While some perturbative results exist in which the stress tensor is treated as a quasiprimary [148, 150–155], what we are after is the full incorporation of
the two-dimensional stress tensor dynamics, which are completely determined by the Virasoro algebra, into
the above picture.

Notwithstanding the important qualitative differences between two and higher dimensions, we will achieve
this by adopting the same “inversion” strategy of branching T-channel conformal blocks into S-channel data.
Remarkably, formulas for a Virasoro \(6j\) symbol, the object that we refer to as the fusion kernel (also called
the crossing kernel), has been known for some time \([105, 156, 157]\). We will also apply the fusion kernel to
various other problems in AdS\(_3\)/CFT\(_2\), demonstrating its power and versatility.

### 6.1.2 The Virasoro fusion kernel

For this section, and the majority of the chapter, we will focus on four-point functions of two pairs of
operators \(O_1, O_2\),

\[
\langle O_1(0)O_2(z, \bar{z})O_2(1)O_1(\infty) \rangle = \sum_s C_{12s}^2 F_S(\alpha_s)\bar{F}_S(\bar{\alpha}_s) = \sum_t C_{11t}C_{22t} F_T(\alpha_t)\bar{F}_T(\bar{\alpha}_t),
\]

where we have written the conformal block expansion in the S-channel, involving the \(O_1O_2\) OPE, and the
T-channel, involving the OPE of identical pairs of operators. In this expression, \(F(\alpha)\) denotes a holomorphic
Virasoro block in the indicated channel, with intermediate holomorphic conformal weights labelled by \(\alpha\), in
a parameterization that we will introduce presently. Dependence on the dimensions of external operators is
suppressed, along with kinematic dependence on locations of operators. In a diagonal basis \(\langle O_iO_j \rangle \propto \delta_{ij}\)
(possible given our assumption that the theory is compact), the identity operator will appear in the T-channel.

As we will soon see, it is natural to express the central charge \(c\) and conformal weight \(h\) in terms of
parameters \(b\) and \(\alpha\), respectively, as

\[
c = 1 + 6Q^2, \quad Q = b + b^{-1}, \quad h(\alpha) = \alpha(Q - \alpha),
\]
along with antiholomorphic counterpart $\bar{\alpha}$ related in the same way to $\bar{h} = h + \ell$, where $\ell$ is the spin. We call $\alpha$ the “momentum,” in analogy with the terminology of the Coulomb gas or linear dilaton theory, or Liouville theory, for which $\alpha$ is related to a target-space momentum (perhaps most familiar from vertex operators of the free boson at $c = 1$). For irrational unitary theories with $c > 1$ and $h, \bar{h} \geq 0$ for all operators, highest-weight representations of the Virasoro algebra fall into two qualitatively different ranges, depending on whether $h$ lies above or below the threshold \((\frac{Q}{2})^2 = \frac{c-1}{24}\). For $h < \frac{c-1}{24}$, we can choose $0 < \alpha < \frac{Q}{2}$, but for $h > \frac{c-1}{24}$, we have complex $\alpha \in \frac{Q}{2} + i\mathbb{R}$; we will denote these the ‘discrete’ and ‘continuum’ ranges, respectively, for reasons that will become clear imminently.

The fusion kernel, which we denote by $F_{\alpha,s,\alpha_t}$, decomposes a T-channel Virasoro block in terms of S-channel blocks:

$$F_T(\alpha_t) = \int_C \frac{d\alpha_s}{2i} F_{\alpha,s,\alpha_t} F_S(\alpha_s)$$

(6.1.3)

We will use an explicit closed-form expression for the kernel $F_{\alpha,s,\alpha_t}$ due to Ponsot and Teschner [156, 157], presented in the next section. Apart from early work [105] that used the kernel to show that Liouville theory solves the bootstrap equation through the DOZZ formula [56, 57], its utility for analytic conformal bootstrap has only recently been appreciated [68, 103, 158–161].

**Support of the fusion kernel**

Without giving the explicit formula for the kernel itself, let us briefly summarize an important property for our purposes, namely its support, i.e. the set of S-channel representations that appear in the decomposition of a T-channel block.

If the external operators are sufficiently heavy that $\text{Re}(\alpha_1 + \alpha_2) > \frac{Q}{2}$, then the contour $C$ in (6.1.3) can be chosen to run along the vertical line $\alpha_s = \frac{Q}{2} + i\mathbb{R}$, meaning that the T-channel block is supported on
S-channel blocks in the continuum range of $\alpha_s$:

$$\mathcal{F}_T(\alpha_t) = \int_0^{\infty} dP \mathcal{F}_{\alpha_s \alpha_t} \mathcal{F}_S(\alpha_s = \frac{Q^2}{2} + iP) \quad (6.1.4)$$

However, $\mathcal{F}_{\alpha_s \alpha_t}$ is a meromorphic function of $\alpha_s$, and has poles at

$$\alpha_m := \alpha_1 + \alpha_2 + mb \quad (6.1.5)$$

that, for sufficiently light external operators, may cross the contour $\alpha_s = \frac{Q^2}{2} + iP$. In the case of unitary external operator weights, when $\alpha_1 + \alpha_2 < \frac{Q^2}{2}$ ($\alpha_i$ necessarily real, in the ‘discrete’ range), the integral acquires additional contributions given by the residues of these poles:

$$\mathcal{F}_T(\alpha_t) = \sum_m -2\pi \text{Res}_{\alpha_s = \alpha_m} \{\mathcal{F}_{\alpha_s \alpha_t} \mathcal{F}_S(\alpha_s)\} + \int_0^{\infty} dP \mathcal{F}_{\alpha_s \alpha_t} \mathcal{F}_S(\alpha_s = \frac{Q^2}{2} + iP) \quad (6.1.6)$$

The sum runs over nonnegative integers $m$ such that the location of the corresponding pole satisfies $\alpha_m < \frac{Q^2}{2}$.\footnote{If $c \leq 25$, for which $b$ is a pure phase, only the $m = 0$ term can be present; this ensures the reality of various results to follow. For $c > 25$ we have chosen the convention $0 < b < 1$.}

The distinct ways in which light and heavy operators appear in this decomposition is why we have dubbed these ranges of dimensions ‘discrete’ and ‘continuous’ respectively.

For $\alpha_t = 0$, corresponding to the exchange of the identity multiplet, $\mathcal{F}_{\alpha_s1}$ has simple poles, so the blocks $\mathcal{F}_S(\alpha_m)$ appear with coefficient given by the residue of the kernel. In other cases ($\alpha_t \neq 0$), they are instead double poles, so the residue of the kernel times the block includes a term proportional to the derivative of the block with respect to $\alpha_s$, evaluated on the pole.

### 6.1.3 Summary of physical results

The main technical work of analysing the fusion kernel using the integral formula is in section 6.2. This includes its analytic structure and residues of poles contributing to (6.1.6): a closed-form, non-integral
expression for T-channel vacuum exchange, given in (6.2.16); and various pertinent limits of the kernel. This also leads straightforwardly to a derivation of the cross-channel behavior of Virasoro blocks. Subsequent sections will use the resulting formulas for various physical applications to CFT\textsubscript{2} and AdS\textsubscript{3} quantum gravity, the main results of which we now summarise.

**Quantum Regge trajectories and stress tensor corrections to MFT**

In every compact CFT, the vacuum Verma module contributes to the T-channel expansion of the four-point function under consideration. Inverting the vacuum Virasoro block by taking $\alpha = 0$, the kernel $F_{\alpha,\tilde{\alpha}}$ (times its antiholomorphic counterpart) is the corresponding “OPE spectral density” of Virasoro primaries in the S-channel, henceforth referred to simply as the spectral density. This spectral density is the finite central charge deformation of MFT double-twist data which takes into account contributions from all Virasoro descendants of the identity. This motivates the term “Virasoro Mean Field Theory” (VMFT) to refer to the OPE data resulting from inversion of the Virasoro vacuum block (including both left- and right-moving halves). Note that, unlike the case for MFT, this does not correspond to a sensible correlation function; in particular the spectrum of VMFT is continuous and contains non-integral spins, so VMFT refers only to a formal, though universal, set of OPE data. The results summarised here are discussed in more detail in section 6.3.1

Because the vacuum block factorizes, we can describe the spectrum in terms of chiral operators, with the understanding that the full two-dimensional spectrum is obtained from products of holomorphic and antiholomorphic operators. This chiral spectrum is the support of the fusion kernel described above. This departs qualitatively from the infinite tower of evenly spaced Regge trajectories in MFT. There is a finite discrete part of the spectrum, coming from the poles in (6.1.6), and a continuous spectrum above $h = \frac{c - 1}{24}$. The discrete contributions appear at

$$\alpha_m = \alpha_1 + \alpha_2 + mb, \quad \text{for } m = 0, 1, \ldots, \left\lfloor b^{-1} \left( \frac{c}{2} - \alpha_1 - \alpha_2 \right) \right\rfloor.$$  

(6.1.7)
Figure 6.1: The twist spectrum of Virasoro double-twist operators arising from inversion of the holomorphic vacuum block, i.e. the spectrum of VMFT. The twists of the external operators are fixed while \( c \) varies. For a given central charge \( c \), there is a discrete number of operators, shown as the solid lines, below the continuum at \( h > \frac{c}{24} - 1 \), shaded in grey. The exact formula for the twists is given in (6.1.8). Each line, when combined with an anti-holomorphic component, forms a quantum Regge trajectory that is exactly linear in spin. At large \( c \), one recovers the integer-spaced operators of MFT.

For the discrete trajectories, inclusion of the Virasoro descendants therefore has the remarkably simple effect that the MFT additivity of dimensions \( h \) is replaced by additivity of momenta \( \alpha \). The dependence of this spectrum on central charge is pictured in figure 6.1.

Writing the result (6.1.7) in terms of the twist\(^3\), we find

\[
h_m = h_1 + h_2 + m + \delta h_m, \quad \text{where} \quad \delta h_m = -2(\alpha_1 + mb)(\alpha_2 + mb) + m(1 + m)b^2 < 0. \quad (6.1.8)
\]

The departure \( \delta h_m \) from the corresponding MFT dimension is the exact anomalous twist due to summation of all multi-traces built from the stress tensor. Note that if we take \( c \to \infty \) (\( b \to 0 \)) with \( h_{1,2} \) fixed, \( \delta h_m \) goes to zero, and the maximal value of \( m \) goes to infinity, recovering MFT. In addition, \( \delta h_m \) is always negative, so the twist is reduced when compared to those of MFT, seen by the monotonicity in figure 6.1.

Including both chiral halves, the discrete set of twists (6.1.8) form what we call a quantum Regge trajec-

\(^3\)In what follows we will use the term ‘twist’ (referring to \( \tau = \Delta - \ell = 2 \min(h, \bar{h}) \)) almost interchangeably with the holomorphic conformal weights \( h \). Likewise, we refer to \( \delta h \) as the ‘anomalous twist’.
teries,” so named to reflect the finite-$c$ summation and their duality to two-particle states in AdS$_3$ quantum gravity with $G_N$ finite in AdS units, as we discuss momentarily. These trajectories are exactly linear in spin. We emphasize the distinction with the analogous problem in $d > 2$, where two-particle dynamics at Planckian energies remain inaccessible.

The data of VMFT is modified by inclusion of other, non vacuum operators in the T-channel. The spectrum is shifted by “anomalous twists”, coming from the double poles in the fusion kernel. These anomalous twists from individual operators, as well as anomalous OPE coefficients, can be formally computed from the coefficients of these poles, with the result given in (6.3.8). This is much the same as inversion of global conformal blocks for non-unit operators, which give corrections to MFT.

**Large spin universality**

The inversion of the T-channel vacuum block, giving the spectrum of VMFT discussed above, is rather formal and not immediately obvious how it relates to physical data of actual CFTs, but in some limits it is in fact universal. Just as MFT governs the large spin OPE of $d > 2$ CFTs, there is a VMFT universality governing the spectrum of $d = 2$ CFTs:

**In a unitary compact CFT$_2$ with $c > 1$ and a positive lower bound on the twists of non-vacuum primaries, the OPE spectral density approaches that of VMFT at large spin.**

This is made more precise in section 6.3.2. In particular, it means that there are Regge trajectories of double-twist operators with twist approaching the discrete values in (6.1.8) at large spin, and that Regge trajectories with twist approaching a different value of $h < \frac{c-1}{24}$ have parametrically smaller OPE coefficients. The continuum for $h > \frac{c-1}{24}$ requires an infinite number of Regge trajectories with $h$ accumulating to $\frac{c-1}{24}$ at large spin, such that any given interval of twists above this value contains an infinite number of operators with spectral density approaching that of VMFT.

We also compute the rate at which the asymptotic twist is approached by including an additional T-
channel operator with momenta \((\alpha_t, \bar{\alpha}_t)\), given by the formula (6.3.15). At large spin \(\ell = \bar{h} - h\), this scales as\(^4\)

\[
\delta h_m^{(\alpha_t, \bar{\alpha}_t)} \approx \exp \left( -2\pi \bar{\alpha}_t \sqrt{\ell} \right).
\]

(6.1.9)

This decays much faster than the power-law suppression \(\ell^{-2h_t}\) one obtains in \(d > 2\), or by ignoring the stress tensor in \(d = 2\) (for example, in computing correlation functions of a QFT in a fixed AdS\(_3\) background). We also compute \(\delta h_m^{(\alpha_t, \bar{\alpha}_t)}\) to leading order in a semiclassical regime of “Planckian spins”, taking \(\ell \to \infty\) and \(c \to \infty\) with \(\ell / c\) and external operator dimensions fixed. The result, given in (6.3.25), has the simple dependence

\[
\delta h_m^{(\alpha_t, \bar{\alpha}_t)} \propto \left( \frac{c}{6\pi} \cosh(\pi \bar{p}_s) \right)^{-2h_t}, \quad \text{with} \quad \bar{p}_s \sim \frac{1}{2} \sqrt{\frac{24\ell}{c} - 1}.
\]

(6.1.10)

This interpolates between the exponential behavior (6.1.9) at \(\ell \gg c\) and the power \(\ell^{-2h_t}\) at \(\ell \ll c\), where the latter is the original result of the lightcone bootstrap [26, 27]. We give this a gravitational interpretation in section 6.5.2.

For T-channel exchanges between identical operators \(O_1 = O_2\) obeying \(\alpha_t < 2\alpha_1\), upon including the coefficient in (6.1.9), the anomalous twist of the leading Regge trajectory \(\delta h_0^{(\alpha_t, \bar{\alpha}_t)}\) is negative. This can be thought of as a Virasoro version of Nachtmann’s theorem [26, 162]: the leading large spin correction to the twist of the first Regge trajectory is negative, so this trajectory is convex.

Because the arguments above ultimately rely on the form of the holomorphic fusion kernel, they imply similar results about OPE asymptotics in limits of large conformal weight, rather than large spin. As an explicit application of this, in section 6.3.4 we give the asymptotic average density of “light-light-heavy” OPE coefficients in any unitary compact 2D CFT with \(c > 1\) and a dimension gap above the vacuum.

\(^4\)Throughout this chapter, we use the notation \(x \sim y\) to denote that \(x / y \to 1\) in the limit of interest, while we use \(x \approx y\) to specify the leading scaling, denoting in particular that \(|\log(x) - \log(y)|\) does not grow in the relevant limit.
Cross-channel Virasoro blocks

While we derive these results purely from the crossing kernel, without direct reference to the four-point function itself, we can relate this to methods of the original derivations of the lightcone bootstrap [26, 27], which solved crossing in the lightcone limit $z \to 1$ of the correlation function. See appendix C.5 for a review of the ‘old-fashioned’ lightcone bootstrap. This requires analysis of the ‘cross-channel’ limit of Virasoro blocks, which we provide in section 6.2.3 for both S- and T-channel blocks.

We note that the lightcone bootstrap is one example of a large class of arguments determining asymptotic OPE data from dominance of the vacuum in a kinematic limit, and crossing symmetry or modular invariance [129, 163–165]. The same strategy of using an appropriate fusion or modular kernel could be applied to streamline these arguments. For this purpose, we note that there also exists a modular kernel for primary one-point functions on the torus [156, 157, 166–168], and along with the fusion kernel this is sufficient to encode any other example [38]. A particularly simple example is Cardy’s formula for the asymptotic density of primary states [65, 169].

Global limit and $1/c$ corrections

In section 6.4 we consider the global limit, in which we fix all conformal weights while taking $c \to \infty$ ($b \to 0$). This decouples the Virasoro descendants, and the Virasoro algebra contracts to its global $sl(2)$ subalgebra. In this limit, the number of discrete Regge trajectories is of order $c$, and the $m$th VMFT trajectory becomes the $m$th MFT trajectory, with twists accumulating to $h_m = h_1 + h_2 + m$ with $m \in \mathbb{Z}_{\geq 0}$. This provides a novel method for the computation of MFT OPE data, including at subleading twist $m \neq 0$. At subleading orders in $1/c$, one can systematically extract the large $c$ expansion of double-twist OPE data due to non-unit operators, by performing the small $b$ expansion near the $m$th pole. As a check, we recover the OPE data of MFT by taking the global limit of the residues of the vacuum kernel, as well as some known results for double-twist anomalous dimensions due to non-unit operators. These matches follow from a correspondence
between the double-twist poles of the Virasoro fusion kernel and a ‘holomorphic half’ of the global $6j$ symbol computed in [148], the precise statement of which can be found in (6.4.5)–(6.4.6).

The data obtained in the $1/c$ expansion is useful for the study of correlation functions of light operators in theories which admit weakly coupled AdS$_3$ duals, especially if the CFT has a sparse light spectrum, whereupon the number of exchanges is parametrically bounded. Expansion of the VMFT OPE data to higher orders in $1/c$ may be performed as desired, for example to extract the anomalous dimension due to multi-graviton states.

**AdS$_3$ interpretation**

The previous results all have an interpretation in AdS$_3$ quantum gravity, which we discuss in section 6.5.1. The discrete quantum Regge trajectories are dual to two-particle bound states, while the large spin continuum at $h = \frac{c-1}{24}$ corresponds to spinning black holes. Heuristically, this dichotomy reflects the threshold for black hole formation at $h = \frac{c-1}{24}$, including the quantum shift $c \to c - 1$ not visible in the classical regime [170]. The finiteness of the tower of discrete trajectories may be viewed as a kind of quantum gravitational exclusion principle, reflecting the onset of black hole formation. The negativity $\delta h_m < 0$ of the VMFT anomalous twist given in (6.1.8) translates into a negative binding energy in AdS$_3$, thus reflecting the attractive nature of gravity at the quantum level. The corrections $\delta h_m^{(\alpha_i, \bar{\alpha}_i)}$ to the VMFT twists $h_m$ are dual to contributions to the two-particle binding energy due to bulk matter. In higher dimensions, the decay $\ell^{-2h}$ of anomalous dimensions reflects the exponential falloff of the $h$-mediated interaction between two particles, with orbit separated by a distance of order $\log \ell$. The $d = 2$ result (6.1.9) actually has precisely the same interpretation, with the apparent discrepancy coming from a gravitational screening effect explained in section 6.5.2. The particles orbiting in AdS come with a dressing of boundary gravitons, and at very large spin this dressing carries most of the energy and angular momentum; this can be removed by a change of conformal frame, corresponding to removing descendants to form a Virasoro primary state.

The addition of momentum $\alpha$ in VMFT has a simple geometric realisation in the semiclassical regime in
In the semiclassical regime, the additivity rule (6.1.7) for discrete momenta of large spin double-twist operators translates into additivity of conical defect angles in AdS$_3$, where the deficit angle is $\Delta \phi = \frac{4\pi}{Q} \alpha$. The leading-twist operator, whose dual conical defect is depicted as the sum of two constituent defects, has momentum $\alpha_1 + \alpha_2$. We have suppressed the $\frac{4\pi}{Q}$ for clarity.

which $O_1$ and $O_2$ are dual to bulk particles that backreact to create conical defect geometries. The deficit angle created by a particle, proportional to its classical mass, is $\Delta \phi_i = \frac{4\pi}{Q} \alpha_i$. The spectrum of discrete twists has the elegant bulk interpretation that the bulk masses, and hence deficit angles, simply add according to (6.1.7). This is depicted in figure 6.2.

Heavy-light semiclassical limit

In section 6.6.1, we study the fusion kernel in the large $c$ heavy-light limit of [66,67], in which the dimension of one pair of external operators scales with $c$, while the other pair have dimensions fixed. This leads to two new derivations of heavy-light Virasoro blocks, one for the vacuum block (recalled in (6.6.7)) for heavy operators above the black hole threshold at $\alpha = \frac{Q}{2}$, and another for non-vacuum heavy-light blocks when the heavy operator is below the black hole threshold. The result is obtained by actually performing the sums (6.1.6) over S-channel blocks, using knowledge of $F_{\alpha,\alpha'}$, and a simplification of the S-channel blocks in this limit. The derivation also gives a new understanding of the emergence of the “forbidden singularities” of the heavy-light blocks, and relates their resolution to the analytic structure of the S-channel blocks.

Late time

One result that follows easily from our analysis is an analytic derivation of the behaviour of the heavy-light Virasoro vacuum block at late Lorentzian times. This block was found numerically to decay exponentially at early times, followed by a $t^{-3/2}$ decay at late times [171,172]. The idea here is as follows: upon using the fusion kernel to write a T-channel block as an integral over S-channel blocks, the time evolution gives a simple phase in the S-channel, and a saddle point computation then yields the $t^{-3/2}$ behaviour. This follows
from the existence of a double zero of the fusion kernel $F_{\alpha,1}$ at $\alpha_s = \frac{Q}{2}$, which is the relevant saddle point for this computation. The more complete expression for the behaviour of the block is in (6.6.20). In fact, this power-law falloff is universal for Virasoro blocks with $\text{Re}(\alpha_1 + \alpha_2) > \frac{Q}{2}$; neither large $Q$ nor the semiclassical limit is required. In the heavy-light case, we also derive the crossover time between the exponential and power law.

Looking forward, we anticipate an array of uses for the results herein, and for the fusion kernel more generally. The validity of our results at finite $c$ provides strong motivation to understand these features of AdS$_3$ quantum gravity directly in the bulk. To begin, it would be nice to perform bulk calculations that match the $1/c$ expansion of the anomalous twist (6.1.8) for light external operators (e.g. using Wilson lines [151, 173–176] or proto-fields [177]). Likewise, the effective theory in [178] may also be capable of reproducing our results. One would also like to reproduce the all-orders result from the bulk: namely, to find a gravitational calculation at large $c$ and finite $h_i/c$, possibly with $m$ of order $c$, that reproduces the complete $\delta h_m$. As we pursue an improved understanding of irrational two-dimensional CFT, perhaps the overarching question suggested by our results is the following: with Virasoro symmetry under more control, can we build a better bootstrap in two dimensions? These developments sharpen the need for an explicit realization of an irrational compact CFT$_2$ to serve as a laboratory for the application of these ideas — a “3D Ising model for 2D.”

### 6.2 Analyzing the fusion kernel

We begin by considering the most general four-point function of primary operators $O_{1,2,3,4}$ in a two-dimensional CFT,

$$G(z, \bar{z}) = \langle O_1(0,0)O_2(z, \bar{z})O_3(1,1)O_4(\infty, \infty) \rangle, \quad (6.2.1)$$

---

5Not to be confused with the 2D Ising model. We point out that the IR fixed point of the coupled Potts model [73] is a potential candidate for such a theory, but its low-lying spectrum has not been conclusively pinned down.
with conformal cross ratios \((z, \bar{z})\). By appropriately taking the OPE between pairs of operators, or, equivalently, inserting complete sets of states in radial quantization, we can write this as a sum over products of three-point coefficients \(C_{ijk}\) with intermediate primary operators, times the Virasoro conformal blocks. There are several choices of which operators to pair in this process, which must give the same result; equating the expansion in the S- and T-channels results in the crossing equation

\[
G(z, \bar{z}) = \sum_s C_{12s}C_{s34} F_{34}^{21}(\alpha_s | z) F_{34}^{21}(\bar{\alpha}_s | \bar{z}) = \sum_t C_{14t}C_{t32} F_{14}^{23}(\alpha_t | 1 - z) \bar{F}_{14}^{23}(\bar{\alpha}_t | 1 - \bar{z}) ,
\]

which imposes strong constraints on the OPE data of the CFT. The block \(F_{34}^{21}(\alpha | z)\) is the contribution to \(G(z, \bar{z})\) of holomorphic Virasoro descendants of a primary of weight \(h(\alpha)\) in the OPE taken between operators \((12), (34)\), normalized such that

\[
F_{34}^{21}(\alpha | z) \sim z^{-h_1-h_2+h(\alpha)} \quad \text{as} \quad z \to 0. \tag{6.2.3}
\]

In this chapter, we will not study this crossing equation directly, but rewrite it as a direct relation between T- and S-channel OPE data.

We will henceforth employ the parameterization in terms of the “background charge” \(Q\) or “Liouville coupling” \(b\) (defined by \(c = 1 + 6Q^2\), \(Q = b + b^{-1}\)), and “momentum” \(\alpha\) (defined by \(h = \alpha(Q - \alpha)\)) introduced in (6.1.2).\(^6\) Note that \(h(\alpha) = h(Q - \alpha)\) and \(Q(b) = Q(b^{-1})\). We will fix the choice of \(b\) by taking \(0 < b < 1\) if \(c > 25\), and by taking \(b\) to lie on the unit circle in the first quadrant if \(1 < c < 25\). For \(c > 1\), unitarity of Virasoro highest-weight representations requires that \(h \geq 0\). Our parameterization naturally

\[\text{There are many reasons why this parameterization is most natural. Minimal model values of } c \text{ correspond to negative rational values of } b^2, \text{ and for any } c, \text{ degenerate representations of the Virasoro algebra occur at } \alpha_{r,s} = -\frac{1}{2}(rb + sb^{-1}), \text{ with } r, s \in \mathbb{Z}_{\geq 0}.\]
splits up this range of dimensions into two distinct pieces:

\[
0 < h < \frac{c - 1}{24} \iff 0 < \alpha < \frac{Q}{2} \quad \text{(discrete)} \quad (6.2.4)
\]

\[
h \geq \frac{c - 1}{24} \iff \alpha \in \frac{Q}{2} + i\mathbb{R} \quad \text{(continuous)} \quad (6.2.5)
\]

We call these the “discrete” and “continuous” ranges because, as we will see, the analytic structure of the crossing kernel implies that T-channel blocks have support on S-channel blocks for a discrete set of dimensions in (6.2.4), but over the whole continuum (6.2.5). This also echoes terminology used in AdS$_3$/CFT$_2$. In Liouville theory, these correspond to dimensions of non-normalizable and normalizable vertex operators respectively.

The defining identity for the fusion kernel is

\[
\mathcal{F}_{14}^{23}(\alpha_t | 1 - z) = \int_C \frac{d\alpha_s}{2i} \mathcal{F}_{\alpha_s,\alpha_t}^{\alpha_1,\alpha_2,\alpha_3,\alpha_4} \mathcal{F}_{34}^{21}(\alpha_s | z), \quad (6.2.6)
\]

for which the contour of integration $C$ will be discussed shortly. It is not obvious that such an object should even exist, but nonetheless a closed form expression for $\mathcal{F}$ has been written down by Ponsot and Teschner [105, 156, 157], which we present (without derivation) in a moment. (See also [179, 180], and [181] for a compact summary.) (6.2.6) is formally defined when all operators have momenta in the continuum. As is the case in studies of the global crossing kernel, we will analytically continue away from the continuum to infer OPE data about four-point functions of arbitrary highest weight representations.

Perhaps a better interpretation is to view the fusion kernel not in terms of blocks, but as a map between S-channel and T-channel OPE data. To do this, define the OPE spectral densities\(^7\) in S- and T-channels,

\[
c_{34}^{21}(\alpha_s, \bar{\alpha}_s) := \sum_p C_{12p} C_{p34} \delta(\alpha_s - \alpha_p) \delta(\bar{\alpha}_s - \bar{\alpha}_p), \quad c_{14}^{23}(\alpha_t, \bar{\alpha}_t) := \sum_p C_{14p} C_{p32} \delta(\alpha_t - \alpha_p) \delta(\bar{\alpha}_t - \bar{\alpha}_p), \quad (6.2.7)
\]

\(^7\)The $\delta$-functions supported at imaginary $\alpha$ may be unfamiliar, but make sense in a space of distributions dual to holomorphic test functions (the only requirement being that the blocks are contained in this space).
which allows us to write the correlation function $G$ as the spectral density integrated against the conformal blocks, in either channel. Replacing the T-channel block using the fusion kernel, then stripping away the S-channel blocks, leads to the following reexpression of the crossing equation:

$$c_{34}^{a_1 a_2}(\alpha_s, \bar{\alpha_s}) = \int_C \frac{d\alpha_t}{2i} \int_C \frac{d\bar{\alpha}_t}{2i} F_{\alpha_s \alpha_t} F_{\bar{\alpha}_s \bar{\alpha}_t} c_{14}^{a_3 a_4}(\alpha_t, \bar{\alpha}_t).$$  \hspace{1cm} (6.2.8)

This can be thought of as an expression for the S-channel spectrum as a linear operator acting on the T-channel spectrum $c_s = F \cdot c_t$ (where we suppress the holomorphic-antiholomorphic factorization)$^8$. In particular, including just a single block in the T-channel, as we will be doing for most of the chapter, $F_{\alpha_s \alpha_t} F_{\bar{\alpha}_s \bar{\alpha}_t}$ simply gives the corresponding S-channel spectral density.

### 6.2.1 Integral form of the kernel

The closed-form expression for the fusion kernel requires the introduction of some special functions, particularly $\Gamma_b(x)$, which we define in appendix C.1, accompanied by a discussion of some of its properties. The salient information is that $\Gamma_b(x)$ is a meromorphic function with no zeros, and poles at $x = -mb - nb^{-1}$ for $m, n \in \mathbb{Z}_{\geq 0}$. In this sense, $\Gamma_b$ can be thought of as analogous to the usual $\Gamma$-function, but adapted to the lattice of points made by nonnegative integer linear combinations of $b, b^{-1}$ rather than just the integers.

Using this function, along with

$$S_b(x) := \frac{\Gamma_b(x)}{\Gamma_b(Q - x)},$$  \hspace{1cm} (6.2.9)

the kernel can be written as

$$F_{\alpha_s \alpha_t} = P(\alpha_t; \alpha_s, \alpha_t) P(\alpha_t; Q - \alpha_s, Q - \alpha_t) \int_C \frac{ds}{i} \prod_{k=1}^4 S_b(s + U_k) S_b(s + V_k),$$  \hspace{1cm} (6.2.10)

\hspace{1cm} $^8$The notation $F_{\alpha_s \alpha_t}$ is chosen as it gives the matrix elements of this linear operator. The blocks (at fixed $z$) can be thought of as elements of the dual space, which explains why the indices in (6.2.6) are transposed relative to what might have been expected. For minimal model values of $c$ and $h$, $F$ becomes an ordinary finite-dimensional matrix.
where

\[
P(\alpha_i; \alpha_s, \alpha_t) = \frac{\Gamma_b(\alpha_s + \alpha_3 - \alpha_4) \Gamma_b(\alpha_s + Q - \alpha_4 - \alpha_3) \Gamma_b(\alpha_s + \alpha_2 - \alpha_3) \Gamma_b(\alpha_s + \alpha_1 + \alpha_2 - Q)}{\Gamma_b(\alpha_t + \alpha_1 - \alpha_4) \Gamma_b(\alpha_t + Q - \alpha_1 - \alpha_4) \Gamma_b(\alpha_t + \alpha_2 - \alpha_3) \Gamma_b(\alpha_t + \alpha_2 + \alpha_3 - Q)} \Gamma_b(2\alpha_t - Q) \Gamma_b(2\alpha_s - Q)
\]

(6.2.11)

and we define \( U_k, V_k \) as follows:

\[
\begin{align*}
U_1 &= \alpha_1 - \alpha_4 & V_1 &= Q - \alpha_s + \alpha_2 - \alpha_4 \\
U_2 &= Q - \alpha_1 - \alpha_4 & V_2 &= \alpha_s + \alpha_2 - \alpha_4 \\
U_3 &= \alpha_2 + \alpha_3 - Q & V_3 &= \alpha_t \\
U_4 &= \alpha_2 - \alpha_3 & V_4 &= Q - \alpha_t
\end{align*}
\]

(6.2.12)

The contour of integration \( C' \) runs from \(-i\infty\) to \( i\infty\), passing to the right of the towers of poles at \( s = -U_i - mb - nb^{-1} \) and to the left of the poles at \( s = Q - V_j + mb + nb^{-1} \), for \( m, n \in \mathbb{Z}_{\geq 0} \).

The analytic structure of the fusion kernel as a function of \( \alpha_s \) will play an important role in our analysis. This is depicted in figures 6.3 and 6.4. For generic external dimensions and operators in the T-channel, the kernel has simple poles in \( \alpha_s \) organized into eight semi-infinite lines extending to the right, and another eight semi-infinite lines extending to the left:

\[
\text{poles at } \alpha_s = \alpha_0 + mb + nb^{-1}, \quad Q - \alpha_0 - mb - nb^{-1}, \text{ for } m, n \in \mathbb{Z}_{\geq 0}
\]

(6.2.13)

with \( \alpha_0 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \) (+ six permutations under reflections \( \alpha_i \rightarrow Q - \alpha_i \)).

Schematically, half of these poles come directly from the special functions in the prefactor, with the other half coming from singularities of the integral. The latter occur when poles of the integrand coincide and pinch the contour of integration between them, namely when \( V_j - U_i - Q = mb + nb^{-1} \). In the important case of pairwise identical external operators, the eight semi-infinite lines of poles in each direction degenerate to four such lines of \emph{double} poles extending in either direction. A notable exception occurs when the internal dimension becomes degenerate \( (\alpha_t, Q - \alpha_t = -\frac{1}{2}(mb + nb^{-1})) \) for \( m, n \in \mathbb{Z}_{\geq 0} \), which requires external
dimensions consistent with the fusion rules [13]. For us, this will be important when $\alpha_t = 0$ with pairwise identical external operators, relevant for vacuum exchange, in which case the kernel has only simple poles.

Figure 6.3: For sufficiently heavy external pairwise identical operators ($\text{Re}(\alpha_1 + \alpha_2) > \frac{Q}{2}$), the fusion kernel has four semi-infinite lines of poles extending in either direction. In the case of vacuum exchange in the T-channel, these poles are simple, otherwise they are double poles. Here we show an example of this in the case that all external operators have weights in the continuum, with $0 < b < 1$. The dashed red curve denotes the contour of integration in the decomposition of the T-channel Virasoro block into S-channel blocks, while the blue and green crosses denote the poles of the fusion kernel.

For external operators with weights in the continuous range ($\alpha \in \frac{Q}{2} + i\mathbb{R}$), the towers of poles extending to the right and left all begin on the line $\text{Re}(\alpha_s) = Q$ and the imaginary axis respectively. In this case the contour $C$ can be taken to run along the line $\alpha_s = \frac{Q}{2} + i\mathbb{R}$ between them, so that only continuum S-channel blocks appear in the decomposition of the T-channel block, illustrated in figure 6.3. This demonstrates (6.1.4).

If some of the external operators have weights in the discrete range $h_i < \frac{c-1}{24}$, certain lines of poles move inward towards the integration contour, and for $\alpha_1 + \alpha_2 < \frac{Q}{2}$ or $\alpha_3 + \alpha_4 < \frac{Q}{2}$, some of these poles cross the contour of integration. To maintain analyticity in the parameters, we must deform the contour to include portions surrounding the relevant poles, contributing a residue. This leads to a finite, discrete sum of S-channel blocks appearing in the decomposition of the T-channel block in addition to the continuum starting.
Figure 6.4: Here we plot the poles of the fusion kernel as a function of \( \alpha_s \) in the case of pairwise identical external operators and \( \text{Re}(\alpha_1 + \alpha_2) < \frac{Q}{2} \), with \( 0 < b < 1 \). In this case, the poles at \( \alpha_s = \alpha_1 + \alpha_2 + mb < \frac{Q}{2} \) (and their reflections) cross the contour of integration and give discrete residue contributions to the T-channel Virasoro block. Note that despite the fact that the external operators have weights lying in the discrete range rather than the continuum we have given \( \alpha_i \) small imaginary parts for the purpose of presentation.

at \( h_s = \frac{c-1}{24} \), as in (6.1.6). For unitary values of the weights, the only momenta that can contribute to this finite sum are

\[
\alpha_s = \alpha_1 + \alpha_2 + mb, \quad \alpha_3 + \alpha_4 + m'b, \quad \text{with} \quad m, m' \in \mathbb{Z}_{\geq 0} \quad (6.2.14)
\]

when these values are less than \( \frac{Q}{2} \). The poles at reflected values of \( \alpha_s \) give identical contributions, and the other lines of poles can never cross the contour for unitary external operators. The contour in this scenario is illustrated in figure 6.4.

Henceforth, we specialize to the case of pairwise identical operators \( \alpha_4 = \alpha_1, \alpha_3 = \alpha_2 \), and for notational brevity omit the labels for the external operators, using the condensed notation

\[
\mathcal{F}_S(\alpha_s) := \mathcal{F}_{21}^{21}(\alpha_s|z), \quad \mathcal{F}_T(\alpha_t) := \mathcal{F}_{11}^{22}(\alpha_t|1-z), \quad \mathbb{F}_{\alpha_s,\alpha_t} := \mathbb{F}_{\alpha_s\alpha_t}^{\alpha_2\alpha_1}. \quad (6.2.15)
\]

A similar phenomenon occurs in the analytic continuation of four-point functions in Liouville theory, whose conformal block decomposition takes the form of the DOZZ structure constants [56,57] integrated against the Virasoro conformal blocks, when poles cross the contour of integration over intermediate dimensions [100,110,182]. However, in that case the poles correspond to scalar operator dimensions, and not chiral dimensions separately.
We also record some results for the cross-channel kernel \( F_{\alpha_1 \alpha_2} := F_{\alpha_1 \alpha_2} \left[ \frac{\alpha_2 \alpha_2}{\alpha_1 \alpha_1} \right] \), which is the inverse operator to \( F_{\alpha_1 \alpha_2} \), in appendix C.2.4.

### 6.2.2 Computing properties of the kernel

In the remainder of the section, we outline how various properties and limits of the fusion kernel are computed, and give the main results, with additional formulas given in appendix C.2. Readers interested only in the physical application of these results may skip to section 6.3.

**Vacuum kernel**

With external operators identical in pairs, the fusion rules allow \( \alpha_t = 0 \), which we will use extensively for exchange of the identity operator, and the corresponding fusion kernel, denoted \( F_{\alpha_1 \alpha_2} \), greatly simplifies. The integral in (6.2.10) becomes singular as \( \alpha_t \to 0 \), since the contour must pass between a double pole at \( s = 0 \) and a single pole at \( s = \alpha_t \); the singular piece can be evaluated simply by the residue of the latter. This gives a simple pole in \( \alpha_t \) (an additional zero at \( s = -\alpha_t \) reduces the strength of the singularity), which is cancelled by a zero in the prefactor. This leaves the following simplified expression for the vacuum fusion kernel:

\[
F_{\alpha_1 \alpha_2} = \frac{\Gamma_b(2Q)}{\Gamma_b(Q)^2 \Gamma_b(2\alpha_1 \alpha_2 - Q) \Gamma_b(2\alpha_1) \Gamma_b(2\alpha_2) \Gamma_b(2Q - 2\alpha_2) \Gamma_b(2Q - 2\alpha_2)} \times (7 \text{ terms with } \alpha \leftrightarrow Q - \alpha).
\]

(6.2.16)

The seven terms not written comprise all possible combinations of reflections of the three momenta \( \alpha_1, \alpha_2, s \).

This simple expression makes the polar structure manifest.\(^{10}\)

\(^{10}\)This idea was also used in [159] to compute a piece of the vacuum kernel.
Singularities

The poles in the fusion kernel encode the coefficients of the discrete sum of blocks appearing in (6.1.6); in this section we compute these coefficients.

For the identity kernel, we can use the expression (6.2.16) to evaluate the residue of the simple poles at \( \alpha_s = \alpha_1 + \alpha_2 + mb \), using identities written in appendix C.1 to simplify the result. The result for general \( m \) is written in (C.2.1), and for \( m = 0 \) gives

\[
\operatorname{Res}_{\alpha_s = \alpha_1 + \alpha_2} F_{\alpha_s,1} = -\frac{1}{2\pi} \frac{\Gamma_b(2Q)\Gamma_b(Q - 2\alpha_1)\Gamma_b(Q - 2\alpha_2)\Gamma_b(2Q - 2\alpha_1 - 2\alpha_2)}{\Gamma_b(2Q - 2\alpha_1)\Gamma_b(2Q - 2\alpha_2)\Gamma_b(Q - 2\alpha_1 - 2\alpha_2)} \left( \frac{\Gamma(1 + b^2 - 2b\alpha_1)\Gamma(1 + b^2 - 2b\alpha_2)}{\Gamma(1 + b^2)\Gamma(1 + b^2 - 2b(\alpha_1 + \alpha_2))} \right) \times (b \leftrightarrow b^{-1}).
\]  

(6.2.17)

In the case of pairwise identical external operators with a non-vacuum primary propagating in the T-channel, the fusion kernel has double poles in \( \alpha_s \). To compute the behaviour at the poles \( \alpha_s = \alpha_1 + \alpha_2 + mb + nb^{-1} \), it is easiest to make use of the kernel’s reflection symmetry as follows. As written in (6.2.10), the contour integral contributes only a simple pole at \( \alpha_s = \alpha_1 + \alpha_2 + mb + nb^{-1} \); combining with the simple pole from the prefactor, we would need to be able to compute the finite part of the contour integral in order to determine the residue. However, the contour integral also contributes a double pole at \( \alpha_s = Q + \alpha_1 - \alpha_2 + mb + nb^{-1} \). Since the kernel is invariant under reflections \( \alpha_i \rightarrow Q - \alpha_i \), we can simply send \( \alpha_2 \rightarrow Q - \alpha_2 \) so that the prefactor is regular at \( \alpha_s = \alpha_1 + \alpha_2 + mb + nb^{-1} \) and isolate the singularities of the contour integral, much like the computation of the vacuum kernel. In this way, we find

\[
F_{\alpha_s,\alpha_t} = \frac{\Gamma_b(\alpha_1 + \alpha_2 - \alpha_s)^2\Gamma_b(Q - \alpha_1 + \alpha_2 - \alpha_s)^2\Gamma_b(\alpha_1 - \alpha_2 - \alpha_s)^2\Gamma_b(2Q - \alpha_1 - \alpha_2 - \alpha_s)^2}{\Gamma_b(2Q - 2\alpha_s)\Gamma_b(Q - 2\alpha_s)}
\]

\[
\times \frac{\Gamma_b(2\alpha_t)\Gamma_b(\alpha_t + \alpha_s - \alpha_1 - \alpha_2)}{\Gamma_b(\alpha_t)^2\Gamma_b(\alpha_t + \alpha_1 - \alpha_2 - \alpha_s)\Gamma_b(\alpha_t + \alpha_2 - \alpha_1)\Gamma_b(\alpha_t + Q - 2\alpha_1)\Gamma_b(\alpha_t + Q - 2\alpha_2)} \times (\alpha_t \rightarrow Q - \alpha_t)
\]

\[
+ \text{(regular at } \alpha_s = \alpha_1 + \alpha_2\text{)}.
\]  

(6.2.18)

This allows the coefficients of the double and simple poles to be read off upon expanding the divergent factor.

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in the numerator $\Gamma_b(\alpha_1 + \alpha_2 - \alpha_s)^2$. A formula that captures the singularities of the non-vacuum kernel at the subleading poles is given in equation (C.2.2).

The coefficient of the non-vacuum kernel at the leading ($m = 0$) double pole is given by

$$d\text{Res}_{\alpha_s = \alpha_1 + \alpha_2} F_{\alpha_s, \alpha_t} = \frac{\Gamma_b(Q)^2 \Gamma_b(Q - 2\alpha_1)^2 \Gamma_b(Q - 2\alpha_2)^2 \Gamma_b(2Q - 2\alpha_1 - 2\alpha_2)}{(2\pi)^2 \Gamma_b(Q - 2\alpha_1 - 2\alpha_2)} \times \left( \frac{\Gamma_b(2\alpha_t)}{\Gamma_b(\alpha_t)^2 \Gamma_b(Q - 2\alpha_1 + \alpha_t) \Gamma_b(Q - 2\alpha_2 + \alpha_t)} \times (\alpha_t \to Q - \alpha_t) \right).$$

(6.2.19)

Here we have introduced the facetious notation $d\text{Res}$ to denote the coefficient of a double pole. The result for general $m$ is given in (C.2.3). The residue at the leading pole may be written in terms of a $b$-deformed digamma function, the logarithmic derivative of $\Gamma_b$

$$\psi_b(z) := \frac{\Gamma_b'(z)}{\Gamma_b(z)} \sim -\frac{1}{z} - \gamma_b + \cdots \text{ as } z \to 0,$$

(6.2.20)

where the $z \to 0$ limit defines a $b$-deformed version of the Euler-Mascheroni constant, $\gamma_b$. We find

$$\frac{\text{Res}_{\alpha_s = \alpha_1 + \alpha_2} F_{\alpha_s, \alpha_t}}{d\text{Res}_{\alpha_s = \alpha_1 + \alpha_2} F_{\alpha_s, \alpha_t}} = 2(\psi_b(\alpha_t) + \psi_b(Q - \alpha_t) + \gamma_b + \psi_b(Q - 2\alpha_1 - 2\alpha_2) - \psi_b(Q - 2\alpha_1) - \psi_b(Q - 2\alpha_2)).$$

(6.2.21)

The result for the residue of the kernel at the subleading poles is given in (C.2.4).

**Large dimension limit**

We here present the asymptotics of the fusion kernel in the limit of large $S$-channel internal weight, with details of the calculations appearing in appendix C.3. These results are important for applications to large spin in particular, as explained in section 6.3.2.

Our main tool in this analysis is the following asymptotic formula for the special function $\Gamma_b(x)$ at large
argument $|x| \to \infty$, derived in appendix C.1.1:

$$\log \Gamma_b(x) = -\frac{1}{2} x^2 \log x + \frac{3}{4} x^2 + \frac{Q}{2} x \log x - \frac{Q^2}{2} x - \frac{Q^2+1}{12} \log x + \log \Gamma_0(b) + O(x^{-1}), \quad (6.2.22)$$

Using this along with the expression (6.2.16) for the vacuum block, the asymptotic form of the vacuum kernel at large internal weight $h_s = \frac{Q^2}{4} + P^2$ is

$$F_{\alpha,1} \sim 2^{-4p^2} e^{-\frac{1}{2} x^2 \log x + \frac{3}{4} x^2} x \log x - \frac{1}{2} x \log x + \frac{Q^2}{2} x - \frac{Q^2+1}{12} \log x + \log \Gamma_0(b) + O(x^{-1}). \quad (6.2.23)$$

The leading exponential piece exactly cancels a similar factor in the large dimension asymptotics of the blocks ($F \approx (16q)^h$, for $q = e^{\pi i \tau}$, $\tau = \frac{i}{2} \pi \frac{1}{2}$), as necessary for correct convergence properties. The formula has a direct interpretation as the asymptotics of 'light-light-heavy' OPE coefficients, discussed in section 6.3.4.

The computation of the asymptotic form of the non-vacuum kernel is similar, but requires a careful evaluation of the integral appearing in (6.2.10) in this limit, discussed in appendix C.3. For $h_t < \frac{c-1}{24}$ (i.e. $\alpha_t < \frac{Q}{2}$), we have

$$F_{\alpha,1} \sim e^{-2\pi \rho} \left[ \frac{\Gamma_b(2Q - 2\alpha_1) \Gamma_b(2Q - 2\alpha_2) \Gamma_b(2Q - 2\alpha_1 - \alpha_t) \Gamma_b(2Q - 2\alpha_2 - \alpha_t)}{\Gamma_b(2Q) \Gamma_b(Q - \alpha_t) \Gamma_b(2Q - 2\alpha_1) \Gamma_b(2Q - 2\alpha_2)} \right] \quad (6.2.24)$$

For heavier operators $h_t \geq \frac{c-1}{24}$ an additional contribution, given by $\alpha_t \to Q - \alpha_t$, must be added.

**Large central charge limits**

In sections 6.4 and 6.5, we will study properties of the fusion kernel in global and semiclassical limits, with large central charge. This requires expansions of the special function $\Gamma_b$ in limits as $b \to 0$. In appendix C.1.2, we derive an all-orders asymptotic series (C.1.29) for $\log \Gamma_b$, with argument scaling as $b^{-1}$. This improves
on the semiclassical form of \( \Upsilon_b(x) := \frac{1}{\Gamma_b(x) \Gamma_b(Q - x)} \) derived in [183]. The leading result reads

\[
\log \Gamma_b(b^{-1} x + \frac{b}{2}) = \frac{1}{2b^2} \left( \frac{1}{2} - x \right)^2 \log b + \frac{2x - 1}{4b^2} \log(2\pi) - \frac{1}{b^2} \int_x^\infty dt \log \Gamma(t) + O(b^2). \tag{6.2.25}
\]

To determine the behaviour of \( \log \Gamma_b \) in the global limit, in which the argument scales like \( b \), one can use this formula with \( x = 1 + O(b^2) \) in conjunction with the recursion relation (C.1.2).

**Virasoro double-twist exchanges**

The kernel also simplifies in the case that the T-channel momentum matches the “Virasoro double twists” of the relevant external operators, namely \( \alpha_t = 2\alpha_1 + mb \) or \( \alpha_t = 2\alpha_2 + mb \). Here, we take the leading double twist \( \alpha_t = 2\alpha_2 \) (with corresponding results for \( m = 0 \) given in appendix C.2.2). The simplification is much the same as for the vacuum kernel given in (6.2.16): the prefactor \( P(\alpha_i; Q - \alpha_s, Q - \alpha_t) \) vanishes at \( \alpha_t = 2\alpha_2 \), but the integral contributes a singularity, so we need only evaluate a pole of the integrand, giving

\[
F_{\alpha_s, 2\alpha_2} = \frac{\Gamma_b(4\alpha_2) \Gamma_b(\alpha_2 + \alpha_1 - \alpha_s)^2 \Gamma_b(\alpha_2 + Q - \alpha_1 - \alpha_s)^2 \Gamma_b(\alpha_1 + \alpha_2 + \alpha_s - Q)^2 \Gamma_b(\alpha_2 - \alpha_1 + \alpha_s)^2}{\Gamma_b(2\alpha_2)^4 \Gamma_b(Q) \Gamma_b(2\alpha_2 + Q - 2\alpha_1) \Gamma_b(2\alpha_2 + 2\alpha_1 - Q) \Gamma_b(2\alpha_s - Q) \Gamma_b(Q - 2\alpha_s)}. \tag{6.2.26}
\]

Note that the kernel still has double poles at the locations of the S-channel Virasoro double-twists, and thus the T-channel Virasoro double-twists contribute to anomalous momenta and double-twist OPE data in the S-channel. This is in contrast with a property of \( d > 2 \) Lorentzian inversion, for which T-channel double-twists give vanishing contribution to the (analytic part of) S-channel OPE data. However, the kernel decays much more rapidly at large dimension, with the leading term in (6.2.24) replaced by \( e^{-2\pi(Q-2\alpha_2)\sqrt{\pi}} \).

**6.2.3  Cross-channel limit of Virasoro blocks**

Our results immediately allow us to read off the behaviour of the T-channel block in the cross-channel limit \( z \to 0 \), because the fusion kernel expresses it in terms of S-channel blocks, which have simple power law
behaviour \(z^{-b_1-h_2+h_z}\). The leading order behaviour is determined by the smallest weight \(h_s\) on which the T-channel block has support in the S-channel, which depends on whether the external dimensions are light enough for the discrete dimensions to be present. Here, we compute these limits for operators identical in pairs, for both S- and T-channel blocks. For the T-channel blocks, there is a qualitative difference between vacuum exchange and other operators if the external operators are sufficiently light that \(\alpha_s = \alpha_1 + \alpha_2\) dominates, since non-vacuum exchange gives a double pole, and hence an additional logarithm.

For \(\text{Re}(\alpha_1 + \alpha_2) < \frac{Q}{2}\), the leading behaviour is controlled by \(\alpha_s = \alpha_1 + \alpha_2\)

\[
\begin{aligned}
\text{Vacuum : } & \mathcal{F}^{22}_{11}(0|1-z) \quad \underset{\alpha_s \rightarrow 0}{\sim} - \left( \text{Res}_{\alpha_s=\alpha_1+\alpha_2} \right) 2\pi f_{\alpha_s} z^{-2\alpha_1 \alpha_2} \\
\text{Non-vacuum : } & \mathcal{F}^{22}_{11}(\alpha_1|1-z) \quad \underset{\alpha_s \rightarrow 0}{\sim} - \left( \text{dRes}_{\alpha_s=\alpha_1+\alpha_2} \right) \left( \log (Q - 2(\alpha_1 + \alpha_2)) \right) z^{-2\alpha_1 \alpha_2} \log z,
\end{aligned}
\]  

(6.2.27) (6.2.28)

where \(\text{Res}\) and \(\text{dRes}\) are given in (6.2.17) and (6.2.19).

For \(\text{Re}(\alpha_1 + \alpha_2) > \frac{Q}{2}\), the leading behaviour is controlled by the bottom of the continuum, \(\alpha_s = \frac{Q}{2}\). By performing a saddle point analysis of the continuum integral over \(\alpha_s\) as \(z \rightarrow 0\), we find

\[
\mathcal{F}^{22}_{11}(\alpha_1|1-z) \quad \underset{\alpha_s \rightarrow 0}{\sim} - \frac{\sqrt{\pi}}{8} \partial^2_{\alpha_s} F_{\alpha_s,\alpha_1} \bigg|_{\alpha_s=\frac{Q}{2}} \left( \log \frac{1}{z} \right)^{-\frac{3}{2}}.
\]  

(6.2.29)

In the case of vacuum exchange, this coefficient can be computed explicitly\(^{11}\)

\[
\partial^2_{\alpha_s} F_{\alpha_s,\alpha_1} \bigg|_{\alpha_s=\frac{Q}{2}} = -32\pi^2 \frac{\Gamma_b(2Q)\Gamma^2_b(\frac{3Q}{2} - \alpha_1 - \alpha_2)\Gamma_b(\frac{Q}{2} + \alpha_1 - \alpha_2)^2\Gamma_b(\frac{Q}{2} - \alpha_1 + \alpha_2)^2\Gamma_b(\alpha_1 + \alpha_2 - \frac{Q}{2})^2}{\Gamma_b(Q)\Gamma_b(2Q - 2\alpha_1)\Gamma_b(2\alpha_1)\Gamma_b(2Q - 2\alpha_2)\Gamma_b(2\alpha_2)}.
\]  

(6.2.30)

As a check, in C.2.3 we explicitly verify (6.2.29) with \(c = 25\) (\(b = 1\)) and \(h_1 = h_2 = 15/16\) (\(\alpha_i = \frac{Q}{2}\)), with arbitrary internal dimension, using the exactly known expression for the relevant conformal blocks [184].

\(^{11}\)The borderline case \(\alpha_1 + \alpha_2 = \frac{Q}{2}\) must be treated separately. We point out that the S-channel Virasoro block with \(\alpha_1 + \alpha_2 = \frac{Q}{2}\), \(\alpha_s = \frac{Q}{2}\) is equal to the T-channel block with \(\alpha_t = 2\alpha_2 = Q - 2\alpha_1\), given simply by a (chiral half of a) Coulomb gas correlation function \(z^{-2\alpha_1 \alpha_2}(1-z)^{-2h_z^2}\), and hence crossing-invariant on its own [110, 159]. Consistent with this, in the limit approaching \(\alpha_1 + \alpha_2 = \frac{Q}{2}\) the kernel (6.2.26) becomes a \(\delta\)-function at \(\alpha_s = \frac{Q}{2}\).
We can use similar methods to determine the cross-channel limit of the S-channel Virasoro blocks by making use of the decomposition

\[
\mathcal{F}^{21}_{21}(\alpha_s|z) = \int_C \frac{d\alpha_t}{2\pi} \tilde{F}_{\alpha_t,\alpha_s} \mathcal{F}^{22}_{11}(\alpha_t|1-z),
\]

(6.2.31)

where we have introduced \( \tilde{F}_{\alpha_t,\alpha_s} = F_{\alpha_t,\alpha_s}^{\alpha_2,\alpha_2/\alpha_1,\alpha_1} \). The analytic structure of \( \tilde{F}_{\alpha_t,\alpha_s} \) as a function of \( \alpha_t \) is slightly different than that of \( F_{\alpha_t,\alpha_s} \) as a function of \( \alpha_s \); in particular it only has simple poles at \( \alpha_t = 2\alpha_1 + mb + nb^{-1} \), \( 2\alpha_2 + mb + nb^{-1} \) and others obtained by reflections \( \alpha_t \to Q - \alpha_t \). There are also quadruple poles at \( \alpha_t = Q + mb + nb^{-1} \) and \( \alpha_t = -mb - nb^{-1} \). The cross-channel limit of these blocks is, in the case that at least one of \( \alpha_{1,2} < Q/4 \), controlled by the leading pole at \( \alpha_t = 2 \min(\alpha_1, \alpha_2) \)

\[
\mathcal{F}^{21}_{21}(\alpha_s|z) \overset{z \to 1}{\sim} - \left( \text{Res}_{\alpha_t=2\alpha_1} \frac{2\pi}{2\pi} \tilde{F}_{\alpha_t,\alpha_s} \right) (1-z)^{h(2\alpha_t)-2h_2}; \alpha_1 < Q/4, \alpha_2
\]

(6.2.32)

with the relevant residues recorded in C.2.4. If neither \( \alpha_1 \) nor \( \alpha_2 \) is less than \( Q/4 \), the leading behaviour in the cross-channel limit is again controlled by the bottom of the continuum

\[
\mathcal{F}^{21}_{21}(\alpha_s|z) \overset{z \to 1}{\sim} - \sqrt{16} \partial^{\alpha_2} \tilde{F}_{\alpha_t,\alpha_s} \bigg|_{\alpha_t=\frac{Q}{4}} (1-z)^{\frac{Q^2}{16}-2h_2} \left( \log \frac{1}{1-z} \right)^{-\frac{1}{2}}; \alpha_1, \alpha_2 > Q/4.
\]

(6.2.33)

The analytic structure of the kernel as a function of \( \alpha_s \) manifestly explains the recent numerical observations [185, 186] that Virasoro blocks have drastically different cross-channel asymptotics depending on whether the external operators are sufficiently heavy (for example, for identical operators, the observed threshold at large central charge was \( h \sim \frac{c}{32} \), corresponding to \( \alpha \sim \frac{Q}{4} \)).
6.3 Extracting CFT data

In this section we discuss some implications of the crossing kernel results from the point of view of “Virasoro Mean Field Theory” outlined in the introduction and corrections to it, and then apply this to give universal results for the spectrum at large spin.

6.3.1 Quantum Regge trajectories and a “Virasoro Mean Field Theory”

As discussed in the introduction, our results for the inversion of the Virasoro vacuum block lead us to coin the nickname Virasoro Mean Field Theory (VMFT) for the non-perturbative incorporation of the stress tensor into ordinary MFT double-trace data. In MFT, correlation functions are computed by Wick contractions of pairs of identical operators (so connected correlation functions are defined to vanish), which for \( G = \langle O_1 O_2 O_2 O_1 \rangle \) gives

\[
G_{\text{MFT}}(z, \bar{z}) = \frac{1}{((1-z)(1-\bar{z}))^{\Delta_2}}. \tag{6.3.1}
\]

Decomposing this in the S-channel, we find a spectrum of quasiprimaries comprising double-trace operators \([O_1 O_2]_{n,\ell} \) for each spin \( \ell \) and each \( n \in \mathbb{Z}_{\geq 0} \). In two dimensions, because the \( z \) and \( \bar{z} \) dependence of \( G_{\text{MFT}} \) factorizes, we can think of the double-traces as products of chiral operators \([O_1 O_2]_m \) for \( m \in \mathbb{Z}_{\geq 0} \), with equally spaced twists \( h_m = h_1 + h_2 + m \), and their antiholomorphic counterparts labelled by \( \bar{m} \):

\[
[O_1 O_2]_{n,\ell} = [O_1 O_2]_m [\bar{O}_1 \bar{O}_2]_{\bar{m}} \quad n = \min(m, \bar{m}), \quad \ell = |m - \bar{m}| \tag{6.3.2}
\]

For VMFT, we replace the exchange of the unit operator in the T-channel by the full Virasoro vacuum block:

\[
G_{\text{VMFT}}(z, \bar{z}) = F_{11}^{\Delta_2}(0|1-z)\bar{F}_{11}^{\Delta_2}(0|1-\bar{z}) \tag{6.3.3}
\]

\( ^{12} \text{In terms of individual operators ('scaling blocks'), rather than quasiprimaries, the MFT double trace data was derived in 1665 [187].} \)
Figure 6.5: The VMFT spectrum, combining holomorphic and anti-holomorphic sectors.

While this does not make sense as a correlation function of a sensible theory\textsuperscript{13} – in particular, it is not single-valued on the Euclidean plane – we can still discuss it at the level of the S-channel data required to reproduce $G_{\text{VMFT}}$ (on the first sheet). The factorization property remains, so for most of this subsection we will just discuss the holomorphic half of the OPE data, with the understanding that the full VMFT data is constructed from products of left- and right-movers, as in figure 6.5. The spectrum of VMFT is the support of the vacuum kernel $F_{\alpha,1}$. The resulting Virasoro primary spectrum contains a continuum starting at $\alpha = Q/2$ and, for operators with $\alpha_1 + \alpha_2 < Q/2$, the following discrete set:

$$\{\mathcal{O}_1\mathcal{O}_2\}_m, \text{ with } \alpha = \alpha_m := \alpha_1 + \alpha_2 + mb,$$

(6.3.4)

with $m \in \mathbb{Z}_{\geq 0}$ small enough that $\alpha_m < \frac{Q}{2}$. The latter are natural analogues of the usual MFT double-twists. Summarizing so far, there are three main properties of VMFT that differ from MFT:

- There is a continuum of double-twists for $h > \frac{c-1}{24}$.
- The discrete part of the spectrum is truncated, and is entirely absent when $\text{Re}(\alpha_1 + \alpha_2) > Q/2$. The spins are generically non-integral.

