# Measures of Irrationality and Vector Bundles on Trees of Rational Curves

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Measures of irrationality and vector bundles on trees of rational curves

A dissertation presented

by

Geoffrey Smith

to

The Department of Mathematics

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Abstract

This thesis is divided into three parts. In the first, we define the covering gonality and separable covering gonality of varieties over fields of positive characteristic, generalizing the definition given by Bastianelli-de Poi-Ein-Lazarsfeld-Ullery for complex varieties. We show that over an arbitrary field a smooth degree $d$ dimension $n$ hypersurface has separable covering gonality at least $d - n$ and a very general such hypersurface has covering gonality at least $\frac{d-n+1}{2}$.

In the second part, a chapter joint with Isabel Vogt, we investigate an arithmetic analogue of the gonality of a smooth projective curve $C$ over a number field $k$: the minimal $e$ such there are infinitely many points $P \in C(\overline{k})$ with $[k(P) : k] \leq e$. Developing techniques that make use of an auxiliary smooth surface containing the curve, we show that this invariant can take any value subject to constraints imposed by the gonality. Building on work of Debarre–Klassen, we show that this invariant is equal to the gonality for all sufficiently ample curves on a surface $S$ with trivial irregularity.

In the final part, we study vector bundles on trees of smooth rational curves. We determine explicit conditions under which a vector bundle on a tree of smooth rational curves can be expressed as the flat limit of some particular vector bundle on $\mathbb{P}^1$. 

Dissertation Advisor: Professor Harris Geoffrey Smith
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1. Background on measures of irrationality

Let $C$ be a smooth geometrically irreducible curve defined over some field $k$. The \textit{gonality} of $C$, denoted $\text{gon}_k(C)$, is defined as the minimal degree of a nonconstant map $C \to \mathbb{P}^1_k$. It is easy to see that $\text{gon}_k(C) = 1$ if and only if $C \cong \mathbb{P}^1_k$, and for curves of positive genus determining the gonality is an important classical question.

If $k$ is algebraically closed and $C$ is a general genus $g$ curve, then it has gonality $\left\lfloor \frac{2g+2}{2} \right\rfloor$ by the Brill-Noether theorem of [21]. But this is merely an upper bound for arbitrary genus $g$ curves; in practice many of the curves one encounters have a far lower gonality, and determining lower bounds for the gonality of curves in special families is an area of active study (such as in [1], [28], and [31]).

Several efforts have been made to generalize the notion of gonality to higher-dimensional varieties. The most natural is the \textit{degree of irrationality} of a variety, defined by Moh and Heinzer in [38] for an irreducible variety $X/k$ of dimension $n$ as the minimum degree of the function field of $X$ over the rational function field $k(t_1, \ldots, t_n)$. Equivalently, this is the minimum degree of a rational map from $X$ to a projective space of the same dimension. This invariant is a generalization of the gonality, in that if $X$ is a smooth curve we have that $\text{irr}(X)$ is the gonality of $X$, and moreover, as the name suggests, measures the failure of $X$ to be rational; we have $\text{irr}(X) = 1$ if and only if $X$ is rational. Since its introduction, the degree of irrationality has been studied for many classes of varieties, for instance in [50], [4], [3], [2], and [37].

The degree of irrationality of a variety is in some sense the most natural generalization of the gonality to higher dimensional varieties. However, it is generally a much more subtle invariant. In particular, it can be incredibly difficult to determine whether a given variety $X$ is rational, or, rephrased, whether $\text{irr}(X) = 1$; historically, it has frequently entailed coming up with new subtle birational invariants of varieties. For instance, Clemens and Griffiths [10] used a subtle birational invariant
related to the intermediate Jacobian of varieties to prove the irrationality of smooth cubic threefolds, and Iskovskikh and Manin \[33\] introduced the method of maximal singularities in the process of showing that smooth quartic threefolds are irrational. Moreover, in many cases, most famously, the cubic fourfold, it is expected that the very general example is irrational, and the search for a proof has spurred enormous quantities of research, discussed extensively in \[29\].

The need for a better behaved birational invariant than rationality inspired the study of uniruled varieties, which over an uncountable algebraically closed field are defined as varieties that are covered by rational curves \[34\]. It is often much easier to determine whether a variety is uniruled than rational; for instance, a variety \(X\) over \(\mathbb{C}\) is uniruled if and only if it contains a single rational curve with globally generated normal bundle.

Like rationality itself, uniruledness also gives rise to quantitative measure of rationality, the covering gonality. This invariant was introduced in \[3\] for varieties defined over \(\mathbb{C}\), and later formally codified in \[6\]; for a complex variety \(X\), its covering gonality \(\text{cvg}(X)\) is defined as the minimum gonality of a curve passing through a general point on \(X\). It has since been bounded for several classes of complex varieties, for instance in \[5\] \[49\], and \[37\].

In Chapter 2 of this dissertation, we will generalize this definition of a covering gonality to varieties defined over fields other than \(\mathbb{C}\). It turns out there are two different invariants, both of which are reasonable choices of generalizations of the covering gonality of a complex variety: we will call these the covering gonality and separable covering gonality of a variety. Formally, the covering gonality of an irreducible proper variety \(X/k\), denoted \(\text{cvg}(X)\), is the minimal \(e\) such that there exists a diagram of
varieties over $k$,

$$
\begin{array}{c}
C \xrightarrow{f} X \\
\downarrow \pi \\
\mathbb{P}^1 \times B
\end{array}
$$

(1.0.1)

where $f$ is a dominant generically finite map and $\pi : C \to \mathbb{P}^1 \times B$ is a finite surjective morphism of degree $e$. The separable covering gonality of $X$, denoted $\text{scvg}_k(X)$, is the minimal $e$ such that such a diagram exists with $f$ separable. If $k = \mathbb{C}$, these two definitions agree with each other and the definition of covering gonality given in [6]. But in positive characteristic their behavior can vary widely from each other, and in Chapter 2 we will give several examples of this phenomenon.

Measures of irrationality can also come from arithmetic. Let $C$ be a smooth and geometrically irreducible curve defined over a number field $k$. By Faltings’ theorem [20], if the genus of $C$ is at least 2, $C$ has only finitely many rational points over $k$ or any finite extension of $k$. However, even though $C$ may only have finitely many rational points over any single number field, it is possible for it to have infinitely many points in degree $d$ extensions, where $e$ is fixed. This observation is the basis of the \textit{arithmetic irrationality} of $C$, denoted $\text{a.irr}_k(C)$. It is defined as the minimal $e$ such that the set

$$
C_e : \bigcup \{C(L) : [L : k] \leq d\}
$$

is infinite. This invariant is 1 for $\mathbb{P}^1$ and in general is bounded above by the gonality of $C$, but determining lower bounds can be tricky and has been the subject of some work, such as [2], [25], and [15].

In Chapter 3 which is a paper [47] written jointly with Isabel Vogt, we show that in a large number of cases, the arithmetic irrationality of a curve $C$ achieves its maximum value, that is $\text{a.irr}_k(C) = \text{gon}_k(C)$.
In this chapter, we generalize the notion of covering gonality as defined in [6] to varieties defined over arbitrary fields. Recall that for an irreducible projective variety $X$ defined over $\mathbb{C}$, the covering gonality of $X$ is the minimum gonality of a curve through a general point on $X$. This quantifies the extent to which $X$ fails to be uniruled. However, over a general field, there are two notions of uniruledness: a variety can be separably uniruled, inseparably uniruled, or not uniruled. Our extension of covering gonality to positive characteristic takes this into account.

**Definition 2.0.1.** Let $X$ be an irreducible proper variety of dimension $n$ over a field $k$. The covering gonality of $X$ over $k$, denoted $\text{cvg}(X)$, is the minimal $e$ such that there exists a diagram of irreducible varieties over $k$,

$$
\begin{array}{ccc}
C & \xrightarrow{f} & X \\
\downarrow{\pi} & & \\
\mathbb{P}^1 \times B
\end{array}
$$

(2.0.1)

where $f$ is a dominant generically finite map and $\pi : C \to \mathbb{P}^1 \times B$ is a finite surjective morphism of degree $e$. The separable covering gonality of $X$, denoted $\text{scvg}_k(X)$, is defined as the minimal $e$ such that such a diagram exists with $f$ separable.

**Remark 2.0.2.** It is essential for this definition that there be a single algebraic family of curves dominating $X$. If we merely require that for any geometric point $p$ of $X$ there exists a gonality $e$ curve passing through $p$, [8] shows there would exist some non-uniruled varieties defined over $\mathbb{F}_p$ having covering gonality 1 over $\mathbb{F}_p$, namely the non-supersingular Kummer surfaces. If $k$ is uncountable and $X$ is projective, this issue does not come up, by Proposition 2.1.4 (5).

Of these two measures, the separable covering gonality is by far the better behaved invariant, and many results proven for the covering gonality of complex varieties hold
for the separable covering gonality as is. For instance, we observe that the proof of Theorem A of [6] is sufficient to prove the following slight generalization.

**Proposition 2.0.3** ([6], Theorem A). Let $X \subset \mathbb{P}^N_k$ be a smooth geometrically irreducible complete intersection of $k$ hypersurfaces of degree $d_1, \ldots, d_k$ with $\sum_i d_i \geq N + 1$. Then we have

$$\text{scvg}_k(X) \geq \sum_i d_i - N + 1$$

The covering gonality is more poorly behaved, as the following example of Shioda and Katsura illustrates.

**Example 2.0.4** ([46], Theorem III). Let $k$ be a field of positive characteristic $p$. Let $r, d$ be positive integers with $r$ even and $d \geq 4$. Then, if there exists an integer $v$ such that

$$p^v \equiv -1 \pmod{d},$$

then the Fermat hypersurface cut out by the homogeneous polynomial $x_0^d + \ldots + x_r^d = 0$ in $\mathbb{P}^{r+1}_k$ is unirational.

So there are smooth hypersurfaces $X$ with $d-n$ arbitrarily large such that $\text{cvg}(X) = 1$ if $k$ has positive characteristic, so Proposition 2.0.3 has no chance of being true without modification for the covering gonality. Instead, we have the following bound.

**Theorem 2.0.5.** Let $X \subset \mathbb{P}^N$ be a very general complete intersection variety of multidegree $(d_1, \ldots, d_k)$ over an uncountable field $k$. Then

$$\text{cvg}_k(X) \geq \frac{\sum_i d_i - N + 2}{2}.$$  

This will be a consequence of a somewhat more general result, Theorem 2.2.4, which connects a certain property of the Chow group of zero-cycles of a variety $V$ to a lower bound on the covering gonality of a very general plane section of $V$. 


This chapter is organized as follows. In Section 2.1 we collect some useful properties of the covering gonality, including an alternate definition which is manifestly a birational invariant. Then in Section 2.2 we prove Proposition 2.0.3 and Theorem 2.2.4, deriving Theorem 2.0.5. Section 2.3 contains proofs of some of the more involved formal properties of covering gonalities asserted in Section 2.1.

2.1. Formal properties of covering gonalities. In this section we prove some basic properties of covering gonalities. To help with this, we introduce an ancillary definition of covering gonality for arbitrary algebraic function fields.

**Definition 2.1.1.** Let $K$ be an algebraic function field over $k$. The covering gonality of $K$, $\text{cvg}_k(F)$, is the minimal $e$ such that there exists two fields $K_C$ and $K_B$ over $k$ and a diagram of finite field extensions

\[
\begin{array}{ccc}
K_C & \xleftarrow{f} & K \\
\pi & \Uparrow & \\
K_B(t) & \end{array}
\]  

such that $[K_C : K_B(t)] = e$. The separable covering gonality of $K$ is the minimal such $e$ such that the diagram \((2.1.1)\) exists with $f$ separable.

As the name suggests, this invariant is closely connected with the covering gonality.

**Proposition 2.1.2.** If $X$ is a proper irreducible variety defined over the field $k$ with function field $K$, then $\text{cvg}(X) = \text{cvg}_k(K)$ and $\text{scvg}_k(X) = \text{scvg}_k(K)$.

Working with covering gonalities for algebraic function fields simplifies some of our discussion. In particular, it makes base change more clear.

**Proposition 2.1.3.** Let $K$ be an algebraic function field over a field $k$. If $L/k$ is an arbitrary field extension of $k$ and $K'$ a field in $K \otimes_k L$, then $\text{cvg}_k(K) = \text{cvg}_L(K')$. If $L/k$ is contained in a separable extension of the field of constants of $K/k$, then $\text{scvg}_k(K) = \text{scvg}_L(K')$.}

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The assumption that $L/k$ be contained in a separable extension of the field of constants of $K$ is necessary; in the unirational hypersurfaces of Example 2.0.4, the separable covering gonality of $X$ is clearly

Together, Propositions 2.1.2 and 2.1.3 indicate the first key properties of the covering gonality; it is a birational invariant of $X$ and (for the most part) does not depend on the field of definition $k$. Their proofs are straightforward, and we defer them to Section 2.3.

We collect here some useful properties of covering gonalities, which echo comparable properties for the invariants defined in [6] over $\mathbb{C}$.

**Proposition 2.1.4.** (1) Both $cvg(X)$ and $scvg_k(X)$ are birational invariants of $X$.

(2) Given an extension $L/k$, if $X_L$ is any irreducible component of $X \times_k \text{Spec}(L)$, then $cvg(X) = cvg(X_L)$. If $L/k$ is contained in a separable extension of the field of constants of $X$, then $scvg(X) = scvg(X_L)$.

(3) $X$ satisfies $cvg(X) = 1$ if and only if $X$ is uniruled. $X$ satisfies $scvg_k(X) = 1$ if and only if $X$ is separably uniruled over $k$.

(4) If $X$ is a smooth irreducible projective curve and $k$ is algebraically closed, then $scvg_k(X) = cvg(X) = \text{gon}_k(X)$.

(5) If $k$ is uncountable and algebraically closed and $X$ is projective, then $cvg(X)$ is equal to the minimum gonality of a curve through a general point of $X$.

(6) If $f : X \to Y$ is a dominant generically finite rational map of projective varieties over $k$, then $cvg(Y) \leq cvg(X)$. If $f$ is also separable, then $scvg_k(Y) \leq scvg_k(X)$.

**Proof.** Proposition 2.1.2 immediately implies (1), because birational varieties have the same function fields. (2) follows from Proposition 2.1.3.
We observe that if $X$ has covering gonality 1, then the diagram (2.0.1) in fact gives a dominant generically finite morphism from $\mathbb{P}^1 \times B$ to $X$, so $X$ is uniruled. The same observation holds for separable covering gonality, proving (3).

(4) follows from Proposition A.1.vii of [40].

The proof of (5) is analogous to the proof of Proposition IV.1.3.5 of [34]. Fix some ample class $H$ on $X$. Let $e$ be the minimum gonality of a curve through a general point of $X$. Because $k$ is uncountable, there must be some fixed degree $d$ such that the set of points $x \in X$ contained in degree $d$ gonality $e$ curves is dense. There is a quasiprojective variety $B = \text{Hom}_d(\mathbb{P}^1, \text{Sym}^e(X))$ parametrizing maps $\phi : \mathbb{P}^1 \to \text{Sym}^e(X)$ such that $\phi^*(H)$ has degree $d$. Moreover, we can take $\mathcal{C}$ to be such that the diagram

$$
\begin{array}{ccc}
\mathcal{C} & \longrightarrow & X \times \text{Sym}^{e-1}(X) \\
\downarrow & & \downarrow \pi \\
\mathbb{P}^1 \times B & \xrightarrow{\text{ev}} & \text{Sym}^e(X)
\end{array}
$$

is a fiber product, where $\pi$ is the obvious symmetrizing map $p, (p_1 \cdots p_{e-1}) \mapsto (p_1 \cdots p_{e-1}p)$.

Composing $\mathcal{C} \to X \times \text{Sym}^{e-1}(X)$ with the projection $X \times \text{Sym}^{e-1}(X) \to X$ gives a map $f : \mathcal{C} \to X$. By the assumption on $d$, some component of $\mathcal{C}$ dominates $X$, and restricting to that component and its image in $\mathbb{P}^1 \times B$ gives the desired covering family of $X$.

For (6), we note that a cover $g : \mathcal{C} \to X$ of $X$ by gonality $e$ curves can be generically composed with $f : X \to Y$ to give a covering family of $Y$, giving the desired inequalities. $\square$

Finally, we note that one can make a couple additional assumptions about covering families. In particular, over algebraically closed fields, we may assume that the covering gonality is computed by a family satisfying the additional properties of covering families assumed in Definition 1.4 of [6].

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Proposition 2.1.5. If $X$ is a smooth geometrically integral proper variety with (separable) covering gonality $e$ defined over the (separably) closed field $k$, then there exists a (separable) covering family $\pi : C \to \mathbb{P}^1 \times B$ of $X$ of degree $e$ such that $B$ and $\pi : C \to B$ are smooth and the general fiber $C_b$ of $\pi$ is birational to its image in $X$.

