Essays on Networks and Behaviors in International Macroeconomics

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Essays on Networks and Behaviors in International Macroeconomics

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Abstract

This dissertation provides new analytical frameworks of network and behavioral theories in order to answer real-world economic questions where more simplified traditional models have struggled to explain. In Chapter 1, I provide a theory of international portfolio investments where international investors have heterogeneous consumption preferences and goods are produced in a global production network. While traditional two-country models can only explain the asset allocation between domestic and foreign assets, my theory explains also the composition of external investments, i.e. how investors should optimally allocate their investments between multiple foreign destinations. I solve in closed form for the optimal equity and bond portfolio investments in a workhorse multi-country real business cycle model and show in Chapter 2 that the network-theory implied portfolio has empirical explanatory power for the actual portfolio holding patterns observed in the data.

In Chapter 3, I provide a behavioral-based explanation of two recent macroeconomic puzzles: (i) the anchoring of inflation and (ii) the flattening of the Phillips Curve. In particular, I introduce endogenous behavioral inattention into the standard New Keynesian model and derive a new Phillips Curve, called the Behavioral-Attention Phillips Curve (BAPC). The BAPC has both an output-gap slope and inflation-expectation slope that decline with inflation uncertainty in the economy. Using data from the Michigan Surveys of Consumers, I estimate the BAPC and show that it provides a better in-sample and out-of-sample fit of the US inflation dynamics since 1978 than traditional versions.

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Introduction

This dissertation contains three chapters that aim to provide new analytical frameworks of networks and behavioral elements to answer economic questions where more simplified traditional models have struggled to explain.

One such open question is “what determines international portfolio investments,” which is the focus of chapter 1. In this chapter, I provide a theory of international portfolio choice where international investors have heterogeneous consumption preferences and countries are linked through the input - output linkage in the global production network. Whereas traditional two-country or N-symmetric-country models could only explain the asset allocation pattern between domestic and foreign assets (the home bias literature), my network framework allows for an analysis of the *composition* of international portfolio investments. In a workhorse international real business cycle model with nondiversifiable labor income risk arising from international production linkages, I solve for the optimal equity and bond portfolios in closed form. I show that despite the complex network environment, a simple measure of international demand exposure, which I call the "International Domar Weights," is key in determining international equity portfolios. This measure extends the closed-economy “Domar weights” to the international setting and capture countries’ interdependence through both direct and indirect trade linkages. Unlike the closed economy Domar weights, which coincide with sales as fraction of the domestic GDP and can be used as sufficient statistics without knowing the shape of the network, the International Domar Weights have no direct measurable empirical objects and must be calculated from the trade networks.

Chapter 2 evaluates the empirical explanatory power of the theory provided in Chapter 1 in explaining the patterns of actual portfolio equity investments in the data. Using the input-output network from the World Input - Output Database (WIOD) and data on portfolio investments from Coordinated Portfolio Investment Survey (CPIS), I apply the framework to a network of 43 major developed and emerging economies. I obtain four main results. First, the theoretical network portfolio is a significant predictor and explains almost half of the variation in international bilateral portfolio investments. The significance of the network portfolio is robust to controlling for gravity factors (market capitalization, distance, EU membership, etc.). Second, including the network-based portfolio in a gravity model for assets resolves the puzzle of why distance matters for asset trade at all. Third, indirect trade linkages matter for portfolio determination, highlighting the need to explicitly account for trade in intermediate inputs. Finally, the model predicts both the levels and the changes in equity home bias that have occurred since 2000.

Chapter 3 seeks to understand two recent phenomena in macroeconomics: (i) the anchoring of inflation, and (ii) the flattening Phillips Curve. In this chapter, I introduce a theory of endogenous uncertainty and attention choice that can jointly account for these phenomena. In particular, I derive a Behavioral Attention Phillips Curve (BAPC) whose slopes on the output gap and inflation expectations decline when inflation is less uncertain. When inflation uncertainty is low, firms find it less costly to misperceive aggregate demand and inflation expectations, thus pay little attention to monetary shocks and change prices less. The dampened price response flattens the Phillips Curve. Inflation becomes more anchored with low uncertainty because costly attention motivates firms to rely more on “rules-of-thumb” such as the 2% inflation target. Using novel measures of inflation uncertainty constructed from surveys of inflation expectation, I show that the new Phillips Curve performs better both in-sample and out-of-sample than the traditional Phillips Curve with constant slope. Particularly, the BAPC does not generate the counterfactual prediction of large disinflation after the 2008-2009 Financial Crisis as does traditional Phillips Curves, resolving the Missing Disinflation Puzzle. Finally, I show that multiple equilibria arise with medium volatility due to the complementarity between pricing and attention choices. This gives rise to a novel policy paradox for the Central Bank and makes it hard to raise inflation in a quiet, low-volatility period.

Chapter 1

A Theory of Optimal International Portfolios with Trade Networks[1]

1.1 Introduction

International portfolio investments are an important part of the global economy, yet we know little about their determinants. To be specific, despite a large literature focusing on the asset allocation problem between domestic versus foreign assets (the "home bias" literature, starting with French and Poterba (1991)), far fewer papers have attempted to explain asset allocation between multiple foreign destinations, i.e. the *composition* of external portfolios. The shortage of empirical work is partly due to the lack of a theory of portfolio choice in an $N-$country setting with large, realistic cross-country asymmetries that are observed in the data.

This paper fills that gap by providing the first theory of international portfolio investments where trade networks play a central role in driving portfolio choices. In particular, I solve in closed form for the optimal equity and bonds portfolio in a workhorse real business cycle (RBC) model with an arbitrary number of countries, arbitrary taste differences, and arbitrary

[1]I am grateful to my advisors - Xavier Gabaix, Kenneth Rogoff, Adi Sunderam, John Campbell, and Robert Barro - for their continuous guidance and support throughout the development of this paper. I want to thank for their comments Emmanuel Farhi, Pol Antras, Jeremy Stein, Elhanan Helpman, Ludwig Straub, Greg Mankiw, Matteo Maggiori, and seminar participants at the Macro, Finance, and International Workshops at Harvard who collectively made the paper better. I thank Shushu Liang, Krishna Dasaratha, Masao Fukui, Hoang Nguyen, Namrata Narain, Maria Voronina, Taehoon Kim, and Robert Siliciano for the many insightful conversations from the onset of this project.

international input-output linkages for the production of intermediate goods.

In this general setting, I show that a measure of international demand exposure, called the "International Domar Weights" (IDWs), is key in determining international equity portfolios. The IDWs extend the closed-economy "Domar weights" to the international setting and capture countries' interdependence through both direct and indirect trade linkages.

In my model, the equity portfolio is chosen optimally to hedge against nondiversifiable labor income risks. The idea that risks to labor income, which is around two-thirds of national income, are significant and may influence portfolio decisions is not new (see, for example, Baxter and Jermann (1997); Heathcote and Perri (2013)). Yet, most analyses in this literature have focused entirely on explaining asset home bias. This paper is the first to explicitly incorporate trade networks and study their implications for portfolios.

In this environment, consumers supply labor to the domestic intermediate good sector. This sector produces an intermediate good that is used in the production of other intermediate and final goods (used for consumption and investments) at home and internationally. When expenditure in a downstream sector changes, the sale and value-added of intermediate producers also change, the extent of which depends on how "exposed" the intermediate producers are to that market.

Hedging against that labor income risk requires an appropriate measure of "exposure." In a model without intermediate good trades, "exposure" is given simply by the level of direct export between two countries. In a model with intermediate good trades, a producer can be exposed to a downstream market which it does not export directly to. For example, a Japanese firm exporting electronic components to China that then assembles them into smartphones to be sold to Vietnam is itself affected by demand for smartphones in Vietnam despite Japan not exporting directly to Vietnam. Therefore, considering only direct bilateral trade likely leads to overlooking important economic relationships.

I show that an "appropriate" measure of demand exposure is a measure of trade in value-added, which I call the "International Domar Weights." The IDW (i, j) measures the fraction of country j's final expenditure (consumption plus investment) that "ultimately" accrues to the value-added of country i.

This theoretical object resembles closely the closed-economy concept of "Domar weights," with important differences. In a celebrated theorem, Hulten (1978) shows that the first-order impact of TFP shock of a sector on aggregate output is well-captured by the sale share - also called the Domar weight - of that sector. While the Domar weights arise endogenously in a closed-economy network model (Long and Plosser, 1983; Gabaix, 2011; Acemoglu *et al.*, 2012), Hulten (1978)'s result shows that one needs only know the sale shares, not the underlying network, to calculate output and welfare effects. In that regard, the network is considered to be "irrelevant."[2] By contrast, despite the similarity in meaning, there is no direct empirical measure of the IDWs. Calculating the IDWs requires knowing the full shape of the input-output linkages and taste differences across countries.

Johnson and Noguera (2012) was the first paper to derive the expression for the IDWs (which they call "trade in value added") in an accounting framework. In this paper, I introduce the IDWs in a fully micro-founded structural macroeconomic model, allowing for a tighter connection between the observable empirical IDWs with unobservable preference parameters, which is useful for calibration and running counterfactual exercises. More importantly, I give economic meanings to the IDWs and show their key role in determining international portfolios.

The real business cycle model considered here is most related to Long and Plosser (1983) and Heathcote and Perri (2013). Long and Plosser (1983) models input-output linkages in a closed-economy RBC model, but do not study portfolio choice. Heathcote and Perri (2013) studies portfolio choice in a two-country RBC model and explains equity home bias using home bias in consumption.

This paper enhances the Heathcote and Perri (2013) result in two ways. First, I model trade networks, which allows for analyzing the composition of external portfolios. The theoretical result emphasizes the IDWs as key equity portfolio drivers, and nests Heathcote and Perri (2013) as a special case. Second, while the equity portfolio in Heathcote and Perri (2013) is fragile to parameter changes (deviating from log utility and Cobb-Douglas production significantly changes the magnitude and even signs of the equity portfolio), the equity portfolio in this paper is robust

[2] One of the few counterarguments to the "irrelevance of networks" result is Baqaee and Farhi (2019), who argue that network matters through large nonlinear, second-order effects.

to parameter changes.