\textsuperscript{13}For a proposal on how this might be upgraded to a full correlation function, see [188].
Whereas MFT is additive in the twists $h_i$, VMFT is additive in the momenta $\alpha_i$.

The discrete set of holomorphic VMFT double-twist operators $\{O_1O_2\}_m$ acquire anomalous twists relative to the MFT operators $[O_1O_2]_m$. Writing $h_m(\alpha_1 + \alpha_2) = h_1 + h_2 + m + \delta h_m$, as in (6.1.8), the additivity of momenta implies that $\{O_1O_2\}_m$ has anomalous twist

$$\delta h_m = -2(\alpha_1 + mb)(\alpha_2 + mb) + m(m + 1)b^2. \quad (6.3.5)$$

We note that this is always negative. The three-point couplings between the (holomorphic) VMFT double-twist operators and the external operators, $C_{O_1O_2(O_1O_2)}_m$, are given by the residues of the vacuum kernel at the double-twist locations:

$$C_{O_1O_2(O_1O_2)}_m = -2\pi \text{Res}_{\alpha_s = \alpha_m} F_{\alpha_s 1} \quad (6.3.6)$$

The explicit expression for the residue of the vacuum kernel is given in (C.2.1).

**Perturbing VMFT**

Starting from MFT, one can perturb the theory by adding non-vacuum operators to the T-channel, leading to anomalous dimensions $\gamma_{n,\ell}$ of the MFT double-twist operators. From the $6j$ symbol, this arises because it acquires double poles at the double-twist locations (and their shadows) [148]. The situation for discrete trajectories in VMFT is entirely analogous. As seen in (6.2.18), $F_{\alpha_s \alpha_t}$ acquires double poles at $\alpha_s = \alpha_m$ for non-vacuum block inversion ($\alpha_t \neq 0$). This leads to a derivative of S-channel blocks with respect to momentum $\alpha_s$, gives an additional logarithm in the cross-channel behaviour of the T-channel block, and thus generates anomalous *momenta* for $\{O_1O_2\}_m$, i.e.

$$\alpha_s = \alpha_m + \delta \alpha_m. \quad (6.3.7)$$
If we formally consider the inversion of a single holomorphic non-vacuum block, we can define a holomorphic anomalous momentum as\(^{14}\)

\[
\delta \alpha_m = C_{11t} C_{22t} \text{Res}_{\alpha_s = \alpha_m} \left( \frac{\bar{f}_{\alpha_s \alpha_t}}{\bar{f}_{\alpha_s \bar{1}}} \right), \tag{6.3.8}
\]

since this is the coefficient of the double pole of \(\bar{f}_{\alpha_s \alpha_t}\) divided by the VMFT OPE coefficient, as follows from (6.3.6). (See section 3.3 of \([148]\) for the analogous method of computation of anomalous dimensions from the global 6j symbol.) The double pole does not, of course, change the momentum by a finite amount, but the derivative of the block with respect to \(\alpha_s\) is interpreted as the first term in a Taylor series, whose coefficient we identify as the anomalous momentum (higher terms must come from sums over infinitely many T-channel operators; see \([25, 28]\) for example). Note that, from (6.2.17) and (6.2.19), the leading twist coefficients (for \(m = 0, \alpha_s = \alpha_1 + \alpha_2 < \frac{Q}{2}\)) are sign-definite:

\[
\text{Res}_{\alpha_s = \alpha_0} \left( \frac{\bar{f}_{\alpha_s \alpha_t}}{\bar{f}_{\alpha_s \bar{1}}} \right) < 0 \tag{6.3.9}
\]

It follows that for identical operators, \(\delta \alpha_0 < 0\). Even if \(\mathcal{O}_1 \neq \mathcal{O}_2\), the anomalous momentum is negative for any T-channel operator which couples with the same sign to \(\mathcal{O}_1\) and \(\mathcal{O}_2\).

These results nicely generalize properties of global inversion and OPE data of MFT. In section 6.4, we show that the MFT results are recovered from the large \(c\) limit of VMFT for fixed operator dimensions \(h\).

**Comparison to \(d > 2\)**

In \(d > 2\) CFTs, the MFT Regge trajectories are the same as those in \(d = 2\): an infinite tower of flat trajectories with twists \(\tau = \Delta_1 + \Delta_2 + 2n\). Incorporating the stress tensor \(T_{\mu\nu}\), of twist \(d - 2\), produces anomalous dimensions \(\gamma_{n,\ell}\) which behave as \(\ell^{-d+2}\) at large spin \(\ell \gg n\). In the opposite regime of large twist,

\(^{14}\)Since all corrections to VMFT come from inverting non-vacuum Virasoro blocks which have both holomorphic and antiholomorphic parts, the full result for the anomalous momentum also depends on the anti-holomorphic kernel, \(\bar{f}_{\alpha_s \alpha_t}\). In the next subsection, we will be more explicit about putting left- and right-movers together to get complete results.
\[ \gamma_{n,\ell} \bigg|_T \approx \frac{n^{d-1}}{c_T} \quad \text{as} \quad n \gg \ell \gg 1 \]  

(6.3.10)

where \( c_T \) is the normalization of the stress-tensor two-point function. In gravitational variables, \( c_T \approx 1/G_N \approx M_{\text{pl}}^{d-1} \). Therefore, the perturbative expansion of \( \gamma_{n,\ell} \) breaks down at or below the Planck scale, \( n \lesssim M_{\text{pl}} \). Arguments from perturbative unitarity \([192]\) and causality \([193]\) place even stronger bounds on this breakdown. However, we do not know how to obtain quantitative understanding of what actually happens to the classical Regge trajectories at Planckian energies. This is due both to technical difficulty, and to non-universality of the \( TT \) OPE. The latter implies that the all-orders stress tensor contribution to the Regge trajectories depends on the details of the CFT: in particular, unlike in \( d = 2 \), the three-point coefficients \( \langle TOO \rangle \), where \( O \) is \( T \) itself or any multi-\( T \) composite, are sensitive to the rest of the CFT data \([193,194]\).

These comments underscore the value of the two-dimensional setting – non-trivial, yet computable – in giving insight into Planckian processes in AdS quantum gravity.

### 6.3.2 Large spin

The OPE data of VMFT has so far been a formal construction, but in this section we will see that it governs a universal sector in physical theories at large spin, applying to all unitary CFTs with \( \mathfrak{sl}(2) \) invariant ground state, and no conserved currents besides the vacuum Virasoro module.\(^{15}\) This means that for any two primary operators, there are associated towers of double-twist operators which asymptote to the VMFT twists (6.1.8) at large spin. Corrections to this come from including T-channel primary operators of positive twist, giving a systematic large spin perturbation theory.

Like analogous results in higher dimensions, this can be derived from solving crossing in the lightcone

\(^{15}\)More precisely, we require that non-vacuum Virasoro primaries have twist bounded away from zero, since it is a logical possibility to have an infinite tower of operators with \( \bar{h} \) accumulating to zero at large \( \bar{h} \). While we can’t rule this out, this seems unlikely to happen in theories of interest. For example, the CFT dual to \( \text{AdS}_3 \times S^3 \times T^4 \) at the symmetric orbifold point has infinitely many higher-spin currents, but when perturbing away from the orbifold point, the anomalous dimensions acquired by the currents seem to grow logarithmically with spin \([195]\). This suggests that away from the orbifold point, the theory has a finite twist gap above the superconformal vacuum descendants.
limit, as briefly reviewed in appendix C.5, but we will argue more directly from (6.2.8). The contribution to the S-channel spectral density from any given T-channel operator is simply the fusion kernel \( F_{\alpha_s, \alpha_t} \).

Now, we take the limit of large spin in the S-channel, at fixed twist:

\[
\alpha_s \text{ fixed, } \bar{\alpha}_s = \frac{Q}{2} + i \tilde{P} \text{ with } \tilde{P} \sim \sqrt{\ell_s} \to \infty
\] (6.3.11)

The relative importance of T-channel operators in this limit is encoded by (6.2.24):

\[
\frac{\bar{F}_{\bar{\alpha}_s, \alpha_t}}{\bar{F}_{\bar{\alpha}_s, 1}} \approx e^{-2\pi \bar{\alpha}_t \tilde{P}}
\] (6.3.12)

This shows that the contribution of operators with \( \text{Re}(\bar{\alpha}_t) > 0 \) is suppressed relative to the vacuum. This suggests that, in a theory with \( \bar{\alpha}_t \) bounded away from zero, the S-channel density is dominated by the inversion of the T-channel vacuum – that is, VMFT – at large spin. (Recall that negative real \( \bar{\alpha}_t \) is forbidden by unitarity.) We make a more careful argument in a moment, after discussing the consequences. We separately discuss large-spin discrete trajectories and the large-spin continuum, before making further comments.\(^\text{16}\)

**Discrete trajectories**

Consider fixing the twist \( \alpha_s \) in the discrete range as we take large spin, with \( \text{Re}(\alpha_1 + \alpha_2) < \frac{Q}{2} \). Let us first review what we found earlier: for each \( m \in \mathbb{Z}_{\geq 0} \) with \( \alpha_m = \alpha_1 + \alpha_2 + mb < \frac{Q}{2} \), there must be a Regge trajectory of operators with \( \alpha_s \to \alpha_m \) as \( \ell_s \to \infty \): these are the Virasoro double-twist families \( \{\mathcal{O}_1 \mathcal{O}_2\}_{m, \ell_s} \) of VMFT. The asymptotics of the corresponding OPE coefficients are determined by the vacuum fusion kernel:

\[
\rho_m^{(0)}(\tilde{P}) = -2\pi i \bar{F}_{\bar{\alpha}_s, 1} \text{ Res}_{\alpha_m = \alpha_s} \bar{F}_{\bar{\alpha}_s, 1}, \text{ with } \bar{\alpha}_s = \frac{Q}{2} + i \tilde{P} \text{ and } \ell_s \sim \tilde{P}^2 \to \infty.
\] (6.3.13)

\(^{16}\)These two large spin sectors correspond to VFMT operators in the discrete \( \times \) continuum and continuum \( \times \) continuum representations, respectively; the discrete \( \times \) discrete operators are bounded in spin due to the upper bound on \( m \), cf. (6.1.7), so they are not part of the large spin universality.
Here, $\rho_m$ is a spectral density of the $m$th Regge trajectory, in terms of $\bar{P}$, so it contributes to the correlation function as

$$G \supset \int_0^\infty d\bar{P} \rho_m^{(0)}(\bar{P}) \mathcal{F}_S(\alpha_m) \bar{F}_S\left(\frac{Q}{2} + i\bar{P}\right).$$  \tag{6.3.14}$$

The superscript denotes that this is the VMFT density. The explicit expression for the residue of the vacuum kernel is given in (6.2.17) for $m = 0$, and (C.2.1) for higher Regge trajectories, and the large internal weight limit of the kernel is in (6.2.23).

Additional T-channel operators give corrections to this, adding a spin-dependent anomalous momentum $\delta\alpha_m^{(\alpha_t, \bar{\alpha}_t)}$ to $\alpha_m$, as well as a correction $\delta\rho_m^{(\alpha_t, \bar{\alpha}_t)}$ to $\rho_m^{(0)}$. Expanding (6.3.14) to first order generates a derivative of the block proportional to $\delta\alpha_m^{(\alpha_t, \bar{\alpha}_t)}$, which can be matched to the coefficient of the double pole in the fusion kernel $\mathcal{F}_{\alpha_s \alpha_t}$ at $\alpha_s = \alpha_m$. The anomalous OPE density $\delta\rho_m^{(\alpha_t, \bar{\alpha}_t)}$ can be read off from the residue at the same point.\footnote{Note that $\delta\rho$ does not translate immediately into a change of spectral density in terms of spin, which is more directly related to anomalous OPE coefficients of individual operators. This picks up a Jacobian factor $2\bar{P} - \delta h'(\bar{P})$ from the anomalous twist, reflecting the fact that the Regge trajectories are no longer exactly linear.}

Altogether, (6.2.8) translates into the following corrections to OPE data:

$$\delta\alpha_m^{(\alpha_t, \bar{\alpha}_t)}(\bar{P}) = C_{11t} C_{22t} \bar{F}_{\bar{\alpha}_t \alpha_t} \alpha_s = \alpha_m \left. \frac{\mathcal{F}_{\alpha_s \alpha_t}}{\mathcal{F}_{\alpha_s t}} \right|_{\bar{\alpha}_s = 0 + i\bar{P}},$$  \tag{6.3.15}$$

$$\delta\rho_m^{(\alpha_t, \bar{\alpha}_t)}(\bar{P}) = -2\pi C_{11t} C_{22t} \bar{F}_{\bar{\alpha}_t \alpha_t} \alpha_s = \alpha_m \left. \mathcal{F}_{\alpha_s \alpha_t} \right|_{\bar{\alpha}_s = 0 + i\bar{P}}.$$

The residue in the first line equals the ratio of (C.2.1) and (C.2.3). Reading off the ratio of the anti-holomorphic kernels appearing in (6.3.15) from (6.2.24) (which also includes the coefficient, suppressed here), we derive the large spin decay of anomalous twist:

$$\delta\alpha_m^{(\alpha_t, \bar{\alpha}_t)} \approx e^{-2\pi i \sqrt{\mathcal{F}_s}} \quad \implies \quad \delta h_m^{(\alpha_t, \bar{\alpha}_t)} \approx e^{-2\pi i \sqrt{\mathcal{F}_s}}.$$  \tag{6.3.17}$$

If $O_1^*$ is the lowest twist operator appearing in the T-channel (necessarily in the discrete range, since at a minimum there are discrete double-twist families $\{O_1 O_1\}$ and/or $\{O_2 O_2\}$), then the leading corrections to
VMFT data at large spin decay as \( e^{-2\pi i \sqrt{\ell_s}} \).\(^{18}\)

**Large-spin continuum**

The continuous sector of VMFT contributes to the four-point function as

\[
G \supset \int_0^\infty dP \int_0^\infty d\bar{P} \rho^{(0)}_{\text{cont.}}(P, \bar{P}) \mathcal{F}_S(Q^2 + iP) \mathcal{F}_S(Q^2 + i\bar{P}),
\]

where the continuum OPE spectral density \( \rho^{(0)}_{\text{cont.}} \) is given by

\[
\rho^{(0)}_{\text{cont.}}(P, \bar{P}) = F_{\alpha_s,1} F_{\bar{\alpha}_s,1} |_{\alpha_s = \frac{Q^2 + iP}{Q_s}, \bar{\alpha}_s = \frac{Q^2 + i\bar{P}}{Q_s}}.
\]

Additional T-channel operators give corrections to this: the contributions of T-channel operators with positive twist,

\[
\delta \rho^{(\alpha_t, \bar{\alpha}_t)}_{\text{cont.}}(P, \bar{P}) = F_{\alpha_t, \alpha_t} F_{\bar{\alpha}_t, \bar{\alpha}_t} |_{\alpha_t = \frac{Q^2 + iP}{Q_s}, \bar{\alpha}_t = \frac{Q^2 + i\bar{P}}{Q_s}},
\]

are exponentially suppressed compared to the VMFT density (6.3.19), again due to (6.3.12). A similar universality in the density of states at large spin, with the same twist gap assumption, follows from a ‘lightcone modular bootstrap’ for the partition function \([101, 199]\); those results require that as \( \ell_s \to \infty \), any interval of twist above this threshold contains infinitely many operators. Arranging operators into analytic families, our requires infinitely many Regge trajectories with \( h_s \) accumulating to \( \frac{c-1}{24} \) at large spin, thus refining the conclusion already reached in \([101, 199]\).

\(^{18}\)In the presence of mixing among degenerate double-twist operators, one must diagonalize the Hamiltonian. See \([25]\) for some useful technology and a worked example involving approximate numerical degeneracy in the 3D Ising model, and \([196–198]\) for double-trace mixing in planar 4D \( \mathcal{N} = 4 \) super-Yang-Mills at strong coupling.
Comments

Asymptotic in what sense? The argument so far only shows that contributions from a finite number of operators in the T-channel are negligible at large spin in the S-channel. This does not rule out a significant contribution to the S-channel spectral density from a sum over infinitely many T-channel operators. Indeed, such contributions must be present to resolve the spectral densities $\rho_m, \rho_{\text{cont.}}$ into sums of delta functions supported at integer spins. The most conservative statement is that the asymptotic formula applies to the integrated spectral density; for example, the sum of OPE coefficients $C_{12s}^2$ for discrete double-twist operators in the $m$th Regge trajectory up to a given spin is asymptotic to the integral of $\rho_m^{(0)}(P)$ up to the corresponding value of $P$. Such a conclusion would follow from a Tauberian argument along the lines of [200,201]. It is likely that a stronger statement holds, at least in sufficiently generic CFTs (obeying an appropriate version of the eigenstate thermalisation hypothesis [202]), in which the asymptotic formula may apply to a microcanonical average over a sufficiently wide range of spin and twist. We leave the questions of how a sum over T-channel operators reproduces a discrete S-channel spectrum, and of more precise and rigorous formulations of the asymptotic formulas, to future work.

Sums over infinite sets of T-channel operators can also lead to S-channel Regge trajectories approaching any twist at large spin, not only those of the VMFT spectrum. However, the twist gap determines that the spectral density of such trajectories is suppressed compared to (6.3.13), by an exponential in $\sqrt{\ell}$ (6.3.12).

Nachtmann from Virasoro For identical operators $\mathcal{O}_1 = \mathcal{O}_2$, we can derive a Virasoro version of Nachtmann’s theorem at large spin from (6.3.15). As long as the leading twist correction comes from $\alpha_t < 2\alpha_1$, we find that the coefficient of $e^{-2\pi \bar{\alpha}_t \sqrt{\ell}}$ in the ratio of antiholomorphic kernels is positive, the residue of the ratio of holomorphic kernels is negative, and the T-channel OPE coefficients appear squared. This implies $\delta \alpha_0^{(\alpha_t, \bar{\alpha}_t)} < 0$. The change in $h$ is determined by $h = \alpha(Q - \alpha)$, which, for $2\alpha_1 < Q$, implies

$$\delta h_0^{(\alpha_t, \bar{\alpha}_t)} < 0.$$  

(6.3.21)
That is, the leading large spin correction to the twist of the first Regge trajectory is negative, so this trajectory is convex.

**Comparison to previous lightcone analyses** The arguments we use make no direct reference to the ‘old-fashioned’ lightcone bootstrap approach of solving the crossing equations (6.2.2) in the lightcone limit in $d > 2$. We comment on the connection in appendix C.5. The most difficult part of this analysis is to evaluate the S-channel blocks in a combined limit of $\bar{h}_s \to \infty$ and $\bar{z} \to 1$, with an appropriate combination held fixed, to reproduce the lightcone singularity by a saddle point in $\bar{h}_s$.

Previous work on the large spin expansion in CFT$_2$ [66] computed the asymptotic twist of the Regge trajectories in a large central charge limit, taking $\frac{h_2}{c}$ fixed in the limit, and $\frac{h_2}{c}$ small. These results are reproduced simply by taking appropriate limits of the addition of momentum variables, $\alpha_s = \alpha_1 + \alpha_2 + mb$. We further explain how our work extends and clarifies that of [66] in section 6.5 where we discuss semiclassical limits.

### 6.3.3 Large spin and large $c$

Our analysis in the previous subsection has focused on the regime of large spin, $\ell_s \gg c$, where there is a universal form for the anomalous momentum due to T-channel exchanges, given in (6.3.15), with the feature that the result decays exponentially in the square root of the spin (6.3.17). This follows from the asymptotic form of the ratio of the non-vacuum to vacuum kernels computed in appendix C.3. In the large-$c$ limit, (C.3.7) reduces to the following

$$\frac{\mathcal{F}_{\alpha_1, \alpha_t}}{\mathcal{F}_{\alpha_1}} \sim \left(\frac{12\pi}{c}\right)^{2\bar{h}_1} \frac{\Gamma(2\bar{h}_1) \Gamma(2\bar{h}_2)}{\Gamma(2\bar{h}_1 - \bar{h}_t) \Gamma(2\bar{h}_2 - \bar{h}_t)} e^{-2\pi \bar{h}_t \sqrt{\frac{mc}{c}}}, \quad (\ell_s \gg c \gg 1). \quad (6.3.22)$$

In this section we will study the anomalous weight $\delta h_m^{(\alpha_1, \alpha_t)}$ in the large-$c$ limit, fixing the ratio $\frac{\ell_s}{c}$.

---

We thank Alex Belin and Davids Meltzer and Poland for suggesting this.
From (6.3.15) we see that we need to be able to compute the ratio of the non-vacuum to vacuum anti-holomorphic fusion kernels in the limit that the S-channel internal weight scales with the central charge. We perform this computation in appendix C.3.3. Recalling that $\bar{h}_s \sim \ell_s$ at fixed $h_s$, we parameterize $\bar{\alpha}_s$ as

$$\bar{\alpha}_s = \frac{Q}{2} + i \bar{p}_s b^{-1}, \quad \text{with } \bar{p}_s \sim \frac{1}{2} \sqrt{\frac{24r}{c} - 1}. \quad (6.3.23)$$

In this limit, we find

$$\frac{\mathcal{F}_{\bar{\alpha}_s, \bar{\alpha}_t}}{\mathcal{F}_{\bar{\alpha}_s}} \sim \frac{\Gamma(2\bar{h}_1)\Gamma(2\bar{h}_2)}{\Gamma(2h_1 - h_t)\Gamma(2h_2 - h_t)} \left( \frac{c}{6\pi} \cosh(\pi \bar{p}_s) \right)^{-2\bar{h}_1} \left( \frac{c}{6\pi} \cosh(\pi \bar{p}_s) \right)^{-2\bar{h}_2} \quad (c \gg 1, \ell_s / c \text{ fixed}). \quad (6.3.24)$$

This interpolates between (6.3.22) in the large spin regime with $\bar{p}_s \sim \sqrt{\frac{m_0}{c}} \to \infty$, and the power law familiar from $d > 2$ at small spin $1 \ll \ell_s \ll c$: in that limit, we have $\bar{p}_s \approx \frac{i}{2} \left(1 - \frac{12r}{c} \right)$, giving $(\cosh(\pi \bar{p}_s))^{-2\bar{h}_1} \sim \ell_s^{-2\bar{h}_1}$.

To compute the anomalous dimension in this limit, we put (6.3.24) together with the holomorphic fusion kernel as in (6.3.15). The appropriate limits of the residues which appear will be obtained in equations (6.4.9) and (6.4.16) in the next section, giving the following result at large $c$, with $\ell_s$ of order $c$:

$$\delta h_m^{(\alpha_t, \bar{\alpha}_t)} \sim -C_{11t} C_{22t} \frac{\Gamma(2\bar{h}_1)\Gamma(2\bar{h}_2)}{\Gamma(2h_1 - h_t)\Gamma(2h_2 - h_t)} \left( \frac{c}{6\pi} \cosh(\pi \bar{p}_s) \right)^{-2\bar{h}_1} \left( \frac{c}{6\pi} \cosh(\pi \bar{p}_s) \right)^{-2\bar{h}_2} \times \frac{\Gamma(h_t)}{\Gamma(h_t)^2} \ _4F_3\left( \begin{array}{c} 1 - h_t, h_t, 2h_1 + 2h_2 - m - 1, -m \\ 1, 2h_1, 2h_2 \end{array} \right) \quad (6.3.25)$$

Focusing on the leading Regge trajectory at $m = 0$, in the limit $1 \ll \ell_s \ll c$ we recover the previously known lightcone bootstrap result for the anomalous dimension (for example, (B.33) of [26], recalling that $\delta h = \gamma / 2$):

$$\delta h_0^{(\alpha_t, \bar{\alpha}_t)} \sim -C_{11t} C_{22t} \frac{\Gamma(2\bar{h}_1)\Gamma(2\bar{h}_2)}{\Gamma(2h_1 - h_t)\Gamma(2h_2 - h_t)} \frac{\Gamma(2h_t)}{\Gamma(h_t)^2} \ell_s^{-2\bar{h}_t}. \quad (6.3.26)$$

The $m$-dependence of (6.3.25) is quite a bit simpler than previous recursive results in the lightcone bootstrap literature in $d \geq 2$ spacetime dimensions.
6.3.4 Large conformal dimension

The data of VMFT is universal at large spin because the fusion kernel for T-channel operators of positive twist is suppressed in this limit compared to the vacuum. The same argument applies to the limit of large dimension $\Delta_s \to \infty$ at fixed spin $\ell_s$ (or, more generally, for any limit where $h_s, \bar{h}_s \to \infty$), in any unitary compact $c > 1$ CFT; in this case, only a gap in conformal dimension above the vacuum is required, not in twist.

One way of expressing this is as a microcanonical average of OPE coefficients, after dividing by the asymptotic density of primary states. This density is similarly universal due to modular invariance, and can be expressed as the modular S-matrix dual to the vacuum, in close analogy to the vacuum fusion kernel for four-point functions. The modular S-matrix decomposes the character of the vacuum module into characters in the modular transformed frame [203]:

$$\chi_1(-1/\tau) = \int_0^\infty dP \, S_{\alpha_1} \chi_P(\tau), \quad S_{\alpha_1} = 4\sqrt{2}\sinh(2\pi bP)\sinh(2\pi b^{-1}P), \quad \alpha = \frac{Q}{2} + iP \quad (6.3.27)$$

At large $P$, $S_{\alpha_1}$ is exponentially larger than the corresponding modular kernel for operators of positive dimension, so is asymptotic to the density of primary states; this is a refined version of Cardy’s formula.

Including this, we find the expression

$$\overline{C_{12s}^2} \sim \frac{\bar{S}_{\alpha_s,1}}{S_{\alpha_s,1}} \times \text{(antiholomorphic)}, \quad \text{where} \quad \alpha_s = \frac{Q}{2} + iP \quad \text{and} \quad P \to \infty \quad (6.3.28)$$

for the square of OPE coefficients, where the bar denotes an average over all primaries with momentum close to $\alpha_s$. The asymptotic form for the fusion kernel is given in (6.2.23), and for the modular S-matrix (6.3.27) we have $S_{\alpha_s,1} \sim \sqrt{2}\pi e^{2\pi z P}$.

This limit was studied in [165] using crossing symmetry for four-point functions with Euclidean kinematics. The calculations there used large internal dimension results for conformal blocks [59] before taking $z \to 1.$
Their results closely resemble (6.3.28), but the discrepancy at subleading orders demonstrates the delicate nature of the order of these two limits of the blocks.\footnote{We thank Shouvik Datta for discussions of this point.}

Finally, we note that in analogy with the above, the results of section 6.3.3 imply analogous averaged OPE asymptotics in the regime of large $\Delta_s$ and large $c$ with fixed $\frac{\Delta_s}{c}$.

### 6.4 Global limit

In this section we will study the global limits of the fusion kernel and the Virasoro double-twist OPE data, for pairwise identical external operators. This is the limit of large central charge $c \to \infty$ with fixed scaling dimensions $h, \bar{h}$, named because the Virasoro conformal blocks reduce to the global $\mathfrak{sl}(2)$ blocks. By including $1/c$ corrections to this limit, it is a simple exercise to extract the large central charge expansion of double-twist OPE data due to non-unit operators.

We rewrite this limit in terms of the momentum by inverting the relation $h = \alpha(Q - \alpha)$ and expanding in the limit $b \to 0$:

$$\alpha(h) = bh + C(h)(b^3 + b^5(2h - 1)) + \mathcal{O}(b^7)$$

(6.4.1)

We have written this in terms of quadratic Casimir for $\mathfrak{sl}(2)$,

$$C(h) \equiv h(h - 1)$$

(6.4.2)

It is interesting to note that to all orders beyond $\mathcal{O}(b)$, the expansion is proportional to $C(h)$.

As the global limit is approached, more and more operators with momenta $\alpha_s = \alpha_1 + \alpha_2 + mb$ (with corresponding weights $h_s = h_1 + h_2 + m + \mathcal{O}(b^2)$) cross the contour in the S-channel decomposition of the T-channel block and give discrete residue contributions. At $b = 0$, these become precisely the global double-twist operators of MFT, and the OPE data from the $m$‘th Virasoro family becomes that of the $m$‘th global
family of MFT.

A natural expectation is that, in the global limit, the fusion kernel should reduce to a “holomorphic half” of the $6j$ symbol for the global two-dimensional conformal group $SO(3,1)$. The $6j$ symbol, which holomorphically factorizes, was recently computed in [148]. We will find that all double-twist OPE data extracted from the Virasoro fusion kernel in the global limit is reproduced by the chiral inversion integral

$$
\Omega^{h_1, h_2, h_3, h_4}_{1-h, h_1, h_2, h_3} = \int_0^1 \frac{dz}{z^2} \left( \frac{z}{1-z} \right)^{h_2+h_3} z^{h_3} h^{h_1, h_2, h_3, h_4}_{2(1-h)} (z) k^{h_3, h_2, h_1, h_4}_{2h_t} (1-z) \quad (6.4.3)
$$

where

$$
k^{h_1, h_2, h_3, h_4}_{2\beta}(z) \equiv z^{\beta} \binom{\beta}{\beta} F_1 (\beta, h_{12}, \beta + h_{34}, 2\beta, z).
$$

(6.4.4)

with $h_{ij} \equiv h_i - h_j$. This integral and notation were introduced in [148], where the $6j$ symbol was computed essentially as $\Omega^{h_1, h_2, h_3, h_4}_{1-h, h_1, h_2, h_3}$ times an anti-holomorphic partner. This integral was evaluated in [148] as a sum of two terms, each of which is a ratio of gamma functions times a $4F_3$ hypergeometric function, and may also be written as a $7F_6$ hypergeometric function [152, 153, 161, 204, 205]. With respect to the Virasoro fusion kernel for pairwise identical operators, the precise statement is that the double-twist poles and residues of $\Omega^{h_1, h_2, h_2, h_1}_{1-h, h_1, 2h_2}$ match those of $F_{\alpha, \alpha_t}$ in the global limit, i.e.

$$
\lim_{b \to 0} \left( \alpha = a_1 + a_2 + mb \right) \left( F_{\alpha, \alpha_t} \right) = \left( \text{Res}_{h=h_1+h_2+m} \Omega^{h_1, h_2, h_2, h_1}_{1-h, h_1, 2h_2} \right) \quad (6.4.5)
$$

and

$$
\lim_{b \to 0} \left( \alpha = a_1 + a_2 + mb \right) \left( b^{-1} F_{\alpha, \alpha_t} \right) = \left( \text{dRes}_{h=h_1+h_2+m} \Omega^{h_1, h_2, h_2, h_1}_{1-h, h_1, 2h_2} \right) \quad (6.4.6)
$$

The double-twist singularities of (6.4.3) come from the region of integration near the origin. We have refrained from discussing the global limit of $F_{\alpha, \alpha_t}$ itself because it does not exist, due to an overall oscillating prefactor.
For example, the following global limit of the vacuum kernel is well-defined away from the poles:

\[
\frac{b \sin(b^{-1} \pi (Q - \alpha_1 - \alpha_2 - \alpha_s))}{\sin(b^{-1} \pi (Q - 2\alpha_s))} \tilde{f}_{\alpha_s, 1} \Gamma(h_1 + h_2 - h_s) \Gamma(h_1 - h_2 + h_s) \Gamma(-h_1 + h_2 + h_s) \Gamma(h_1 + h_2 + h_s - 1) \\
\times \frac{\Gamma(h_1 + h_2 - h_s) \Gamma(h_1 - h_2 + h_s) \Gamma(-h_1 + h_2 + h_s) \Gamma(h_1 + h_2 + h_s - 1)}{2\pi \Gamma(2h_s - 1) \Gamma(2h_1) \Gamma(2h_2)}
\]

However, the ratio of sines in the prefactor simplifies to give a \((-1)^m\) at double-twist values \(\alpha_s = \alpha_1 + \alpha_2 + mb\), so the residues of the kernel at these poles have a well-defined limit, even though the kernel itself does not. An analogous statement applies to the non-vacuum kernel; this is consistent with the cross-channel decomposition of the global blocks, which only depends on the (d)residues (see (6.4.16)-(6.4.18) for the explicit expressions).

We will support (6.4.5)–(6.4.6) with several calculations. This match is a non-trivial check of our use of the fusion kernel and the global 6j symbols to extract double-twist data in unitary CFTs: the fusion kernel and the global 6j symbol require analytic continuation away from different ranges of conformal weights,\(^{21}\) but at large \(c\), the double-twist data so extracted do match. We have not determined the correspondence between regular terms of the Virasoro and global 6j symbols in the \(b \to 0\) limit, but it would be worth doing so.

### 6.4.1 Vacuum kernel

The exact vacuum kernel was given in (6.2.16). The double-twist residues admit the small \(b\) expansion

\[
2\pi \text{ Res}_{\alpha_s = \alpha_m} F_{\alpha_s, 1} = R_{12}^{(0)}(m) + b^2 R_{12}^{(2)}(m) + \mathcal{O}(b^4)
\]

\(^{21}\)Recall that the kernel is formally defined only for operators in the continuum, \(\alpha = \frac{Q}{2} + iP\) with \(P \in \mathbb{R}\). Similarly, the global 6j symbol is formally defined only for operators in the principal series \(h = \frac{1}{2} + is\) with \(s \in \mathbb{R}\). For any \(c > 1\), these two ranges do not match. However, we note that at large \(c\) (small \(b\)), the momentum for a continuum operator with \(P \approx b^{-1}p\) becomes \(\alpha \sim b^{-1} \left(\frac{1}{2} + ip\right)\), i.e. \(b^{-1}\) times the principal series conformal weight. Why there is such a correspondence between the momentum and the weight is not completely clear to us. Similar phenomena were observed in [161].
First let us extract the MFT OPE data by studying the residues of the kernel in the \( b \to 0 \) limit. The global limit of the residues (C.2.1) is given by the following

\[
R_{12}^{(0)}(m) \equiv -\frac{(2h_1)_m(2h_2)_m(2h_1 + 2h_2 - 1)_m}{4^m m!(h_1 + h_2 - \frac{1}{2})_m(h_1 + h_2)_m} \tag{6.4.9}
\]

The result (6.4.9) can be seen to satisfy (6.4.5). The relation to MFT OPE coefficients, which were derived in [206], is

\[
(C_{12[12]_{n,\ell}}^{2})_{\text{MFT}} = (-1)^\ell R_{12}^{(0)}(n) R_{12}^{(0)}(n + \ell) \tag{6.4.10}
\]

Equivalently, if one treats the chiral Virasoro index \( m \) as the spin, \((C_{12[12]_{0,\ell}}^{2})_{\text{MFT}} = (-1)^{\ell+1} R_{12}^{(0)}(\ell)\).

Moving beyond leading order, expanding the vacuum kernel near the \( m \)'th pole gives corrections to MFT OPE data for the double-twist operators \([O_1 O_2]_{m,\ell}\) in the small \( b \) – that is, \( 1/c \) – expansion. Corrections at \( \mathcal{O}(b^2) \sim \mathcal{O}(c^{-p}) \) are due to exchange of composite operators in the Virasoro vacuum module, of schematic form :\( T^q : \) with \( q \leq p \). These are dual to multi-graviton states in the bulk.

At first non-trivial order, corrections are due to \( T \) exchange. The first correction to the MFT OPE coefficients may be read off from the global limit of (C.2.1), keeping in mind the fact that \( \alpha_i \sim h_i b + C(h_i)b^3 + \mathcal{O}(b^5) \):

\[
R_{12}^{(2)}(m) = R_{12}^{(0)}(m)
\begin{pmatrix}
m^2 - 4h_1 h_2 + 2 \frac{C(h_1) + C(h_2)}{2h_1 + 2h_2 + 2m - 1} - 2 \frac{C(h_1) + C(h_2)}{2h_1 + 2h_2 + m - 1} \\
- 2C(h_1)(\psi(2h_1) + \psi(2h_1 + 2h_2 + 2m) - \psi(2h_1 + m) - \psi(2h_1 + 2h_2 + m)) \\
- 2C(h_2)(\psi(2h_2) + \psi(2h_1 + 2h_2 + 2m) - \psi(2h_2 + m) - \psi(2h_1 + 2h_2 + m))
\end{pmatrix} \tag{6.4.11}
\]

At \( m = 0 \), this is simply \( R_{12}^{(2)}(0) = 4h_1 h_2 \). More physically interesting are the anomalous dimensions in the small \( b \) expansion, for which we only need to expand our result \( \delta h_m = -2(\alpha_1 + mb)(\alpha_2 + mb) + m(m + 1)b^2 \), cf (6.3.5), to any desired order. It is slightly more enlightening to expand the location of the \( m \)'th pole in

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small $b$:

$$\frac{1}{\alpha_s - (\alpha_1 + \alpha_2 + mb)} = \frac{b^{-1}}{h_s - (h_1 + h_2 + m)} + \frac{b(C(h_1) + C(h_2) - C(h_s))}{(h_s - (h_1 + h_2 + m))^2} + O(b^3) \quad (6.4.12)$$

Plugging in the location of the pole gives the anomalous twist due to $T$ exchange,

$$\delta h_m|_T = b^2(C(h_1) + C(h_2) - C(h_1 + h_2 + m)) = b^2(-2h_1h_2 + m(1 - m - 2h_1 - 2h_2)) \quad (6.4.13)$$

Again, (6.4.13) can be seen to match the result derived from (6.4.3) and (6.4.5)–(6.4.6), and matches previous results of $[148, 151]$.

The appearance of the Casimirs was recently observed in $[151]$ as a curiosity. We now give a different angle on this: the Casimir emerges naturally from the Virasoro vacuum kernel in the global limit thanks to (6.4.1). Note that due to the all-orders appearance of $C(h)$ in (6.4.1),

$$\delta h_0 \propto C(h_1 + h_2) - C(h_1) - C(h_2) \quad (6.4.15)$$

to all orders in $b$.

A notable feature of the results (6.4.13) and (6.4.14) is their spin-independence: that is, $\gamma_{n,\ell}(0, h)$ depends only on the twist $n$. This follows from three facts: currents have vanishing twist and give a constant contribution at large spin; $\gamma_{n,\ell}$ is analytic in spin, up to contributions not captured by the Lorentzian inversion formula; and analytic functions of a single complex variable that are constant at infinity are constant everywhere. Viewing the anomalous twist (6.1.8) as the resummation of an infinite number of twist-zero stress tensor composite contributions, this explains the linearity of the VMFT Regge trajectories.

---

22 One can use equation (3.55) of $[148]$ to write down the contribution of a $T$-channel block for holomorphic current exchange, with conformal weights $(0, h)$ where $h \in \mathbb{Z}$:

$$\gamma_{\tau,\ell}(0, h) = -\frac{2\Gamma(2h)}{\Gamma(h)^2} C^2_{\phi\phi} \quad (6.4.14)$$

where we have taken identical external scalars $\phi$ for simplicity. Using $h = 2$, $C^2_{\phi\phi} = 2h^2/c \sim b^2h^2/3$ and the fact that the total anomalous dimension equals twice the change in $h$ (i.e. $\gamma_{\tau,\ell}|_T = 2\delta h_0|_T$), we find agreement with (6.4.13). The fact that we take the current to have weights $(0, h)$ rather than $(h, 0)$ follows from a choice of convention in [148].
6.4.2 Non-vacuum kernel

Let us also perform the small $b$ expansion of the non-vacuum kernels. We focus on the discrete poles. The double pole coefficients are given by the expansion of (C.2.3) in the global limit

\[
\frac{d\text{Res}}{\alpha_s = \alpha_m} F_{\alpha_s \alpha_t} = \sum_{n=0}^{m} \frac{\Gamma(2h_t)\Gamma(h_t + n)\Gamma(2h_1 + m)\Gamma(2h_2 + m + n - 1)}{2\pi n!\Gamma(h_t)^2\Gamma(h_t - n)\Gamma(2h_1 + n)\Gamma(2h_2 + n)\Gamma(2h_1 + 2h_2 + 2m - 1)} b + O(b^3)
\]

\[
= \frac{\Gamma(2h_t)(2h_1)_m(2h_2)_m}{2\pi m!\Gamma(h_t)^2(2h_1 + 2h_2 + m - 1)_m} {}_4F_3 \left( \begin{array}{c} 1 - h_t, h_t, 2h_1 + 2h_2 + m - 1, -m \\ 1, 2h_1, 2h_2 \end{array} \right) b + O(b^3) \tag{6.4.16}
\]

\[
\equiv \beta_m^{(2)}(h_1, h_2; h_t) b + O(b^3).
\]

This is computed as a finite sum. Similarly, the residues may be extracted by expanding (C.2.4):

\[
\text{Res}_{\alpha_s = \alpha_m} G_{\alpha_s \alpha_t} = \sum_{n=0}^{m} \left( \partial_n + \partial_n \right) \frac{\Gamma(2h_t)\Gamma(h_t + n)\Gamma(2h_1 + m)\Gamma(2h_2 + m + n - 1)}{2\pi (n!)^2(2h_t)^2(2h_t - n)\Gamma(2h_1 + n)\Gamma(2h_2 + n)\Gamma(2h_1 + 2h_2 + 2m - 1)} + O(b^2)
\]

\[
= \beta_m^{(1)}(h_1, h_2; h_t) + O(b^2). \tag{6.4.17}
\]

From (6.1.6), we see that these residues serve as the coefficients in the decomposition of the global $\mathfrak{sl}(2)$ blocks into global double-twist blocks (and derivatives thereof) in the cross-channel:

\[
F_T(h_t|1 - z) = -2\pi \sum_{m=0}^{\infty} \left[ \beta_m^{(1)}(h_1, h_2; h_t) F_S(h_1 + h_2 + m|z) + \beta_m^{(2)}(h_1, h_2; h_t) \partial_{h_s} F_S(h_s|z)|_{h_s = h_1 + h_2 + m} \right], \tag{6.4.18}
\]

where we have introduced the following notation for the $\mathfrak{sl}(2)$ blocks:

\[
F_T(h_t|1 - z) = (1 - z)^{h_1 - 2h_2} 2 F_1(h_1, h_t; 2h_t; 1 - z)
\]

\[
F_S(h_s|z) = z^{h_s - h_1 - h_2} 2 F_1(h_s - h_1 + h_2, h_s - h_1 + h_2; 2h_s; z). \tag{6.4.19}
\]
These results give corrections to $m > 0$ MFT OPE data in a compact form. An advantage relative to global conformal approaches is that in the latter, one needs to subtract descendant contributions of $m' < m$ that mix with the $m$th subleading quasiprimary; in the present Virasoro approach, where Virasoro primaries become global primaries in the global limit, this unmixing is not required. At $m = 0$, we can quickly check these results against the chiral inversion integral (6.4.3). To extract the $m = 0$ double-twist data, we extract the singularities near $z = 0$, where

$$
\Omega^{h_1,h_2,h_3,h_4}_{1-h_1,h_2,2h_3} \sim -\frac{\Gamma(2h_t)}{\Gamma(h_t)^2} \int_0^1 \frac{dz}{z} z^{-h+h_1+h_2} (\log z + (2\psi(h_t) - 2\psi(1)))
$$

$$
= \frac{1}{(h-h_1-h_2)^2 \Gamma(h_t)^2} + \frac{1}{h-h_1-h_2} \frac{\Gamma(2h_t)}{\Gamma(h_t)^2} (2\psi(h_t) - 2\psi(1))
$$

(6.4.20)

These coefficients match (6.4.16) and (6.4.17) after accounting for the $2\pi$.\(^{23}\)

**Special case: T-channel double-twists**

Finally, we recall that we were also able to give the closed-form expression in (6.2.26) for the fusion kernel when the T-channel operator sits exactly at a Virasoro double-twist momentum, $\alpha_t = 2\alpha_2$. The fusion kernel given in (6.2.26) has double poles at $\alpha_s = \alpha_1 + \alpha_2 + mb$ with coefficients that remain finite in the global limit, corresponding to the MFT double-twists.\(^{24}\) One can again check that (6.4.5)–(6.4.6) is satisfied.

---

\(^{23}\)Treating $T$ as a quasiprimary, this formula reproduces (6.4.14) upon plugging in $h_t = 2$, using the fact that the OPE data is minus the coefficients obtained by Lorentzian inversion [33], and that $\gamma = 2\delta h$. It also reproduces $R^{(2)}_{12}(0) = 4h_1h_2$ upon using $C_{12T}^{(2)} = 2h_1h_2/c \sim b^3 h_1h_2/3$ and $\psi(2) - \psi(1) = 1$.

\(^{24}\)Following the discussion around (C.3.6), this should have subleading asymptotics at large $h_s$ relative to the general case $\alpha_t \neq 2\alpha_2$, so as to be consistent with the emergence of zeroes in the global 6j symbol for the inversion of a T-channel double-twist operator [148]. One can show that indeed, this is suppressed exponentially. See (C.3.6) for the leading asymptotics for general $\alpha_t$, noting in particular the zero at $\alpha_t = 2\alpha_2$. 

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6.5 Gravitational interpretation of CFT results

6.5.1 Generic $c$

MFT has an interpretation as the dual of free fields on a fixed AdS background. In the absence of interactions, energies of composite states, and hence conformal dimensions $\Delta$, add. In VMFT, we have added all multi-trace stress tensor exchanges, taking into account the complete contribution of multi-graviton exchanges to two-particle binding energies. We have found that this has the remarkably simple effect that momentum $\alpha$ becomes an additive quantity, and nonlinearities are all included in the relation $h = \alpha(Q - \alpha)$. We give this a gravitational interpretation in one particular circumstance below.

This additivity law holds until reaching the threshold $\alpha = \frac{Q}{2}$, or $h = \frac{c-1}{24}$, which has the gravitational interpretation of reaching the threshold for black hole formation. At large $c$, classical BTZ black holes exist for a range of energy and spin corresponding to $\min(h, \bar{h}) > \frac{c}{24}$, at the edge of which lie the extremal rotating black holes; our results are not the first to suggest a quantum shift of the extremality bound to $\frac{c-1}{24}$ [65, 170, 211]. Above this, the VMFT spectrum becomes continuous, so it captures only some coarse-grained aspects of the physics. This chimes with the expectation that, while the thermodynamics of black holes are determined by IR data, resolving a discrete spectrum of microstates in a complete theory depends on detailed knowledge of UV degrees of freedom.

The universality of MFT at large spin in $d > 2$ comes from superposing two highly boosted states, which

\[ h = \frac{c-1}{24} \]

[203] As an aside, it should be pointed out that the Virasoro vacuum block does not capture the full gravitational dressing of the MFT four-point function as computed from bulk effective field theory. One sign of this is that the Virasoro vacuum block alone does not give a single-valued Euclidean correlation function, in contrast to computations from bulk gravitational effective field theory, order-by-order in $G_N$. To first order, the former includes only the stress-tensor global block, whereas the latter is given by a tree-level Witten diagram for graviton exchange, which also includes the exchange of $[O_1O_1]$ and $[O_2O_2]$ double traces [149, 207–209]. This makes it clear what VMFT is missing from the CFT point of view in order to comprise a viable CFT correlator, namely the exchange of double- and, at higher orders, multi-trace operators required for consistency (for example, integer quantized spins). See [151] for a related discussion. In purely gravitational language, this exclusion of multi-traces can be interpreted as neglecting the overlap of the wavefunctions of the different external particles (though including the quantum ‘fuzziness’ of the wavefunction). This was made more precise in [210], with a proposal that Virasoro blocks capture all orders in the two-parameter perturbation theory of first-quantized particles coupled to gravity, while omitting contributions which are nonperturbative in this particular expansion.
move close to the boundary confined by the AdS potential on opposite sides of AdS, and are separated by a large proper distance of order $L_{\text{AdS}} \log \ell$ [26,27]. Since interactions fall off exponentially with distance (even if they are strong or even nonlocal on AdS scales), these states become free, and hence well-approximated by the double-traces of MFT. In AdS$_3$, the situation is qualitatively different, because the gravitational potential does not fall off with distance, giving rise to a finite binding energy even at large spin. Because gravity couples universally to energy-momentum, the binding energy is determined by only the dimensions of the contributing operators and $c$, and our result (6.1.8) for the anomalous twist computes this exactly for the discrete Regge trajectories with $\alpha = \alpha_1 + \alpha_2 + mb$. The binding energy (6.1.8) is always negative, giving a fully quantum version of the attractive nature of gravity.

MFT is not only a good approximation at large spin, but also for large $N$ theories with weakly coupled bulk duals. In AdS$_3$, the same applies to our results in a $c \to \infty$ limit with operator dimensions fixed, in which they reduce to MFT as shown in section 6.4; but VMFT also reproduces results from classical gravity when $h$ scales with $c$, so that gravitational interactions are strong. The clearest demonstration of this is the classical computation of the energy of the lightest two-particle bound state. A heavy scalar particle, with action given by its mass $m \gg L_{\text{AdS}}^{-1}$ times the worldline proper time, $S = m \int d\tau$, backreacts on the metric by sourcing a conical defect with deficit angle

$$\Delta \phi = 8\pi m G_N$$

where we assumed $mG_N \ll 1$. Translating to CFT variables using $2h \sim mL_{\text{AdS}}$ and $c = \frac{3L_{\text{AdS}}}{2G_N} \sim 6Q^2$, the momentum is proportional to the particle mass, $\alpha \sim \frac{mL_{\text{AdS}}}{2Q}$, and the deficit angle may be written as

$$\Delta \phi = \frac{4\pi \alpha}{Q}.$$ 

The classical solution for two particles superposed at the centre of AdS simply adds their masses $m$, so the energy of the lightest two-particle state is given, to leading order in $c$, by the addition of momentum $\alpha$, as
in VMFT. The relation (6.5.2) holds even for finite \( m G_N \), where the deficit angle is

\[
\Delta \phi = 2\pi \left( 1 - \sqrt{1 - 8m G_N} \right).
\]

(6.5.3)

This shows that the addition of defect angles, depicted in figure 6.2, is the classical, large \( c \) version of the fully quantum finite \( c \) addition of momenta \( \alpha \).\(^{26}\)

### 6.5.2 Gravitational interpretation of anomalous twists

The anomalous twist \( \delta h_m^{(\alpha t, \bar{\alpha} t)} \) due to non-vacuum T-channel exchange maps, in the bulk, to corrections to the VMFT Regge trajectories due to couplings of the double-twist constituents to other bulk matter. For identical operators \( \mathcal{O}_1 = \mathcal{O}_2 \), the negativity of the anomalous twist due to primaries above the vacuum, \( \delta h_0^{(\alpha t, \bar{\alpha} t)} < 0 \), shown in (6.3.21), implies that these matter couplings further decrease the binding energy of the leading-twist operator (at least in the case \( \alpha_t < 2\alpha_1 \), where this negativity applies), interpreted as the attractive nature of gravity.

However, the result (6.3.17) for the large spin asymptotics is in striking contrast to the analogous result taking only global primaries into account, familiar from \( d > 2 \) [26, 27]:

\[
\delta h_s \approx \ell^{-2h_t} \quad \text{(global primaries, } d > 2) \tag{6.5.4}
\]

\[
\delta h_s \approx e^{-2\pi \sqrt{\ell s}} \quad \text{(Virasoro primaries, } d = 2) \tag{6.5.5}
\]

The scaling for \( d > 2 \) (also valid for \( 1 \ll \ell_s \ll c \) in \( d = 2 \), as shown in section 6.3.3) is interpreted as the long-distance propagator of an exchanged field, decaying as \( e^{-2h_t r} \), where \( r \sim \log \ell_s \) is the separation between particles in a two-particle global primary state of orbital angular momentum \( \ell_s \). In fact, an identical

\(^{26}\)Note that the definition of \( \alpha \) includes the shift \( c \to c - 1 \), thus including some quantum corrections. This, and the validity of the additivity rule for momenta at finite \( c \), suggests that in the AdS\(_{3}\) quantum theory, there is some (perhaps non-geometric) notion of conical defect associated to CFT local operators with large spin and low twist, with “angle” \( 4\pi \alpha / Q \). These would be analogous to putative quantum black holes whose microstates are dual to CFT local operators above threshold.
explanation is true for our $d = 2$ result, with the discrepancy explained by the fact that we must consider Virasoro primaries in the S-channel, which modifies the relation between the CFT operator spin $\ell_s$ and the bulk orbital spin, $\ell_{\text{orb}}$. This requires performing a conformal transformation on the naive bulk two-particle state such that it is dual to a Virasoro primary; it will turn out that the orbital angular momentum carries most of its spin in descendants. We now demonstrate this in a context of weak gravitational interactions, namely at large $c$ with fixed external conformal weights, but allowing the spin of the two-particle state to take any value.

We first construct classical states of two particles orbiting in the AdS potential at large separation, which will be dual to coherent superpositions of double-twist operators of large angular momentum. A single particle state at the centre of global AdS is created in radial quantization by inserting the corresponding operator at the origin $z = \bar{z} = 0$; to change the trajectory of the particle we apply a global conformal transformation, which moves the operator insertion to some other $z$, $\bar{z}$, where we take $1 - |\bar{z}| \ll 1$ to give large angular momentum. The two-particle state simply inserts two such operators, in such a way that the particles are well-separated:

$$|\Psi\rangle = \phi_1(z_1, \bar{z}_1 = e^{-\epsilon_1})\phi_2(z_2, \bar{z}_2 = -e^{-\epsilon_2})|0\rangle$$

(6.5.6)

The dependence on left-moving coordinates $z_1, z_2$ won’t play any significant part here. The norm $\langle \Psi | \Psi \rangle$ is precisely the lightcone limit of the four-point function at small $\epsilon_{1,2}$, and the particles can be interpreted as weakly interacting when this is well-approximated by the product of two-point functions, or in other words dominated by the vacuum operator in the T-channel. In particular, we will suppress the gravitational interactions by taking large $c$; including them would require the holomorphic Virasoro vacuum block.

Under this assumption of weak interactions, the stress-tensor expectation values can now be computed by analyticity and the OPE. In the cylinder frame parameterized by $w = i \log z$, this is given by the Casimir
energy on the circle, plus sums of contributions from each particle:

\[
\langle T(\bar{w})\rangle_\Psi = \frac{c}{24} - \bar{h}_1 \left( \frac{\sinh \epsilon_1}{\cosh \epsilon_1 - \cos \bar{w}} \right)^2 - \bar{h}_2 \left( \frac{\sinh \epsilon_2}{\cosh \epsilon_2 + \cos \bar{w}} \right)^2.
\]  

(6.5.7)

It will be sufficient for us to demand only a classical version of the primary state condition, namely that the one-point function of the stress-tensor 

\[
\langle T(\bar{w})\rangle_\Psi = \langle \Psi | T(\bar{w}) | \Psi \rangle
\]

is consistent with \(|\Psi\rangle\) being primary (in our limit only the antiholomorphic version will be important).

Demanding only the quasiprimary condition requires that the expectation values of \(\bar{L}_{\pm 1}\) vanish, and at small \(\epsilon_i\) we have \(\langle \bar{L}_{\pm 1} \rangle_\Psi \sim \frac{\bar{h}_1}{\epsilon_1} - \frac{\bar{h}_2}{\epsilon_2}\) so this is achieved by taking \(\epsilon_i \sim \frac{2\bar{h}_i}{\ell_{\text{orb}}}\) with \(\ell_{\text{orb}} \gg 1\). The spin of the state \(|\Psi\rangle\) is then computed by \(\langle \bar{L}_0 \rangle_\Psi \sim \frac{\bar{h}_1}{\epsilon_1} + \frac{\bar{h}_2}{\epsilon_2} \sim \ell_{\text{orb}}\), so \(\ell_{\text{orb}}\) is the ‘orbital angular momentum’. It therefore determines the separation of the particles, and hence the forces between them, in the same way as for higher dimensions, giving \(\delta h_s \sim \ell_{\text{orb}}^{-2\bar{h}_i}\).

Now we impose the Virasoro primary condition \(\langle \bar{L}_n \rangle_\Psi = 0\) for all \(n \neq 0\), or in other words that \(\langle \bar{T}(\bar{w})\rangle\) is a constant. To achieve this, we act on the state \(|\Psi\rangle\) with a conformal transformation, described by a diffeomorphism of the \(\bar{w}\) circle, to give a new state \(|\Psi'\rangle\). Choosing \(\bar{w}' \in \text{Diff}(S^1)\) (so that \(\bar{w}'\) is a smooth, monotonic \(2\pi\) periodic function of \(\bar{w}\)), the stress tensor expectation value after the conformal transformation is

\[
\langle \bar{T}(\bar{w}')\rangle_\Psi = \left( \frac{d\bar{w}'}{d\bar{w}} \right)^{-2} \left[ \langle \bar{T}(\bar{w})\rangle_\Psi - \frac{c}{12} S(\bar{w}'; \bar{w}) \right] = \frac{c}{24} - \langle \bar{L}_0 \rangle_\Psi' \quad (\text{constant}),
\]

(6.5.8)

where \(S(f(w); w) = \frac{f'''(w)}{f'(w)} - \frac{3}{2} \left( \frac{f''(w)}{f'(w)} \right)^2\) is the Schwarzian derivative. The diffeomorphism \(\bar{w}'\) is uniquely determined (up to rigid rotations) by the condition that the transformed stress tensor is constant. We will not directly compute the required conformal transformation \(\bar{w}'(\bar{w})\), but rather will indirectly determine the relevant information by considering the following ODE:

\[
\psi''(\bar{w}) + \frac{6}{c} \langle \bar{T}(\bar{w})\rangle_\psi(\bar{w}) = 0
\]

(6.5.9)
For us, we need only the mathematical fact that this is preserved under \( \text{Diff}(S^1) \) if \( \psi \) transforms as a weight \(-\frac{1}{2}\) primary, \( \psi'(\bar{w}') = \left( \frac{dw'}{d\bar{w}} \right)^{\frac{3}{2}} \psi(\bar{w}) \) (though it is not coincidental that the same equation is familiar from semiclassical computations of Virasoro conformal blocks, as well as solutions of Liouville’s equation and Einstein’s equations in AdS\(_3\), e.g. \([66, 183, 212, 213]\)). In particular, the monodromy of the ODE around the \( w \) circle (as parameterized by the trace of the monodromy matrix \( \text{Tr} M \)) is invariant under transformations (6.5.8) of the stress tensor. In the ‘classically primary’ state \( |\Psi'\rangle \) this is easily computed:

\[
\langle \hat{L}_0 \rangle_{\Psi'} = \frac{c}{24} (1 + 4\bar{p}_s^2) \sim \ell_s \implies \text{Tr} M = 2 \cosh(2\pi \bar{p}_s) \quad (6.5.10)
\]

Now it remains only to compute the monodromy of (6.5.9) with the stress tensor as given in (6.5.7), and to equate that with (6.5.10). This can be done when \( \ell_{\text{orb}} \gg 1 \), so \( \epsilon_i \ll 1 \) (without assumptions on the relative size of \( \ell_{\text{orb}} \) and \( c \) required), by patching together solutions in different regions, as explained in appendix C.6.

The result is

\[
\text{Tr} M = 2 \cosh(2\pi \bar{p}_s) \sim \left( \frac{12\pi}{c} \right)^2 \frac{4\bar{h}_1 \bar{h}_2}{\epsilon_1 \epsilon_2} - 2 \implies \ell_{\text{orb}} \sim \frac{c}{6\pi} \cosh(\pi \bar{p}_s) \quad (6.5.11)
\]

Note that, depending on the size of \( \ell_{\text{orb}} \) relative to \( c \), the monodromy could be large or of order one. We have now determined the ‘orbital’ angular momentum \( \ell_{\text{orb}} \) as a function of the spin of the primary state \( \ell_s \), as parameterized by \( \bar{p}_s \). This determines the anomalous dimensions as for \( d > 2 \):

\[
\delta h_s \approx \ell_{\text{orb}}^{-2\tilde{h}_s} \sim \left( \frac{c}{6\pi} \cosh(\pi \bar{p}_s) \right)^{-2\tilde{h}_s} \quad (6.5.12)
\]

This precisely matches the result computed from the fusion kernel in the appropriate limit, in section 6.3.3.

Not only do we reproduce the anomalous dimensions at large spin, but also interpolate smoothly to the result at \( \ell_s \ll c \), which gives the results of a global analysis applied to \( d = 2 \):

\[
\ell_s \ll c \implies \ell_{\text{orb}} \sim \ell_s \quad (6.5.13)
\]

\[
\ell_s \gg c \implies \ell_{\text{orb}} \sim \frac{c}{12\pi} e^{\pi \sqrt{2\ell_s}}. \quad (6.5.14)
\]
6.6 Semiclassical limits and late-time physics

6.6.1 Heavy-light limit

In this subsection we study the fusion kernel in the semiclassical ‘heavy-light’ limit. Denoting \( h_1 = H \) and \( h_2 = h \), the limit is defined as

\[
\text{Heavy-light limit: } H \to \infty, \quad c \to \infty \quad \text{with} \quad \frac{H}{c}, h \text{ fixed.} \tag{6.6.1}
\]

Our present goal is to compute the T-channel heavy-light blocks of [66,67] and to understand their corrections from a new perspective. We will mostly discuss the vacuum in the T-channel, because in some appropriate limits the vacuum block dominates other contributions to the full correlation function of theories with bulk duals\(^{27}\).

First, we take the fusion kernel for T-channel vacuum exchange, parameterized as

\[
\alpha_1 = \frac{Q}{2} + ib^{-1}p, \quad \alpha_2 = bh, \quad \alpha_s = \frac{Q}{2} + ib^{-1}p_s, \quad \alpha_t = 0 \tag{6.6.2}
\]

The limit (6.6.1) corresponds to the \( b \to 0 \) limit with \( p, p_s, h \) fixed. In terms of the dimensions, this means that

\[
H \sim \frac{c-1}{24}(1 + 4p^2), \tag{6.6.3}
\]

so \( p \in \mathbb{R} \) if the heavy operator is above the black hole threshold. The momentum \( p \) is often parameterized by an effective inverse “temperature” \( p = \pi/\beta \). It is straightforward to take the limit of the expression (6.2.16),

\(^{27}\)Dominance of the vacuum block alone requires several assumptions. In particular, unless \( h \sqrt{2} \gg 1 \), even if the light operator is dual to a bulk free field, its double-trace contributions are not suppressed, but are required to give the sum over images for the full two-point function in the background created by the heavy operator.
using the semiclassical limits of the special function $\Gamma_b$ in section 6.2.2. To leading order, we find

$$\lim_{b \to 0} b^2 \log F_{\alpha_s} \approx \frac{J(2p) + J(2p_s) - 2J(p - p_s) - 2J(p + p_s) - 2\pi(\abs{p} - \abs{p_s})\Theta(\abs{p} - \abs{p_s})}{2\pi},$$

where $J(x) := \int_1^{1+ix} \log \Gamma(t) \, dt + \int_1^{1-ix} \log \Gamma(t) \, dt = F(1 + ix) + F(1 - ix) - 2F(1)$. (6.6.4)

The main properties of $J$ that will be important are that it is even, analytic on the real line (with branch cuts on the imaginary axis starting at $\pm i$), and vanishes at the origin. The resulting expression is even in $p$ and $p_s$, as required by reflection symmetry, so we can restrict attention to $p, p_s > 0$. The kernel is maximal at $p_s = p$, where the leading semiclassical exponent above vanishes, and has a kink there coming from the final term, with derivative $2\pi$ on the left and zero on the right, as shown in figure 6.6. This kink can be understood as a consequence of the poles at $\alpha_s = \alpha_1 + \alpha_2 + mb$ (and a reflected line) accumulating close to the $\Re \alpha_s = \frac{Q}{2}$ axis. In the semiclassical limit, these poles coalesce into a branch cut, and in the heavy-light limit, two such cuts meet at $p_s = p$ giving rise to the kink. In a moment we will take an alternative limit where the poles become individually visible, and give them a gravitational interpretation.