Proof. We prove the result for separable covering gonalities; the result for cvg is analogous. Let $C \to B$ be a covering family of $X$ as in (2.0.1). Since $C$ is irreducible and $f : C \to X$ is separable, we have that $C$ is geometrically integral, so $B$ is as well. Let $C'$ be the closure of the image of $C$ in $B \times X$. We have that $C'$ is geometrically integral because $C$ is, and $C' \to B$ is proper. Moreover, the generic fiber of $C' \to B$ has gonality at most $e$ by Proposition 2.1.4 (5). So $C'$ possesses a rational map to $\mathbb{P}^1$ such that $C' \to B \times \mathbb{P}^1$ has degree at most $e$. Removing the image of the indeterminacy locus from $B$, we then have a family $C' \to B \times \mathbb{P}^1$ such that every fiber over $B$ is isomorphic to its image in $X$. Let $C''$ be the normalization of $C'$. This variety is smooth in codimension 1, and so the map $C'' \to B$ is generically smooth, so restricting $B$ to some open set gives a smooth family of curves over smooth variety, such that the general fiber $C_b$ is birational to its image in $X$. 

2.2. Bounding covering gonalities. The two notions of the covering gonality given in this chapter satisfy similar formal properties to each other, as we saw above. But as a practical matter, existing methods of computing covering gonality mostly compute scvg in positive characteristic. For instance, Proposition 2.0.3 was first stated and proved in [6] for the covering gonality of complex varieties, but its proof holds in positive characteristic, essentially as is.

Proof of Proposition 2.0.3. By Proposition 2.1.4 (2), we may assume the field $k$ is separably closed. By adjunction we have that the canonical bundle $\omega_X$ on $X$ is given by $\omega_X \cong O(mH)$, where $m := \sum_i d_i - N - 1$. Suppose $X$ has separable covering
gonality $e$, and that this gonality is achieved by the separable covering family

$$\begin{array}{c}
\mathcal{C} \xrightarrow{f} X \\
\downarrow \pi \\
\mathbb{P}^1 \times B
\end{array}$$

satisfying all the properties listed in Proposition 2.1.5. By adjunction, we have the isomorphism

$$\omega_{\mathcal{C}} \cong f^*(\omega_X) \otimes \mathcal{O}(R),$$

where $R$ is the ramification divisor of $f$. Let $b \in B$ be general, so the fiber $C_b$ of $\mathcal{C}$ over $b$ is smooth and birational to its image in $X$. By adjunction, we have $\omega_{C_b} \cong \omega_{\mathcal{C}|_{C_b}}$. Moreover, since $b$ is general, we have that $C_b$ is not contained in the ramification locus of $f$, so we have an isomorphism

$$\omega_{C_b} \cong f^*(\omega_X)|_{C_b} + R',$$

where $R'$ is some effective divisor on $C_b$. Let $D = p_1 + \ldots + p_e$ be a general fiber of the map $\pi : C_b \to \mathbb{P}^1$, so in particular we may assume every $p_i$ is contained in the open subset of $C_b$ isomorphic to its image in $X$. By the Riemann-Roch theorem, we have $H^0(C_b, \omega_{C_b}(-p_1 - \ldots - p_e)) > H^0(C_b, \omega_{C_b}) - e$, so for some $j \leq e - 1$ we must have an equality

$$H^0(C_b, \omega_{C_b}(-p_1 - \ldots - p_j)) = H^0(C_b, \omega_{C_b}(-p_1 - \ldots - p_{j+1})).$$

But, supposing $j \leq m$, there is a degree $m$ hypersurface in $\mathbb{P}^N$ passing through $f(p_1), \ldots, f(p_j)$ but not $f(p_{j+1})$, so the points $p_1, \ldots, p_{j+1}$ impose independent conditions on $f^*(\omega_X)|_{C_b}$ and hence on $\omega_{C_b}$. So the equality above implies $e - 1 > m$, giving

$$\text{scvg}_k(X) \geq \sum_i d_i - N + 1,$$

as was to be shown. \qed
As Example 2.0.4 demonstrates, we cannot hope that Proposition 2.0.3 holds for the ordinary covering gonality. However, over the course of the remainder of this paper, we will use a different, Chow-theoretic method to prove the lower bound Theorem 2.0.5. In fact, we will prove a somewhat more general statement, Theorem 2.2.4, from which we will derive Theorem 2.0.5 using some known results. To state the theorem, we first introduce a convenient definition.

**Definition 2.2.1.** An irreducible proper variety $X$ defined over an algebraically closed field is **Chow nondegenerate** if for all divisors $Z$ on $X$ the map of Chow groups of zero-cycles

$$\text{CH}_0(Z) \to \text{CH}_0(X)$$

induced by inclusion is not surjective. We will call $X$ Chow degenerate if it is not Chow nondegenerate.

Uniruled varieties are always Chow degenerate, as given a uniruling $f : \mathbb{P}^1 \times B \to X$ and any closed point $p \in \mathbb{P}^1$, we have that $\text{CH}_0(X)$ is generated by points in the union of the image of $p \times B$ and the exceptional locus of $f$ in $X$. So Chow nondegenerate varieties always satisfy $\text{cvg}(X) \geq 2$. Riedl and Woolf [44] observed this, and to give examples of non-uniruled varieties they proved the following.

**Theorem 2.2.2** ([44]). If $X/\overline{k}$ is a general complete intersection of multidegree $(d_1, \ldots, d_k)$ in $\mathbb{P}^N$ and $\sum_i d_i - N - 1 \geq 0$, then $X$ is Chow nondegenerate.

**Remark 2.2.3.** This theorem in this exact format does not appear in [44], but follows from its Theorem 3.3—a result on the coniveau filtration of complete intersections originally due to [16] Exposé XXI—and Proposition 3.7.

What we will show is that a very general plane section of a Chow nondegenerate variety has a covering gonality that increases with the codimension. More precisely, we prove the following.
Theorem 2.2.4. Let $X$ be a projective Chow nondegenerate variety over an uncountable field $k = \overline{k}$. Fix an embedding $X \hookrightarrow \mathbb{P}^n$. Then a very general codimension $c$ plane section of $X$ has covering gonality at least $\frac{3+c}{2}$.

Theorems 2.2.4 and 2.2.2 immediately imply Theorem 2.0.5 so it only remains to prove Theorem 2.2.4.

Proof of Theorem 2.2.4. Fix some $e$. If the very general codimension $c$ plane section of $X$ has covering gonality at most $e$, then there is a family $\pi : C \to B$ of gonality $e$ curves on $X$ such that a general codimension $c$ plane section $X'$ of $X$ is dominated by fibers of $\pi$ that map entirely into $X'$. In particular, the general codimension $c$ plane section is dominated by a family of gonality at most $e$ curves of some fixed Hilbert polynomial $P_0(t)$. For each $0 \leq c' \leq c$, let $X_{c'}$ be the set of triples $(\Lambda, Y, p)$, with $\Lambda$ a codimension $c'$ plane in $\mathbb{P}^n$, $Y$ a divisor on $\Lambda$ of degree $2e - 1$, and $p$ a geometric point of $(\Lambda \cap X) \setminus Y$. Let $R_0^{c'}$ be the subset of this locus containing triples $(\Lambda, Y, p)$ where there exists a reduced curve $C$ on $\Lambda \cap Y$ with Hilbert polynomial $P_0(t)$ passing through $p$ such that $p$ is Chow-equivalent on $C$ to a divisor supported on $C \cap Y$, and let $R_{c'}$ be the closure of this locus.

We first show that $R_{c}$ has codimension at most $2e - 2$ in $X_{c}$. Fixing $\Lambda$ and $p$ general, there exists a curve $C$ through $p$ on $\Lambda \cap X$ with Hilbert polynomial $P_0(t)$ having a pencil of degree at most $e$ by hypothesis, and by the generality assumption on $p$ we may assume this pencil is unramified at $p$. Then the space of divisors $Y$ passing through the other $e - 1$ points of the fiber of the pencil containing $p$ and entirely containing the $e$ points of some other fiber of the pencil, but not $p$, is a nonempty codimension $\leq 2e - 2$ set.

Now we bound the codimension of all the $R_{c'}$ from below using the fact that $X$ is Chow nondegenerate. To start, note that $R_0$ is a strict subset of $X_{0}$. For given any divisor $Y$ on $X$, there is some $p$ on $X$ not Chow equivalent to any divisor on $Y$. In
particular $p$ cannot be carried to $Y$ by a pencil on any $C$ with Hilbert polynomial $P_0(t)$ lying in $X$, or the flat limit of such curves.

We now apply the following lemma, which is a variation of Proposition 3.5 of [45].

**Lemma 2.2.5.** Assume $k$ is algebraically closed. Let $S_{c'}$ and $S_{c'+1}$ be closed subsets of $\mathcal{X}_{c'}$ and $\mathcal{X}_{c'+1}$ satisfying the following property:

If $(\Lambda, Y, p)$ is in $S_{c'+1}$, then for any codimension $c'$ plane section $\Lambda'$ of $\mathbb{P}^n$ containing $\Lambda$ and divisor $Y'$ on $\Lambda'$ such that $Y'|_{\Lambda} = Y$, we have $(\Lambda', Y', p) \in S_{c'}$.

Then if every irreducible component of $S_{c'}$ has codimension at least $\epsilon > 0$, every irreducible component of $S_{c'+1}$ has codimension at least $\epsilon + 1$.

Applying this lemma to the $R_{c'}$, we have that $R_{c'}$ has codimension at least $c' + 1$. Looking at the codimension of $R_{c'}$, we then have $c + 1 \leq 2e - 2$, so $e \geq \frac{3 + c}{2}$, as was to be shown. $\square$

**Proof of Lemma 2.2.5.** We may clearly assume that $S_{c'+1}$ is irreducible. It suffices to work in the case where $S_{c'}$ is also irreducible, as given the irreducibility of $S_{c'+1}$, the locus of triples in $\mathcal{X}_{c'}$ with hyperplane section in $S_{c'+1}$ is itself irreducible. Say $S_{c'}$ has codimension $\epsilon$. Let $\Phi \subset S_{c'} \times S_{c'+1}$ be the incidence correspondence between $(\Lambda', Y', p) \in S_{c'+1}$ and $(\Lambda, Y, p) \in S_{c'}$ with $(\Lambda', Y', p)$ as a hyperplane section. $\Phi$ is irreducible as a fiber bundle with irreducible fibers over $S_{c'+1}$. It has dimension $\dim(S_{c'+1}) + \dim(F)$, where $F$ is any fiber of the projection $\Phi \to S_{c'+1}$. We note that there is an $n - c' - 1$ dimensional family of hyperplane sections of a given triple $(\Lambda, Y, p) \in \mathcal{X}_{c'}$, so $\Phi$ has dimension strictly less than $\dim(S_{c'}) + n - c' - 1$, as by Lemma 3.6 of [45] $S_{c'+1}$ cannot contain every hyperplane section of $S_{c'}$. So we have

$$\dim(S_{c'+1}) < \dim(S_{c'}) + n - c' - 1 - \dim(F).$$
Using $\dim(F) + \dim(\mathcal{X}_{c+1}) = \dim(\mathcal{X}_c) + n - c' - 1$, we then have

$$\dim(S_{c+1}) < \dim(S_c) + \dim(\mathcal{X}_{c+1}) - \dim(\mathcal{X}_c)$$

proving the desired result.

2.3. Covering gonality for algebraic function fields. In this section we prove Propositions 2.1.2 and 2.1.3 on covering gonalities for algebraic function fields. In this section, we will use the term *diagram of field extensions* of $K$ to refer to a diagram

$$
\begin{array}{c}
K_C \\
\downarrow \pi \\
K_B(t)
\end{array}
\xleftarrow{f} K
$$

satisfying the properties of Definition 2.1.1.

We first prove Proposition 2.1.3 in two parts: we first show that (separable) covering gonalities are preserved after replacing $k$ with the field of constants of $K/k$ as Lemma 2.3.1, we then show in Lemma 2.3.2 that replacing that field with any (separable) extension preserves (separable) covering gonalities.

**Lemma 2.3.1.** If $k^e$ is the field of constants of $K/k$, then

$$cvg_k(K) = cvg_{k^e}(K) \text{ and } scvg_k(K) = scvg_{k^e}(K).$$

**Proof.** Let $K_C/K$ and $K_C/K_B(t)$ be extensions of fields as in Definition 2.1.1. The map $f : K \hookrightarrow K_C$ restricted to $k^e$ gives $K_C$ the structure of a field extension of $k^e$. Then the subfield of $K_C$ generated by $K_B + L$ is a field $K_{B'}$ defined over $k^e$, the natural map $K_{B'}(t) \hookrightarrow K_C$ is an extension of fields defined over $k^e$ and we have $[K_C : K_{B'}(t)] \leq [K_C : K_B(t)]$. So $cvg_{k^e}(K) \leq cvg_k(K)$ and $scvg_{k^e}(K) \leq scvg_k(K)$.

On the other hand, given any covering diagram (2.1.1) of field extensions of $K/k^e$, regarding $K_C$ and $K_B$ as field extensions of $k$ gives a diagram of field extensions of
Lemma 2.3.2. Let $K/k$ be an algebraic function field with field of constants $k$. Then if $L$ is any field extension of $k$ then

$$\text{cvg}_L(K \otimes_k L) = \text{cvg}_k(K).$$

If $L/k$ is separable, then

$$\text{scvg}_L(K \otimes_k L) = \text{scvg}_k(K).$$

Proof. We only prove the first assertion; the second follows analogously. Any extension of $k$ is an algebraic extension of a purely transcendental extension of $k$, so we handle those two cases separately. First suppose $L/k$ is algebraic. Given a diagram of field extensions

$$
K_C \xleftarrow{f} K \\
\pi \downarrow \quad , \quad K_B(t)$$

defined over $k$, let $K'_C$ be a quotient of $K_C \otimes_k L$ by some maximal ideal, which will be 0 if $K_C$ has field of constants $k$. Then the diagram

$$
K'_C \xleftarrow{f} K \otimes_k L \\
\pi \downarrow \\
(K_B \otimes_k L \cap K'_C)(t)$$

is a diagram of extensions as in Definition 2.1.1 and we have

$$[K'_C : (K_B \otimes_k L \cap K'_C)(t)] \leq [K_C : K_B(t)].$$
So \( \text{cvg}_L(K \otimes_k L) \leq \text{cvg}_k(K) \). For the other direction, suppose

\[
\begin{array}{c}
K_C \leftarrow^f K \otimes_k L \\
\pi \downarrow \quad \quad \quad \quad \\
K_B(t)
\end{array}
\]

is a diagram of extensions over \( L \) as in Definition 2.1.1. Because \( K_C \) and \( K_B \) are finitely generated, there is some finitely generated extension \( L'/k \) contained in \( L \) such that \( K_B \) admits a presentation \( K_B \cong L(b_1,\ldots,b_k)/(r_1,\ldots,r_\ell) \) where the \( r_i \) are polynomials in the \( b_j \) with coefficients in \( L' \) and \( K_C \) admits a presentation \( K_B(t)(c_1,\ldots,c_k')/(q_1,\ldots,q_{k'}) \) where the \( q_i \) are polynomials in the \( c_j \) with coefficients in \( L'(b_1,\ldots,b_k,t) \). Moreover, we can assume that the generators \( c_i \) include the image in \( K_C \) of a set of generators of \( K/k \).

Let \( K'_B := L'(b_1,\ldots,b_k)/(r_1,\ldots,r_\ell) \) and \( K'_C := K'_B(t)(c_1,\ldots,c_k')/(q_1,\ldots,q_{k'}) \). Since the \( c_i \) include a set of generators of \( K/k \), \( K'_C \) is an extension of \( K \), and \( K'_C/K \) is finite because it is finitely generated and \( K'_C \) is contained in the algebraic extension \( K_C \) of \( K \). And, noting \( K_C \cong K'_C \otimes_{L'} L \) and \( K_B \cong K'_B \otimes_{L'} L \), we have \([K_C : K_B(t)] = [K'_C : K_B(t')]\). So we have \( \text{cvg}_L(K \otimes_k L) \geq \text{cvg}_k(K) \), hence \( \text{cvg}_L(K \otimes_k L) = \text{cvg}_k(K) \).

Now suppose \( L \) is purely transcendental over \( k \). \( \text{cvg}_L(K \otimes_k L) \leq \text{cvg}_k(K) \) is clear by the same argument as above. Now suppose we have a diagram

\[
\begin{array}{c}
K_C \leftarrow^f K \otimes_k L \\
\pi \downarrow \quad \quad \quad \quad \\
K_B(t).
\end{array}
\]

By picking some finite sets of generators and relations for \( K_C/K_B(t) \), \( K_C/K \) and \( K_B/L \), and replacing \( L \) by the field generated over \( k \) by all coefficients in the relations defining these extensions, we may assume \( L \) is finitely generated, so inductively we may assume \( L = k(x) \). In addition, by the result for algebraic extensions, we may assume that \( k \) is infinite. So for the rest of the proof, assume \( k \) is infinite and \( L = k(x) \).
First, given a diagram of algebraic function fields over $k$ as in (2.1.1), taking the tensor product of the diagram with $k(x)$ gives a covering diagram of $K \otimes_k k(x)$ of the same degree as that for $K$. So $\text{cvg}_K(K) \geq \text{cvg}_{k(x)}(K \otimes_k k(x))$.