I attain this robustness by allowing bonds to be a part of the asset menu. The optimal bond portfolio plays a central role in hedging income risks arising from fluctuation in relative prices (the terms of trade and exchange rates), allowing equity to just hedge against labor income risks. Coeurdacier and Gourinchas (2016) emphasizes the role of bonds in hedging against relative price fluctuations. I show that indeed bonds are useful in a dynamic model with investment, and solve for the optimal bond portfolios as functions of the trade network matrices.

1.2 Related literature

This paper contributes to three strands of literature.

First, it is related to a large literature on international risk-sharing and diversification motive of international portfolio investments. In a seminal paper, Lucas (1982) argues that investors with identical preferences can perfectly diversify consumption risk by having each country holding half of the other country's output stock in a two-country model. French and Poterba (1991) started a large literature on home bias and noted that equity portfolios in reality are strongly biased towards domestic equity, contradicting the Lucas (1982) argument.

Some authors view this empirical evidence as a lack of international risk-sharing caused by either trade costs for goods (Obstfeld and Rogoff (2001), Fitzgerald (2012)) or assets (Portes and Rey (2005)). Lane and Milesi-Ferretti (2008) generalizes the Obstfeld and Rogoff (2001) model to an $N-$country setting and is one of very few studies that explains bilateral equity positions. In their model, equity investment is tightly linked to bilateral imports, because both are affected by trade costs in goods. I argue in this paper that not only bilateral trade but also indirect linkages matter.

Heathcote and Perri (2013) argues that perfect risk-sharing is precisely achieved with a home-biased equity portfolio if endogenous investment demand is biased towards domestic goods. The Heathcote and Perri (2013) portfolio changes significantly, however, as the model deviates from log utility and Cobb-Douglas production. Coeurdacier and Gourinchas (2016) and Engel and Matsumoto (2009), in two-country models without investment and trade networks, emphasize the role of bonds in hedging against relative price fluctuations. This paper contributes by providing the first theory of portfolio choice with trade networks and studies external portfolio composition. I also provide a joint analysis of equities and bonds in a model with investment, unifying the fragmented frameworks in this literature.

The second strand of related literature is on the production network in closed economies. Domar (1961) and Hulten (1978) pioneered this literature and found that in efficient economies, the effect of an industry-level productivity shock on aggregate output is fully captured by the sale share of GDP of the industry origin of the shock. Building on Hulten (1978)'s theorem, Gabaix (2011) proves how a sufficiently fat-tailed distribution of firm sizes can make idiosyncratic firm-level shocks have first-order aggregate effects. Long and Plosser (1983) and Acemoglu *et al.* (2012) solve for the sale shares endogenously as a function of the production network. Acemoglu *et al.* (2012) gives a similar argument to Gabaix (2011) but instead focusing on the distribution of node degrees of the input-output network. Baqaee and Farhi (2019) moves beyond Hulten's first-order result and studies the nonlinear impact of TFP shocks that the Domar weights (sale shares) fail to capture.

Finally, this literature is related to an emerging literature on networks in finance. In the closed-economy setting, Herskovic (2018) shows that beta-sorted portfolios using network concentration and sparsity factors generates excess returns that cannot be explained by traditional models. Herskovic *et al.* (2020) shows empirical evidences that production networks also explain the evolution of firm size and volatility distributions in the data. Gofman *et al.* (2020) analyzes a firm-level supplier-customer dataset and shows that a measure of a firm's distance from consumption-good producers predicts stock returns and exposure to aggregate shocks. Babus and Kondor (2018) studies learning and information diffusion in the network of trade in over-the-counter markets.

In the open-economy finance literature, Using data on sovereign CDS prices, Chang *et al.* (2020) provides empirical evidences that trade network matters for shock propagation in global markets. Richmond (2019) shows that countries that are more central in the international trade network tend to have lower interest rates and currency risk premia. Jiang and Richmond (2019) study how the trade networks affect international transmission of shocks and international asset prices in a model without investment. This paper contributes by looking at portfolio ("quantity") instead of asset prices. Section 1.6.3 also shows how asset pricing can be studied in closed form in a model with investment.

1.3 Model

There are N countries, indexed by $i \in \mathcal{N} = \{1,2,...,N\}$. Each country produces a country-specific intermediate good which can be traded abroad and a final good which is used only for domestic consumption and investment.

Production

The intermediate good in country i is produced using domestic labor L_t^i, capital K_t^i, and a bundle of intermediate goods X_t^i using a Cobb-Douglas production function:

$$Y_t^i = \exp\left(z_t^i\right)\left(\left(K_t^i\right)^{\alpha}\left(L_t^i\right)^{\theta}\right)^{1-\gamma_i}\left(X_t^i\right)^{\gamma_i}. \tag{1.1}$$

The intermediate input share is γ_i, with $1-\gamma_i$ being the share of value added in production (the contribution from capital and labor). The share of capital in value added is α, while the labor share is $\theta = 1-\alpha$. The intermediate bundle X^i of country i consists of intermediate goods produced at home and imported from abroad:

$$X_t^i = \left[\sum_{j\in\mathcal{N}} \omega_{ij}^{\frac{1}{\varepsilon}}\left(X_{jt}^i\right)^{1-\frac{1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}}, \quad \sum_{j\in\mathcal{N}} \omega_{ij} = 1 \ \forall i.$$

The parameter ε is the elasticity of substitution between different intermediate goods, and ω_{ij}'s are parameters that indicate the preference of producers in country i for the intermediate good of country j.

Each country also produces a final good,[3] which is a CES basket of the intermediate goods

[3]The final sector here is isomorphic to a preference aggregator from a modeling's perspective. The only production technology is in the intermediate sector, so any productivity shock that affects production technology is placed in the intermediate sector.

with the same elasticity of substitution:[4]

$$G_t^i = \left[\sum_{j \in \mathcal{N}} \xi_{ij}^{1/\varepsilon} \left(G_{jt}^i \right)^{1-\frac{1}{\varepsilon}} \right]^{\frac{\varepsilon}{\varepsilon-1}}, \quad \sum_{j \in \mathcal{N}} \xi_{ij} = 1 \ \forall i. \tag{1.2}$$

Similarly, ξ_{ij}'s are parameters that indicate the preference of consumers in country i for the intermediate good of country j.

Define Ξ_t to be a matrix whose ij-th element is the share of country i's total expenditure on final goods that is spent on intermediate goods from country j. Similarly, let Ω_t be a matrix whose ij-th element denotes the share of country i's total expenditure on intermediate input that is spent on intermediate goods from country j:

$$\Xi_{ij,t} \equiv \frac{P_{jt}^i G_{jt}^i}{G_t^i}, \quad \Omega_{ij,t} \equiv \frac{P_{jt}^i X_{jt}^i}{P_{Xt}^i X_t^i}. \tag{1.3}$$

where P_t^i is the local currency price of intermediate good i, and P_{jt}^i is the price of good j when sold in country i in currency i. By construction, each row of Ξ and Ω sums to one.

The final good is used as a numéraire in each country, so its price is 1. The Law of One Price is assumed to hold, so that the price of good j in country i, denoted by P_{jt}^i, is given by its producer-currency price converted to the destination's currency: $P_{jt}^i = P_t^j / \mathcal{E}_{jt}^i$. The convention used in this paper is that a higher value of $\mathcal{E}_{jt}^i$ corresponds to an appreciation of currency i. Throughout the paper, I refer to the currency of country 1 as "dollar", and use the short-hand notation $\mathcal{E}^i$ to indicate $\mathcal{E}_1^i$, i.e. "in dollars." Since intermediate goods are the only exports/imports in the model, their relative prices are also often referred to as the Terms of Trade (TOT).

Optimal demand for intermediate input in both sectors requires:

$$\Xi_{ij,t} = \xi_{ij} \left(P_{jt}^i \right)^{1-\varepsilon}, \quad \Omega_{ij,t} = \omega_{ij} \left(P_{jt}^i / P_{Xt}^i \right)^{1-\varepsilon}, \tag{1.4}$$

where $P_{Xt}^i \equiv \left[\sum_{j \in \mathcal{N}} \omega_{ij} \left(P_{jt}^i \right)^{1-\varepsilon} \right]^{\frac{1}{1-\varepsilon}}$ is the price index for country i's intermediate input bundle, and $1 = \left[\sum_{j \in \mathcal{N}} \xi_{ij} \left(P_{jt}^i \right)^{1-\varepsilon} \right]^{\frac{1}{1-\varepsilon}}$ is the price of its final good.

[4] The assumption that the intermediate and final sectors share the same elasticity of substitution is made for the simplicity of exposition and can easily be relaxed.

Capital accumulation

In each country, there is a capital sector which accumulates capital and leases to domestic intermediate good firms. Every period, this sector purchases I_t^i units of final good i and transforms them into $Z_{It}^i I_t^i$ units of new capital. Z_{It}^i represents the efficiency of transforming consumption goods into capital goods, commonly called the *investment-specific technology* shock.[5] The capital stock in period $t+1$ is given by:

$$K_{t+1}^i = (1-\delta)K_t^i + Z_{It}^i I_t^i.$$

The price of capital in terms of consumption goods is $P_{Kt}^i = 1/Z_{It}^i$.

In period t, the capital sector earns a revenue $r_t^i K_t^i$ and makes an excess profit (revenue less investment) of $\Pi_t^i = r_t^i K_t^i - I_t^i$. This excess profit is assumed to pay out in full as dividend. The value of the capital sector is:

$$V(K_t^i) = \max_{I_t^i} \left\{ r_t^i K_t^i - I_t^i + \mathbb{E}_t \left[\Theta_{t+1}^i V((1-\delta)K_t^i + Z_{It}^i I_t^i) \right] \right\},$$

where Θ_{t+1}^i denotes the stochastic discount factor of domestic residents.[6] Normalizing the number of shares of each country's capital stock to 1, $V(K)$ is both the market capitalization and the cum-dividend stock price. The ex-dividend stock price is given by $P_{St}^i \equiv \mathbb{E}_t \left[\Theta_{t+1}^i V(K_{t+1}^i) \right]$.