In order to compute the T-channel block, we integrate $F_{\alpha,1}$ against S-channel blocks. At this point, it may seem that we gain nothing by using the fusion kernel, since we must have control over the S-channel blocks. However, since the fusion kernel is largest for $p_s$ close to $p$, the integral is dominated by internal S-channel

![Figure 6.6: The order $b^{-2}$ piece of log $F_{\alpha_s}$, with $p = 1$, as a function of $p_s$.](image-url)
dimensions close to the heavy operator dimension. To zoom in on this regime, we choose \( \alpha_s = \alpha_H + ibx \) with \( x \) of order one, so that \( h_s - H \sim 2px \) in the limit. As it turns out, these S-channel blocks simplify as much as one could ever hope for: as observed in appendix E of \[66,28\], if \( H - h_s, h \ll \sqrt{c} \sim b^{-1} \), the S-channel block reduces to the ‘scaling block’, the contribution of the single primary operator alone:

\[
F_S(\alpha_s|z) \sim z^{-h_1-h_2+h_s} \sim z^{2px-h} \tag{6.6.5}
\]

Now, to evaluate the identity T-channel block in this limit, we will integrate this S-channel block against the fusion kernel. Writing \( \alpha_s = \alpha_H + ibx \) and taking the large \( c \) limit with \( x \) fixed, we have

\[
F_{\alpha_s} \sim b^{-1}(2p)^{2h}e^{\pi x} \frac{1}{2\pi} \frac{\Gamma(h-ix)\Gamma(h+ix)}{\Gamma(2h)} \quad \text{as } b \to 0. \tag{6.6.6}
\]

In particular, we see that this is consistent with the earlier result, since for large \( |x| \), we have \( \Gamma(h-ix)\Gamma(h+ix) \approx e^{-\pi|x|} \), the exponential scaling with \( x \) joining smoothly onto either side of the kink at \( p_s = p \). The accumulating poles at \( \alpha_s = \alpha_1 \pm (\alpha_2 + mb) \) that caused the kink are now individually visible.

Now that we have the blocks and the kernel for this region of internal dimensions, we can evaluate this part of the integral,

\[
F_T(1|1-z) \sim \int_{-x_0}^{x_0} dx (2p)^{2h}e^{\pi x} \frac{1}{2\pi} \frac{\Gamma(h-ix)\Gamma(h+ix)}{\Gamma(2h)} z^{2px-h} \tag{6.6.7}
\]

for some \( x_0 \) consistent with the parametric regime articulated above. To begin, we take the bounds \( x_0 \to \infty \). To compute the resulting integral, we choose \( z = e^{it} \) for \( t \) with a positive real part, allowing us to close the contour in the upper half-plane, where we pick up the residues

\[
2\pi i \operatorname{Res}_{x=i(h+n)} \frac{1}{2\pi} \frac{\Gamma(h-ix)\Gamma(h+ix)}{\Gamma(2h)} = \binom{-2h}{n}, \tag{6.6.8}
\]

\(^{28}\)All descendants are suppressed by powers of \( \frac{H-h_s}{2} \) and \( \frac{b^2}{c} \): projecting the four-point function onto the \( h_s \) representation by inserting complete sets of states \( L_{-n_1} \cdots L_{-n_k}|h_s\rangle \), the matrix elements appearing are of order \( h^k, (H-h_s)^k \), but the normalization coming from the inverse Kac matrix suppresses by \( c^{-k} \).
These sum to a binomial series, yielding

\[ \mathcal{F}_T(1|1-z) \sim z^{-h} \left( \frac{2ip}{z-ip-z^*p} \right)^{2h} \]  \quad (6.6.9)

Continuity allows us to extend this to real \( z \). This result (6.6.9) is precisely the limit computed in [66] using the monodromy method for large \( h \), and extended to finite \( h \) in [67]. In fact, in [66] the inverse Laplace transform of the heavy-light block was used to derive precisely the spectral density (6.6.6), the inverse of the computation in (6.6.7).

The poles here have a nice gravitational interpretation as quasinormal modes of the BTZ black hole (after combining with antiholomorphic counterparts) [66, 214]. Here it is clear that, while their presence dominates the integral over intermediate states, the contour runs between them along real \( p_s \), so the poles should not be interpreted as part of the physical spectrum of states. This is important for preservation of unitarity, since they correspond to complex scaling dimensions.

This computation applies when the heavy operator lies above the black hole threshold, but the same form for the semiclassical limit of the vacuum block holds when \( \frac{24H}{c} < 1 \), simply taking \( p \to ip \) in (6.6.7). We derive this in a similar way from the fusion kernel in section 6.6.1, while generalizing to allow for a light non-vacuum exchange operator in the T-channel.

**Forbidden singularities**

In the computation above, we have been rather cavalier, taking \( x_0 \to \infty \) and assuming that the integral over S-channel intermediate dimensions is dominated by a particular region. However, for certain kinematics, we already see a sure sign that this cannot possibly hold: for some values of \( z \), the integral (6.6.7) diverges! Since \( \Gamma(h - ix)\Gamma(h + ix) \approx e^{-\pi|x|} \) for large \( |x| \), convergence of the integral requires that \( e^{-\pi} < |z| < 1 \). This divergence causes a seemingly paradoxical property of the heavy-light blocks, dubbed the ‘forbidden singularities’ [215], also studied in [172, 216]. To see this most clearly, write \( z = e^{-\tau} \), with \( \tau \) interpreted as
imaginary time in radial quantization, and \( p = \pi / \beta \):

\[
\mathcal{F}_T(1|1 - e^{-\tau}) \sim e^{h\tau} \left( \frac{\beta}{\pi} \sin \frac{\pi \tau}{\beta} \right)^{-2h}
\]  

(6.6.10)

This (excepting the prefactor, which can be removed by making a conformal transformation to the cylinder) is periodic in \( \tau \) with period \( \beta \); indeed, this (times its antiholomorphic counterpart) equals the universal CFT\(_2\) result for a two-point function on the line at finite temperature. As noted in [215], though, the singularity as \( \tau \to 0 \) required by the OPE limit gives rise to further singularities at \( \tau = n\beta \), which cannot be present in a correlation function or conformal block (e.g. because of convergence in the unit disk in the nome variable \(|q| [59]\)).

We now have a simple explanation for this phenomenon: as we get too close to the first singularity at \( \tau = \pi / p \), the suppression of the fusion kernel at smaller values of intermediate dimension is no longer sufficient to overwhelm the S-channel blocks, and the integral (6.6.7) starts to pick up significant contributions from large negative \( x \), where our approximation for the kernel and S-channel block no longer apply. The full integral over \( p_s \) still converges, and the block is finite, but now the integral is dominated by a saddle-point at smaller values of intermediate dimension, and is therefore enhanced exponentially in \( c \). This is an example of the equivalence between ensembles breaking down for certain observables: the canonical and microcanonical results are only close for choices of observable which do not significantly shift the saddle-point over energies. Here we see very explicitly (albeit in the context of an individual block, rather than a full microcanonical correlation function) that when the Euclidean time becomes too close to the effective temperature, the intermediate energies (represented by \( p_s \)) shift to a new saddle point.

**Conical defects and non-vacuum exchange**

We now address the case where we remain in the heavy-light limit, but with \( \frac{H}{c} < \frac{1}{24} \). Simultaneously, we generalize to allow for non-vacuum exchange in the T-channel, with \( h_t \) of order one. From the gravitational
point of view, in this range of dimensions the heavy operator is dual to a particle creating a conical defect, rather than a black hole; this qualitative difference is visible from our calculation because the fusion kernel contains discrete contributions from poles at $\alpha_s = \alpha_m$, which dominate in the semiclassical limit.

We now parameterize the external weights as

$$\alpha_1 \sim \eta b^{-1}, \ 0 < \eta < \frac{1}{2}, \ \alpha_2 \sim h b,$$  \hspace{1cm} (6.6.11)

so $H \sim \frac{2}{5} \eta (1 - \eta)$. We compute the coefficients of the poles at $\alpha_s = \alpha_m$ with $m$ of order one in this limit. For the vacuum T-channel, the residues match those found above in the calculation of (6.6.7) with $p \to i(\eta - \frac{1}{2})$; for nonzero $h_t$, the coefficients of the double pole, from the expressions in appendix C.2, have the following small $b$ limits:

$$d \text{Res}_{\alpha_s = \alpha_m} \mathbb{F}_{\alpha_s, \alpha_t} \sim \sum_{n=0}^{m} \frac{(1 - 2\eta)^{2h - h_t} \Gamma(2h_t) \Gamma(h_t + n) \Gamma(2h + m)}{2\pi(n!)^2 \Gamma(1 + m - n) \Gamma(h_t) ^2 \Gamma(h_t - n) \Gamma(2h + n)} b $$

$$\text{Res}_{\alpha_s = \alpha_m} \mathbb{F}_{\alpha_s, \alpha_t} \sim \sum_{n=0}^{m} (\partial_m + \partial_n) \frac{(1 - 2\eta)^{2h - h_t} \Gamma(2h_t) \Gamma(h_t + n) \Gamma(2h + m)}{2\pi(n!)^2 \Gamma(1 + m - n) \Gamma(h_t) ^2 \Gamma(h_t - n) \Gamma(2h + n)}. \hspace{1cm} (6.6.12)$$

In the large $c$ limit, we can simply sum the contributions of these residues for $m \in \mathbb{Z}_{\geq 0}$ (the number of poles that have crossed the contour is of order $c$), and furthermore use the scaling block in place of the S-channel block as above, since the poles lie at $h_s - H$ of order one:

$$\mathcal{F}_T(\alpha_t|1-z) = \int \frac{d\alpha_s}{2i} \mathbb{F}_{\alpha_s, \alpha_t} \mathcal{F}_S(\alpha_s|z)$$

$$\sim -2\pi z^{-2\eta} \sum_{m=0}^{\infty} \left( \text{Res}_{\alpha_s = \alpha_m} \mathbb{F}_{\alpha_s, \alpha_t} + b^{-1} (1 - 2\eta) d \text{Res}_{\alpha_s = \alpha_m} \mathbb{F}_{\alpha_s, \alpha_t} \log z \right) z^{m(1-2\eta)}.$$

This gives a power series in $w := z^{1-2\eta}$, whose coefficients (6.6.12) match the expansion of the T-channel
global block $k_{2h_1}(1 - w)$, but in terms of the new $w$ variable:

$$
\mathcal{F}_T(\alpha_t|1 - z) \sim (1 - 2\eta)^{2h_t - h_s} z^{-2\eta} (1 - w)^{h_t - 2h_2} F_1(h_t, h_t; 2h_t; 1 - w). \tag{6.6.14}
$$

This is the same as the result computed by [67], who directly summed descendants in a basis of Virasoro generators adapted to the $w$ coordinate. The gravitational interpretation is natural here, as the conformal transformation to the $w$ coordinate maps locally pure AdS onto the conical defect geometry. In CFT language, the $w$ coordinate is the choice such that the stress tensor expectation value in the presence of the heavy operator vanishes. In our calculation, the defect angle comes from the spacing between the VMFT double-twist poles.

The technical details of the calculation are similar to the $\frac{H}{c} > \frac{1}{24}$ case, but the interpretation is rather different: the poles we are summing over should now be interpreted as physical operators which will appear in the S-channel spectrum (albeit with some corrections in the exact correlation function), double-twist composites of one heavy and one light particle.

**Connection to large spin analysis**

The semiclassical heavy-light limit of the block was used by [66] for an analysis at large spin, which we now put into the context of our finite $c$ results of section 6.3.2. Their computation involved decomposing the T-channel heavy-light identity block into the S-channel, interpreting the result as the asymptotic twist of Regge trajectories at large spin. Our derivation of the heavy-light block – for example, (6.6.7) – is precisely the inverse of this, so it is clear that their result gives the relevant large $c$ limit of the fusion kernel, with $h_1 - h_s$ of order one. Certain aspects of their result become clearer with our new perspective. They decomposed into S-channel quasiprimaries, but saw no sign of Virasoro descendants; we now see that this is a consequence of the suppression of S-channel descendants in the relevant limit. For $\frac{h_1}{c} < \frac{1}{24}$ they saw an infinite number of discrete Regge trajectories, but their result is reliable only for $m \ll \sqrt{c}$, and similarly, for $\frac{h_1}{c} > \frac{1}{24}$ they see
a continuum in twist, but are only really sensitive to the range $|h_s - h_1| \ll \sqrt{c}$. Their results for the large $\ell$
asymptotic twist when $\frac{24h_1}{c} < 1$, namely

$$h_s = h_1 + \sqrt{1 - \frac{24h_1}{c}(h_2 + m)}, \quad (6.6.15)$$

follow simply from a large $c$ expansion of $\alpha_s = \alpha_1 + \alpha_2 + mb$, as long as $m \ll \sqrt{c}$. Taking in addition $\frac{h_1}{c} \ll 1$, this reduces to a ‘Newtonian limit’ with $\frac{h_1 h_2}{c}$ fixed as $c \to \infty$, in which the Virasoro vacuum block becomes the exponential of the stress-tensor contribution, and we find, for the leading Regge trajectory,

$$h_s = h_1 + h_2 - \frac{12h_1 h_2}{c} \quad \text{(Newtonian limit)} \quad (6.6.16)$$

We also reproduce this result from the Lorentzian inversion formula in appendix C.4.

### 6.6.2 Late time

One interesting choice of kinematics for the four-point function we have been considering gives the a time-ordered two-point function of the light operator $O = O_2$ on the Lorentzian cylinder, in the heavy state $|\psi\rangle$ corresponding to $O_1$:

$$\langle \psi | O(t, \phi) O(0) | \psi \rangle = G(z = e^{-i(t+\phi)}, \bar{z} = e^{-i(t-\phi)}) \quad (6.6.17)$$

As we evolve in time, we find singularities when $t \pm \phi$ goes through $2\pi$ times an integer, due to the operators passing through their mutual lightcones. To regulate this (and give the proper time-ordering), $t$ should be given a small negative imaginary part so that $z, \bar{z}$ lie inside the unit circle. Since we will just consider chiral blocks here, the dependence on angle $\phi$ will be suppressed.

It is natural to write this as an expansion over conformal blocks in the T-channel, $OO \to \psi\psi$, since summing over light operators in this channel reproduces the results of gravitational effective field theory; in particular, the heavy-light vacuum block (6.6.9), with $z = e^{-it}$, equals the result computed from a free
particle in a planar BTZ background. But this decays exponentially for all time, as $e^{-2pht}$ (coming from the slowest quasinormal mode, the leading pole in (6.6.7)), which is in tension with an S-channel expansion as a discrete sum over operators weighted by a phase $e^{-i\Delta_s t}$. This implies that the correlation function cannot decay forever, but must eventually (at times longer than the inverse level-spacing) fluctuate around a value of order $e^{-S}$, a version of the information paradox \cite{9}.

Here, we will see to what extent the decay is corrected within a single T-channel block once finite $c$ corrections are included, a problem studied numerically in \cite{172}. We restrict to the case $\text{Re}(\alpha_1 + \alpha_2) > \frac{Q}{2}$, where $Q$ is arbitrary. In the S-channel, Lorentzian time evolution simply gives a phase $e^{-ith_s}$. Restricting for ease of presentation to time differences $t \in 2\pi\mathbb{Z}$, we have

$$F_T(\alpha | t_0 + t) = \int \frac{d\alpha_s}{2i} F_S(\alpha_s | t_0) F_{\alpha_s \alpha_\ell} e^{-it\alpha_s (Q - \alpha_s)}$$

$$= \int_{c-1/24}^{\infty} \frac{dh_s}{2\sqrt{h_s - c-1/24}} F_S(\alpha_s | t_0) F_{\alpha_s \alpha_\ell} e^{-ith_s},$$

(6.6.18)

(6.6.19)

where $\alpha_s = \frac{Q}{2} + iP_s$. The last equality makes it clear that this is a Fourier transform with respect to $h_s$. At very late times, this will be controlled by the least smooth feature in $h_s$, and since the blocks and fusion kernel are analytic functions of $\alpha_s$, this must be at $h_s = c-1/24$, $\alpha_s = \frac{Q}{2}$. For generic external dimensions, the fusion kernel has a double zero at this location in $\alpha_s$, coming from factors of $\Gamma_b (Q - 2\alpha_s) \Gamma_b (2\alpha_s - Q)$ in the denominators of (6.2.10) and (6.2.16). Since $dh/d\alpha = Q - 2\alpha$, this becomes a simple zero in $h_s$. This means that the spectral density in terms of $h_s$ begins at $c-1/24$ with a square root edge, and after taking the Fourier transform, the result is that the block decays as $t^{-3/2}$. Refining this slightly, we can also find the coefficient of this late time decay analytically, at least for the vacuum block. Writing $\alpha_s = \frac{Q}{2} + iP_s$, the calculation of the integral at late time is most naturally done by stationary phase, dominated by $P_s = 0$. Taking into
account the double zero of the kernel, we find

\[ \mathcal{F}_T(\alpha_t|t + t_0) = \int_0^\infty dP_s \mathcal{F}_S(\alpha_s|t_0) \mathcal{F}_{\alpha_s, \alpha_t} e^{-i\frac{Q^2}{2} + P^2} \]

\[ \sim \left. \frac{1}{2} \partial^2_{P_s} \mathcal{F}_{\alpha_s, \alpha_t} \right|_{\alpha_s = \alpha_t = \frac{Q}{2}} \mathcal{F}_S \left( \frac{Q}{2}|t_0 \right) e^{-i\frac{Q^2}{2} \frac{\sqrt{\pi}}{4} (it)^{-3/2}}. \]  

(6.6.20)

This is valid for all \( Q \). This power law was observed numerically by [172], with precisely this explanation proposed. This behaviour is familiar from random matrix theory where the semicircle eigenvalue distribution has the same square root edge, and hence the same \( t^{-3/2} \) power law decay of the spectral form factor at times sufficiently early that fluctuations away from the average eigenvalue density are not resolved.

For the identity operator, it is simple to use \( (6.2.16) \) to evaluate the first term in the prefactor; this was done in \( (6.2.30) \). In the heavy-light limit \( (6.6.1) - (6.6.3) \), that result becomes

\[ \frac{1}{2} \partial^2_{P_s} \mathcal{F}_{\alpha_s, \alpha_t} \left|_{\alpha_s = \frac{Q}{2}} \right. \sim \frac{16\pi^2}{\sinh(2\pi P) \Gamma(2\pi)^2} \left( \frac{p}{b} \right)^{4h-1} e^{-b^{-2}(2\pi p + 4J(p) - J(2p))}, \]

(6.6.21)

The exponent matches the earlier semiclassical result evaluated at \( p_s = 0 \). The dependence on the initial S-channel block is somewhat trickier to compute, but if we choose \( t_0 = -i\epsilon \) to be a small imaginary time regulator, we have \( z = e^{-it_0} \sim 1 - \epsilon \) for small \( \epsilon \), so are in the regime where this can be evaluated by the cross-channel limit \( (6.2.32) \). This gives (assuming \( \alpha_2 < \frac{Q}{2}, \alpha_1 \))

\[ \mathcal{F}_S \left( \frac{Q}{2}|t_0 \right) = \frac{\Gamma_b(Q)^3 \Gamma_b(2\alpha_1 - 2\alpha_2) \Gamma_b(2Q - 2\alpha_1 - 2\alpha_2) \Gamma_b(Q - 2\alpha_2)^4}{\Gamma_b(Q - 4\alpha_2) \Gamma_b\left( \frac{Q}{2} + \alpha_1 - \alpha_2 \right)^4 \Gamma_b\left( \frac{3Q}{2} - \alpha_1 - \alpha_2 \right)^4} e^{-2h_2 + h(2\alpha_2)} (1 + O(\epsilon)) \]

\[ \sim \sqrt{\frac{(\pi P)^3 \cosh(\pi P)}{\sinh^3(\pi P)}} \left( \frac{\tanh(\pi P)}{\pi P} \right)^{2h} e^{b^{-2}(4J(p) - J(2p))} e^{-2b^2 h^2}, \]

where in the second line we have again taken the heavy-light limit. Ignoring phases and subleading factors (where we fix \( e^{-1} \ll \epsilon \ll 1 \) in the large \( c \) limit), we find the following late-time behaviour:

\[ \mathcal{F}_T(1|t) \approx e^{-2\pi P b^2} t^{-3/2} \]

(6.6.22)
The same power law was also seen in the contribution of the vacuum Virasoro characters to the spectral form factor [217]. This can be explained as the same square-root edge of the modular-S matrix [65] (the analogue of the fusion kernel for Virasoro characters), but in this case the identity (for which null descendants must be subtracted) is qualitatively different, with non-vacuum characters only decaying as $t^{-1/2}$, since their dual channel spectral densities diverge as $(h_s - \frac{c-1}{24})^{-1/2}$.

While the power law behaviour (6.6.20) holds for any sufficiently heavy (generic) external dimensions and any $c$, in the semiclassical heavy-light regime it does not set in until a parametrically late time. Before times of order $c$ there is exponential decay, as visible from the semiclassical limit of the block (6.6.7), (6.6.10). There is in fact a sharp transition where this exponential decay ceases, since the block behaves simply as a sum of the exponential decay $e^{-2pht}$ with order one coefficient, and the power law with exponentially small coefficient $e^{-2\pi \tilde{c} p}$, and these terms exchange dominance at a crossover time which is sharply defined at large $c$,

$$t_c = \frac{\pi c}{6h}, \quad (6.6.23)$$

precisely the result observed in numerical studies [172]. To see that there is a sharp transition, take times of order $c$ where the stationary phase approximation and large $c$ saddle-point approximation combine into a steepest descent analysis; it turns out that there are always two separate saddle-points, those producing the semiclassical result and late-time behaviour, and they lie on separate steepest descent contours. A logical alternative was to find a single saddle-point which moved from $\alpha_s = \alpha_1$ to $\alpha_s = \frac{Q}{2}$ as time increased, giving a smooth function of $\frac{t}{c}$ as opposed to a sharp transition.

Finally, we point out that for light external operators $\alpha_1 + \alpha_2 < \frac{Q}{2}$, while there is still a piece giving a $t^{-3/2}$ decay coming from the integral over the continuum of $\alpha_s$, it is not dominant, because the poles at

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29While this crossover time is order $c$, it does not grow with the energy of the heavy state, so is not of order the entropy, suggesting that it is not caused by resolving nontrivial eigenvalue statistics. From its derivation, it is clear that this is the case, since it is simply a feature of the coarse-grained spectral density, so true signs of discreteness of the S-channel spectrum can only come from sums over infinitely many internal T-channel operators. Nonetheless, it would be interesting to clarify if this feature is present in correlation functions of theories with gravitational duals once T-channel blocks beyond the vacuum are added, and to attempt to give a semiclassical description, perhaps in terms of near-threshold black holes.
\[ \alpha_s = \alpha_1 + \alpha_2 + mb \] contribute phases \( e^{i\theta_s} \). At large \( c \), these poles can give a very large number of fluctuating contributions of comparable amplitude but with irrationally related periods, and are the source of the erratic behaviour observed in [151].

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Universal dynamics of heavy operators in CFT$_2$

7.1 Introduction and discussion

Two dimensional conformal field theories are among the most important and interesting quantum field theories. They describe important condensed matter and statistical mechanics systems at criticality and, remarkably, possess an infinite dimensional group of symmetries related to local conformal transformations [13]. In this chapter we will be interested in irrational CFTs with $c > 1$ and an infinite number of primary states. Although these theories are not exactly solvable, they are nevertheless under much greater analytic control than their higher dimensional cousins. In this chapter we will describe a particular example of this fact: the dynamics of heavy (i.e. high dimension) operators is universal in two dimensional CFTs, in the
sense that these dynamics are determined only by the central charge and not by any other details of the theory.

The basic dynamical data that defines a CFT is a list of primary operators $O_i$, along with

- Their scaling dimensions $\Delta_i \equiv h_i + \bar{h}_i$ and spins $J_i \equiv h_i - \bar{h}_i$, and
- The operator product expansion (OPE) coefficients $C_{ijk}$.

These data, along with the central charge $c$, uniquely determine the correlation functions of the theory in flat space as well as on an arbitrary surface. Ideally one would like to solve the constraints of unitarity and conformal invariance to determine the possible allowed values of the $\{h_i, \bar{h}_i, C_{ijk}\}$, and hence completely classify two dimensional CFTs. In the absence of such a complete classification, however, we will ask a more modest question: which features of this data are universal (i.e. true in any conformal field theory) and which are theory dependent?

A simple example of a universal feature is the dimension and spin of the identity operator:

$$h_1 = 0 = \bar{h}_1 \quad (7.1.1)$$

which is the same in every CFT. A second and somewhat more subtle universal feature is Cardy’s formula for the growth of the high energy density of primary states [169]:

$$\rho(h, \bar{h}) \approx \exp\left\{4\pi \left(\sqrt{(c - 1) \frac{h}{24}} + \sqrt{(c - 1) \frac{\bar{h}}{24}}\right)\right\} \quad \text{when } h, \bar{h} \to \infty. \quad (7.1.2)$$

Equation (7.1.2) is true in any compact CFT with $c > 1$, and is universal in the sense that it depends only on the central charge $c$ and not on any other details of the theory. In fact, these two universal features (7.1.1)

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1. We restrict our attention in this chapter to unitary, compact CFTs, defined to have a discrete spectrum with a unique $\mathfrak{sl}(2)$-invariant ground state. The same approach will, however, apply more generally with some modest modifications. We focus on theories with $c_L = c_R = c$ for simplicity, but the modification of our results to theories with $c_L \neq c_R$ is straightforward.

2. Throughout this chapter we use the notation $a \sim b$ to denote that $\frac{a}{b} \to 1$ in the limit of interest. We will also use the notation $a \approx b$ to denote that $a$ and $b$ have the same leading scaling in the limit of interest.

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and (7.1.2) are closely related: they are “dual,” in the sense that they are related by modular invariance. Cardy’s formula is the statement that the identity operator has dimension zero, albeit interpreted in a dual channel in the computation of the torus partition function.

Every unitary, compact CFT possesses an additional universal feature: the identity operator will appear in the fusion of any operator with itself. In terms of the OPE data, this means that

$$C_{ij2} = 1$$ (7.1.3)

for any operator $O_i$. This leads to the following natural question: what is the corresponding dual universal feature? In other words, what universal feature do the three point coefficients obey which plays the same role to equation (7.1.3) as Cardy’s formula (7.1.2) does to equation (7.1.1)?

We will answer this question in this chapter. The result is a universal asymptotic formula for the average value of the OPE coefficients:

$$\overline{C_{ijk}}^2 \sim C_0(h_i, h_j, h_k)C_0(\bar{h}_i, \bar{h}_j, \bar{h}_k)$$ (7.1.4)

where

$$C_0(h_i, h_j, h_k) = \frac{1}{\sqrt{2}} \frac{\Gamma_b(2Q)}{\Gamma_b(Q)^3} \prod_{\pm \pm \pm} \Gamma_b \left( \frac{Q}{2} \pm iP_a \pm iP_j \pm iP_k \right) \prod_{a \in \{i,j,k\}} \Gamma_b(Q + 2iP_a) \Gamma_b(Q - 2iP_a).$$ (7.1.5)

In this equation $\prod_{\pm}$ denotes a product of eight terms with all possible sign permutations. Here rather than using the central charge $c$ and dimensions $h$ and $\bar{h}$ to write our formula, we have used the “Liouville parameters”

$$c = 1 + 6Q^2 = 1 + 6(b + b^{-1})^2, \quad h = \alpha(Q - \alpha), \quad \alpha = \frac{Q}{2} + iP.$$ (7.1.6)

Just as with Cardy’s formula, this result is universal in the sense that it is true in any (compact, unitary) CFT, and the only free parameter appearing in this formula is the central charge $c$.

In interpreting this formula, a few comments are in order. The first is that equation (7.1.4) is an expression

3We have chosen a basis of operators such that the two-point function is diagonal and canonically normalized, $\langle O_i(0)O_j(1) \rangle = \delta_{ij}$. 223
for the average OPE coefficient, with the heavy operator weight(s) averaged over all Virasoro primary operators, which is valid for any finite \( c > 1 \). In this sense, our result differs from most of the previous results in the literature. The second is that, although we have only written one formula, equation (7.1.4) is secretly three different formulas hiding in one. In particular, this formula is valid in three different regimes, and is derived using three types of crossing symmetry. Equation (7.1.4) holds:

- When two operators are taken to be fixed and the third is taken to be heavy, in which case it follows from the crossing symmetry of four-point functions with pairwise identical external operators.
- When one operator is fixed and the other two are heavy, in which case it follows from the modular covariance of torus two-point functions of identical operators.
- When all of the operators are taken to be heavy, in which case it follows from modular invariance of the genus two partition function.

In each case, the averaging taken in equation (7.1.4) should be understood as an average over the heavy operator(s), but not over the other operators which are held fixed.\(^4\) The surprising result is that we obtain exactly the same formula in each case.

Various authors have previously considered the asymptotic behaviour of three point coefficients in each of these three separate limits [68, 103, 129, 163–165, 218–228]. The asymptotic formulas which were obtained generally relied on detailed computations of the conformal blocks, and – while correct – required assumptions about the behaviour of the blocks in certain kinematic regimes or the simplification of large central charge. Our single asymptotic formula (7.1.5) unifies all of these previous results, and in the darkness binds them. Moreover, it holds for any finite value of the central charge \( c > 1 \), and interpolates between all of the previously known results in the literature.

Before describing the details of our derivation, in the remainder of the introduction we will describe the

\(^4\)As we will elaborate on below, “heavy” in this context means that \( h \) and \( \bar{h} \) are much larger than both the central charge and the dimensions of the other operators which are held fixed. For this reason the three different regimes described above are distinct, and there is a-priori no reason to expect to get the same result in each regime.
strategy underlying our derivation and comment in more detail on the interpretation of this result.

7.1.1 The strategy: bootstrap without the blocks

In order to illustrate our basic strategy, consider the following simple example where one extracts the asymptotic behaviour of OPE coefficients from crossing symmetry of four point functions. Consider the four point function of an operator $O$

\[
\langle O(0)O(x)O(1)O(\infty)\rangle = \sum_{s} |C_{s}\rangle \langle s|^{2} x^{h_{s} - 2h_{O}} x^{\tilde{h}_{s} - 2\tilde{h}_{O}} = \sum_{t} |C_{t}\rangle \langle t|^{2} (1 - x)^{h_{t} - 2h_{O}} (1 - \tilde{x})^{\tilde{h}_{t} - 2\tilde{h}_{O}}
\]

where in first line and second lines we have expanded in a basis of intermediate operators in the $S$-channel and $T$-channel, respectively. In this simple version of the computation the sums run over all operators in the theory, both primaries and descendants, and we are not organizing the states into representations of the conformal group. The functions $x^{h_{s} - 2h_{O}}$ and $(1 - x)^{h_{t} - 2h_{O}}$ play the role of conformal blocks in the $S$- and $T$-channel, respectively. This four point function has a pole at $x = 1$ coming from the operator $1$ in the $T$-channel, which allows us to determine the asymptotic behaviour of the $S$-channel expansion coefficients $|C_{s}\rangle \langle s|$ when $h_{s}$ is large. We do so by expanding the $T$-channel conformal block of the identity operator into $S$-channel blocks:

\[
\frac{1}{(1 - x)^{2h_{O}}} = \sum_{n=0}^{\infty} (-1)^{n} \binom{-2h_{O}}{n} x^{n} = \sum_{n=0}^{\infty} \binom{2h_{O} + n - 1}{n} x^{n}
\]

The binomial coefficient $\binom{2h_{O} + n - 1}{n}$ appearing in this expression is a simple example of a crossing kernel: the coefficients which appear when we expand a conformal block in one channel in terms of conformal blocks in a dual channel.\(^5\) Comparing the two channel decompositions of our correlation function, we see that our

\(^5\)Note however that this crossing kernel is only supported on a discrete set of intermediate operator weights (namely $h_{s} = 2h_{O} + n$ for $n$ a non-negative integer); this is similar to the situation for global $SL(2, \mathbb{R})$ conformal blocks, which can be expanded as a sum over double-twist blocks and their derivatives in the cross channel (see [229] for an explicit
crossing kernel must equal the average value of the OPE coefficients at \( h_s = 2 h_O + n \) in the limit where the operator \( \mathcal{O}_s \) is heavy:

\[
\left| C_{\mathcal{O}_s \mathcal{O}_s} \right|_{\text{scaling}}^2 \sim \left( \frac{h_s - 1}{h_s - 2 h_O} \right) \left( \frac{\bar{h}_s - 1}{\bar{h}_s - 2 \bar{h}_O} \right) \frac{h_s^{2 h_O - 1} \bar{h}_s^{2 \bar{h}_O - 1}}{\Gamma(2 h_O) \Gamma(2 \bar{h}_O)}, \quad h_s, \bar{h}_s \to \infty \quad (7.1.9)
\]

The subscript ‘scaling’ reminds us that, as we did not organize into representations of the conformal group, the average here is over all heavy operators \( \mathcal{O}_s \) – both primaries and descendants – of dimensions \( h_s, \bar{h}_s \). We have also not specified the exact nature of the average which is being taken, i.e. over how wide a range of operators one must average in order for the result \((7.1.9)\) to hold. We will return to this subtlety below.

In order to determine the asymptotic behaviour of primary operator OPE coefficients we must improve this computation by organizing the sum over intermediate states into a sum over representations of the conformal group. This is accomplished by taking \( \mathcal{O}_s \) and \( \mathcal{O}_t \) above to be primary operators and replacing the functions \( x^{h_s - 2 h_O} \) and \( (1 - x)^{h_t - 2 h_O} \) by the appropriate conformal blocks. We then expand the identity block for the \( T \)-channel in terms of the \( S \)-channel blocks for heavy operators, exactly as in \((7.1.8)\). The average value of the primary operator OPE coefficients is then given by the analog of the binomial coefficient appearing in this expansion. As conformal blocks for Virasoro symmetry are not known analytically one might think that this computation is impossible. Remarkably, this is not the case, as Ponsot and Teschner obtained explicit (but complicated) expressions for the crossing kernel of Virasoro blocks for four-point functions \([156, 157]\).\(^6\)

However, when we take the operator in the \( T \)-channel to be \( 1 \) these crossing kernels simplify considerably, and they are essentially given by our expression \((7.1.5)\).

This computation will be carried out in more detail below, but already several features are apparent. The first is that, as conformal blocks are purely kinematic objects – i.e. they depend on central charge and the dimensions of the operators under consideration but not on which theory we are studying – the crossing decomposition). This is unlike the case of Virasoro blocks that will be the subject of this chapter, as the cross-channel decomposition of the Virasoro block will typically involve a continuum.

\(^6\)The higher-dimensional analog of the Virasoro fusion kernel is the \( 6j \) symbol for the principal series representations of the Euclidean global conformal group \( SO(d + 1, 1) \) \([148]\), which serves as a crossing kernel for conformal partial waves.
kernels are purely kinematic as well. This guarantees that our resulting asymptotic formula will be universal, in the sense that it depends only on the central charge but not on any other details of the theory. The second is that, from this point of view, conformal blocks can be bypassed altogether and one can work directly with crossing kernels. In particular, as long as one is interested in understanding the constraints that crossing symmetry imposes on the dynamical data of a CFT (the spectrum and OPE coefficients) the conformal blocks represent an unnecessary complication. Blocks are only needed if one wishes to extract an observable, such as a correlation function, from this basic dynamical data.

The above discussion shows that crossing symmetry of four point functions will determine the asymptotic behaviour of OPE coefficients in the limit where one operator is taken to be heavy and the others are held fixed. In order to obtain other constraints, we must consider crossing symmetry and modular invariance for more general observables. The most general observable is an $n$-point correlation function of Virasoro primaries on a Riemann surface of genus $g$, which we will denote $G_{g,n}(\{q_i\})$, where the $q_i$ are a set of continuous variables which parameterize the moduli of the Riemann surface as well as the locations of the insertion points of these primary operators. We then expand this observable as a sum over intermediate operators propagating in a particular channel, as

$$G_{g,n}(\{q_i\}) = \sum_{\{\mathcal{O}_j\}} \mathcal{C}_{\{\mathcal{O}_j\}} \mathcal{F}(\{P_j\}|\{q_i\})$$

$$= \int [dP_j] \rho(\{P_j\}) \mathcal{F}(\{P_j\}|\{q_i\}).$$

(7.1.10)

Here the $\{\mathcal{O}_j\}$ are the internal operators which contribute to this observable, and the $\mathcal{C}_{\{\mathcal{O}_j\}}$ are the corresponding products of OPE coefficients. We are organizing into conformal families, and the conformal block $\mathcal{F}(\{P_j\}|\{q_i\})$ encodes the contribution of all descendants of the operators $\{\mathcal{O}_j\}$. As the conformal blocks are kinematic, they depend only on the spins and dimensions of the operators $\{\mathcal{O}_j\}$, which we are writing in terms of the parameters $\{P_j\}$ defined by equation (7.1.6). In order to keep the notation compact, in this formula $\{P_j\}$ and $\{q_i\}$ denote both the holomorphic and anti-holomorphic weights of the internal operators and moduli of the punctured Riemann surface, and the block $\mathcal{F}(\{P_j\}|\{q_i\})$ includes contributions from both left-
and right-moving descendants. For simplicity we have suppressed the dependence on the external operators. In the last line we have introduced a “density of OPE coefficients”

\[
\rho(P_j) = \sum_{\langle O_j \rangle} c_{\langle O_j \rangle} \prod_j \delta(P_j - P_{O_j}) \delta(\bar{P}_j - \bar{P}_{O_j})
\]

(7.1.11)

which is a function only of the \( P_j \).\(^7\)

In (7.1.10) we have reduced the correlation function to a sum of products of OPE coefficients. On a higher genus Riemann surface this is an in principle complicated procedure, as one must decompose the Riemann surface into pairs-of-pants and then sum over internal operators which propagate through the cuffs of these pairs of pants. This makes the computation of the conformal blocks quite difficult. The advantage of our approach is that by working directly with crossing kernels rather than conformal blocks, almost all of the details of this construction are irrelevant. Thus it is possible to understand the constraints of modular invariance and crossing symmetry without the need to explicitly construct the Riemann surface.

We now wish to compare this to the expansion of our observable in another channel:

\[
G_{g,n}(\{q_i\}) = \sum_{\langle O_k \rangle} \tilde{C}_{\langle O_k \rangle} \tilde{\mathcal{F}}(\{R_k\}||\{\tilde{q}_i\})
\]

(7.1.12)

\[
= \int [dR_k] \rho(\{R_k\}) \tilde{\mathcal{F}}(\{R_k\}||\{\tilde{q}_i\}).
\]

Here we denote the OPE coefficients, the Virasoro conformal blocks, and the OPE spectral density in this alternate channel with a tilde. We have also denoted the moduli on which the conformal blocks depend with a tilde to emphasize that the blocks in different channels typically admit perturbative expansions in different parameterizations of the moduli. In general the relationship between the two coordinate systems \( q_j \) and \( \tilde{q}_i \) on moduli space is quite complicated. Our strategy of working entirely with crossing kernels ensures, however, that we never need to determine this relationship explicitly.

\(^7\)Strictly speaking \( \rho \) is a distribution rather than a function. Moreover, the \( P_i \) will be either real or purely imaginary depending on dimensions and spins of the operators \( O_j \), and the definition of the integral in (7.1.10) includes contributions from all states.
Associativity of the operator product expansion implies that our two different operator product expansions must agree. We then compare these two different expansions by introducing the crossing kernel $K$ defined by:

$$F(\{P_j\}|\{q_i\}) = \int [dR_k] K_{\{R_k\}\{P_j\}} \tilde{F}(\{R_k\}|\{\tilde{q}_j\}).$$  \hfill (7.1.13)

Plugging this into equation (7.1.10) and comparing with (7.1.12) gives us the crossing equation.

$$\tilde{\rho}(\{R_k\}) = \int [dP_j] K_{\{R_k\}\{P_j\}} \rho(\{P_j\}).$$  \hfill (7.1.14)

In cases where the same OPE data appears in both channels, the solutions to the crossing equation are the unit eigenvectors of the crossing kernel.

We now wish to extract universal features of the OPE coefficients $C_{\{O_j\}}$ by considering limits where the identity operator dominates in one channel. In particular, we would like to consider cases where the right hand side of the crossing equation (7.1.14) is dominated by the identity operator (i.e. dominated by the term with all $O_j = 1$) when the internal weights $R_k$ are taken to infinity. This will occur when

$$\frac{K_{\{R_k\}\{P_j\}}}{K_{\{R_k\}\{1\}}} \to 0 \quad \text{as} \quad R_k \to \infty. \hfill (7.1.15)$$

In this limit the density of OPE coefficients is just given by the corresponding crossing kernel of the identity operator:

$$\tilde{\rho}(\{R_k\}) \approx K_{\{R_k\}\{1\}} \quad \text{as} \quad R_k \to \infty. \hfill (7.1.16)$$

This is the generalization of our earlier result (7.1.9), that the crossing kernel of the identity operator serves as the universal asymptotic behaviour of the OPE coefficients for heavy states.

We emphasize that, although we have phrased it more abstractly, this is equivalent to the familiar strategy where one studies the crossing equation in a kinematic regime in which the exchange of the identity operator dominates in one channel. For example, in the case of the four-point function the limit we are considering
is equivalent to the one where the cross ratio $x \to 1$. Similarly, the application of this strategy to the torus partition function gives Cardy’s formula. A final example is the lightcone bootstrap \([26, 27]\), where the spectrum and OPE data of CFT\(_{d>2}\) approaches that of mean field theory at large spin. However these arguments typically require the detailed knowledge of conformal blocks in certain Lorentzian kinematic regimes, which in the Virasoro case is out of reach except in the simplest cases. The advantage of our approach is that we only require the crossing kernel, bypassing the need to compute the conformal blocks explicitly.

### 7.1.2 The Moore-Seiberg construction of crossing kernels

We now wish to apply this construction to constrain the asymptotics of the squared OPE coefficients $|C_{ijk}|^2$. To begin, recall that $C_{ijk}$ is the correlation function $\langle \mathcal{O}_i \mathcal{O}_j \mathcal{O}_k \rangle_{S^2}$ on the sphere, with the operators inserted at three points. Thus to study $|C_{ijk}|^2$ we must consider observables obtained by sewing together two copies of the sphere at these insertion points. For example, the four point function on the sphere is obtained by sewing together these two spheres at a single point – say, the insertion point of the operators $\mathcal{O}_k$ – to give:\(^8\)

$$\langle \mathcal{O}_i(0) \mathcal{O}_j(x, \bar{x}) \mathcal{O}_j(1) \mathcal{O}_i(\infty) \rangle_{S^2} = \sum_{\mathcal{O}_k} |C_{ijk}|^2 \mathcal{F}(P_k | x) \mathcal{F}(\bar{P}_k | \bar{x})$$

(7.1.17)

where $\mathcal{F}(P_k | x)$ is an appropriate holomorphic conformal block. Applying the crossing arguments of the previous section will then lead to an asymptotic formula for the $|C_{ijk}|^2$ in the limit where $\mathcal{O}_k$ is taken to be heavy but the operators $\mathcal{O}_i$ and $\mathcal{O}_j$ are held fixed. Similarly, we can sew together the spheres at a pair of points, the locations of the operators $\mathcal{O}_j$ and $\mathcal{O}_k$, to obtain the two point function on the torus:

$$\langle \mathcal{O}_i(v, \bar{v}) \mathcal{O}_i(0) \rangle_{T^2} = \sum_{\mathcal{O}_j, \mathcal{O}_k} |C_{ijk}|^2 \mathcal{F}(P_j, P_k | \tau, v) \mathcal{F}(\bar{P}_j, \bar{P}_k | \bar{\tau}, \bar{v})$$

(7.1.18)

---

\(^8\)The notation $\mathcal{O}'(\infty)$ means $\lim_{z \to \infty} z^{2h_\mathcal{O}} \bar{z}^{2\bar{h}_\mathcal{O}} \mathcal{O}(z, \bar{z})$.  

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where \( \mathcal{F}(P_j, P_k | \tau, v) \) is now a conformal block for two point functions on the torus. This will lead to an asymptotic formula for \( |C_{ijk}|^2 \) in the limit where both \( \mathcal{O}_j \) and \( \mathcal{O}_k \) are heavy and \( \mathcal{O}_i \) is fixed. Finally, sewing together all three insertion points gives the genus two partition function:

\[
Z_{g=2}(q, \bar{q}) = \sum_{\mathcal{O}_i, \mathcal{O}_j, \mathcal{O}_k} |C_{ijk}|^2 \mathcal{F}(P_i, P_j, P_k | q) \mathcal{F}(\bar{P}_i, \bar{P}_j, \bar{P}_k | \bar{q}) \tag{7.1.19}
\]

where \( q \) is a collection of genus two modular parameters and \( \mathcal{F}(P_i, P_j, P_k | q) \) is a holomorphic genus two conformal block. This will lead to an asymptotic formula which is valid when all of the operators are taken to be heavy.

The strategy described above is only useful, however, if we can accomplish two things: we first need to find a dual channel where the identity operator dominates, and we must then compute the relevant crossing kernels. To accomplish this we will follow the strategy of Moore and Seiberg [37], who argued that all of the constraints of the associativity of the OPE are completely captured by crossing symmetry of four point functions on the sphere and modular covariance of one-point functions on the torus. This is because any crossing transformation for any observable can be constructed by composing “elementary” crossing transformations: four point crossing on the sphere (or fusion), and modular transformations for one-point functions on the torus (along with braiding, which we will not use in this chapter). The crossing kernels for these elementary crossing transformations were written down explicitly in [105, 156, 157, 166]. Thus, by assembling these together using the Moore-Seiberg construction, we can obtain explicit formulas for general crossing transformations — such as those on higher genus Riemann surface — without ever computing a conformal block.

We will write this down very explicitly below, but the general strategy is easy to understand. The two elementary crossing transformations we use can be represented pictorially as in figure 7.1. The first of these is the crossing transformation for four point functions on the sphere, where we have chosen to represent the four external operators by holes rather than infinitesimal points. The \( S \)- and \( T \)-channel decompositions of the four point function then correspond to the two different ways of constructing this four-holed sphere.

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Figure 7.1: The elementary crossing transformations: sphere four-point crossing between $S$ and $T$ channels, and torus one-point crossing between the $\tau$ and $(1/\tau)$ frames.

**Figure 7.2:** Example of a crossing transformation on the torus two-point function.

as two pairs-of-pants glued together shown above. Similarly, the second picture in figure 7.1 describes the crossing transformation between two different channels for a one-point function on the torus.

We can now construct crossing transformations for two point functions on the torus by composing these elementary transformations, as in figure 7.2. We recognize the first of these as the modular $S$ transformation for one point functions on the torus, and the second as the fusion move for four point functions on the sphere. The result is an expression for this more complicated crossing kernel as a product of these two elementary kernels. Indeed, we recognize the channel on the far right as precisely the one which gives the square of the OPE coefficients in equation (7.1.18), where $O_j$ and $O_k$ are the operators which propagate through the two blue circles. Our asymptotic formula for $|C_{ijk}|^2$ is then obtained by considering the kinematic limit which is dominated by the identity operator $1$ propagating in the channels (marked by yellow circles) on the far left.

We can construct the crossing transformations at genus two in a similar manner, as in figure 7.3: we have first done two crossing moves for torus one point functions, followed by a four-point crossing move on the sphere. Again, the channel on the far right gives the square of the OPE coefficients considered in equation (7.1.19) where the operators $O_i, O_j$ and $O_k$ propagate through the three blue circles. The asymptotic formula

**Figure 7.3:** Example of a crossing transformation on the $g = 2$ partition function.
for \(|C_{ijk}|^2\) when these three operators are taken to be heavy is found by considering the limit where the identity operator 1 dominates in the channel decomposition depicted on the far left. This formula is given in terms of a genus two crossing kernel which – by construction – is a product of the elementary crossing kernels which were written down by Ponsot and Teschner.

The result is an asymptotic formula for the averaged OPE coefficients \(|C_{ijk}|^2\) in the three limits described above, where either one, two or all three operators are taken to be heavy, and only the heavy operators are averaged over. For example, in the case where the differences between the heavy operator dimensions and all spins \(J_i\) are held fixed in the large-dimension limit, we can state all of our asymptotic formulas as follows:\(^9\)

\[
\frac{C^2_{O_1O_2O}}{C^2_{O_1O_2O}} \approx 16^{-\Delta} e^{-2\pi \sqrt{\frac{\Delta}{c}} \Delta^2(\Delta_1+\Delta_2)-\frac{\Delta_1}{4}}, \quad \Delta \gg c, J, \Delta_1, J_i
\]  
(7.1.20)

\[
\frac{C^2_{O_0O_1O_2}}{C^2_{O_0O_1O_2}} \approx e^{-4\pi \sqrt{\frac{\Delta}{c}} \Delta_1 \Delta_0}, \quad \Delta_1, \Delta_2 \gg c, J_1, \Delta_0, J_0, |\Delta_1 - \Delta_2|
\]  
(7.1.21)

\[
\frac{C^2_{O_1O_2O_3}}{C^2_{O_1O_2O_3}} \approx \left(\frac{27}{16}\right)^{3\Delta_1} e^{-6\pi \sqrt{\frac{\Delta}{c}} \Delta_1 \Delta_0 |\Delta_i - \Delta_j|}, \quad \Delta_1, \Delta_2, \Delta_3 \gg c, J_i, |\Delta_i - \Delta_j|
\]  
(7.1.22)

In addition to these, there are other distinct asymptotic limits, for example fixing the ratios of \(\Delta_i\) instead of differences as in (7.4.12) and (7.4.20), which are also controlled by (7.1.4). Remarkably, all of these formulas (appearing in equations (7.4.4), (7.4.12), (7.4.13), (7.4.20) and (7.4.21)) are realized as limits of the same underlying formula (7.1.5). This is perhaps the most surprising feature of our result, and is a consequence of the Moore-Seiberg procedure which constructs all of these different crossing kernels from the same elementary building blocks.

### 7.1.3 Generalizations to other observables

We emphasize that, although we have applied our strategy to the computation of the asymptotics of the \(|C_{ijk}|^2\), this argument works much more generally. Whenever one can find a kinematic limit where the identity block dominates a CFT observable, there is a corresponding universal formula for the OPE data

\(^9\)Here, our notation with the \(\approx\) symbol means that we have omitted the order one coefficients appearing in these formulas. These coefficients can be found in equations (7.4.4), (7.4.13) and (7.4.21).
in the dual channel – it is just a matter of assembling the appropriate crossing kernel. In this sense our strategy should be regarded as defining an entire class of CFT asymptotic formulas which govern the universal dynamics of heavy operators in two dimensional CFTs. It would clearly be worth exploring these dynamics in more detail.

In addition, while our main focus is on universal asymptotic formulas – namely those which are constructed only from the propagation of the identity operator in a cross channel – one can also consider non-universal quantities which are constructed from other light operators propagating in a cross channel. For example, the leading corrections to the universal formulas described above will come from the other light operators in the theory, and one can obtain improved (but non-universal) asymptotic formulas which depend on the data (such as the spectrum and OPE coefficients) of whatever light operators are present in the theory.

The most interesting example of this type would be one where the contribution from $1$ in the cross channel vanishes, in which case the asymptotic behaviour would be non-universal and depend on the light data of the theory. The prototypical example is the average value of the Light-Heavy-Heavy OPE coefficient $C_{ijj}$, where the state $i$ is heavy and averaged over, while the $j$ is held fixed. This is determined by considering the modular covariance of one point functions $\langle O_j \rangle_{T^2(\tau)}$ on the torus in the limit $\tau \to 0$ [129]. The contribution from the identity operator propagating in the dual channel (i.e. taking $\tau \to 1/\tau$) is just the one-point function of $O_j$ on the plane, which vanishes. The first non-vanishing contribution will come from the lightest operator $\chi$ which has $C_{j\chi\chi} \neq 0$. Previous results have either worked only at large central charge, or have organized into scaling blocks or global blocks, rather than full conformal blocks (so that the average in $C_{ijj}$ is an average over quasi-primaries or over all states in the theory, rather than over Virasoro primaries) [129]. We can now write down the complete answer at finite central charge, where the average is taken only over primaries; this will be discussed in section 7.7.
7.1.4 Large central charge limit

One important special case is the large central charge limit, which is relevant for holographic theories with an AdS gravity dual. In this case a generic heavy state is interpreted as a microstate of a BTZ black hole. The observation that the average OPE coefficients take a universal form then has a natural physical interpretation, as the emergence of a semi-classical black hole geometry which arises upon coarse-graining over heavy states. That our formula depends only on the central charge and the dimensions and spins of the operators reflects the fact that this semi-classical configuration is purely geometric: the holographically computed OPE coefficient depends on Newton’s constant and the masses and spins of the objects under consideration, but not on any other details of the state. Our formulas can thus be regarded as an extrapolation of the usual gravitational “no hair” theorems to CFT. Indeed, various limits of our formula have already been shown to reproduce the classical dynamics of particles in black hole backgrounds \( [103,129,218-220] \), and appear in closely related gravitational computations of semiclassical conformal blocks \( [188,230] \). We note that from the point of view of classical gravity it is not at all obvious that there should be a single formula that interpolates between the three different limits we are considering (where either one, two or all three of the operators are taken to be heavy). Indeed, our formula reflects this: it smoothly interpolates between these three limits at finite \( c \), but not after taking a \( c \to \infty \) limit.

Perhaps the most important point to emphasize here is that, as we take \( c \to \infty \), the “heavy” operators appearing in our formula should still be understood to have dimension large compared to \( c \). This is necessary in order for the identity operator to still dominate in the dual channel. Such a state, however, will be interpreted as a black hole whose horizon area is very large in AdS units. A black hole whose size is order one in AdS units would correspond to an operator whose dimension is order \( c \). It is therefore natural to ask under what circumstances the regime of validity of our asymptotic formulas could be extended to operators with finite \( h/c \) in the large \( c \) limit. Generically, this will only happen if we impose severe restrictions on the “light” data in our theory. For example, the regime of validity of Cardy’s formula can be extended all the way down to dimensions of order \( c \) only if the density of states of the light spectrum is sufficiently sparse \( [64] \).
It would be interesting to ask whether similar considerations could be applied to our asymptotic formulas.
We expect that the corresponding sparseness constraint will be considerably more subtle, however, and may require more than just a constraint on the density of OPE coefficients of light operators – see [230,231] for discussions of this in the context of higher genus partition functions of symmetric product orbifolds and holographic CFTs.

7.1.5 Chaos, integrability and eigenstate thermalization

Our results have an important role to the play in the study of chaos in two dimensional CFTs. To see this, we first note that while we have written formulas of the form

\[ C_{ijk} \sim C_0(h_i, h_j, h_k)C_0(\bar{h}_i, \bar{h}_j, \bar{h}_k) \]  

(7.1.23)

we have not yet stated precisely what range of states one must average over. The weakest possible statement would be that our asymptotic formula is true only in an integrated sense, where rather than averaging over a small window of states one simply sums over all states below some (large) cutoff. We expect, however, that a much stronger version is true, where one needs to integrate only over a small window; results that establish this kind of behaviour go under the general name of Tauberian theorems (see e.g. [200,201,225,232–234] for recent applications of Tauberian theorems in this context). In the present case we would require new results for several variables, adapted to the Virasoro crossing transforms. This is an important avenue for future research, which is not merely a mathematical subtlety but a question of important physical interest.

In particular, our expectation is that in a generic, chaotic theory one would need to average only over very small window in order to obtain the asymptotic result (7.1.23). In other words, in a chaotic theory the typical OPE coefficient should be rather close to the average one. In an integrable theory, however, many OPE coefficients will vanish due to selection rules, so any average result is obtained only by including many different states in the average. We expect that in a chaotic theory one would need to average over a window
of size not much larger than $e^{-S}$, where $S$ is the microcanonical entropy, while in an integrable theory one must average over a window of some fixed width rather than one that is exponentially small at high energies. It is important to emphasize that all of our results are derived from crossing and modular constraints which hold in any CFT. Thus our result (7.1.23) will be equally true in integrable and chaotic theories. The crucial difference will be in the way in which this average is realized. Indeed, we would propose that the size of the window one must average over should be used as a sharp criterion for chaos in conformal field theory: a chaotic theory is one where one needs to average only over windows of size $O(e^{-S})$. It would be interesting to compare this to other proposed characterizations of chaos in quantum field theory.

Indeed, our asymptotic formulas also play an important role in the eigenstate thermalization hypothesis (ETH) [202,235], which states that in a chaotic theory the matrix elements of an operator $O$ should obey

$$
\langle i|O|j \rangle \approx f^O(\Delta_i)\delta_{ij} + g^O(\Delta_i, \Delta_j)R_{ij}
$$

(7.1.24)

for states $i$ and $j$ of fixed energy density in a large volume thermodynamic limit. Here, $f^O$ and $g^O$ are smooth functions of energy related to the microcanonical one- and two-point functions, and $R_{ij}$ is a random variable of zero mean and unit variance; if the one- and two-point functions are of order one, then $f^O$ is of order one and $g^O$ of order $e^{-S/2}$. In a scale-invariant theory, the large volume thermodynamic limit is equivalent to a large energy limit at fixed volume, which is the heavy limit we have been studying. When $O$ is a local operator, ETH is a statement about the statistics of structure constants (see [228,236–249] for more detailed discussion of ETH in the context of conformal field theories).

In a two dimensional CFT it is natural to take this to be a statement about primary operator OPE coefficients; descendant state OPE coefficients are completely determined by Ward identities, and hence by definition do not provide any information about the chaotic dynamics of the theory. Indeed, dynamics within a particular Virasoro representation will never thermalize due to the infinitude of conserved quantities. At infinite central charge this distinction is largely irrelevant, as the typical high energy state is – if not a primary state itself – then very close to one. For finite $c$ CFTs, however, these considerations become important and
the most sensible definition of ETH is one where (7.1.24) is interpreted as a statement about the statistics of primary operators.

In this case our asymptotic formulas for \( \overline{C_{Oii}} \) and \( \overline{|C_{Oij}|^2} \) determine the functions \( f^O \) and \( g^O \):

\[
\overline{C_{Oii}} = f^O(\Delta_i), \quad \overline{|C_{Oij}|^2} = (g^O(\Delta_i, \Delta_j))^2
\]  

Thus our formulas provide a precise formulation of ETH for CFTs with finite central charge \( c \). It is important to emphasize that our asymptotic formulas predict the form of the smooth functions \( f^O \) and \( g^O \) (and provide the consistency check that \( |C_{Oij}|^2 \sim e^{-S} \)), but say nothing about the statistics of the remainder term \( R_{ij} \).

The statement that \( R_{ij} \) has zero mean and unit variance, severely constraining the fluctuations of matrix elements, is an important component of ETH and one which is invisible using the techniques of this chapter. Indeed, all CFTs are crossing invariant, so no argument based on crossing symmetry alone can distinguish between a chaotic and an integrable theory. Our arguments establish the universal behaviour of averaged OPE asymptotics, and so are not sensitive to the fine-grained statistics of individual eigenstates. Some additional input must be included in order to use crossing arguments to probe this more refined structure of ETH. One might hope that assuming no additional currents would be sufficient to ensure the theory is chaotic, but while we make use of this assumption to establish universal formulas that apply at large spin, it is not clear how to use it to say more about statistics of OPE coefficients relevant for ETH.

An important feature of the ETH formula is that it is expected to govern the statistics of OPE coefficients in the Heavy-Heavy-Light limit, where the operators \( i \) and \( j \) are heavy but \( O \) is fixed. On the other hand, our asymptotic formulas for OPE coefficients smoothly interpolate between this limit and the Light-Light-Heavy and Heavy-Heavy-Heavy regimes. This immediately suggests that the ETH conjecture (7.1.24) should be generalized to these regimes as well. It also suggests that a version of ETH should hold not just at large dimension, but also for operators with large spin at fixed twist. We expect this extended regime of validity to be a special feature of CFTs (where there is a state-operator correspondence) rather than general QFTs. One intriguing aspect of this conjecture is that while the Heavy-Heavy-Light version of ETH has a natural
thermodynamic interpretation – it captures the intuitive notion that in a chaotic theory every state should be approximately thermal in the thermodynamic limit – the interpretation of equation (7.1.24) in this extended regime is much more mysterious.

A second important point is that the behaviour of the two functions $f^O$ and $g^O$ is quite different in two dimensional CFTs from their behaviour in higher dimensions. In a higher dimensional theory the diagonal terms in the OPE coefficients are exponentially larger than the off-diagonal terms: $f^O$ is of order one, while $g^O \approx e^{-\frac{1}{2}S(\Delta_i + \Delta_j)}$ is exponentially suppressed. In a two dimensional CFT this behaviour is modified, as $f^O$ itself is exponentially small. This can be seen by noting that at high temperature a thermal one point function becomes a one point function on the cylinder $S^1 \times \mathbb{R}$, which is – by the usual radial quantization map – conformally equivalent to the plane. Hence thermal one point functions will be exponentially small at high temperature, with exponent determined by the dimension of the lightest operator which couples to the operator $O$. Thus we expect that the off-diagonal terms for a generic primary operator $O$ will be exponentially suppressed relative to the diagonal terms, but with an exponent that is not $e^{-\frac{1}{2}S(\Delta_i + \Delta_j)}$ but rather is determined by the size of the gap in the theory. This is a consequence of the strange fact that in CFT$_2$ thermal one point functions vanish at high temperature, while thermal two point functions do not.