In the other direction, first note that we can assume $k$ is infinite by replacing it as needed by some infinite extension; this does not change covering gonalities by the result for algebraic extensions above. Now suppose we have a covering diagram of $K \otimes_k k(x)$ over $k(x)$ as in (2.1.1). Let $C$, $B$, and $X$ be varieties over $k$ with function fields $K_C$, $K_B$ and $K$, so we have a diagram of rational maps

$$
\begin{array}{ccc}
C & \xrightarrow{f} & X \times \mathbb{P}^1 \\
\downarrow{\pi} & & \downarrow{\pi} \\
\mathbb{P}^1 \times B & & 
\end{array}
$$

Replacing $C$ by a smaller variety as needed, we may assume $f$ and $\pi$ are regular. Moreover, $C$ and $B$ both possess compatible rational maps to $\mathbb{P}^1$, where on $C$ the map is given by composing $f$ with the projection $X \times \mathbb{P}^1 \to \mathbb{P}^1$. Because $f$ and $\pi$ are generically finite, for all but finitely many $k$-points $p$ in $\mathbb{P}^1$ the maps of fibers

$$
\begin{array}{ccc}
C_p & \xrightarrow{f} & X \\
\downarrow{\pi} & & \downarrow{\pi} \\
\mathbb{P}^1 \times B_p & & 
\end{array}
$$

are generically finite and $B_p$ and $C_p$ are both integral, and moreover the restriction of $\pi$ to $C_p \to \mathbb{P}^1 \times B_p$ has unchanged degree. The function fields of one of these fibers form a covering diagram of $X/k$ of degree equal to $[K_C : K_B(t)]$, so $\text{cvg}_k(K) \leq \text{cvg}_{k(x)}(K \otimes_k k(x))$. 

\[\square\]

**Proof of Proposition 2.1.3.** Suppose $L/k$ is an extension of $k$ contained in a separable extension of $k^c$. Let $L^c$ be the field of constants of $K \otimes_{k^c \cap L} L$ over $L$, so $L^c$ is separable
over $k^c$. Then by Lemmas 2.3.1 and 2.3.2 we have

$$\text{scvg}_k(K) = \text{scvg}_{k^c}(K) = \text{scvg}_{L^c}(K \otimes_{k^c} L^c) = \text{scvg}_L(K \otimes_{k^c} L^c)$$

and we have $K \otimes_{k^c} L^c$ is isomorphic to any field contained in $K \otimes_k L$. The proof for covering gonalities is analogous.

\[\square\]

**Proposition 2.3.3.** If $X$ is a proper irreducible variety defined over the field $k$ with function field $K$, then $\text{cvg}(X) = \text{cvg}_k(K)$ and $\text{scvg}_k(X) = \text{scvg}_k(K)$.

**Proof.** One direction here is clear; if we have a diagram of irreducible varieties

$$\begin{array}{ccc}
C & \xrightarrow{f} & X \\
\downarrow & & \downarrow \\
\mathbb{P}^1 \times B & & \\
\end{array}$$

as in Definition 2.0.1, taking function fields of every variety gives a covering diagram of $K$ in the sense of Definition 2.1.1. So $\text{cvg}(X) \geq \text{cvg}_k(K)$ and $\text{scvg}(X) \geq \text{scvg}_k(K)$.

We now prove the inequality $\text{cvg}(X) \leq \text{cvg}_k(K)$. Suppose we have a diagram of field extensions of $K$,

$$\begin{array}{ccc}
K_C & \xrightarrow{f} & K \\
\uparrow & & \uparrow \\
K_B(t) & & \\
\end{array}$$

We will construct $C$ and $B$ with function fields $K_C$ and $K_B$ respectively that fit into a covering diagram (2.0.1) of $X$ that produces the above diagram upon taking function fields.

Let $\mathcal{C}_\eta$ be the unique regular projective curve defined over $K_B$ with function field $K_C$. Then over some variety $B'/k$ with function field $K_B$ there is a variety $\mathcal{C}'$, a
proper map \( \pi : C' \to B' \) and a fiber diagram

\[
\begin{array}{ccc}
C_\eta & \xleftarrow{\pi} & C' \\
\downarrow{\pi} & & \downarrow{\pi} \\
\text{Spec}(K_B) & \xleftarrow{\cdot} & B'.
\end{array}
\]

This can be shown by embedding \( C_\eta \) in some projective space over \( K_B \) and letting \( B' \) be affine with fraction field \( K_B \) such that \( C_\eta \) is cut out by homogeneous polynomials with coefficients in \( H^0(B', \mathcal{O}_{B'}) \). Moreover, since \( C' \) has function field \( K_B \), the function \( t \) on \( C' \) defines a rational map \( C' \to \mathbb{P}^1 \). This is defined except on some closed subset \( Z \) of \( C' \) of codimension at least 2. In addition, let \( Z' \) be the locus in \( B' \) of \( b \) such that \( t \) is constant on some irreducible component of \( C'|_{\pi^{-1}(b)} \); this is not all of \( B \) since \( t \) is transcendental over \( K_B \). Define \( B'' = B' \setminus (\pi(Z) \cup Z') \), and let \( C'' \) be the open subset of \( C' \) lying over \( B'' \). Finally, we have a rational map \( f : C'' \to X \), which is again defined except on a closed subset \( Z'' \) of codimension at least 2 on \( C'' \). Then, defining \( B = B'' \setminus \pi(Z'') \) and \( C = \pi^{-1}(B) \), we have exactly the desired diagram

\[
\begin{array}{ccc}
C & \xrightarrow{f} & X \\
\downarrow{\pi} & & \\
\mathbb{P}^1 \times B
\end{array}
\]

where both \( \pi \) and \( f \) are regular by the construction, and \( \pi \) is finite surjective because we removed the locus \( Z' \) over which it failed to be finite surjective. Since \( \pi \) is induced by the field extension \( K_C/K_B(t) \), it has the same degree, and we conclude \( \text{cvg}(X) \leq \text{cvg}_k(K) \), giving the desired equality.

If we, throughout this process, assumed \( f \) was separable, no aspect of this construction would change, so we also have \( \text{scvg}_k(X) \leq \text{scvg}_k(K) \). \( \square \)

3. Low degree points on curves

This chapter is joint work with Isabel Vogt, and appears in the paper [47].
Let $C$ be a nice (smooth, projective and geometrically integral) curve over a number field $k$. For an algebraic point $P \in C(\overline{k})$, the degree of $P$ is the degree of the residue field extension $[k(P) : k]$. In this chapter we investigate the sets

$$C_e := \{ P \in C(\overline{k}) : \deg(P) \leq e \} = \bigcup_{[F:k] \leq e} C(F)$$

of algebraic points on $C$ with residue degree bounded by $e$.

When $e = 1$, this is the set of $k$-rational points on $C$. If the genus of $C$ is 0 or 1, then there is always a finite extension $K/k$ of the base field over which $(C_K)_1 = C(K)$ is infinite. On the other hand, if the genus of $C$ is at least 2, then for all finite extensions $K/k$, Faltings’ theorem guarantees that the set $(C_K)_1$ is finite [21]. While understanding the set of rational points is an interesting and subtle problem, here we will be primarily concerned with the infinitude of the sets $C_e$ as $e$ varies. Define the arithmetic degree of irrationality to be

$$\text{a.irr}_k(C) := \min(e : C_e \text{ is infinite}).$$

This invariant is not preserved under extension of the ground field, so we also define

$$\text{a.irr}_\mathbb{F}(C) := \min(e : \text{there exists a finite extension } K/k \text{ with } (C_K)_e \text{ infinite}).$$

As is implicit in the notation, this notion depends only upon the $\overline{k}$-isomorphism class of $C$, see Remark 3.6.3. The situation for $k$-points can therefore be summarized as:

$$\text{a.irr}_k(C) = 1 \iff \text{genus of } C \leq 1$$

For $e \geq 2$, the situation for higher genus curves is more interesting. Recall that the $k$-gonality of $C/k$,

$$\text{gon}_k(C) := \min(e : \text{there exists a dominant map } C \to \mathbb{P}^1_k \text{ of degree } e),$$

20
is a measure of the “geometric degree of irrationality” of $C$. This notion is also not
invariant under extension of the base field (e.g., a genus 0 curve has $k$-gonality 1 if
and only if it has a $k$-point). For that reason we also define the geometric gonality to
be $\text{gon}_{\bar{k}}(C) := \text{gon}_{\bar{k}}(C_{\bar{k}})$, which is stable under algebraic extensions. If $f : C \to \mathbb{P}^1_k$ is
dominant of degree at most $e$, then $f^{-1}(\mathbb{P}^1(k)) \subset C_e$. Therefore we always have the
upper bound

$$\text{a.irr}_k(C) \leq \text{gon}_k(C).$$

This bound need not always be sharp: if $f : C \to E$ is a dominant map of de-
gree at most $e$ onto a positive rank elliptic curve $E$, then $f^{-1}(E(k)) \subset C_e$ is also
infinite. When $e = 2$ (resp. $e = 3$) then Harris–Silverman and Hindry [25, 30]
(resp. Abramovich–Harris [2]) showed

$$\text{a.irr}_{\bar{k}}(C) = e \iff e$$

is minimal such that $C_{\bar{k}}$ is a degree $e$ cover of a curve of genus $\leq 1$.

Debarre–Fahlaoui [14] gave examples of curves lying on projective bundles over an el-
liptic curve that show the analogous result is false for all $e \geq 4$. The arithmetic degree
of irrationality is therefore a subtle invariant of a curve, capturing more information
than only low degree maps.

Implicit in the work of Abramovich–Harris [2] and explicit in a theorem of Frey
[23], is the fact that Faltings’ theorem implies that if $C_e$ is infinite, then $C$ admits a
map of degree at most $2e$ onto $\mathbb{P}^1_k$. Therefore we have an inequality in both directions

$$\text{gon}_k(C) / 2 \leq \text{a.irr}_k(C) \leq \text{gon}_k(C).$$

In this chapter, we develop and apply geometric techniques to compute $\text{a.irr}_k(C)$
and $\text{gon}_k(C)$ when $C$ lies on a smooth auxiliary surface $S$. The first result in this
direction is that the inequalities in (3.0.2) are sharp, and that subject to these bounds,
we may decouple $\text{a.irr}_k(C)$ and $\text{gon}_k(C)$. 21
Theorem 3.0.1. Given any number field $k$ and a pair of integers $\alpha, \gamma \geq 1$, there exists a nice curve $C/k$ such that

\begin{equation}
(3.0.3) \quad \text{a.irr}_k(C) = \text{a.irr}_k(C) = \alpha, \quad \text{gon}_k(C) = \text{gon}_k(C) = \gamma
\end{equation}

if and only if $\gamma/2 \leq \alpha \leq \gamma$. In fact, for $\gamma \geq 4$, the equalities (3.0.3) are satisfied for all smooth curves in numerical class $(\gamma, \alpha)$ on $S = E \times \mathbb{P}^1_k$, where $E/k$ is a positive-rank elliptic curve.

Using these geometric techniques, we next describe classes of curves where the arithmetic and geometric degrees of irrationality agree; that is, where there are as few points as allowed by the gonality. In such cases, we have the strongest finiteness statements on low degree points.

The first explicit examples of this kind were given by Debarre and Klassen for smooth plane curves $C/k$ of degree $d$ sufficiently large. Max Noether calculated the gonality for $d \geq 2$:

(1) If $C(k) \neq \emptyset$, then $\text{gon}_k(C) = d - 1$, and all minimal degree maps are projection from a $k$-point of $C$, and

(2) If $C(k) = \emptyset$, then $\text{gon}_k(C) = d$.

For smooth plane curves of degree $d \geq 8$, Debarre–Klassen [15] prove an arithmetic strengthening of this result:

(1) If $C(k) \neq \emptyset$, then $C_{d-2}$ is finite, and so $\text{a.irr}_k(C) = \text{gon}_k(C) = d - 1$. Furthermore, all but finitely many points of degree $d - 1$ come from intersecting $C$ with a line over $k$ through a $k$-point of $C$.

(2) If $C(k) = \emptyset$, then $C_{d-1}$ is finite, and so $\text{a.irr}_k(C) = \text{gon}_k(C) = d$.

We generalize this result to smooth curves on other surfaces $S$. The key property of $\mathbb{P}^2$ that we need in general is that it has discrete Picard group; i.e., in the classical language of surfaces, it has irregularity 0. The explicit condition $d \geq 8$ can be
replaced by requiring that the class of $C$ is “sufficiently positive” in the ample cone in the sense that it is sufficiently far from the origin, and sufficiently far from the boundary of the ample cone.

**Theorem 3.0.2.** Let $S/k$ be a nice surface with $h^1(S, \mathcal{O}_S) = 0$. If $C/k$ is a smooth curve in an ample class on $S$, then

\[
a.\text{irr}_k(C) \geq \min \left( \text{gon}_k(C), \frac{C^2}{9} \right).
\]

In particular, let $P$ be a very ample divisor on $S$, and define the set

\[
\text{Exc}_P := \{ \text{integral classes } H \text{ in } \text{Amp}(S) \text{ such that } H^2 \leq 9(H \cdot P - 1) \}.
\]

1. If $C \subset S$ is a smooth curve with class $[C] \in \text{Amp}(S) \setminus \text{Exc}_P$, then $a.\text{irr}_k(C) = \text{gon}_k(C)$.

2. For any closed subcone $N \subset \text{Amp}(S)$, the set $\text{Exc}_P(N) := \text{Exc}_P \cap N$ is finite.

As an immediate consequence, we obtain an effective generalization of the Debarre-Klassen result to other surfaces with $h^1(S, \mathcal{O}_S) = 0$.

**Corollary 3.0.3.** Suppose that $C$ embeds in a nice surface $S/k$ having $h^1(S, \mathcal{O}_S) = 0$, with $\mathcal{O}_S(1)$ very ample and $C \in |\mathcal{O}_S(\alpha)|$. If

\[
\alpha \geq \begin{cases} 
8 & : \mathcal{O}_S(1)^2 = 1 \\
9 & : \text{otherwise},
\end{cases}
\]

then $a.\text{irr}_k(C) = \text{gon}_k(C)$.

**Corollary 3.0.4.** Under the hypotheses of Corollary 3.0.3, if $S$ satisfies $\text{Pic}(S_{\overline{k}}) = \mathbb{Z} \cdot \mathcal{O}_S(1)$, then there are finitely many points of degree strictly less than $(\alpha - 1)\mathcal{O}_S(1)^2$ on $C_K$ for any finite extension $K/k$. 23
Proof. By Lemma 4.4, \((\alpha - 1)\mathcal{O}_S(1)^2 \leq \gon_k(C) \leq \gon_k(C)\). Therefore by Corollary 3.0.3,
\[(\alpha - 1)\mathcal{O}_S(1)^2 \leq \a.irr_k(C).\]

Corollary 3.0.3 combined with Theorem 3.1 is enough to deduce the analogous result for most complete intersection curves in \(\mathbb{P}_k^n\), generalizing in another direction Debarre and Klassen’s original result when \(n = 2\):

**Corollary 3.0.5.** Let \(C/k\) be a smooth complete intersection curve in \(\mathbb{P}_k^n\), \(n \geq 3\), of type \(9 \leq d_1 < d_2 \leq \cdots \leq d_{n-1}\). Then
\[\a.irr_k(C) = \gon_k(C).\]

In particular, by Lazarsfeld’s computation of the minimal gonality of such a curve Exercise 4.12], there are finitely many points of degree strictly less than \((d_1 - 1)d_2 \cdots d_{n-1}\) on \(C_K\) for any finite extension \(K/k\).

For any surface \(S\) and any finite polyhedral subcone \(N \subseteq \Amp(S)\), the set \(\Exc_P(N)\) in Theorem 3.0.2 is effectively computable. Given some particular surface \(S\), our techniques are amenable to explicit computations, and can sometimes yield a full computation of all classes \([C] \in \Amp(S)\) for which \(\a.irr_k(C)\) is strictly less than \(\gon_k(C)\). For example:

**Theorem 3.0.6.** Let \(C\) be a nice curve of type \((d_1, d_2)\), with \(1 \leq d_1 \leq d_2\), on \(\mathbb{P}_k^1 \times \mathbb{P}_k^1\). Then if \((d_1, d_2) \neq (2, 2)\) or \((3, 3)\), we have that \(\a.irr_k(C) = \gon_k(C) = d_1\).