As I assume that there is no adjustment cost for investment, Tobin's Q is 1, and there is no difference between the outside value of capital (P_{St}^i) and inside value (K_{t+1}^i). As a result, the ex-dividend stock price is simply the current value of future capital:

$$P_{St}^i = P_{Kt}^i K_{t+1}^i = K_{t+1}^i / Z_{It}^i.$$

The value of the capital sector is linear in the capital stock:

$$V_t^i = \left((1-\delta) P_{Kt}^i + r_t^i \right) K_t^i. \tag{1.5}$$

[5] Justiniano *et al.* (2010) estimates a new neoclassical synthesis model and finds that this form of investment shock explains most of the variability of output and hours at business-cycle frequencies.

[6] Note that the Stochastic Discount Factor (SDF) used to discount future dividends of firm i is that of domestic residents. This may be unreasonable, given that foreign residents likely will hold firm i's stocks as well. However, this is not a concern in this model. As I will show later, with only equity claims, markets will be complete in this model, and the SDF will be unique for the residents of all countries.

Optimal capital accumulation satisfies an Euler equation:

$$1 = \mathbb{E}_t \left[\Theta^i_{t,t+1} \left((1-\delta) \frac{P^i_{Kt+1}}{P^i_{Kt}} + \frac{r^i_{t+1}}{P^i_{Kt}} \right) \right], \tag{1.6}$$

where $r^i_t \equiv (1-\gamma)\alpha P^i_t Y^i_t / K^i_t$. The first term inside the round bracket represents the capital gain on undepreciated capital, while the second term is the marginal product of capital.

Preference

Each country i has a representative household-investor who wishes to maximize their lifetime utility:

$$U^i = \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u\left(C^i_t\right) \right], \tag{1.7}$$

where utility is assumed to take the CRRA form with a coefficient of relative risk aversion of $1/\sigma$. The representative household supplies inelastically $\overline{L}^i$ units of labor and earns labor income $W^i_t \overline{L}^i$.

We assume that the assets available are *country-specific equities*, modeled as claims on the capital sector, and *perpetual bonds*, modeled as claims on the numéraire (the final good) in each country.[7] Investors everywhere can invest in all assets and can adjust their portfolios costlessly.

Country i's portfolio at time t is denoted by $\left(\Lambda^i_{j,t}, B^i_{j,t}\right)_{j\in\mathcal{N}}$, where Λ^i_j is the fraction of shares of country j owned by i, and B^i_j is the corresponding bond position. Let Λ be an $N \times N$ matrix whose ij-th element is the equity share Λ^i_j, and B is the corresponding square matrix for bonds.

The household's problem is to maximize (1.7) subject to the budget constraint:

$$C^i_t + \sum_{j=1}^{N} \mathscr{E}^j_{it} \Lambda^i_{j,t+1} P^j_{St} + \sum_{j=1}^{N} \mathscr{E}^j_{it} P^j_{Bt} B^i_{j,t+1} = W^i_t \overline{L^i} + \sum_{j=1}^{N} \mathscr{E}^j_{it} \Lambda^i_{jt} \left(\Pi^j_t + P^j_{St}\right) + \sum_{j=1}^{N} \mathscr{E}^j_{it} B^i_{jt} \left(1 + P^j_{Bt}\right). \tag{1.8}$$

Market clearing conditions

The final good i is used for domestic consumption of households and investment by domestic firms:

$$G^i_t = C^i_t + I^i_t, \quad i = 1, 2, \ldots, N. \tag{1.9}$$

[7]Short-term bonds can also be added to the menu of assets available to investors. However, as I prove in Proposition 1, markets will be complete with only trades in equity and long-term bonds, making the short-term bonds redundant assets.

Country i's intermediate good is used by other intermediate and final good producers:

$$Y_t^i = \sum_j (G_{it}^j + X_{it}^j), \quad i = 1, 2, ..., N. \tag{1.10}$$

Finally, clearing the asset markets requires that all the holding of each stock sums to one, and holding of each bond sums to zero:

$$\sum_i \Lambda_{jt}^i = 1, \qquad \text{and} \qquad \sum_i B_{jt}^i = 0 \qquad j = 1, 2, ..., N$$

Tilde notations

Throughout this paper, to economize on notations, I will use $\widetilde{X}_t$ to indicate "the dollar value for variable X at time t" ("dollar" is our name for currency 1). To be specific, if X^i is a quantity in country i with price P^i in the currency of country i, then $\widetilde{X}^i = \mathcal{E}^i P^i X^i$.

1.4 Measuring demand exposure: the International Domar Weights

Before presenting the optimal portfolio result, let us consider how shocks propagate through the trade network and affect relative national incomes.

Country-specific productivity shocks $\{z_t^i\}_{i,t}$ and investment-specific technology shocks $\{z_{It}^i\}_{i,t}$ are the main sources of risks in this model. These shocks affect cross-country relative national incomes through two channels: fluctuations in investment expenditure $\{\widetilde{I}_t^i\}_{i,t}$ and the relative prices of intermediate goods $\{\mathcal{E}_t^i P_t^i\}_{i,t}$ (the *terms of trade*).

In particular, a rise in investment spending of country i, induced by either a positive technology shock in the production of the intermediate good or the capital good, will generate higher labor incomes for production sectors with high value-added content in good i. Similarly, a rise in the price of intermediate good i raises the marginal cost of downstream sectors that rely, directly or indirectly, on intermediate input i. The changes in relative prices will induce expenditure switching and, as a result, changes in national incomes. It is worth noting that the terms of trade effect also includes the effect on the relative prices of consumption: the consumption good whose main component has become expensive also becomes expensive. This also leads to changes in national incomes as agents desire to hedge against their real exchange rate risk.

In equilibrium, since bond returns are highly correlated with the changes in the TOT, bonds will be used to hedge against the TOT and RER channel above. Having hedged against income risks induced by relative prices, the equity portfolio shares will be chosen optimally to hedge against the investment demand channel. Solving for the optimal equity portfolios boils down to measuring how one country is exposed to demand fluctuations of another downstream country.

Despite the complex network structure, a theoretical object called the "International Domar Weights" (IDWs) summarily captures both direct and indirect demand exposure for any country pair. The IDWs have a simple int erpretation: the $(ij)-$element of this matrix is the fraction of country j's final expenditure that accrues to country i's capital and labor income. To put it differently, M_{ij} is the fraction of country j's consumption, measured in nominal value, that is produced by factors of country i.

I give the formal definition for the IDWs in Definition 1, and describe its economic meaning in Lemma 1.

Definition 1 (International Domar weights)**.** *Let Ω_t and Ξ_t be the expenditure share matrices for intermediate and final goods, respectively. The International Domar Weights (IDWs) matrix is given by*

$$M_t = (1-\gamma)\left[\mathbf{I} - \gamma \Omega_t^{'}\right]^{-1} \Xi_t^{'}. \tag{1.11}$$

Lemma 1. *The IDWs satisfy:*

$$\widetilde{VA}_t = (1-\gamma)\widetilde{Y}_t = M_t \widetilde{G}_t, \tag{1.12}$$

where $\widetilde{VA}_t$ a vector of value-added, $\widetilde{Y}_t$ is a vector of country output, and $\widetilde{G}_t$ a vector of country final expenditure.

Proof. Substitute the expenditure shares (1.3) into the market clearing condition for intermediate goods:

$$\widetilde{Y}_t^i = \sum_{j=1}^{N} \left(\widetilde{G}_{it}^j + \widetilde{X}_{it}^j\right) = \sum_{j=1}^{N} \left(\Xi_{ij,t} \widetilde{G}_t^j + \gamma \Omega_{ij,t} \widetilde{Y}_t^j\right).$$

Again, the tilde notation denotes nominal amounts (quantities evaluated at their dollar prices).

Put in matrix form and simplify, we have:

$$\widetilde{VA}_t = (1-\gamma)\widetilde{Y}_t = M_t \widetilde{G}_t, \qquad \text{where } M_t \equiv (1-\gamma)\left[\mathbf{I} - \gamma \Omega_t^{'}\right]^{-1} \Xi_t^{'}.$$

□

The IDWs in Definition 1 and Lemma (1) are derived purely from accounting identities tracing payments from from one sector to another. Up to assumptions about sectoral structure, Lemma (1) holds true regardless of the underlying models. Johnson and Noguera (2012) first derived the IDWs in an accounting framework and called them "trade in value-added." Here I connect the empirical concepts with structural parameters in a micro-founded model and give the economic implications of IDWs for shock transmission and portfolio determination.

The IDWs measure how downstream demand fluctuations affect income of upstream producers. It is easy to see this when production functions take the Cobb-Douglas form. In that case, the expenditure shares (Ω_t, Ξ_t) are constant and equal to the Cobb-Douglas preference parameters. The IDW matrix is also a constant matrix $M_t = M$. Totally differentiating equation (1.12) and using hat notation to denote deviation from the steady state gives us the following relationship between value added and final good expenditures, linked by the IDWs:

$$\widehat{\widetilde{VA}}_t = M \widehat{\widetilde{G}}_t. \tag{1.13}$$

Ceteris paribus, a shock that raises final expenditure of country j by 1 dollar raises the value-added of country i by M_{ij} dollars.

When the production functions take a CES form with an elasticity of substitution different from 1, the expenditure shares themselves vary over time. In this case, the IDWs still measure the expenditure-income relationship through a generalized version of equation (1.13). Linearizing equation (1.12) around the nonstochastic steady state, we have:

$$\widehat{\widetilde{VA}}_t = \overline{M}\widehat{\widetilde{G}}_t - \overline{M}_P(\widehat{p}_t + \widehat{e}_t), \tag{1.14}$$

where $\overline{M}$ denotes the IDWs in the steady state, and $\overline{M}_P$ a matrix that captures the expenditure switching effect.

Equation (1.14) decomposes the impact of any shock on cross-country value-added into two components: the impact of changing final expenditure $\widetilde{G}$ holding fixed expenditure shares, and the impact of changes in expenditure shares induced by relative prices changes. The first component in (1.14) has the same interpretation as before: *holding fixed relative prices*, a 1 dollar increase in final expenditure of country j raises the value-added of country i by $\overline{M}_{ij}$ dollars. The second term in (1.14) represents the *expenditure switching* channel: consumers substitute one good for another when their relative prices change.