In the extreme case – where the size of the gap in the theory is sufficiently large – the off-diagonal terms will be the same size as the diagonal terms. We will clarify this statement in section 7.7 and show that this will occur when the lightest non-vacuum primary that couples to $O$ has dimension greater than or equal to $\frac{c}{16}$ (in the case that this lightest operator is a scalar). This fact will be a simple consequence of the structure of the corresponding crossing kernels. A theory with a gap of size $O(c)$ would be interpreted as a theory of pure gravity in AdS$_3$ in the large $c$ limit, as the spectrum of perturbations around empty AdS would include only boundary gravitons (i.e. descendants of the identity operator). We therefore come to a remarkable conclusion – a theory of pure gravity in AdS$_3$ is precisely one where the off-diagonal terms in ETH are not suppressed relative to the diagonal ones. This provides an intriguing link between black hole dynamics and quantum chaos. A similar conclusion was recently reached for JT gravity in two dimensions.
7.1.6 Connection to Liouville theory

Our universal OPE coefficient formula (7.1.5) closely resembles the DOZZ formula for the structure constants of Liouville theory [56, 57]. However, our universal asymptotic formulas do not apply to Liouville theory, since it is not compact (the spectrum does not include an $sl(2)$-invariant ground state). We here explain the similarity of the formulas by noting that they both follow from Virasoro representation theory, and contrast their interpretation.

The spectrum of Virasoro primary states of Liouville theory is continuous, consisting of scalars of dimension $h = \bar{h} = c - 1 - \frac{2}{P^2}$. Their three-point coefficients are given by the DOZZ formula $C_{DOZZ}(P_1, P_2, P_3)$, which is related to our formula (7.1.5) by

$$C_0(P_1, P_2, P_3) \propto \frac{C_{DOZZ}(P_1, P_2, P_3)}{\left(\prod_{k=1}^3 S_0(P_k)\rho_0(P_k)\right)^{\frac{1}{2}}}, \quad (7.1.26)$$

with a proportionality constant independent of $P_1, P_2, P_3$, and $S_0$ is the ‘reflection coefficient’ defining the normalisation of the vertex operators through the two-point function\(^\text{10}\)

$$\langle V_{P_1}(0)V_{P_2}(1) \rangle = 2\pi\delta(P_1 - P_2)S_0(P_1). \quad (7.1.29)$$

Since the theory is noncompact, there is in fact no canonical normalisation of operators, and only the combination (7.1.26) (up to the $P$-independent normalisation) is unambiguously determined from the bootstrap.

\(^\text{10}\)The proportionality constant is

$$\frac{(\pi\mu\gamma(b^2)b^{2-2\nu})^{\frac{P}{2}}}{2\frac{\pi}{4}} \frac{\Gamma_b(2Q)}{\Gamma_b(Q)} \quad (7.1.27)$$

and the reflection coefficient is

$$S_0(P) = \frac{(\pi\mu\gamma(b^2)b^{2-2\nu})^{-2iP/b}}{\Gamma_b(Q + 2iP/\Gamma_b(Q - 2iP)) \Gamma_b(Q + 2iP/\Gamma_b(-2iP))}, \quad (7.1.28)$$

where $\mu$ is the Liouville cosmological constant and $\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}$. 

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The denominator can be understood as a change of measure on the space of states, from the one defined by (7.1.29) to a natural one proportional to $dP \rho_0(P)$ (see footnote 16).

Given this relation, one might be tempted to interpret our result as describing the precise sense in which Liouville theory captures the universal dynamics of heavy operators, a point of view that has been advocated in the context of holographic theories in $[158,251]$. We should not, however, interpret this too literally, since $C_{\text{DOZZ}}$ has a very different interpretation to $C_0$. In particular, Liouville theory has only scalar primary operators, with OPE coefficients $C_{\text{DOZZ}}$, whereas our results give OPE coefficients for all spins, from a product of two copies of $C_0$ (left- and right-moving). Indeed, a unitary compact CFT with $c > 1$ will necessarily contain primary operators with arbitrarily large spin $[110]$, and Liouville theory falls outside the scope of our asymptotic formula precisely because it is not compact. Rather, we regard the relation (7.1.26) as a consequence of the fact that Liouville dynamics is governed by precisely the same Virasoro representation theory that determines our asymptotic formula, as we now explain.

Liouville theory is distinguished by having only scalar Virasoro primary states. In this sense, it is analogous to the A-series or diagonal minimal models which exist for degenerate values of $c < 1$, and have a spectrum of scalar primaries (finitely many in that case). The restriction to scalars is sufficient to uniquely specify the theory, since it determines a unique solution to the bootstrap (up to normalisation of operators and a decoupled TQFT). Furthermore, this solution is given explicitly in terms of the identity fusion kernel by a relation precisely of the form (7.1.26), which is determined by representation theoretic considerations. We give an argument that can be applied both to four-point crossing symmetry and to modular covariance of torus one-point functions. This type of argument for four-point crossing is not new (see $[252]$, for example), but the version for torus one-point functions is novel, as far as we are aware.$^{11}$ We sketch the arguments here, giving more detailed explanations of the relevant identities in section 7.5.

To outline the argument for uniqueness, we first write the crossing equation (7.1.14) including left- and

$^{11}$ We thank S. Ribault for correspondence.
right-moving dependence explicitly as

$$\rho'(P', \bar{P}') = \int [dPd\bar{P}] K_{\rho' P} \rho(\bar{P}, P) \rho(P, \bar{P}).$$

(7.1.30)

Here, the densities $\rho, \rho'$ denotes a spectral density for internal operators in either the four-point function or the torus one-point function, and $\mathbb{K}$ is either a fusion kernel $\mathbb{F}$ or a modular $S$-transform $\mathbb{S}$, as discussed in sections 7.3.2 and 7.3.3 respectively. We can schematically write this as a matrix equation

$$\rho' = \mathbb{K} \rho \mathbb{K}^\dagger,$$

(7.1.31)

where the rows and columns of $\rho$ are labelled by $P, \bar{P}$ respectively, and similarly for $\rho'$. Now, if we assume that the spectrum contains only scalars, then $\rho$ and $\rho'$ are diagonal (nonzero only for $P = \bar{P}$). In that case, we can choose to use a different normalisation for the conformal blocks, and hence fusion kernel, that absorbs factors of $\rho^{1/2}, (\rho')^{-1/2}$ into the columns and rows of $\mathbb{K}$: $\hat{\mathbb{K}} = (\rho')^{-1/2} \mathbb{K} \rho^{1/2}$. With this normalisation, the crossing equation becomes $\hat{\mathbb{K}} \hat{\mathbb{K}}^\dagger = 1$, so that $\hat{\mathbb{K}}$ is unitary (after restricting to the support of $\rho, \rho'$). Such a normalisation exists for the fusion kernel [156], thus determining a scalar solution of crossing. This solution reproduces the DOZZ formula up to the ambiguities of normalisation. Moreover, the only way that this solution can fail to be unique is if $\hat{\mathbb{K}}$ is block diagonal in the $P$ basis.\(^\text{12}\)

For the final step, we must relate the unitary normalisation of $\mathbb{K}$ to the identity fusion kernel. For four-point crossing, such a relation follows from a special case of the pentagon identity satisfied by the fusion kernel. The identity representation is picked out by its simple fusion rule, which implies that the fusion kernel with an external identity operator is trivial. For the torus one-point function, we have a similar identity relating the modular $S$-matrix and fusion kernel. We give the explicit forms of these identities and their derivations in section 7.5, along with arguments explicitly verifying them from the closed-form

\(^{12}\)In fact, the Virasoro fusion kernel is block diagonal, since the degenerate representations form an invariant subspace. If we relax the assumption of unitarity, this leads to a second solution to crossing, namely the ‘generalized minimal model’ [252].
expressions [156,157,166] for the Virasoro fusion and modular kernels.

7.1.7 Discussion

Before moving on to a derivation of our formula, we discuss a few final interesting features of our result.

While our asymptotic formula (7.1.5) might look arbitrary, it is in fact extremely highly constrained if we assume analyticity. In fact, equation (7.1.5) is almost completely determined by its analytic structure and simple physical considerations. To see this, we note that $C_0(P_i, P_j, P_k)$ is a meromorphic function of its arguments which has

- Zeroes when $P_i = i \frac{Q^2}{2} + \frac{i}{2} (rb + sb^{-1})$ with $r, s \in \mathbb{Z}_{\geq 0}$,
- Poles when $P_i = P_j + P_k \pm i \frac{Q^2}{2} + i (mb + nb^{-1})$ with $m, n \in \mathbb{Z}_{\geq 0}$,

and is invariant under reflections $P_i \rightarrow -P_i$ and permutations of the $(P_i, P_j, P_k)$. These zeros occur precisely when $O_i$ has a null Virasoro descendant at level $rs$. The poles occur precisely when the weights of $O_i$ are equal to the weights of a double twist operator built out of $O_j$ and $O_k$ [229]. A meromorphic function is uniquely determined by its poles and zeroes, up to the exponential of a polynomial. Thus in retrospect, once one postulates the existence of a meromorphic function that interpolates between the asymptotic regimes, one could have completely determined $C_0(P_i, P_j, P_k)$ up to the exponential of a polynomial in the $(P_i, P_j, P_k)$, simply by demanding the existence of zeros at null states and poles at double twist operators. One might even argue that this polynomial must be a constant in order to guarantee the convergence of the operator product expansion (although this argument is subtle because we are varying the $(P_i, P_j, P_k)$ as complex variables independently). This suggests that the function $C_0(P_i, P_j, P_k)$ can be completely determined by analyticity and simple physical constraints.

We will now move on to the derivation of our result. We begin in section 7.2 with a detailed warm-up exercise, where we describe the derivation of various versions of Cardy’s formula using the crossing kernel for modular transformations. We then proceed to discuss the Moore-Seiberg procedure in more detail in
section 7.3.1, before turning to the elementary crossing kernels in sections 7.3.2 and 7.3.3. We apply this to compute higher genus crossing kernels and OPE asymptotics in section 7.4. Large central charge limits, and comparisons to the literature, are discussed in section 7.6. Section 7.7 discusses the computation of the average value of the light-heavy-heavy OPE coefficients using the modular covariance of torus one-point functions. We relegate some details of the elementary crossing kernels and their asymptotics to the appendices.

7.2 Cardy’s formula from crossing kernels

To illustrate the main idea of this chapter, we first revisit the derivation of the Cardy formula for primary states (and its large-spin version [229, 253–256]) using the modular S-matrix, a strategy which we will generalize in later sections. We follow the presentation and notation of [256], which contains some more details and applications. The relationship between the Cardy formula and the modular S-matrix was first elucidated in [211].

7.2.1 Natural variables for Virasoro representation theory

As a preliminary, we introduce a parameterization of the CFT data that turns out to be natural for the representation theory of the Virasoro algebra. The central charge $c$ can be written in terms of a “background charge” $Q$ or “Liouville coupling” $b$ as

$$c = 1 + 6Q^2 = 1 + 6(b + b^{-1})^2. \quad (7.2.1)$$

We will make the choice that $c > 25$ corresponds to $0 < b < 1$, while $1 < c < 25$ corresponds to $b$ a pure phase in the first quadrant. To label Virasoro representations we use a variable $P$, or sometimes the equivalent
\[ \alpha = \frac{Q}{2} - iP, \text{ which is related to the more usually seen conformal weight by} \]

\[ h = \left( \frac{Q}{2} \right)^2 + P^2 = \alpha(Q - \alpha), \quad (7.2.2) \]

and similarly \( \bar{P} \) or \( \bar{\alpha} \) in place of \( \bar{h} \). Two things about this parameterization should be noted. First, it is redundant, being invariant under the reflection reflections \( P \to -P \) (or \( \alpha \to Q - \alpha \)). Secondly, it naturally splits unitary values of the weights (\( h \geq 0 \)) into two distinct ranges: \( h \geq \frac{c-1}{24} \) corresponds to real \( P \) (or \( \alpha \in \frac{Q}{2} + i\mathbb{R} \)), and \( 0 \leq h < \frac{c-1}{24} \), which corresponds to imaginary \( P \) (or \( \alpha \in (0, \frac{Q}{2}) \)).

### 7.2.2 The partition function and density of primary states

Now consider the torus partition function of a compact\(^{13}\) CFT with \( c > 1 \). The partition function encodes the spectrum of the theory, admitting a decomposition into Virasoro characters:

\[ Z(\tau, \bar{\tau}) = \chi_1(\tau) \bar{\chi}_1(\bar{\tau}) + \sum_i \chi_{P_i}(\tau) \bar{\chi}_{\bar{P}_i}(\bar{\tau}) \quad (7.2.3) \]

The sum runs over Virasoro primary states labelled by \( i \), with conformal weights labelled by \( P_i, \bar{P}_i \), and the nondegenerate Virasoro characters \( \chi_P \) packaging together all states in a conformal multiplet are given by

\[ \chi_P(\tau) = \frac{q^{P^2}}{\eta(\tau)}, \quad (7.2.4) \]

where \( q = e^{2\pi i \tau} \). The identity character \( \chi_1 \) is distinguished because the corresponding representation is degenerate (\( L_{-1} \) annihilates the vacuum state), so

\[ \chi_1(\tau) = \chi_{\frac{1}{2}(b-1+\bar{b})}(\tau) - \chi_{\frac{1}{2}(b-1-\bar{b})}(\tau) = \frac{q^{-\frac{Q^2}{2}}(1-q)}{\eta(\tau)}. \quad (7.2.5) \]

\(^{13}\)By “compact,” we mean a CFT with a normalizable \( SL(2, \mathbb{C}) \)-invariant vacuum state and a discrete spectrum of Virasoro primary operators.
If there are any other conserved currents (operators with $h = 0$ or $\bar{h} = 0$) in the theory, we should similarly use this degenerate character for either the left- or right-moving half.

We can rewrite the character decomposition of the partition function in terms of a density of primary states $\rho$, writing

$$Z(\tau, \bar{\tau}) = \int \frac{dP}{2} \frac{d\bar{P}}{2} \rho(P, \bar{P}) \chi_P(\tau) \bar{\chi}_{\bar{P}}(\bar{\tau}),$$

(7.2.6)

where $\rho$ is a distribution given by a sum of delta-functions $\delta(P - P_i)\delta(\bar{P} - \bar{P}_i)$ for each primary. Using the reflection symmetry, we make the choice that $\rho$ is an even distribution, so each primary contributes four terms related by reflections in $P, \bar{P}$, and we introduce the factors of $\frac{1}{2}$ in the integrals to avoid overcounting.

It is also convenient to always use nondegenerate characters in the expansion, so for the identity (and other currents, if present), $\rho$ includes delta-functions with negative weight at $P, \bar{P} = \pm \frac{1}{2}(b^{-1} - b)$ to subtract the null descendants. Finally, we note that $\rho$ is a somewhat unconventional distribution, since it has support at imaginary values for operators with $h, \bar{h} < \frac{c-1}{24}$. This is nonetheless rigorously defined if we integrate against analytic test functions, of which the characters should form a complete set in an appropriate topology (see [256] for more details).

### 7.2.3 The modular S-transform

Locality of a CFT implies invariance of the torus partition function under the modular S-transform

$$Z(-1/\tau, -1/\bar{\tau}) = Z(\tau, \bar{\tau}),$$

(7.2.7)

which in turn constrains the allowed CFT spectrum. We will reformulate this constraint directly on the density of states $\rho(P, \bar{P})$. To do this, first note that the modular S-transformation $\tau \to -1/\tau$ acts on
individual characters as a Fourier (cosine) transform in the momentum:

\[ \chi_P(-1/\tau) = \int \frac{dP'}{2} \chi_{P'}(\tau) S_{P'P}[\mathbb{1}] \]

\[ S_{P'P}[\mathbb{1}] = 2\sqrt{2}\cos(4\pi PP') \quad (7.2.8) \]

The kernel of this integral transform is the ‘modular S kernel’ \( S_{P'P}[\mathbb{1}] \), where the \([\mathbb{1}]\) label indicates that the partition function is a trivial example of the torus one-point function of the identity operator, with the generalization to nontrivial operators to follow. The notation emulates the situation in rational CFTs, where there are a finite number representations, so the modular kernel \( S[\mathbb{1}] \) becomes a finite-dimensional matrix.

Given a function \( Z(\tau, \bar{\tau}) \) expanded in characters using a density of primary states as in (7.2.6), we can take a modular S-transform and use the kernel (7.2.8) to rewrite the transformed characters:

\[ Z(-1/\tau, -1/\bar{\tau}) = \int \frac{dP}{2} \frac{dP'}{2} \frac{d\bar{P}}{2} \frac{d\bar{P}'}{2} S_{P'P}[\mathbb{1}] S_{\bar{P}'\bar{P}}[\mathbb{1}] \rho(P, \bar{P}) \chi_{P'}(\tau) \bar{\chi}_{P'}(\bar{\tau}) \]

\[ (7.2.9) \]

Exchanging order of integration between the primed and unprimed variables, we can interpret this as an expansion (7.2.6) of the modular transformed function with a transformed density of primary states:

\[ \tilde{\rho}(P', \bar{P}') = \int \frac{dP}{2} \frac{d\bar{P}}{2} S_{P'P}[\mathbb{1}] S_{\bar{P}'\bar{P}}[\mathbb{1}] \rho(P, \bar{P}) \]

\[ (7.2.10) \]

Since the partition function uniquely determines the spectrum, this equation expresses the modular S-transform as a Fourier transform acting on the density of primary states \( \rho \).\(^{14}\) In particular, a physical spectrum corresponding to a modular invariant theory is invariant under this Fourier transform:

\[ \text{Modular invariance} \iff \tilde{\rho}(P, \bar{P}) = \rho(P, \bar{P}) \]

\[ (7.2.11) \]

\(^{14}\text{We can strip off the characters since, by assumption, they are complete in the relevant space of test functions. This just means that a distribution is defined by its integral against all characters, i.e. its corresponding partition function. The same applies for the more complicated transforms we encounter later.}\)
From (7.2.10), we can think of the modular S-matrix as the contribution of a single operator to the density of states in the transformed channel. The only exception to this is the degenerate representations with $\hbar = 0$ (or $\bar{\hbar} = 0$), so we introduce an ‘identity S-matrix’

$$S_{P1}[1] = S_{P^{'1}{(b - 1) + b}}[1] - S_{P^{'1}{(b - 1) - b}}[1] = 4\sqrt{2}\sinh(2\pi b P)\sinh(2\pi b^{-1} P),$$

(7.2.12)

which encodes the contribution of such a degenerate state. The density of states $S_{P1}[1]S_{\bar{P}_1}[1]$ dual to the vacuum will be of central importance for us.

### 7.2.4 Cardy formulas

The density of states $\rho(P, \bar{P})$ is a sum of delta-functions for each primary operator, so for a modular invariant spectrum, by taking the S-transform we can instead write it as a sum over modular S-matrices:

$$\rho(P, \bar{P}) = S_{P1}[1]S_{P1}[1] + \sum_i S_{PP}[1]S_{\bar{P}_i}[1]$$

(7.2.13)

We have not explicitly included any nontrivial primary currents, which would contribute the identity S-matrix in $P$ and the nondegenerate S-matrix in $\bar{P}$ or vice versa. If such currents are present, it is most natural to organize the states into multiplets of an extended algebra, under which all currents are descendants of the vacuum, and use the modular S-matrix pertaining to the extended algebra.

Now consider this sum in the limit of large $P$ and/or $\bar{P}$. In this limit, the relative importance of the terms is determined by $P_i, \bar{P_i}$: for a state with $0 < h < \frac{c - 1}{24}$, the relevant S-matrix is exponentially suppressed relative to the vacuum:

$$\frac{S_{PP}[1]}{S_{P1}[1]} \sim \begin{cases} e^{-4\pi \alpha' P} & \alpha' = \frac{Q}{2} + iP' \in (0, \frac{Q}{2}) \\ 2\cos(4\pi PP')e^{-2\pi QP} & P' \in \mathbb{R} \end{cases}$$

as $P \to \infty$

(7.2.14)

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From this, we find (at least naively; we revisit this more carefully at the end of the section) that the density of states at large \( P, \bar{P} \) asymptotically approaches the vacuum S-matrix:

\[
\rho(P, \bar{P}) \sim \rho_0(P)\rho_0(\bar{P}) \quad \text{as} \quad P, \bar{P} \to \infty, \quad \text{where} \quad \rho_0(P) := \mathbb{S}_{P^4[1]} \sim \sqrt{2}e^{2\pi Q P}
\]  

(7.2.15)

This is of course nothing but Cardy’s formula for the asymptotic density of primary states at large dimension, correct up to corrections exponential in \( \sqrt{h}, \sqrt{\bar{h}} \) coming from the lightest non-vacuum primary state.\(^{15}\)

With this derivation, it becomes clear that the Cardy formula (7.2.15) is also valid in a ‘large spin’ regime where we fix \( h \) and take \( \bar{h} \to \infty \) \([229,253–256]\). In this limit, the relative suppression (7.2.14) of non-vacuum blocks is controlled by ‘barred’ dimension only, so we require the additional assumption of a ‘twist gap’ (\( \bar{h} \) is bounded away from zero for all non-vacuum operators, so in particular there are no extra conserved currents).

In this limit, for any fixed \( h > \frac{c-1}{24} \), the density of states grows with spin \( \ell \) as \( e^{2\pi \sqrt{\frac{c-1}{24}} \ell} \), with a prefactor determined by \( \rho_0(P) \); for any \( h < \frac{c-1}{24} \), this prefactor is formally zero, which means that the density grows more slowly (perhaps still exponentially in \( \sqrt{\ell} \), but with a smaller coefficient).

We therefore find that the asymptotic spectrum of CFTs is quite generally determined by the simple formula

\[
\rho_0(P) = \mathcal{S}_{P^4[1]} = 4\sqrt{2}\sinh(2\pi b P)\sinh(2\pi b^{-1} P), 
\]  

(7.2.16)

which we refer to as the ‘universal density of states’ for \( c > 1 \) compact CFTs without extended current algebras. Our derivation emphasizes that this object comes from the representation theory of the Virasoro algebra, describing the decomposition of the trivial representation after modular transformation.\(^{16}\) In the remainder of the chapter, we will show that another representation theoretic object similarly controls the OPE coefficients in a variety of limits.

\(^{15}\)This is the density in the \( P, \bar{P} \) variables, so a Jacobian is required to convert to density in \( h, \bar{h} \). For an asymptotic formula in dimension \( \Delta = h + \bar{h} \) only, insensitive to spin, one simply integrates (7.2.15) over the circle \( P^2 + \bar{P}^2 = \Delta - \frac{c}{12} \), obtaining the Bessel function formulas of, for example, [110,232].

\(^{16}\)In fact, \( \rho_0 \) has a purely representation theoretic characterization: it is the Plancherel measure on the space of representations of the Virasoro algebra [156].
Now, our argument for the asymptotic formula (7.2.15) was very imprecise, and indeed the result is simply false if interpreted literally, so we briefly discuss the sense in which it holds. The equation (7.2.13) expressing the density of states as a sum of modular S kernels does not converge in the usual sense (and uniform convergence would be necessary for our argument to apply immediately), and since $\rho$ is a sum of delta functions, it does not have smooth asymptotic behaviour. Rather, the sum converges in the sense of distributions (it should converge when integrated against any test function), which requires some ‘smearing’, and the asymptotic formulas should be interpreted accordingly. The most conservative statement is that the formula applies in an integrated sense: the total number of states below a given energy or spin is asymptotic to the integral of the Cardy formula (see [232–234] for a more detailed discussion and rigorous results). In the particular case of the Cardy formula, a very interesting recent paper [232] has shown that if the averaging window is of fixed width in the large dimension limit, corrections due to the finite size of the averaging window only affect the order-one term in the expansion of the logarithm of the density of states at large dimension. For chaotic theories, we expect the far stronger statement that the asymptotic formula applies to a microcanonical density of states averaged over a small window (we require only that the window contains parametrically many states, so its width can shrink as fast as $e^{-S}$); this is a consequence of the eigenstate thermalization hypothesis [202, 235]. The exact interpretation of our asymptotic formulas is not the focus of this chapter, so we will henceforth leave this aspect for future study.

### 7.3 Crossing equations for general correlation functions

We now extend this formulation of modular invariance as a transform on the density of states, discussed in section 7.2, to its most general context as a similar formulation of all consistency conditions of CFT$_2$. 
7.3.1 The Moore-Seiberg construction

In two dimensional CFTs, the most general correlation function of local operators, comprising $n$ operators $O_1, \ldots, O_n$ on a surface $\Sigma_g$ of arbitrary genus $g$ (which we denote by $G_{g,n}$), can be formulated entirely in terms of the basic data of the theory, namely the spectrum and OPE coefficients of primary operators.\footnote{This excludes correlation functions on surfaces with boundaries and/or nonorientable surfaces, both of which require additional data.} Note that this is far better than the situation in higher dimensions, where it is unclear how to determine general correlation functions, even on conformally flat manifolds such as the torus $(S^1)^d$, in terms of data of the theory on $\mathbb{R}^d$. Here, we review the construction of general correlation functions, and the crossing relations required to consistently formulate the theory on an arbitrary surface.

The basic strategy is to break the surface into simple constituent pieces, separated by circular boundaries, and insert a complete set of states along each boundary. First, we insert a circle surrounding each operator insertion; by the state-operator correspondence, the operator insertion is equivalent to deleting a disc to produce a boundary, and projecting onto the corresponding state on that boundary. Label the resulting $n$ boundaries by an index $e \in \mathcal{E}$ (for ‘external’) and let $k_e$ denote the operator on each boundary, falling in Virasoro representations $P_{k_e}, \bar{P}_{k_e}$.

We are then left with a genus $g$ surface with $n$ boundaries, which we can decompose into $2g + n - 2$ pairs of pants (that is, topological 3-holed spheres, occasionally called ‘trinions’), which we label by indices $t \in \mathcal{T}$, by cutting along a further $3g + n - 3$ circles. Along each of these $3g + n - 3$ ‘cuffs’ where the pants are joined to one another, labelled by an index $i \in \mathcal{I}$ (for ‘internal’), we insert a complete set of states. Each term in the sum over states is then a product of amplitudes for each pair of pants, which can be conformally mapped to sphere three-point functions, and thus is fixed by the structure constants of the corresponding Virasoro primaries.

The contribution of descendants propagating along each cuff is completely fixed by Virasoro symmetry, proportional to the OPE coefficients of the primaries from which they descend. We may therefore package
together the contribution of all descendants of a particular set of primaries (labelled by \( \{ k_i \}_{i \in \mathcal{I}} \)) together, into a ‘conformal block’. In other words, this is the sum over states described above, but restricting the states along each cuff \( i \) to some chosen multiplet of the symmetry, in the representation \( P_{k_i}, \bar{P}_{k_i} \). By construction, the blocks are purely kinematic, depending on the surface \( \Sigma_g \) and the pair of pants decomposition, the locations of operator insertions, the central charge, and the conformal weights \( P_{k_e}, \bar{P}_{k_e} \) and \( P_{k_i}, \bar{P}_{k_i} \) labelling the representations of the \( n \) external and \( 3g+n-3 \) internal operators. Since the conformal algebra factorizes into holomorphic and antiholomorphic sectors, the blocks also factorize in this way, so we can write them as a product \( \mathcal{F} \bar{\mathcal{F}} \): \( \mathcal{F} = \mathcal{F}[P_e](P_\sigma) \) depends on the \( n \) external representations \( P_e \) (for \( e \in \mathcal{E} \)), the \( 3g+n-3 \) internal representations \( P_i \) (for \( i \in \mathcal{I} \)), and kinematic variables collectively labelled by \( \sigma \); we similarly have \( \bar{\mathcal{F}} = \bar{\mathcal{F}}[\bar{P}_\ell](\bar{P}_\bar{\sigma}) \). For Euclidean correlation functions, the kinematic variables \( \sigma \) are (once a conformal frame has been specified) \( 3g-3+n \) complex numbers parameterizing the complex structure moduli of \( \Sigma_g \) and complex coordinates of the locations \( x_e \) of operator insertions, and \( \bar{\sigma} \) are complex conjugates of \( \sigma \); more generally, \( \sigma \) and \( \bar{\sigma} \) need not be related in this way (for example, for Lorentzian kinematics they often become independent and real ‘lightcone’ coordinates).

The dynamical data of the theory appears through the spectrum of operators, and the OPE coefficients \( C_{\partial t} \) for each pair of pants \( t \in \mathcal{T} \), where \( \partial t \) denotes a triple of indices \( k_e \) or \( k_i \) labelling the primary operators propagating in the three cuffs bounding \( t \). The result is an expression of the following form for the correlation

---

18 The blocks (and the correlation functions) depend on the metric on the surface in two distinct ways. Firstly, there are finitely many moduli (the \( 3g+n-3 \) complex parameters \( \sigma \)) determining the metric and operator locations up to equivalence under diffeomorphisms and Weyl transformations \( g \mapsto e^{2\pi i g} \), upon which the correlation function and blocks depend nontrivially. Secondly, there is the choice of metric within each such conformal class, which changes the correlation function only by kinematic factors: the conformal anomaly, and local conformal factors for each operator.

19 In fact, the decomposition into pairs of pants is not quite sufficient to determine the blocks. A Dehn twist, a relative rotation by angle \( 2\pi \) around a cuff, introduces phases \( e^{2\pi i (h-c/24)} \) and \( e^{-2\pi i (\bar{h}-\bar{c}/24)} \) in \( \mathcal{F} \) and \( \bar{\mathcal{F}} \) respectively, so extra topological data is needed to keep track of these relative phases. When we combine blocks into the product \( \mathcal{F} \bar{\mathcal{F}} \) with \( c-\bar{c} \in 24\mathbb{Z} \) (here, we always have \( c = \bar{c} \)) and integer spin \( (\hat{h}-h \in \mathbb{Z}) \), this ambiguity cancels. We also require this extra data to fix an ambiguity in ordering of OPE coefficients, which pick up a sign under odd permutations of indices if the total spin is odd: \( C_{\pi(1)\pi(2)\pi(3)} = \text{sgn}(\pi)^{\ell_1+\ell_2+\ell_3} C_{123} \) for \( \pi \in S_3 \). Relatedly, note that the condition for unitarity is \( C_{123} C_{321} \geq 0 \), so for total odd spin \( \ell_1 + \ell_2 + \ell_3 \), \( C_{123} \) is pure imaginary.
Figure 7.4: A conformal block decomposition of the torus two-point function $G_{1,2}$, where the kinematic parameters $\sigma$ consist of a complex structure $\tau$ for the torus, and a separation $w$ between operators. We sum over representations in the internal cuffs; for the yellow cuff $i_1$, this corresponds to the operators appearing in the OPE of external operators $e_1, e_2$, and for the blue cuff $i_2$, an insertion of a complete set of states in the thermal trace.

$$G_{1,2}(\tau, w, \bar{\tau}, \bar{w}) = \sum_{i_1} \sum_{i_2} C_{e_1 e_2 i_1} C_{i_1 i_2 i_2} \mathcal{F}[P_{e_1}, P_{e_2}](P_{i_1}, P_{i_2}|w, \tau) \mathcal{F}[\bar{P}_{e_1}, \bar{P}_{e_2}](\bar{P}_{i_1}, \bar{P}_{i_2}|\bar{w}, \bar{\tau})$$

The OPE coefficients $C_{e_1 e_2 i_1}, C_{i_1 i_2 i_2}$ are associated with the pairs of pants labelled $A, B$ respectively, with $\partial A = (e_1, e_2, i_1)$ and $\partial B = (i_1, i_2, i_2)$.

The last line defines a ‘spectral density’ $\rho_{\text{spec}}$ analogous to the density of states in (7.2.6), now with several internal operators, weighted by OPE coefficients; the ‘reflections’ refers to an additional three terms with $P_{k_i} \rightarrow -P_{k_i}$ and/or $\bar{P}_{k_i} \rightarrow -\bar{P}_{k_i}$ so that $\rho_{\text{spec}}$ is an even function of these variables. This general case is rather abstract, but we will ultimately be interested in a few simple instances, for which we write concrete versions of (7.3.1) in later sections; for now, one illustrative example is shown in figure 7.4.

While our quick argument is sufficient to demonstrate that the conformal blocks exist, and are determined by Virasoro symmetry, it is another matter entirely to actually compute them. Closed form expressions are known only in very special cases. The most efficient way to compute them numerically is via recursion relations [59, 60, 114, 138], but even these are organized using different kinematic parameters and conformal
frames for different channels, so it remains a challenging task to formulate crossing symmetry using them. The technical obstacles remain formidable even with the simplification of large central charge, where there are still few analytic results, and one must also confront the possibility of Stokes phenomena that are not well understood [163,229,241]. Fortunately, we will see later that for our purposes, it is not required to know anything about the blocks directly!

While we have a systematic procedure for constructing the correlation functions by sewing pairs of pants, it is far from unique, since there are infinitely many distinct ways to decompose a surface into pairs of pants. We refer to a choice of decomposition as a “channel”, each channel giving rise to a corresponding conformal block decomposition of the correlation function. Consistency requires that the conformal block decompositions (7.3.1) give the same result for the correlation function, whichever channel we choose to use. This is a generalized statement of crossing symmetry or modular invariance, which imposes strong constraints on the data of the CFT.

To formulate this notion of crossing symmetry more directly in terms of the data of the CFT, we must first consider how to relate the block decompositions in different channels. Following the work of Moore and Seiberg [37,38], we can relate any two of the infinite collection of possible channels by repeated composition of a small number of elementary ‘moves’, which can be described by purely topological relationships between pairs of pants decompositions. We will make use of two such moves, ‘fusion’ and ‘modular S’ (or just S), illustrated and described in figure 7.5, along with an example where the two are composed.\(^{20}\)

Now, we may informally think of the set of conformal blocks in any particular channel, labelled by the set of internal representations \(\{P_i\}_{i \in \mathcal{I}}\), as forming a basis for correlation functions. Given a second channels, with a new set of internal cuffs \(\mathcal{I}'\), there should be a change of basis matrix to the new variables \(\{P'_{i'}\}_{i' \in \mathcal{I}'}\), relating the two corresponding sets of blocks. From this point of view, it is plausible that the conformal

\(^{20}\)For a complete set of moves, we also require ‘braiding’, which acts on any two joined pairs of pants by adding a half twist to the separating cycle. The extra topological data required to fix the phases from footnote 19 is also necessary to uniquely prescribe the fusion/braiding moves among the infinitely many ways to split a sphere with four boundaries into two pairs of pants. It was only recently proved in [257] that fusion, braiding and S moves form a complete set of generators to relate any channels. We are grateful to Xi Yin for bringing [257] to our attention.
Figure 7.5: The elementary crossing moves relate different pair-of-pants decompositions of the four-punctured sphere and the once-punctured torus, or more generally anywhere that these appear as pieces of any decomposition of a surface. The associated crossing kernels relate Virasoro conformal blocks in the corresponding channels. The fusion kernel (top) relates sphere four-point Virasoro blocks in the S- and T-channels, and the modular kernel (middle) relates torus one-point blocks in modular S-transformed frames. In the final line, we show an example relating two channels in the torus two-point function $G_{1,2}$ by composing these moves.
blocks in any two channels can be related by an integral transform, with some ‘crossing kernel’ $K$:

$$F[P_i](P_i|\sigma) = \int \left( \prod_{i \in I} \frac{dP_{i'}}{2} \right) F'[P_{i'}](P_{i'}|\sigma')K_{P_i, P_{i'}}[P_i]$$  (7.3.2)

We allow for a change of kinematic variables $\sigma \rightarrow \sigma'$ because natural variables (e.g. those appropriate for recursion relations) may be different in each channel. This equation is a generalization of the relationship (7.2.8) between characters in channels related by a modular transform, where the kernel $K[P_{\sigma}]$ was given by the modular S-matrix $S[\sigma]$. Furthermore, if we relate two channels by a composition of the elementary moves described above and in figure 7.5, the crossing kernel itself can be built by composing the kernels for the elementary moves.\textsuperscript{21} Remarkably, not only do these kernels exist, but for the elementary moves they are known in closed form! This is surprising and powerful when we consider how little analytic control we have regarding the conformal blocks. We will introduce these elementary kernels in the following subsections.

If the blocks are to be regarded as basis vectors, then the corresponding components of any particular correlation function are the OPE coefficients, as encoded in the spectral densities $\rho_{\text{spec}}$. Given a change of basis matrix $K$, we can therefore relate the spectral densities in two channels by an integral transform with kernel $K$, generalizing (7.2.10):\textsuperscript{22}

$$\rho'_{\text{spec}}(P_{i'}, \bar{P}_{i'}) = \int \left( \prod_{i \in \mathcal{I}} \frac{dP_i}{2} \frac{d\bar{P}_i}{2} \right) K_{P_i, P_{i'}}[P_i][\bar{P}_i, \rho_{\text{spec}}(P_i, \bar{P}_i)]$$  (7.3.3)

This is a direct statement of crossing or modular invariance, which makes no reference to the correlation function, the kinematics or the conformal blocks. As a corollary to the Moore-Seiberg construction, invariance under elementary moves implies invariance in complete generality, so four-point function crossing symmetry and torus one-point modular invariance for all operators suffice to prove consistency of a theory formulated on

\textsuperscript{21}For this, it is important that the same kernels apply for the elementary moves when the external operators are descendants of a given primary (which we sum over when these external legs become internal legs for a more complicated correlation function). This follows because descendant correlators can be obtained by acting with differential operators which are independent of the channel decomposition.

\textsuperscript{22}This requires that the space of blocks is not overcomplete, so there is no nontrivial linear combination of blocks that gives the zero correlation function. This is extremely plausible; for example, the short distance behaviour of the correlator should be determined by the minimal dimension on which the spectral density has support.
any surface. Nonetheless, more complicated correlation functions encode an infinite set of these constraints in a natural way, so more general crossing relations are still useful to learn about the theory, as we will see.

The elementary moves do not act freely on the space of channels, so they themselves are also highly constrained by the relations between moves. For example, we can consider a five-point function, made up of three pairs of pants, joined with two internal cuffs. Applying fusion moves alternately on each of the cuffs, we return to the original channel after five moves, and imposing that this combination of five \( F \)'s acts trivially gives us the ‘pentagon identity’. Assuming analyticity of the kernels, along with properties of degenerate representations, such identities suffice to determine the kernels uniquely \([105, 156, 157]\).

The considerations we have described here have been understood and exploited for several decades, but largely in the context of rational models, for which only finitely many representations appear, so the kernel \( K \) is a finite-dimensional matrix (for a review, see \([258, 259]\)). When applied to irrational theories, the technicalities are somewhat more subtle, and our aims must be more modest (we should certainly not hope to classify and solve all theories!), but this point of view nonetheless seems to be the most powerful way to formulate the constraints of crossing, even for irrational CFTs.

For the remainder of the section, we move beyond the abstract discussion to discuss more concretely the kernels for the elementary fusion and S moves, and their salient properties.

### 7.3.2 Elementary crossing kernels 1: fusion

The first of our elementary crossing moves arises when we consider the sphere four-point function

\[
G_{0,4}(z, \bar{z}) = \langle \mathcal{O}_1(z) \mathcal{O}_2(\bar{z}) \mathcal{O}_3(1) \mathcal{O}_4(\infty) \rangle_{S^2},
\]

where \( z, \bar{z} \) denote the conformal cross ratios. By successively taking the OPE between pairs of operators (corresponding to inserting a complete set of states in radial quantization), this can be written as sum over
products of three-point functions of pairs of the external operators and intermediate operators:

\[ G_{0,4}(z, \bar{z}) = \sum_{\mathcal{O}_s} C_{12s} C_{34s} \mathcal{F} \left[ \begin{array}{c} \rho_2 \\ \rho_1 \\ \rho_4 \\ \rho_3 \end{array} \right] (P_{s}|z) \bar{\mathcal{F}} \left[ \begin{array}{c} \rho_2 \\ \rho_1 \\ \rho_3 \\ \rho_4 \end{array} \right] (\bar{P}_s|\bar{z}) \]

\[ = \int \frac{dP_s}{2} \frac{d\bar{P}_s}{2} \rho_s(P_s, \bar{P}_s) \mathcal{F} \left[ \begin{array}{c} \rho_2 \\ \rho_1 \\ \rho_3 \\ \rho_4 \end{array} \right] (P_s|z) \bar{\mathcal{F}} \left[ \begin{array}{c} \rho_2 \\ \rho_1 \\ \rho_3 \\ \rho_4 \end{array} \right] (\bar{P}_s|\bar{z}), \tag{7.3.5} \]

where \( \mathcal{F} \left[ \begin{array}{c} \rho_2 \\ \rho_1 \\ \rho_3 \\ \rho_4 \end{array} \right] (P|z) \) are the S-channel Virasoro blocks. In the second line we have written this decomposition as an integral against the S-channel ‘spectral density’ \( \rho_s \) (leaving implicit the dependence on external operators), which for a discrete spectrum is a sum of delta-functions weighted by the OPE coefficients \( C_{12s} C_{34s} \); this is an example of the general decomposition (7.3.1), analogous to (7.2.6) for the partition function.

For this expression, we have chosen to take the OPE between operators \( \mathcal{O}_1 \) and \( \mathcal{O}_2 \), giving the S-channel expansion (equivalently, we decompose the four-holed sphere into two pairs of pants, with cuffs \( 1, 2, s \) and \( s, 3, 4 \)). But the result must be the same if we instead choose to use the T-channel expansion, taking the OPE of operators \( \mathcal{O}_2 \) and \( \mathcal{O}_3 \). This associativity of the OPE leads to the crossing equation:

\[ \int \frac{dP_s}{2} \frac{d\bar{P}_s}{2} \rho_s(P_s, \bar{P}_s) \mathcal{F} \left[ \begin{array}{c} \rho_2 \\ \rho_1 \\ \rho_3 \\ \rho_4 \end{array} \right] (P_s|z) \bar{\mathcal{F}} \left[ \begin{array}{c} \rho_2 \\ \rho_1 \\ \rho_3 \\ \rho_4 \end{array} \right] (\bar{P}_s|\bar{z}) \]

\[ = \int \frac{dP_t}{2} \frac{d\bar{P}_t}{2} \rho_t(P_t, \bar{P}_t) \mathcal{F} \left[ \begin{array}{c} \rho_2 \\ \rho_1 \\ \rho_3 \\ \rho_4 \end{array} \right] (P_t|1 - z) \bar{\mathcal{F}} \left[ \begin{array}{c} \rho_2 \\ \rho_1 \\ \rho_3 \\ \rho_4 \end{array} \right] (\bar{P}_t|1 - \bar{z}), \tag{7.3.6} \]

The T-channel spectral density \( \rho_t \) appearing here is similar to \( \rho_s \), but weighted by different OPE coefficients \( C_{41t} C_{23t} \). This is the crossing relation between the two pair of pants decompositions of the four-holed sphere pictured on the top line of figure 7.5.

Continuing to follow the philosophy we applied to modular invariance in section 7.2 and generalized in 7.3.1, we will rewrite the crossing equation directly as a transform relating S- and T-channel spectral densities. To do this, we require an object expressing the decomposition of the T-channel Virasoro blocks in terms of
S-channel blocks. This is the *fusion kernel* (or crossing kernel, or $6j$ symbol), with the defining relation

$$
\mathcal{F}\left[\begin{array}{c}
p_2
\hline
p_1
\end{array}\right]
\left(\begin{array}{c}
p_3
\hline
p_4
\end{array}\right)
(P_1 | 1 - z) = \int \frac{dP_s}{2} \mathcal{F}_{P_s P_s}[P_2 P_1] \mathcal{F}_{P_3 P_4}[P_s | z),
\tag{7.3.7}
$$

which is analogous to the relation (7.2.8) between the modular S-matrix and characters.

It is not a priori obvious that such an object should even exist, but it is a remarkable fact that it does, and an even more remarkable fact that it has been explicitly constructed by Ponsot and Teschner [105, 156, 157]. A closed form expression is given in (D.1.2) in appendix D.1, which contains the necessary technical results, many of which were derived in [229]. We discuss the most relevant properties in a moment.

With the fusion kernel $\mathcal{F}$ in hand, we can now write the crossing equation as a transform relating the spectral density in each channel, just as in (7.2.10):

$$
\rho_s(P_s, \bar{P}_s) = \int \frac{dP_t}{2} \frac{d\bar{P}_t}{2} \mathcal{F}_{P_s P_t} \rho_t(P_t, \bar{P}_t)
\tag{7.3.8}
$$

Here we have suppressed the notation labelling the external operators, but it should be borne in mind that the kernel of this transform depends on the external operator dimensions $P_{1,2,3,4}$.\textsuperscript{23}

Like the modular transform of the vacuum (7.2.16) was the most important object in section 7.2, the fusion transform of the vacuum will play a correspondingly central role for our new asymptotic formulas. This can only appear in the case that the external operator dimensions are equal in pairs, $P_1 = P_4$ and $P_2 = P_3$ (in the T-channel). In that case, the fusion kernel simplifies\textsuperscript{24} [229], and we find it convenient to

\textsuperscript{23}There is a similar transform to write the S-channel spectral density in terms of U-channel data (with density weighted by OPE coefficients $C_{13u}C_{u,24}$) using the braiding kernel. This is a fusion kernel conjugated by phases, which become signs for integer spins:

$$
\rho_s(P_s, \bar{P}_s) = \int \frac{dP_u}{2} \frac{d\bar{P}_u}{2} (-1)^{\ell_1 + \ell_4 + \ell_u + \ell_s} \mathcal{F}_{P_s P_u}[P_2 P_1] \mathcal{F}_{P_3 P_4}[P_u P_s] \rho_u(P_u, \bar{P}_u)
\tag{7.3.9}
$$

The resulting signs for odd spins are much the same as for U-channel inversion in [33], for example.

\textsuperscript{24}Unlike for the modular S-matrix in section 7.2, the fusion kernel for the identity can be obtained as a continuous $\epsilon_t \to 0$ limit of the generic fusion kernel (with external operators identical in pairs). This occurs because the null states continuously decouple (their OPE coefficients go to zero continuously as $\epsilon_t \to 0$). See footnote 27 for a more detailed comparison.
write it as
\[ F_{P_1;P_2}^{P_1P_2} = \rho_0(P_s)C_0(P_1, P_2, P_s), \] (7.3.10)
where \( \rho_0(P) \) is the density of states appearing as the modular S-transform of the vacuum (7.2.16). It turns
out that \( C_0 \) is then symmetric under the exchange of all three of its arguments, and has a simple explicit
expression in terms of the special function \( \Gamma_b \):
\[ C_0(P_1, P_2, P_3) = \frac{1}{\sqrt{2}} \frac{\Gamma_b(2Q) \prod_{\pm} \Gamma_b \left( \frac{Q}{2} \pm iP_1 \pm iP_2 \pm iP_3 \right)}{\prod_{k=1}^{3} \Gamma_b(Q + 2iP_k) \Gamma_b(Q - 2iP_k)} \] (7.3.11)
The \( \prod \) in the numerator denotes the product of the eight combinations related by the reflections \( P_k \rightarrow -P_k \).
The function \( \Gamma_b \) is a ‘double’ gamma function, which is meromorphic, with no zeros, and with poles at
argument \(-mb - nb^{-1}\) for nonnegative integers \( m, n \) (similarly to the usual gamma function, which has poles
at nonpositive integers).

If external operators are sufficiently light (specifically, \( \alpha_1 + \alpha_2 \leq \frac{Q}{2} \) or \( \alpha_3 + \alpha_4 \leq \frac{Q}{2} \)), the fusion kernel
has a new subtlety, arising from poles in \( P_s \) that cross the real axis. In order to maintain analyticity in the
parameters, the contour in the decomposition (7.3.7), which is implicitly taken to run along the real \( P_s \) axis,
must be deformed. We can take the deformed contour to run along the real \( P_s \) axis, but must additionally
include circles surrounding the poles which have crossed the axis, contributing residues. This gives rise to
a finite sum of S-channel operators with imaginary \( P_s \) \( (h_s < \frac{c-1}{24}) \) in the decomposition of the T-channel
conformal block. See [229] for more details. We can describe this by including a sum of \( \delta \)-functions supported
at imaginary \( P_s \) in the kernel \( F \) [256].

The non-vacuum kernels with T-channel dimension \( h_t > 0 \) will be important for us only to compare their
asymptotic contribution to the S-channel. The key result, established in [229], is precisely analogous to
(7.2.14) for the modular S-matrix:

\[
\frac{\mathcal{F}_{P_s P_t}}{\mathcal{F}_{P_s}} \approx \begin{cases} 
\alpha_t = \frac{Q}{2} + iP_t \in (0, \frac{Q}{2}) 
\quad & \text{as } P_s \to \infty 
\alpha_t \quad & P_t \in \mathbb{R} 
\end{cases}
\]

This result is accurate up to a factor independent of \(P_s\), see equation (D.2.3).

### 7.3.3 Elementary crossing kernels 2: modular S

The second elementary move is a modular transform applied to one-point functions of Virasoro primary operators on the torus

\[
G_{1,1}(-\tau, \bar{\tau}) = \langle \mathcal{O}_0 \rangle_{T^2(\tau, \bar{\tau})},
\]

where \(\tau\) labels the complex structure of the torus, and the conformal weight of the external operator is

\[
h_0 = \left(\frac{Q}{2}\right)^2 + P_0^2 = \alpha_0(Q - \alpha_0).
\]

The translation invariance of the torus means that the correlation function is independent of the location of the operator.

Generalizing the modular invariance of the torus partition function (which is the special case where the external operator \(\mathcal{O}_0\) is the identity), \(G_{1,1}\) transforms covariantly under modular transformations, in particular the S-transform \(\tau \mapsto -1/\tau\):

\[
G_{1,1}(-1/\tau, -1/\bar{\tau}) = \tau^{h_0} \bar{\tau}^{h_0} G_{1,1}(\tau, \bar{\tau})
\]

The factor \(\tau^{h_0} \bar{\tau}^{h_0} = |\tau|^\Delta e^{-i\ell_0 \text{arg} \tau}\) comes from rescaling and rotating the torus so the thermal circle becomes the spatial circle\(^{25}\). It occurs because the definition of the one-point function implicitly makes a choice of metric on the torus, namely the flat metric in which the spatial circle has length \(2\pi\); after modular transform, the cycle interpreted as the spatial circle changes, and hence the metric is rescaled. The discussion of

\(^{25}\)Performing this transform twice corresponds to rotating the torus through an angle \(\pi\) and gives a factor \((-1)^{\ell_0}\), from which we conclude that \(G_{1,1}\) is zero for operators with odd spin, since any nonzero expectation value would break this \(Z_2\) symmetry.
subsection 7.3.1 implicitly assumed that we use the same metric for every channel, so there were no such factors.

We can write this correlation function in terms of the usual CFT data by inserting a complete set of states on the spatial circle, and collecting the contributions from each Virasoro representation into torus conformal blocks $F[P_0](P|\tau)$ with internal primary weight $P$:

$$G_{1,1}(\tau, \bar{\tau}) = \sum_\mathcal{O} C_{\mathcal{O}\mathcal{O}_0} F[P_0](P|\tau) \bar{F}[\bar{P}_0](\bar{P}|\bar{\tau})$$

In the second line we have defined the thermal spectral density $\rho[\mathcal{O}_0]$ for the external operator $\mathcal{O}_0$, consisting of $\delta$-functions for each internal operator with coefficient $C_{\mathcal{O}\mathcal{O}_0}$, analogously to (7.2.6) and (7.3.5), and another special case of (7.3.1).

Reprising the same strategy, we will recast modular covariance as invariance of $\rho[\mathcal{O}_0](P, \bar{P})$ under an $S$-transform, directly generalizing (7.2.10) for the density of states. To do this, we introduce the torus one-point kernel, the object which decomposes torus one-point conformal blocks into the modular-$S$ transformed frame:

$$\tau^{h_0} F[P_0](P|\tau) = \int \frac{dP'}{2} F[P_0](P'| - 1/\tau) S_{P'\bar{P}}[P_0]$$

(7.3.16)

Given this object, the modular $S$ transformation acts on the spectral density as

$$\tilde{\rho}[\mathcal{O}_0](P', \bar{P}') = \int \frac{dP}{2} \frac{d\bar{P}}{2} S_{P'\bar{P}}[P_0] S_{P'\bar{P}}[\bar{P}_0] \rho[\mathcal{O}_0](P, \bar{P}),$$

(7.3.17)

and modular covariance of $G_{1,1}$ is stated as $\tilde{\rho}[\mathcal{O}_0] = \rho[\mathcal{O}_0]$.

Once again, we are fortunate to have an explicit expression for the modular $S$-kernel due to Teschner [166] (see also [167, 168]). We reproduce the precise formula in (D.1.8) of appendix D.1, where we demonstrate

\footnote{Explicitly, $F[P_0](P|\tau) = \text{Tr}_P(e^{2\pi i \tau L_0} \mathcal{O}_0)$, where the trace is taken over the representation of the Virasoro algebra with weight labelled by $P$, normalizing the expectation value of $\mathcal{O}_0$ in the lowest weight state to unity.}

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various important properties of the kernel, the most salient of which we now state.

Most important for us is that, like the fusion kernel, the modular S-kernel simplifies when the external operator is the identity, taking $h_0 \to 0$ ($P_0 \to \frac{iQ}{2}$). In this limit, we find that

$$S_{PP'}[P_0] \to S_{PP'}[1] = 2\sqrt{2} \cos (4\pi P P'),$$  \hspace{1cm} (7.3.18)

recovering the modular S-matrix for non-degenerate torus characters (7.2.8) from section 7.2. Note that the kernel relevant for inversion of the vacuum character, namely

$$S_{P1}[1] = 4\sqrt{2} \sinh(2\pi b P) \sinh(2\pi b^{-1} P)$$  \hspace{1cm} (7.3.19)

as in equation (7.2.12), is not recovered by a straightforward $\alpha' \to 0$ limit of (7.3.18), because the degenerate vacuum character is not given simply by the $h' \to 0$ limit of the non-degenerate character. This is unlike the fusion kernel, where the identity kernel is obtained by an $\alpha' \to 0$ limit of the generic kernel with external operators identical in pairs: in that case the null descendants continuously decouple in the $h_0 \to 0$ limit.

The second important property for us will be the behaviour of the kernel in the large dimension limit $P \to \infty$, which we normalize by the vacuum S-matrix $S_{P1}[1] \approx e^{2\pi Q P}$ for comparison:

$$\frac{S_{PP'}[P_0]}{S_{P1}[1]} \approx \begin{cases} 
    e^{-4\pi \alpha' P} P^{h_0} & \alpha' = \frac{Q}{2} + iP' \in (0, \frac{Q}{2}) \quad \text{as } P \to \infty \\
    e^{-2\pi Q P} \cos(4\pi P P') P^{h_0} & P' \in \mathbb{R}
\end{cases}$$  \hspace{1cm} (7.3.20)

These formulas, derived in appendix D.2.2, are accurate up to a constant (that is, independent of $P$) factor.

---

27Since $\langle k'|O_1L_1L_{-1}|h'\rangle = \frac{2k'+h_0(h_0-1)}{2^{k'}} \langle h'|O_0|h'\rangle$, we can take a vacuum limit in which the null descendant is decoupled by fixing $h' = -\frac{1}{2} h_0(h_0-1) \sim \frac{1}{2} h_0$ and taking $h_0 \to 0$. Indeed, taking a limit $\alpha_0, \alpha' \to 0$ with $\alpha' \sim \frac{1}{4} \alpha_0$, one can explicitly check that $S_{PP'}[P_0] \to S_{P1}[1]$ (for a derivation, see (D.1.14) and surrounding discussion). In contrast, for the fusion kernel we can take a more direct limit because the matrix elements $\langle h_1|O_1L_1L_{-1}|h_2\rangle \langle h_1|O_2|h_2\rangle = \frac{h_0}{2} \langle h_1|O_1|h_1\rangle \langle h_2|O_2|h_2\rangle$ go to zero as $h_1 \to 0$. 

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when the external operator is the identity.

7.4 OPE asymptotics from crossing kernels

Now that we have formulated the consistency conditions as statements about transforms of spectral densities, it is simple to repeat the arguments of section 7.2, which led to the Cardy formula, in a variety of new situations. Specifically, we study crossing for the three correlation functions which decompose into two pairs of pants, and extract asymptotic formulas for squares of OPE coefficients.

7.4.1 Sphere four-point function: heavy-light-light

For our first example, we study the constraints of crossing symmetry for the four-point function of pairwise identical operators. We have already introduced all the required definitions and results in subsection 7.3.2; in particular, we have the fusion transformation (7.3.7) relating S- and T-channel spectral densities,

$$\rho_s(P_s, \bar{P}_s) = \int \frac{dP_t}{2} \frac{d\bar{P}_t}{2} F_{P_s P_t} F_{\rho_s \rho_t}(P_t, \bar{P}_t),$$  \hspace{1cm} (7.4.1)

and the result (7.3.12) that the fusion kernel for operators of positive dimension $h_t > 0$ is exponentially suppressed compared to the identity at large $P_s$. This is precisely the same situation we had for the modular S-matrix when we derived the Cardy formula (7.2.15), so repeating that argument gives us an analogous result for the S-channel spectral density:

$$\rho_s(P_s, \bar{P}_s) \sim \tilde{F}_{P_s} \left[ \begin{array}{c} P_2 \\ P_3 \\ P_1 \end{array} \right] \tilde{F}_{P_s \bar{P}_s} \left[ \begin{array}{c} P_2 \\ P_3 \\ P_1 \end{array} \right], \hspace{1cm} P_s, \bar{P}_s \to \infty.$$  \hspace{1cm} (7.4.2)

This finding is not new, but was one of the main results of [229]. The focus of that paper was the large spin limit of fixed $P_s$ and $P_s \to \infty$, but we here emphasize that this also holds for large dimension (both $P_s, \bar{P}_s \to \infty$), in fact more generally since we need not assume existence of a twist gap in that case.
In higher dimensional CFTs, the analogous operation of expanding the T-channel identity block (which is simply the product of two-point functions) into the S-channel defines the spectrum and OPE coefficients of ‘double trace’ operators of mean field theory (MFT). The identity fusion kernel can therefore be thought of as a deformation of MFT to include Virasoro symmetry, and the corresponding spectral data was accordingly dubbed “Virasoro mean field theory” (VMFT) in \[229\]. The large-spin universality of the identity kernel is the \( d = 2 \) analogue of the result for \( d > 2 \) that there exist ‘double-twist’ operators whose dimensions and OPE coefficients approach those of MFT at large spin \[26,27\].

The analogy with double-twist operators in higher dimensions is sharpest for \( h < \frac{c - 1}{24} \). If the external operators \( \mathcal{O}_1, \mathcal{O}_2 \) have sufficiently low twist, then there are a finite number of trajectories that asymptote at large spin to discrete values of \( h < \frac{c - 1}{24} \); see [229] for details. There is also a continuum starting at \( h = \frac{c - 1}{24} \) described by the smooth VMFT OPE density, which has no known analog in higher dimensions.

For \( h > \frac{c - 1}{24} \), either fixed in the large spin limit or taken to be large simultaneously with \( \bar{h} \), the asymptotic spectrum encoded in the fusion kernel is a smooth function of \( P, \bar{P} \). Just as for the Cardy formula explained in section 7.2, (7.3.5) should then be interpreted as a microcanonical statement about the asymptotic spectral density integrated over a window of energies. We can translate the result to a microcanonical average of OPE coefficients, by dividing by the Cardy formula (7.2.15) giving the asymptotic density of primary states \( \rho(P_s, \bar{P}_s) \sim \rho_0(P_s) \rho_0(\bar{P}_s) \) in the relevant limits. Writing the identity fusion kernel in the form (7.3.10) of the universal density \( \rho_0(P_s) \) times \( C_0(P_1, P_2, P_s) \), we find that \( C_0 \) gives the microcanonical average of the OPE coefficients:

\[
[|C_{12s}|^2] \sim C_0(P_1, P_2, P_s) C_0(P_1, \bar{P}_2, \bar{P}_s), \quad P_s, \bar{P}_s \to \infty.
\] (7.4.3)

This result is valid for any two fixed operators \( \mathcal{O}_1, \mathcal{O}_2 \), averaging over operators \( \mathcal{O}_s \) in either a large dimension or large spin limit.
The asymptotic form of $C_0$ in this limit was computed in [229]:

$$C_0(P_1, P_2, P_3) \sim 2^{-4P_3^2} e^{-\pi Q P_3 \Delta_{P_1 + P_2}} \frac{2^{2Q^2-2} \Gamma_0(b) \Gamma_b(2Q)}{\Gamma_b(Q) \Gamma_b(Q + 2ip_1) \Gamma_b(Q - 2ip_1) \Gamma_b(Q + 2iP_2) \Gamma_b(Q - 2iP_2)}$$

(7.4.4)

where $\Gamma_0(b)$ is a special function that appears in the large-argument asymptotics of $\Gamma_b$; see appendix A of [229] for more details. The first factor exactly cancels a similar factor in the conformal blocks ($F_{(16q)h}$ [59]), ensuring that the block expansion has the correct domain of convergence. A formula of this form for the asymptotics of the averaged heavy-light-light structure constants was first obtained in [165]. In that paper, the authors used the asymptotics of the Virasoro four-point blocks in the heavy limit $h_s \to \infty$ [59], subsequently taking a $z \to 1$ limit to reproduce the OPE singularity from the T-channel identity operator. Their result matches the leading asymptotics of our formula (7.4.4) when written in terms of the conformal weights and central charge (as in equation (7.1.20)); we find new terms appearing at subleading order arising from a subtlety in the order of $h_s \to \infty$ and $z \to 1$ limits. Working directly with the spectral densities allows us to avoid such difficulties in studying conformal blocks.

### 7.4.2 Torus two-point function: heavy-heavy-light

For our second example, we study the two-point function of identical Virasoro primaries on the torus:

$$G_{1,2}(\tau, \bar{\tau}; w, \bar{w}) = \langle \mathcal{O}_0(w, \bar{w}) \mathcal{O}_0(0,0) \rangle_{\mathcal{T}_2(\tau, \bar{\tau})} \quad (7.4.5)$$

There are two qualitatively distinct ways to decompose such a correlation function into conformal blocks. Firstly, we can take the OPE between the two operators and insert a single complete set of states around a cycle of the torus, which we call the OPE channel. Secondly, we can insert two complete sets of states
The sequence of Moore-Seiberg moves to express the OPE channel torus two-point block in terms of necklace channel blocks: a modular $S$, followed by a fusion move.