In particular, this lets us compute:

\[
\a.irr_k(C) = \begin{cases} 
1 & : \ d_1 \leq 1 \ or \ (d_1, d_2) = (2, 2), \\
2 & : \ d_1 = 2 \ and \ d_2 \geq 3, \ or \ (d_1, d_2) = (3, 3) \ and \ C \ bielliptic, \\
d_1 & : \ otherwise.
\end{cases}
\]
Remark 3.0.7. We say that a point $P \in C(\mathbb{F})$ is sporadic if $[k(P) : k] < \text{a.\text{irr}}_k(C)$. From the perspective of the arithmetic of elliptic curves, there is much interest in understanding sporadic points on modular curves, e.g. the classical $X_1(N)$, since these indicate “usual” level structure. Since $X_1(N)(\mathbb{Q}) \neq \emptyset$, $X_1(N)$ is always a subvariety of its Jacobian variety $J_1(N)$. And whenever $\#J_1(N)(\mathbb{Q})$ is finite, we have that $\text{a.\text{irr}}_\mathbb{Q}(X_1(N)) = \text{gon}_\mathbb{Q}(X_1(N))$. In particular this holds for $N \leq 55$ and $N \neq 37, 43, 53$ by work of Derickx and van Hoeij [18]; in the same article, they compute the gonality (and therefore the arithmetic degree of irrationality when $N \neq 37$) for all $N \leq 40$. It is our hope that the geometric techniques we develop here might prove useful for specific curves of arithmetic interest.

As in previous work, the proofs of these results begin by translating the problem of understanding degree $e$ points on $C$ to understanding rational points on $\text{Sym}^e C =: C^{(e)}$, which is a parameter space for effective divisors of degree $e$ on $C$. There is a natural map

$$C^{(e)} \to \text{Pic}^e C,$$

sending an effective divisor $D$ to the class of the line bundle $\mathcal{O}(D)$. We denote the image of this map $W_e(C)$. We now have two problems: understand the infinitude of rational points on the fibers of $C^{(e)} \to \text{Pic}^e C$ (which is related to the dimension of the space of sections of the corresponding line bundle), and understand the infinitude of rational points on the image $W_e(C)$ (which, by Faltings’ Theorem, is related to positive-dimensional abelian varieties in $W_e(C)$).

The majority of this chapter is therefore devoted to proving purely geometric results over $\mathbb{C}$ about nonexistence of positive-dimensional abelian varieties in $W_e(C)$ for appropriate $e$. Using the theory of stability conditions on vector bundles, we show that such an abelian variety in $W_e C$ forces the existence of a certain type of effective divisor on $S$. Given a particular surface $S$, we can often use the geometry of $S$ to obtain a contradiction; this is how we proceed with Theorem 3.0.1. When the surface
is not explicitly given, the fact that such a divisor class does not move in a positive
dimensional family (from $h^1(S, \mathcal{O}_S) = 0$) allows us to construct an embedding of the
abelian variety into $W_{f}C$ for smaller $f$ and eventually obtain a contradiction.

3.1. Abelian Varieties in $W_eC$. In this section we prove purely geometric results
(Theorems 3.3.1 and 3.4.1) about nonexistence of abelian subvarieties that will imply
our main theorems. Therefore the basefield is assumed to be $\mathbb{C}$, unless otherwise
noted, and $\text{gon}(C) := \text{gon}_{\mathbb{C}}(C)$ denotes the geometric gonality.

The proofs of these results will proceed by contradiction: the existence of a positive-
dimensional abelian variety $A \subset W_eC$ will force the existence of a family of effective
divisors of moderately low degree moving in basepoint-free pencils. We will then use
a geometric lemma proved in Section 3.2 to produce interesting effective divisors on
an auxiliary surface containing the curve. The proof ideas bifurcate here: when the
auxiliary surface is specified explicitly, we may then directly use the geometry to
obtain a contradiction. When the surface is simply known to have $h^1(O) = 0$, we use
the interesting effective divisor to inductively produce such a family of effective line
bundles on $C$ of even lower degree that will force a contradiction for all but finitely
many possible starting classes of curves $C$.

The first step in this procedure relies on the following observation, due originally
to Abramovich and Harris [2, Lemma 1], and whose consequence for the gonality
of $C$ was noted by Frey [23]. Assume that $A \subset W_eC$ is a translate of an abelian
variety of dimension at least 1 and $A \not\subset x + W_{e-1}C$ for any $x \in C$. Let $A_2$ denote
the image of $A \times A$ under the addition map $W_eC \times W_eC \rightarrow W_{2e}C$. Note that $A_2$ is
(noncanonically) isomorphic to $A$: a choice of basepoint in $A$ induces an isomorphism
$\text{Pic}^eC \simeq \text{Jac}_C$, under which the addition map on $W_eC$ agrees with the group law on
$\text{Jac}_C$ and $A \subset \text{Jac}_C$ is an abelian subvariety.
Lemma 3.1.1. The line bundle $L_p$ corresponding to a point $p \in A_2 \subseteq W_{2e}C \subseteq \text{Pic}^{2e} C$ is basepoint-free and has

$$h^0(C, L_p) - 1 \geq \dim A.$$\\

Proof. Let $C^{(e)}$ denote the $e$th symmetric power of $C$. We have the following commutative diagram

$$
\begin{array}{ccc}
C^{(e)} \times C^{(e)} & \xrightarrow{\text{finite}} & C^{(2e)} \\
\downarrow & & \downarrow \\
W_eC \times W_eC & \xrightarrow{\oplus} & W_{2e}C \\
\downarrow & & \downarrow \\
A \times A & \xrightarrow{\oplus} & A_2
\end{array}
$$

Given a point $p \in W_{2e}C$, the fibers of the map $C^{(2e)} \rightarrow W_{2e}C$ are of dimension $h^0(C, L_p) - 1$. As the fibers of the bottom map $A \times A \rightarrow A_2$ are $(\dim A)$-dimensional, we see that if $p \in A_2$, the fiber of $C^{(2e)} \rightarrow W_{2e}C$ over $p$ must be at least this large. Furthermore, if $x \in C$ is in the base locus of this $(\dim A)$-dimensional linear system, then it would necessarily be the case that $x$ is always in the linear system parameterized by the points of $A$. This is impossible, as we assumed that $A$ is not contained in a translate of $W_{e-1}C$. \hfill \Box

3.2. Linear series of low degree. In this section we prove the key geometric input on linear series of moderately low degrees on curves $C$ whose class is ample on a surface $S$. This is a purely geometric result over an algebraically closed field of characteristic 0.

We first recall some of the basic theory of torsion-free coherent sheaves on varieties over algebraically closed fields $k = \overline{k}$. Let $F$ be a torsion-free coherent sheaf on a nice variety $X$ of dimension $m$. Given an ample class $H$ on $X$, we define the slope of $F$ with respect to $H$

$$
\mu_H(F) := \frac{c_1(F) \cdot H^{m-1}}{\text{rk}(F)}. 
$$
In what follows, we leave the reliance on \( H \) implicit, and just refer to the slope as \( \mu(F) \).

The sheaf \( F \) is called \( \mu \)-unstable \((\text{with respect to } H)\) if there exists a coherent sheaf \( E \subseteq F \) such that
\[
\mu(E) > \mu(F).
\]
Otherwise we say that \( F \) is \( \mu \)-semistable \((\text{with respect to } H)\).

The \( \mu \)-semistable sheaves are the building blocks of torsion-free coherent sheaves on \( X \). More precisely, for \( F \) any torsion-free coherent sheaf, by \cite{[32], Theorem 1.6.7} there exists a unique \textit{Harder-Narasimhan filtration} of \( F \),
\[
0 = F_0 \subset F_1 \subset \ldots \subset F_n = F,
\]
which is characterized by the following properties

1. Each quotient \( G_i := F_i/F_{i-1} \) is a torsion free \( \mu \)-semistable sheaf.
2. If \( 1 \leq i < j \leq n \), then \( \mu(G_i) > \mu(G_j) \).

In particular, we will use the fact that given an unstable torsion free coherent sheaf \( F \), there is a unique nonzero subsheaf \( E \subseteq F \) such that \( F/E \) is semistable and torsion free, and \( \mu(E) \) is maximal among subsheaves of \( F \). We call this \( E \) the \textit{maximal destabilizing subsheaf} of \( F \).

\textit{Remark 3.2.1.} If \( X \) is a curve (i.e., \( m = 1 \)), then a vector bundle \( F \) is unstable if and only if it is destabilized by a subbundle \( E \subseteq F \), since the saturation of a destabilizing subsheaf will yield a destabilizing subbundle. However, saturation does not in general yield a subbundle (though, by the fact that the quotient is torsion free, the maximal destabilizing subsheaf of a vector bundle is always saturated \cite[Definition 1.1.5]{[32]}). If \( X \) is a surface (i.e., \( m = 2 \)) and \( F \) is a vector bundle, then the maximal destabilizing subsheaf is itself also a vector bundle (though not in general a subbundle), as we now show. By \cite[Corollary 1.4]{[27]}, any sheaf \( E \) on \( X \) that is \textit{reflexive} (i.e. the natural
map \( E \to E^\vee \) is an isomorphism) is locally free. Furthermore, a saturated subsheaf \( E \subseteq F \) of a locally free sheaf is reflexive (as \( E^\vee / E \to F / E \) would otherwise be a torsion subsheaf, see also [27, Corollary 1.5]). The maximal destabilizing subsheaf is saturated, and hence reflexive, and hence locally free.

We also define the discriminant of a coherent sheaf \( F \) on a smooth complex projective surface in terms of Chern characters as the quantity

\[
\Delta(F) := 2 \text{ch}_0(F) \text{ch}_2(F) - \text{ch}_1(F)^2.
\]

The following fundamental theorem of Bogomolov [32, Theorem 3.4.1] implies that the property of \( \mu \)-stability of sheaves on surfaces is numerical.

**Theorem 3.2.2** (Bogomolov inequality). Let \( S \) be a smooth complex projective surface. If \( F \) is a \( \mu \)-semistable torsion-free coherent sheaf on \( S \) with respect to some ample class, then \( \Delta(F) \leq 0 \).

**Remark 3.2.3.** Once one knows that stability is a numerical property, the fact that \( \Delta(F) \) is the precise combination of Chern classes capturing this follows from the fact that it is the minimal polynomial in the Chern classes that is invariant under twisting by line bundles.

We now apply Bogomolov’s Inequality to prove a geometric result that will ultimately produce the bounds we desire. Results of this kind where first obtained by Reider [13], see Prop. 2.10, Remark 2.11 and Cor. 1.40.

**Proposition 3.2.4.** Let \( S \) be a smooth projective surface and \( C \subset S \) a smooth curve such that \( \mathcal{O}_S(C) \) is ample. If \( \Gamma \) is a divisor on \( C \) that moves in a basepoint-free pencil, satisfying

\[
\deg \Gamma < C^2/4,
\]

then there exists a divisor \( D \) on \( S \) satisfying the following four conditions
(1) \(h^0(S, D) \geq 2\),
(2) \(C \cdot D < C^2/2\),
(3) \(\deg \Gamma \geq D \cdot (C - D)\).
(4) If \(E\) is any divisor on \(S\) such that

\[(3.2.1) \quad h^0(\mathcal{O}_C(E|_C - \Gamma)) = 0 \quad \text{and} \quad E \cdot C < C^2,\]

then \(h^0(\mathcal{O}_S(E - D)) = 0\). In particular, \(h^0(\mathcal{O}_C(D|_C - \Gamma)) > 0\).

**Proof.** As \(\Gamma\) moves in a basepoint-free pencil, there is a choice of two sections generating the line bundle and hence giving a surjection \(\mathcal{O}_C^{\oplus 2} \to \mathcal{O}_C(\Gamma)\). This map fits into an exact sequence

\[(3.2.2) \quad 0 \to \mathcal{O}_C(-\Gamma) \to \mathcal{O}_C^{\oplus 2} \to \mathcal{O}_C(\Gamma) \to 0.\]

Let \(i: C \hookrightarrow S\) be the inclusion map. Then \(i_*\mathcal{O}_C(\Gamma)\) is a torsion sheaf on \(S\). We define the coherent sheaf \(F\) on \(S\) via the exact sequence of coherent sheaves

\[(3.2.3) \quad 0 \to F \to \mathcal{O}_S^{\oplus 2} \to i_*\mathcal{O}_C(\Gamma) \to 0,\]

where the right map factors through the surjection \(\mathcal{O}_S^{\oplus 2} \to \mathcal{O}_C^{\oplus 2}\).

As the only associated point of \(i_*\mathcal{O}_C(\Gamma)\) is the generic point of the divisor \(C \subseteq S\), the sheaf \(F\) is reflexive [27, Corollary 1.5], and hence locally free [27, Corollary 1.4].

Set \(e := \deg \Gamma\). Using Grothendieck-Riemann-Roch to calculate the Chern classes of the pushforward \(i_*\mathcal{O}_C(\Gamma)\), we may compute the discrete invariants of \(F\) from the exact sequence (3.2.3):

\[
\begin{align*}
\text{ch}_0(F) &= \text{rk}(F) = \text{rk}(\mathcal{O}_S^{\oplus 2}) = 2 \\
\text{ch}_1(F) &= c_1(F) = c_1(\mathcal{O}_S^{\oplus 2}) - c_1(i_*\mathcal{O}_C(\Gamma)) = -[C] \\
\text{ch}_2(F) &= \text{ch}_2(\mathcal{O}_S^{\oplus 2}) - \text{ch}_2(i_*\mathcal{O}_C(\Gamma)) = C^2/2 - c_2(i_*\mathcal{O}_C(\Gamma)) = C^2/2 - e.
\end{align*}
\]
The vector bundle $F$ therefore has Chern character $\text{ch}(F) = (2, -[C], C^2/2 - e)$ and hence has discriminant $\Delta(F) = C^2 - 4e$. Therefore, by assumption, $\Delta(F) > 0$, so $F$ is $\mu$-unstable with respect to any ample class on $S$; we will use $C$ as the ample class on $S$.

Let $L$ to be the maximal destabilizing subsheaf of $F$, which by Remark 3.2.1 is locally free and hence a line bundle. Write $L \cong O(-D)$, where $D$ is some divisor on $S$. We show this $D$ satisfies properties (1)–(4) of the proposition.

- **Property (2):** $C \cdot D < C^2/2$:

  By the definition of the maximal destabilizing subsheaf, we have

  $$\mu_C(L) = (-D) \cdot C > (-C) \cdot C/2 = \mu_C(F),$$

  which is equivalent to property (2).

- **Property (3):** $e \geq D \cdot (C - D)$:

  In the exact sequence

  $$0 \to O(-D) \to F \to Q \to 0,$$

  the quotient $Q$ is $\mu$-semistable with respect to $C$.

  Therefore $\Delta(Q) \leq 0$, which is equivalent to

  $$e \geq D \cdot (C - D).$$

- **Property (1):** $h^0(S, D) \geq 2$:

  Dualizing the inclusion $O(-D) \to O_S^{\oplus 2}$, it suffices to show that the map $H^0(S, O_S^{\oplus 2}) \to H^0(S, O(D))$ is injective. If it is not injective, then we may assume that one map $H^0(S, O_S) \to H^0(S, O(D))$ is zero, and hence (since $O_S$ and $O(-D)$ are reflexive), the original inclusion must factor $O(-D) \to O_S \to O_S^{\oplus 2}$. In particular, $D$ is effective and the quotient of the inclusion $O(-D) \to O_S^{\oplus 2}$ is isomorphic to $O_S \oplus O_D$. 

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Because $C \cdot D < C^2/2$ and $C$ is integral and ample, we have that $D \cap C$ must be zero-dimensional. Hence $\text{Hom}(\mathcal{O}_D, i_*\mathcal{O}_C(\Gamma)) = 0$. Furthermore, $\mathcal{O}_S$ admits no surjective maps onto $i_*\mathcal{O}_C(\Gamma)$. Therefore $\mathcal{O}_D \oplus \mathcal{O}_S$ does not surject onto $i_*\mathcal{O}_C(\Gamma)$. This is a contradiction, as the inclusion $\mathcal{O}(-D) \hookrightarrow \mathcal{O}_S^{\oplus 2}$ factors through $F \hookrightarrow \mathcal{O}_S^{\oplus 2}$, and $\mathcal{O}_S^{\oplus 2}/F = i_*\mathcal{O}_C(\Gamma)$. Therefore, we must have $h^0(S, \mathcal{O}(D)) \geq 2$.