The expenditure switching effect is captured by a matrix $\overline{M}_P$, which is also a function of the trade matrices (Ω, Ξ) and defined formally in Definition 2.

Definition 2. *Let $(\overline{\Omega}, \overline{\Xi})$ be the steady-state expenditure share matrices. Let Φ_Y and Φ_G be diagonal matrices of steady-state outputs and final expenditures, respectively. The expenditure-switching matrix $\overline{M}_P$ is given by*

$$\overline{M}_P = (\varepsilon - 1)(1 - \gamma)\left[\mathbf{I} - \gamma\overline{\Omega}'\right]^{-1}\left(\overline{\Xi}'\Phi_G\overline{\Xi} + \gamma\overline{\Omega}'\Phi_Y\overline{\Omega} - \Phi_Y\right). \tag{1.15}$$

In the case of Cobb-Douglas production, $\varepsilon = 1$ and there is no expenditure switching. As the elasticity increases above (decreases below) 1, a rise in the relative price of an intermediate good induces expenditure switching away from (into) that good. The first two terms in the last bracket of (1.15) represent reduction in sales to downstream final and intermediate sectors, respectively. The last term represents a valuation effect: holding quantities constant, an increase in price increases sale.

In summary, regardless of elasticity, the steady-state IDW $\overline{M}$ capture the effect of downstream demand on upstream income. As the equity portfolio will be important for hedging against labor income risks, $\overline{M}$ will be the key determinant for the optimal equity portfolio. On the other hand, as elasticity deviates from 1, there will be significant risks coming from changing relative prices. Since bonds play an important role to hedge against relative price fluctuations, the matrix $\overline{M}_P$ is key in determining the optimal bond portfolio.

Illustration of IDWs

Table 1.1 illustrates the IDWs in the data and shows how measures of direct bilateral trade underestimate actual trade linkage. The first row of Table 1.1 shows that while Japan accounts

Origin	Destination	Intermediate Share	Final Share	Scaled IDWs
Japan	China	1.61%	2.31%	5.57%
Japan	US	0.99%	0.49%	2.64%
Japan	Australia	1.00%	0.10%	1.50%
Japan	Korea	0.73%	0.30%	1.28%
Japan	Russia	0.50%	0.04%	1.11%

Table 1.1: *International Domar weights versus Bilateral Trade Shares for Japan*

Data: World Input - Output Table. The intermediate import share matrix Ω and final import share matrix Ξ are imputed directly from WIOT. The international Domar weights are calculated as $M = (1-\gamma)\left[\mathbf{I} - \gamma\Omega'\right]^{-1}\Xi'$. Calculations assume an intermediate share of $\gamma = 0.45$.

for only 1.61% of all Chinese intermediate imports and 2.31% of Chinese imported final goods, the corresponding scaled IDW is 5.57%. Thus, in terms of factor content, Japanese factors account for 5.57% of Chinese final consumption, and the bulk of Japanese factor exports to China is via intermediate goods produced in other countries.[8] For an investor who wants to ask the question "how exposed is Japan to Chinese market?", the more meaningful measure will be the IDW, not direct trade shares.

Relationship with closed-economy Domar weights

This definition of IDW closely mirrors its closed-economy counterpart. Seminal works by Hulten (1978) and Long and Plosser (1983) showed that in a closed economy with N sectors, intermediate input trade network Ω, and a representative agent with preferences described by the utility function $\ln\mathscr{C} = \sum_{j=1}^{N}\xi_j \ln C_j$ (with $\sum_j \xi_j = 1$), the equilibrium industry sale shares are given by:

$$m = (1-\gamma)\left[\mathbf{I} - \gamma\Omega'\right]^{-1}\xi. \tag{1.16}$$

[8]If there were only direct trade, the Domar weight will be a weighted average between the intermediate import share and final import share. A Domar weight that is larger than both import shares suggest that the country engage in large indirect exports, as is the case for the five top export destinations of Japan in Table 1.1.

These sales shares are called the "Domar weights"[9] because Domar (1961) first proposed using sale shares as weights to aggregate sectoral TFP shocks. Not only useful for aggregation, Hulten (1978) showed that the Domar weights also capture the first-order impact of sectoral TFP shocks on aggregate output. Recent works by Gabaix (2011), Acemoglu *et al.* (2012), and Baqaee and Farhi (2019) provide important insights into how the Domar weights, their distribution, and the underlying networks can account for "macro" fluctuations using "micro" shocks (idiosyncratic shocks at firm-level or industry-level).

There are two important differences between the IDWs and the closed-economy version, however.

First, the heterogeneity of final preferences makes the IDWs a two-dimensional concept. In other words, since preferences are given by the square matrix Ξ instead of the vector ξ, the IDWs matrix $\boldsymbol{M}$ is also two-dimensional.

Second, while the closed-economy Domar weights are observed directly as sale shares, the IDWs are not directly observable and must be calculated from the expenditure share matrices. This is an important theoretical point because it points to the empirical relevance of the network structure. The need to know the shape of the network here contrasts with other theoretical exercises where shape of the network is "irrelevant." A typical example is Hulten's theorem, where one needs only the sale shares to know the impact of sectoral TFP shocks on aggregate output, not information regarding how sectors are linked together.

1.5 Optimal portfolios

Solution method

The portfolio determination problem is well-known for its intractability and/or lack of closed form solutions (Devereux and Sutherland, 2011; Okawa and van Wincoop, 2012).[10] This is true

[9]Sale shares are defined by $m_i = \frac{\text{Sales}_i}{\sum_j \text{Sales}_j}$, where $\left(\sum_j \text{sales}_j\right)/\text{GDP} = 1/(1-\gamma)$. Traditionally, the Domar weights are calculated with GDP in the denominator, so Domar weight$_i = m_i/(1-\gamma)$. Thus, the Domar weights in this paper is equal to the Domar weights in the literature up to a normalization so that $\sum m_i = 1$.

[10]Macroeconomists are used to dealing with complex environments by taking the first-order approximation of the model around a deterministic steady state. However, in the linearized model, the covariance structure necessary to pin down optimal portfolios is absent, making the portfolio problem indeterminate. To resolve this problem,

for a simple two-symmetric-country model, much less an N-country model with complex trade networks. Let S_t be a vector of state variables, and let $\Lambda(S_t)$ be the optimal portfolio decision rules. Perform a Taylor expansion on this function:

$$\Lambda(S_t) = \Lambda(\overline{S}) + \Lambda_S(S_t)(S_t - \overline{S}) + \text{higher order terms.}$$

Samuelson (1970) shows that only the term $\Lambda(\overline{S})$ matters for first-order dynamics of the economy. This term corresponds to the steady-state portfolio, i.e., the portfolio that prevails when the variance of model shocks approaches zero. Devereux and Sutherland (2011) developed a general computational algorithm to solve for $\Lambda(\overline{S})$ in a wide class of macroeconomic models.[11] However, the Devereux and Sutherland (2011) solution portfolio depends on solution matrices after solving the DSGE state-space models computationally, thus removing hope for analytical insights.

In contrast, analytical solutions will be possible in this paper due to the fact that, in this model, the zeroth-order portfolio is able to complete markets to the first-order of approximation. In the linearized model, the $2N$ independent shocks (N productivity shocks z_t^i and N investment-specific technology shocks z_{It}^i) have a linear effect on equilibrium variables, and can be perfectly hedged using $2N$ linearly independent assets (N stocks and N bonds).

When markets are complete, the task of solving for portfolios is simplified significantly: one needs only impose efficient allocation conditions, then back out the portfolios that would support such allocations.[12] Furthermore, complete markets will also imply that the optimal portfolio is independent of the covariance structure of shocks (Devereux and Sutherland, 2011). While the exogenous correlation of shocks likely matters for portfolio choices, this environment helps us focus on endogenous correlations induced by the trade networks.

Samuelson (1970) observes that because portfolio holding only enters the model through multiplicative terms, only the zeroth-order term of the optimal portfolio matters for the first-order movement of all non-portfolio variables.

[11]In particular, the nth-order portfolio can be solved by approximating the non-portfolio equations to the $(n+1)$-th order, and the portfolio choice equations to the $(n+2)$-th order.

[12]This approach to solve for portfolio in complete markets settings is common, see for example Coeurdacier and Rey (2013); Coeurdacier and Gourinchas (2016).

Optimal portfolios

I now present the main theoretical result of the paper: the closed form solution of the optimal equity and bond portfolios. In mentioning the bond portfolio below, the (ij)-element of a bond portfolio $\widetilde{B}$ has the meaning "i's dollar position of currency-j bond, evaluated at steady-state exchange rate." A bond portfolio without tilde notation means the number of bond units (not converted to the same currency).

Proposition 1 (Portfolio with General Elasticities)**.** *For any observed network of trade* $(\overline{\Omega},\overline{\Xi})$*:*

1. *The competitive equilibrium with bonds and equities is locally Pareto-efficient.*

2. *The optimal equity portfolio hedges against labor income risk and is given by:*

$$\Lambda = \theta\overline{M}\left[\mathbf{I} - \alpha\overline{M}\right]^{-1}, \tag{1.17}$$

   where $\overline{M} = (1-\gamma)\left[\mathbf{I} - \gamma\overline{\Omega}'\right]^{-1}\overline{\Xi}'$ *is the matrix of International Domar Weights, defined in Definition 1.*

3. *The optimal bond portfolio has two components:* $\widetilde{B}^{RER}$*, which hedges against RER risk, and* $\widetilde{B}^{TOT}$*, which hedges against TOT risk. They are given by:*

$$\widetilde{B} = \widetilde{B}^{RER} + \widetilde{B}^{TOT}$$

$$\widetilde{B}^{RER} = (1-\sigma)[\mathbf{I} - \Lambda]\Phi_C$$

$$\widetilde{B}^{TOT}\overline{\Xi} = [\theta\mathbf{I} + \alpha\Lambda]M_P$$

   where Λ *is the optimal equity portfolio,* Φ_C *is a diagonal matrix of steady-state consumptions, and* M_P *is the matrix that captures expenditure-switching effect as defined in Definition 2.*

4. *The competitive equilibrium allocation is the same as the Social Planner allocation with a vector of Pareto weights* χ *that satisfies* $(\mathbf{I} - \Lambda)\chi = 0$*.*

Proof. See Appendix A.2 □

I now provide the intuition for the optimal equity and bond portfolios. Readers who are more interested in the empirical relevance of the theory portfolios may wish to skip to Section **??**.