Figure 7.6: The sequence of Moore-Seiberg moves to express the OPE channel torus two-point block in terms of necklace channel blocks: a modular $S$, followed by a fusion move.

The second and fourth lines define ‘necklace’ and ‘OPE’ spectral densities $\rho_N$, $\rho_{\text{OPE}}$. We have written the blocks using different kinematic variables, since the natural parameters (for recursion relations, for example [138]) are different in the two channels. In the necklace channel, $q_1$ and $q_2$ encode a Euclidean time evolution, between the two operator insertions, and then round the torus back to the first operator insertion; in the OPE channel, there is only one such parameter $q$, along with a separation $v$ between the operators controlling the OPE. These parameters can be related to one another, but all our results are derived without explicit reference to any kinematics.

We will consider the crossing kernel that decomposes torus two-point blocks for identical operators in the OPE channel (with internal Liouville momenta $P'_1, P'_2$) into two-point blocks in the necklace channel (in the modular $S$-transformed frame). This sewing procedure is illustrated in figure 7.6, from which we see that the required kernel is simply given by the product of the torus one-point kernel and the sphere four-point
In an appropriate limit, the necklace channel data will be dominated by the identity propagating in both internal cuffs of the OPE channel, described by the identity kernel

\[
\rho_N(P_1, P_2, \bar{P}_1, \bar{P}_2) = \int \frac{dP'_1}{2} \frac{dP'_2}{2} \frac{d\bar{P}'_2}{2} \frac{d\bar{P}'_1}{2} \mathbb{K}_{P_1; P_2; \bar{P}'_2} \mathbb{F}_{P_1; P'_2} \rho_{\text{OPE}}(P'_1, P'_2, \bar{P}'_1, \bar{P}'_2) \]  

(7.4.7)

Once again, the asymptotics of \( C_0 \) universally governs the asymptotics of OPE coefficients, this time in a ‘heavy-heavy-light’ limit, where one operator is fixed, and the other two operators are taken to have large dimensions. Corrections to this identity contribution due to the exchange of non-vacuum primaries in the OPE channel are exponentially suppressed when we take \( P_1, P_2 \) to be large, just as we have seen before. The technical result required to show this is

\[
\mathbb{K}_{P_1; P_2; \bar{P}'_2} \sim e^{-2\pi \alpha'_1 P_1} \]  

(7.4.9)

in the limit \( P_1, P_2 \to \infty \), with either the ratio or difference of \( P_1 \) and \( P_2 \) held fixed. This result is asymmetric in \( P_1 \) and \( P_2 \) because the OPE channel does not treat operators symmetrically\(^{28}\); it guarantees suppression of all non-vacuum blocks because \( \alpha'_2 \) cannot be nonzero unless \( \alpha'_1 \) is also nonzero. See the discussion in appendix D.2.1 for more details.

As in the case of the sphere four-point function, this result means that the necklace channel spectral density is well approximated by exchange of the vacuum Verma module in the OPE channel when the

\(^{28}\) We could make the derivation symmetric in \( P_1, P_2 \) by including an extra fusion move, so that we are relating two different OPE channels. Starting with the identity block, this extra fusion move is ‘free’ (that is, the necklace identity block is equal to the OPE identity block), since there are external operators for \( \mathbb{F} \) in the identity representation. However, this extra move makes the argument for suppression of non-vacuum operators more technically challenging.
internal weights are taken to be heavy:

\[
\rho_{\text{necklace}}^{(P_0,P_0)}(P_1,\bar{P}_1;P_2,\bar{P}_2) \approx \mathcal{K}_{P_1,P_2;01}^{01}[P_0]\mathcal{K}_{\bar{P}_1,\bar{P}_2;12}[\bar{P}_0], \ P_1,P_2,\bar{P}_1,\bar{P}_2 \to \infty
\]  

(7.4.10)

Thus the kernel corresponding to propagation of the identity in the OPE channel (7.4.9) encodes an asymptotic formula for OPE coefficients in the heavy-heavy-light regime, averaged over the heavy operators, and for any fixed light operator. Stripping off the density of states of the heavy operators, we have

\[
|C_{012}|^2 \sim C_0(P_0,P_1,P_2)C_0(\bar{P}_0,\bar{P}_1,\bar{P}_2), \ P_1,P_2,\bar{P}_1,\bar{P}_2 \to \infty.
\]  

(7.4.11)

As in the case of the sphere four-point function, in the presence of a nonzero twist gap the above asymptotic formula also holds in the large-spin regime when only \(P_1, P_2\) or \(\bar{P}_1, \bar{P}_2\) are taken to be large.

Now that there are multiple internal weights, there are several distinct ways to take the large-weight limit. First, we can take the weights to infinity at fixed ratio \(\frac{P_2}{P_1}\), assuming without loss of generality that \(P_1 > P_2\). We will take this limit by writing \(P_i = x_i P\), with \(x_i\) fixed in the large-\(P\) limit. One finds:

\[
\log C_0(P_0, x_1 P, x_2 P)
\]

\[
= (-4x_1^2 \log(2x_1) - 4x_2^2 \log(2x_2) + 2(x_1 + x_2)^2 \log(x_1 + x_2) + 2(x_1 - x_2)^2 \log(x_1 - x_2)) P^2
\]

\[- \pi Q(x_1 + x_2)P + \left(\frac{2Q^2}{3} + 4P_0^2 - \frac{1}{3}\right) \log P
\]

\[+ \log \frac{2^{\frac{1}{2}(2Q^2-1)}(x_1 x_2)^{\frac{1}{2}(Q^2+1)}(x_1^2 - x_2^2)^{\frac{1}{2}(Q^2+12P_0^2-2)}\Gamma_0(b)^4\Gamma_0(2Q)}{\Gamma_b(Q)^3\Gamma_b(Q - 2iP_0)\Gamma_b(Q + 2iP_0)} + O(P^{-1}).
\]

(7.4.12)

The other interesting limit takes the difference \(P_1 - P_2 = 2\delta\) to be fixed, with the average \(P \to \infty\). Note that in terms of dimensions \(h\), this means that \(h_1 - h_2\) is of order \(\sqrt{h}\). In this limit one finds the following
Several recent papers have studied asymptotics of the averaged off-diagonal heavy-heavy-light structure constants in CFT\(_2\), including [220–222]. The most directly comparable result is equation (2.33) of [220], which studied these OPE asymptotics by considering the torus two-point function in a particular kinematic limit, imposing modular covariance, and performing an inverse Laplace transform to extract the spectral density. While the first line of our result (7.4.13) reproduces the entropic suppression \(e^{-S/2}\) expected from the eigenstate thermalization hypothesis, there appears to be a nontrivial difference between our subleading terms (written in terms of the dimensions and the central charge in equation (7.1.21)) and those of [220]. Again, we would like to emphasize the technical simplicity of our argument, which does not rely on carefully establishing the behaviour of conformal blocks in simultaneous large-weight and kinematic limits.

### 7.4.3 Genus-two partition function: heavy-heavy-heavy

The final constraint from crossing we will study arises from modular invariance of the genus two partition function \(G_{2,0}\). We will relate the conformal block decomposition in two channels, which we call ‘sunset’ and
We have here suppressed the dependence of that relates dumbbell channel genus-two Virasoro blocks to those in the sunset channel. From figure 7.7, we see that, like the crossing kernel for the torus two-point function, this kernel is simply a product of sphere four-point and torus one-point kernels:

\[
G_{2,0} = \sum_{\mathcal{O}_1} \sum_{\mathcal{O}_2} \sum_{\mathcal{O}_3} C_{123}^2 \mathcal{F}(\text{sunset})(P_1, P_2, P_3) \mathcal{F}(\text{sunset})(\bar{P}_1, \bar{P}_2, \bar{P}_3)
\]

\[
= \int \left( \prod_{j=1}^{3} \frac{dP_j}{2} \frac{d\bar{P}_j}{2} \right) \rho_{\text{sunset}}(P_1, P_2, P_3, \bar{P}_1, \bar{P}_2, \bar{P}_3) \mathcal{F}(\text{sunset})(P_1, P_2, P_3) \mathcal{F}(\text{sunset})(\bar{P}_1, \bar{P}_2, \bar{P}_3)
\]

\[
= \sum_{\mathcal{O}_1'} \sum_{\mathcal{O}_2'} \sum_{\mathcal{O}_3'} C_{1'2'3'} \mathcal{F}(\text{dumbbell})(P_1', P_2', P_3') \mathcal{F}(\text{dumbbell})(\bar{P}_1', \bar{P}_2', \bar{P}_3')
\]

\[
= \int \left( \prod_{j=1}^{3} \frac{dP_j}{2} \frac{d\bar{P}_j}{2} \right) \rho_{\text{dumbbell}}(P_1, P_2, P_3, \bar{P}_1, \bar{P}_2, \bar{P}_3) \mathcal{F}(\text{dumbbell})(P_1, P_2, P_3) \mathcal{F}(\text{dumbbell})(\bar{P}_1, \bar{P}_2, \bar{P}_3).
\]

(7.4.14)

We have here suppressed the dependence of \(G_{2,0}\) and the blocks on the moduli, since by now it is hopefully clear that we have no need of them. This is fortunate, because for \(g \geq 2\) the description of the moduli spaces and relations between different channels becomes technically very challenging, and in particular, we must contend more directly with the factors arising from the conformal anomaly.

To study the consequences of the genus-two modular crossing equation, we will employ the crossing kernel that relates dumbbell channel genus-two Virasoro blocks to those in the sunset channel. From figure 7.7, we see that, like the crossing kernel for the torus two-point function, this kernel is simply a product of sphere four-point and torus one-point kernels:

\[
\mathcal{K}_{P_1, P_2; P'_1, P'_2; P_3, P'_3} = \mathcal{S}_{P_1, P'_1} [P'_2] \mathcal{S}_{P_2, P'_2} [P'_3] \mathcal{F}_{P_3, P'_3}
\]

\[
\rho_{\text{sunset}}(P_1, \bar{P}_1) = \int \left( \prod_{i=1}^{3} \frac{dP_i}{2} \frac{d\bar{P}_i}{2} \right) \mathcal{K}_{P_i, P'_i} \rho_{\text{dumbbell}}(P'_i, \bar{P}'_i)
\]

(7.4.15)
Once again, we will find that in appropriate limits, the spectral density in the sunset channel is dominated by the contribution of the identity in all internal cuffs of the dumbbell channel. The corresponding spectral density is given by the following identity kernel:

$$K[p_1, p_2, p_3; 111] = \mathbb{S}_{P_{11}}[1] \mathbb{S}_{P_{21}}[1] F_{P_{21}} \left[ \begin{array}{c} p_1 \\ p_1 \\ p_3 \end{array} \right]$$

(7.4.16)

Thus, once again, the asymptotic behaviour of the OPE coefficients, now when all three operators are heavy, is determined by the asymptotics of the universal object $C_0(P_1, P_2, P_3)$. Precisely as in (7.4.9), corrections to this asymptotic formula due to the exchange of non-vacuum primaries in the dumbbell channel are exponentially suppressed by the ratio

$$\frac{K[p_1, p_2, p_3; 111]}{K[\bar{p}_1, \bar{p}_2, \bar{p}_3; 111]} \approx e^{-2\pi(\alpha_1' P_1 + \alpha_2' P_2 + \alpha_3' P_3)}$$

(7.4.17)

in the limit where the ratios or differences between the $P_i$ are held fixed. In the original dumbbell channel, $\alpha_2'$ cannot be nonzero unless both $\alpha_1'$ and $\alpha_3'$ are nonzero, so this is always exponentially small. More details are contained in appendix D.2.1.

The conclusion is that the sunset channel OPE density is well-approximated by the exchange of the vacuum Verma module in the dumbbell channel when the internal weights all become heavy:

$$\rho_{sunset}(P_1, \bar{P}_1; P_2, \bar{P}_2; P_3) \approx K[p_1, p_2, p_3; 111] K[\bar{p}_1, \bar{p}_2, \bar{p}_3; 111], \quad P_i, \bar{P}_i \to \infty$$

(7.4.18)

Thus the kernel (7.4.16) encodes an asymptotic formula for OPE coefficients in the heavy-heavy-heavy regime, averaged over the weights of all three heavy operators

$$|C_{123}|^2 \sim C_0(P_1, P_2, P_3) C_0(\bar{P}_1, \bar{P}_2, \bar{P}_3), \quad P_i, \bar{P}_i \to \infty.$$  

(7.4.19)
As before, in the presence of a nonzero twist gap this formula holds at large spin in which only the left-moving
momenta $P_1, P_2, P_3$ or the right-moving momenta $\bar{P}_1, \bar{P}_2, \bar{P}_3$ are taken to be large.

We can now recover asymptotic formulas for the microcanonical average of all heavy OPE coefficients
from the relevant asymptotics of $C_0$. For example, if we fix ratios of $P_i$, parameterizing as $P_i = x_i P$
with $x_i > 0$ fixed and $P \to \infty$, we have

$$
\log C_0(x_1P, x_2P, x_3P) = \left( -4 \sum_{i=1}^{3} x_i^2 \log(2x_i) + \sum_{\epsilon_2, \epsilon_3 = \pm} (x_1 + \epsilon_2 x_2 + \epsilon_3 x_3)^2 \log |x_1 + \epsilon_2 x_2 + \epsilon_3 x_3| \right) P^2
$$

$$
- \pi Q(x_1 + x_2 + x_3) P + \left( \frac{5Q^2 - 1}{6} \right) \log P
$$

$$
+ \log \frac{2^{\frac{Q^2}{2}} (x_1 x_2 x_3)^{\frac{1}{2}(Q^2+1)} \prod_{\epsilon_2, \epsilon_3 = \pm} |x_1 + \epsilon_2 x_2 + \epsilon_3 x_3|^\frac{1}{2}(Q^2-2) \Gamma_0(b)^2 \Gamma_b(2Q)}{\Gamma_b(Q)^3} + O(P^{-1}).
$$

In the case where $|P_1 - P_j|$ is fixed in the limit, we instead have

$$
\log C_0(P + \delta_1, P + \delta_2, P - \delta_1 - \delta_2)
$$

$$
= 3P^2 \log \frac{27}{16} - 3\pi Q P + \frac{1}{6} (5Q^2 - 1) \log(P)
$$

$$
+ \log \frac{2^{\frac{Q^2}{2} - 8(\delta_1^2 + \delta_2^2 + \delta_2^2)} 3^{\frac{Q^2-2}{2}} \Gamma_0(b)^2 \Gamma_b(2Q)}{\Gamma_b(Q)^3} + O(P^{-1}).
$$

These limits were studied using genus 2 modular invariance in [163], using conformal block techniques.

This analysis used the same underlying crossing relation, relating the heavy blocks in the sunset channel to
the identity in the OPE channel (or, equivalently, a different necklace channel, obtained by an additional
fusion move; the identity blocks in these two channels are identical). Results were only obtained for large
c, where additional techniques to analyse conformal blocks are available, only included terms up to order
$P \sim \sqrt{h}$ in $\log C_0$, and did not have a complete result for the term scaling exponentially in $P^2 \sim h$ (the first
line of (7.4.21)) valid at general ratios of operator dimensions. Nonetheless, all our formulas match those
in [163], including confirming a conjectured correction $c \to c - 1$ from finite central charge. Our new method,
with far less work, extends these results to higher orders and finite central charge.

7.5 On the relation to Liouville theory

In section 7.1.6, we observed the relation between our universal object $C_0$ and the DOZZ formula for the structure constants of Liouville theory,

$$C_0(P_1, P_2, P_3) \propto \frac{C_{DOZZ}(P_1, P_2, P_3)}{\left(\prod_{k=1}^3 S_0(P_k) \rho_0(P_k)\right)^{\frac{3}{2}}}.$$  \hfill (7.5.1)

We then sketched an argument which explained this relation from a common origin in representation theory. We here give more details of that argument, explaining why the DOZZ formula must be constructed from the identity fusion kernel, as the unique solution to crossing built from only scalar Virasoro primaries.

To this end, we give general arguments for the identities which establish that the identity fusion kernel provides a solution to crossing with scalar primaries, applicable for any chiral algebra. Many of the methods are familiar in the context of rational CFTs. Secondly, we explicitly demonstrate that the relevant identities hold for the Virasoro crossing kernels of [156, 157, 166], which is a consistency check that these arguments extend to this non-rational situation.

We perform this analysis for two cases. First, we study four-point crossing, where our arguments are very similar to those given in [252], for example. Secondly, we give similar arguments for modular S-invariance of the torus one-point function.

7.5.1 Four-point crossing symmetry

Following the general arguments of section 7.1.6, the four-point crossing equation (7.3.8) with only scalar primaries becomes

$$\mathcal{F}_{P_1 P_s}^{-1} \rho_s(P_s) = \mathcal{F}_{P_s P_t} \rho_t(P_t).$$ \hfill (7.5.2)
where we have inverted one of the fusion kernels to move it to the left hand side. We can write this relation with explicit dependence on the external operators as follows:

\[
\mathcal{F}_{ts}^{-1} \begin{bmatrix} 41 \\ 32 \end{bmatrix} \rho_s = \mathcal{F}_{st} \begin{bmatrix} 21 \\ 34 \end{bmatrix} \rho_t .
\]  

(7.5.3)

We have used the fact that the inverse fusion kernel is the same as the fusion kernel with a permutation of external operators. Here and in the following, for brevity of notation we have suppressed momentum labels by replacing \( P_i \) simply with \( i \); in particular, the external operator labelled by 1 is not to be confused with the identity representation, denoted by \( \mathbb{1} \). Our aim in the following is to find an identity of the form (7.5.3), and hence a solution to crossing. Note that if we have one solution to this equation, any other solution is related by multiplying \( \rho_s, \rho_t \) by the same constant (independent of \( P_s, P_t \), but not the external operators since we cannot fix their normalisations). The only exception to this occurs when the fusion kernel is block diagonal, in which case there is an independent solution for each block.

To proceed, we make use of a consistency condition satisfied by fusion kernels, the famous pentagon identity, which in our notation reads

\[
\sum_r \mathcal{F}_{rp} \begin{bmatrix} 1q \\ 23 \end{bmatrix} \mathcal{F}_{sq} \begin{bmatrix} 15 \\ r4 \end{bmatrix} \mathcal{F}_{tr} \begin{bmatrix} 2s \\ 34 \end{bmatrix} = \mathcal{F}_{tg} \begin{bmatrix} p5 \\ 34 \end{bmatrix} \mathcal{F}_{sp} \begin{bmatrix} 15 \\ 2t \end{bmatrix} .
\]  

(7.5.4)

We have written this with a sum over \( r \), as appropriate for the fusion matrix in rational CFTs. For the \( c > 1 \) Virasoro fusion kernels of [156,157] with continuous families of representations, the sum becomes an integral with the appropriate measure. The identity follows from considering two possible sequences of fusion moves applied to the five-point conformal blocks, sketched in figure 7.8, which must act in the same way.

We only require a special case of the identity, taking \( q = \mathbb{1} \), which also sets \( s = 4 \) and \( p = 3 \) so that the blocks and fusion kernels are well-defined (otherwise, they become infinite signalling the disallowed fusion).
Figure 7.8: The sequence of crossing moves applied to the sphere five-point conformal block leading to the pentagon identity (7.5.4) for the fusion kernel.

The first fusion move then becomes trivial, giving a δ-function that sets $r = 1$

$$\delta_{1r}.$$  \hspace{1cm} (7.5.5)

This relation can be explicitly verified for the Ponsot-Teschner fusion kernel (D.1.2) by taking the appropriate limit: the kernel vanishes at generic $P_r$ in the limit, but a delta-function $\delta(P_r - P_1)$ is produced by two poles which pinch the contour, with finite residue in the limit. This leaves us with an identity without an internal sum,

$$\delta_{1r} = \delta_{1r},$$  \hspace{1cm} (7.5.6)

which one can check from the explicit form of the Ponsot-Teschner fusion kernel (D.1.2). To see this, we note that we can rewrite the desired equality (7.5.6) as

$$\delta_{1r} = \delta_{1r},$$  \hspace{1cm} (7.5.7)

where by the tilded entries $\tilde{i}$, we mean that we replace $P_i \rightarrow -P_i$, an operation under which the fusion kernel is invariant. Written in this form, (7.5.7) is immediate from the expressions (D.1.2) after a shift of
the variable in the contour integrals, and using $S_b(x) = S_b(Q - x)^{-1}$.

Now, by permuting labels in (7.5.6) $(1 \rightarrow t \rightarrow s \rightarrow 1)$, we have

$$F_{14}^{\tau_4 \tau_4} F_{st}^{21 34} = F_{s1}^{34 34} F_{13}^{t_4 t_4}, \quad (7.5.8)$$

where we recognize one term on the left as the fusion kernel of interest in (7.5.3). By another permutation of labels, swapping $2 \leftrightarrow 4$ and $t \leftrightarrow s$ in (7.5.8), we find an identity involving the inverse fusion kernel of interest,

$$F_{14}^{s_2 s_2} F_{ts}^{41 32} = F_{t1}^{32 32} F_{13}^{s_2 s_2}, \quad (7.5.9)$$

Now, since the fusion kernels are invariant under exchange of rows or columns, the $F_{13}$ kernels appearing in the two identities are the same, so we can combine (7.5.8) and (7.5.9) to find

$$F_{st}^{34 34} F_{14}^{s_2 s_2} F_{ts}^{41 32} = F_{st}^{21 34} F_{14}^{t_4 t_4} F_{t1}^{32 32}, \quad (7.5.10)$$

This is an identity precisely of the form (7.5.3) and hence a scalar solution to four-point crossing, with

$$\rho_s = k F_{s1}^{34 34}, \quad \rho_t = k F_{t1}^{t_4 t_4}, \quad (7.5.11)$$

where $k$ is independent of $P_s, P_t$, but otherwise arbitrary. Using the expression (7.3.10) for the identity fusion kernel in terms of our universal functions $C_0$ and $\rho_0$, we can write this solution as

$$\rho_s(P_s) = k \rho_0(P_s) C_0(P_1, P_2, P_s) C_0(P_s, P_3, P_4), \quad \rho_t = k \rho_0(P_t) C_0(P_1, P_4, P_t) C_0(P_t, P_2, P_3), \quad (7.5.12)$$

where a factor of $\rho_0(P_1)$ has been absorbed into $k$. From the relation (7.1.26) between $C_0$ and the DOZZ formula, we see that $\rho_s$ and $\rho_t$ are precisely the S- and T-channel spectral densities in Liouville theory, making an appropriate choice of $k$. 277
7.5.2 S-invariance of torus one-point functions

We now make a similar argument to show that Liouville theory provides the unique modular covariant torus one-point functions for scalars. We begin by writing the equation for one-point S-invariance (7.3.17) for scalars in a form analogous to (7.5.3), using the fact that $S$ is its own inverse:

\[ S_{ts}[\mathcal{O}]\rho_{\mathcal{O}}(s) = S_{st}[\mathcal{O}]\rho_{\mathcal{O}}(t). \]  

(7.5.13)

Here, the same torus one-point spectral density $\rho_{\mathcal{O}}$ (where $\rho_{\mathcal{O}}(s)$ is the density of internal states $\rho(s)$ times OPE coefficients $C_{\mathcal{O}ss}$) appears on both sides.

To find a relation of the form (7.5.13), we require an identity for the modular S-kernel $S$ to play an analogous role to the pentagon identity in the above. Such an identity arises from consistency of torus two-point functions, where two different sequences of moves applied to a vacuum block must be equivalent:

\[ F_{\mathcal{O}1}[tt] S_{st}[\mathcal{O}] = S_{s1}[1] \sum_u e^{2\pi i(h_s+h_t-h_u-h_{\mathcal{O}}/2)} F_{u1}[st] F_{\mathcal{O}u}[ss]. \]  

(7.5.14)

This identity is well-known for rational theories [38, 39], but also applies to the Virasoro kernels at generic central charge [260], with the sum over $u$ replaced by an integral with appropriate measure. For rational theories, this identity is the key to proofs of the Verlinde formula [38, 39], so these considerations can be applied to explore analogues of the Verlinde formula for irrational theories [260].

For us, the most important feature of this identity is that the right-hand side is symmetric under swapping $s \leftrightarrow t$, except for the factor of the identity S-matrix $S_{s1}[1]$. From this observation, we find the simple relation

\[ S_{st}[\mathcal{O}] S_{ts}[1] F_{\mathcal{O}1}[tt] = S_{ts}[\mathcal{O}] S_{st}[1] F_{\mathcal{O}1}[ss]. \]  

(7.5.15)
This identity is precisely of the desired form (7.5.13), with
\[
\rho_0(P) \propto S_{P_4}[\mathbb{F}^{\mathbb{C}_4}] = \rho_0(P_0)\rho_0(P)C_0(P_0, P, P).
\] (7.5.16)

Up to a $P$-independent normalisation constant, this is precisely the torus one-point spectral density for Liouville theory constructed from the DOZZ formula (7.1.26).

### 7.6 Semiclassical limits

Throughout this chapter we have emphasized that our asymptotic formulas apply in any two-dimensional irrational CFT for any $c > 1$, providing universal results in a kinematic limit of large dimension or spin. However, it is natural to expect our results to be particularly powerful in holographic theories with a weakly coupled $\text{AdS}_3$ dual, and to have a corresponding gravitational interpretation. The basic reason for this is simple: the corrections to the asymptotic formula come from the lightest operators in the theory, and existence of a holographic dual requires having few such operators (a sparse light spectrum) \[64, 149, 208\]. For example, in higher dimensions generic theories contain double-twist operators with anomalous dimensions suppressed at large spin \[26, 27\]; in holographic theories, the ‘t Hooft limit extends this to double-trace operators with anomalous dimensions suppressed at large $N$, now at finite spin. The corresponding gravitational interpretation involves two-particle states in AdS, which generically are weakly interacting only with very large orbital angular momentum, when the particles are widely separated, but in holographic theories also interact weakly at finite separation. An example in $d = 2$ is the density of states, which for holographic theories is given by the Cardy formula not just for very heavy operators, but also at large $c$ for energies of order $c$ \[64\], interpreted as the Bekenstein-Hawking entropy of BTZ black holes \[58\].

With this in mind, in this section we will give gravitational interpretations of our universal OPE coefficients $C_0$ in various large $c$ limits. We will not attempt here to pin down precisely when these formulas apply, in terms of constraints on the theory and regime of operator dimensions; see \[226\] for recent work in this
Regardless, it is simpler to interpret and understand this regime in the gravitational description. Since
our formulas come from expanding an identity block in an alternative channel, we can interpret our formulas
as a microcanonical version of ‘vacuum block domination’, giving the density of states in a regime where
a correlation function is well-approximated by only the identity Virasoro block in the appropriate channel
[261–264]. At large $c$, an identity block is given by the gravitational action of a particular locally empty AdS
solution (which could be a BTZ black hole or handlebody at higher genus), along with worldlines of particles
propagating between external operator insertions [210, 218, 230, 265, 266]:

$$F_1 \approx e^{-c S_{Grav}}$$

We therefore expect our formulas to be applicable when the gravitational path integral is dominated by such
a solution, up to loop corrections.\textsuperscript{29} This holds for a kinematic regime of parametrically low temperature
or small cross-ratios, but for holographic theories is expected to extend to a regime of kinematics which are
fixed in the large $c$ limit. The question is how far this regime extends before encountering a phase transition.
The simplest such phase transitions are first-order ‘Hawking-Page’ transitions, where an identity block in
different channel dominates. However, note that even for local, weakly coupled gravitational theories, there
need not be any channel in which the vacuum dominates: for example, there may be a phase in which a scalar
field condenses after a second-order phase transition [230, 231]. Vacuum dominance potentially particularly
subtle for correlation functions in kinematic regimes such as those with operators out of time order [267].

We now give our examples of gravitational interpretations of the universal OPE coefficients $C_0$ in various
limits. These are all explored in more detail elsewhere, but we present them here together as consequences
of the same formula, emphasizing the unifying nature of our results. Furthermore, the list may well not
be exhaustive, since we have not included all possible semiclassical limits, and our understanding of the
\textsuperscript{29}Note that the identity block itself need not be a larger contribution than any other block. Corrections at one-loop
order change the coefficient of $e^{-c S_{Grav}}$, and come from light operator exchanges of the same order as $F_1$.\textsuperscript{280}
connections to gravity is far from complete.

### 7.6.1 Spectral density of BTZ black holes

For our first example, we take a large $c$ limit of $C_0$ which probes the physics of BTZ black holes. We take two operators to be heavy, with dimensions $h_1, h_2$ scaling with $c$, to correspond to black hole states, but with similar dimensions, $h_1 - h_2$ fixed as $c \to \infty$. The third operator, acting as a probe of the geometry, has $h$ fixed in the limit. In terms of the momentum variables $\mathbf{P}$, we take

$$
P_1 = b^{-1} p + b \delta, \quad P_2 = b^{-1} p - b \delta, \quad P_3 = i \left( \frac{Q}{2} - bh \right),
$$

and fix $p, \delta, h$ in the $b \to 0$ limit. We can then interpret $C_0$ as governing the matrix elements $\langle \text{BH}_2 | \mathcal{O} | \text{BH}_1 \rangle$ of the probe operator $\mathcal{O}$ of dimension $h$ between black hole states of nearby energies.

This limit of the fusion kernel was studied in [229], with the result

$$
\rho_0(b^{-1}p)C_0(P_1, P_2, P_3) \sim \frac{(2p)^{2h}}{2\pi b} \frac{\Gamma(h + 2i\delta)\Gamma(h - 2i\delta)}{\Gamma(2h)}.
$$

This is the left-moving half spectral density associated to free matter propagating an a BTZ black hole background\[^{30}\] [214]. In particular, the poles at imaginary $\delta$ are associated with the frequencies of quasinormal modes governing the approach to equilibrium. This result is sufficient to recover the ‘heavy-light’ limit of conformal blocks [66, 67]; see [229] for more details.

### 7.6.2 Near-extremal BTZ and the Schwarzian theory

Our second example (based on [268]) is similar to the first, but treats the distinct case where the black hole of interest is very close to extremality.

\[^{30}\]Strictly speaking, this holds in a case where we are insensitive to the compactness of the spatial circle, either large black holes or heavy external operators.
Rotating BTZ black holes exist for dimensions above the extremality bound $h > \frac{c-1}{24}$, and we will tune our operators close to this, with $h - \frac{c-1}{24}$ of order $c^{-1}$. Our third operator will remain a light probe. This means we have

\[ P_1 = bk_1, \quad P_2 = bk_2, \quad P_3 = i \left( \frac{Q}{2} - bh \right), \tag{7.6.4} \]

where we fix $k_1, k_2, h$ and take $b \rightarrow 0$.

In this limit, our universal density of states $\rho_0$ and OPE coefficients $C_0$ are given by

\[ \rho_0(bk) \sim 8\sqrt{2}\pi b^2 k \sinh(2\pi k) \tag{7.6.5} \]

\[ C_0 \left( bk_1, bk_2, i \left( \frac{Q}{2} - bh \right) \right) \sim \frac{b^{4h}}{\sqrt{2}(2\pi b)^3} \prod_{\pm} \frac{\Gamma(h \pm ik_1 \pm ik_2)}{\Gamma(2h)}, \tag{7.6.6} \]

where the $\prod_{\pm}$ refers to a product of four terms with all possible sign combinations. These expression may be familiar from the Schwarzian theory, which governs the dynamics of weakly broken conformal symmetry [161, 269, 270]. This theory arises in near-extremal black holes, which have a near-horizon AdS$_2$ region with dynamics governed by Jackiw-Teitelboim gravity [270, 271]. Specifically, $\rho_0$ is proportional to the density of states for the Schwarzian theory, and $C_0$ to a transition amplitude appearing in calculations of correlation functions [161, 272, 273].

The appearance of these quantities is a sign that there is a universal sector of large $c$ CFTs which knows about quantum geometry, where the metric fluctuations are not suppressed. The connection between the Schwarzian theory, near-extremal BTZ and universality in CFT is explored in more detail in [268].

### 7.6.3 Conical defect action

Finally, we consider a regime where all three operators have dimensions scaling with $c$. If we take $\frac{24h}{c} > 1$ in this limit, as required for asymptotic formulas, $C_0$ should be interpreted as giving a three-point function of black hole microstates. It is unclear whether there is a direct calculation of this quantity, giving the semiclassical limit of $C_0$ as an on-shell action. However, perhaps surprisingly, if we fix $\frac{24h}{c} < 1$ and take
\( c \to \infty \), there is such an interpretation, shown in [103]. Those authors computed the vacuum fusion kernel in a large central charge limit,

\[
\alpha_i = b^{-1}\eta_i, \quad b \to 0, \quad \text{fixed } \eta_i, \quad i = 1, 2, 3,
\]

(7.6.7)

and equated it to a suitably regularized on-shell action of a geometry corresponding to three heavy particles running between the asymptotic boundary and a trivalent vertex. The action in this case is Einstein-Hilbert, plus an action \( m_i L_i \) for each particle, where \( L_i \) is a regularized proper length of the particle’s worldline and \( m_i \sim \frac{c}{\eta_i} \) is its mass. Since the particles have masses of order \( c \), they backreact to form three conical defects in the geometry, meeting at the vertex.\(^{31}\)

In our notation, we can express the result of [103] as a limit of \( C_0 \):

\[
\log C_0 \sim b^{-2} \left( -\frac{1}{2} S_{\text{grav}}(\eta_1, \eta_2, \eta_3) + i\theta(\eta_1, \eta_2, \eta_3) \right),
\]

\[-\frac{1}{2} S_{\text{grav}} = (F(2\eta_1) - F(\eta_2 + \eta_3 - \eta_1)) + (1 - 2\eta_1) \log(1 - 2\eta_1) + (2 \text{ permutations}) \]

\[+ F(0) - F(\eta_1 + \eta_2 + \eta_3) - 2(1 - \eta_1 - \eta_2 - \eta_3) \log(1 - \eta_1 - \eta_2 - \eta_3) \]

\[\theta = \pi(\eta_1 + \eta_2 + \eta_3 - 1),\]

(7.6.8)

where \( F(z) = I(z) + I(1 - z) \) for \( I(z) = \int_{\frac{1}{2}}^{z} dy \log \Gamma(y) \). The action \( b^{-2} S_{\text{grav}} \) appearing here is precisely the gravitational action for the conical defect network described above. When left- and right-moving sectors are combined, for scalars the phase \( \theta \) cancels.

When conformal blocks are computed at large \( c \) as an on-shell gravitational action, this conical defect action, and hence this limit of \( C_0 \), appear as the natural normalization of the blocks [188, 230]. While the relation with our universal asymptotic formulas is suggestive, it remains rather mysterious from that point of view, and deserves to be better understood.

\(^{31}\)No particle action was included in [103], but they also included no singular contribution to the Einstein Hilbert action localized on the worldline. These two terms are equal and opposite, so the results are equivalent.
7.7 Torus one-point functions & the eigenstate thermalization hypothesis

Although the primary focus of this chapter is on the asymptotic behaviour of the $C_{ijk}^2$, similar techniques can be applied to other observables in two-dimensional conformal field theory. For example, by studying the modular covariance of the torus one-point function of an operator $O_0$ one obtains an asymptotic formula for diagonal heavy-heavy-light structure constants $C_{OHH}$, where we average over the heavy operator $H$. This was discussed in [129], who found

$$C_{OHH} \approx N_0 C_{0xx} \left( \Delta_H - \frac{c-1}{12} \right)^{\Delta_0/2} \exp \left[ -\frac{\pi}{3} \left( 1 - \sqrt{1 - \frac{12\Delta_\chi}{c-1}} \right) \sqrt{\frac{12\Delta_H}{c-1}-1} \right], \quad (7.7.1)$$

in the limit that $\Delta_H \to \infty$. Here $\chi$ is the lightest operator to which $O_0$ couples (i.e. for which $C_{0xx} \neq 0$), and is assumed to be sufficiently light, $\Delta_\chi < \frac{c-1}{12}$. The normalization factor $N_0$ depends only on $c$, $\Delta_\chi$ and $\Delta_0$. This analysis was performed at the level of the scaling blocks in [129] and was generalized to include the contribution of global blocks in [218]. When regarded as a formula for the average value of the primary operators, however, equation (7.7.1) is true only at leading order in $1/c$; the inclusion of Virasoro blocks provides corrections which are only subleading at large $c$.

We can now write down the finite $c$ version of this formula using the modular S kernel introduced in section 7.3.3 for torus one point functions. Following the same logic that led to our other asymptotic formulas, we conclude that

$$C_{OHH} \approx C_{0xx} \frac{S_{P_H}P_0[P_0][P_0]}{\rho_0(P_H)\rho_0(P_H)}, \quad P_H, \tilde{P}_H \to \infty, \quad (7.7.2)$$

provided that $\chi$, the lightest operator that couples to $O_0$, is sufficiently light ($\alpha_\chi$ lies in the discrete range in the sense of [229]) and that there exists a gap above this lightest operator so that corrections due to the inversion of the contributions of other operators in the original channel are indeed suppressed. The large $P$ asymptotics of this formula are straightforward to find by taking the large $P_H$ limit of the modular S kernel,
namely
\[ \frac{\mathbb{S}_{P_H P_x} [P_b]}{\rho_0 (P_H)} \approx e^{-4\pi \chi P_H P_{h_0}}. \]  
(7.7.3)

This reproduces the earlier result (7.7.1) in the appropriate limit.

We would like to emphasize two important qualitative differences between this formula and our other asymptotic formulas. The first is that it is not universal in the same sense as our other formulas, as it explicitly depends on the lightest operator that couples to \( O_0 \), both through its conformal weights and OPE coefficient (this is because the vacuum Verma module cannot propagate as an intermediate state in either channel of the torus one-point function). Second, its derivation is on even less rigorous footing than our other asymptotic formulas because the structure constants that appear in the conformal block decomposition of the torus one-point function need not be positive, and so the spectral densities \( \rho [O_0], \tilde{\rho} [O_0] \) do not in general have definite sign and may oscillate when integrated. This is unlike the product of structure constants that appear in the necklace channel conformal block decomposition of the torus two-point function of identical operators or the sunset channel of the genus-two partition function, which are positive in a unitary CFT. In fact, if the lightest operator that couples to \( O \) is sufficiently heavy (in particular, if it has twist \( > \frac{c-1}{12} \)), then one cannot even argue that the asymptotics of the structure constants are universal as corrections due to the propagation of other operators in the original channel are not parametrically suppressed.

As discussed in section 7.1.5, the fact that the averaged diagonal heavy-heavy-light OPE coefficients are exponentially suppressed (via e.g. (7.7.3)) implies a different hierarchy of suppression between the averaged diagonal and non-diagonal heavy-heavy-light structure constants than would naively have been expected from the usual statement of the eigenstate thermalization hypothesis, where \( f^O \) is order one and \( g^O \approx e^{-\frac{1}{2} S(\Delta)} \). Indeed, if the lightest operator that couples to \( O_0 \) satisfies \( \text{Re}(\alpha_\chi + \bar{\alpha}_\chi) \geq \frac{Q}{2} \) (for scalars, this corresponds to dimension \( \Delta_\chi \geq \frac{c-1}{16} \)), then there is no suppression whatsoever of the averaged off-diagonal structure constants compared to the diagonal, and indeed the diagonal terms may be even smaller than the off-diagonal in this regime. This may be seen by comparing equation (7.7.3) with equation (7.4.13). This contrast is particularly sharp in holographic theories with a large gap in the spectrum of primary operators.
with only Planckian degrees of freedom. Indeed the dual of a theory of “pure” quantum gravity in $AdS_3$ is in a sense one where the averaged diagonal heavy-heavy-light structure constants are smallest.

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8

Pure gravity and conical defects

8.1 Introduction

One of the simplest possible theories of gravity is three dimensional general relativity with negative cosmological constant, governed by the Einstein Hilbert action

\[ S = \frac{1}{16\pi G_N} \int d^3x \sqrt{g} \left( R + \frac{2}{L^2} \right). \]  

(8.1.1)

Although this theory has no local gravitons and all classical solutions are locally AdS$_3$ with radius $L$, it nevertheless has a rich structure including black hole solutions [44]. With asymptotically AdS boundary
conditions, the theory possesses an infinite dimensional asymptotic Virasoro symmetry [43] with central charge

\[ c = \frac{3L}{2G_N} + \mathcal{O}(1). \]  

(8.1.2)

This highly constrains the theory, and allows the use of the analytical technology of two dimensional conformal field theory to study foundational questions about quantum gravity in a potentially exactly solvable setting.

There is a sense in which every 2D CFT defines a theory of three-dimensional quantum gravity. However, for the bulk dual to resemble semiclassical Einstein gravity, the central charge must be large and the spectrum of light operators must be sufficiently sparse [63,149]. Under these assumptions, semiclassical Einstein gravity has many features that are universal. For example, the Cardy formula for the asymptotic density of states in 2D CFT [169], which reproduces the Bekenstein-Hawking entropy of black holes with an \( AdS_3 \) factor in the near-horizon geometry [58,274], has an extended regime of validity under these assumptions [64], reflecting the validity of gravitational effective field theory in \( AdS_3 \).\(^1\)

Indeed, the low energy spectrum of a pure theory of gravity – namely one with only metric degrees of freedom – includes only Virasoro descendants of the vacuum state. These are interpreted as “boundary gravitons,” analogous to the edge modes of a quantum hall system, which correspond to perturbations of the \( AdS_3 \) metric which are not pure gauge only because of the existence of the asymptotic boundary. The BTZ black holes solutions correspond to states with mass above the Planck scale – or dimensions \( h \) and \( \bar{h} \) greater than \( c/24 \) in the CFT language. The natural expectation, therefore, is that pure quantum gravity should be a maximally sparse theory, whose spectrum includes only these boundary gravitons as well as heavy states with \( h, \bar{h} > c/24 \) which describe the microstates of these black holes [63]. Indeed, the question of how sparse the light spectrum can be consistent with the constraints of conformal symmetry, unitarity and locality of the dual CFT (or equivalently, consistency of quantum gravity in the bulk) is a topic of ongoing research without a definite conclusion (see e.g. [54,55,101,275,276]).

\(^1\)The analysis in [64] actually only established the extended regime of validity of the Cardy formula for dimensions \( \Delta \geq \frac{c}{12} \) in the case of large central charge and a sparse light spectrum; the gravitational effective field theory argument would suggest that it should hold for \( \Delta \geq \frac{c}{12}(1 + \epsilon) \), where \( \epsilon \) is an order-one number.
In \cite{65,277}, the authors attempt to compute the partition function of pure three-dimensional gravity via an explicit sum over known saddles in the Euclidean gravitational path integral. In CFT terms, the sum over gravitational instantons amounts to a sum over $PSL(2, \mathbb{Z})$ images of the Virasoro vacuum character

\[ Z_{MWK}(\tau, \bar{\tau}) = \sum_{\gamma \in \mathbb{Z} \setminus PSL(2, \mathbb{Z})} |\chi_{\text{vac}}(\gamma \tau)|^2, \]  

a procedure generally known as a Poincaré series. The partition function thus obtained is a finite, modular-invariant function, but has the following unphysical features:

1) The spectrum of twists at fixed spin $j$ is continuous rather than discrete.

2) At $h = \bar{h} = \frac{c-1}{24}$, the spectrum has a degeneracy of $-6$.

3) In \cite{255}, it was pointed out that at any odd spin, for sufficiently low energy, the density of states is negative.

While the first feature is perhaps in conflict with the conventional interpretation of pure 3d gravity as dual to a single quantum system,\(^2\) the last two features signal non-unitarity are more problematic. In this chapter we will describe a simple modification of \cite{277} which is positive everywhere. We will moreover provide a natural physical interpretation of the new contributions that render the spectrum unitary.

Several other proposed resolutions of problems (2) and (3) have been considered in the literature. In \cite{65}, it was noted that problem (2) can be cured with the addition of six $c = 1$ free boson partition functions to the spectrum.\(^3\) In \cite{255}, problem (3) was cured with the addition of operators with sufficiently low twist (in particular with $\min(h, \bar{h}) \leq \frac{c-1}{32}$). In this chapter we will calculate the minimal operator content needed to cure (3). In particular, we will attempt to maximize the dimension and minimize the degeneracy of each

\(^2\)This is more problematic than the continuous spectra encountered in familiar examples of non-compact CFTs such as Liouville theory and the dual of type IIB string theory on $AdS_{3} \times S^{3} \times M_{4}$ supported purely by NS-NS three-form fluxes \cite{278,279} because here the vacuum is a normalizable state.

\(^3\)An entirely different – although highly non-minimal – proposal to cure these problems was considered in \cite{277}, which is to sum separately over two sets of $PSL(2, \mathbb{Z})$ instantons, which act on left- and right-moving sectors separately. The bulk interpretation of this is not entirely clear, however, as these lead to saddles which – although smooth as $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ Chern-Simons connections – are both complex and singular from the metric point of view.
additional operator we add. It turns out that at large $c$ the minimal prescription involves the addition of operators with conformal dimensions\footnote{We use the standard notation where the left- and right-moving conformal dimensions are related to the conformal dimension and spin by $\Delta = h + \bar{h}$ and $J = h - \bar{h}$.}

\[ h = \bar{h} = \frac{c}{24} \left( 1 - \frac{1}{N^2} \right) + \mathcal{O}(c^0), \quad N = 2, 3, \ldots. \] (8.1.4)

We will discover that these states have a beautiful interpretation in terms of AdS$_3$ gravity. In particular, equation (8.1.4) is precisely the correct conformal dimension to be interpreted as a conical defect geometry which is a quotient of AdS$_3$ by $\mathbb{Z}_N$. From the bulk point of view, this is a massive particle which produces a deficit angle of $2\pi(1 - \frac{1}{N})$. We will see that these orbifolds, with positive integer values of $N$, can be consistently included in the Euclidean path integral of three dimensional general relativity.

We will then proceed to construct the full spectrum of the theory by including these as saddle points in the sum over geometries that computes the torus partition function. We will argue that the gravitational path integral is one-loop exact (just as for the smooth saddle points which contribute to the torus path integral in [277]). We will compute the resulting spectrum that arises from the sum over geometries. We show that the inclusion of a finite number of orbifolds in the sum leads to a density of states that is finite and positive everywhere except at the threshold value of $h = \bar{h} = \frac{c - 1}{24}$. This negativity at threshold was encountered previously in the literature [65] and is easily cured with the addition of a free boson partition function. The result is a spectrum that is finite and continuous (in the regime $h, \bar{h} > \frac{c - 1}{24}$), but is everywhere positive for sufficiently large central charge. The fact that the spectrum is continuous reflects that it cannot be dual to a single, unitary, compact CFT, but it is possible that it can be interpreted as some kind of “averaged” partition function, as is known to occur in two dimensions [280,281]. We also consider the more general case where the infinite family of orbifolds with $N = 2, 3, \ldots$ are included in the sum over geometries. At first sight it might appear that the result must diverge, as the states (8.1.4) accumulate at $h = \bar{h} = \frac{c - 1}{24}$, resulting in an infinite number of states at this critical value of the dimension. We will, however, see that this is precisely cancelled by a divergence (coming with the opposite sign) that arises from the sum over
modular images. In this way the contributions of two infinite families of instantons – the \( PSL(2, \mathbb{Z}) \) saddles and the orbifolds – combine to give a candidate pure gravity partition function that is finite, although the resulting density of states does not appear to be well-defined. Another possibility is that one must truncate the sum over orbifolds in a central charge dependent way so that the full set of states is only included at infinite \( c \); a natural choice would be at \( N \sim \mathcal{O}(c^{1/2}) \), c.f. (8.1.4).

This chapter is organized as follows. In section 8.2, we review the partition function of \([65, 277]\) and rewrite the spectrum as a sum over modular crossing kernels. We also write down a minimal addition of operators to render the spectrum positive everywhere. In section 8.3 we provide a physical interpretation of the extra states we add as singular orbifolds that contribute to the path integral of pure gravity in AdS\(_3\). We will argue that this path integral must be one-loop exact. In section 8.4, we perform a direct one-loop computation to see the \( 1/c \) of effect of these orbifold geometries to the path integral. Finally in section 8.5 we conclude and discuss possible interpretations of our partition function.

### 8.2 Non-unitarity of the pure gravity partition function

We will start by reviewing the construction of the so-called MWK partition function of pure AdS\(_3\) gravity, which was first discussed in \([277]\) and studied further in \([65]\). This partition function is obtained by computing a sum over Euclidean geometries with torus boundary, including in the sum all metrics that are continuously connected to a smooth saddle point. This sum includes thermal \( AdS_3 \), the Euclidean BTZ black hole and the so-called \( PSL(2, \mathbb{Z}) \) family of black holes. The \( PSL(2, \mathbb{Z}) \) family of black holes can be thought of as generalizations of the BTZ black hole with different linear combinations of the boundary cycles taken to be contractible. At the level of the partition function of the boundary theory, this amounts to a sum over \( PSL(2, \mathbb{Z}) \) images of the Virasoro vacuum character. So while the construction of the boundary partition function via a Poincaré series may seem ad hoc from the point of view of the CFT, it has a natural physical interpretation as the sum over saddles in the Euclidean gravitational path integral. As discussed in \([255]\), the
microcanonical density of states one obtains from this procedure becomes negative for odd spin primaries of high spin and low twist. One way to cure this negativity is by the addition of states with twist at or below $\frac{c-1}{16}$ to the seed spectrum. In this section we consider the “minimal” set of operators that can be added to render the spectrum unitary. Although in this chapter we focus on the large $c$ limit, we note that many of the results in this section only require $c > 1$.

### 8.2.1 Review of the MWK spectrum

The MWK partition function is obtained by taking the Virasoro vacuum character – which is interpreted as the partition function of a gas of boundary gravitons in thermal $\text{AdS}_3$ – and summing over $\text{PSL}(2, \mathbb{Z})$ images. Although the sum diverges, $[65, 277]$ used zeta-function regularization to give a finite, modular-invariant answer with a continuous density of states at each integer spin. We will give a slightly different derivation of this result, where rather than working with the partition function we will work directly with the density of states. Thus instead of computing the partition function as a sum over $\text{PSL}(2, \mathbb{Z})$ images of the vacuum character, we compute directly the contribution to the density of states coming from one of the $\text{PSL}(2, \mathbb{Z})$ images of the vacuum state in terms of a modular crossing kernel, and then sum over $\text{PSL}(2, \mathbb{Z})$.

The modular crossing kernel $K^{(\gamma)}$ for an element $\gamma$ of $\text{PSL}(2, \mathbb{Z})$ expresses the modular transform of a Virasoro character $\chi_h(\gamma \tau)$ in a basis of untransformed characters:

$$\chi_k(\gamma \tau) = \int_{\frac{2\pi i}{c \tau}}^{\infty} dh' K_h^{(\gamma)} \chi_{h'}(\tau),$$

(8.2.1)

where $\gamma \tau = \frac{a \tau + b}{c \tau + d}$ for $\begin{pmatrix} a & b \\ s & d \end{pmatrix} \in \text{PSL}(2, \mathbb{Z})$.\(^5\) The Virasoro character is

$$\chi_h(\tau) = \frac{q^h - q^{-h}}{\eta(\tau)},$$

(8.2.2)

\(^5\)We denote the element of $\text{PSL}(2, \mathbb{Z})$ as $\begin{pmatrix} a & b \\ s & d \end{pmatrix}$ to avoid confusion with the central charge $c$. 

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where \( q = e^{2\pi i \tau} \) and the explicit expression for the modular crossing kernel for \( s > 0 \) is\(^6\)

\[
K_{\gamma}^{(\gamma)}(h, h') = \epsilon(\gamma) \sqrt{\frac{2}{\pi}} e^{\frac{2\pi i}{s}(a(h - \frac{c-1}{24}) + d(h' - \frac{c-1}{24}))} \cos \left( \frac{4\pi}{s} \sqrt{(h - \frac{c-1}{24})(h' - \frac{c-1}{24})} \right). \quad (8.2.3)
\]

Here \( \epsilon(\gamma) \) is a \( h, h' \)-independent phase that will be unimportant since we will be considering products of holomorphic and antiholomorphic Virasoro characters.

The MWK partition function of \([277]\) and \([65]\) is obtained by summing over \( PSL(2, \mathbb{Z}) \) images of the Virasoro vacuum character

\[
\chi_{\text{vac}}(\tau, \bar{\tau}) = (\chi_0(\tau) - \chi_1(\tau))(\bar{\chi}_0(\bar{\tau}) - \bar{\chi}_1(\bar{\tau})) = \frac{q^{-\frac{c-1}{24}}(1 - q)^2}{|\eta(\tau)|^2}. \quad (8.2.4)
\]

Aside from the vacuum Verma module itself, the spectrum has support only on primaries above the BTZ threshold, \( \min(h, \bar{h}) \geq \frac{c-1}{24} \). Thus one can write the density of states of the MWK partition function simply as a \( PSL(2, \mathbb{Z}) \) sum over the modular crossing kernels, as:

\[
\rho^{MWK}(h, \bar{h}) = \sum_{\gamma \in \mathbb{Z} \setminus PSL(2, \mathbb{Z})} \left[ K_{h_0}^{(\gamma)} \bar{F}_{h_0}^{(\gamma)} - K_{h_0}^{(\gamma)} F_{h_1}^{(\gamma)} - K_{h_1}^{(\gamma)} F_{h_0}^{(\gamma)} + K_{h_1}^{(\gamma)} \bar{F}_{h_1}^{(\gamma)} \right]. \quad (8.2.5)
\]

In appendix E.1 we show that the density of states (8.2.5) can easily be decomposed into sectors with integer

\(^6\)We thank Henry Maxfield for bringing the explicit expression for the general \( PSL(2, \mathbb{Z}) \) modular crossing kernel to our attention. See also appendix D of [255] for the kernels with \( b = 0 \).
spin. We then arrive at the following density of states (see also (4.11) of [255])\(^7\)

\[
\rho_{j}^{\text{MWK}}(t) = \frac{2}{\sqrt{t(t+j)}} \sum_{s=1}^{\infty} \frac{1}{s} \left[ S(j,0;s) \cosh \left( \frac{4\pi}{s} \sqrt{\frac{c-1}{24}(t+j)} \right) \cosh \left( \frac{4\pi}{s} \sqrt{\frac{c-1}{24}t} \right) 
- S(j,-1;s) \cosh \left( \frac{4\pi}{s} \sqrt{\frac{c-1}{24}(t+j)} \right) \cosh \left( \frac{4\pi}{s} \sqrt{\frac{c-25}{24}t} \right) 
- S(j,1;s) \cosh \left( \frac{4\pi}{s} \sqrt{\frac{c-25}{24}(t+j)} \right) \cosh \left( \frac{4\pi}{s} \sqrt{\frac{c-1}{24}t} \right) 
+ S(j,0;s) \cosh \left( \frac{4\pi}{s} \sqrt{\frac{c-25}{24}(t+j)} \right) \cosh \left( \frac{4\pi}{s} \sqrt{\frac{c-25}{24}t} \right) \right],
\]

(8.2.6)

where \(S(j, J; s)\) is a Kloosterman sum

\[
S(j, J; s) = \sum_{0 \leq d < s, \gcd(s,d)=1} \exp \left( 2\pi i \frac{dj}{s} + \left( d^{-1}\right)s J \right),
\]

(8.2.7)

and we have written the density of states in terms of what we will refer to as the \textit{reduced twist}

\[
t \equiv \min(h, \bar{h}) - \frac{c-1}{24}.
\]

(8.2.8)

This density of states (8.2.6) is only defined for \(t \geq 0\), i.e. it only has support on twists above the BTZ threshold.

The sum over \(\text{PSL}(2, \mathbb{Z})\) described above is known as a Poincaré series, and the procedure described above is an algorithm that takes a seed state and computes a modular invariant spectrum that is otherwise only supported above the BTZ threshold \(t \geq 0\) (except for the seed state, of course). As we will see, however, unitarity of the resulting spectrum is not guaranteed. See figure 8.1 for a sketch of the spectrum.

More generally, we can consider the contribution to the density of states of the \(\text{PSL}(2, \mathbb{Z})\) sum of the contribution of an arbitrary light seed operator with reduced twist \(T < 0\) and spin \(J\). The above procedure

\(^7\)In what follows, we will often make use of parity-invariance and assume without loss of generality that \(j \geq 0\).
Figure 8.1: The spectrum of the MWK partition function. We start with the vacuum state, at \( h = \hbar = 0 \) and get a continuous spectra at all integer spins, with \( h, \hbar \geq \frac{c}{24} \). The spectrum is not unitarity; the red regions represent the parts of the spectrum where the density of states is negative.

leads to the following density of states in the spin-j sector for \( jJ \geq 0 \)

\[
\rho_j^{(T,J)}(t) = \frac{2}{\sqrt{t(t + |j|)}} \sum_{s=1}^{\infty} \frac{1}{s} S(j, J; s) \cosh \left( \frac{4\pi}{s} \sqrt{-T + |J|} (t + |j|) \right) \cosh \left( \frac{4\pi}{s} \sqrt{-T} t \right). 
\] (8.2.9)

On the other hand, if \( jJ \leq 0 \) we have

\[
\rho_j^{(T,J)}(t) = \frac{2}{\sqrt{t(t + |j|)}} \sum_{s=1}^{\infty} \frac{1}{s} S(j, J; s) \cosh \left( \frac{4\pi}{s} \sqrt{-T + |J|} t \right) \cosh \left( \frac{4\pi}{s} \sqrt{-T} (t + |j|) \right). 
\] (8.2.10)

The sums (8.2.6), (8.2.9) and (8.2.10) are divergent, as the summands behave like \( \mathcal{O}(s^{-1}) \) at large \( s \). In appendix E.1 we describe a regularization procedure that renders them finite, analogous to the zeta function regularization of [65,277]. We also show that this regularized sum of modular kernels is precisely the MWK spectrum [65,277]. The regularization procedure produces a negativity at threshold in the scalar sector of

\[ \text{[In the recent paper [282], a different prescription based on a technique by Rademacher was described that does not have this divergence in } s. \text{ One of the cosh factors is replaced with a sinh, leading to a different finite, modular-invariant partition function with the same censored (twist below the black hole threshold) part of the spectrum. In the case of a scalar seed, the Rademacher expansion can be interpreted as a Poincaré series where the seed is taken to be the original character plus a continuous density of scalar characters above the black hole threshold. It is not obvious to us whether the Rademacher expansion with a non-scalar seed can also be interpreted as a Poincaré series. We thank L. F. Alday for discussions on this point.}\]
the theory at $\hbar = \frac{\bar{c} - 1}{24}$ discussed in [65, 277]:

$$\rho_{j=0}^{\text{MWK}}(t) = -6\delta(t) + \text{(a continuous function of } t\text{)}, \quad (8.2.11)$$

which we re-derive in appendix E.1 using the modular kernels.

In [65] an ad hoc prescription was offered to cure this negativity, namely via the addition of six copies of the compact boson partition function to the pure gravity partition function. Besides this discrete negative degeneracy in the scalar sector, the MWK density of states in each spin sector is a smooth function of the twist supported only above the BTZ threshold $t > 0$.

The derivation described above allows us to easily understand the negativity in the near-extremal, large-spin regime recently discussed in [255]. From (8.2.6), we see that there is a simple expression for the near-extremal ($t \to 0$), large-spin ($j \to \infty$) limit of the MWK density of states

$$\rho_j^{\text{MWK}}(t) \sim \frac{8\pi^2}{\sqrt{j}} \sqrt{t} \exp \left( 4\pi \sqrt{\left( \frac{c - 1}{24} \right) j} \right) + \sum_{s=2}^{\infty} \left[ \frac{S(j, 0; s) - S(j, -1; s)}{s \sqrt{j} t} \exp \left( \frac{4\pi}{s} \sqrt{\left( \frac{c - 1}{24} \right) j} \right) \right] + \frac{S(j, 0; s) - S(j, 1; s)}{s \sqrt{j} t} \exp \left( \frac{4\pi}{s} \sqrt{\left( \frac{c - 25}{24} \right) j} \right). \quad (8.2.12)$$

The first term in (8.2.12) exhibits the usual square-root edge in the reduced twist near extremality at large spin, as noted in [255, 256]. Note that when the twist is sufficiently small, in particular for $t \gtrsim \frac{1}{8\pi} e^{-2\pi \sqrt{\left( \frac{c - 1}{24} \right) j}}$, the second and third terms will dominate over the first. We see that if the coefficient $S(j, 0; s) - S(j, -1; s)$ is negative, then there may be a regime where the density of states is negative.\(^9\)

To expand on this point, we compare the above expression to the contribution to the partition function of a Poincaré series starting with a seed of reduced twist $T < 0$ and spin $J = 0$. This contributes to the density of states as

$$\rho_j^{(T,J=0)}(t) \sim \sum_s \frac{S(j, 0; s)}{s \sqrt{j} t} \exp \left( \frac{4\pi}{s} \sqrt{-Tj} \right). \quad (8.2.13)$$

\(^9\)The asymmetry between $S(j, 0; s) - S(j, -1; s)$ and $S(j, 0; s) - S(j, 1; s)$ in this expression arises because we take $j > 0$. If we instead took $j < 0$, the role of the two terms would be exchanged.
in the limit under consideration. The generalization to seed states with \( J \neq 0 \) is straightforward, using \( (8.2.9) \) and \( (8.2.10) \).

We conclude that if \( S(j, 0; n) - S(j, -1; n) \) is negative for some \( j, n \), it contributes a negativity to the partition function of the same magnitude as a state with reduced twist \( T_n \) such that

\[
T_n = -\frac{c - 1}{24n^2}.
\]  

\[ (8.2.14) \]

Similarly, if \( S(j, 0; n) - S(j, 1; n) \) is negative, for some \( j, n \), it contributes a negativity to the partition function of the same magnitude as a state with twist \( \tilde{T}_n \) such that

\[
4\pi \sqrt{-\tilde{T}_n} j = \frac{4\pi}{n} \sqrt{\left(\frac{c - 25}{24}\right) j},
\]

\[ (8.2.15) \]

which means

\[
\tilde{T}_n = -\frac{c - 25}{24n^2} = T_n + \frac{1}{n^2}.
\]

\[ (8.2.16) \]

A crucial point, and one which will be described in more detail in section 8.3, is that – to leading order in \( c \) – the reduced twists \( T_n \) and \( \tilde{T}_n \) precisely correspond to the orbifold geometries discussed in section 2 of [277].