- **Property (4):** A divisor $E$ satisfying equation (3.2.1) also satisfies $h^0(\mathcal{O}_S(E - D)) = 0$:

  Let $E$ be a divisor on $S$ such that

  $$h^0(\mathcal{O}_C(E|_C - \Gamma)) = 0 \quad \text{and} \quad E \cdot C < C^2.$$  

  By the projection formula we have

  $$\mathcal{O}_S(E) \otimes i_*\mathcal{O}_C(\pm \Gamma) \simeq i_*\mathcal{O}_C(E|_C \pm \Gamma).$$

  We therefore have the diagram with exact rows

  $$
  \begin{array}{cccccc}
  0 & \longrightarrow & F \otimes \mathcal{O}_S(E) & \longrightarrow & \mathcal{O}_S(E)^{\oplus 2} & \longrightarrow & i_*\mathcal{O}_C(\Gamma) \otimes \mathcal{O}_S(E) & \longrightarrow & 0 \\
  & & \downarrow \text{res} & & \| & & \downarrow & & \\
  0 & \longrightarrow & i_*\mathcal{O}_C(E|_C - \Gamma) & \longrightarrow & i_*\mathcal{O}_C(E|_C)^{\oplus 2} & \longrightarrow & i_*\mathcal{O}_C(E|_C + \Gamma) & \longrightarrow & 0
  \end{array}
  $$

  By assumption we have $E \cdot C < C^2$; therefore as $C$ is ample, $h^0(\mathcal{O}_S(E - C)) = 0$, and so the vertical map res is injective on global sections. Combined with the assumption that $h^0(C, E|_C - \Gamma) = 0$, we have that $h^0(S, F \otimes \mathcal{O}_S(E)) = 0$. Tensoring (3.2.4) with $\mathcal{O}_S(E)$ and taking global sections, this implies $h^0(S, E - D) = 0$ as desired.

  Since $C \cdot D < C^2/2 < C^2$, if $h^0(C, D|_C - \Gamma) = 0$, then we could take $E = D$ and obtain the contradiction $h^0(S, \mathcal{O}_S) = 0$. Hence we must have that $\mathcal{O}_C(D|_C - \Gamma)$ is effective. \qed
Remark 3.2.5. The use of Bogomolov’s inequality as a tool for proving the existence of divisors satisfying nice positivity properties originates in Reider’s proof [12] of Reider’s theorem, and has been developed by Lazarsfeld [36] and others. In particular, see [31, 41] for recent applications in the Picard rank 1 case.

3.3. Examples: curves on $E \times \mathbb{P}^1$. As a first example, let us now see how these techniques apply when $C$ is a smooth curve on $S = E \times \mathbb{P}^1$. We denote the projection maps

\[
\begin{array}{ccc}
E \times \mathbb{P}^1 & \overset{\pi_1}{\longrightarrow} & E \\
\downarrow & & \downarrow \\
E & \overset{\pi_2}{\longrightarrow} & \mathbb{P}^1
\end{array}
\]

to $E$ and $\mathbb{P}^1$, respectively. Then

\[
\text{Pic } S = \pi_1^* \text{Pic } E \oplus \pi_2^* \text{Pic } \mathbb{P}^1.
\]

As is standard, if $\mathcal{L}_1$ is a line bundle on $E$ and $\mathcal{L}_2$ is a line bundle on $\mathbb{P}^1$, we write $\mathcal{L}_1 \boxtimes \mathcal{L}_2$ for $\pi_1^* \mathcal{L}_1 \otimes \pi_2^* \mathcal{L}_2$. Furthermore, the Néron-Severi group is $\text{NS}(S) = \mathbb{Z} \oplus \mathbb{Z}$, spanned by the classes $F_1$ and $F_2$ of fibers of the first and second projections, respectively. These satisfy the intersection relations

\[
F_1^2 = 0, \quad F_2^2 = 0, \quad F_1 \cdot F_2 = 1.
\]

We will denote the numerical class $xF_1 + yF_2$ of a divisor by $(x, y)$. The effective cone of $S$ is then the set of all classes with $x, y \geq 0$, and the ample cone is the set of all classes with $x, y > 0$.

The following geometric result is the main ingredient in the proof of Theorem 3.0.1.

**Theorem 3.3.1.** Let $C$ be a smooth curve on $S = E \times \mathbb{P}^1$ in numerical class $(\gamma, \alpha)$ for

\[2 \leq \gamma/2 \leq \alpha \leq \gamma.\]

Then $C$ satisfies the following properties.
(a) \( \text{gon}(C) = \gamma \).
(b) \( W_{\alpha}C \) contains an elliptic curve isogenous to \( E \).
(c) If \( e < \alpha \), then \( W_{e}C \) does not contain any positive-dimensional abelian varieties.

**Remark 3.3.2.** Note that Bertini’s theorem guarantees that there exist smooth curves in numerical class \((\gamma, \alpha)\) once \( \gamma \geq 2 \) and \( \alpha \geq 1 \), as the linear equivalence class is necessarily basepoint free.

**Proof.** We have \( \mathcal{O}_{S}(C) \cong \mathcal{O}_{E}(\gamma e) \boxtimes \mathcal{O}_{\mathbb{P}^{1}}(\alpha) \) for some point \( e \in E \); then \( C^{2} = 2\alpha\gamma \).

The two projection maps exhibit \( C \) as a \( \gamma \)-sheeted cover of \( \mathbb{P}^{1} \), and an \( \alpha \)-sheeted cover of \( E \). Therefore \( \text{gon}(C) \leq \gamma \). Furthermore, we have a nonconstant map \( E \to W_{\alpha}(C) \) sending \( x \in E \) to \( \mathcal{O}(\pi_{1}^{-1}(x)) \), proving part (b).

(a) Suppose to the contrary that \( \Gamma \) is a divisor on \( C \) of degree at most \( \gamma - 1 \) that moves in a basepoint free pencil. Then

\[
\text{deg} \Gamma \leq \gamma - 1 < \alpha\gamma/2 = C^{2}/4,
\]

as \( \alpha \geq 2 \). So by Proposition 3.2.4, there exists an effective divisor \( D \) on \( S \) with at least 2 sections, satisfying

\[
\text{(3.2.4)(2)}: C \cdot D < C^{2}/2; \\
\text{(3.2.4)(3)}: D \cdot (C - D) \leq \text{deg} \Gamma.
\]

The divisor \( D \) is in numerical class \( xF_{1} + yF_{2} \) for some \( x \geq 0 \) and \( y \geq 0 \), and so these numerical conditions translate into

\[
\alpha x + \gamma y < \alpha\gamma \\
\alpha x + \gamma y - 2xy \leq \Gamma < \gamma.
\]

Upon rearrangement we have:

\[
\text{(2’)}: \alpha(\gamma/2 - x) + \gamma(\alpha/2 - y) > 0, \\
\text{(3’)}: (\gamma/2 - x)(\alpha/2 - y) > (\gamma/2)(\alpha/2 - 1) \geq 0,
\]
as \( \alpha \geq 2 \). Therefore both \( \gamma/2-x \) and \( \alpha/2-y \) have to be positive. Furthermore, we have

\[
(\gamma/2)(\alpha/2 - y) \geq (\gamma/2 - x)(\alpha/2 - y) > (\gamma/2)(\alpha/2 - 1),
\]

so \( y = 0 \). Plugging \( y = 0 \) back into inequality \( (3') \), we see

\[
(\alpha/2)(\gamma/2 - x) > (\gamma/2)(\alpha/2 - 1)
\]

and so \( x < \gamma/\alpha \leq 2 \). So \( x \) is 0 or 1. But every divisor of numerical class 0 or \( F_1 \) has at most 1 section, which is a contradiction.

(c) Suppose to the contrary that there exists a positive dimensional abelian variety \( A \hookrightarrow W_eC \) for \( e \leq \alpha - 1 \); and further that \( e \) is minimal for this property. Then by Lemma 3.1.1 the points \( p \in A_2 \) parameterize basepoint free linear systems \( \Gamma_p \). On the other hand,

\[
\deg \Gamma_p = 2e \leq 2\alpha - 2 < \alpha \gamma/2 = C^2/4,
\]

since \( \gamma \geq 4 \). Proposition 3.2.4 produces a divisor \( D_p \) on \( E \times \mathbb{P}^1 \), say in numerical class \( xF_1 + yF_2 \), satisfying the following properties:

- **3.2.4(1):** \( h^0(D_p) \geq 2 \);
- **3.2.4(2):** \( \alpha x + \gamma y < \alpha \gamma \);
- **3.2.4(3):** \( \alpha x + \gamma y - 2xy \leq \deg \Gamma_p \leq 2\alpha - 2 \leq 2\gamma - 2 \).
- **3.2.4(4):** \( \mathcal{O}_C(D_p|_C - \Gamma_p) \) is effective.

We may write the two inequalities as

\[
\alpha(\gamma/2 - x) + \gamma(\alpha/2 - y) > 0, \quad (\gamma/2 - x)(\alpha/2 - y) > \alpha(\gamma/4 - 1) \geq 0,
\]
with the rightmost inequality coming from our assumption that \( \gamma \geq 4 \). Therefore both \( \gamma/2 - x \) and \( \alpha/2 - y \) must be positive. We have

\[
(\gamma/2 - x)(\alpha/2) \geq (\gamma/2 - x)(\alpha/2 - y) > \alpha(\gamma/4 - 1),
\]

and so \( x < 2 \). Similarly for \( y \) we obtain \( y < 2\alpha/\gamma \leq 2 \). Combining this with the requirement that \( D_p \) move in a pencil on \( E \times \mathbb{P}^1 \), we see that it must be in numerical class \( F_2 \) or \( F_1 + F_2 \).

Let \( D \) be a divisor on \( S \) in numerical class \( F_1 + F_2 \) that contains \( D_p \). Let \( q \in E \) be such that \( \mathcal{O}_S(D) \simeq \mathcal{O}_E(q) \otimes \mathcal{O}_{\mathbb{P}^1}(1) \). Then

\[
D|_C - \Gamma_p \geq D_p|_C - \Gamma_p \geq 0,
\]

by 3.2.4(4). By the K"unneth formula

\[
H^0(S, D) = H^0(E \times \mathbb{P}^1, \mathcal{O}_E(q) \otimes \mathcal{O}_{\mathbb{P}^1}(1)) \simeq H^0(E, \mathcal{O}_E(q)) \otimes H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))
\]

and so every divisor in \( |D| \) is reducible, the union of the fiber \( \pi_1^{-1}(q) \) and some fiber of \( \pi_2 \). As

\[
\mathcal{O}_S(D - C) \simeq \mathcal{O}_E(q - \gamma e) \otimes \mathcal{O}_{\mathbb{P}^1}(1 - \alpha),
\]

and both \( H^0(E, \mathcal{O}_E(q - \gamma e)) \) and \( H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1 - \alpha)) \) are 0, the K"unneth formula implies that \( h^0(S, D - C) = h^1(S, D - C) = 0 \). Therefore the map \( H^0(E \times \mathbb{P}^1, D) \to H^0(C, D|_C) \) is an isomorphism. We therefore have \( h^0(C, D|_C) = 2 \) and every divisor on \( C \) linearly equivalent to \( D|_C \) is the union of \( \pi_1^{-1}(q) \cap C \) and \( \pi_2^{-1}(z) \cap C \) for some \( z \in \mathbb{P}^1 \). The linear system \( |D|_C \) has base locus exactly \( \pi_1^{-1}(q) \cap C \).

By assumption \( |\Gamma_p| \) is a basepoint free sub-linear series of \( |D|_C \). As such, a general element of \( |\Gamma_p| \) cannot pass through the basepoints of \( |D|_C \), and so must be supported in a fiber of the second projection \( \pi_2 : C \to \mathbb{P}^1 \). Therefore
\[ \pi_2^*(\mathcal{O}_{\mathbb{P}^1}(1))|_C - \Gamma_p \text{ is effective. Since } \Gamma_p \text{ is basepoint free,} \]
\[ \deg(\Gamma_p) \geq \text{gon}(C) = \gamma, \]
by part (a). We also have \( \deg(\pi_2^*(\mathcal{O}_{\mathbb{P}^1}(1))|_C) = \gamma \), which forces \( \Gamma_p = \pi_2^*(\mathcal{O}_{\mathbb{P}^1}(1))|_C \)
for all \( p \). Since \( \Gamma_p \) is independent of \( p \), the dimension of \( A_2 \) is 0. This contradicts the fact that \( A \) has positive dimension. \( \square \)

3.4. Nonexistence of abelian subvarieties. We now show how Lemma 3.1.1 in combination with Proposition 3.2.4 can prove the nonexistence of positive-dimensional abelian subvarieties in \( W_eC \) when \( C \) lies on an arbitrary smooth surface with \( h^1(S, \mathcal{O}_S) = 0 \) and \( e \) is small.

**Theorem 3.4.1.** Let \( S/\mathbb{C} \) be a nice surface with \( h^1(S, \mathcal{O}_S) = 0 \), and let \( C \) be a smooth ample curve on \( S \). Then for \( e < C^2/9 \), the locus \( W_eC \) contains no positive-dimensional abelian varieties.

**Proof.** Suppose to the contrary that for some \( e < C^2/9 \), there exists a positive-dimensional abelian variety \( A \) contained in \( W_eC \). Choose \( e \) minimal. By Lemma 3.1.1 if \( p \) is in \( A_2 \subseteq W_{2e}C \), then the corresponding effective line bundle \( \mathcal{O}(\Gamma_p) \) moves in a base point free pencil.

By our hypothesis on \( e \), we have that \( 2e < C^2/4 \). Applying Proposition 3.2.4 to the divisor \( \Gamma_p \) on \( C \), there exists a divisor \( D_p \) on \( S \) satisfying:

1. **3.2.4(2):** \( C \cdot D_p < C^2/2 \);
2. **3.2.4(3):** \( D_p \cdot (C - D_p) \leq \deg \Gamma_p = 2e; \)
3. **3.2.4(4):** \( H^0(D_p|_{C - \Gamma_p}) \neq 0. \)

For each \( p \), let \( (A_2)_p \) be the locus of \( q \in A_2 \) such that \( \mathcal{O}_C(D_p|_{C - \Gamma_q}) \) is effective. Then
\[
\bigcup_{p \in A_2} (A_2)_p = A_2,
\]
by Proposition \ref{3.2.4}. Further, by the upper semicontinuity of $\dim H^0$, the locus $(A_2)_p$ is closed for any particular $p$, and since $h^1(S, \mathcal{O}_S) = 0$, we have that $\text{Pic}(S)$ is discrete and countable. As a result, there must be some single $p$ such that $(A_2)_p = A_2$.

Let $D = D_p$, so $\mathcal{O}_C(D|C - \Gamma_q)$ is effective for all $q \in A_2$.

The map $A_2 = (A_2)_p \to W_{C: D - 2e}C$ sending a point $p \in A_2$ to the effective divisor class $D|C - \Gamma_p \in W_{C: D - 2e}C$ is an embedding. Therefore, $W_{C: D - 2e}C$ contains an abelian subvariety, and so by minimality of $e$ we conclude that $C \cdot D - 2e \geq e$, and hence

\begin{equation}
C \cdot D \geq 3e.
\end{equation}

Set $m_0 = D^2/(C \cdot D)$. As the curve $C$ is ample, the Hodge index theorem implies

\begin{equation}
C^2D^2 \leq (C \cdot D)^2,
\end{equation}

and so $m_0C^2 \leq C \cdot D$. Combining this with inequalities \ref{3.2.4}(2) and \ref{3.2.4}(3), respectively, we get

\begin{equation}
m_0 \leq \frac{C \cdot D}{C^2} < \frac{1}{2},
\end{equation}

\begin{equation}
m_0C^2(1 - m_0) \leq C \cdot D(1 - m_0) = D \cdot (C - D) \leq 2e.
\end{equation}

Furthermore, combining inequality \ref{3.4.1} with \ref{3.2.4}(3) we have

\begin{equation*}
3e(1 - m_0) \leq C \cdot D(1 - m_0) \leq 2e,
\end{equation*}

and so together with \ref{3.4.3}, we have $1/3 \leq m_0 < 1/2$.

The function $m_0(1 - m_0)$ is monotonically increasing in the range $[1/3, 1/2]$, and so \ref{3.4.4} gives

\begin{equation*}
\frac{2C^2}{9} \leq m_0(1 - m_0)C^2 \leq 2e,
\end{equation*}

so we conclude $C^2 \leq 9e$, which contradicts our hypothesis. \hfill \Box
For a very ample divisor $P$ on $S$, recall that we define the *exceptional subset* with respect to $P$ to be

$$\text{Exc}_P := \{\text{integral classes } H \in \text{Amp}(S) \text{ such that } H^2 \leq 9(H \cdot P - 1)\}.$$ 

**Corollary 3.4.2.** Let $S/k$ be a smooth projective surface with $h^1(S, \mathcal{O}_S) = 0$. For any very ample divisor $P$, if $C \subset S$ is a smooth curve with class $[C] \in \text{Amp}(S) \setminus \text{Exc}_P$, then for all $e < \text{gon}_k(C)$, the locus $W_e C$ does not contain any positive-dimensional abelian varieties.