1.5.1 Intuition for optimal equity portfolio

As mentioned above, a key channel through which productivity shocks affect relative country income is through changes in country investment expenditures. This section explains how holding the "right" equity portfolio can perfectly hedge against non-diversifiable labor income risk.

Let $\widetilde{\boldsymbol{I}}_t$ be the vector of country investment expenditures. This vector will be a function of state variables $\boldsymbol{S}_t$, which include the realized TFP shocks. A positive, persistent domestic TFP shock will raise the marginal product of capital today and in future periods, encouraging higher investment. As discussed in Section 1.4, changes in investment expenditures affect value-added through the IDWs: $\widehat{\widetilde{\boldsymbol{VA}}}_t = \overline{\boldsymbol{M}}\widehat{\widetilde{\boldsymbol{I}}}_t$. This changes labor income (fraction θ of value-added) by $\theta\widehat{\widetilde{\boldsymbol{VA}}}_t = \theta\overline{\boldsymbol{M}}\widehat{\widetilde{\boldsymbol{I}}}_t$, and capital income by $\alpha\overline{\boldsymbol{M}}\widehat{\widetilde{\boldsymbol{I}}}_t$.[13] Since making investment lowers dividends (capital income minus investment) one-for-one, the net impact of investment on capital income is $\left(\alpha\overline{\boldsymbol{M}} - \mathbf{I}\right)\widehat{\widetilde{\boldsymbol{I}}}_t$.

Given portfolio shares Λ, the total income change induced by investment is:

$$\underbrace{\theta\overline{\boldsymbol{M}}\widehat{\widetilde{\boldsymbol{I}}}_t}_{\text{labor income}} + \underbrace{\Lambda\left(\alpha\overline{\boldsymbol{M}} - \mathbf{I}\right)\widehat{\widetilde{\boldsymbol{I}}}_t}_{\text{financial dividend}} . \tag{1.18}$$

The income change in equation (1.18) does not include income arising from capital gains. This happens because a higher stock price simultaneously increases financial wealth and makes purchasing future shares more costly. Maintaining a constant portfolio Λ requires using all the capital gain income for purchasing future shares. In other words, the capital gain component of financial income net of share purchase will be exactly zero $\sum_j \mathcal{E}_t^j P_{St}^j \Lambda_{jt}^i - \sum_j \mathcal{E}_t^j P_{St}^j \Lambda_{j,t+1}^i = 0$. Conditional on holding a constant portfolio Λ, the remaining component of equity returns that is used for hedging is dividend income.

Given the linear form, it is easy to see that there is a unique matrix Λ that makes financial dividend exactly hedge against labor income for any changes in investment expenditure $\widehat{\widetilde{\boldsymbol{I}}}$:

$$\theta\overline{\boldsymbol{M}} + \Lambda\left(\alpha\overline{\boldsymbol{M}} - \mathbf{I}\right) = 0 \Rightarrow \Lambda = \theta\overline{\boldsymbol{M}}\left[\mathbf{I} - \alpha\overline{\boldsymbol{M}}\right]^{-1}.$$

[13] "Capital income" refers to the income earned by capital (not taking into account ownership). This income is then repatriated to capital owners depending on equity portfolio shares. The income earned by ultimate investor - not the underlying capital - is called "financial income" (taking into account ownership).

To understand the optimal equity portfolio better, perform a Taylor expansion on the optimal portfolio:

$$\Lambda = \theta \overline{M} + \theta \alpha \overline{M}^2 + \theta \alpha^2 \overline{M}^3 + \ldots \tag{1.19}$$

To the first order, the optimal portfolio Λ is proportional to the IDW matrix $\boldsymbol{M}$ - our measure of demand exposure. In words, country $\boldsymbol{i}$ should hold a larger share of country $\boldsymbol{j}$'s equity market if $\boldsymbol{i}$ is a key supplier of $\boldsymbol{j}$'s final good (via both direct and indirect trade).

It may sound counterintuitive that a country should hold more shares of its key trade partner, as this would imply that returns to equities are high precisely in the states of high labor income. However, as previously discussed, in a setting where a constant portfolio supports efficient allocations, the relevant correlation is the correlation between labor income and equity dividend income, which is negative in this model due to the presence of endogenous investment.

Heathcote and Perri (2013) made this point in a two-country special case with log utility and Cobb-Douglas production, and the argument holds true in this general framework. The equity portfolio in Heathcote and Perri (2013), however, is fragile to changes in parameters, particularly the coefficient of relative risk aversion and elasticity of substitution. The main reason is due to a lack of assets available to investors, which leads the equity portfolio shouldering the burden of TOT and RER hedging as well. In this paper, I add bonds to the asset menu, allowing investors to separately hedge against relative price fluctuations using bonds and making the equity portfolio robust to parameter changes.[14]

Relationship with network centrality measures

We can also understand the optimal portfolio choice from a network perspective. In particular, consider a directional, weighted network described by an adjacency matrix $\boldsymbol{A}$. Let $\boldsymbol{\alpha}$ be a scale vector that describes the importance of network linkages, and $\boldsymbol{b}$ a vector *external importance*. The notion of "external importance" changes depending on the context of the network, but in general it has the meaning of "importance without network considerations." This stands in contrast with *endogenous* importance arising from being connected to an important node in

[14]The usage of bonds to remove relative-price hedging motives for equity portfolios has been advocated in the literature. See, for example, Coeurdacier and Gourinchas (2016) and Engel and Matsumoto (2009).

the network, which is called the Bonacich measure of centrality. In this setting, the Bonacich centrality measure is defined as:[15]

$$\mathscr{B}(\mathbf{A}, \alpha, \mathbf{b}) \equiv [\mathbf{I} - \alpha A]^{-1} b.$$

We can interpret the optimal equity portfolio in terms of the Bonacich centrality measure. To see this, consider the transpose of the equity portfolio matrix given in equation (1.17):

$$\Lambda' = \left[\mathbf{I} - \alpha M'\right]^{-1} (\theta M').$$

Column i of Λ' (row i of Λ) is the portfolio holding of a country i. This equity portfolio is precisely a vector of Bonacich centrality measure:

$$\Lambda_{i,:} = \mathscr{B}(M', \alpha, \theta M_{\text{row } i}).$$

Intuitively, country i should hold higher equity shares in countries that are more central (in the Bonacich sense) from the perspective of country i. My theoretical contribution is to show that the relevant network for portfolio consideration is not the trade network, but the network based on the IDWs matrix.

The optimal TOT-hedging bond portfolio

In section (1.4), I showed that productivity shocks also can also affect output and value-added through their effect on the relative prices of intermediate goods (the TOT channel). Intuitively, a positive TFP shock lowers the price of the corresponding intermediate good. This effect trickles downstream through the network: the sectors that use an input that has become cheaper have lower marginal costs and will also set lower prices. If $\varepsilon > 1$, the goods that become relatively cheaper have higher revenues and value-added (expenditure switching).

---

[15]Suppose $\mathbf{b}$ is a vector that measures *external* or *exogenous* importance, where high b_i implies i is more important (without taking into account network interactions yet). Then, given the network structure Ω where Ω_{ij} indicates the strength of the directional edge from i to j. Then node i can be *central* if either it is externally important, or connects to another node that is central. Let αdenotes the strength of the latter channel relative to the former, we have:

$$\mathscr{B}_i = \alpha \sum_j \Omega_{ij} \mathscr{B}_j + \mathbf{b}_i \Rightarrow \mathscr{B} = \alpha \Omega \mathscr{B} + \mathbf{b} \Rightarrow \mathscr{B} = [\mathbf{I} - \alpha \Omega]^{-1} b.$$

The effect of productivity shock on income through the TOT channel is given by:

$$\frac{\partial \text{income}}{\partial \text{TOT}} = -\underbrace{[\theta \mathbf{I} + \alpha \Lambda]}_{\frac{\partial \text{income}}{\partial \text{value-added}}} \underbrace{\overline{M}_P}_{\frac{\partial \text{value-added}}{\partial \text{TOT}}} . \tag{1.20}$$

The second term in equation (1.20), $\overline{M}_P$, is defined in (2) and captures the expenditure-switching effect. The $(ij)-$element of this matrix describes the revenue loss (in dollars) of the intermediate good sector i if the price of intermediate good j increases by 1 percent. The first term on the RHS of (1.20) describes how value-added in the intermediate sector translates into domestic income: a fraction θ of value-added goes directly to domestic residents as labor income, while a fraction $\alpha\Lambda$ of value added is accrued as financial income.[16]

The optimal RER-hedging bond portfolio has payoff that exactly hedges again the TOT risk:

$$\widetilde{B}^{TOT}\widehat{e} - [\theta \mathbf{I} + \alpha\Lambda]\overline{M}_P(\widehat{p} + \widehat{e}) = 0. \tag{1.21}$$

To the first order, the prices of final goods $\widehat{e}$ are simply linear combinations of the intermediate good prices $\widehat{p} + \widehat{e}$, with weights given by the taste matrix:

$$\widehat{e} = \overline{\Xi}(\widehat{p} + \widehat{e}). \tag{1.22}$$

Substituting (1.22) into (1.21), we find that the bond portfolio that exactly hedges against TOT risk satisfies:

$$\widetilde{B}^{TOT}\overline{\Xi} = [\theta \mathbf{I} + \alpha\Lambda]\overline{M}_P. \tag{1.23}$$

The system of equations (1.23) has $(N-1)\times(N-1)$ independent variables and $(N-1)\times(N-1)$ independent equations, so the portfolio is uniquely determined.

In general, the TOT-hedging portfolio has positive diagonal elements, implying that investors should take a long position on domestic bonds. Intuitively, since the TOT and RER co-move positively, the states in which the TOT appreciates and lowers demand for home goods are also the states in which home bonds pay high returns. This makes domestic bonds a natural hedge against the loss of income induced by TOT movements.

[16] The multiplication by the equity portfolio share Λ is because part of domestic capital is owned by foreigners.