**8.2.2 Minimalist spectrum**

In this section we will describe the minimal number of states needed to add to the MWK spectrum to render it positive. For simplicity we will only add scalar seeds. We will (first) maximize the dimension and (second) minimize the degeneracy of every additional scalar operator we add. Remarkably at sufficiently large central charge, it appears that adding a *finite* number of operators is enough to render the entire spectrum positive above threshold. In the notation of the \( (8.2.14), (8.2.16) \), we only need to add operators up at \( T_2, \tilde{T}_2, T_3, T_4, T_5, \tilde{T}_5, T_6, \) and \( T_7 \). In appendix E.2 we will explicitly show this calculation; here we will
Table 8.1: Minimum number of scalars we need to add to render the MWK partition function finite.

<table>
<thead>
<tr>
<th>State</th>
<th>Conformal weights</th>
<th>Minimal “degeneracy”</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_2$</td>
<td>$\hbar = \bar{\hbar} = \frac{c-1}{32}$</td>
<td>1</td>
</tr>
<tr>
<td>$\bar{T}_2$</td>
<td>$\hbar = \bar{\hbar} = \frac{c+7}{32}$</td>
<td>1</td>
</tr>
<tr>
<td>$T_3$</td>
<td>$\hbar = \bar{\hbar} = \frac{c-1}{27}$</td>
<td>1</td>
</tr>
<tr>
<td>$T_4$</td>
<td>$\hbar = \bar{\hbar} = \frac{5(c-1)}{128}$</td>
<td>1</td>
</tr>
<tr>
<td>$T_5$</td>
<td>$\hbar = \bar{\hbar} = \frac{c-1}{25}$</td>
<td>$\frac{5+\sqrt{5}}{10}$</td>
</tr>
<tr>
<td>$\bar{T}_5$</td>
<td>$\hbar = \bar{\hbar} = \frac{c}{25}$</td>
<td>$\frac{5-\sqrt{5}}{10}$</td>
</tr>
<tr>
<td>$T_6$</td>
<td>$\hbar = \bar{\hbar} = \frac{35(c-1)}{864}$</td>
<td>1</td>
</tr>
<tr>
<td>$T_7$</td>
<td>$\hbar = \bar{\hbar} = \frac{2(c-1)}{49}$</td>
<td>$\frac{3+4 \sin \left(\frac{3\pi}{14}\right)}{7}$</td>
</tr>
</tbody>
</table>

simply write down the result.\textsuperscript{10}

In Table 8.1, we write down the degeneracy and weight of every seed scalar operator needed. To each term in Table 8.1, we perform a Poincaré sum as in (8.2.9) and add it to the MWK partition function. Finally we add the partition function of $12 + \frac{3+4 \sin \left(\frac{3\pi}{14}\right)}{7}$ self-dual free boson partition functions (to cancel the delta function negativities at threshold in the scalar sector). In appendix E.2 we will show that if the central charge is sufficiently large, this spectrum is positive everywhere; numerically we find that the cutoff is $c \gtrsim 3237.7$.

Of course the degeneracies in Table 8.1 are not integers, so this spectrum cannot be interpreted as a single compact, unitary CFT. On the other hand, the spectrum above the BTZ threshold is already continuous so we have already abandoned compactness.\textsuperscript{11} We will comment more on possible interpretations in section 8.5.\textsuperscript{12}

\textsuperscript{10}This table of degeneracies has been independently discovered by L. F. Alday and J.-B. Bae [282].

\textsuperscript{11}By non-compact we simply mean that the spectrum is not given by a discrete sum of delta functions. However, the spectrum still has a normalizable $PSL(2,\mathbb{C})$ invariant vacuum, unlike other non-compact CFTs such as Liouville theory.

\textsuperscript{12}If we demand that the degeneracies of all the $T_n$, $\bar{T}_n$ states are integers, we can have 1 state at $T_2$, $\bar{T}_2$, $T_3$, $T_4$, and $T_5$, as well as 11 free boson partition functions, and the resulting partition function will be positive for $c > 1465.4$.  

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8.3 A geometric interpretation for the missing states

In this section we will discuss how these orbifold states can be included directly in the path integral formulation of pure gravity in AdS$_3$. We will follow the construction of MWK, who computed the partition function of pure gravity by enumerating the known saddle points in the gravitational path integral, along with an appropriate series of quantum corrections:

$$Z(\tau, \bar{\tau}) \equiv \int_{\partial M = T^2} Dg \ e^{-S} = \sum_{g_0} e^{-cS(g_0) + S^{(1)}(g_0) + \ldots}$$  \hspace{1cm} (8.3.1)

We are computing here a sum over geometries whose boundary is a torus with modular parameter $\tau$, so that the partition function can be used to extract the density of states $\rho(h, \bar{h})$. Here $S(g_0)$ denotes the classical action of a classical saddle $g_0$, $S^{(1)}$ the one-loop correction to this action, and $\ldots$ the infinite series of subleading perturbative corrections. The basic conjecture of MWK is that if the complete set of classical solutions $g_0$, along with the infinite series of perturbative corrections $\ldots$ are included in the sum, then the saddle point approximation (8.3.1) is exact. This amounts to an assumption about the nature of the path integral of quantum gravity – that it includes only metrics which are continuously connected to saddle points. This is a familiar feature of many quantum field theories (especially Chern-Simons and topological field theories) whose partition function can be computed exactly, but should nevertheless be regarded as an assumption.

In the original MWK computation, the classical saddle points which contribute to this sum were taken to be the smooth Euclidean geometries which solve Einstein’s equation (so are locally Euclidean AdS$_3$) and have torus boundary. These geometries were completely classified, and can be interpreted physically as thermal AdS$_3$ and the Euclidean continuations of the $PSL(2, \mathbb{Z})$ family of BTZ black holes [283, 284]. These latter saddle points are, in Euclidean signature, related to the thermal AdS saddle by $PSL(2, \mathbb{Z})$ modular transformations. The perturbative corrections to these saddles can be computed by noting that in pure gravity the thermal AdS partition function will, at the perturbative level, receive contributions only from
multi-graviton states which are entirely fixed by Virasoro symmetry. The result is that the perturbative partition function is one-loop exact, and one can perform the sum over geometries explicitly. However, as reviewed in the previous section, the resulting spectrum is not unitary, as the density of states is negative in the near-extremal, large-spin regime.

Here, we will advocate a small departure from the above construction: we will relax the assumption that the saddle point geometries \( g_0 \) must be manifolds. We will instead consider the inclusion of orbifold geometries, which take the form of quotients of AdS\(_3\). We will see that the orbifolds with torus boundary can be completely classified, just as in the previous discussion. The result will be that the sum over geometries now includes additional gravitational instantons corresponding to \( \mathbb{Z}_N \) quotients of thermal AdS, along with their images under PSL(2, \( \mathbb{Z} \)) modular transformations. We will also see that the perturbative corrections to these classical saddle points are one-loop exact. The resulting gravity partition function can then be computed, and has a completely positive density of states in the dangerous near-extremal, large-spin regime.

In \([277]\) the classical solutions to the equations of motion of three dimensional gravity with negative cosmological constant and torus boundary were classified. Here we briefly review that classification, before generalizing to include orbifolds. Any solution \( M \) is a quotient \( M = \text{AdS}_3/\Gamma \), where \( \Gamma \) is a subgroup of \( SO(1,3) \) that acts discretely on \( \text{AdS}_3 \subset \text{AdS}_3 \).\(^{13}\) The conformal boundary of \( M \), topologically a torus, can then also be written as a quotient

\[
\Sigma = U/\Gamma, \tag{8.3.2}
\]

where \( U \subset \mathbb{CP}^1 \) is the conformal boundary of \( \text{AdS}_3 \), which is acted on by \( SO(1,3) = SL(2,\mathbb{C}) \) in the usual way by fractional linear Mobius transformations. One can then show \([277]\) that the fundamental group of \( U \) must be

\[
\pi_1(U) \cong \mathbb{Z}. \tag{8.3.3}
\]

In this case \( U \) is topologically a cylinder and \( \Gamma \) is a discrete subgroup of the group of diagonal matrices. The

\(^{13}\)Here we slightly abuse notation by using \( \text{AdS}_3 \) to denote Euclidean Anti-de Sitter space (i.e. \( \mathbb{H}_3 \)) rather than Lorentzian AdS.
semiclassical solutions then split into two qualitatively different classes, which we briefly describe below.

In the first class $\Gamma \cong \mathbb{Z}$, which was the case primarily considered in [65, 277]. This class of solutions corresponds to thermal $AdS_3$ and the $PSL(2,\mathbb{Z})$ family of BTZ black holes [283]. In this case $\Gamma$ is (up to an overall conjugation) generated by the diagonal matrix

$$X = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix},$$  \hspace{1cm} (8.3.4)$$

where $q = e^{2\pi i \tau}$ is the modulus of the boundary torus $\Sigma$, with $|q| < 1$. Of course, $\tau$ only defines inequivalent boundary Riemann surfaces up to $PSL(2,\mathbb{Z})$ transformations

$$\tau \rightarrow \frac{a\tau + b}{s\tau + d} \in PSL(2,\mathbb{Z}).$$  \hspace{1cm} (8.3.5)$$

Furthermore, as pointed out by [277], the family of three-manifolds solving the equations of motion are actually only labelled by a pair of coprime integers $(s, d)$; $(a, b)$ are only uniquely fixed by the $PSL(2,\mathbb{Z})$ condition $ad - bs = 1$ up to integer shifts of $(a, b)$ by $(s, d)$ since the resulting $q$ is invariant under such shifts.

Following [277], we refer to such three-manifolds as $\mathcal{M}_{s,d}$. They can be thought of as generalizations of the BTZ black hole with different linear combinations of the boundary cycles taken to be contractible.

It will be instructive to briefly describe the geometry of $\mathcal{M}_{0,1}$ in more detail. The Euclidean $AdS_3$ metric is given by

$$ds^2 = dr^2 + \cosh^2 r \, dT^2 + \sinh^2 r \, d\phi^2,$$  \hspace{1cm} (8.3.6)$$

where $r \in [0, \infty)$, $T \in (-\infty, \infty)$ and $\phi \in [0, 2\pi)$, with the conformal boundary at $r = \infty$. To quotient by the action of $X$ as in (8.3.4), the geometry is cut at $T = 0$ and $T = 2\pi \tau_2$ and glued upon making a rotation by the angle $2\pi \tau_1$. The resulting geometry is what we refer to as thermal $AdS_3$. In section 8.4 we will find it convenient to make use of the alternate coordinates $\rho = e^T$ and $\csc^2 \theta = \cosh^2 r$, so that $\rho \in [1, e^{2\pi \tau_2}]$ and
\[ \theta \in (0, \frac{\pi}{2}] . \]

In the other class of solutions \( \Gamma \cong \mathbb{Z} \times \mathbb{Z}_N \), for \( N \) an integer greater than or equal to 2. In this case \( \Gamma \) is generated by the following matrix in addition to (8.3.4)

\[
Y = \begin{pmatrix}
e^{\frac{2\pi i}{N}} & 0 \\
0 & e^{-\frac{2\pi i}{N}}
\end{pmatrix}.
\] (8.3.7)

This class of solutions includes geometries that are (singular) \( \mathbb{Z}_N \) orbifolds of thermal \( AdS_3 \) and of the \( PSL(2, \mathbb{Z}) \) family of black holes. The possibility of including these in the gravitational path integral was briefly considered in [277], before being discarded in favor of the “minimal” strategy of including only smooth manifolds. The \( \mathbb{Z}_N \) generators act on the \( AdS_3 \) bulk with fixed points, leading to a codimension-2 singularity in the interior and a deficit angle

\[
\Delta \phi_N = 2\pi (1 - N^{-1}).
\] (8.3.8)

The metric of the \( \mathbb{Z}_N \) orbifolds of thermal \( AdS_3 \), \( \mathcal{M}_{0,1}/\mathbb{Z}_N \), is given by (8.3.6), but the quotient by \( Y \) as in (8.3.7) leads to the further identification \( \phi \sim \phi + \frac{2\pi}{N} \). See figure 8.2 for a cartoon of the Euclidean geometry.

\[ \text{Figure 8.2: Left:} \] The Euclidean geometry is asymptotically \( AdS_3 \) but has an orbifold singularity in the interior (shown in green). \[ \text{Right:} \] The orbifold singularity corresponds to a conical defect of \( 2\pi(1 - 1/N) \).

The conventional wisdom regarding such singularities in three-dimensional gravity is that they signal the presence of massive particles with Planckian mass that backreact on the geometry and source a conical defect. In the classical limit, the deficit angle \( \Delta \phi \) sourced by a massive scalar particle is related to its mass by [285]

\[
\Delta \phi \approx 8\pi G_N m. \] (8.3.9)
The masses \( m_N \) corresponding to the \( Z_N \) conical defects lead to the following conformal weights in the classical limit

\[
h_N \approx \frac{c}{24} (1 - N^{-2}). \tag{8.3.10}
\]

In \cite{229} a proposal was made for a (generically non-geometric) notion of conical defect states at finite central charge associated to local CFT operators, based on the observation that the large-spin spectrum is additive in the Liouville momentum \( \alpha \), which corresponds in the classical limit to the additivity of the conical deficit angles at large separation. The proposal was\(^\text{14}\)

\[
\Delta \phi = \frac{4\pi \alpha}{Q} = 2\pi \left( 1 - \sqrt{1 - 8GNm} \right). \tag{8.3.11}
\]

This provides a plausible definition of the conical defect states in the quantum regime. The finite \( c \) value of the reduced twist corresponding to the \( Z_N \) deficit angle (8.3.8) is then given precisely by (8.2.14), one of the values of the twist whose addition to the gravitational path integral may cure the large-spin non-unitarity of the pure gravity spectrum.

Here, we will consider such singular geometries as equally valid contributions to the gravitational path integral. In particular, we will consider the addition of states with reduced twist (8.2.14) along with their \( PSL(2, \mathbb{Z}) \) images. This corresponds to including the singular orbifold saddles, with all possible linear combinations of the boundary cycles taken to be contractible, in the gravitational path integral with torus boundary.

Although such a modification to the gravitational path integral may violate the aesthetic ideals of “pure” quantum gravity, we would like to point out that the addition of these states is qualitatively different at the level of the spectrum of the dual CFT than the addition of a generic light operator, in the following sense. In \cite{229} and \cite{253}, it was shown that an irrational CFT with \( c > 1 \) and nonzero twist gap has the following property: if the spectrum includes two light Virasoro primary operators with twists \( h_1 \) and \( h_2 \),

\(^{14}\)The Liouville variables \( \alpha(h), Q(c) \) are defined below and in appendix E.1.
then the spectrum also contains infinitely many primary operators arranged into discrete Regge trajectories, with twists that accumulate to the following "Virasoro double-twist" values at large spin

\[ h_m = h_1 + h_2 + m + \delta h_m, \quad \text{for each integer } m \geq 0 \text{ such that } h_m < \frac{c-1}{24}, \quad (8.3.12) \]

where

\[ \delta h_m = -2(\alpha(h_1) + mb(c))(\alpha(h_2) + mb(c)) + m(m + 1)b^2(c) < 0 \quad (8.3.13) \]

is the anomalous twist resulting from the exact summation of the exchange of all multi-stress tensor composites. Here, \( \alpha(h) \) is a function of the weight defined in terms of the Liouville variables\(^{15}\) as

\[ \alpha(h) = \frac{Q}{2} + iP(h) \in \left(0, \frac{Q}{2}\right) \quad \text{for } 0 < h < \frac{c-1}{24}. \quad (8.3.14) \]

In the bulk, these infinite towers of states have the interpretation of multi-particle bound states in \( AdS_3 \), with the large-spin anomalous dimension (8.3.13) corresponding to the gravitational binding energy due to the totality of (multi-)graviton exchanges, which is nontrivial even at infinite separation in \( AdS_3 \). Precisely above the value \( h = \frac{c-1}{12} \) (corresponding to the \( Z_2 \) orbifold seed state), there can be no such discrete towers of composite operators built out of the seed operator; this is because \( 2\alpha(\frac{c-1}{12}) = \frac{Q}{2} \). Correspondingly, there are no infinite towers of multi-particle bound states built out of the particle that sources the conical defect. See figure 8.3 for a qualitative comparison between the pure gravity spectrum endowed with conical defect states and the spectrum of an irrational CFT with a generic light operator. We thus view our proposed modification as a minimal violation of the conceptual ideals of pure quantum gravity, in the sense that the new states have purely geometrical interpretations, and do not lead to infinite towers of multi-particle bound states in \( AdS_3 \).

We note in passing that the need to include subleading saddles corresponding to orbifold geometries in the near-horizon \( AdS_2 \) functional integral in order to reproduce non-perturbative corrections to the entropy of

\(^{15}\)See (E.1.3) and (E.1.4) for explicit definitions of \( P, Q, b \) in terms of the usual CFT\(_2 \) variables.
Figure 8.3: **Left:** The spectrum of Virasoro primary operators of the pure gravity partition function endowed with the conical defect states. There are no discrete Regge trajectories with an asymptotic twist below the black hole threshold, and the spectrum of twists is continuous above the black hole threshold. **Right:** A rough sketch of the spectrum of Virasoro primary operators of a generic irrational CFT with a light scalar operator with weight $h_0$ below $\frac{c}{16}$. In this case there are a finite, discrete set of infinite towers of multi-twist operators with asymptotic twist below the black hole threshold. This sketch is meant to be schematic, and merely capture the existence of the discrete multi-twist Regge trajectories below the black hole threshold — in a genuine CFT, the Regge trajectories are not exactly flat (i.e. the multi-twist operators have anomalous dimensions at finite spin) and the spectrum of twists above the black hole threshold is only continuous at large spin. Furthermore, the presence of the multi-twist Regge trajectories will serve to modify the black hole threshold in a spin-dependent way; see [256] for more details.

Four-dimensional supersymmetric black holes from localization has been previously observed in [286–288].\textsuperscript{16} However, these orbifold geometries are typically non-singular from the point of view of the full near-horizon geometry.

### 8.4 The one-loop determinant on orbifold backgrounds

We will now consider the perturbative corrections to the orbifold geometries described above. In this section we will study the one-loop partition function for gravitons on backgrounds that are $\mathbb{Z}_N$ orbifolds of thermal $AdS_3$, namely $H_3/(\mathbb{Z} \times \mathbb{Z}_N)$, where $H_3$ is hyperbolic three-space. For the purposes of this computation of the gravitational one-loop determinant of the conical defect geometries it is essential that $N$ is an integer.\textsuperscript{16}

\textsuperscript{16}We are grateful to Xi Yin for bringing this to our attention.
We will closely follow the approach of [118], where the one-loop gravity partition function on Euclidean $AdS_3$ was computed using the heat kernel method and the method of images. There, it was shown that the one-loop determinant on Euclidean $AdS_3$ yielded an answer that was consistent with the Virasoro vacuum character

$$Z_{\text{1-loop}}(\tau, \bar{\tau}) = (q \bar{q})^{-\frac{c}{24}} \prod_{m=2}^{\infty} \frac{1}{|1 - q^m|^{2}}, \quad c = \frac{3L}{2G_N} + {\mathcal{O}}(1), \quad (8.4.1)$$

where $L$ is the $AdS_3$ radius.

Since our backgrounds of interest are orbifolds of thermal $AdS_3$, we will simply apply the method of images to the results of [118]. We expect to recover the non-degenerate Virasoro character

$$Z_{\text{1-loop}}^{(N)}(\tau, \bar{\tau}) = (q \bar{q})^{-\hat{T}} \prod_{m=1}^{\infty} \frac{1}{|1 - q^m|^{2}}, \quad (8.4.2)$$

where the reduced twist $T$ ought to agree with the values (8.2.14) and (8.2.16) required to render the pure gravity spectrum positive, at least to leading order in $c$. We emphasize that equation (8.4.2) is one-loop exact, as it is completely fixed by the central charge and the structure of the Virasoro algebra. Thus, although we will not explicitly compute the two-loop and higher corrections, we expect that their only effect will be to renormalize the effective value of the central charge.

### 8.4.1 Review of the gravitational one-loop determinant on thermal $AdS_3$

We will start by briefly reviewing the computation of [118] for thermal $AdS_3$. The heat kernel $K(t; x, x')$ is a function of two spacetime positions $x, x'$ and an auxiliary time variable $t$. Suppose one is interested in computing the one-loop contribution to a partition function $Z$

$$Z = \int D\phi \exp \left( -h^{-2} \int d^4x \sqrt{g} \Delta \phi \right), \quad (8.4.3)$$
where $\Delta$ is a Laplacian and $h$ is a perturbative coupling. The heat kernel is typically defined in terms of eigenfunctions of the Laplacian, $\Delta_x \phi_n(x) = \lambda_n \phi_n(x)$ as

$$K(t; x, x') = \sum_n e^{-\lambda_n t} \phi_n(x) \phi_n(x'),$$

(8.4.4)

where the eigenfunctions $\phi_n$ have been normalized in a convenient way, $\sum_n \phi_n(x) \phi_n(x') = \delta^d(x - x')$ and $\int d^d x \sqrt{g} \phi_n(x) \phi_n(x) = \delta_{nm}$. The one-loop contribution to the partition function is computed by the logarithm of the determinant of the Laplacian\(^{17}\)

$$S^{(1)} = -\frac{1}{2} \sum_n \log \lambda_n.$$  

(8.4.5)

This quantity can be conveniently computed by the trace of the heat kernel $K$

$$S^{(1)} = \frac{1}{2} \int_0^\infty \frac{dt}{t} \int d^3 x \sqrt{g} K(t; x, x).$$

(8.4.6)

Luckily for us, the heat kernels for gravitons on $H_3$ and $H_3/Z$ have been computed by [118], and we may simply proceed to use the method of images in order to compute the gravitational one-loop determinant on the orbifold backgrounds. The one-loop determinant on $H_3/Z$ takes the form [118]

$$S^{(1)} = \text{vol}(H_3/Z) \int_0^\infty \frac{dt}{t} \frac{1}{4 \pi t^{3/2}} \left( e^{-t} (1 + 8t) - e^{-4t} (1 + 2t) \right)$$

$$+ \sum_{n \neq 0} \int_0^\infty \frac{dt}{t} \int_{H_3/Z} d^3 x \sqrt{g} K^{\pm H_3(t, r(x, \gamma^n x))}.$$  

(8.4.7)

Many comments are in order. The first is that the first term is divergent and will require regularization; there is an IR divergence due to the infinite volume of $H_3/Z$ and a UV divergence due to the $t \to 0$ behaviour of the integrand, both of which can be dealt with by a local counterterm. Here, $Z$ is generated by $\gamma$, which

\(^{17}\)This simplified discussion does not directly apply to the case of interest because the relevant geometries are non-compact, leading to a divergent contribution to $S^{(1)}$ proportional to the volume of spacetime. Furthermore in the non-compact case the Laplacian typically does not have a discrete spectrum of eigenvalues.
acts on \( \mathbb{H}_3 \), parameterized by the metric

\[
    ds^2 = \frac{dy^2 + dzd\bar{z}}{y^2},
\]  

as

\[
    \gamma(y, z, \bar{z}) = (|q|^{-1}y, q^{-1}z, \bar{q}^{-1}\bar{z}),
\]

where \( q = e^{2\pi i \tau} \) is the modulus of the torus boundary. The geodesic distance \( r \) that enters the heat kernel is given by

\[
    r(x, x') = \text{arccosh} \left( 1 + \frac{(y - y')^2 + |z - z'|^2}{2yy'} \right).
\]  

It is often convenient to make use of polar coordinates

\[
    y = \rho \sin \theta, \\
    z = \rho \cos \theta e^{i\phi}.
\]

These coordinates range from \( 1 \leq \rho < e^{2\pi \tau_2} \), \( 0 \leq \phi < 2\pi \) and \( 0 \leq \theta < \frac{\pi}{2} \) on \( \mathbb{H}_3/\mathbb{Z} \). In terms of these coordinates, the geodesic distance in the second term in (8.4.7) is given by

\[
    r(x, \gamma^n x') = \text{arccosh} \left( 1 + 2 \sinh^2(n\pi \tau_2) + 2 \cot^2 \theta |\sin(n\pi \tau)|^2 \right),
\]

and the corresponding volume element is given by

\[
    d^3x\sqrt{g} = d\phi d\rho dr \frac{\sinh r}{4|\sin \pi n\tau|^2}.
\]  

The end result of the computation in [118], the details of which we will not attempt to reproduce here, is
the following for the gravitational one-loop determinant on thermal $AdS_3$

\begin{align}
S^{(1)} &= \text{vol}(H_3/Z) \int_0^\infty \frac{dt}{t} \frac{1}{4\pi^3/2} \left( e^{-t(1+8t)} - e^{-4t(1+2t)} \right) \\
&\quad + \int_0^\infty \frac{dt}{t} \sum_{n=1}^{\infty} \frac{2\pi^2 \tau_2}{|\sin \pi n \tau|^2} \frac{e^{-12\pi^2 n \tau_1^2}}{4\pi^{3/2} t} \left( e^{-t \cos 4\pi n \tau_1} - e^{-4t \cos 2\pi n \tau_1} \right) \\
&= -\frac{13}{6\pi} \text{vol}(H_3/Z) - \sum_{j=2}^{\infty} \log |1 - q^j|^2.
\end{align}

The first term is computed by analytically continuing the integral over $t$ by using the integral representation of the gamma function. After holographically renormalizing the infinite volume of $H_3/Z$ and combining with the classical piece, the one-loop gravity partition function on $H_3/Z$ then takes the form of the Virasoro vacuum character \((8.4.1)\) with

\begin{equation}
c = \frac{3L}{2G_N} + 13.
\end{equation}

This one-loop shift of the central charge has recently been emphasized in \[178\]. Although it can be removed by a local counterterm, we will keep it for the purposes of comparison to the orbifold computation, where we will no longer have the freedom to renormalize the cosmological constant.

### 8.4.2 The gravitational one-loop determinant on the conical defect backgrounds

The heat kernel on $H_3/(Z \times Z_N)$ can be obtained from that on $H_3$ by the method of images

\begin{equation}
K^{H_3/(Z \times Z_N)}(t, x, x') = \sum_{m=1}^{N} \sum_{n \in \mathbb{Z}} K^{H_3}(t, r(x, \gamma_N^{n,m} x')),
\end{equation}

where

\begin{equation}
\gamma_N^{n,m}(y, z, \bar{z}) \to (|q|^{-n/N} y, q^{-n/N} e^{-2\pi im/N} z, \bar{q}^{-n/N} e^{2\pi im/N} \bar{z}),
\end{equation}

where $n \in \mathbb{Z}$ and $m \in \{1, 2, \ldots, N\}$. Note that throughout this computation we have rescaled $\tau$ by $N^{-1}$ so that the complex structure of the boundary torus is the same as in the $N = 1$ case. The geodesic distance
that enters the heat kernel via the method of images then takes the form

\[ r(x, \gamma_n^m x) = \operatorname{arccosh}\left(1 + 2 \sinh^2\left(\frac{\pi \tau_2}{N}\right) + 2 \cot^2 \theta \left| \sin\left(\frac{\pi}{N} (n \tau + m)\right)\right|^2\right). \] (8.4.18)

We are now well-positioned to evaluate the gravity one-loop determinant on \( \mathbb{H}_3/(\mathbb{Z} \times \mathbb{Z}_N) \), which is given by

\[
S^{(1)} = \text{vol}(\mathbb{H}_3/(\mathbb{Z} \times \mathbb{Z}_N)) \int_0^\infty \frac{dt}{t} \frac{1}{4\pi t^{3/2}} \left( e^{-t}(1 + 8t) - e^{-4t}(1 + 2t) \right) \\
+ \sum_{m=1}^{N-1} \int_0^\infty \frac{dt}{t} \int_{\mathbb{H}_3/(\mathbb{Z} \times \mathbb{Z}_N)} d^3x \sqrt{g} K^{\mathbb{H}_3}(t, r(x, \gamma_0^m x)) \\
+ \sum_{m=1}^{N} \sum_{n \neq 0} \int_0^\infty \frac{dt}{t} \int_{\mathbb{H}_3/(\mathbb{Z} \times \mathbb{Z}_N)} d^3x \sqrt{g} K^{\mathbb{H}_3}(t, r(x, \gamma_n^m x)) \\
\equiv \delta_1 + \delta_2 + \delta_3,
\] (8.4.19)

where

\[
\delta_1 = -\frac{13}{6\pi} \text{vol}(\mathbb{H}_3/(\mathbb{Z} \times \mathbb{Z}_N)) \\
\delta_2 = \frac{1}{2} \sum_{m=1}^{N-1} \int_0^\infty \frac{dt}{t} \frac{2\pi^2 \tau_2}{N^2 \sin(\pi m/N)^2} 4\pi^{3/2} \sqrt{t} \left[ e^{-t} \cos(4\pi m/N) - e^{-4t} \cos(2\pi m/N) \right] \\
\delta_3 = \frac{1}{2} \sum_{m=1}^{N} \sum_{n=1}^\infty \int_0^\infty \frac{dt}{t} \frac{2\pi^2 \tau_2 e^{-\frac{12\pi m \tau_2}{N}}}{4\pi^{3/2} N^2 \sqrt{t}} \left[ \frac{e^{-t} \cos\left(\frac{4\pi}{N} (n \tau_1 + m)\right) - e^{-4t} \cos\left(\frac{2\pi}{N} (n \tau_1 + m)\right)}{\left| \sin\left(\frac{2\pi}{N} (n \tau + m)\right)\right|^2} + (n \rightarrow -n) \right].
\] (8.4.20)

Both \( \delta_1 \) and \( \delta_2 \) are formally divergent. The divergence in \( \delta_1 \) is due to the non-compactness of spacetime, while the divergence in \( \delta_2 \) from the \( t \rightarrow 0 \) part of the integrand is due to the fixed points of the \( \mathbb{Z}_N \) action. As we will see, both of these divergences can be regularized, although in the latter case we do not have an a priori physical justification for the choice of regularization scheme.

We will start by considering the final term. Borrowing some manipulations from [118], it can be written
as the following

\[\delta_3 = \sum_{m=1}^{N} \sum_{n=1}^{\infty} \sum_{n=1}^{2N} \frac{1}{2nN} \left[ \frac{e^{4\pi im/N} q^{\frac{2n}{N}}}{1 - e^{2\pi im/N} q^{\frac{2n}{N}}} + \frac{e^{-4\pi im/N} \bar{q}^{\frac{2n}{N}}}{1 - e^{-2\pi im/N} \bar{q}^{\frac{2n}{N}}} + (m \rightarrow -m) \right] \]

\[= \sum_{m=1}^{N} \sum_{n=1}^{\infty} \frac{1}{nN} \cos(2\pi m(\ell + 2)/N) \left( q^{\frac{n(\ell+2)}{N}} + \bar{q}^{\frac{n(\ell+2)}{N}} \right) \]

\[= \sum_{n=1}^{\infty} \sum_{\ell=0}^{\infty} \frac{\delta^{(N)}_{\ell+2,0}}{n} \left( q^{\frac{n(\ell+2)}{N}} + \bar{q}^{\frac{n(\ell+2)}{N}} \right) \]

\[= - \sum_{j=1}^{\infty} \log |1 - q_j|^2, \]

where \(\delta^{(N)}_{m,n} = 1\) if \(m \equiv n \pmod{N}\), and vanishes otherwise. This is precisely of the form of the non-degenerate Virasoro character that encodes contributions from descendants on a torus with modular parameter \(q\). Notice that the effect of the sum over \(\mathbb{Z}_N\) images is to produce a sum that starts at \(j = 1\) rather than \(j = 2\) as in (8.4.14).

Let’s now move on to the second term, for which the \(t\) integral will require regularization, due to the fixed points in the \(\mathbb{Z}_N\) action. As in the \(\mathfrak{sl}_3/\mathbb{Z}\) example, we will analytically continue using the integral representation of the gamma function. We have

\[\delta_2 = - \sum_{m=1}^{N-1} \frac{1}{N^2 \sin(\pi m/N)^2} \left[ \cos(4\pi m/N) - 2 \cos(2\pi m/N) \right] = +4\pi \tau_2 \frac{N^2 - 13}{24N^2}. \]

This almost takes the form of the one-loop correction to the dimension for scalar operators with reduced twist \(T = T_N\), except the shift in the numerator is by \(-13\) instead of the expected \(-1\). However, this contribution from the \(\mathbb{Z}_N\) images of the singular locus is potentially ambiguous due to the need to regularize, and may be subject to further ambiguities in the choice of boundary conditions for the graviton at the singular locus.\(^{18}\)

Although we do not have an a priori physical justification for our choice of regularization scheme, we simply choose to regularize in the same way as in thermal AdS\(_3\), where the analytic continuation of the integral

\(^{18}\)We thank Hirosi Ooguri, Yifan Wang, and Xi Yin for discussions on this point.
representation of the gamma function yields the one-loop correction to the central charge given in (8.4.15).

Combining the one-loop results (8.4.21) and (8.4.22) with the classical action and renormalizing the volume of the orbifold spacetime as in the thermal $AdS_3$ computation, we arrive at the following expression for the gravitational one-loop partition function on the orbifold backgrounds

$$Z_{1\text{-loop}}^{(N)} = e^{-\left(\frac{3L}{2\sqrt{N}} + 13\right)\frac{1}{6\pi} \text{vol}(H_3/(Z \times Z_N))} (q \bar{q})^{N^2 - 13 \over 24N^2} \prod_{m=1}^{\infty} \frac{1}{|1 - q^m|^2}$$

$$(8.4.23)$$

$$= (q \bar{q})^{N^2 - 13 \over 24N^2} \prod_{m=1}^{\infty} \frac{1}{|1 - q^m|^2}$$

where, following [178], we have renormalized the volume of spacetime as

$$\text{vol}(H_3/(Z \times Z_N)) = \frac{\pi^2 \tau_2}{N^2}.$$  

Given that the contribution from the thermal $AdS_3$ saddle is a Virasoro vacuum character with central charge $c = \frac{3L}{2\sqrt{N}} + 13$, this suggests that the twist of the primary corresponding to the Virasoro character we obtain from computing the gravitational path integral over the singular orbifold geometries is given by

$$T = -\frac{c - 13}{24N^2} = -\frac{c - 1}{24N^2} + \frac{1}{2N^2}.$$  

$$(8.4.25)$$

We have arrived at the result that the twist of the corresponding non-degenerate Virasoro character appears to be halfway between $T_N$ and $\bar{T}_N$. We do not have a physical explanation for this result, although it is perhaps suggestive that the resulting twist is the arithmetic mean of the two values of the twist whose inclusion in the partition function is suggested by the modular bootstrap argument.
8.5 Discussion

In this chapter we have reviewed the torus partition function of pure three-dimensional gravity. We have described a natural modification to [65,277] that is positive everywhere that maximizes the dimensions of extra operators added while minimizing their degeneracies. These operators added have the correct leading behavior in $c$ to be interpreted as singular orbifolds of $AdS_3$ by $\mathbb{Z}_N$. We pause here to make a few comments and list interesting questions.

The first comment we make is that the spectrum of the resulting theory is continuous. Moreover, the one-loop correction in a $1/c$ expansion of the orbifold geometries does not precisely match those of the operators that we add. A conservative viewpoint would be that these facts are evidence that a pure theory of gravity in $AdS_3$ does not exist, even when these singular orbifolds are included in a sum over geometries. Indeed this is similar to the philosophy advocated in [277]. An alternative possibility is that the spectrum we write down is not dual to a single CFT, but rather an ensemble average of CFTs in a similar spirit as [280,281]. This would allow for a continuous density of states at each spin that is in an appropriate sense the “averaged value” of the partition functions of the ensemble of CFTs. We also note that the regularized one-loop correction to the orbifold saddles is a non-degenerate Virasoro character with a twist (8.4.25) that is precisely the average value of the two possible twists whose addition is suggested by the modular bootstrap argument: $T_N, \tilde{T}_N$ in (8.2.14) and (8.2.16), although the precise order-one value of the twist may be subject to ambiguities due to the need to regularize.

The second comment we make is about how many orbifold geometries to include in the sum over geometries. As we saw in section 8.2.2, unitarity only requires a finite number of orbifold geometries to include. One possibility, then, is that only a finite number of orbifolds are included in the path integral sum, since they are all that are needed to cancel the negativities in the spinning sector of the partition function. On the other hand, it may seem unnatural to only include some but not all of the orbifold geometries. If we include all orbifold geometries and interpret them as states with zero spin, there would be an accumulation point.
in dimension at finite $c$. This may be an indication that some regularization procedure of these states can be used to cancel the delta function negativity at $\hbar = \hbar = \frac{c-1}{24}$ from the $PSL(2,\mathbb{Z})$ sum (which currently is removed via the rather ad hoc procedure of adding six compact free boson partition functions). Another possibility is that there is a $c$-dependent cutoff for the number of orbifolds one needs to include in the sum, with the number of orbifolds included going to infinity in the limit of large $c$. A third possibility is we give each orbifold an $O(1)$ spin. For instance for each state $T_N$ and $\bar{T}_N$, we include two copies: one at spin $N-1$ and one at spin $-(N-1)$, and cut off $N$ as a function of $c$. This would both remove the negativity at $\hbar = \hbar = \frac{c-1}{24}$, and avoid the accumulation point at finite energy.

Let us comment more on the sum over orbifolds. Remarkably, if one includes all the orbifold geometries (as scalars), even though there is an accumulation point at finite dimension, the partition function remains finite. The reason is the following. Consider the $PSL(2,\mathbb{Z})$ sum of a nondegenerate Virasoro character at reduced twist $T < 0$ and spin $J = 0$:

$$\sum_{\gamma \in \mathbb{Z} \setminus PSL(2,\mathbb{Z})} \frac{(q, \bar{q})^T}{\eta(q, \eta(\bar{q}))},$$

(8.5.1)

In (8.5.1), the sum over $PSL(2,\mathbb{Z})$ should be interpreted with the regularization of [277] which we reviewed in section 8.2.1 and appendix E.1. Performing this sum (i.e. taking the Laplace transform of (8.2.9) and (E.1.8)) gives:\footnote{See also sections 3.1 and 3.2 of [65].}

$$Z_T(\tau, \bar{\tau}) = \frac{1}{\eta(q)\eta(\bar{q})} \left( (q, \bar{q})^T - 1 + \sum_{m=1}^{\infty} \frac{\zeta(2m)\Gamma(m)(-16\pi T)^m}{\zeta(2m+1)(2m)!\tau_2^m} \right) + \sum_{j=1}^{\infty} \left( e^{2\pi ij\tau_1} + e^{-2\pi ij\tau_1} \right) \sum_{m=1}^{\infty} \left( \frac{(-16\pi^2 T)^m \sigma_{2m}(j)}{(2m)!j^m \zeta(2m+1) K_m(2\pi j \tau_2)} \right),$$

(8.5.2)

where $\sigma_n(j) = \sum_d d^n$ is the divisor function and $K_m$ is the modified Bessel function of the second kind.
If we do a sum over orbifolds for each $\mathbb{Z}_N$, then we will essentially be summing (8.5.1) with $T$ scaling as

$$T^{(N)} = \frac{1}{N^2} \left( -\frac{c}{24} + \mathcal{O}(1) \right) \quad (8.5.3)$$

where the $\mathcal{O}(1)$ correction in (8.5.3) is a $c$-independent piece that depends on if we take $T_N, \tilde{T}_N$, etc. that will be irrelevant for the following discussion. It turns out that (8.5.2) at small $T$ and finite $\tau, \bar{\tau}$, the $T$-dependence in the sum (8.5.1) scales linearly with $T$:

$$Z_T(\tau, \bar{\tau}) \sim T, \quad T \text{ small.} \quad (8.5.4)$$

The intuition for (8.5.4) is that the first and second term in (8.5.2) cancel at small $T$. Thus the negativity from the delta function is key to arrive at (8.5.4). At large $N$, the partition function sum scales as

$$\sum_{N=1}^{\infty} Z_T^{(N)}(\tau, \bar{\tau}) \sim \sum_{N=1}^{\infty} \frac{1}{N^2} \left( -\frac{c}{24} + \mathcal{O}(1) \right) \quad (8.5.5)$$

which is finite (in (8.5.4), (8.5.5) we are suppressing the $\tau, \bar{\tau}$ dependence). This means that the partition function after summing over all orbifold geometries is finite, despite the accumulation point in dimension\(^{20}\). Roughly, the divergence due to the accumulation point at extremality is cancelled by the infinite number of “$-1$” subtractions (the second term in (8.5.2), corresponding to the negative delta function precisely at extremality arising from the zeta function regularization as discussed in appendix E.1).

In fact, the partition function resulting from the Poincaré series of all of the $\mathbb{Z}_N$ conical defect states with

\(^{20}\)Interestingly if one repeats the same calculation but instead uses the Rademacher expansion rather than the Poincaré series, corresponding to a slightly different “seed” spectrum, as in [282], the partition function at small $T$ would scale as $\sqrt{T}$, not $T$, which would lead to a divergence in the sum over $N$. 

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reduced twist $T_N$ can be written in the following compact way

$$
\sum_{N=1}^{\infty} Z_{T_N}(\tau, \bar{\tau}) = \frac{1}{\eta(q) \eta(q)} \left( \sum_{m=1}^{\infty} \frac{(4\pi \xi \tau_2)^m}{m!} \zeta(2m) + \sum_{m=1}^{\infty} \frac{\Gamma(m)(16\pi \xi)^m \zeta(2m)^2}{\zeta(2m+1)(2m)! \tau_2^m} \right)
+ \sum_{j=1}^{\infty} 2 \left( e^{2\pi ij \tau_1} + e^{-2\pi ij \tau_1} \right) \sum_{m=1}^{\infty} \frac{(16\pi^2 \xi)^m \sigma_{2m}(j) \zeta(2m)}{(2m)!j^m \zeta(2m+1)} K_m(2\pi j \tau_2),
$$

(8.5.6)

where $\xi = \frac{c-1}{24}$. Similarly the sum over $\bar{T}_N$ is given by the same expression (8.5.6) but with $\xi$ replaced with $\xi - 1 = \frac{c-25}{24}$. Finally we subtract out the $PSL(2, \mathbb{Z})$ sum of a seed spectrum with $T = -\frac{c-1}{24}, J = \pm 1$, corresponding to the null descendants of the identity (see the results in sections 3.3 and 3.4 of [65], which we rewrite for convenience in appendix E.3).

It would be extremely interesting if there were a physical interpretation for the finiteness of this partition function, perhaps as an ensemble average of CFT partition functions since it does not appear to admit a standard decomposition into Virasoro characters. Moreover the density of states when integrated against a test function with support at $h = \bar{h} = \frac{c-1}{24}$ can become negative, since we have not added the free boson partition functions in order to preserve finiteness.\footnote{We thank H. Maxfield for emphasizing this point.} We leave this question for future work.

Going forward, one ought to study not just the spectrum but also the dynamics of pure gravity in $AdS_3$. The extreme sparseness of the light spectrum of the quantum theory of pure gravity means that the universal asymptotic formulas for OPE data recently derived in [289] (corrections to which are controlled by the low-lying operators in the spectrum) ought to have a maximally-extended regime of validity in the large central charge limit, analogous to the case of the Cardy formula studied in [64]. We also note that the universal structure constant $C_0(h_1, h_2, h_3)$ can be straightforwardly analytically continued for conformal weights precisely as low as $h_i = \frac{c-1}{32}$. Perhaps the most straightforward route to access the structure constants would be to study the pure gravity genus-two partition function. We leave this interesting line of inquiry for future study.

\footnote{We thank H. Maxfield for emphasizing this point.}
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Appendices to chapter 3

A.1 Zamolodchikov’s recurrence relation

The Virasoro block for a four-point function \((\mathcal{O}_1(z)\mathcal{O}_2(0)\mathcal{O}_3(1)\mathcal{O}_4(\infty))\) with central charge \(c\), external weights \(h_i\), and internal weight \(h\) can be represented as

\[
\begin{align*}
F_{c}^{\text{Vir}}(h; z) &= [16q(z)]^{h - \frac{c-1}{24} z \frac{c-1}{24} - h_1 - h_2 (1 - z) \frac{c-1}{24} - h_1 - h_3} \\
&\times [\theta_3(q(z))]^{\frac{c-1}{12} - \frac{1}{3(h_1 + h_2 + h_3 + h_4)} H(\lambda_i^2, h|q(z))} 
\end{align*}
\)  

(A.1.1)
where the nome $q(z)$ is defined as

$$q(z) \equiv \exp(i\pi \tau(z)), \quad \tau(z) \equiv \frac{iF(1-z)}{F(z)}, \quad F(z) \equiv {_2F_1}(1/2, 1/2, 1|z). \quad (A.1.2)$$

If we define

$$c = 1 + 6Q^2, \quad Q = b + \frac{1}{b}, \quad h_{m,n} = \frac{Q^2}{4} - \lambda_{m,n}^2, \quad \lambda_{m,n} = \frac{1}{2}(\frac{m}{b} + nb), \quad (A.1.3)$$

then $H(\lambda_i^2, h|q(z))$ satisfies Zamolodchikov's recurrence relation

$$H(\lambda_i^2, h|q(z)) = 1 + \sum_{m,n \geq 1} \frac{[q(z)]^m R_{m,n}(\{\lambda_i\})}{h - h_{m,n}} H(\lambda_i^2, h_{m,n} + mn|q(z)), \quad (A.1.4)$$

where $h_{m,n}$ are the weights of degenerate Virasoro representations, and $R_{m,n}(\{\lambda_i\})$ are

$$R_{m,n}(\{\lambda_i\}) = 2 \frac{\prod_{r,s}(\lambda_1 + \lambda_2 - \lambda_{r,s})(\lambda_1 - \lambda_2 - \lambda_{r,s})(\lambda_3 + \lambda_4 - \lambda_{r,s})(\lambda_3 - \lambda_4 - \lambda_{r,s})}{\prod_{k,\ell} \lambda_{k,\ell}}. \quad (A.1.5)$$

The product of $(r, s)$ is over

$$r = -m + 1, -m + 3, \ldots, m - 1, \quad (A.1.6)$$
$$s = -n + 1, -n + 3, \ldots, n - 1,$$

and the product of $(k, \ell)$ is over

$$k = -m + 1, -m + 2, \ldots, m, \quad (A.1.7)$$
$$\ell = -n + 1, -n + 2, \ldots, n,$$

excluding $(k, \ell) = (0, 0)$ and $(k, \ell) = (m, n)$. 

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A.2 Liouville CFT and DOZZ structure constants

The Liouville CFT is parameterized by the central charge \( c = 1 + 6Q^2 \), where \( Q = b + b^{-1} \), and a cosmological constant \( -\mu < 0 \). It is governed by the action

\[
S_{\text{Liouville}} = \frac{1}{4\pi} \int d^2z \sqrt{g} \left( g^{mn} \partial_m \phi \partial_n \phi + QR\phi + 4\pi \mu e^{2\phi} \right).
\] (A.2.1)

To study Liouville theory on the sphere, one typically works with a flat reference metric \( g_{mn} \) supplemented with the boundary condition

\[
\phi(z, \bar{z}) = -2Q \log |z| + \mathcal{O}(1), \quad |z| \to \infty.
\] (A.2.2)

The field \( \phi(z, \bar{z}) \) is not a primary operator under holomorphic coordinate transformations \( z \to w(z) \). In this case one must take care to regulate the action and introduce boundary terms to ensure that the action is finite and invariant under conformal transformations.

The Hilbert space consists of a continuous spectrum of scalar primary operators \( \mathcal{V}_\alpha \) with \( \alpha \in \frac{Q}{2} + i\mathbb{R}_{\geq 0} \) and conformal dimension \( \Delta = 2\alpha(Q-\alpha) \). Operators with \( \alpha \) outside this range, such as the identity operator, do not correspond to normalizable states and thus do not belong to the Hilbert space. Making use of a somewhat nonstandard convention (the reason for which will become clear soon), we normalize the primaries so that in the asymptotic regime where the Liouville potential vanishes (the \( \phi \to -\infty \) limit) they take the form

\[
\mathcal{V}_\alpha \sim S(\alpha)^{-\frac{1}{2}} e^{2\alpha\phi} + S(\alpha)^{\frac{1}{2}} e^{2(Q-\alpha)\phi},
\] (A.2.3)

where \( S(\alpha) \) is the reflection amplitude

\[
S(\alpha) = -\frac{\pi \gamma(b^2)^{(Q-2\alpha)/b} \Gamma(1-(Q-2\alpha)/b)\Gamma(1-(Q-2\alpha)b)}{\Gamma(1+(Q-2\alpha)/b)\Gamma(1+(Q-2\alpha)b)}.
\] (A.2.4)

\(^{1}\)In the literature on the Liouville CFT, usually considered are operators with the asymptotic form \( \mathcal{V}_\alpha \sim e^{2\alpha\phi} \) and which do not have standard two-point functions.
The torus partition function is the same as that of a single non-compact free scalar. The sphere two-point function of primary operators is

$$
\langle \mathcal{V}_{\alpha_1}(z, \bar{z}) \mathcal{V}_{\alpha_2}(0) \rangle = \frac{\delta(\alpha_1 - \alpha_2)}{|z|^{\Delta_1 + \Delta_2}}.
$$

(A.2.5)

Note that with our choice of conventions the two-point function is canonically normalized. The sphere three-point function is given by the DOZZ structure constants \([56, 57]\)

$$
\langle \mathcal{V}_{\alpha_1}(z_1, \bar{z}_1) \mathcal{V}_{\alpha_2}(z_2, \bar{z}_2) \mathcal{V}_{\alpha_3}(z_3, \bar{z}_3) \rangle = \left( \prod_{j=1}^{3} S(\alpha_j)^{-\frac{1}{2}} \right) \frac{C(\alpha_1, \alpha_2, \alpha_3)}{|z_{12}|^{\Delta_1 + \Delta_2 - \Delta_3} |z_{23}|^{\Delta_2 + \Delta_3 - \Delta_1} |z_{31}|^{\Delta_3 + \Delta_1 - \Delta_2}}.
$$

\[C(\alpha_1, \alpha_2, \alpha_3) = \left[ \pi \mu^2 \gamma(b^2) b^{-2b^2} \right]^{(Q - \Sigma_i \alpha_i)/b} \]

\[\times \frac{\Upsilon_b'(0) \Upsilon_b(2\alpha_1) \Upsilon_b(2\alpha_2) \Upsilon_b(2\alpha_3)}{\Upsilon_b(\Sigma_i \alpha_i - Q) \Upsilon_b(\alpha_1 + \alpha_2 - \alpha_3) \Upsilon_b(\alpha_2 + \alpha_3 - \alpha_1) \Upsilon_b(\alpha_3 + \alpha_1 - \alpha_2)}.\]

(A.2.6)

The special functions are given by the following

$$
\gamma(x) = \frac{\Gamma(x)}{\Gamma(1 - x)}
$$

\[\log \Upsilon_b(x) = \int_0^\infty dt \ t^{-1} \left[ \frac{Q}{2} - x \right]^2 \ e^{-t} - \frac{\sinh^2 \left[ \frac{(\frac{Q}{2} - x) t}{b} \right]}{\sinh \frac{bt}{b} \ sinh \frac{t}{b}} \], \ 0 < \text{Re}(x) < \text{Re}(Q).

(A.2.7)

Note in particular that the upsilon function satisfies \(\Upsilon_b(Q - x) = \Upsilon_b(x)\), which implies that \(\Upsilon_b(\frac{Q}{2} + iP)\) is a real function of \(P\). To extend \(\Upsilon_b(x)\) beyond the range of its definition, one notes the following identities

$$
\Upsilon_b(x + b) = \gamma(bx) b^{1 - 2bx} \Upsilon_b(x)
$$

$$
\Upsilon_b(x + b^{-1}) = \gamma(b^{-1}x) b^{\frac{b-1}{2b}} \Upsilon_b(x),
$$

(A.2.8)

which can be proven by considering an integral representation of \(\log \Gamma(x)\). The function \(\Upsilon_b(x)\) has simple zeros at \(x = 0, x = Q\) as well as \(x = mb + \frac{n}{b}\) when \(m\) and \(n\) are both non-positive integers, and when \(m\) and \(n\) are both positive integers. It is instructive to rewrite the Liouville three-point function coefficient as a manifestly real function of the Liouville momenta \(P_i = -i(\alpha_i - \frac{Q}{2})\), since \(P\) takes non-negative values for
operators in the physical Hilbert space:

\[
\mathcal{C}(P_1, P_2, P_3) \equiv \left[ \pi \mu \gamma (b^2)^{2-2b^2} \right]^{\frac{2}{b}} \left( \prod_{j=1}^{3} S(\alpha_j)^{-\frac{1}{2}} \right) C(\alpha_1, \alpha_2, \alpha_3) \\
= \frac{\mathcal{Y}_b'(0)}{\mathcal{Y}_b(\frac{Q}{2} + i \sum_j P_j)} \times \left( \frac{(\mathcal{Y}_b(2iP_1)\mathcal{Y}_b(-2iP_1))^{\frac{1}{2}}}{\mathcal{Y}_b(\frac{Q}{2} + i(P_2 + P_3 - P_1))} \times (2 \text{ permutations}) \right),
\]

(A.2.9)

where we have used that the reflection amplitude can also be written as

\[
S(\alpha) = \left[ \pi \mu \gamma (b^2)^{2-2b^2} \right]^{(2-2\alpha)/b} \frac{\mathcal{Y}_b(2\alpha)}{\mathcal{Y}_b(2\alpha - Q)}.
\]

(A.2.10)

The statement that the formula for the structure constants (A.2.6) proposed by \([56, 57]\) satisfies crossing symmetry was established in \([105]\). There, the existence of the fusion transformations of Virasoro conformal blocks was substantiated, leading to a derivation of the formula (A.2.6). The four-point function is constructed from the DOZZ structure constants as

\[
(V_{\alpha_1}(z, \bar{z})V_{\alpha_2}(0)V_{\alpha_3}(1)V_{\alpha_4}(\infty))
= \left( \prod_{j=1}^{4} S(\alpha_j)^{-\frac{1}{2}} \right) \int_0^\infty \frac{dP}{\pi} C(\alpha_1, \alpha_2, \frac{Q}{2} + iP)C(\alpha_3, \alpha_4, \frac{Q}{2} - iP) \left| F_{c_\text{Vir}} \left( \frac{\Delta_i}{2}, \frac{\Delta_\alpha}{2}, z \right) \right|^2
\]

(A.2.11)

Note that the OPE coefficients \(C(P_1, P_2, P)\) are real for real Liouville momenta \(P_1, P_2, P\) provided \(c > 1\), even if \(b\) is complex (when \(1 < c < 25\)). The \(\mu\)-dependent prefactor can be absorbed by redefining the normalization of sphere correlators as well as that of the primary operators themselves.

Although modular invariance demands that the Liouville momentum \(P\) is real for all primary operators \(\mathcal{V}_\alpha\) in the Hilbert space (this is also seen directly from canonical quantization of Liouville theory on the cylinder), we may analytically continue \(\alpha_i\) to purely imaginary \(P_i\). The analytically continued (A.2.6) continues to obey the crossing equation and unitarity, provided that poles of \(C(\alpha_1, \alpha_2, \frac{Q}{2} + iP)C(\alpha_3, \alpha_4, \frac{Q}{2} - iP)\) in \(P\) do
not cross the $P$-integration contour. If a pole crosses the integration contour, the crossing invariant 4-point function would pick up a residue contribution which may violate unitarity. This is indeed the case, as seen in section 3.2.2.

### A.3 The BTZ spectral density

In this section we will evaluate the modular spectral function of perturbative pure gravity, including one-loop corrections. Restricting to $\tau = i\beta$, we can write the BTZ contribution to the reduced partition function as

The BTZ spectral density

In this section we will evaluate the modular spectral function of perturbative pure gravity, including one-loop corrections. Restricting to $\tau = i\beta$, we can write the BTZ contribution to the reduced partition function as

$$\hat{Z}_{\text{BTZ}}(\beta) = \beta^{-1/2} e^{\frac{4\pi}{\beta^2} \rho_{\text{BTZ}}(\Delta)}$$

where $\rho_{\text{BTZ}}(\Delta)$ is the BTZ spectral density.

Applying an inverse Laplace transform, we can derive the BTZ spectral density

$$\rho_{\text{BTZ}}(\Delta) = 2\pi \sum_{n=0}^{2} C_n I_0 \left( 4\pi \sqrt{2\xi n (\Delta - 2\xi)} \right) .$$

where $C_0 = 1, C_1 = -2, C_2 = 1$. Note that the other known saddle points of the gravitational path integral, related by $SL(2,\mathbb{Z})$ transformations, are always non-perturbatively suppressed for purely imaginary $\tau$.

Thus the perturbative pure gravity modular spectral function, obtained from the thermal $AdS_3$ and Euclidean BTZ saddle points in the gravitational path integral, can be written as

$$f^\text{BTZ}_{\text{mod}}(\Delta_*) = \frac{1}{Z_{\text{AdS}_3}(\beta) + Z_{\text{BTZ}}(\beta)} \left[ \hat{Z}_{\text{AdS}_3}(\beta) + \int_{2\xi}^{\Delta_*} d\Delta \beta^{1/2} e^{-2\pi\beta(\Delta-2\xi)} \rho_{\text{BTZ}}(\Delta) \right] \bigg|_{\beta=1}$$

We are interested in the behaviour of this function for $\Delta_*$ in a window of size $\sim \sqrt{c}$ about $\xi \frac{c}{\beta}$ in the semiclassical limit. From the asymptotic form of the Bessel function, it is easy to see that for $y \sim O(1)$, we
have
\[ \rho_{\text{BTZ}}(\Delta_*) = \frac{c}{6} + y\sqrt{c} \approx 2(1 - e^{-2\pi})^2 \sqrt{\frac{3}{2c}} e^{2\pi(\frac{c}{6} + y\sqrt{c} - 3y^2)} + \mathcal{O}(c^{-1}). \quad (A.3.4) \]

Defining \( \tilde{f}_{\text{BTZ}}^{\text{mod}}(y_*) = f_{\text{mod}}^{\text{BTZ}}(\Delta_* = \frac{c}{6} + y_*\sqrt{c}) \), we end up with the modular spectral function
\[ \tilde{f}_{\text{mod}}^{\text{BTZ}}(y_*) \approx \frac{3}{4} + \frac{1}{4} \text{Erf}(\sqrt{6\pi} y_*), \quad (A.3.5) \]
where we have kept only the leading terms in the semiclassical approximation. This is the same as the spectral function one would obtain from applying the “naive” Cardy formula (3.5.1).

A.4 Details of the numerical computations

A.4.1 Details of the solution of the semidefinite problem

Here we provide some details of the numerical computations of the bounds on the spectral functions, implemented using the SDPB package [79]. In practice, there are several truncations that must be made. First, we must restrict to a finite basis of linear functionals acting on the crossing equation, of total derivative order \( N \). We must also approximate the Virasoro conformal blocks; we can only compute the blocks to a finite order \( d_q \) in the elliptic nome \( q(z) \) from Zamolodchikov’s recurrence relation (reviewed in Appendix A.1). Finally, recall that the upper and lower bounds on the spectral function are derived as the minimal coefficients such that a certain set of positivity conditions (for instance (3.2.7) or (3.4.7)) can be satisfied by a linear combination of derivatives of the conformal blocks or characters evaluated at the crossing symmetric point. In practice, we can only impose the positivity conditions on the blocks or characters of a finite set of spins in the spectrum; we denote the maximal spin considered by \( s_{\text{max}} \).

The truncation on spin means that for a fixed derivative order \( N \), we will not have taken into account all inequalities that the coefficients in (3.2.7) or (3.4.7) must satisfy to constitute a bound on the spectral
function, leading to bounds that are in principle too strong. Meanwhile, the truncation to finite $d_q$ introduces a controlled error into the computation of the (derivatives of the) conformal blocks evaluated at the crossing-symmetric point. Thus to derive bounds at a fixed $N$, we must ensure that both $s_{\text{max}}$ and $d_q$ are sufficiently large so that a bound exists and is stable against further increasing these parameters to within our numerical precision. It is worth emphasizing that while the truncations to finite $s_{\text{max}}$ and $d_q$ are controlled approximations, when these parameters are sufficiently large the bounds derived using a fixed derivative order $N$ are rigorous. Of course, the optimal bounds are obtained in the $N \to \infty$ limit.

Let us begin by discussing the bounds on the sphere four-point spectral function in the case that there are only scalar primaries in the spectrum. Of course in this case we need not worry about the spin truncation. We should note that in practice, the inequalities we feed into semidefinite programming are not quite of the form (3.2.7, 3.2.9), for the simple reason that the Virasoro blocks are not polynomials in the dimension of the internal primary. To illustrate the procedure, we write

$$\mathcal{F}_{12;0,\Delta}(z, \bar{z}) = (256q\bar{q})^{\Delta^2/2 - \xi} P_{12}(\Delta; q, \bar{q}) + \mathcal{O}(q^{d_u}, \bar{q}^{d_u})$$

(A.4.1)

where $P_{12}(\Delta; q, \bar{q})$ is a binomial in $q, \bar{q}$ with $\Delta$-dependent coefficients. Derivatives of the blocks can then be cast in terms of

$$\frac{\partial^n}{\partial z^m}(q\bar{q})^{\Delta^2/2 - \xi} P_{12}(\Delta; q, \bar{q}) \bigg|_{z=\bar{z}=1/2} = \frac{(16e^{-\pi})^\Delta}{Q(\Delta)} P_{n,m}(\Delta)$$

(A.4.2)

where $P_{n,m}(\Delta)$ is a polynomial in $\Delta$ and

$$Q(\Delta) = \prod_i (\Delta - \Delta_i)^2,$$

(A.4.3)

where $\Delta_i$ are the locations of the poles kept at the given order of approximation in the computation of the Virasoro block. Importantly, the prefactor $(16e^{-\pi})^\Delta Q^{-1}(\Delta)$ is non-negative for unitary values of the
internal dimension. The positivity conditions (3.2.7), (3.2.9) then amount to the following

\[(y_{0,0} - \Theta(\Delta_* - \Delta))P_{0,0}(\Delta) + \sum_{1 \leq m+n \leq N, \text{ odd}} y_{m,n}P_{m,n}(\Delta) \geq 0, \ \Delta \geq 0\]

\[(w_{0,0} + \Theta(\Delta_* - \Delta))P_{0,0}(\Delta) + \sum_{1 \leq m+n \leq N, \text{ odd}} w_{m,n}P_{m,n}(\Delta) \geq 0, \ \Delta \geq 0.\]  

(A.4.4)

In the case that there are only scalar primaries in the spectrum, we must take particular care to ensure that \(d_q\) is sufficiently large, for the reason that the four-point function when decomposed into Virasoro blocks truncated at a finite order in \(q\) would appear to have contributions from primaries of nonzero spin. Upper and lower bounds on the spectral function in this case can only be found numerically at a fixed derivative order \(N\) when \(d_q\) is sufficiently large. Empirically, for central charges and derivative orders in the ranges considered in section 3.2.2, we find that \(d_q = 4N\) is sufficient to compute stable bounds on the spectral function. To illustrate the convergence of the bounds as the derivative order is increased, figure A.1 shows the upper bound on the spectral function \(f^+_N(\Delta_* = \frac{7}{12})\) as a function of \(N^{-1}\) for \(c = 8\) with the external operator dimensions at the Liouville threshold. It is clear that we have not been able to access \(N\) sufficiently large so that extrapolation to the \(N \to \infty\) limit can be reliably performed.