**Proof.** Suppose that $e < \text{gon}_k(C)$. Then $\text{gon}_k(C) \leq P \cdot C$, as exhibited by projection from a codimension 2 plane in $\mathbb{P}H^0(C, P|_C)^\vee$. Therefore $e \leq P \cdot C - 1$. As $[C] \not\in \text{Exc}_P$, we have that $P \cdot C - 1 < C^2/9$. Combining these inequalities gives $e < C^2/9$. Therefore, by Theorem 3.4.1, $(W_e C)_C$ (and hence $W_e C$) does not contain positive-dimensional abelian varieties. \[\square\]

To imply the result stated in the introduction, we need the following elementary results about the intersection

$$\text{Exc}_P(N) := \text{Exc}_P \cap N,$$

for $N$ a closed subcone of the ample cone.

**Lemma 3.4.3.** For any closed subcone $N$ and any very ample divisor $P$ on $S$, the set $\text{Exc}_P(N)$ of exceptional classes in $N$ with respect to $P$ is finite.

We will deduce this from the following elementary result.

**Lemma 3.4.4.** Suppose that $N \subset \mathbb{R}^n$ is a closed cone and let $f: N \to \mathbb{R}$ be a continuous function taking positive values away from 0. Let $\Lambda$ be any lattice in $\mathbb{R}^n$.  

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If for all $H \in N$ and all $\lambda \geq 0$, we have
\[
f(\lambda H) = \lambda f(H),
\]
then for any $c \in \mathbb{R}$, the set
\[
\{H \in N : f(H) \leq c\} \cap \Lambda
\]
is finite.

Proof. Let $\mathbb{S}$ be the unit sphere in $\mathbb{R}^n$. Set $c_{\text{min}} = \inf\{f(H) | H \in \mathbb{S} \cap N\}$. Since $\mathbb{S}$ is compact and $N$ is closed, the intersection $\mathbb{S} \cap N$ is compact. So this minimum is achieved by $f$ on $\mathbb{S} \cap N$, and in particular $c_{\text{min}} > 0$. By the hypothesis $f(\lambda H) = \lambda f(H)$, we then have that $f(H) > rc_{\text{min}}$ for all $H \in N \setminus B_r$, where $B_r$ is the closed ball of radius $r$. Then for any $c > 0$, the set
\[
\{H \in N | f(H) \leq c\}
\]
is a closed set contained in the compact set $B_{c/c_{\text{min}}}$, and is hence compact. So its intersection with the discrete set $\Lambda$ is finite. \qed

Proof of Lemma 3.4.3. If $H^2 \leq 9(P \cdot H - 1)$, then $H^2 < 9H \cdot P$; hence it suffices to show that there are finitely many such integral classes $H$ in $N$. Let $f : N \setminus \{0\} \to \mathbb{R}$ be the continuous function
\[
H \mapsto f(H) = \frac{H^2}{9P \cdot H}.
\]
As $H$ and $P$ are both ample, the function $f$ is positive and clearly satisfies $f(\lambda H) = \lambda f(H)$. Therefore by Lemma 3.4.4 there are only finitely many integral classes $H$ for which
\[
f(H) = \frac{H^2}{9P \cdot H} < 1. \quad \Box
\]
3.5. Example: curves on $\mathbb{P}^1 \times \mathbb{P}^1$. When the divisor structure on $S$ is sufficiently well understood, our techniques allow one to explicitly compute the exceptional set for the entire ample cone. We present one example here.

**Proposition 3.5.1.** Let $S = \mathbb{P}^1 \times \mathbb{P}^1$, and let $C$ be a smooth curve of any bidegree $(d_1, d_2)$ with $d_1 \leq d_2$ and $(d_1, d_2) \neq (3, 3)$ or $(2, 2)$. Then, for $e < d_1$, $W_e C$ contains no positive-dimensional abelian varieties.

**Remark 3.5.2.** The assumption that $(d_1, d_2) \neq (3, 3)$ or $(2, 2)$ in the proposition is necessary, as we now explain. The smooth $(3, 3)$ curves on $\mathbb{P}^1 \times \mathbb{P}^1$ (the complete intersection of a quadric and a cubic surface under the embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$ by $\mathcal{O}(1, 1)$) are canonical curves of genus 4, and there exist bielliptic genus 4 curves. Explicitly, if the cubic surface is the cone over a smooth plane cubic and the quadric is general, then projection from the cone point gives a degree 2 map from the curve to the cubic plane curve.

Likewise, if $(d_1, d_2) = (2, 2)$, then $C$ is elliptic and $W_1 C = C$.

**Proof.** The cases of curves with $d_1 \leq 1$ are trivial. If $d_1 = 2$, by assumption $d_2 \geq 3$, so the genus of the curve is $d_2 - 1 > 1$, and $W_1 C$ contains no positive-dimensional abelian varieties. So we may assume $d_1 \geq 3$ and $d_2 \geq 4$. Let $C$ be a smooth curve of bidegree $(d_1, d_2)$ with $d_1 \geq 3$ and $d_2 \geq 4$, and suppose the conclusion of the proposition fails for $C$. Let $e$ be minimal such that $W_e C$ contains a positive dimensional abelian variety $A$ so $e < d_1$. By Lemma [3.1.1], the points $p$ of $A_2$ give rise to basepoint free pencils $\Gamma_p$ of degree $2e$. Then

$$\deg \Gamma_p = 2e < 2d_1 \leq d_1d_2/2 = C^2/4.$$

So we apply Proposition [3.2.4] to guarantee the existence of an effective divisor $D$, say of class $(x, y)$ with $x, y \geq 0$, satisfying

$$3.2.4(2): d_1 y + d_2 x = C \cdot D < C^2/2 = d_1d_2;$$
3.2.4(3): \( d_1 y + d_2 x - 2xy = D \cdot (C - D) \leq 2e < 2d_1 \leq 2d_2 \).

In exactly the same way as in the proof of Theorem 3.3.1, this forces \( x, y \leq 1 \). Thus \((x, y)\) is \((0, 1)\), \((1, 0)\), or \((1, 1)\). As in the proof of Theorem 3.4.1, there is a single choice of divisor class \( D \) such that \( \mathcal{O}_C(D|_C - \Gamma_p) \) is effective for all \( p \). In the first two cases, sending \( \Gamma_p \) to the effective divisor of class \( D|_C - \Gamma_p \), whose degree is \( D \cdot C - 2e \leq D^2 = 2xy = 0 \) by 3.2.4(3), induces an isomorphism between \( A_2 \) and \( W_0(C) = \text{pt} \), which contradicts that \( A \) is positive-dimensional.

Now we consider the case \((x, y) = (1, 1)\). By 3.2.4(3) we have the inequality

\[ d_1 + d_2 - 2 \leq 2e. \]

Combining this with

\[ 2e \leq 2d_1 - 2 \leq d_1 + d_2 - 2 \]

shows that equality must hold everywhere. Therefore \( d_1 = d_2 \) and \( e = d_1 - 1 \).

Now \( D \cdot C - \deg \Gamma_p \leq D^2 = 2 \), so we have an inclusion \( A_2 \to W_2C \), so \( W_2C \) contains a positive-dimensional abelian variety. This is a contradiction since we have \( e = d_2 - 1 \geq 3 \) and assumed \( e \) was minimal such that \( W_eC \) contains an abelian variety. \( \square \)

3.6. Number-Theoretic Consequences. Lang’s general conjecture [35, §3, Statement 3.6] on rational points is known in its entirety for subvarieties of abelian varieties by the work of Faltings.

**Theorem 3.6.1** (Faltings [22]). Let \( k \) be a number field. Let \( X \subset A \) be a subvariety of an abelian variety \( A \) over \( k \). Then there exist finitely many translates of abelian subvarieties

\[ Z_i = z_i + B_i, \quad (B_i \subset A \text{ abelian subvariety}) \]

that contain all of the rational points of \( X \). In particular, if \( X(k) \) is infinite, then \( X \) contains a translate of a positive-dimensional abelian subvariety of \( A \).
Recall that $C^{(e)}$ is the $e$th symmetric power of $C$, and $W_e C$ is the image of $C^{(e)} \to \text{Pic}^e C$. We will apply Faltings’ theorem to $W_e C$.

**Lemma 3.6.2.** Let $e_0 \leq \text{gon}_k(C)$ be some positive integer. If for all $e < e_0$, the subvariety $W_e C_k \subseteq \text{Pic}^e C_k$ does not contain any positive-dimensional abelian varieties, then

$$\text{a.irr}_k(C) \geq e_0.$$ 

In particular, if this holds with $e_0 = \text{gon}_k(C)$, then $\text{a.irr}_k(C) = \text{gon}_k(C)$.

**Proof.** If $e < \text{gon}_k(C)$, then $C^{(e)}(k) \to W_e C(k)$ is injective, so $C^{(e)}(k)$ is finite if and only if $W_e C(k)$ is finite. As $W_e C$ is a subvariety of the torsor $\text{Pic}^e C$ of the abelian variety $\text{Pic}^0 C$, Faltings’ theorem implies that the set of points $W_e C(L)$ is finite for all finite extensions $L/k$ if and only if $W_e C_k$ does not contain any positive-dimensional abelian varieties.

**Remark 3.6.3.** Lemma 3.6.2 shows that $\text{a.irr}_k(C) \leq e$ if and only if $\text{gon}(C_k) \leq e$ or $W_f C_k$ contains a positive-dimensional abelian subvariety for some $f \leq e$. Thus $\text{a.irr}_k(C)$ depends only on $C_k$.

**Proof of Theorem 3.0.1.** Suppose that $\gamma, \alpha$ are such that $0 < \gamma/2 \leq \alpha \leq \gamma$. It suffices to find a nice curve $C$ over $\mathbb{Q}$ such that $\text{a.irr}_Q(C) = \text{a.irr}_Q(C) = \alpha$ and $\text{gon}_Q(C) = \text{gon}_Q(C) = \gamma$.

For $\gamma = 1$, then $\alpha = 1$ and we take $C = \mathbb{P}^1_Q$. If $\gamma = 2$ and $\alpha = 1$, we may take $C$ to be an elliptic curve over $\mathbb{Q}$ of positive rank. For $\gamma = 2$ and $\alpha = 2$, we may take $C$ to be any hyperelliptic curve of genus at least 2. For $\gamma = 3$ and $\alpha = 2$, we may take any non-hyperelliptic curve that is a double cover of a positive-rank elliptic curve (see Remark 3.5.2 for a construction in genus 4). For $\gamma = 3$ and $\alpha = 3$, we may take any non-hyperelliptic, non-bielliptic trigonal curve by the work of Harris-Silverman [25, Corollary 3] and Lemma 3.6.2 (e.g., a canonical curve of genus 4 that is non-bielliptic).
Therefore, we may assume that \( \gamma \geq 4 \) (and so \( \alpha \geq 2 \)). Let \( E \) be a positive rank elliptic curve over \( \mathbb{Q} \). By Theorem 3.3.1 a smooth curve \( C \) on \( E \times \mathbb{P}^1_{\mathbb{Q}} \) in numerical class \((\gamma, \alpha)\) has \( \text{gon}_E(C) = \gamma \) and \( \text{a.irr}_E(C) \geq \alpha \). As \( C \) has a map \( \pi_1 \) of degree \( \alpha \) to \( E \), we further have \( \pi_1^{-1}(E(\mathbb{Q})) \subseteq C_\alpha \), and so \( \text{a.irr}_E(C) \leq \alpha \); therefore equality holds. \( \square \)

**Proof of Theorem 3.0.2.** Suppose that \( C \hookrightarrow S/k \) is a smooth ample curve. Let 
\[
e_0 := \min \left( \text{gon}_k(C), \frac{C^2}{9} \right).
\]
Then by Theorem 3.4.1 \( W_e C \) contains no positive-dimensional abelian varieties for \( e < e_0 \leq \text{gon}_k(C) \). Therefore, by Lemma 3.6.2 we have that \( \text{a.irr}_k(C) \geq e_0 \). Now let \( P \) be any choice of very ample divisor on \( S \). If \( 9C \cdot P \leq C^2 \) (i.e. \( [C] \notin \text{Exc}_P \)), then by Corollary 3.4.2 \( W_e C \) contains no positive-dimensional abelian varieties for \( e < \text{gon}_k(C) \). Therefore \( \text{a.irr}_k(C) = \text{gon}_k(C) \) by Lemma 3.6.2.

On the other hand, for any closed subcone \( N \subseteq \text{Amp}(S) \), Lemma 3.4.3 guarantees that \( \text{Exc}_P \cap N \) is finite. \( \square \)

**Proof of Corollary 3.0.3.** If \( \text{Pic}(S_k) \cong \mathbb{Z} \cdot \mathcal{O}_S(1) \) for a very ample line bundle \( \mathcal{O}_S(1) \) and \( \mathcal{O}_S(C) \cong \mathcal{O}_S(\alpha) \), then \( [C] \notin \text{Exc}_{\mathcal{O}_S(1)} \) is equivalent to \( \alpha \geq 9 \). \( \square \)

**Proof of Corollary 3.0.3.** The gonality of any complete intersection curve \( C \subset \mathbb{P}^r_k \) of type \((d_1, d_2, \ldots, d_{r-1})\) is at most \( \deg C = d_1d_2 \cdots d_{r-1} \). It therefore suffices by Lemma 3.6.2 to show the nonexistence of abelian subvarieties in \( W_e(C_C) \) for \( e < d_1d_2 \cdots d_{r-1} \). By [31] Theorem 3.1, if \( 4 \leq d_1 < d_2 \leq \cdots \leq d_{n-1} \), then \( C_C \) lies on a smooth complete intersection surface \( S/C \) of type \((d_2, \cdots, d_{n-1})\) with \( \text{Pic} S \cong \mathbb{Z} \cdot [\mathcal{O}_S(1)] \) and \( [C] \) equals \( \mathcal{O}_S(d_1) \). Therefore, the result follows from Theorem 3.0.3 with \( P = \mathcal{O}_S(1) \), as \( P \cdot C = d_1d_2 \cdots d_{r-1} \). \( \square \)

**Proof of Theorem 3.0.6.** Let \( C \) be a nice curve of type \((d_1, d_2)\) on \( \mathbb{P}^1_k \times \mathbb{P}^1_k \) with \( 1 \leq d_1 \leq d_2 \). Then we claim that \( \text{gon}_k(C) = \text{gon}_{\mathbb{F}}(C) = d_1 \). The upper bound is provided
by the projection
\[ C \hookrightarrow \mathbb{P}^1_k \times \mathbb{P}^1_k \to \mathbb{P}^1_k \]
on onto the first factor. As the tensor product of a \( p \)-very ample and a \( q \)-very ample bundle is \((p + q)\)-very ample, the line bundle
\[ K_C = \mathcal{O}_C(d_1 - 2, d_2 - 2) = \mathcal{O}_C(d_1 - 2, d_1 - 2) \otimes \mathcal{O}_C(0, d_2 - d_1) \]
is \((d_1 - 2)\)-very ample (as \( \mathcal{O}_C(0, d_2 - d_1) \) is either trivial or base point free.) Therefore we have the lower bound \( \text{gon}_{K}(C) \geq d_1 \) (in fact even a weaker statement is true, see \[7, \text{Lemma 1.3}]\). By Proposition 3.5.1, \( W_k C \) contains no positive-dimensional abelian subvarieties for \( e < d_1 \) as long as \((d_1, d_2) \neq (2, 2) \) or \((3, 3) \). Therefore \( C_{d_1-1} \) is finite for all such \((d_1, d_2) \) and \( \text{a.irr}_k(C) = \text{gon}_k(C) = d_1 \). If \((d_1, d_2) = (2, 2) \), then \( C_K \) is an elliptic curve and so \( \text{a.irr}_K(C) = 1 \). If \((d_1, d_2) = (3, 3) \), then \( C \) is a canonical curve of genus 4, and in particular is not hyperelliptic. If \( C \) is bielliptic, then \( (C_K)_2 \) is infinite for any finite extension \( K \) of \( k \) over which the underlying genus 1 curve acquires infinitely many \( K \)-points, so \( \text{a.irr}_K(C) = 2 \). If \( C \) is not bielliptic, then the work of Harris-Silverman \[25, \text{Corollary 3}] \) implies that \( (C_K)_2 \) is finite for every finite extension \( K \) of \( k \), so
\[ \text{a.irr}_K(C) \geq 3 = \text{gon}_K(C) \geq \text{a.irr}_K(C), \]
and \( \text{a.irr}_K(C) = 3 \). \( \square \)

4. Vector bundles on trees of smooth rational curves

Let \( C \) be a connected nodal curve of arithmetic genus 0 defined over an algebraically closed field \( k \). \( C \) admits a simple geometric description, as its irreducible components are each isomorphic to \( \mathbb{P}^1 \), and its dual graph is a tree; in what follows, we will refer to it as a tree of smooth rational curves, following the convention of \[34, \text{II.7.4}] \). Likewise, vector bundles on \( C \) are relatively easy to describe explicitly. Given the
normalization $\tilde{C}$ of $C$, a vector bundle on $C$ is specified by a vector bundle on $\tilde{C}$ and the choice of gluing data on the nodes of $C$. That said, it is much less clear what the vector bundles on families of trees of smooth rational curves can look like.