The optimal RER-hedging bond portfolio

The third channel of risk through which productivity shocks affect country incomes is through their effect on the real exchange rate, i.e. the relative prices of final goods.

Let Φ_C denote the diagonal matrix that contains the nominal consumption levels of all countries in the steady state. Suppose productivity shocks have made some final goods more expensive than others, captured by the vector $\hat{e}$. Complete markets imply that the relative increase in consumption expenditure is given by $(1-\sigma)\Phi_C\hat{e}$.[17] In particular, if agents are more risk-averse than implied by log utility ($\sigma < 1$), they want to shift consumption expenditure into states where the consumption good has become more expensive (the opposite is true for $\sigma > 1$).

The changes in consumption expenditures will have consequences for relative country income:

$$\frac{\partial \text{income}}{\partial \text{consumption}} = \underbrace{[\theta \mathbf{I} + \alpha\Lambda]}_{\frac{\partial \text{income}}{\partial \text{value-added}}} \underbrace{\overline{M}}_{\frac{\partial \text{value-added}}{\partial \text{consumption}}}$$

The effect of consumption expenditure on value added is captured by the IDWs $\overline{M}$, as discussed extensively in Section 1.4. Changes in value-added translate into income changes as described by the matrix $\theta\mathbf{I} + \alpha\Lambda$, previously discussed in Section 1.5.1.

The optimal RER-hedging bond portfolio has payoff that compensates exactly for the budget surplus / shortage induced by changes in desired consumption:

$$\widetilde{B}^{RER}\hat{e} + (1-\sigma)\left[(\theta\mathbf{I} + \alpha\Lambda)\overline{M} - \mathbf{I}\right]\Phi_C\hat{e} = 0.$$

This gives the portfolio:

$$\widetilde{B}^{RER} = (1-\sigma)\left[\mathbf{I} - (\theta\mathbf{I} + \alpha\Lambda)\overline{M}\right]\Phi_C = (1-\sigma)[\mathbf{I} - \Lambda]\Phi_C,$$

where the last equality follows from the optimal equity portfolio (1.17).

[17] Complete markets imply that consumption satisfies $\hat{\tilde{c}}_t = \hat{\tilde{c}}_t^1 \mathbf{1} + (1-\sigma)\hat{e}_t$. This is the multi-country version of the well-known Backus-Smith condition. There is a common component in consumption changes, represented by the term $\hat{\tilde{c}}_t^1\mathbf{1}$, that does not affect relative country income. The second term, $(1-\sigma)\hat{e}_t$, represents the consumption changes that are country-specific and affect relative income.

The element ij of this portfolio takes the form:

$$\frac{\widetilde{B}_{ij}^{RER}}{\widetilde{\overline{C}}_j} = (1-\sigma)\left\{1_{i=j} - \Lambda_{ij}\right\}.$$

Let us suppose $\sigma < 1$. In this case, risk-averse investors long domestic bond with a position (percent of destination consumption) that is proportional to the equity foreign bias 1-Λ_{ii}.

Intuitively, domestic bond pays high returns precisely when domestic consumption is expensive, making it a natural hedge. The bond position is proportional to the equity foreign bias because a country that holds more foreign equity is one that sources more inputs from abroad. An increase in consumption spending in such country increases foreign wealth more relative to home. Thus, financing an increase in consumption spending when the RER appreciates requires more wealth transfer than if the production structure were concentrated on domestic sectors. As a result, the long position needs to be larger.

It is easy to see that the investors will want to short foreign bonds ($\widetilde{B}_{ij}^{RER} = -(1-\sigma)\Lambda_{ij} \leq 0$ for $j \neq i$). This is because an increase in foreign consumption expenditure is a positive income shock to home, and this happens precisely in the states when foreign bond pays high returns.

Illustration: Optimal portfolios with two symmetric countries

I now illustrate the optimal portfolio result using a simple case of two symmetric countries with home bias in consumption. In particular, assume the following trade matrices:

$$\overline{\Omega} = \overline{\Xi} = \begin{bmatrix} \omega & 1-\omega \\ 1-\omega & \omega \end{bmatrix}, \quad \omega > \frac{1}{2}.$$

Simple calculations give the following matrix of International Domar Weight:

$$\overline{M} = (1-\gamma)\left[\mathbf{I} - \gamma\overline{\Omega}'\right]^{-1}\overline{\Xi}' = \begin{bmatrix} m & 1-m \\ 1-m & m \end{bmatrix}$$

with

$$m = \frac{\omega - \gamma(2\omega-1)}{1-\gamma(2\omega-1)} \in (\frac{1}{2}, 1].$$

The composite parameter m represents country i's exposure to its own domestic demand. This exposure parameter reduces with γ (less reliant on domestic value-added in production) and increases with the home bias measure ω. Note that $m > \frac{1}{2}$ as long as there is home bias $\omega > \frac{1}{2}$.

Using the result in Proposition 1, the equity portfolio is given by:

$$\Lambda = \theta \overline{M} \left[\mathbf{I} - \alpha \overline{M} \right]^{-1} = \begin{bmatrix} \lambda & 1-\lambda \\ 1-\lambda & \lambda \end{bmatrix}, \tag{1.24}$$

with the level of home bias in equity given by:

$$\lambda = \frac{m - \alpha(2m-1)}{1-\alpha(2m-1)} = \frac{1}{2} + \frac{1}{2}\frac{(2m-1)(1-\alpha)}{1-\alpha(2m-1)} > \frac{1}{2}.$$

The first term ($\frac{1}{2}$) is the Lucas diversification term, which states that in a symmetric setting, countries can hedge against idiosyncratic output shocks by holding equal shares of each country's output. The second term is the equity home bias term, which takes into account the fact that heavier exposure to domestic demand implies higher domestic labor income risk. Without intermediate goods ($\gamma = 0$), the equity portfolio (1.24) is the same as the Heathcote and Perri (2013) equity portfolio.

Turning to bonds, using Proposition 1, the RER-hedging bond portfolio is given by

$$\frac{\widetilde{B}^{RER}}{\overline{C}} = (1-\sigma) \begin{bmatrix} 1-\lambda & -(1-\lambda) \\ -(1-\lambda) & 1-\lambda \end{bmatrix},$$

while the TOT-hedging bond portfolio is given by:

$$\frac{\widetilde{B}^{TOT}_{ii}}{\overline{Y}} = \frac{1-2\alpha(1-\lambda)}{2\omega - 1} \cdot m_p, \quad m_p \equiv (\varepsilon - 1)(1-\gamma)(\omega + m - 2\omega m).$$

The composite parameter m_p represents the strength of the expenditure switching effect. Any calibration with $\varepsilon > 1$ (sufficiently strong substitution force) and $\alpha(1-\lambda) < \frac{1}{2}$ (which is likely to hold as the capital share α and the foreign equity bias $1-\lambda$ are both significantly lower than $\frac{1}{2}$) will have $\widetilde{B}^{TOT}_{ii} > 0$, i.e. going long on domestic bonds. Intuitively, when the elasticity of substitution is sufficiently high ($\varepsilon > 1$), a shock that makes home intermediate good expensive reduces Home income as all buyers substitute into other intermediate goods. Given the high correlation between the terms of trade and the exchange rate, a natural hedge against this

Elas. Subs. ε	Coef. RRA $1/\sigma$	Equity HB λ	Bond TOT (%GDP) $\widetilde{B}_{11}^{TOT}/GDP$	Bond RER (%GDP) $\widetilde{B}_{11}^{RER}/GDP$
1	1	0.73	0	0
2	1	0.73	0.44	0
5	1	0.73	1.75	0
5	2	0.73	1.75	0.11
5	4	0.73	1.75	0.17

Table 1.2: *Optimal portfolio for two symmetric countries with home bias*

This table calculates the optimal portfolio equity and bond holding for the case with two symmetric countries and home bias in final good. The key parameters used in this calibration are: $\omega = 0.85$ (consumption home bias), $\gamma = 0.45$ (share of intermediate inputs in production), $\theta = 1 - \alpha = \frac{2}{3}$ (labor share), $\beta = 0.96$ (annual discount factor), $\delta = 5\%$ (annual depreciation).

negative income shock will be domestic bonds.

Table (1.2) presents a simple calibration of this two-country model. In the baseline case of log utility and Cobb-Douglas production ($\sigma = \varepsilon = 1$), there is no TOT and RER hedging motive, and the equity portfolio itself is enough to achieve full risk sharing. The level of home bias is 73%, which is a good description of the average equity home bias in the data.[18]

When $\varepsilon \neq 1$, it becomes necessary to hedge against TOT risks. In particular, with $\varepsilon = 2$, the optimal portfolio prescribes a long position on domestic bonds equal 44% of GDP (and shorting an equal amount of foreign bonds). With an even higher elasticity $\varepsilon = 5$, domestic investors need to hold domestic bonds at 175% of GDP. Furthermore, as investors become more risk-averse than log utility, investors need to hold even more domestic bonds to hedge against real exchange rate shocks, albeit much more muted in magnitude (around 11-17% of GDP).

[18] Strictly speaking, this means that 73% of Home equity is owned by domestic investors. In this symmetric setting, it also means that Home portfolio puts 73% weight on domestic assets, but the two concepts do not coincide more generally.

1.6 Extensions

1.6.1 Model with multiple intermediate sectors

So far, I have assumed that each country has a unique country-specific intermediate good. I now show that the analysis can be more "granular" and extend the model to allow for many intermediate industries within a country.

Let $\mathcal{K}_i$ be the set of intermediate industries in country i, and $\mathcal{K} = \cup_{i \in \mathcal{N}} \mathcal{K}_i$ is the set of intermediate industries of the world. Each industry is now given as a nation-industry pair, e.g. "USA, Agriculture" versus "Japan, Agriculture." For exposition, let us assume that $\varepsilon = 1$, but this assumption is unnecessary. The production function of intermediate sector s is now given by:

$$Y_t^s = \exp\left(z_t^s\right)\left(K_t^s\right)^\alpha \left(L_t^s\right)^\theta \Pi_{j \in \mathcal{K}} \left(X_{jt}^s\right)^{\gamma \omega_{sj}}, \quad \forall k \in \mathcal{K}.$$

The final good i is now produced with technology:

$$G_t^i = \Pi_{s \in \mathcal{K}} \left(G_{st}^i\right)^{\xi_{is}}, \quad \forall i \in \mathcal{N}.$$

Let Ω and Ξ be the matrix of trade shares as before. Ω now has dimension $S \times S$, with $S = |\mathcal{K}|$ is the total number of intermediate sectors in the world; Ξ has dimension $N \times S$.