![Figure A.1: The upper bounds \(f^+_N(\Delta_* = \frac{7}{12})\) for \(c = 8, \ \Delta_\phi = \frac{7}{12}\) as a function of the inverse derivative order.](image)

Since the \(q\)-truncation order is the bottleneck for the speed of the numerical computations, this limits the range of derivative orders we are able to consider in computing bounds on the spectral function. For this
reason, it is convenient to consider bounds obtained by further truncating the basis of linear functionals to
\[ \partial^m \partial^n \bigg|_{z = \bar{z} = 1/2} \] with \( m + n \leq N \) and either \( m \leq 1 \) or \( n \leq 1 \). This basis leads to weaker bounds at a fixed \( N \),
but renders bounds at larger \( N \) accessible. For instance, as shown in section A.4.3, we are able to compute
bounds on the spectral function up to \( N = 25 \) with \( d_q = N + 9 \) using linear functionals in this reduced basis.
However, we caution that it is not clear that the \( N \to \infty \) limit of the bounds obtained using this reduced
basis of linear functions converges to that of the full-basis bounds.

We now turn to the bounds on the modular spectral function. The implementation of the positivity
conditions (3.4.7) and (3.4.9) proceeds similarly as in deriving bounds on the four-point spectral function;
here, for each spin one simply factors out \( q^{\Delta_{+}^{2} - \xi} \) \( q^{\Delta_{-}^{2} - \xi} \bigg|_{z = \bar{z} = 1/2} \) to reduce derivatives of the reduced Virasoro
characters to polynomials in the primary operator dimension. Now, although the Virasoro characters are
known exactly, one must contend with the fact that the positivity conditions can only be imposed on a finite
set of spins. Empirically, for values of the central charge up to those considered in section 3.4.3 (\( c \sim \mathcal{O}(10^2) \)),
the truncation \( s_{\text{max}} = N + 10 \) is sufficient to ensure stable bounds on the modular spectral function.

A.4.2 Details of the solution of the linear problem

In this subsection we discuss some details of the numerical evaluation of the inner products relevant for
solving the linear problem in section 3.3.1. Note that the norm of \( v \) is given by

\[ \langle v, v \rangle = \int_0^\infty d\Delta C^4_{12:0,\Delta} \frac{f_p(\Delta)}{f_v(\Delta)} \] (A.4.5)

Finiteness of this norm requires the spectrum to be continuous and \( C^4_{12:0,\Delta} \frac{f_p(\Delta)}{f_v(\Delta)} \) to be locally integrable and
decaying sufficiently quickly. If \( C^4_{12:0,\Delta} \) gives a convergent OPE expansion for \( |z| < 1 \), it should decay at
infinity at least as \( 16^{ -2\Delta} \). The decay condition is therefore automatically satisfied if \( \frac{f_p(\Delta)}{f_v(\Delta)} \) grows slower than
\( 16^{ 2\Delta} \).
We need also to ensure that \( p_{n,m} \) have finite norm. The norm is

\[
\langle p_{n,m}, p_{n,m} \rangle = \int_0^\infty \left( \partial_z^n F_{12,0,\Delta} \right)^2 \left( \partial_{\bar{z}}^m F_{12,0,\Delta} \right)^2 \frac{f_v(\Delta)}{f_p(\Delta)} \, d\Delta,
\]

(A.4.6)

the integrand behaves as

\[
\text{polynomial} \times (16q)^{-2\Delta} \frac{f_v(\Delta)}{f_p(\Delta)},
\]

(A.4.7)

where \( q = e^{-\pi} \), and we get that the ratio \( \frac{f_v(\Delta)}{f_p(\Delta)} \) should grow slower than \((16q)^{-2\Delta}\). We then choose this ratio to be

\[
\frac{f_v(h)}{f_p(h)} = (16q)^{-2\Delta} e^{-\lambda \Delta},
\]

(A.4.8)

where \( \lambda \in (0, 2\pi) \). We will set \( \lambda = \pi \).

Let us describe the details of the calculation. We want to compute the Gram matrix \( \langle p_{n,m}, p_{n',m'} \rangle \). For this, we need to be able to compute integrals with the Virasoro conformal blocks. Recalling (A.4.1), the inner product \( \langle p_{n,m}, p_{n',m'} \rangle \) is given by

\[
\langle p_{n,m}, p_{n',m'} \rangle = \int_0^\infty P_{n,m}(\Delta) P_{n',m'}(\Delta) e^{-\lambda \Delta} \frac{Q^2(\Delta)}{Q^2(\Delta)} \, d\Delta.
\]

(A.4.9)

It is a standard fact that such integrals can be evaluated in terms of incomplete gamma functions. However, we want to do this efficiently, since \( P \) and \( Q \) are high-degree polynomials.\(^2\) The computation of products \( P_{n,m}(\Delta) P_{n',m'}(\Delta) \) can be optimized by means of fast Fourier transform. After the product is computed, it suffices to compute the integrals

\[
\int_0^\infty \frac{\Delta^k e^{-\lambda \Delta}}{Q^2(\Delta)} \, d\Delta.
\]

(A.4.10)

To do that, we first write

\[
Q^{-2}(\Delta) = \sum_i \sum_{k=1}^4 \frac{\alpha_{i,k}}{(\Delta - \Delta_i)^k},
\]

(A.4.11)

\(^2\)For example, in order to do calculations for \( c = 8, \Delta_\phi = \frac{11}{12} \), \( N = 25 \), it is necessary to compute the \( q \)-expansion of the conformal blocks to the order \( q^{100} \) (see below). At this order \( \deg P_{25,0} = 679 \) and \( Q \) has 327 zeros.
thus reducing the problem to the integrals of the form

\[ \int_0^\infty \frac{\Delta^n e^{-\lambda \Delta}}{(\Delta - \Delta_i)^k} d\Delta. \] (A.4.12)

Reduction to incomplete gamma function is immediate if we shift \( \Delta \) by \( \Delta_i \). However, in this case we need to expand \((\Delta + \Delta_i)^n\) which produces \( n \) terms, and \( n \) can be large. Instead we write

\[ \int_0^\infty \frac{\Delta^n e^{-\lambda \Delta}}{(\Delta - \Delta_i)^k} d\Delta = e^{-\lambda \Delta_i} \frac{\Gamma(n + 1)}{\Gamma(k)} \sum_{l=0}^{k-1} \binom{k-1}{l} (-\lambda)^{k-1-l} \Delta_i^{l-n} \Gamma(l - n; -\Delta_i \lambda). \] (A.4.13)

Here \( k \) is bounded by 4, so we get a compact sum. The parameter \( n \) does enter into the incomplete gamma function, but it satisfies a recursive relation which allows one to compute it as \( n \) increases, effectively making the complexity of computation of this integral \( O(1) \) for every value of \( n \).

Having found the Gram matrix, it is immediate to find the coefficients of the expansion of \( v_N \) in the basis \( p_{n,m} \); they are given by the first row of the inverse of the Gram matrix. Computation of \( P_N \theta_{\Delta^*} \) proceeds similarly, except that now we need to know all the inner products \( \langle p_{n,m}, \theta_{\Delta^*} \rangle \). These can be computed just as above, shifting everything by \( \Delta^* \). In practice, we find that the basis of \( p_{n,m} \) is ill-conditioned and thus we need to know the Gram matrix to a high precision. This typically demands a large \( q \)-truncation order \( d_q \). For example, in figure 3.3(a) and figure 3.4 in section 3.3.1, values of \( d_q \) between 60 and 100 were used. In figure 3.3(b), however, we needed to go to \( d_q = 200 \). In general, the required value of \( d_q \) grows with \( N \), similarly to what we observed in semidefinite problems. On the other hand, the linear method computes faster than the semidefinite one, which allows us to study much higher values of \( N \).

There is a small subtlety in the computation of \( v_N \) for the mixed correlator, due to the fact that \( Q(\Delta) \) has a double zero at \( \Delta = 0 \), which in the case \( \Delta_1 \neq \Delta_2 \) is not canceled by zeros of \( P_{n,m} \). In this case (e.g. in the figure 3.4) we have tried two approaches. The first approach is introducing lower bound on the intermediate scaling dimension \( \Delta_{gap} \) below the Liouville threshold. The second approach is to modify equation (A.4.8)
as
\[
\frac{f_v(h)}{f_p(h)} = (16q)^{-2\Delta}e^{-\lambda\Delta^4}. \tag{A.4.14}
\]

While the obtained results differ slightly at small \(N\), already at \(N = 13\) both provide equally good approximations for the Liouville spectral functions in figure 3.4.

**Numerical checks of completeness**

Here we consider the question of completeness of the systems

\[
B_{\text{even}} = \{ \partial^n \partial^m \partial_{z=\frac{1}{2}} F_{12;\Delta} |_{z=\frac{1}{2}}, n, m \in \mathbb{Z}_{\geq 0}, n \text{ or } m \text{ even} \}, \tag{A.4.15}
\]

\[
B_{\text{odd}} = \{ \partial^n \partial^m \partial_{z=\frac{1}{2}} F_{12;\Delta} |_{z=\frac{1}{2}}, n, m \in \mathbb{Z}_{\geq 0}, n \text{ or } m \text{ odd} \} \cup \{ \partial_{F_{12;\Delta}} |_{z=-\frac{1}{2}} \} \tag{A.4.16}
\]

with respect to the measure described above. In both cases we attempt to approximate the step functions \(\theta_{\Delta_*} \simeq P_N \theta_{\Delta_*}\), where \(P_N\) is the projection onto the subspace spanned by elements of either system with \(n + m \leq N\), and compute the residual errors

\[
E_N = \frac{|(1 - P_N)\theta_{\Delta_*}|^2}{|\theta_{\Delta_*}|^2}. \tag{A.4.17}
\]

We do this for a range of \(\Delta_*\) in the case of the mixed correlator with \(c = 8\), \(\Delta_1 = \Delta_0\), \(\Delta_2 = \frac{12}{7}\Delta_0\). The results are shown in figure A.2 for \(B_{\text{even}}\) and in figure A.3 for \(B_{\text{odd}}\), consistent with the completeness of both bases. In the plots of \(E_N\) as a function of \(N^{-1}\), we have rescaled \(E_N(\Delta_*)\) by an \(N\)-independent factor for each sample value of \(\Delta_*\) (denoted by \(\bar{E}_N\)) so that for all \(\Delta_*\) the slope of the linear fit with \(N^{-1}\) (which appears to be valid asymptotically for large \(N\)) is approximately 1.
A.4.3 Bounds from a reduced basis of linear functionals

Here we consider the bounds on the scalar-only spectral function using the following reduced basis of linear functionals

$$\frac{\partial^n \partial^m}{\partial \bar{z}^n \partial \bar{\bar{z}}^m} \big|_{z=\bar{z}=\bar{\bar{z}}=\frac{1}{2}}, \quad m \leq 1 \text{ or } n \leq 1, \quad m + n \leq N. \quad (A.4.18)$$

Figure A.4 shows the reduced-basis bounds for $c = 8$ and $c = 30$ with $\Delta_\phi = \Delta_0$. Clearly at fixed $N$ the bounds obtained using the reduced basis would be weaker, but due to the simplicity of the reduced basis it is
Figure A.4: Upper and lower bounds on the spectral function from linear functionals in the reduced basis, assuming only scalar primaries and $\Delta \varphi = c - \frac{1}{12}$ for $c = 8$ (left) and $c = 30$ (right). The black curve denotes the corresponding spectral function of Liouville theory.

Now possible to access bounds at higher $N$ within the same computing time. This provides a useful arena to study the convergence of the bounds at large derivative orders, with the caveat that the $N \to \infty$ limit of the bounds obtained from the reduced basis are likely weaker than the optimal bounds from the most general linear functionals. If the latter is the case, one may eventually need to relax the restriction on $\min(m, n)$ in (A.4.18).
B.1 Virasoro Ward identities

The Virasoro algebra is defined by

\[ [L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12}n(n^2 - 1)\Delta_{n,-m}. \]  

(B.1.1)

Ward identities for 3-point functions of Virasoro descendants (4.2.3) were used extensively in [105,111,119]. Here we summarize the relevant results in our notation. The \( z \)-dependence of the 3-point function \( \rho \) takes
the form

\[ \rho(\xi_3, \xi_2, \xi_1 | z) = z^{L_0(\xi_3) - L_0(\xi_2) - L_0(\xi_1)} \rho(\xi_3, \xi_2, \xi_1 | 1), \]  

(B.1.2)

where \( L_0(\xi_i) \) is the holomorphic conformal weight of the descendant \( \xi_i \). Recall that \( \xi_1 \) is inserted at \( z = 0 \) and \( \xi_3 \) at \( z = \infty \). We have the following Ward identities:

\[ \rho(\xi_3, L_{-1}\xi_2, \xi_1 | z) = \partial_z \rho(\xi_3, \xi_2, \xi_1 | z), \]

\[ \rho(\xi_3, L_n\xi_2, \xi_1 | z) = \sum_{m=0}^{n+1} \binom{n+1}{m} (-z)^m (\rho(L_{m-n}\xi_3, \xi_2, \xi_1 | z) - \rho(\xi_3, \xi_2, L_{n-m}\xi_1 | z)), \]

\[ n > -1, \]

\[ \rho(\xi_3, L_{-n}\xi_2, \xi_1 | z) = \sum_{m=0}^{\infty} \binom{n-2+m}{n-2} \left[ z^m \rho(L_{n+m}\xi_3, \xi_2, \xi_1 | z) + (-)^n z^{-n+1-m} \rho(\xi_3, \xi_2, L_{m-1}\xi_1 | z) \right], \]

\[ n > 1, \]

\[ \rho(L_{-n}\xi_3, \xi_2, \xi_1 | z) = \rho(\xi_3, \xi_2, L_n\xi_1 | z) + \sum_{m=-1}^{l(n)} \binom{n+1}{m+1} z^{n-m} \rho(\xi_3, L_m\xi_2, \xi_1 | z). \]

(B.1.3)

In the last line, \( l(n) = n \) for \( n \geq -1 \), and \( l(n) = \infty \) otherwise. In particular, to move \( L_m \) acting on \( \xi_1 \) through a primary \( \nu_2(z) \) of weight \( d_2 \), we can use the commutator

\[ [L_m, \nu_2(z)] = z^m (z \partial_z + (m+1)d_2) \nu_2(z). \]

(B.1.4)

### B.2 Factorization of 3-point functions

Here we explain the factorization properties of generic 3-point functions of three descendants involving null states, as given in (4.4.1). Using the Ward identities (B.1.3), we can move the Virasoro generators \( L_{-M} \) in the second entry of \( \rho \) on the LHS of (4.4.1) to the first and third entries. Thus it suffices to consider
the case where \( L_M \) is the empty chain. To give a flavor of the derivations, we shall prove (4.3.5) which is one of the identities in (4.4.1) with \( M = 0 \). Suppose \( L_A \) in (4.3.5) is a Virasoro chain of length \( m \), i.e. \( L_A = L_{-m} L_{-m-1} \ldots L_{-1} \). The \( m = 0 \) case is easy to prove and was given in [119]. We will induct on \( m \): suppose the property holds for \([A] = m\), and now consider the \([A] = m + 1\) case. Repeatedly applying the commutation relation (B.1.4), we have

\[
\rho(L_B \chi_{rs}, \nu_2, L_{-m+1} \ldots L_{-1} \nu_1) = \rho(L_{m+1} L_B \chi_{rs}, \nu_2, L_{-m} \ldots L_{-1} \nu_1)
\]

\[
- z^{-m+1} ((-a_{m+1} + 1) d_2 + z \partial_z) \rho(L_B \chi_{rs}, \nu_2, L_{-m} \ldots L_{-1} \nu_1)
\]

\[
= \rho(L_{m+1} L_B \nu d_{rs}, \nu_2, L_{-m-1} \ldots L_{-1} \nu_1) \rho(\chi_{rs}, \nu_2, \nu_1)
\]

\[
- z^{-m+1} ((-a_{m+1} + 1) d_2 + z \partial_z) \rho(L_B \chi_{rs}, \nu_2, L_{-m} \ldots L_{-1} \nu_1)
\]

\[
= \rho(L_B \nu d_{rs} + \nu_2, L_{-m-1} \ldots L_{-1} \nu_1) \rho(\chi_{rs}, \nu_2, \nu_1)
\]

The other identities in (4.4.1) can be proven similarly, by repeatedly applying (B.1.4) and using the property that the null state \( \chi_{rs} \) behaves as a primary.

Note importantly that the second factors on the RHS of (4.4.1) are fusion polynomials,

\[
P^{rs} \begin{bmatrix} d_2 \\ d_1 \end{bmatrix} = \rho(\chi_{rs}, \nu_2 | \nu_1) = \rho(\nu_1, \chi_{rs}, \nu_2 | 1) = \rho(\nu_2, \nu_1, \chi_{rs} | 1),
\]

\[
\rho(\chi_{rs}, \nu_1, \chi_{rs}) = P^{rs} \begin{bmatrix} d_1 \\ d_1 \end{bmatrix} P^{rs} \begin{bmatrix} d_1 \\ d_{rs} + d_1 \end{bmatrix}.
\]
C.1 Special functions

In this appendix we discuss the different properties of the special functions necessary to derive our results for the fusion kernel. Throughout this appendix (and chapter 6), $m$ and $n$ are non-negative integers and $Q = b + b^{-1}$. 
Definition of $\Gamma_b(x)$

The main function that appears is the Barnes double gamma function $\Gamma_b(x)$, having the property $\Gamma_b = \Gamma_{b^{-1}}$ and satisfying the functional equation

$$\Gamma_b(x + b) = \sqrt{2\pi b^{b x - \frac{1}{2}}} \Gamma_b(x), \quad (C.1.1)$$

along with a similar equation with $b \to b^{-1}$. It is a meromorphic function with no zeroes and simple poles at $x = -mb - nb^{-1}$. Its normalization is fixed by $\Gamma_b\left(\frac{Q}{2}\right)$. The functional relation (C.1.1) can be used repeatedly to derive the following shift relations

$$\Gamma_b(x + mb + nb^{-1}) = (2\pi)^{m+n} \prod_{\ell=0}^{m-1} \Gamma(xb + \ell b^2)^{-1} \prod_{k=0}^{n-1} \Gamma(xb^{-1} + kb^{-2} + m)^{-1} \frac{b^{\frac{n-m}{2}} - mn + x(mb - nb^{-1}) + \frac{1}{2}m(m-1)b^2 - \frac{1}{2}n(n-1)b^{-2}}{\Gamma_b(x)}. \quad (C.1.2)$$

The double gamma function admits the following integral representation, convergent for $x$ in the right half-plane [181]:

$$\log \Gamma_b(x) = \int_0^\infty \frac{dt}{t} \left[ \frac{e^{-xt} - e^{-Qt/2}}{(1 - e^{-bt})(1 - e^{-b^{-1}t})} - \frac{1}{2} \frac{(Q/2 - x)^2 e^{-t} - Q/2 - x}{t} \right] \quad (C.1.3)$$

Along with the shift relation, this defines the function everywhere.

Residues

The integral representation (C.1.3) fixes the residue of $\Gamma_b(x)$ at $x = 0$ to

$$\text{Res}_{x \to 0} \Gamma_b(x) = \frac{\Gamma_b(Q)}{2\pi}. \quad (C.1.4)$$

1We will often implicitly assume that $b^2$ is not a rational number.
We furthermore have the following Laurent expansion

\[ \Gamma_b(x) \sim \frac{\Gamma_b(Q)}{2\pi} \left( \frac{1}{x} - \gamma_b + \mathcal{O}(x) \right), \quad x \to 0, \tag{C.1.5} \]

where \([159,290]\)

\[ \gamma_b = -\frac{3}{2}b^{-1} \log b + \left( \gamma - \frac{1}{2} \log(2\pi) \right)b^{-1} + b \log b + \frac{\gamma}{2} \]

\[ -ib \int_0^\infty dy \frac{\psi(1 + ib^2 y) - \psi(1 - ib^2 y)}{e^{2\pi y} - 1}, \tag{C.1.6} \]

and \(\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}\) is the digamma function. We can use the shift relation \((C.1.2)\) to find the residues at the locations of the other poles

\[ \text{Res}_{x=-mb-nb^{-1}} \Gamma_b(x) = \frac{b^{-\frac{3}{2}} + \frac{1}{2}m(m+1)b^2 - \frac{1}{2}n(n+1)b^{-2}}{(2\pi)^{\frac{3}{2}}} \left( \prod_{k=1}^m \Gamma(-kb^2) \right) \left( \prod_{l=1}^n \Gamma(-lb^{-2}) \right) \text{Res}_{x=0} \Gamma_b(x). \tag{C.1.7} \]

**Asymptotics**

Here we list the main asymptotic formulae that are used in chapter 6. Some of the derivations are detailed in the following subsections.

Starting from the integral representation \((C.1.3)\), one can show that \(\Gamma_b\) has the following asymptotic behaviour for fixed \(b\) as \(|x| \to \infty\) for \(x\) in the right half-plane

\[ \log \Gamma_b(x) \sim -\frac{1}{2}x^2 \log x + \frac{3}{4}x^2 + \frac{Q}{2}x \log x - \frac{Q}{2}x - \frac{Q^2}{12} \log x + \log \Gamma_0(b) + \mathcal{O}(x^{-1}). \tag{C.1.8} \]

Here, we have introduced the following function

\[ \log \Gamma_0(b) = -\int_0^\infty \frac{dt}{t} \left( \frac{e^{-Qt/2}}{(1-e^{-bt})(1-e^{-b^{-1}t})} - \frac{1}{t^2} - \frac{Q^2 - 2}{24} e^{-t} \right). \tag{C.1.9} \]
The semiclassical limit, which corresponds to taking $b \to 0$ with arguments scaling like $b^{-1}$, is given by

$$
\log \Gamma_b(b^{-1}x + \frac{1}{2}) \sim \frac{1}{2b^2} \left( \frac{1}{2} - x \right)^2 \log b + \frac{2x - 1}{4b^2} \log(2\pi) - \frac{1}{b^2} \int_{\frac{1}{2}}^{x} dt \log \Gamma(t) - \sum_{n=0}^{\infty} c_{n+1} b^{4n+2} \left( \psi^{(2n)}(x) - \psi^{(2n)}(1/2) \right),
$$

(C.1.10)

with the $c_n$'s defined in section C.1.2.

The global limit means taking the small $b$ limit but with argument scaling like $b$ this time. It can be derived from the semiclassical one by using the shift relation and the result is

$$
\Gamma_b(bx) \sim \frac{1}{8} b^{-2} \log b - \frac{1}{2} F(0)b^{-2} + \frac{3}{4}(2x - 1) \log b + \log \Gamma(x) + \frac{2x - 3}{4} \log(2\pi) + O(b^2),
$$

(C.1.11)

where $F(x = 0) = \int_{\frac{1}{2}}^{0} dy \ \log \frac{\Gamma(y)}{\Gamma(1-y)} = \frac{1}{12} \log 2 - 3 \log A$ where $A$ is Glaisher's constant.

Results for $S_b(x)$

It is often convenient to define the function

$$
S_b(x) = \frac{\Gamma_b(x)}{\Gamma_b(Q - x)}.
$$

(C.1.12)

$S_b(x)$ is a meromorphic function with poles at $x = -mb - nb^{-1}$ and zeroes at $x = Q + mb + nb^{-1}$. It satisfies the following shift relations

$$
S_b(x + mb + nb^{-1}) = (-)^{mn} 2^{m+n} \left( \prod_{\ell=0}^{m-1} \sin(\pi b(x + \ell b)) \right) \left( \prod_{k=0}^{n-1} \sin(\pi b^{-1}(x + kb^{-1})) \right) S_b(x)
$$

(C.1.13)

and admits the following integral representation in the strip $0 < \text{Re}(x) < Q$

$$
\log S_b(x) = \int_{0}^{\infty} \left( \frac{\sinh((\frac{Q}{2} - x)t)}{2 \sinh(\frac{Q}{2}) \sinh(\frac{t}{2b})} - \frac{Q - 2x}{t} \right) dt.
$$

(C.1.14)
It is useful to record the asymptotics of $S_b(x)$ as $|x| \to \infty$ for fixed $b$. The following formula is valid for $x$ in the upper half-plane

$$
\log S_b(x) \sim -\frac{i\pi}{2} x^2 + \frac{i\pi}{2} Qx - \frac{i\pi}{12} (Q^2 + 1) + \mathcal{O}(x^{-1}).
$$

(C.1.15)

This follows directly from the asymptotics (C.1.8). The asymptotic formula for $x$ in the lower half-plane can be deduced by noting that $S_b(x) = \frac{1}{S_b(Q-x)}$.

The global and semiclassical limits can be derived directly from those of $\Gamma_b$ so we will not write them here.

**Results for $\Upsilon_b(x)$**

We also define the upsilon function

$$
\Upsilon_b(x) = \frac{1}{\Gamma_b(x) \Gamma_b(Q-x)},
$$

(C.1.16)

which is an entire function of $x$ with zeroes at $x = -mb - nb^{-1}$ and $x = Q + mb + nb^{-1}$. It satisfies the shift relations

$$
\Upsilon_b(x + mb + nb^{-1}) = \left( \prod_{\ell=0}^{m-1} \frac{\Gamma(xb + \ell b^2 + n)}{\Gamma(1 - xb - \ell b^2 - n)} \right) \left( \prod_{k=1}^{n-1} \frac{\Gamma(xb^{-1} + kb^{-2})}{\Gamma(1 - xb^{-1} - kb^{-2})} \right) b^{m-n-2mn-2x(mb-nb^{-1})-m(m-1)b^2+n(n-1)b^{-2}} \Upsilon_b(x)
$$

(C.1.17)

and admits the following integral representation in the strip $0 < \text{Re}(x) < Q$

$$
\log \Upsilon_b(x) = \int_0^\infty \frac{dt}{t} \left[ \left( \frac{Q}{2} - x \right)^2 e^{-t} - \frac{\sinh^2((Q-x)\frac{t}{2})}{\sinh(\frac{t}{2}) \sinh(\frac{Q}{2})} \right].
$$

(C.1.18)
The asymptotics of $\Upsilon_b(x)$ for large argument, derived mainly from (C.1.8), are given by the following formula for $x$ in the upper half-plane

$$\log \Upsilon_b(x) \sim x^2 \log x - \left(\frac{3}{2} + \frac{i\pi}{2}\right) x^2 - Qx \log x + \left(1 + \frac{i\pi}{2}\right) Qx + \left(\frac{Q^2 + 1}{6}\right) \log x$$

$$- \frac{i\pi}{12} (Q^2 + 1) - 2 \log \Gamma_0(b) + O(x^{-1}).$$

The global and semiclassical limit are again easy to derive so we don’t write them explicitly.

**C.1.1 Derivation of large argument asymptotics**

This section is dedicated to deriving the large argument asymptotics (C.1.8) for the double gamma function. The main idea to derive asymptotic formulae is to massage the integrals into a form where we can apply Watson’s lemma. This says that for a function $f$ which is smooth near 0, and is bounded by some exponential, we can write an asymptotic formula by Taylor expanding $f$ and integrating term-by-term:

$$\int_0^\infty dt \ e^{-tx} f(t) \sim \sum_{n=0}^\infty f^{(n)}(0)x^{-n-1} \text{ as } x \to \infty$$

(C.1.20)

In fact, the Riemann-Lebesgue lemma implies that this is valid for $|x| \to \infty$ anywhere in the right half-plane $Re \ x > 0$, under slightly stronger assumptions on $f$. In our examples, we won’t be able to do this immediately, because the function multiplying $e^{-tx}$ has a pole at $t = 0$. The trick will be to subtract the polar piece using some simple functions, and what remains will be integrals that we can evaluate, typically in terms of simple powers and logarithms.

For this first case, we start with the integral representation (C.1.3). The interesting $x$ dependence of this function comes from the piece of the integrand proportional to $e^{-xt}$. This term in the integrand alone is singular at $t = 0$, but by subtracting some simple pieces, we can create something for which Watson’s lemma is applicable:

$$\int_0^\infty \frac{dt}{t} \left(\frac{1}{(1-e^{-t})(1-e^{-t/b})} - \frac{1}{t^2} - \frac{Q}{2t} - \frac{Q^2 + 1}{12}\right) e^{-xt} \sim \frac{Q}{24x} + \cdots$$

(C.1.21)
Now we can start to group the remaining terms by their $x$ dependence. A piece independent of $x$ is given by

$$\log \Gamma_0(b) = -\int_0^\infty \frac{dt}{t} \left( \frac{e^{-Qt/2}}{1-e^{-bt}} \left(1 - \frac{1}{2} x e^{-t} \right) - \frac{1}{t^2} + \frac{Q^2}{24} \right). \tag{C.1.22}$$

Note that the exponential in the final subtraction is required for convergence at infinity. The integral of the remaining terms can be evaluated as follows:

$$I_3(x) = -\int_0^\infty \frac{dt}{t} \left( \frac{e^{-xt}}{t^2} - \frac{1}{t^2} + \frac{x}{t} - \frac{1}{2} x^2 e^{-t} \right), \quad I_3(0) = 0, \quad I_3'(x) = I_2(x)$$

$$I_2(x) = \int_0^\infty \frac{dt}{t} \left( \frac{e^{-xt}}{t} - \frac{1}{t} + x e^{-t} \right), \quad I_2(0) = 0, \quad I_2'(x) = I_1(x)$$

$$I_1(x) = \int_0^\infty \frac{dt}{t} (e^{-t} - e^{-tx}), \quad I_1(1) = 0, \quad I_1'(x) = I_0(x) = \int_0^\infty e^{-xt} dt = \frac{1}{x}$$

$$\implies I_1(x) = \log x, \quad I_2(x) = x \log x - x, \quad I_3(x) = \frac{1}{2} x^2 \log x - \frac{3}{4} x^2.$$

Putting this all together, we have

$$\log \Gamma_0(x) \sim -\frac{1}{2} x^2 \log x + \frac{3}{4} x^2 + \frac{Q}{2} x \log x - \frac{Q^2}{2} x - \frac{Q^2 + 1}{12} \log x + \Gamma_0(b) + \cdots \tag{C.1.23}$$

where $\Gamma_0(b)$ is given by (C.1.22), and the ellipsis can be expanded asymptotically from (C.1.21), by Taylor expanding the contents of the brackets and integrating term by term.

This applies for $|x| \to \infty$ in the region $\text{Re}(x) \geq 0$. In fact, we can extend this range by using the recursion formula, under which the asymptotic expansion remains invariant but we find an integral that converges in a wider regime. We can use the same formula taking $|x| \to \infty$ in the region $\text{Re}(x) \geq X$, for any fixed $X \in \mathbb{R}$.

### C.1.2 Derivation of semiclassical limit

Now we consider the limit where we take $b \to 0$, with parameter proportional to $b^{-1}$. We can make life a little easier by differentiating first, since we can use $\log \Gamma_b \left( \frac{Q}{2} \right) = 0$ to fix the constant when we integrate
back up. It will be convenient to actually choose the argument to be $b^{-1}x + b/2$, and it is straightforward to remove this shift at the end. After substituting for this argument, and rescaling the integration variable $t$ by $b$, we have

$$\frac{\Gamma'(b^{-1}x + \frac{b}{2})}{\Gamma(b^{-1}x + \frac{b}{2})} = \int_0^\infty \frac{dt}{t} \left[ \frac{-bte^{-xt-b^2t/2}}{(1-e^{-t})(1-e^{-b^2t})} + \frac{1}{2b} \frac{1}{(1-2x)e^{-bt} + \frac{1}{bt}} \right]$$

(C.1.24)

$$= \int_0^\infty \frac{dt}{t} \left[ \frac{-bte^{-xt}}{2(1-e^{-t}) \sinh \left( \frac{b^2t}{2} \right)} + \frac{1}{2b} \frac{1}{(1-2x)e^{-t} + \frac{1}{bt}} \right] - \frac{1}{2b} (1-2x) \log b.$$ 

The second line here is designed to remove any $b$ dependence from exponentials in the integrand. Once this has been achieved, if we Taylor expand the integrand in $b$, the result is integrable term by term.\(^2\) Writing the expansion in terms of coefficients of the Taylor series $\frac{1}{2 \sinh(n/2)} = u^{-1} \sum_{n=0}^{\infty} c_n u^{2n}$ (in closed form, $c_n = -(1-2^{-2n-1}) \frac{B_{2n}}{(2n)!}$), we have

$$\frac{\Gamma'(b^{-1}x + \frac{b}{2})}{\Gamma(b^{-1}x + \frac{b}{2})} \sim -\frac{1}{2b} (1-2x) \log b + \frac{1}{b} \int_0^\infty \frac{dt}{t} \left[ \frac{-e^{-xt}}{1-e^{-t}} + \frac{1}{2} (1-2x)e^{-t} + \frac{1}{t} \right]$$

$$- \sum_{n=1}^{\infty} c_n b^{n-1} \int_0^\infty \frac{dt}{t} \frac{12n-1}{2} e^{-xt}$$

(C.1.25)

All these integrals can be evaluated in closed form. The main ingredient we will need is an integral expression for $\log \Gamma(z)$:

$$\log \Gamma(z) = \int_0^\infty \frac{dt}{t} \left[ \frac{e^{-zt} - e^{-t}}{1-e^{-t}} + (z-1)e^{-t} \right], \quad \text{Re}(z) > 0.$$  

(C.1.26)

Differentiating this many times gives the polygamma functions

$$\psi^{(m)}(z) = (-1)^{m+1} \int_0^\infty \frac{dt}{t} \frac{t^m e^{-zt}}{1-e^{-t}}, \quad \text{Re}(z) > 0, \quad m > 0.$$  

(C.1.27)

\(^2\)The series in $b$ doesn’t converge uniformly, so we won’t get a convergent series. But truncating the expansion at a given $n$, the remainder is bounded by a constant times $b^{n+1}$, so integrating term by term does give an asymptotic series.
This allows us to immediately evaluate all the integrals:

\[
\frac{\Gamma'(b^{-1} x + \frac{b}{2})}{\Gamma(b^{-1} x + \frac{b}{2})} \sim \frac{1}{2b} (2x - 1) \log b + \frac{1}{b} \log \sqrt{2\pi} \psi^{(2n-1)}(x) - \sum_{n=1}^{\infty} c_n b^{4n-1} \psi^{(2n-1)}(x). \tag{C.1.28}
\]

The only slightly challenging aspect here is to evaluate the integral giving the constant \(\log \sqrt{2\pi}\).

We finally only have to integrate this up, using knowledge of the value at \(x = \frac{1}{2}\). Integration with respect to the argument is the same as integration with respect to \(x\), after division by \(b\), and gives us

\[
\log \Gamma_0(b^{-1} x + \frac{b}{2}) \sim \frac{1}{2b^2} \left( \frac{1}{2} - x \right)^2 \log b + \frac{2x - 1}{4b^2} \log(2\pi) - \frac{1}{b^2} \int_{\frac{1}{2}}^{x} dt \log \Gamma(t) - \sum_{n=0}^{\infty} c_{n+1} b^{4n+2} \left( \psi^{(2n)}(x) - \psi^{(2n)}(1/2) \right). \tag{C.1.29}
\]

We can also take the parameter \(x\) to be large, and match this to a small \(b\) expansion of the previous section. These expansions agree perfectly, including matching the leading order asymptotics of the constant term

\[
\log \Gamma_0(b) \sim \frac{1}{24b^2} \log b + \frac{1}{2b^2} \left( \log A - \frac{1}{12} \log 2 \right) + \cdots. \tag{C.1.30}
\]

### C.2 Further results for the fusion kernel

In this appendix we will record some lengthy technical results for properties of the fusion kernel omitted from the main text.

#### C.2.1 Residues at subleading poles

In section 6.4, we derived MFT OPE data and corrections due to non-vacuum exchange for subleading double-twists by studying the global limit of the residues of the kernel at its subleading poles. Furthermore, in section 6.4, we derived the heavy-light semiclassical Virasoro blocks (in the case that the heavy operator
is dual to a conical defect in the bulk) by summing over residues of the kernel in this semiclassical limit. In this subsection we will present the finite-c values for the residues from which these results were derived.

The residue of the fusion kernel with the vacuum exchanged in the T-channel at its subleading poles is given by

$$
\text{Res}_{\alpha_s=\alpha_1+\alpha_2+mb} F_{\alpha_s,1} = -\frac{\Gamma_b(2Q)\Gamma_b(Q+mb)}{2\pi(1+\frac{m}{2})\Gamma_b(Q)^2} b^{\frac{m}{2}+\frac{1}{2}m(m+1)b^2} \left(\prod_{\ell=0}^{m-1} \Gamma(b^2(\ell-m))\right)
\Gamma_b(2Q-2\alpha_1-2\alpha_2-2mb)\Gamma_b(2\alpha_1+2\alpha_2-Q+2mb) \left(\frac{\Gamma_b(Q-2\alpha_1-mb)\Gamma_b(2\alpha_1+mb)}{\Gamma_b(2Q-2\alpha_1)\Gamma_b(2\alpha_1)}\right) \times (\alpha_1 \to \alpha_2).
$$

(C.2.1)

This can be reduced to an expression involving only normal gamma functions using the shift relations (C.1.2), but we do not find it particularly illuminating to do so.

In (6.2.18), we gave a formula for the non-vacuum kernel that captured all singularities at the leading pole. To compute the residues at the subleading poles, we will need to sum over multiple contributions from the contour integral in (6.2.10). In particular, after taking $\alpha_2 \to Q - \alpha_2$ so that all the relevant singularities come form the integral, the residues of the integrand at $s = Q - V_2 + \ell b$ for $\ell \leq m$ contribute to the singularity at $\alpha_s = \alpha_1 + \alpha_2 + mb$. At the end of the day, one finds

$$
F_{\alpha_s,\alpha_t} = \sum_{\ell=0}^{m} \left\{ \prod_{k=0}^{\ell-1} \frac{1}{2\sin(\pi b(Q+kb))} \right\}
\frac{S_b(Q+\alpha_1-\alpha_2-\alpha_s+\ell b)S_b(Q-\alpha_1+\alpha_2-\alpha_s+\ell b)S_b(\alpha_1+\alpha_2-\alpha_s+\ell b)^2}{S_b(2Q-2\alpha_s+\ell b)S_b(Q+\alpha_1+\alpha_2-\alpha_s+\ell b)S_b(\alpha_1+\alpha_2-\alpha_s+\ell b)}
\frac{\Gamma_b(\alpha_1-\alpha_2+\alpha_s)\Gamma_b(-\alpha_1+\alpha_2+\alpha_s)\Gamma_b(Q+\alpha_1-\alpha_2-\alpha_s)\Gamma_b(Q-\alpha_1+\alpha_2-\alpha_s)}{\Gamma_b(Q-2\alpha_s)\Gamma_b(2\alpha_s-Q)}
\frac{\Gamma_b(2Q-\alpha_1-\alpha_2-\alpha_s)^2\Gamma_b(Q-\alpha_1-\alpha_2+\alpha_s)^2\Gamma_b(2Q-2\alpha_t)\Gamma_b(2\alpha_t)\Gamma_b(\alpha_t)^2}{\Gamma_b(Q-2\alpha_1+\alpha_t)\Gamma_b(Q-2\alpha_2+\alpha_t)\Gamma_b(2Q-2\alpha_1-\alpha_t)\Gamma_b(2Q-2\alpha_2-\alpha_t)}
+ \text{(regular at } \alpha_s = \alpha_1 + \alpha_2 + mb).}
$$

(C.2.2)

The coefficient of the fusion kernel with non-vacuum exchange in the T-channel at its subleading double
poles is given by

\[
\text{Res}_{\alpha_s = \alpha_1 + \alpha_2 + mb} \Gamma_{\alpha_s, \alpha_t} = \sum_{n=0}^{\infty} \left( \prod_{k=0}^{n-1} \frac{1}{2 \sin(\pi b(Q + kb))} \right) \frac{\Gamma_b(Q)^2 b^{n(1+(a+b)/2)}}{(2\pi)^{n+2}} \left( \prod_{a=0}^{n-1} \Gamma(b^2(a-n))^2 \right)
\]

\[
\times \frac{\Gamma_b(2\alpha_t)\Gamma_b(2Q - 2\alpha_t)\Gamma_b(\alpha_t + nb)\Gamma_b(\alpha_t + nb)\Gamma_b(\alpha_t)^2}{\Gamma_b(Q - 2\alpha_1 + \alpha_t)\Gamma_b(Q - 2\alpha_2 + \alpha_t)\Gamma_b(2\alpha_1 - \alpha_t)\Gamma_b(2\alpha_2 - \alpha_t)}
\]

\[
\times \frac{\Gamma_b(2\alpha_1 + mb)\Gamma_b(2\alpha_2 + mb)\Gamma_b(Q - 2\alpha_1 - mb)\Gamma_b(Q - 2\alpha_2 - mb)\Gamma_b(Q - 2\alpha_1 - nb)\Gamma_b(Q - 2\alpha_2 - nb)}{\Gamma_b(2\alpha_1 + nb)\Gamma_b(2\alpha_2 + nb)\Gamma_b(Q - \alpha_t - nb)\Gamma_b(\alpha_t - nb)}
\]

\[
\times \frac{\Gamma_b(Q + mb)^2\Gamma_b(2Q - 2\alpha_1 - 2\alpha_2 - mb)^2\Gamma_b(2\alpha_1 + 2\alpha_2 + (m + n)b - Q)}{\Gamma_b(Q + nb)^2\Gamma_b(2Q - 2\alpha_1 - 2\alpha_2 - (m + n)b)\Gamma_b(Q - 2\alpha_1 - 2\alpha_2 - 2mb)\Gamma_b(2\alpha_1 + 2\alpha_2 + 2mb - Q)}
\]

\[
(2.23)
\]

Similarly, the residue of the non-vacuum kernel at the subleading poles takes the hideous form

\[
\text{Res}_{\alpha_s = \alpha_1 + \alpha_2 + mb} \Gamma_{\alpha_s, \alpha_t} = \sum_{n=0}^{\infty} \left( \prod_{k=0}^{n-1} \frac{1}{2 \sin(\pi b(Q + kb))} \right) \frac{\Gamma_b(Q)^2 b^{n(1+(a+b)/2)}}{(2\pi)^{n+2}} \left( \prod_{a=0}^{n-1} \Gamma(b^2(a-n))^2 \right)
\]

\[
\times \frac{\Gamma_b(2\alpha_t)\Gamma_b(2Q - 2\alpha_t)\Gamma_b(\alpha_t + nb)\Gamma_b(\alpha_t + nb)\Gamma_b(\alpha_t)^2}{\Gamma_b(Q - 2\alpha_1 + \alpha_t)\Gamma_b(Q - 2\alpha_2 + \alpha_t)\Gamma_b(2\alpha_1 - \alpha_t)\Gamma_b(2\alpha_2 - \alpha_t)}
\]

\[
\times \frac{\Gamma_b(2\alpha_1 + mb)\Gamma_b(2\alpha_2 + mb)\Gamma_b(Q - 2\alpha_1 - mb)\Gamma_b(Q - 2\alpha_2 - mb)\Gamma_b(Q - 2\alpha_1 - nb)\Gamma_b(Q - 2\alpha_2 - nb)}{\Gamma_b(2\alpha_1 + nb)\Gamma_b(2\alpha_2 + nb)\Gamma_b(Q - \alpha_t - nb)\Gamma_b(\alpha_t - nb)}
\]

\[
\times \frac{\Gamma_b(Q + mb)^2\Gamma_b(2Q - 2\alpha_1 - 2\alpha_2 - mb)^2\Gamma_b(2\alpha_1 + 2\alpha_2 + (m + n)b - Q)}{\Gamma_b(Q + nb)^2\Gamma_b(2Q - 2\alpha_1 - 2\alpha_2 - (m + n)b)\Gamma_b(Q - 2\alpha_1 - 2\alpha_2 - 2mb)\Gamma_b(2\alpha_1 + 2\alpha_2 + 2mb - Q)}
\]

\[
\times \left[ 2\alpha_t b + 2nb \log(b) - 2b \sum_{a=0}^{n-1} \psi(b^2(a-n)) + 2\psi_b(Q + mb) - 2\psi_b(Q + nb) - \psi_b(Q - 2\alpha_1 - mb) - \psi_b(2\alpha_1 + mb) - \psi_b(2\alpha_1 + nb) + \psi_b(Q - 2\alpha_1 - nb) - \psi_b(2\alpha_2 + nb) - \psi_b(2\alpha_1 + mb) - \psi_b(Q - 2\alpha_1 - mb) - \psi_b(2\alpha_2 + nb)
\]

\[
- 2\psi_b(Q - 2\alpha_2 - mb) - 2\psi_b(Q - 2\alpha_1 - 2\alpha_2 - mb) + 2\psi_b(2\alpha_1 + 2\alpha_2 + (m + n)b - Q) - 2\psi_b(2\alpha_1 + 2\alpha_2 + 2mb - Q)
\]

\[
- \psi_b(Q - 2\alpha_2 - mb) + \psi_b(Q - 2\alpha_1 - nb) + \psi_b(2\alpha_1 + 2\alpha_2 + (m + n)b - Q) + \psi_b(Q - \alpha_t - nb) + \psi_b(Q - \alpha_t + nb) + \psi_b(\alpha_t - nb) + \psi_b(\alpha_t + nb)
\]

\[
(2.24)
\]
C.2.2 Exchange of subleading Virasoro double-twists

In the main text, we presented the exact formula for the fusion kernel in the case of exchange of the leading Virasoro double twist in the T-channel. Here we record the (more complicated) form of the kernel in the case of exchange of subleading double-twists

\[
\Gamma_{\alpha_s,2\alpha_2+mb} = \frac{m!}{\prod_{k=0}^{m-n-1} \sin(\pi b (Q + k b)) \Gamma(-b^2(m-k))} b^{-\frac{1}{2}(m-n)(1+(1+m)n)b^2} 
\]

\[
\Gamma_{b}(2Q - 4\alpha_2 - 2mb)\Gamma_{b}(Q - 2\alpha_2 - nb)^2\Gamma_{b}(2Q - 2\alpha_1 - 2\alpha_2 - nb)\Gamma_{b}(4\alpha_2 + (m+n)b - Q) 
\]

\[
\Gamma_{b}(Q + nb)\Gamma_{b}(Q - 2\alpha_2 - nb)^2\Gamma_{b}(2Q - 2\alpha_1 - 2\alpha_2 - mb)\Gamma_{b}(2\alpha_2 + mb)^2\Gamma_{b}(2\alpha_2 + nb)^2 
\]

\[
\Gamma_{b}(2Q - \alpha_1 - \alpha_2 - \alpha_s)\Gamma_{b}(Q - \alpha_1 + \alpha_2 - \alpha_s)^2\Gamma_{b}(Q - \alpha_1 + \alpha_2 + \alpha_s - Q)\Gamma_{b}(\alpha_1 + \alpha_2 + \alpha_s + nb - Q) 
\]

\[
\Gamma_{b}(Q - \alpha_1 - 2\alpha_2 + \alpha_s)\Gamma_{b}(Q - \alpha_1 + \alpha_2 + \alpha_s)^2\Gamma_{b}(Q - \alpha_1 - \alpha_2 - \alpha_s - nb)\Gamma_{b}(Q - \alpha_1 - \alpha_2 + \alpha_s - nb) 
\]

Notice that only certain terms in the sum contribute to a double pole at \( \alpha_s = \alpha_1 + \alpha_2 + m'b. \)

C.2.3 Cross-channel blocks at \( c = 25 \)

The conformal blocks at \( c = 25 \) with \( h_1 = h_2 = 15/16 \) and arbitrary \( h \) are known due to Zamolodchikov [184]:

\[
\mathcal{F}_{11}^{22}(\alpha|z) = (16q)^{\alpha(2-\alpha)-1}(z(1-z))^{-7/8}\theta_3^{-3}(q) \quad (Q = 2, h_1 = h_2 = \frac{15}{16}) \quad (C.2.6)
\]

where \( \theta_3(q) = \sum_{n \in \mathbb{Z}} q^{n^2} \) is the Jacobi theta function, and \( q = \exp[-\pi^{2/3} \frac{(1+1,1-z)^{1/3}}{2(1+z,1,z)}]. \) Using (e.g. as quoted in [236])

\[
q \sim e^{-\frac{\pi^2}{6} \frac{z^2}{1-1/z}} , \quad \theta_3(q) \sim \sqrt{\frac{\pi}{1-q}} \quad \text{as} \quad z \to 1 \quad (C.2.7)
\]

leads to

\[
\mathcal{F}_{11}^{22}(0|1-z) \sim \frac{\pi^2}{16} z^{-\frac{3}{2}} \left( \log \frac{1}{z} \right)^{-\frac{3}{2}} \quad (Q = 2, h_1 = h_2 = \frac{15}{16}) \quad (C.2.8)
\]
This matches (6.2.29) and (6.2.30) upon plugging in the relevant values.

### C.2.4 Branching from S- to T-channel

In the main text we analyzed the crossing kernel when branching T-channel blocks for pairwise identical operators (11 $\to$ 22) into S-channel blocks (12 $\to$ 12). In this section we do the opposite. The object of study is the kernel $\tilde{f}_{\alpha_1 \alpha_2}$ for $f_{\alpha_1 \alpha_2}$, For sufficiently light external primaries, the residues of this kernel at its leading poles controls the $z \to 1$ asymptotics of the S-channel Virasoro blocks (given by (6.2.32)).

The kernel $\tilde{f}_{\alpha_1 \alpha_2}$ is of course still a meromorphic function of $\alpha_s$, now with simple poles at $\alpha_t = 2\alpha_1 + mb + nb^{-1}, 2\alpha_2 + mb + nb^{-1}, 2Q - 2\alpha_1 + mb + nb^{-1}, 2Q - 2\alpha_2 + mb + nb^{-1}$ and reflections (in $\alpha_t$), as well as quadruple poles at $\alpha_t = Q + mb + nb^{-1}$ and $\alpha_s = -mb - nb^{-1}$. It is straightforward to compute e.g. the residues at the poles $\alpha_t = 2\alpha_1 + mb, 2\alpha_2 + mb$ using tools that we have previously developed. For example, the contour integral contributes a pole at $\alpha_t = 2\alpha_1 + mb$, so for the purposes of extracting the residue at the leading pole, we may write the kernel as

$$\tilde{f}_{\alpha_1 \alpha_2} = \frac{\Gamma_b(Q - \alpha_t)^4 \Gamma_b(2\alpha_1 - \alpha_t) \Gamma_b(2Q - 2\alpha_1 - \alpha_t) \Gamma_b(2\alpha_2 - \alpha_t) \Gamma_b(2Q - 2\alpha_2 - \alpha_t)}{\Gamma_b(Q - 2\alpha_t) \Gamma_b(2Q - 2\alpha_t) \Gamma_b(Q + \alpha_1 - \alpha_t + \alpha_s) \Gamma_b(Q + \alpha_2 - \alpha_t + \alpha_s) \Gamma_b(Q + \alpha_1 - \alpha_t - \alpha_s) \Gamma_b(Q + \alpha_2 - \alpha_t - \alpha_s) \Gamma_b(\alpha_1 + \alpha_2 - \alpha_s) \Gamma_b(\alpha_1 + \alpha_2 - \alpha_s) \Gamma_b(\alpha_1 - \alpha_2 + \alpha_s - Q) \Gamma_b(\alpha_1 - \alpha_2 + \alpha_s - Q) \Gamma_b(Q - \alpha_1 - \alpha_2 + \alpha_s - Q) \Gamma_b(Q - \alpha_1 - \alpha_2 + \alpha_s - Q) \Gamma_b(Q - \alpha_1 - \alpha_2 + \alpha_s - Q) \Gamma_b(Q - \alpha_1 - \alpha_2 + \alpha_s - Q) \Gamma_b(\alpha_1 - \alpha_2 + \alpha_s - Q) \Gamma_b(\alpha_1 - \alpha_2 + \alpha_s - Q) \Gamma_b(\alpha_1 + \alpha_2 - \alpha_s - Q) \Gamma_b(\alpha_1 + \alpha_2 - \alpha_s - Q) \Gamma_b(\alpha_1 + \alpha_2 - \alpha_s - Q)}$$

$$+ \text{(regular at } \alpha_t = 2\alpha_1) \quad (C.2.9)$$

where we recall $\alpha_{12} = \alpha_1 - \alpha_2$. The residue of the kernel at the leading pole is then simply expressed as the following

$$\text{Res}_{\alpha_t = 2\alpha_1} \tilde{f}_{\alpha_1 \alpha_2} = -\frac{\Gamma_b(Q)}{2\pi} \times$$

$$\frac{\Gamma_b(Q - 2\alpha_1)^4 \Gamma_b(-2\alpha_{12}) \Gamma_b(2Q - 2\alpha_1 - 2\alpha_2) \Gamma_b(2Q - 2\alpha_s) \Gamma_b(2\alpha_s)}{\Gamma_b(Q - 4\alpha_1) \Gamma_b(-\alpha_{12} + \alpha_s)^2 \Gamma_b(Q - \alpha_1 - \alpha_2 + \alpha_s)^2 \Gamma_b(Q - \alpha_1 - \alpha_2 + \alpha_s)^2 \Gamma_b(Q - \alpha_1 - \alpha_2 + \alpha_s)^2 \Gamma_b(Q - \alpha_1 - \alpha_2 + \alpha_s)^2 \Gamma_b(Q - \alpha_1 - \alpha_2 + \alpha_s)^2} \quad (C.2.10)$$

Similarly, to compute the residue at $\alpha_t = 2\alpha_2$ we send $\alpha_2 \to Q - \alpha_2$ so that the contour integral rather than the prefactor contributes the singularity at $\alpha_t = 2\alpha_2$. The end result is of course the same as (C.2.10), but
with $\alpha_1 \leftrightarrow \alpha_2$.

## C.3 Large internal weight asymptotics

In this appendix we take the large internal weight asymptotics of the expressions derived in the main text for pairwise identical external operators. The final results are in (C.3.2) and (C.3.6).

### C.3.1 Vacuum kernel

To study the large-internal dimension asymptotics of the vacuum kernel, we write $\alpha_s = \frac{Q}{2} + iP$ so that we have

$$
\mathcal{F}_{\alpha_s} = \frac{\Gamma_b(2Q)}{\Gamma_b(Q)^3 \Gamma_b(2\alpha_1) \Gamma_b(2Q - 2\alpha_1) \Gamma_b(2\alpha_2) \Gamma_b(2Q - 2\alpha_2)} \frac{\Gamma_b \left( \alpha_1 + \alpha_2 - \frac{Q}{2} + iP \right) \times (3 \text{ terms with } \alpha \leftrightarrow Q - \alpha)}{\Gamma_b(2iP)} \times (P \leftrightarrow -P).
$$

(C.3.1)

For the purposes of computing the asymptotic density of OPE coefficients (cf (6.2.8)), we will be interested in the $P \to \infty$ limit of this quantity.

To make sense of the large $P$ limit, we need to consult (C.1.8) for the asymptotics of $\Gamma_b(x)$. This gives

$$
\mathcal{F}_{\alpha_s} \sim 2^{-4P^2} e^{\pi \frac{\sigma}{\pi} P} P^{2(h_1 + h_2) - \frac{1}{2} \frac{\pi}{P} + \frac{\pi}{2 \pi} \Gamma_0(0)b} \frac{\Gamma_b(2Q)}{\Gamma_b(Q)^3 \Gamma_b(2\alpha_1) \Gamma_b(2Q - 2\alpha_1) \Gamma_b(2\alpha_2) \Gamma_b(2Q - 2\alpha_2)}.
$$

(C.3.2)

### C.3.2 Non-vacuum kernel

The asymptotics of the non-vacuum kernel are slightly trickier to study than the vacuum case, since we need to work out the asymptotics of the $s$ integral. First, writing $\alpha_s = \frac{Q}{2} + iP$, we look at the integrand in the limit $P \to \infty$ with the ratio $\sigma \equiv s/P$ fixed. For this we need the asymptotics of $S_b(x)$; see equation
(C.1.15). Since $S_b$ has different asymptotic expansions in the upper and lower half-planes, the integrand has four different regions depending on the imaginary part of $\sigma$. Keeping terms only to the leading order in $P$ with nontrivial $\sigma$ dependence, we have the following:

$$\log \left( \prod_{k=1}^{4} \frac{S_b(s + U_k)}{S_b(s + V_k)} \right)$$

$$\sim \begin{cases} 
-i\pi P^2 + 2\pi i Q\sigma P + \mathcal{O}(P^0) & \text{Im} \sigma > 1 \\
-i\pi(\sigma^2 - 2i\sigma)P^2 + 2\pi(-\alpha_2 + \alpha_1 + i\sigma(Q - \alpha_2 + \alpha_1))P + \mathcal{O}(P^0) & 0 < \text{Im} \sigma < 1 \\
i\pi(\sigma^2 + 2i\sigma)P^2 + 2\pi(-\alpha_2 + \alpha_1 - i\sigma(Q - \alpha_2 + \alpha_1))P + \mathcal{O}(P^0) & -1 < \text{Im} \sigma < 0 \\
i\pi P^2 - 2i\pi Q\sigma P + \mathcal{O}(P^0) & \text{Im} \sigma < -1 
\end{cases}$$

(C.3.3)

Looking at the region $0 < \text{Im} \sigma < 1$, there is a saddle-point at the edge, where $\sigma = i$, which is where the contour of integration passes to the left of a pole at $s \approx \alpha_s$ (from $V_1$). We can then take the contour to follow the path of steepest descent away from this, along the line $\text{Re} \sigma + \text{Im} \sigma = 1$ between $\sigma = i$ and $\sigma = 1$. This part of the contour, along with a piece that can be taken to run to infinity in the positive imaginary direction, contributes a term of size $e^{-2\pi QP}$ (times an order one piece and a phase $e^{-i\pi P^2}$). Similarly, a piece of the contour running from negative imaginary infinity, to $\sigma = -i$, and then on to $\sigma = 1$, contributes a term of the same size.

We cannot, however, simply take these two pieces of contour to join near $\sigma = 1$ to form the complete integration contour, since there are lines of poles starting near $\sigma = 0$, and going to the right, to which the contour must pass to the left. To include this piece, we only need to take the residues of the integrand at the poles; furthermore, as can be seen from the real part of the asymptotic expansion above on the real axis, the poles further to the right will be exponentially suppressed in $P$, so only the leftmost pole(s) are needed to leading order.

More explicitly, we can take a different limit with $s$ fixed and $P \to \infty$, with the only relevant terms in
the integrand giving
\[ \frac{1}{S_b(s + V_1)S_b(s + V_2)} \sim e^{-2\pi(s + \alpha_2 - \alpha_1)P}, \quad (C.3.4) \]
and now to evaluate the integral we need only find the residues of poles at \( s = Q - V_3 \) or \( s = Q - V_4 \), whichever is further to the left, including this exponential factor which suppresses poles lying further to the right. For operators below the threshold \( h_t < \frac{c-1}{24} \), we can always choose \( \text{Re} \alpha_t < \frac{Q}{2} \) using the reflection symmetry, so the relevant pole is at \( s = \alpha_t \). If this pole alone dominates, the integral asymptotically gives the following:
\[
\int_{-i\infty}^{i\infty} ds \sum_{k=1}^4 \frac{S_b(s + U_k)}{S_b(s + V_k)} \sim \frac{S_b(\alpha_t + U_1)S_b(\alpha_t + U_2)S_b(\alpha_t + U_3)S_b(\alpha_t + U_4)}{S_b(2\alpha_t)} e^{-2\pi(\alpha_t + \alpha_2 - \alpha_1)P}. \quad (C.3.5)
\]
This pole is more important than the other pieces, of order \( e^{-2\pi Q_P} \), as long as \( \text{Re}(\alpha_t + \alpha_2 - \alpha_1) < Q \); this is always true for unitary operator dimensions. Finally, if \( h_t > \frac{c-1}{24} \), there are two poles that give contributions of the same size, so we must add a second term, which simply takes \( \alpha_t \to Q - \alpha_t \). The special case \( \alpha_t = Q/2 \) can be found as a limit of the sum of both terms (each of which will diverge, but with the sum approaching a finite limit).

Incidentally, a nice thing is that when we evaluated the identity block, the integral was given by the same pole, so the \( \alpha_t \to 0 \) limit should return us smoothly to the vacuum result above. This is a useful check.

Now we need only include the prefactor. The pieces depending on \( \alpha_s \) get expanded just as for the vacuum block, and the pieces depending on \( \alpha_t \) combine nicely with the integrand, giving
\[
\Gamma_{\alpha_s,\alpha_t} \sim 2^{-4P^2} e^{\pi(Q-2\alpha_s)P} P^4(h_1 + h_2) - \frac{c+5}{2} 2\pi \Gamma_0(b)^6 \frac{\Gamma_b(2Q - 2\alpha_t)\Gamma_b(Q - 2\alpha_t)}{\Gamma_b(Q - \alpha_t)^4\Gamma_b(2\alpha_1 - \alpha_t)^4\Gamma_b(2Q - 2\alpha_1 - \alpha_t)\Gamma_b(2\alpha_2 - \alpha_t)^4\Gamma_b(2Q - 2\alpha_2 - \alpha_t)}. \quad (C.3.6)
\]
The quantity relevant for the anomalous momenta (6.3.15) is the following ratio

\[
\frac{F_{\alpha, \alpha_t}}{G_{\alpha, 1}} \sim e^{-2\pi \alpha_t P} \frac{\Gamma_b(2\alpha_1)\Gamma_b(2Q - 2\alpha_1)}{\Gamma_b(2\alpha_1 - \alpha_t)\Gamma_b(2Q - 2\alpha_1 - \alpha_t)} \frac{\Gamma_b(2\alpha_2)\Gamma_b(2Q - 2\alpha_2)}{\Gamma_b(2\alpha_2 - \alpha_t)\Gamma_b(2Q - 2\alpha_2 - \alpha_t)} \times \frac{\Gamma_b(2Q - 2\alpha_t)\Gamma_b(Q - 2\alpha_t)\Gamma_b(Q)^3}{\Gamma_b(2Q)\Gamma_b(Q - \alpha_t)^4},
\]

where recall that in the large-weight limit \( P \approx \sqrt{\hbar_s} \). This reduces to the vacuum result when we take \( \alpha_t \) to vanish. Recall that as written this applies for \( h_t < \frac{c-1}{24} \).