In this chapter, we begin an investigation into the behavior of vector bundles on trees of smooth rational curves. For the most part, we work in the setting of a family of curves $\mathcal{C}$ over a base scheme $\Delta = \text{Spec}(R)$, where $R$ is a DVR with residue field $k$ and some fraction field $K$. In this setting, the main question we address is the following.

**Question 4.0.1.** Let $C$ and $C'$ be trees of smooth rational curves over some base field, and let $E$ and $E'$ be vector bundles on $C$ and $C'$ respectively. Under what circumstances does $E'$ specialize to $E$? That is, when is there a flat family $\pi : C \to \Delta$, where $\Delta := \text{Spec}(k[[t]])$, and a vector bundle $\mathcal{E}$ on $\mathcal{C}$ such that $\mathcal{C}_0 \cong C$ and $\mathcal{E}|_{\mathcal{C}_0} \cong E$, while $\mathcal{C}_\eta \cong C \times_k K$ and $\mathcal{E}|_{\mathcal{C}_\eta} \cong E$ for $\eta$ the generic point?

**Remark 4.0.2.** Coskun considered a similar question, and in [11, Section 4] gave conditions under which a surface scroll specializes to a given tree of scrolls. This closely corresponds with the rank two case of this question.

The value of this question largely lies in the theory of jumping curves. Given a vector bundle $E$ on a projective variety $X \subset \mathbb{P}^n$, in many cases a key step in understanding the behavior of $E$ involves understanding the restriction of $E$ to lines contained in $X$; in this context, a *jumping line* is a line $\ell$ where the splitting type of $E|_{\ell}$ is more special than at the general point. There is a more extensive discussion of jumping lines and their applications to vector bundles on projective space in [39].

The theory of jumping lines has been partially extended to higher degree curves on varieties, in, for instance, [12]. One obstruction present in the higher-degree case, however, is the fact that rational curves can degenerate to reducible curves; in this setting, an answer to Question 4.0.1 clarifies what sorts of jumping curve can exist.
It is clear that $E$ and $E'$ must have the same degree and rank for $E'$ to specialize to $E$. Beyond this, the best-known obstruction to $E'$ specializing to $E$ is a failure of the upper semicontinuity condition for cohomology of coherent sheaves. Every line bundle of degree $d$ on $C$ deforms to $\mathcal{O}(d)$ on $\mathbb{P}^1$, so the upper semicontinuity condition [26, Theorem 12.8] states that if $E'$ specializes to $E$, then the inequalities

\begin{equation}
(4.0.1) \quad h^0(C, E \otimes L) \geq h^0(\mathbb{P}^1, E'(\deg L))
\end{equation}

and

\begin{equation}
(4.0.2) \quad h^1(C, E \otimes L) \geq h^1(\mathbb{P}^1, E'(\deg L))
\end{equation}

hold for all line bundles $L$ on $C$. These inequalities are in fact equivalent to each other, because the flatness of $\mathcal{E}$ over $\Delta$ implies $\chi(C, E \otimes L) = \chi(\mathbb{P}^1, E'(\deg L))$.

In the case $C \cong C' \cong \mathbb{P}^1$, it is well-established that these are the only obstructions. By the Birkhoff-Grothendieck theorem, vector bundles of degree $d$ and rank $r$ on $\mathbb{P}^1$ are in bijection with splitting types, which are collections of integers $(d_1, \ldots, d_r)$ with $d_1 \geq d_2 \geq \cdots \geq d_r$ and $\sum_{1 \leq i \leq r} d_i = d$. In this case, the semicontinuity condition 4.0.1 reduces to statement that if the splitting type $(d'_1, \ldots, d'_r)$ specializes to $(d_1, \ldots, d_r)$, then we have

$$\sum_{1 \leq i \leq k} d'_i \leq \sum_{1 \leq i \leq k} d_i$$

for all $k$. By Theorem 14.7 of [19], this semicontinuity condition is sufficient for a vector bundle with splitting type $(d'_1, \ldots, d'_r)$ to specialize to one with splitting type $(d_1, \ldots, d_r)$.

We will analogously show that these are the only obstructions for any $C'$ if we have $C' \cong \mathbb{P}^1$. 
Theorem 4.0.3. In the setup of Question 4.0.1, a vector bundle $E'$ on $C' \cong \mathbb{P}^1$ specializes to $E$ if and only if $E$ and $E'$ have the same degree and rank and the semicontinuity condition holds for all line bundles on $C$.

To prove Theorem 4.0.3, we ideally would construct the desired vector bundle on the family $C$ as an iterated extension of line bundles on $C$. However, this is not always possible, as the following example illustrates.

Example 4.0.4. Let $C$ be the union of two smooth projective curves intersecting at a node. Let $\mathcal{O}(a, b)$ denote the line bundle on $C$ that has degree $a$ on the first $\mathbb{P}^1$ and degree $b$ on the second. Then the vector bundle $E := \mathcal{O}(2, 0) \oplus \mathcal{O}(0, 2)$ is allowed by semicontinuity to be the limit of the vector bundle $E' := \mathcal{O}(1) \oplus \mathcal{O}(3)$, but $E$ has no line subbundle of degree 3.

To overcome this obstacle, in the proof of Theorem 4.0.3 we will replace $C$ with a larger tree of smooth rational curves $C''$, such that the desired vector bundle $\mathcal{E}$ on a family $C'$ with special fiber $C''$ can be constructed as a deformation of an iterated extension of line bundles. We then produce the desired specializations by blowing down $C'$.

Remark 4.0.5. Deopurkar [17] notes that in the example above, the fact that $E$ has no line subbundle of degree 3 means that $E'$ cannot specialize to $E$ if we impose the additional condition that the total space $C$ is regular. Indeed, the construction we use to prove Theorem 4.0.3 in this case produces a surface $C$ with an ordinary double point at the node of $C$.

This chapter is organized as follows. In Section 4.1 we construct the moduli stacks of vector bundles on prestable curves of genus $g$ and establish their basic properties, in the process establishing that specializations of vector bundles compose (Proposition 4.1.5), and establish Theorem 4.0.3 for line bundles. Then, in section 4.2 we prove Theorem 4.0.3.
4.1. Preliminaries.

4.1.1. The moduli stack of vector bundles on trees of smooth rational curves. Throughout, stacks will be stacks on $\text{Aff}_k$ with the fppf topology, where $k$ is an algebraically closed field, following the convention of [13], on which we rely.

Recall that a prestable curve is a nodal connected curve. Let $\text{Curves}_{ps,g}$ denote the moduli stack of prestable curves of arithmetic genus $g$, in the sense of [18, Definition 0E6T]. This stack is algebraic and smooth and is an open substack of the moduli stack of all genus $g$ curves ([18, 0E6U, 0E6W]). In addition, we will require the fact that $\text{Curves}_{ps,g}$ is quasi-separated and locally of finite presentation over $k$, which follows from [18, Lemma 0DSS].

Let $X_{g,r}$ be the stack over $k$ of pairs $C, E$, where $C$ is a connected nodal curve of arithmetic genus $g$ and $E$ is a vector bundle on $C$ of rank $r$. More precisely, $X_{g,r}$ is the fibered category over $\text{Aff}_k$ with objects over the scheme $S/k$ given by $(\pi : C \to S, E)$ with $\pi$ a flat morphism from an algebraic space $C$ which is proper, locally of finite presentation, and nodal of relative dimension 1 with genus $g$ fibers such that $\pi_*(\mathcal{O}_C) \cong \mathcal{O}_S$, and $E$ is a vector bundle on $C$, and with morphisms coming from the obvious maps of pairs. This stack admits a map

$$F : X_{g,r} \to \text{Curves}_{ps,g}$$

forgetting the data of the vector bundle. We have the following.

**Proposition 4.1.1** ([13], Proposition 3.6). $X_{g,r}$ is an algebraic stack with locally finitely presented, separated diagonal.

**Proof.** We observe that the moduli stack of vector bundles of rank $r$ on curves over $k$ is just the stack $\text{CurveMaps}(BGL_r)$ of [13, Definition 3.5]. There is a natural map $F : \text{CurveMaps}(BGL_r) \to \text{Curves}$, which by [13, Proposition 3.6] is representable by algebraic stacks. Then, since $\text{Curves}_{ps,g}$ is an open algebraic substack of $\text{Curves}$, the
fiber product \( \text{CurveMaps}(BGL_r) \times_{\text{Curves}} \text{Curves}_{ps,g} \), which is \( X_{g,r} \) is an algebraic stack. Then, since \( \text{CurveMaps}(BGL_r) \) has locally finitely presented, separated diagonal, we have its open substack \( X_{g,r} \) has the same properties.

We have a decomposition \( X_{g,r} = \bigcup_{d \in \mathbb{Z}} X_{g,r,d} \), where \( X_{g,r,d} \) is the moduli stack of pairs \((C, E)\) with \( E \) of rank \( r \) and degree \( d \). Each \( X_{g,r,d} \) is of course an algebraic stack with locally finitely presented, separated diagonal, but more is true.

**Proposition 4.1.2.** The forgetful map \( F : X_{g,r,d} \rightarrow \text{Curves}_{ps,g} \) is quasi-separated and \( X_{g,r,d} \) is quasi-separated.

**Proof.** We must show that the diagonal map \( \Delta : X_{g,r,d} \rightarrow X_{g,r,d} \times_{\text{Curves}_{ps,g}} X_{g,r,d} \) is quasi-compact. The quasi-compactness of \( \Delta \) is equivalent to the statement that for any family of prestable genus \( g \) curves \( \pi : C \rightarrow U \) over an affine base \( U \) and any pair of rank \( r \) degree \( d \) vector bundles \( E_1, E_2 \) on \( C \) the scheme \( \text{Isom}_U(E_1, E_2) \) is quasi-compact. But \( \pi \) is finitely presented and \( E_1, E_2 \) are flat over \( U \), so by [18, Proposition 08K9], \( \text{Isom}_U(E_1, E_2) \) is affine and hence quasi-compact.

We have that \( \text{Curves}_{ps,g} \) is quasi-separated, so \( X_{g,r,d} \) is quasi-separated because \( F \) is.

This result allows us to give a modified form of Question 4.0.1.

**Question 4.1.3.** Let \( p \) be a \( k \)-point of \( X_{g,r,d} \) whose associated pair \((C, E)\) has automorphism group \( G \). Let \( Z_p \) be the substack \( p/G \) of \( X_{g,r,d} \) corresponding to this point. Which points \( Z_{p'} \) are in the closure of \( Z_p \)?

Question 4.1.3 has one advantage over Question 4.0.1 in that it is clear that if \( Z_{p''} \) is in the closure of \( Z_{p'} \) and \( Z_{p'} \) is in the closure of \( Z_p \), then \( Z_{p''} \) is in the closure of \( Z_p \). But its connection to Question 4.0.1 is not obvious, and is provided by the following result.
Lemma 4.1.4. Let \((C, E)\) be a pair associated to a \(k\)-point \(p\) of \(X_{g,r,d}\). If the point \(p'\) corresponding to the pair \((C', E')\) is in the closure of \(Z_p\), then there is a flat family of curves \(C \to B\) over an irreducible affine base curve \(B \ni 0\) and vector bundle \(E\) on \(C\) with fiber \((C', E')\) over 0 and fiber \((C, E)\) over any nonzero \(k\)-point of \(B\).

Proof. Let \(p\) be a \(k\)-point of \(X_{g,r,d}\) and given a smooth atlas \(V \to X_{g,r,d}\), let \(U \subset V\) be a finitely presented affine open subscheme containing some preimage \(\tilde{p}\) of \(p\). Suppose \(p\) is in the closure of some other \(k\)-point \(q\). We construct the fiber product.

\[
\begin{array}{ccc}
F & \longrightarrow & U \\
\downarrow & & \downarrow \\
q & \longrightarrow & X_{g,r,d}
\end{array}
\]

Without further assumptions, \(F\) may be an algebraic space, so let \(\tilde{F} \to F\) be an étale cover by a scheme, and let \(f : \tilde{F} \to U\) be the resulting composition. Since \(p\) is in the closure of \(q\), a lift \(\tilde{p}\) of \(p\) in \(U\) will be in the closure of the image of \(f\). Moreover, \(f\) is finitely presented since \(X_{g,r,d}\) has finitely presented diagonal. So \(f\) has constructible image in \(U\). \(\tilde{p}\) is in the closure of this image, and in particular there is some integral locally closed subscheme \(Z\) of \(U\) contained in the image of \(f\) such that we have \(\tilde{p} \in \overline{Z}\).

Then the map \(\overline{Z} \to X_{g,r,d}\) corresponds to a family of vector bundles on a family of curves \(C_{\overline{Z}} \to \overline{Z}\) with fiber over a \(k\)-point of \(\overline{Z}\) the pair \(q\) and fiber over \(\tilde{p}\) the pair \(p\). We have that \(\overline{Z}\) is an integral affine scheme of finite type over a field, that is, an affine variety. Then there is some affine curve \(B\) in \(\overline{Z}\) whose set-theoretic intersection with \(\overline{Z} \setminus Z\) is exactly \(\tilde{p}\). The family \(C \to B\) associated to \(B\) satisfies all the properties in the proposition. \(\square\)

Lemma 4.1.4 is a key step in producing compositions of specializations. We will apply it to prove the following result.

Proposition 4.1.5. If \((\mathbb{P}^1, E_1)\) specializes to \((\mathbb{P}^1, E_2)\) and \((\mathbb{P}^1, E_2)\) specializes to \((C, E_3)\) in the sense of Question 4.0.1, then \((\mathbb{P}^1, E_1)\) specializes to \((C, E_3)\).
Proof. From the specialization hypotheses, we have that \((C, E_3)\) is in the closure of \((\mathbb{P}^1, E_1)\) in \(X_{0,r,d}\). Then there exists an affine curve \(B\) with marked point 0, a family of curves \(\pi : \mathcal{C} \to B\), and a vector bundle \(\mathcal{E}\) on \(\mathcal{C}\) such that \(\mathcal{C}_0 \cong C, \mathcal{C}_p \cong \mathbb{P}^1_k\) for \(k\)-points \(p \neq 0\) of \(B\), and \(\mathcal{E}|_{\mathcal{C}_0} \cong E_3\) and \(\mathcal{E}|_{\mathcal{C}_p} \cong E_1\) for all \(p \neq 0\). By replacing \(B\) with a normalization, we may moreover assume \(B\) is regular, so the formal neighborhood of 0 in \(B\) is isomorphic to \(\text{Spec}(k[[t]])\).

The map \(\pi : \mathcal{C} \setminus \mathcal{C}_0 \to B \setminus 0\) is an étale-locally trivial \(\mathbb{P}^1\) bundle, which is Zariski-locally trivial by Tsen’s theorem. And the restriction of \(\mathcal{E}\) to each fiber over a nonzero closed point \(p\) of \(b\) is isomorphic to \(E_1\). Semicontinuity then gives that the restriction of \(\mathcal{E}\) to the generic fiber of \(\mathcal{C} \to B\) is \(E_1 \times_k K_B\). Restricting to a formal neighborhood of 0 in \(B\) gives the desired specialization. \(\square\)

4.1.2. Specialization of line bundles. The only specializations we will construct directly in this chapter are specializations of line bundles. These specializations will be constructed using the following.

**Proposition 4.1.6.** Let \(\pi : \mathcal{C} \to \Delta\) be a flat family of trees of rational curves with \(\Delta\) the spectrum of a DVR \(R\) over \(k\) with generic point \(\eta\) and closed point \(p\). Suppose \(\mathcal{C}_\eta \cong \mathbb{P}^1_{K_R}\). If \(\mathcal{C}\) is smooth, then a line bundle \(L\) on \(\mathcal{C}_p\) can be expressed as the degeneration of a line bundle \(\mathcal{O}(d)\) on \(\mathcal{C}_\eta\) if and only if \(\text{deg}(L) = d\).