Note that each intermediate sector has its own idiosyncratic TFP shock z_t^s for $s \in \mathcal{K}$. Investors can buy claims on capital for each industry, so the number of assets still matches the number of shocks. Let Λ_{is} denote the fraction of shares of industry s owned by investors of country i, and $\Lambda_{N \times S}$ be the portfolio matrix.

Define the following "modified identity matrix":[19]

$$E = \begin{bmatrix} 1_{1\times|\mathcal{K}_1|} & 0 & \dots & 0 \\ 0 & 1_{1\times|\mathcal{K}_2|} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1_{1\times|\mathcal{K}_N|} \end{bmatrix} \tag{1.25}$$

[19]E serves the role of an aggregating matrix. Suppose $\widetilde{Y}^*$ is a $|\mathcal{K}| \times 1$ vector of output for each industry, then $\widetilde{Y} = E\widetilde{Y}^*$ is an $N \times 1$ vector of output for each country, with $\widetilde{Y}_i = \sum_{k \in \mathcal{K}_i} \widetilde{Y}_k$.

The matrix of International Domar Weights (IDWs) are still exactly as before:

$$\boldsymbol{M} = (1-\gamma)\left[\mathbf{I} - \gamma\overline{\Omega}'\right]^{-1}\overline{\Xi}'.$$

Since there are now multiple industries within a given country, the appropriate measure of country-level international Domar weights is given by $\boldsymbol{B} \equiv \boldsymbol{E}\boldsymbol{M}$. The optimal equity portfolio is now given by a modified version of the portfolio in (1.17).

Proposition 2. *The optimal portfolio when there are multiple industries within each country is given by:*

$$\Lambda = \theta \boldsymbol{E}\boldsymbol{M}\boldsymbol{E}\left[\mathbf{I} - \alpha \boldsymbol{M}\boldsymbol{E}\right]^{-1}.$$

where $\boldsymbol{E}$ *is the modified identity matrix (defined in* (1.25)*) and* $\boldsymbol{M} \equiv (1-\gamma)\left[\mathbf{I} - \gamma\overline{\Omega}'\right]^{-1}\overline{\Xi}'$ *is the modified IDWs.*

Proof. See Appendix A.2 □

When $|\mathcal{K}_i| = 1$ for all i, $\boldsymbol{E}$ is the identity matrix, and we recover the optimal portfolio as in the baseline formula (1.17).

1.6.2 Model with nontraded sector

The model so far has abstracted from a nontraded sector. Adding a nontraded sector may be desirable in quantitative models that attempt to match low international consumption correlations or deviations from the Purchasing Power Parity (PPP). I now show that the baseline model in this paper can incorporate in nontraded sectors without complicating the portfolio analysis.

Suppose that a share ι of final consumption and investment now comes from the nontraded sector:

$$G_{it} = \left[\left(\sum_j \xi_{ij}^{1/\varepsilon}\left(G_{jt}^i\right)^{1-\frac{1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}}\right]^{1-\iota} D_{it}^{\iota}$$

For simplicity, suppose that the nontraded good in country i is produced using labor only: $\overline{D}_t^i = z_{it}^{NT} L_{it}^{NT}$. In equilibrium, since all nontraded goods must be consumed domestically, we have $D_t^i = \overline{D}_t^i$. On the asset side, investors in all countries can buy claims on the nontraded sector in other countries.

Proposition 3. *The optimal equity portfolio when nontraded goods are present is given by:*

$$\Lambda = \left[\iota\mathbf{I} + (1-\iota)\theta\overline{M}\right]\left[\mathbf{I} - (1-\iota)\alpha\overline{M}\right]^{-1},$$

where ι is the nontraded good expenditure share, and $\overline{\boldsymbol{M}}$ is the steady-state IDWs.

Incorporating nontraded goods into the model tends to increase equity home bias. When $\iota = 1$ (only nontraded good), then we have perfect equity home bias $\Lambda = \mathbf{I}$. When $\iota = 0$ (no nontraded good), we get back to the original optimal equity portfolio in Proposition 1. The portfolio here is a weighted average of the two cases.

The increase in equity home bias is because domestic investors will end up holding all of the equities of nontraded sectors (Obstfeld and Rogoff, 2001). Because (i) returns have to be paid in tradable goods and (ii) nontraded goods enter separably in utility, no further risk-sharing can be achieved using the equities of nontraded sectors.

1.6.3 Network Implication for International Asset Prices

Up until now, the focus has been on portfolio quantity. I now show how the model can be used to analyze the network implications for international asset prices.

In general, it is hard to describe the behavior of asset prices in closed form in models with endogenous investment and general elasticities. Recent works by Richmond (2019) and Jiang and Richmond (2019) study the network implications for international asset prices in a model without investment and Cobb-Douglas production function. Here, I show that it is possible to study asset prices in an environment with networks and investment together, in closed form, if we assume (i) log utility, (ii) Cobb-Douglas production, and (iii) full depreciation. This specification follows after Brock and Mirman (1972), who made the same assumptions to attain a constant saving rate in a closed-economy model.

In the following, I use $\otimes$ to denote the Kronecker product, and $vec(\cdot)$ is the vectorization operator (stacking a matrix into a column).

Proposition 4. *Suppose that $\delta = \varepsilon = \sigma = 1$, and that TFP follows a random walk: $\Delta z_t = \varepsilon_t \sim \mathcal{N}(0, \Sigma)$. Then:*

1. *The covariance matrix for equity returns, $\Sigma_r = cov(r_S)$, is given by:*

$$vec(\Sigma_r) = \frac{1}{(1-\gamma)^2}\left[\mathbf{I} - \alpha^2 M' \otimes M'\right]^{-1} M' \otimes M' vec(\Sigma)$$

2. *The covariance matrix of exchange rate is given by:*

$$vec(\Sigma_e) = (A \otimes A) vec(\Sigma_r)$$

with

$$A = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & -1 & 0 & \dots & 0 \\ 1 & 0 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & -1 \end{bmatrix}.$$

Proof. See Appendix A.2.3. □

This model nests the network model with consumption only by Jiang and Richmond (2019) by setting $\alpha = 0$ (no capital). In that case, the returns on equity are given by:

$$r_t^{JR} = \rho + \frac{M'}{1-\gamma}\Delta z_t.$$

Equity in Jiang and Richmond (2019) is modeled as claims on consumption stream. Innovations to productivity increase output and lower consumption price, raising stock returns. The multiplication by the transpose of IDWs represents *supply effect*: how productivity shocks upstream affecting output downstream.

In this paper, capital is present in the model. This adds another intertemporal transmission channel of productivity shocks. Since capital is built one period in advance, a productivity shock that stimulates investment increases the stock of capital, lowering the cost of capital next period, increasing output next period. Equity returns in this model are given by:

$$r_t = \rho + \frac{M'}{1-\gamma}\Delta z_t - \alpha M' \Delta \mu_{t-1}. \tag{1.26}$$

The last term in equation (1.26) represents the intertemporal channel as described: a lowering in the price of investment yesterday boosts equity returns today. The higher the capital share, the more pronounced is the effect of this channel.

Note that the network matters for the intertemporal channel as well: an investment booms yesterday in a country affects not only the return of that country, but that of its key trade partners as well. This network investment effect has implications for the correlation of returns that is captured by the terms $\left[\mathbf{I}-\alpha^2 M' \otimes M'\right]^{-1}$ in Proposition 4.

In this section, I have described how the model with investment can be used to study network implication for asset prices analytically. Evaluating quantitatively whether the model prediction changes significantly when we depart from the Brock and Mirman (1972) specification, as well as quantifying the strength of the intertemporal network channel are interesting problems for future research.

1.7 Conclusion

This paper was motivated by the lack of a theory that can explains the rich heterogeneity of country equity and bond portfolios that we observe in the data. I proposed the first theory of international portfolios that accounts for international production linkages and taste differences, all in a standard real business cycle setting. The generality of the framework allows me to study not just the allocation problem between Home versus Foreign (home bias), but also the composition of external portfolio as well. Empirically, I showed that the network-theory portfolio explains well the data on portfolio holding, and its significance is robust to controlling for a large variety of bilateral gravity factors.

I anticipate three directions for future research that can greatly improve our understanding of the role of networks for international portfolios. First, while I focused on the portfolio problem in a real setting without nominal frictions or nominal shocks, the paper is not meant to evaluate or understate the importance of such factors. Studying how networks interact with pricing regimes and nominal frictions will be interesting for future research. Second, this paper studies portfolios in an environment with complete markets in order to obtain insights from analytical solutions. It is important to study (perhaps quantitatively) how the optimal portfolios change when we depart from complete markets. Finally, the focus on tractability given a complex production structure has come at the cost of model ingredients that are useful in matching well moments of asset prices. Incorporating those ingredients and checking to what extent the optimal portfolios change would be a very useful exercise.

Chapter 2

Optimal International Portfolios with Trade Networks: An Empirical Assessment

2.1 Introduction

This chapter aims to evaluate the empirical usefulness of the theory presented in Chapter 1 in explaining the actual patterns of portfolio equity investments that we observe in the data. To implement the theory, I rely on the international input - output relationship from the World Input - Output Database (WIOD) to calculate the theory-implied optimal portfolios. These portfolios will then be compared to the actual portfolio holding, with data coming from the Coordinated Portfolio Investment Survey (CPIS). The analysis is limited to the 43 countries in the sample of WIOD, which includes both the most important developed and emerging economies. I obtain four main results.