Properties

For light T-channel operators, we just have exponential decay \( e^{-2\pi \alpha_t P} \) with \( \alpha_t \in \mathbb{R} \). For heavy T-channel operators, there is an exponential decay at a fixed rate \( e^{-\pi Q P} \), along with oscillations. For light external operators, (6.2.24) has a definite sign (always positive) when \( \alpha_t \) is lighter than T-channel double-twists.

Another feature is the set of zeros in (6.2.24) at \( \alpha_t = 2\alpha_{1,2} \). This is a nice result that dovetails with a similar result in the Lorentzian inversion of T-channel global conformal blocks, as we now explain. In the global case, if the T-channel block is for exchange of a spin-\( J \) operator whose twist is precisely equal to \( 2h_1 \) or \( 2h_2 \), the \( 6j \) symbol vanishes [33,148]: i.e. writing a \( d \)-dimensional T-channel global conformal block as \( G_{\Delta,J}(1 - z, 1 - \bar{z}) \),

\[
dDisc_T(G_{2h_1 + J,J}(1 - z, 1 - \bar{z})) = 0
\]

where \( dDisc_T \) means that the operators are taken around \( \bar{z} = 1 \). Thus, the Lorentzian inversion gives a vanishing result. The total result for the cross-channel decomposition involves non-analytic contributions at low spin which are not captured by the Lorentzian inversion formula; however, at large spin, these pieces can be neglected.

Returning to the present Virasoro case, then, recall that we evaluated the kernel \( F_{\alpha, \alpha_t} \) for \( \alpha_t = 2\alpha_1 \) or \( \alpha_t = 2\alpha_2 \) in (6.2.26). It is nonzero. There is no reason for it to vanish, in part because this is a chiral object, insensitive to the spin of the intermediate operator. However, in the limit of large S-channel twist taken
above, we see that zeroes emerge. This mimics the result in the global case, providing yet another analogy between global double-twist operators \([O_1 O_2]_{m,\ell}\) with \(h = h_1 + h_2 + m\) and the Virasoro double-twists \(\{O_1 O_2\}_{m,\ell}\) with \(\alpha = \alpha_1 + \alpha_2 + mb\).

**C.3.3 Heavy internal weight in the large-\(c\) limit**

In the main text we compute the anomalous momenta in a large \(c\) limit in which the spin is scaled with the central charge. The main technical ingredient needed for this computation is the ratio of the non-vacuum to vacuum kernels in this limit (as seen in (6.3.15)). We will parameterize this limit by taking

\[
\alpha_s = \frac{Q}{2} + ip_s b^{-1}
\]

and sending \(b \to 0\) while keeping all other weights held fixed. As we will see, the computation will be similar to the limit in which the internal weight is parametrically larger than the central charge.

Scaling \(s\) with \(b^{-1}\) as \(s = S b^{-1}\), the integrand in the kernel (6.2.10) behaves in this limit as

\[
\log \left( \prod_{i=1}^{4} \frac{S_b(s + U_i)}{S_b(s + V_i)} \right) \sim \left[ S - 1 + 2I(1 - S) - I(S) - I(S - 1) + (1 - S) \log(1 - S) - \frac{3}{2} \log(2\pi) 
+ I(\frac{1}{2} - ip + S) + I(\frac{1}{2} + ip + S) - I(\frac{1}{2} - ip - S) - I(\frac{1}{2} + ip - S) \right] b^{-2} + O(b^0),
\]

(C.3.10)

where \(I(x) = \int_{\frac{x}{2}}^{x} dt \log \Gamma(t)\). As a function of \(S\), the integrand should have poles extending to the right at \(S \approx \frac{1}{2} \pm ip, 0, 1\) (up to corrections of order \(b^2\)). However there are also poles extending to the left at \(S \approx -1, 0, 1\). To evaluate the integral will then require us to for example pick up the residues of the poles extending to the right at \(S \approx 0\).

Noting that the poles of interest occur for \(s \sim O(b)\), the part of the integrand dependent on \(p_s\) takes the
following form in the large-$c$ limit

$$\log \left( \frac{1}{S_b(s + V_1)S_b(s + V_2)} \right) \sim \log \left( \frac{\Gamma\left(\frac{1}{2} + ip_s\right)\Gamma\left(\frac{1}{2} - ip_s\right)}{2\pi} \right) \left( \frac{2s}{b} - 2(h_1 - h_2) + O(b) \right).$$  \hfill (C.3.11)

Since $b^{-1}$ is the large parameter and $\alpha_t \sim h_t b$, we in principle have to care about the subleading poles at $s = \alpha_t + mb$, since these are not a priori parametrically suppressed compared to the leading pole. Evaluating the contributions of these poles to the integral, we have

$$\int_{C'} ds \prod_{i=1}^4 \frac{S_b(s + U_i)}{S_b(s + V_i)} \sim \sum_{m=0}^{\infty} \frac{(-m)^m}{(2\pi b^2)^m m!} \left( \frac{\Gamma\left(\frac{1}{2} + ip_s\right)\Gamma\left(\frac{1}{2} - ip_s\right)}{2\pi} \right)^{2(h_t + m - h_1 + h_2)} \frac{S_b(\alpha_t + mb)^2 S_b(Q - 2\alpha_1 + \alpha_t + mb) S_b(2\alpha_2 + \alpha_t - Q + mb)}{S_b(2\alpha_t + mb)},$$  \hfill (C.3.12)

where in the second line we will only be keeping the leading terms in the small $b$ expansion (recall that $\alpha_i \sim h_i b$). We can now combine with the prefactor and divide by the same limit of the vacuum kernel to arrive at

$$\frac{\mathcal{F}_{\alpha_+,\alpha_t}}{\mathcal{F}_{\alpha_+,1}} \sim \sum_{m=0}^{\infty} \frac{(-m)^m (2\pi b^2)^{2(h_t + m)}}{m!} \frac{\Gamma(2h_1)\Gamma(2h_2)(h_t)_m^2(2h_1 + h_t - 1)_m}{\Gamma(2h_1 - h_t - m)\Gamma(2h_2 - h_t)(2h_t)_m} \left( \frac{\Gamma\left(\frac{1}{2} + ip_s\right)\Gamma\left(\frac{1}{2} - ip_s\right)}{2\pi} \right)^{2h_t + 2m}.$$

We see that the terms coming from the subleading poles do end up being suppressed relative to the leading pole in this particular semiclassical limit. So in the large $c$ limit we are left with

$$\frac{\mathcal{F}_{\alpha_+,\alpha_t}}{\mathcal{F}_{\alpha_+,1}} \sim \frac{\Gamma(2h_1)\Gamma(2h_2)}{\Gamma(2h_1 - h_t)\Gamma(2h_2 - h_t)} \left( \frac{c}{6\pi} \cosh(\pi p_s) \right)^{-2h_t}.$$

\hfill (C.3.14)

In the $p_s \sim \sqrt{\frac{6h_t}{c}} \to \infty$ limit, this reproduces the large-spin result (6.3.22). Similarly, as discussed in section 6.3.3, in the limit that the spin is parametrically smaller than the central charge, this also reproduces both the scaling with spin and the precise coefficient of the anomalous weights due to T-channel exchange familiar from the usual lightcone bootstrap [26, 27].
C.4 Lorentzian vacuum inversion in the Newtonian limit

In this appendix we derive the result for the leading anomalous twist due to the Virasoro vacuum module, \( \delta h_0 = -2\alpha_1\alpha_2 \), directly from the Lorentzian inversion formula, in the “Newtonian” limit

\[
(h_1, h_2, c) \to \infty , \quad \frac{h_1 h_2}{c} \text{ fixed} .
\]  

(C.4.1)

In terms of momenta \( \alpha_i \), the product \( \alpha_1 \alpha_2 \) behaves as \( \mathcal{O}(b^0) \) (for instance, taking \( \alpha_i \sim bh_i \) with \( h_i \sim b^{-1} \).)

In this limit, the T-channel Virasoro vacuum block is known to take the form [66]

\[
\mathcal{F}_{\text{vac}}(1-z) = \exp \left[ \frac{2h_1 h_2}{c} k_4(1-z) \right] .
\]  

(C.4.2)

To extract the anomalous twist \( \delta h_0 \), we plug into the chiral inversion formula,

\[
\int_0^1 \frac{dz}{z^2} k_{2(1-h)}(z) \left( \frac{z}{1-z} \right)^{h_1 + h_2} \mathcal{F}_{\text{vac}}(1-z)
\]  

(C.4.3)

and extract the leading singularity near \( z = 0 \). Using

\[
k_4(1-z)|_{z \ll 1} \sim -6 \log z ,
\]  

(C.4.4)

the inversion integral becomes

\[
\int_0^1 \frac{dz}{z^{h_1 + h_2 - \frac{12h_1 h_2}{c}}} \sim \frac{1}{h - (h_1 + h_2 - \frac{12h_1 h_2}{c})}
\]  

(C.4.5)

\[
\sim \frac{1}{h - (h_1 + h_2 - 2\alpha_1 \alpha_2)}
\]

which is the desired result. Subleading anomalous twists \( \delta h_m \) may be extracted from the subleading behaviour near \( z = 0 \).
C.5  Large spin analysis from old-fashioned lightcone bootstrap

Here we connect the large spin results of section 6.3.2 – derived without the use of conformal blocks themselves – with the kinematic method using conformal blocks.

Let us consider crossing symmetry 6.2.2 for the case of interest where external operators are identical in pairs

\[
\sum_s (C_{1s})^2 \mathcal{F}^{21}_{21} (\alpha_s | z) \bar{\mathcal{F}}^{21}_{21} (\bar{\alpha}_s | \bar{z}) = \sum_t C_{1t} C_{22} \mathcal{F}^{22}_{11} (\alpha_t | 1 - z) \bar{\mathcal{F}}^{22}_{11} (\bar{\alpha}_t | 1 - \bar{z}),
\]

(C.5.1)

Many of our results can be derived, with somewhat more work, by considering the \( \bar{z} \to 1 \) limit in which the insertions of operators become null separated. Recalling our convention \( \mathcal{F}^{11}_{34} (\alpha | z) \sim z^{h_1 - h_2} \), the \( \bar{z} \to 1 \) limit is dominated in the T-channel by the operator with the lowest \( \bar{h}_t \), which is the identity,

\[
\sum_s (C_{1s})^2 \mathcal{F}^{21}_{21} (\alpha_s | z) \bar{\mathcal{F}}^{21}_{21} (\bar{\alpha}_s | \bar{z} \to 1) \sim \mathcal{F}^{22}_{11} (0|1 - z) (1 - \bar{z})^{-2\bar{h}_2}
\]

(C.5.2)

No individual S-channel block is sufficiently singular in the \( \bar{z} \to 1 \) limit to reproduce the appropriate \((1 - \bar{z})^{-2\bar{h}_2}\) singularity of the T-channel, so we need an infinite number of them. The only way to solve crossing is then by including infinite families of operators at large \( \bar{h}_s \).

The coefficient of the singularity depends on \( z \) through the T-channel vacuum block \( \mathcal{F}^{22}_{11} (0|1 - z) \). We can most easily relate this \( z \)-dependence to S-channel twists \( h_s \) by subsequently taking the additional limit \( z \to 0 \), with \( z \gg 1 - \bar{z} \), sometimes called the “double lightcone limit”. We can simply read off the S-channel twists from the powers that appear in the small \( z \) expansion of \( \mathcal{F}^{22}_{11} (0|1 - z) \). We derived the leading small \( z \) behaviour, appropriate for the \( m = 0 \) Regge trajectory, in section 6.2.3, essentially by reverse-engineering the known result for the double-twist spectrum from the fusion kernel. For \( \alpha_1 + \alpha_2 < \frac{Q}{2} \) this goes as \( z^{-2\alpha_1 \alpha_2} \), coming simply from the leading pole of the fusion kernel. Therefore, the twists must accumulate to \( h_s = h_1 + h_2 - 2\alpha_1 \alpha_2 \), which manifestly reproduces our results (6.1.8) for \( m = 0 \) in a more laborious and less transparent way. Non-vacuum T-channel exchanges have the same power, but with an additional \( \log z \),
so including them as subleading terms gives rise to anomalous twists. For Re(\(\alpha_1 + \alpha_2\)) > \(\frac{Q}{2}\), the \(z \to 0\) limit of the T-channel vacuum block may be read off from (6.2.29). Finally, extracting the spin dependence of the spectral density of Virasoro primaries requires understanding the antiholomorphic S-channel blocks in an appropriate combined cross-channel, large dimension limit \(\tilde{z} \to 1\), \(\tilde{h}_s \to \infty\).

### C.6 Computations for interpretation of anomalous twists

In section 6.5.2, we gave an interpretation for the behavior of anomalous twists, due to an AdS interaction familiar from previous work, but after changing to a new conformal frame in which the relevant two-particle states are Virasoro primaries. In this appendix, we give details of the calculation of the spin as measured in the new primary frame.

Mathematically, as explained in the text, the main step is to compute the monodromy of the differential equation (6.5.9):

\[
\psi''(w) + \left[ \frac{1}{4} - \frac{1 - \nu_1^2}{4} \left( \frac{\sinh \epsilon_1}{\cosh \epsilon_1 - \cos w} \right)^2 - \frac{1 - \nu_2^2}{4} \left( \frac{\sinh \epsilon_2}{\cosh \epsilon_2 + \cos w} \right)^2 \right] \psi(w) = 0 \quad \text{(C.6.1)}
\]

The \(\nu\) parameters are related to conformal dimensions as \(\bar{h} = \frac{c}{24} (1 - \nu^2)\), so in particular, if \(\bar{h} \ll c, \nu\) is close to unity.

For generic \(w\), as \(\epsilon_i \to 0\), the terms with nontrivial \(w\) dependence in the ODE are unimportant, so a basis of solutions to the ODE can be well-approximated by \(\cos w\) and \(\sin w\). However, when \(w\) is of order \(\epsilon_1\), we must take into account the first nontrivial term; writing \(w = \epsilon_1 x\) and taking \(\epsilon_1 \to 0\) with fixed \(x\), the resulting limit of the ODE has simple solutions:

\[
\frac{d^2 \psi}{dx^2} - \frac{1 - \nu^2}{(1 + x^2)^2} \psi = 0 \quad \Longrightarrow \quad \psi \sim \begin{cases} 
\sqrt{\epsilon_1^2 + w^2} \cos \left( \nu_1 \arctan \frac{w}{\epsilon_1} \right) & (w = \epsilon_1 x) \\
\sqrt{\epsilon_1^2 + w^2} \sin \left( \nu_1 \arctan \frac{w}{\epsilon_1} \right) & 
\end{cases} \quad \text{(C.6.2)}
\]
Now we can find the coefficients of $\cos \frac{w}{2}$ and $\sin \frac{w}{2}$ to which these solutions match, by expanding at $|w| \gg \epsilon_1$ and reading off the constant and the coefficient of $\frac{w}{2}$ respectively. From this, we can write a monodromy matrix which starts with a solution in the $(\cos \frac{w}{2}, \sin \frac{w}{2})$ basis for $-\pi < w < 0$, solves through the $w \approx 0$ region, and reexpresses the result in the $(\cos \frac{w}{2}, \sin \frac{w}{2})$ basis for $0 < w < \pi$:

$$M_0 = \left( \begin{array}{cc} \nu_1 \sin \left( \frac{\nu_1}{2} \right) \epsilon_1 & -\nu_1 \cos \left( \frac{\nu_1}{2} \right) \epsilon_1 \\ 2 \cos \left( \frac{\nu_1}{2} \right) & 2 \sin \left( \frac{\nu_1}{2} \right) \end{array} \right) \left( \begin{array}{cc} \nu_1 \sin \left( \frac{\nu_1}{2} \right) \epsilon_1 & \nu_1 \cos \left( \frac{\nu_1}{2} \right) \epsilon_1 \\ -2 \cos \left( \frac{\nu_1}{2} \right) & 2 \sin \left( \frac{\nu_1}{2} \right) \end{array} \right)^{-1}$$

$$= \left( \begin{array}{cc} \nu_1 \sin \left( \frac{\nu_1}{2} \right) \epsilon_1 & -\nu_1 \cos \left( \frac{\nu_1}{2} \right) \epsilon_1 \\ -\cos(\pi \nu_1) & \nu_1 \sin(\pi \nu_1) \\ 2 \epsilon_1 \sin(\pi \nu_1) & -\cos(\pi \nu_1) \end{array} \right)$$

We then repeat the analysis near $w = \pi$ to get another monodromy matrix for passing through that point:

$$M_\pi = \left( \begin{array}{cc} \cos(\pi \nu_2) & \frac{2}{\epsilon_2 \nu_2} \sin(\pi \nu_2) \\ -\frac{\nu_2 \epsilon_2}{2} \sin(\pi \nu_2) & \cos(\pi \nu_2) \end{array} \right)$$

Finally, we combine these to find the trace of the monodromy around the entire circle:

$$\text{Tr} M = \text{Tr} M_0 M_\pi \sim \frac{4}{\epsilon_1 \epsilon_2 \nu_1 \nu_2} \sin(\pi \nu_1) \sin(\pi \nu_2) - 2 \cos(\pi \nu_1) \cos(\pi \nu_2)$$

Here, we have used $\epsilon_i \ll 1$ to drop one term; however, we do not drop the last term, because in some regimes of interest the remaining terms will be of comparable magnitude.

To obtain the result (6.5.11) quoted in the text, it remains only to replace $\nu_i$ with conformal dimensions, assumed much less than $c$, in which case we have $\frac{\sin(\pi \nu)}{\nu} \sim \frac{12 \pi h}{c}$, $\cos(\pi \nu) \sim -1$:

$$\text{Tr} M \sim \left( \frac{24 \pi c}{c} \right)^2 \frac{h_1 h_2}{\epsilon_1 \epsilon_2} - 2$$

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D.1 Explicit forms of elementary crossing kernels

In this section we will review the explicit forms of the elementary crossing kernels used in chapter 7, with a focus on the analytic structure of the kernels as a function of the intermediate weights.

D.1.1 Sphere four-point

We will start by reviewing the explicit form of the fusion kernel, which implements the fusion transformation relating sphere four-point Virasoro conformal blocks in different OPE channels (see equation (7.3.7)). The
fusion kernel was worked out in explicit detail by Ponsot and Teschner [156, 157]. The expression involves the special functions $\Gamma_b(x)$, which is a meromorphic function with no zeros that one may think of as a generalization of the ordinary gamma function, but with simple poles at $x = -(mb + nb^{-1})$ for $m, n \in \mathbb{Z}_{\geq 0}$, and

$$ S_b(x) = \frac{\Gamma_b(x)}{\Gamma_b(Q - x)}. \quad (D.1.1) $$

Many properties of these special functions, including large argument and small $b$ asymptotics, were summarized in [229] (see in particular appendix A of that paper). The explicit expression for the kernel involves a contour integral and is given by

$$ \mathcal{F}_{P_s, P_t} \left[ \frac{P_{P_2 P_3 P_4}}{P_{P_1}} \right] = P_b(P_i; P_s, P_t)P_b(P_i; -P_s, -P_t) \int_{C'} \frac{ds}{i} \prod_{k=1}^{4} \frac{S_b(s + U_k)}{S_b(s + V_k)}, \quad (D.1.2) $$

where the prefactor $P_b$ is given by

$$ P_b(P_i; P_s, P_t) = \frac{\Gamma_b\left(\frac{Q}{2} + i(P_s + P_3 - P_4)\right)\Gamma_b\left(\frac{Q}{2} + i(P_s - P_3 - P_4)\right)\Gamma_b\left(\frac{Q}{2} + i(P_s + P_2 - P_4)\right)\Gamma_b\left(\frac{Q}{2} + i(P_s + P_4 - P_2)\right)}{\Gamma_b\left(\frac{Q}{2} + i(P_t + P_1 - P_4)\right)\Gamma_b\left(\frac{Q}{2} + i(P_t - P_1 - P_4)\right)\Gamma_b\left(\frac{Q}{2} + i(P_t + P_2 - P_3)\right)\Gamma_b\left(\frac{Q}{2} + i(P_t + P_3 - P_2)\right)} \times \frac{\Gamma_b(Q + 2iP_i)}{\Gamma_b(2iP_s)} \quad (D.1.3) $$

and the arguments of the special functions in the integrand are

$$ U_1 = i(P_1 - P_4) \quad V_1 = Q/2 + i(-P_s + P_2 - P_4) $$

$$ U_2 = -i(P_1 + P_4) \quad V_2 = Q/2 + i(P_s + P_2 - P_4) \quad (D.1.4) $$

$$ U_3 = i(P_2 + P_3) \quad V_3 = Q/2 + iP_t $$

$$ U_4 = i(P_2 - P_3) \quad V_4 = Q/2 - iP_t $$

The contour $C'$ runs from $-i\infty$ to $i\infty$, traversing between the towers of poles running to the left at $s = -U_i - mb - nb^{-1}$ and to the right at $s = Q - V_j + mb + nb^{-1}$ in the complex $s$ plane, for $m, n \in \mathbb{Z}_{\geq 0}$. 360
Viewed as a function of the internal weight $P_s$, the kernel (D.1.2) has eight semi-infinite lines of poles extending to both the top and bottom of the complex plane

$$\mathbb{F}_{P_sP_t}\left[\frac{P_2P_3}{P_sP_t}\right]: \text{simple poles at } P_s = \pm i \left( \frac{Q}{2} + iP_0 + mb + nb^{-1} \right), \text{ for } m, n \in \mathbb{Z}_{\geq 0}.$$

(D.1.5)

where $P_0 = P_1 + P_2, P_3 + P_4$ (and six permutations under reflection $P_i \rightarrow -P_i$).

Roughly, half of these poles are explicit singularities of special functions in the prefactor (D.1.3), while the other half arise from singularities of the contour integral, which occur when poles of the integrand pinch the contour. In the case particularly relevant for chapter 7 of pairwise identical operators $P_4 = P_1, P_3 = P_2$, these singularities are enhanced to double poles, although there is an exception when the T-channel internal weight $P_t$ is degenerate ($P_t = \pm \frac{1}{2}((m+1)b+(n+1)b^{-1}), m, n \in \mathbb{Z}_{\geq 0}$), in which case the poles remain simple when the external operators have weights consistent with the fusion rules.

In most cases, the contour of integration over the internal weight $P_s$ in the fusion transformation (7.3.7) can be taken to run along the real axis. However, as emphasized in [229, 253], when the external operators are sufficiently light, in particular when

$$\text{Re}(i(P_1 + P_2)) < -\frac{Q}{2} \text{ or } \text{Re}(i(P_3 + P_4)) < -\frac{Q}{2}$$

(D.1.6)

then some poles of the fusion kernel (D.1.5) cross the real $P_s$ axis and the contour must be deformed, leading to a finite number of discrete residue contributions to the S-channel decomposition of the T-channel Virasoro block. These correspond to the Virasoro analog of double-twist operators [229].

In the special case of pairwise identical operators with T-channel exchange of the identity, the contour integral can be computed very explicitly and the fusion kernel takes the following simple form, which makes
the analytic structure manifest

\[ F_{P_s} \left[ \frac{P_2 P_1}{P_2 P_1} \right] = \frac{\Gamma_b(2Q) \Gamma_b(Q + i(P_1 + P_2 - P_s)) \times (7 \text{ permutations under reflection } P \rightarrow -P)}{\Gamma_b(Q)^3 \Gamma_b(2iP_s) \Gamma_b(-2iP_s) \Gamma_b(Q + 2iP_1) \Gamma_b(Q - 2iP_1) \Gamma_b(Q + 2iP_2) \Gamma_b(Q - 2iP_2)} \]

\[ = \rho_0(P_s) C_0(P_1, P_2, P_s). \] (D.1.7)

**D.1.2 Torus one-point**

The crossing kernel that implements the modular S transformation on torus one-point Virasoro blocks (see equation (7.3.16)) was worked out by Teschner [166]. Similarly to the fusion kernel, its explicit form involves a contour integral and is given by

\[ S_{PP'}[P_0] = \frac{\rho_0(P) \Gamma_b(Q + 2iP') \Gamma_b(Q - 2iP') \Gamma_b(Q + i(P - P_0)) \Gamma_b(Q - i(P + P_0))}{\Gamma_b(Q + 2iP) \Gamma_b(Q - 2iP) \Gamma_b(Q + 2i(P + P_0)) \Gamma_b(Q - 2i(P - P_0))}
\]

\[ \times \int_C \frac{d\xi}{i} e^{-4\pi i P' \xi} \frac{S_b(\xi + \frac{Q}{2} + i(P + \frac{1}{2} P_0)) S_b(\xi + \frac{Q}{2} - i(P - \frac{1}{2} P_0))}{S_b(\xi + \frac{3Q}{2} + i(P - \frac{1}{2} P_0)) S_b(\xi + \frac{3Q}{2} - i(P + \frac{1}{2} P_0))} \] (D.1.8)

\[ = Q_b(P, P', P_0) \int_C \frac{d\xi}{i} e^{-4\pi i P' \xi} T_b(\xi, P, P_0). \]

This integral representation only converges when

\[ \frac{1}{2} \text{Re}(\alpha_0) < \text{Re}(\alpha') < \text{Re} \left( Q - \frac{1}{2} \alpha_0 \right). \] (D.1.9)

Outside of this range, the kernel is defined via analytic continuation, using the fact that it satisfies a shift relation that we will make explicit shortly.

The integral contributes the following series of poles in the P plane, one extending to the top and the other extending to the bottom

integral: poles at \( P = \pm \frac{i}{2} \left( \frac{Q}{2} + iP_0 + mb + nb^{-1} \right), m, n \in \mathbb{Z}_{\geq 0}. \) (D.1.10)
Together with the prefactor, the full kernel has the following polar structure in the $P$ plane

$$S_{PP'}[P_0] : \text{ poles at } P = \frac{i}{2} \left( \frac{Q}{2} - iP_0 + mb + nb^{-1} \right), \quad m, n \in \mathbb{Z}_{\geq 0}, \quad \text{& all reflections (in } P, P_0). \quad (D.1.11)$$

One can think of these poles as arising in the case that the external operator is a (Virasoro) double-twist of the internal operator. Unlike the case of the fusion kernel, for unitary values of the weights none of these poles can cross the contour of integration $\text{Im}(P) = 0$.

Similarly to the case of the fusion kernel, the modular $S$ kernel can be straightforwardly evaluated in the case that the external operator is the identity, $P_0 = i\frac{Q}{2}$. In this case, the prefactor vanishes and so we only need to extract the singularities of the contour integral. By carefully studying this limit, one finds

$$S_{PP'}[\mathbb{I}] = 2\sqrt{2} \cos(4\pi PP'), \quad (D.1.12)$$

precisely reproducing the non-degenerate modular $S$ matrix for the Virasoro characters $(7.2.8)$. To study the limit in which the internal operator in the original channel is also the identity one must be more careful, for the simple reason that the Virasoro vacuum character is not the same as the $h' \to 0$ limit of the non-degenerate Virasoro character; in the latter case, there are null states that do not decouple continuously.

To study this limit more carefully, we note that the modular kernel satisfies the following shift relation (see e.g. [167])

$$2 \cosh(2\pi bP)S_{PP'}[P_0] = \left( \frac{\Gamma(b(Q + 2iP'))\Gamma(2ibP')}{\Gamma(b(Q + i(2P' - P_0)))\Gamma(b(Q + i(2P' + P_0)))} S_{PP' - i\frac{b}{2}}[P_0] \right. \left. + \frac{\Gamma(b(Q - 2iP'))\Gamma(-2ibP')}{\Gamma(b(Q - i(2P' - P_0)))\Gamma(b(Q - i(2P' + P_0)))} S_{PP' + i\frac{b}{2}}[P_0] \right). \quad (D.1.13)$$

Now consider the limit $P' \to i\frac{b^{-1} - 1}{2}$ of this equality. The first term on the right-hand side will be singular unless we take $P_0$ to $i\frac{Q}{2}$ at the same time. To facilitate the study of this limit, we write $P' = \frac{i}{2}(b^{-1} - \epsilon)$,
\[ P_0 = i \left( \frac{Q}{2} - \epsilon \right), \text{ and take } \epsilon \to 0. \] Taking the limit, we find

\[
\lim_{\epsilon \to 0} S_{P, \frac{i}{2}(Q - \epsilon)} \left[ i \left( \frac{Q}{2} - \epsilon \right) \right] = 2 \cosh(2\pi b P) S_{P, \frac{i}{2}(Q - \epsilon)} \left[ i \frac{Q}{2} \right] - 2S_{P, \frac{i}{2}(Q - \epsilon)} \left[ i \frac{Q}{2} \right]
\]

\[
= 4\sqrt{2} \sinh(2\pi b P) \sinh(2\pi b^{-1} P),
\]

precisely reproducing the modular S matrix for the inversion of the Virasoro vacuum character (7.2.12). Note that one cannot recover this by taking the appropriate limit of (D.1.8), as \( \alpha_0 = 2\alpha' \) is at the boundary of the regime of convergence of the integral representation.

### D.2 Asymptotics of crossing kernels

In this section we will collect results for the asymptotic form of the elementary crossing kernels when some of the weights are taken to be heavy. These results are important for establishing both the form of our asymptotic formulas and their validity, via the suppression of corrections due to the propagation of non-vacuum primaries.

#### D.2.1 Sphere four-point

In [229], the asymptotic form of the fusion kernel when the S-channel internal weight \( P_s \) was taken to be heavy with fixed external weights was extensively studied. The main result of that analysis was the following asymptotic form of the vacuum fusion kernel (D.1.7) with pairwise identical operators, which follows directly from the asymptotics of the special function \( \Gamma_b \) that were established in that paper

\[
\tilde{\mathcal{F}}_{P_s} \left[ \begin{array}{c} P_2 \ P_3 \\ P_2 \ P_3 \end{array} \right] \sim 2^{-4P_s^2} e^{\pi Q P_s P_s^4(b_1 + h_2) - \frac{\pi^2}{2}} \\
\times \frac{2^{-2P_s^{2}G_0(b)6\Gamma_b(2Q)}}{(Q)4\Gamma_b(Q + 2iP_1)\Gamma_b(Q - 2iP_1)\Gamma_b(Q + 2iP_2)\Gamma_b(Q - 2iP_2)}, \quad P_s \to \infty
\]
appears in the large-argument asymptotics of $\Gamma_b(x)$.

By carefully studying the asymptotics of the contour integral in the definition of the fusion kernel, in [229] it was also established that the fusion kernel with non-zero T-channel weight is exponentially suppressed at large $P_s$ compared to the vacuum kernel

\[
\mathcal{F}_{P_2 P_3} \begin{pmatrix}
  P_2 & P_1 \\
  P_2 & P_1 \\
\end{pmatrix} \sim e^{-2\pi \alpha_s P_s} \left( \frac{\Gamma_b(Q + 2iP_1)\Gamma_b(Q - 2iP_1)}{\Gamma_b(\frac{Q}{2} + i(2P_1 - P_t))\Gamma_b(\frac{Q}{2} - i(2P_1 + P_t))} \times (P_1 \to P_2) \right) 
\]

\[
\times \frac{\Gamma_b(Q - 2iP_1)\Gamma_b(-2iP_1)\Gamma_b(Q)^3}{\Gamma_b(2Q)\Gamma_b(\frac{Q}{2} - iP_t)^4}, \quad P_s \to \infty.
\]

Thus we learn that corrections to the heavy-light-light asymptotic formula (7.4.4) due to the exchange of non-vacuum primaries in the T-channel are exponentially suppressed.

**With heavy external operators**

In order to establish the validity of the off-diagonal HHL and HHH asymptotic formulas, we need to ensure that the propagation of non-vacuum primaries is suppressed compared to that of the vacuum. The only nontrivial step is establishing the suppression of

\[
\mathcal{F}_{P_2 P_3} \begin{pmatrix}
  P_2 & P_1 & P_1 \\
  P_2 & P_1 & P_1 \\
\end{pmatrix}
\]

when one or both of the external operators $P_1, P_3$ are taken to be heavy along with the S-channel internal weight $P_2$.

Let’s start with the case relevant for the torus two-point kernel. For simplicity and clarity of presentation,
we will explicitly present the case where $\alpha_1, \alpha_2 = \frac{Q}{2} + iP, P \to \infty$, with $\alpha_3 \equiv \alpha_0$ and $\alpha'_2$ fixed. Focusing on the contour integral involved in the definition of the four-point kernel and writing the integration variable as $s = \sigma P$, we have the following asymptotics of the integrand

$$
\log \prod_{k=1}^{4} \frac{S_b(s + U_k)}{S_b(s + V_k)}
\sim \begin{cases} 
2\pi (\alpha_0 + iQ\sigma)P - \pi i((Q - \alpha_0)^2 + h'_2) + \mathcal{O}(P^{-1}), & \text{Im}(\sigma) > 2 \\
-2\pi (\alpha_0 - iQ\sigma + i\alpha_0\sigma)P - \pi i((Q - \alpha_0)^2 - h_0 + h'_2) + \mathcal{O}(P^{-1}), & 0 < \text{Im}(\sigma) < 2 \\
-2\pi (\alpha_0 + iQ\sigma)P + \pi i((Q - \alpha_0)^2 + h'_2) + \mathcal{O}(P^{-1}), & \text{Im}(\sigma) < 0
\end{cases}
$$

(D.2.5)

The integrand decays exponentially at $\sigma = \pm i\infty$ and no poles cross the contour so we evaluate the integral using these leading approximations for the integrand. In this way one finds

$$
\int \frac{ds}{i} \prod_{k=1}^{4} \frac{S_b(s + U_k)}{S_b(s + V_k)} \sim \text{(order-one)} e^{-2\pi \alpha_0 P},
$$

(D.2.6)

so that all together we have

$$
\mathbb{F}_{P_2P'_2} \left[ \begin{array}{c} p_0 \\ p_1 \\ p_0 \\ p_1 \end{array} \right] \sim \text{(order-one)} (P)^{2h_0 - h'_2},
$$

(D.2.7)

and corrections due to the propagation of non-vacuum primaries with $0 < \alpha'_2 < \frac{Q}{2}$ are encoded by the ratio

$$
\frac{\mathbb{F}_{P_2P'_2} \left[ \begin{array}{c} p_0 \\ p_1 \\ p_0 \\ p_1 \end{array} \right]}{\mathbb{F}_{P_2P'_2} \left[ \begin{array}{c} p_0 \\ p_1 \\ p_0 \\ p_1 \end{array} \right]} \sim \text{(order-one)} P^{-h'_2}.
$$

(D.2.8)

The analysis is similar for corrections to the HHH asymptotics due to propagation of non-vacuum primaries in the dumbbell channel. One finds the following for the asymptotics of the integrand when all three weights
\( \alpha_1, \alpha_2, \alpha_3 = \frac{Q}{2} + iP \) are taken to be heavy and we scale the integration variable with \( P \) as before.

\[
\begin{align*}
\log \prod_{k=1}^{4} \frac{S_b(s + U_k)}{S_b(s + V_k)} \\
&\sim \begin{cases}
3\pi P^2 + 2\pi iQ\sigma P - \frac{n}{4}(Q^2 + 4h'_2) + \mathcal{O}(P^{-1}), & \text{Im}(\sigma) > 2 \\
-\pi i(1 + 4i\sigma - \sigma^2)P^2 + \pi Q(-2 + i\sigma)P - \frac{n}{4}(-2 + Q^2 + 12h'_2) + \mathcal{O}(P^{-1}), & 1 < \text{Im}(\sigma) < 2 \\
2\pi \sigma P^2 + \pi Q(-2 + i\sigma)P - \pi i h'_2 + \mathcal{O}(P^{-1}), & 0 < \text{Im}(\sigma) < 1 \\
c.c., & \text{Im}(\sigma) < 0
\end{cases}
\end{align*}
\]

In this case the dominant contribution turns out to be of the form

\[
\int \frac{ds}{s} \prod_{k=1}^{4} \frac{S_b(s + U_k)}{S_b(s + V_k)} \sim (\text{order-one})e^{-2\pi QP},
\]

leading to

\[
\mathbb{F}_{P_2 P'_2} \left[ \begin{array}{c} p_1 \\ p_3 \\ p_1 \end{array} \right] \sim (\text{order-one}) \left( \frac{27}{16} \right)^{3P^2} e^{-\pi QP} P^{-2h'_2 + \frac{5Q^2 - 1}{4}}.
\]

Thus non-vacuum corrections to the HHH asymptotic formula are suppressed via the ratio

\[
\frac{\mathbb{F}_{P_2 P'_2} \left[ \begin{array}{c} p_1 \\ p_3 \\ p_1 \end{array} \right]}{\mathbb{F}_{P_2} \left[ \begin{array}{c} p_1 \\ p_3 \end{array} \right]} \sim (\text{order-one}) P^{-2h'_2}.
\]

D.2.2 Torus one-point

In order to establish the validity of the heavy-heavy-light and heavy-heavy-heavy universal formulas, we also need to study the asymptotics of the torus one-point kernel in the limit that the internal weight in one of the
channels becomes heavy, namely the limit $P \to \infty$. In this limit, the prefactor $Q_b$ reduces to the following

\[
\log Q_b(P, \alpha', P_0) \sim 2\pi(Q - \alpha_0)P + h_0 \log(2P) \\
+ \log \sqrt{2} \frac{\Gamma_b(Q + 2iP')\Gamma_b(Q - 2iP')}{S_b\left(\frac{Q}{2} + iP_0\right)\Gamma_b\left(\frac{Q}{2} - i(2P' - P_0)\right) + O(P^{-1})}
\]

(D.2.13)

To study the asymptotics of the contour integral, we start by considering scaling the integration variable with $P$, i.e. $\xi = \sigma P$. Then the integrand behaves in the following way at large $P$ depending on the imaginary part of $\sigma$

\[
\log T_b(\sigma P, P, P_0) \sim \begin{cases} 
2\pi i\sigma(Q - \alpha_0)P + O(P^{-1}), & \text{Im}(\sigma) > 1 \\
-2\pi(Q - \alpha_0)P + O(P^{-1}), & -1 < \text{Im}(\sigma) < 1 \\
-2\pi i\sigma(Q - \alpha_0)P + O(P^{-1}), & \text{Im}(\sigma) < -1
\end{cases}
\]

(D.2.14)

In this limit, there are poles extending to the left and right at $\text{Im}(\sigma) = \pm 1$ pinching the contour.

For $\alpha'$ in the discrete range, we cannot evaluate the integral by deforming the contour and summing over residues e.g. in the $\xi$ right half-plane since the integrand does not decay exponentially along the arc at infinity. However, so long as the internal weight $\alpha'$ obeys the condition (D.1.9), the integral along the contour $\text{Re}(\xi) = 0$ converges nicely and the integral in this limit can easily be computed by using the asymptotics (D.2.14). When $\alpha' \in (0, \frac{Q}{2})$, we have

\[
\int_C \frac{d\xi}{i} e^{-4\pi \xi P'} T_b(\xi, P, P_0) \approx \frac{\frac{Q}{2} - iP_0}{2\pi(-2iP')(\frac{Q}{2} + i(2P' - P_0))} e^{-2\pi P'\left(\frac{Q}{2} + i(2P' - P_0)\right)}.
\]

(D.2.15)

Combining with the asymptotics of the prefactor, we recover the following asymptotics

\[
S_{PP'}[P_0] \approx \left(\frac{\frac{Q}{2} - iP_0}{\sqrt{2\pi(-2iP')(\frac{Q}{2} + i(2P' - P_0))} S_b\left(\frac{Q}{2} + iP_0\right)\Gamma_b\left(\frac{Q}{2} + i(2P' - P_0)\right)\Gamma_b\left(\frac{Q}{2} - i(2P' + P_0)\right)}\right) \\
\times e^{-4\pi iP'\left(2P\right)^{h_0}}
\]

(D.2.16)

To compute the kernel when $\alpha'$ is outside of the regime (D.1.9), we can make use of the shift relations
So, if \( \alpha' + \frac{n-1}{2} \text{Re}(b) > \frac{1}{2} \text{Re}(\alpha_0) \) (but \( \alpha' + \frac{n-1}{2} \text{Re}(b) < \frac{1}{2} \text{Re}(\alpha_0) \)), then in this limit we have

\[
S_{PP'}[P_0] \approx \left( \prod_{k=1}^{n} f(P' - i k \frac{b}{2}, P_0) \right) e^{2 \pi n b P} S_{P,P'-i\frac{b}{2}}[P_0],
\]

where

\[
f(P', P_0) = \frac{\Gamma(b(\frac{Q}{2} - i(2P' + P_0)))\Gamma(b(\frac{Q}{2} + i(-2P' + P_0)))}{\Gamma(b(Q - 2iP'))\Gamma(-2ibP')}.
\]

Notice that the exponential part of the prefactor cancels the different exponential asymptotics of the shifted kernel \( S_{P,P'-i\frac{b}{2}} \) so that the overall asymptotics are preserved.
Appendices to chapter 8

E.1 MWK regularization

In this appendix we will show that the regularized sum over modular kernels (8.2.5) precisely reproduces the MWK spectrum. We start with (8.2.5), which we can rewrite as.

\[
\rho_{\text{MWK}}^{\text{MWK}}(h, \bar{h}) = \frac{1}{2 \pi} \sum_{s=1}^{\infty} \sum_{d' \in \mathbb{Z}/s\mathbb{Z}} \sum_{n=-\infty}^{\infty} \frac{2}{s P} e^{2 \pi i (d' + n s) j} \left[ \cosh \frac{2 \pi Q P}{s} \cosh \frac{2 \pi Q \bar{P}}{s} - e^{-\frac{2 \pi i (d' - 1) s}{s}} \cosh \frac{2 \pi Q P}{s} \cosh \frac{2 \pi Q \bar{P}}{s} \right. \\
- e^{-\frac{2 \pi i (d' - 1) s}{s}} \cosh \frac{2 \pi \bar{Q} P}{s} \cosh \frac{2 \pi \bar{Q} \bar{P}}{s} + \cosh \frac{2 \pi \bar{Q} P}{s} \cosh \frac{2 \pi \bar{Q} \bar{P}}{s} \right].
\]  

(E.1.1)
Following [277], we have decomposed the sum over $d$ in the $PSL(2, \mathbb{Z})$ sum by writing $d = d' + ns$, for $n \in \mathbb{Z}$ and $d' \in (\mathbb{Z}/s\mathbb{Z})^*$, which denotes the subset of $\mathbb{Z}/s\mathbb{Z}$ with a multiplicative inverse. We recognize the sum over $n$ as the Dirac comb, so we have

$$
\rho_{MWK}^{MWK}(h, \bar{h}) = \sum_{\ell = -\infty}^{\infty} \sum_{s=1}^{\infty} \frac{2}{sPP} \delta(j - \ell) \left[ S(j, 0; s) \cosh \frac{2\piQP}{s} \cosh \frac{2\piQ\bar{P}}{s} - S(j, -1; s) \cosh \frac{2\piQP}{s} \cosh \frac{2\piQ\bar{P}}{s} - S(j, 1; s) \cosh \frac{2\piQP}{s} \cosh \frac{2\piQ\bar{P}}{s} + S(j, 0; s) \cosh \frac{2\piQP}{s} \cosh \frac{2\piQ\bar{P}}{s} \right],
$$

(E.1.2)

where the Liouville momenta $P, \bar{P}$ are defined in terms of the weights $h, \bar{h}$ as

$$
P = \sqrt{h - \frac{c-1}{24}}, \quad \bar{P} = \sqrt{\bar{h} - \frac{c-1}{24}},
$$

(E.1.3)

and the background charge is defined in terms of the central charge as

$$
Q(c) = b(c) + b^{-1}(c) = \sqrt{\frac{c-1}{6}}, \quad \bar{Q}(c) = b(c) - b^{-1}(c) = \sqrt{\frac{c-25}{6}}.
$$

(E.1.4)

Finally, $S(j, J; s)$ is the Kloosterman sum, as in (8.2.7), and $j = h - \bar{h}$ is the spin. The sum is over $PSL(2, \mathbb{Z})/\mathbb{Z}$ because the seed, the Virasoro vacuum character (8.2.4), is independent of integer shifts of $\tau$.

Decomposing into spin sectors, we reproduce (8.2.6), which we rewrite below for convenience:

$$
\rho_{j}^{MWK}(t) = \frac{2}{\sqrt{t(t+j)}} \sum_{s=1}^{\infty} \frac{1}{s} \left[ S(j, 0; s) \cosh \left( \frac{4\pi}{s} \sqrt{\frac{c-1}{24}(t+j)} \right) \cosh \left( \frac{4\pi}{s} \sqrt{\frac{c-25}{24} t} \right) - S(j, -1; s) \cosh \left( \frac{4\pi}{s} \sqrt{\frac{c-1}{24}(t+j)} \right) \cosh \left( \frac{4\pi}{s} \sqrt{\frac{c-25}{24} t} \right) - S(j, 1; s) \cosh \left( \frac{4\pi}{s} \sqrt{\frac{c-25}{24}(t+j)} \right) \cosh \left( \frac{4\pi}{s} \sqrt{\frac{c-1}{24} t} \right) + S(j, 0; s) \cosh \left( \frac{4\pi}{s} \sqrt{\frac{c-25}{24}(t+j)} \right) \cosh \left( \frac{4\pi}{s} \sqrt{\frac{c-1}{24} t} \right) \right].
$$

(E.1.5)

To do the sum over $s$ we will need to regularize the modular kernel. Implicit in the definition of the modular kernel (8.2.3) is the fact that the Dedekind eta function transforms with weight one-half under $PSL(2, \mathbb{Z})$. 371
For the purposes of analytic continuation, we now introduce a regularized crossing kernel corresponding to modular forms of generic weight \( w \). In this case, the appropriate crossing relation is

\[
(-i(s\tau + d))^{-w} e^{2\pi i(\gamma\tau)(h - \frac{c+1}{24})} = \int_{\frac{c-1}{24}}^{\infty} dh' \, g^{(w,\gamma)}_{k'h} \, e^{2\pi i\tau(h' - \frac{c+1}{24})},
\]

(E.1.6)

where the regularized modular crossing kernel is given by

\[
g^{(w,\gamma)}_{k'h} = \epsilon(w, \gamma) \left( \frac{2\pi}{s} \right)^w (h' - \frac{c-1}{24})^{w-1} e^{2\pi i\left(a(h - \frac{c+1}{24}) + d(h' - \frac{c+1}{24})\right)} \frac{_{\,0}F_{\,1}(w; \frac{4\pi^2 (\frac{c}{24} - h)(h' - \frac{c+1}{24})}{s})}{\Gamma(w)},
\]

(E.1.7)

with \( w = \frac{1}{2} \) being the case of physical interest. \( \epsilon(w, \gamma) \) is again a phase that will be unimportant in what follows.

Consider, for example, the regularized contribution to the scalar \((j = 0)\) density of states of a scalar \((J = 0)\) seed

\[
\rho^{(T,J=0)}_{j=0}(t) = \lim_{w \rightarrow \frac{1}{2}} \sum_{s=1}^{\infty} \left( \frac{2\pi}{s} \right)^{2w} t^{2w-2} \Gamma(w)^{-2} S(0,0;s) + \sum_{s=1}^{\infty} \frac{2}{st} \left[ \cosh^2 \left( \frac{4\pi}{s} \sqrt{-Tt} \right) - 1 \right] S(0,0;s) + \sum_{s=1}^{\infty} \frac{2}{st} \left[ \cosh^2 \left( \frac{4\pi}{s} \sqrt{-Tt} \right) - 1 \right] S(0,0;s)
\]

(E.1.8)

where we obtained the delta function coefficient by integrating \( t \) from 0 to any finite positive number, and then taking the limit in \( w \). This reproduces the negative delta function at \( h = \bar{h} = \frac{c+1}{24} \), and the remaining sum converges in \( s \).

Similarly, the regularized contribution to the scalar \((j = 0)\) density of states from a seed state with reduced
twist $T < 0$ and spin $J > 0$ is given by

$$
\rho_{j=0}^{(T,J)}(t) = \lim_{w \to 1} \sum_{s=1}^{\infty} \left( \frac{2\pi}{s} \right)^{2w} t^{2w-2} \Gamma(w)^{-2} S(0, J; s) + \sum_{s=1}^{\infty} \frac{2}{st} \left[ \cosh \left( \frac{4\pi}{s} \sqrt{-Tt} \right) \cosh \left( \frac{4\pi}{s} \sqrt{-T + J}t \right) - 1 \right] S(0, J; s)
$$

(E.1.9)

Thus we see that the regularized MWK density of states

$$
\tilde{\rho}_{j}^{\text{MWK}}(t) = \tilde{\rho}_{j}^{(T=-\frac{c-1}{24}, J=0)}(t) - \tilde{\rho}_{j}^{(T=-\frac{c+1}{24}, J=1)}(t) - \tilde{\rho}_{j}^{(T=-\frac{c-1}{24}, J=-1)}(t) + \tilde{\rho}_{j}^{(T=-\frac{c+23}{24}, J=0)}(t)
$$

(E.1.10)

contains a discrete negative degeneracy at extremality in the scalar sector

$$
\tilde{\rho}_{j=0}^{\text{MWK}}(t) = -6\delta(t) + (\text{a continuous function of } t).
$$

(E.1.11)

### E.2 A systematic analysis of the large-spin negativities

All of the states we will need to add are at reduced twist $T_n = -\frac{c-1}{24n^2}$ and $\bar{T}_n = T_n + \frac{1}{n^2}$. Let us now calculate how many states we will need to add for each $n$.

#### n = 2

Since 2 is prime the only term we need to worry about is the term at twist 0, aka the vacuum. The term

$$
\frac{S(j, 0; 2) - S(j, -1; 2)}{2}
$$

(E.2.1)

is most negative for $j \equiv 1 \pmod{2}$, and evaluates to $-1$. Therefore we need to add a scalar with twist $\frac{3}{4} \left( \frac{c-1}{24} \right)$ with degeneracy 1. When we look at $n = 4, 6, \ldots$, we only need to consider odd spins.
For \( j \equiv 1 \) (mod 2),
\[
\frac{S(j, 0; 2) - S(j, 1; 2)}{2} = -1
\] (E.2.2)
which means we also need to add a single state at \( T_2 \).

\( n = 3 \)

Since 3 is prime the only term we need to worry about is the term at twist 0, aka the vacuum. The term
\[
\frac{S(j, 0; 3) - S(j, -1; 3)}{3}
\] (E.2.3)
is most negative for \( j \equiv 1 \) (mod 3), and evaluates to \(-1\). Therefore we need to add a scalar with twist \( \frac{8}{9} \left( \frac{c-1}{24} \right) \) with degeneracy 1. When we look at \( n = 6, 9, \ldots \), we need to only consider spins 1 (mod 3). At \( j \equiv 1 \) (mod 3),
\[
\frac{S(j, 0; 3) - S(j, 1; 3)}{3} = 0,
\] (E.2.4)
so no states at reduced twist \( T_3 \) are necessary.

\( n = 4 \)

This is the first nontrivial example. We need to worry about the vacuum, and the one state we added at twist \( \frac{c-1}{32} \). From the vacuum, we learned that the only states that have a chance of being negative at high spin are the ones that have odd spin (and also spins that are 1 (mod 3) but that is irrelevant since 3 and 4 are coprime).

From the vacuum we need to consider
\[
\frac{S(j, 0; 4) - S(j, -1; 4)}{4}
\] (E.2.5)
For \( j \equiv 1 \pmod{4} \), this is \(-\frac{1}{2}\) and for \( j \equiv 3 \pmod{4} \) this is \( \frac{1}{2} \). From the twist \( \frac{-1}{32} \) state we need to consider

\[
\frac{S(j, 0; 2)}{2} \tag{E.2.6}
\]

For odd \( j \) this is \(-\frac{1}{2}\). Therefore the states that are \( 1 \pmod{4} \) currently are negative; we need to add a state with twist \( \frac{15}{16} \left( \frac{-1}{24} \right) \) with degeneracy \( \frac{1}{2} + \frac{1}{2} = 1 \) to cancel this. When we look at future \( s = 8, 12, 16, \ldots \), we need to only consider \( 1 \pmod{4} \) for negativity. For states \( j \equiv 1 \pmod{4} \),

\[
\frac{S(j, 0; 4) - S(j, 1; 4)}{4} + \frac{S(j, 0; 2)}{2} = 0, \tag{E.2.7}
\]

so no states with reduced twist \( \tilde{T}_4 \) are necessary.

\( n = 5 \)

The term

\[
\frac{S(j, 0; 5) - S(j, -1; 5)}{5} \tag{E.2.8}
\]

is most negative for \( j \equiv 1 \pmod{5} \), where it evaluates to \(-\frac{5 + \sqrt{5}}{10} \sim -0.7236 \). We therefore need to add \( \frac{5 + \sqrt{5}}{10} \) states with reduced twist \( T_5 \). For \( j \equiv 1 \pmod{5} \),

\[
\frac{S(j, 0; 5) - S(j, 1; 5)}{5} = -\frac{5 - \sqrt{5}}{10} \sim -0.2764. \tag{E.2.9}
\]

We therefore also need to add \( \frac{5 - \sqrt{5}}{10} \) states at reduced twist \( \tilde{T}_5 \).

\( n = 6 \)

From the \( n = 2, 3 \) calculations, we only need to consider states of spin \( j \equiv 1 \pmod{6} \). The term \( T_2 \) contributes \( \frac{S(j, 0; 3)}{3} \) and the term \( T_3 \) contributes \( \frac{S(j, 0; 2)}{2} \) to the density of states with order \( \frac{\exp \left( \frac{i}{\sqrt{3}} \sqrt{\frac{3}{\pi j}} \right)}{\sqrt{j \left( j - \frac{1}{2} \right)}} \).
If \( j \equiv 1 \pmod{6} \) then
\[
\frac{S(j,0;6) - S(j,-1;6)}{6} + \frac{S(j,0;3)}{3} + \frac{S(j,0;2)}{2} = -1.
\] (E.2.10)

Thus we add 1 state at reduced twist \( T_6 \). Similarly the state \( \bar{T}_2 \) contributes \( \frac{S(j,0;3)}{3} \) to the density of states with order \( \frac{\exp \left( \frac{4\pi \sqrt{(\bar{k}+\pi)} j}{\sqrt{j(b-\frac{2\pi}{2})}} \right)}{\sqrt{j(b-\frac{2\pi}{2})}} \). If \( j \equiv 1 \pmod{6} \) then
\[
\frac{S(j,0;6) - S(j,1;6)}{6} + \frac{S(j,0;3)}{3} = 0
\] (E.2.11)

so no states are needed with reduced \( T_6 \).

\( n = 7 \)

The term
\[
\frac{S(j,0;7) - S(j,-1;7)}{7}
\] (E.2.12)

is most negative for \( j \equiv 2 \pmod{7} \), where it evaluates to \( -\frac{3+4\sin \left( \frac{3\pi}{7} \right)}{7} \sim -0.7849 \). Thus we need to add \( \frac{3+4\sin \left( \frac{3\pi}{7} \right)}{7} \) states with reduced twist \( T_7 \).

However, if \( j \equiv 2 \pmod{7} \), then
\[
\frac{S(j,0;7) - S(j,1;7)}{7} = -1 + 4 \cos \left( \frac{\pi}{7} \right) - 2 \sin \left( \frac{3\pi}{7} \right) \sim 0.1938 > 0.
\] (E.2.13)

Thus we do not need to add any states at reduced \( T_7 \).

So far our analysis shows that the addition of the states above renders the spectrum positive for all large spins \( j \). For sufficiently large central charge, this moreover is sufficient to render the spectrum positive for all finite spins. If the central charge is large enough, the negativity coming from the \( s = 8, 9, \ldots \) terms will be suppressed; since \( \frac{3+4\sin \left( \frac{3\pi}{7} \right)}{7} > -2 \frac{S(j,0;7) - S(j,1;7) - S(j,-1;7)}{7} \) for all \( j \), the term added is sufficient to render the density of states at all spins mod 7 positive.
In practice we find numerically that for central charge $c > c^*$, the spectrum is positive everywhere, with $c^* \sim 3237.7$.  

(E.2.14)

### E.3 Full orbifold sum

In this appendix, we write the full, finite partition function that includes a sum over all orbifolds $T_N$ and $\tilde{T}_N$. As discussed in section 8.5, the sum over all states with reduced twist $T_N, \tilde{T}_N$ results in a convergent, finite, modular invariant partition function. The sum over $T_N$ becomes

$$\sum_{N=1}^{\infty} Z_{T_N}(\tau, \bar{\tau}) = \frac{1}{\eta(q)\bar{\eta}(q)} \left( \sum_{m=1}^{\infty} \frac{(4\pi \xi \tau_2)^m}{m!} \zeta(2m) + \sum_{m=1}^{\infty} \frac{\Gamma(m)(16\pi^m \xi(2m)^2}{\zeta(2m+1)(2m)!}\right.$$ 

$$+ \sum_{j=1}^{\infty} 2 \left( e^{2\pi i j \tau_1} + e^{-2\pi i j \tau_1} \right) \sum_{m=1}^{\infty} \frac{(16\pi^m \xi)^m \sigma_{2m}(j) \zeta(2m)}{(2m)!j^m \zeta(2m+1)} K_m(2\pi j \tau_2) \right) ,$$

(E.3.1)

with $\xi = \frac{c-1}{24}$, while the sum over $\tilde{T}_N$ becomes the same equation but with $\xi \to \xi - 1 = \frac{c-23}{24}$. The $N = 1$ terms in these expressions also correspond to two of the four terms in the vacuum character. Finally, the remaining two terms are given by a $PSL(2, \mathbb{Z})$ sum of a seed with spin $J = \pm 1$, and reduced twist $T = -\frac{c-1}{24}$.

This sum is given in [65], which we rewrite in terms of hypergeometric functions as

$$\frac{1}{\eta(q)\bar{\eta}(q)} \left[ 2e^{2\pi \tau_2 E} \cos(2\pi \tau_1) + 4 + \sum_{m=1}^{\infty} \frac{4 \pi^m m}{m!} T_m(E) \tau_2^{-m} \zeta(2m+1) \right. + \sum_{m=1}^{\infty} \sum_{j=1}^{\infty} \frac{4 \sqrt{\pi} \tau_2^m j^m \cos(2\pi j \tau_1)}{\Gamma(m+\frac{1}{2})}$$

$$\times \left( Z_{j,1}(m + \frac{1}{2}) \sum_{a=0}^{m} (-1)^a + m \frac{1}{a!} F_1(-a, 1 - 2a + m, -4\pi \tau_2 j E) K_a(2\pi \tau_2 j) \right)$$

$$+ Z_{-j,1}(m + \frac{1}{2}) \sum_{a=0}^{m} (-1)^a \frac{1}{a!} F_1(-a, 1 - 2a + m, 4\pi \tau_2 j E) K_a(2\pi \tau_2 j)$$

$$+ \sum_{j=1}^{\infty} 4 \cos(2\pi j \tau_1) \left( Z_{j,1}(\frac{1}{2}) + Z_{-j,1}(\frac{1}{2}) \right) K_0(2\pi \tau_2 j) \right] ,$$

(E.3.2)
where $T_m$ is a Chebyshev polynomial. In (E.3.2), we set
\[ E = \frac{c - 13}{12} \]  
(E.3.3)
and the function $Z_{j,J}(m)$ is a Kloosterman zeta function defined as
\[ Z_{j,J}(m) = \sum_{c=1}^{\infty} c^{-2m} S(j, J; c). \]  
(E.3.4)

The expression (E.3.4) is convergent for every line in (E.3.2) except the last; there it has to be defined via analytic continuation. For convenience we can rewrite (E.3.2) in terms of a generalization of the non-holomorphic Eisenstein series:
\[ \frac{1}{\sqrt{\tau_2 \eta(q) \bar{\eta}(\bar{q})}} \sum_{m=0}^{\infty} \frac{(2\pi E)^m}{m!} \hat{E} \left( m + \frac{1}{2}, \tau, \bar{\tau} \right) \]  
(E.3.5)
where
\[ \hat{E} \left( m + \frac{1}{2}, \tau, \bar{\tau} \right) = 2 \cos (2\pi \tau_1) \tau_2^{m+\frac{1}{2}} + \frac{2\sqrt{\pi} \Gamma(2m+1)}{\Gamma(m+\frac{1}{2})} \tau_2^{\frac{1}{2}-m} \]
\[ \quad + \sum_{a=1}^{\infty} \frac{2(-1)^{a} \pi \Gamma(a+m) \zeta(4a+2m+1)}{\Gamma(a+1) \Gamma(2a+2m+\frac{1}{2}) \tau_2^{\frac{1}{2}-m-2a}} \]
\[ \quad + \sum_{j=1}^{\infty} \sum_{a=0}^{\frac{a}{2}} \sum_{b=0}^{\infty} \frac{4(-1)^{b} \beta \gamma \Pi \zeta(2\pi j \tau_2) \tau_2^{-b} K_{b+m} (2\pi j \tau_2)}{\Gamma(a-2b+1) \Gamma(a+m+\frac{1}{2}) \Gamma(b+1)} \]
\[ \quad \times \left[ (-1)^{a} Z_{j,1}(m+a+\frac{1}{2}) + Z_{j,-1}(m+a+\frac{1}{2}) \right]. \]  
(E.3.6)

Each of the $\hat{E}$ functions in (E.3.6) are the $PSL(2, \mathbb{Z})$ sum of $2\tau_2^{m+\frac{1}{2}} \cos(2\pi \tau_1)$ and are by construction modular invariant. It would be straightforward to generalize this to a $PSL(2, \mathbb{Z})$ sum of $\tau_2^{m+\frac{1}{2}} e^{2\pi i J_2}$ for any integer $J$, but we will do not so here.

The final partition function is the sum of (E.3.1) and (E.3.1) with $\xi$ replaced with $\xi - 1$, minus (E.3.2).


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hep-th/0005003.


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