**Proof.** A line bundle \(\mathcal{L}\) on \(\mathcal{C}\) must have the same degree over each of the fibers of \(\pi\), so the requirement \(\text{deg}(L) = d\) is necessary. Suppose \(\text{deg}(L) = d\). The closure in \(\mathcal{C}\) of a \(K_R\) point \(p \in \mathcal{C}_\eta\) is a divisor of degree 1 on the fibers of \(\pi\). Letting \(L_1\) be the line bundle associated to this divisor, by replacing \(L\) with \(L \otimes (L_1|_{\mathcal{C}_p})^{\otimes -d}\) and twisting \(\mathcal{O}(d)\) down to \(\mathcal{O}\), we may assume that we want to specialize \(\mathcal{O}_{\mathcal{C}_\eta}\) to some degree 0 line bundle \(L\). \(L\) is determined by the degree of its restrictions to each \(\mathbb{P}^1\) contained in \(\mathcal{C}_p\), and the sum of these degrees is 0. Moreover, since \(\mathcal{C}\) is smooth, the Weil divisor on \(\mathcal{C}\) given by the class of a smooth subcurve \(\mathcal{C}'\) of \(\mathcal{C}_p\) is Cartier, and the
associated line bundle \( \mathcal{O}(C') \) has degree 1 on each \( C'' \) intersecting \( C' \) and degree \(-e\) on \( C'' \), where \( e \) is the number of irreducible curves intersecting \( C' \) in \( C_p \). We observe then that given any two intersecting smooth subcurves \( C' \) and \( C'' \) on \( C_p \), there is a line bundle of degree 1 on \( C' \) and \(-1\) on \( C'' \), given by

\[
\mathcal{O}(C'') \bigoplus_{\text{side of } C' \cap C''} \mathcal{O}(C''').
\]

Restrictions of these line bundles clearly generate all line bundles of degree 0 on \( C_p \), because \( C_p \) is connected. The result follows.

\[\square\]

4.2. **Proof of Theorem 4.0.3.** In this section, we prove Theorem 4.0.3. Naively, we might hope that the specializations we desire on a family \( C \) may be produced by iterated extensions of line bundles constructed using Proposition 4.1.6. As demonstrated in Example 4.0.4, this is optimistic in general; though we hope that \( \mathcal{O}(1) \oplus \mathcal{O}(3) \) specializes to \( E \) in the example, \( E \) evidently has no line subbundles of degree 3, and so there is no hope of describing the desired vector bundle on a family as an extension of line bundles.

Instead, we will show that such an extension can be constructed if \( C \) is replaced by an **enlargement** \( C' \), defined as a surjective map of trees of smooth rational curves \( f : C' \to C \) such that the restriction of \( f \) to any irreducible component of \( C' \) is an isomorphism or a constant map. Given an enlargement \( C' \) of \( C \) and a vector bundle \( E \) on \( C \), we will also call the pair \( (C', f^*(E)) \) an enlargement of \( (C, E) \).

Allowing for enlargements will turn out to eliminate the obstruction present in Example 4.0.4. The following lemma presents a way to construct a maximal-degree line subbundle on an enlargement of \( (C, E) \).

**Lemma 4.2.1.** Let \( C \) be a tree of smooth rational curves and \( E \) a vector bundle on \( C \) such that \( H^0(C, E(L)) \neq 0 \) for all line bundles \( L \) of degree 0 on \( C \), but \( H^0(C, E(L)) = 0 \) for all line bundles of degree 0 on \( C \). Then there exists a line subbundle \( \mathcal{L} \) of \( E \) of degree 1 on each irreducible component of \( C \).
0 for some line bundle $L$ of degree $-1$. Then there is an enlargement $(C', E')$ of $(C, E)$ such that $E'$ has a line subbundle of degree at least 0.

Proof. In what follows, a coconnected subtree of $C$ is defined to be a union $C_1$ of irreducible components in $C$ such that $C_1$ and the complement of $C_1$ in $C$ is connected.

We proceed by induction on the number of irreducible components of $C$. If $C$ has one component, then $C \cong \mathbb{P}^1$ and the lemma is trivial. Otherwise, let $L_{-1}$ be a line bundle on $C$ such that $E \otimes L_{-1}$ has no global sections. Set $E' := E \otimes L_{-1}$. Let $S$ be the set of coconnected subtrees $C_1$ of $C$ such that $H^0(C_1, E'|_{C_1} \otimes L_0) \neq 0$ for all line bundles $L_0$ of degree 0 on $C_1$.

We first show that for any coconnected subtree $C_1$ of $C$, either $C_1$ or its complement $C_2 := C \setminus C_1$ is in $S$. We proceed by contradiction. Suppose that both $C_1$ and $C_2$ are not in $S$. Let $L_1$ and $L_2$ be degree 0 line bundles on each of $C_1$ and $C_2$ such that $H^0(C_1, E'|_{C_1} \otimes L_1) = H^0(C_2, E'|_{C_2} \otimes L_2) = 0$, let $L_0$ be the line bundle on $C$ that restricts to $L_1$ and $L_2$ on $C_1$ and $C_2$ respectively, and let $L$ be the line bundle that has degree 1 on the irreducible component of $C_1$ that contains $C_1 \cap C_2$, and otherwise has degree 0. Then we have an exact sequence

$$0 \to H^0(C_1, E'|_{C_1} \otimes L_1) \to H^0(C, E'(L_0 \otimes L)) \to H^0(C_2, E'|_{C_2} \otimes L_2)$$

and $H^0(C, E'(L_0 \otimes L)) = 0$, contradicting our hypothesis on $E$. So either $C_1$ or $C_2$ must be in $S$.

Now let $C_1$ be an irreducible component of $C$. Let $C_2$ be an irreducible component of $C$ intersecting $C_1$ such that the coconnected subtree of $C$ containing $C_1$ but not $C_2$ is in $S$, if such a curve exists. Repeat to produce $C_3, C_4$ and so on.

One of the following cases then occur.

(1) This process never terminates. In this case, we must have that for some $i$ this process “turns around” after producing $C_i$, in the sense that $C_{i-1} = C_{i+1}$. We
then have that both the coconnected subtree $C_1$ containing $C_{i-1}$ but not $C_i$ and its complement $C_2 := C \setminus C_1$ are in $S$.

(2) This process terminates at some $C_i$. Then we have that every connected component of $C \setminus C_i$ is in $S$.

In the first case, noting $C \notin S$ by the definition of $E'$, possibly after performing suitable enlargements on $C_1$ and $C_2$ we have that by the inductive hypothesis there are line subbundles $L_1$ and $L_2$ of $E'|_{C_1}$ and $E'|_{C_2}$ respectively, each of degree at least 0. Enlarge $C$ by inserting a $\mathbb{P}^1$ at the intersection $C_1 \cap C_2$; call the new pair $(\tilde{C}, \tilde{E}')$. Then there is a degree at least $-1$ line subbundle of $\tilde{E}'$ that restricts to $L_1$ and $L_2$ on $C_1$ and $C_2$ respectively, and restricts to $O(-1)$ on the curve between them. After twisting by the enlargement of $L^*_{-1}$, this produces a degree at least 0 line subbundle of $\tilde{E}$.

In the second case, let $L$ be the line bundle of degree 1 on $C_i$ and degree 0 otherwise. Then $E'(L)$ has a global section $s$ which can only vanish on a union of coconnected subtrees $C_1 \cup \ldots \cup C_k$ of $(C \setminus C_i)$. If $k = 0$, we are done, so suppose $k \geq 1$. Let $C'$ be the subtree of $C$ on which $s$ is generally nonvanishing—we have that $s$, by vanishing at $k$ points of $C'$, gives a line subbundle $L'$ of degree at least $k$ of $E'(L)|_{C'}$. Moreover, by the inductive hypothesis and the fact that every $C_j$ is in $S$, there are nonnegative degree line subbundles $L_j$ of $E'|_{C_j}$ for each $j$. Enlarge $C$ to a tree $\tilde{C}$ by inserting a $\mathbb{P}^1$ at each point $C_j \cap C'$. There is a line subbundle $\tilde{L}$ of $\tilde{E}$ on $\tilde{C}$ that restricts to $L_j$ on each $C_j$, restricts to $L'$ on $C'$, and restricts to $O(-1)$ on each new $\mathbb{P}^1$. Then $\tilde{L}$ is a line subbundle of $\tilde{E}$ of nonnegative degree.

Remark 4.2.2. In fact, under the conditions of the lemma above the line subbundle $L$ must have degree exactly 0. For if $L$ has positive degree, all degree $-1$ twists of it would have sections, so $E'$ would have sections.
Example 4.2.3. Let $C$ be a tree of two smooth rational curves intersecting in a node, and let $E := \mathcal{O}(2,0) \oplus \mathcal{O}(0,2)$ be as in Example 4.0.4. We have that every degree $-3$ twist of $E$ has global sections, but $E' := E \otimes \mathcal{O}(-2,-2) \cong \mathcal{O}(0,-2) \oplus \mathcal{O}(-2,0)$ has no global sections. The set $S$ associated to $E'$ in the proof of Lemma 4.2.1 then contains every coconnected subtree of $C$ except $C$ itself; so case (1) of the proof applies. Enlarging $C$ by inserting a smooth rational curve at the node, we then have that the line bundle of degree 0 on the two original components of $C$ and degree $-1$ on the new component is a line subbundle of the enlargement of $E'$, as we wanted.

Lemma 4.2.4. Let $\pi : \mathcal{C} \to \Delta$ be a family of trees of smooth rational curves over $\Delta$ an affine $k$-scheme, and let $\mathcal{E}$ and $\mathcal{G}$ be vector bundles on $\mathcal{C}$. Let $p$ be a $k$-point of $\Delta$, let $C_p$ be the fiber of $\pi$ over $p$, and let $F$ be a vector bundle on $C_p$ given as an extension

$$0 \to E \to F \to G \to 0, \quad (4.2.1)$$

where $E = \mathcal{E}|_{C_p}$ and $G = \mathcal{G}|_{C_p}$. Then there exists an extension of vector bundles on $\mathcal{C}$,

$$0 \to E \to F \to G \to 0,$$

that restricts to (4.2.1) on $C_p$.

Proof. We must show that the map

$$\text{res} : \text{Ext}^1(\mathcal{G}, \mathcal{E}) \to \text{Ext}^1(\mathcal{G}, E)$$

given by restricting vector bundles to $C_p$ is surjective. To see this, we use the diagram

$$\begin{array}{ccc}
\text{Ext}^1(\mathcal{G}, \mathcal{E}) & \xrightarrow{\text{res}} & \text{Ext}^1(\mathcal{G}, E) \\
\downarrow \cong & & \downarrow \cong \\
H^1(\mathcal{C}, \mathcal{G}^* \otimes \mathcal{E}) & \longrightarrow & H^1(\mathcal{C}, \mathcal{G}^* \otimes E),
\end{array}$$

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where the bottom map is the map on cohomology coming from the restriction of $\mathcal{G}^* \otimes \mathcal{E}$ to $C_p$. This diagram is commutative, and the bottom map is a surjection because $H^2(C, \mathcal{G}^* \otimes \mathcal{E} \otimes \mathcal{I}_p) = 0$ since $\pi$ has fiber dimension 1 over an affine base.

Remark 4.2.5. This lemma gives no control over what the restriction of $\mathcal{F}$ to $\pi^{-1}(\Delta \setminus p)$ looks like. In practice, when we use the lemma, we will assume that there are no nonsplit extensions of $\mathcal{G}$ by $\mathcal{E}$ on $\pi^{-1}(\Delta \setminus p)$, so the restriction of $\mathcal{F}$ to $\pi^{-1}(\Delta \setminus p)$ will be

$$\mathcal{E}|_{\pi^{-1}(\Delta \setminus p)} \oplus \mathcal{G}|_{\pi^{-1}(\Delta \setminus p)}.$$ 

We now have every tool we need to prove Theorem 4.0.3. It will be a corollary of the following result.

Theorem 4.2.6. Suppose $E$ is a vector bundle on $C$ and $E'$ a vector bundle on $\mathbb{P}^1$ such that $E$ and $E'$ have the same degree and rank and the semicontinuity condition 4.0.1 holds for every line bundle $L$ on $C$. Then there exists an enlargement $f : C' \to C$ of $C$, a specialization $E''$ of $E'$ on $\mathbb{P}^1$, a flat family $C \to \Delta$ over $\Delta = \text{Spec}(k[[t]])$ with regular total space, general fiber isomorphic to $\mathbb{P}^1_{k((t))}$, and special fiber isomorphic to $C'$, and a vector bundle $\mathcal{E}$ on $C$ that restricts to $f^*(E)$ on the special fiber and to $E''$ on the generic fiber.

Proof. We first note that given any tree of smooth rational curves $C'$, it is easy to construct a family of trees of smooth rational curves $\pi : C \to \text{Spec}(k[[t]])$ with generic fiber $\mathbb{P}^1$ and special fiber $C'$ such that $C$ is regular. Starting with the trivial family $\mathbb{P}^1 \times \text{Spec}(k[[t]])$, the desired family can be constructed by iteratively blowing up points on the central fiber.

To construct the vector bundle $\mathcal{E}$ on this regular total space $C$, we proceed by induction on the rank of $E$. For $E$ a line bundle, the result follows from Proposition 4.1.6. Now suppose that $E$ and $E'$ are vector bundles of rank at least 2 satisfying the hypotheses of the theorem. Suppose $d$ is maximal such that $H^0(C, E(L_d)) \neq 0$ for all
line bundles $L_{-d}$ of degree $-d$ on $C$. Then by Lemma [4.2.1] possibly after replacing $C$ with an enlargement $f : C' \to C$, the vector bundle $f^*(E)$ has a line subbundle $L_d$ of degree $d$, while by semicontinuity no direct summand of $E'$ has degree greater than $d$. Let $E''$ be the unique vector bundle that satisfies

$$h^0(\mathbb{P}^1, E''(e)) = \max(h^0(\mathbb{P}^1, \mathcal{O}(d + e)), h^0(\mathbb{P}^1, E'(e)))$$

for all $e$. Clearly, $E''$ is a specialization of $E'$. Moreover, $E''$ satisfies the semicontinuity property (4.0.1) with respect to $f^*(E)$ for all choices of line bundles, because for any line bundle $L$ we have $h^0(C, E \otimes L) \geq h^0(\mathbb{P}^1, E'_{0}(\deg L))$ by hypothesis, and we have the chain of inequalities

$$h^0(C', f^*(E) \otimes L) \geq h^0(C', L_d \otimes L) \geq h^0(\mathbb{P}^1, \mathcal{O}(d + \deg L))$$

that are satisfied for any $L$. We now show that some specialization of $E''$ on $\mathbb{P}^1$ specializes to an enlargement of $f^*(E)$. Let $Q' := E''/\mathcal{O}(d)$. We have that $Q'$ has the same degree and rank as $Q := f^*(E)/L_d$, and we have

$$h^0(\mathbb{P}^1, Q'(e)) = \begin{cases} h^0(\mathbb{P}^1, E''(e)) - d - 1 - e & \text{if } e \geq -d - 1 \\ 0 & \text{if } e < -d - 1 \end{cases},$$

so $Q$ satisfies the semicontinuity condition (4.0.1) with respect to $Q'$ for all line bundles. So by the induction hypothesis there is an enlargement $f' : C'' \to C'$ and a vector bundle $Q$ on a smoothing $C \to \Delta$ of $C''$ that restricts to some specialization $Q''$ of $Q'$ on the general fiber and $Q$ on the special fiber. Let $L_d$ be the line bundle on $C$ that restricts to $f'^*(L_d)$ on the special fiber. By Lemma [4.2.4] there is then a vector bundle $E$ on $C$ that is given by an exact sequence

$$0 \to L_d \to E \to Q \to 0$$

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and restricts to the exact sequence

\[ 0 \to f''(L_d) \to f''(f^*(E)) \to f''(Q) \to 0 \]

on \( C'' \). Such an extension automatically restricts to \( \mathcal{O}(d) \oplus Q'' \) on the general fiber, because there are no nontrivial extensions of \( Q'' \) by \( \mathcal{O}(d) \) on \( \mathbb{P}^1 \). \( \mathcal{O}(d) \oplus Q'' \) is a specialization of \( E'' \) and therefore of \( E \), so \( \mathcal{E} \) is the vector bundle giving the desired specialization.

\begin{proof}
By Theorem 4.2.6, we have that there exists a specialization \( E'' \) of \( E' \) on \( \mathbb{P}^1 \), an enlargement \( (\tilde{C}, \tilde{E}) \) of \( (C, E) \), a flat family \( \tilde{C} \to \text{Spec}(k[[t]]) \) with regular total space, and a vector bundle \( \tilde{E} \) on \( \tilde{C} \) such that \( C_\eta \cong \mathbb{P}^1_{k(t)} \), \( C_p \cong \tilde{C} \), and the restriction of \( \tilde{E} \) to \( C_\eta \) and \( C_p \) is \( E'' \) and \( \tilde{E} \) respectively. The enlargement map \( f : \tilde{C} \to C \) extends to a blow-down map \( \phi : \tilde{C} \to C \) which contracts every \( \mathbb{P}^1 \) subtree contracted by \( f \). Moreover \( \tilde{E} \) is trivial on the components of \( \tilde{C} \) contracted by \( f \), so likewise \( \tilde{E} \) is trivial on such components. Then the direct image \( \phi_* (\tilde{E}) \) is a vector bundle on \( C \) giving a specialization from \( E'' \) to \( E \). Then, by Proposition 4.1.5, we have that there exists a family of trees of smooth rational curves \( \pi : C \to \Delta \) and a vector bundle \( \mathcal{E} \) on \( C \) that restricts to \( (\mathbb{P}^1, E') \) on the generic fiber and to \( (C, E) \) on the special fiber, as we were to show.

\end{proof}

\textbf{References}