First, I show that the equity portfolio predicted by trade networks (henceforth, the *network portfolio*) explains equity holdings data better than other models that assume symmetry and models without input-output linkages. By itself, the network portfolio explains almost half of the variation of all bilateral equity shares. A regression of data shares against theory shares has an adjusted R^2 of 0.45. This model performs better than a simple model where investors hold the world portfolio, so that the portfolio share is proportional to the market capitalization of the destination country (adj. $R^2 = 0.41$). A gravity model with other covariates (distance, contiguity, EU membership, common language, etc.) has more explanatory power (adj. $R^2 = 0.57$), but at the cost of parsimony. Importantly, controlling for these bilateral gravity variables, the network portfolio is still a significant predictor of the data portfolio, explaining 10% of the variation that cannot be explained by gravity factors.

The second finding is that the network portfolio helps resolve the "distance puzzle" for assets. In particular, a gravity equation for assets without the network portfolio yields a large negative coefficient on distance (from -0.6 to -1). It is unclear why distance would have a large effect on the trade of assets. Portes and Rey (2005) found that the effect of distance is reduced on including measures of information frictions, but it remains significant. I show that including the network-based portfolio in a gravity equation for assets makes the effect of distance statistically indistinct from zero.

Third, I show that the portfolio which disregards intermediate input linkages is less successful in explaining the data, highlighting the necessity of considering the whole trade networks structure for portfolio determination.

Finally, turning focus to the diagonal elements of the portfolio matrix, I show that the model explains cross-country differences in equity home bias, both in level and in long-run changes.

Which real world assets correspond to the theory?

The model in chapter 1 has predictions for both the country equity and bond portfolios. In the model, equities are country-specific risky claims on capital, and bonds are real, perpetual claims on the final goods in each country. Up to the assumptions that would admit country-level representative firm, equities in the model correspond to the claims on the entire equity market of each country.

In comparing the model's equity portfolio with CPIS data on aggregate equity holdings across countries, I make an implicit assumption that investors buy the entire market when investing in a country. This assumption is not restrictive, however. In section 1.6.1, I present an optimal portfolio result that allows for an arbitrary number of "sectors" (can be "firms") in each country, which should allow future researchers to enhance this analysis when more refined data becomes available. Evaluating the composition effect within a country's equity market is outside the scope of this paper and will not be pursued here.

It is somewhat harder to connect the theoretical bond portfolios with real-world assets, because there are other motives for holding bonds, e.g. the desire to save/borrow to trade off consumption intertemporally, that are not present here. Taking the model literally requires having high-quality data on bond portfolios that are used for hedging motives (e.g. data on currency forward contracts). Without that data available, I decide to focus the empirical evaluation of the theory on the equity portfolio in this paper.

I perform the analysis in two steps: I first focus on the composition of the external portfolio in section 2.2, before turning to explaining asset home bias in section 2.2.6.

2.2 Explaining the composition of the external portfolio

2.2.1 Data and Empirical Strategy

In this section, I compare the optimal portfolio presented in Proposition 1 (henceforth, *network portfolio*) with the bilateral portfolio equity investment recorded in the Coordinated Portfolio Investment Surveys (CPIS) dataset.

Calculating the network portfolio requires knowing the trade shares matrices for intermediate goods (Ω) and final consumption / investment (Ξ). I calculate these two matrices from the World Input-Output Database (WIOD). The WIOD dataset records annual total export flows from a country-sector pair to another country-sector pair, decomposed into usage for intermediate versus final consumption. The dataset covers 43 countries, including 28 EU countries, 14 other major advanced and emerging economies, and an aggregation called Rest of World (ROW) for the remaining countries. Importantly, WIOD contains the decomposition of export flows for intermediate uses (56 sectors) and for final uses (5 sectors). The data is available for 1997, and every year from 2000 to 2014. Sectoral data is aggregated the national level to map directly to the baseline model, but a more granular analysis at the sector level can be done using the result in section 1.6.1.

In particular, the trade networks (Ω, Ξ) are calculated by:

$$\Omega_{ij} = \frac{\text{Export from } i \text{ for use as intermediate input in } j}{\sum_k \text{Export from } k \text{ for use as intermediate input in } j}.$$
$$\Xi_{ij} = \frac{\text{Export from } i \text{ for use as final consumption in } j}{\sum_k \text{Export from } k \text{ for use as final consumption in } j}.$$

Having calculated (Ω, Ξ), the network portfolio rule is given by Proposition 1:

$$\Lambda = \theta M [\mathbf{I} - \alpha M]^{-1}, \qquad \text{with } M \equiv \left[\mathbf{I} - \gamma \Omega'\right]^{-1} \Xi'.$$

Note that Λ_{ij} is the portfolio fraction of j's equity market cap owned by i. The portfolio share of country j in the external equity portfolio of country i is given by:

$$\text{Theory equity share}_{ij} = \frac{\Lambda_{ij} P_j^S}{\sum_{k \neq i} \Lambda_{ik} P_k^S}, \tag{2.1}$$

where P_j^S is the stock market cap of country j in the steady state.[1]

The portfolio shares (2.1) is then compared to its data counterpart, which is calculated from the IMF's Coordinated Portfolio Investment Surveys (CPIS) data set. The CPIS data set contains bilateral equity and bond investment data for more than 200 countries over the period 2001-2018. Since the WIOD country sample is more restrictive, I aggregate the non-WIOD countries into a Rest of World (ROW) block. Being a data set of external investments, CPIS does not contain investments into own country.

The data equity share between origin i and destination j for a given year is given by:

$$\text{CPIS equity share}_{ij} = \frac{\text{Portfolio equity investment from } i \text{ to } j}{\text{Total portfolio equity investment from } i \text{ to all countries } k}. \tag{2.2}$$

In the analysis, I drop the country pairs for which the CPIS equity share is extremely small (less than 0.001%), which leaves a sample of 720 country pairs. The theory portfolio is close to zero but still strictly positive for many country pairs that we observe strictly zero CPIS shares. This fact indicates that there is perhaps some fixed cost of investing that is not present in my model. Accounting for the zeros in the asset holding matrix will be a useful future exercise.

[1] In the steady state, the stock market cap of each country can be shown to be the eigenvector of the optimal portfolio rule matrix Λ corresponding to an eigenvalue of 1.

Restating CPIS from residency to nationality

Using CPIS data is subject to the "residency" vs. "nationality" problem (Coppola *et al.*, 2020). In particular, since global firms sometimes finance via shell companies in tax havens or financial centers, CPIS data often obscures the true nationality of the issuers or investors. For example, Coppola *et al.* (2020) shows that US investment into China is understated while investments into tax havens are overstated. To partially resolve the data, I use the equity portfolio reallocation matrices provided by Coppola *et al.* (2020) to "restate CPIS." The reallocation matrices are only available for the year 2017. I assume that the entries of the reallocation matrix are stable over the years, and apply to all years in my sample.

Furthermore, the reallocation matrix are not available for all origin countries. For countries that do not have a reallocation matrix, I retain the original CPIS values. Finally, Coppola *et al.* (2020) provides one reallocation matrix for the entire European Monetary Union (EMU). Since I have EMU countries as separate data points in my analysis, I assume that the reallocation matrices for outward investments from the EMU are similar across countries. Having better nationality adjustment of portfolio data will greatly enhance the analysis in this paper.

Parameters

In calculating portfolios, I calibrate the capital share (of value added), labor share (of value added), and intermediate input shares to their standard values in the literature: $\alpha = 1/3$, $\theta = 2/3$, $\gamma = 0.45$.[2]

2.2.2 Fit of network portfolio

To evaluate the performance of the network portfolio, I run a simple baseline regression:

$$\log\left(\text{CPIS portfolio share}_{ij}\right) = \alpha + \beta \log\left(\text{Network portfolio share}_{ij}\right) + \varepsilon_{ij}. \tag{2.3}$$

Here, the indices (i,j) denotes a distinct country-pair $(i \neq j)$. If the theory is a "perfect" explanation of the data, one expects $\alpha = 0$ and $\beta = 1$ and an R^2 of 1. Data for the year 2005 is

[2] The equity portfolio (1.17) can be modified to allow α, β, γ to vary by countries. These shares can then be imputed from the data, and the calculation of equity portfolios can be done with zero free parameters.

used for the empirical analysis, but similar results are obtained for other years and reported in the online appendix.

This regression is reported in column (1) of Table 2.1. The network portfolio share is a statistically significant predictor of the actual portfolio share in the data with a estimated coefficient $\widehat{\beta} = 1.179$. That $\widehat{\beta} > 1$ suggests that the model slightly under-predicts the data. The adjusted $R-$squared of the regression is 0.45, indicating that the network portfolio by itself explains close to half of variation in bilateral portfolio holdings.

Figure 2.1 plots the network portfolio shares on the x-axis and the CPIS portfolio shares on the y-axis. It is easy to see that the network portfolio fits the data well, reflecting the high R^2 of the regression.

There are two problems with this graph, however. First, there are a few outliers on the upper-left corner. The model recognizes that Korea and Mexico have little demand exposure to Luxembourg, thus should have small Domar weights and low equity shares. The data, however, shows that investments to Luxembourg account for more than 10% of the Korean and Mexican external portfolios. This is again the "nationality vs. residency" symptom that plagues CPIS.[3]

The second concern is that there are confounding factors that may over-state the explanatory power of the network portfolio. On the upper right corner, the network portfolio predicts a high investment share from Mexico and Canada into the US, reflecting the importance of US consumers to producers in these two countries. However, a simple alternative explanation may simply be that geographical proximity lowers the information friction between North American countries, increasing portfolio shares. Furthermore, the high investment shares from Mexico to Canada into the US perhaps reflects the "size factor," i.e. simply the US is a large economy.

To show the explanatory power of the network portfolio, I benchmark its performance against two other parsimonious models: size and distance. The "size model" is a simple model in which all investors hold the "world portfolio" regardless of their origin. This can occur in a model where all investors have identical consumption baskets and do not face idiosyncratic labor income risks, so that their portfolio optimization problems are identical. In equilibrium, this implies

[3] Coppola *et al.* (2020) restatement of CPIS does not include reallocation matrix for investments originating from Korea and Mexico.

that the portfolio share of country j in country i's external portfolio is proportional to country j's share of the world market capitalization excluding i:

$$\text{Portfolio share}_{ij} = \frac{\text{Market cap}_j}{\text{World market cap} - \text{market cap}_i}.$$

To test this simple theory of bilateral holding, I regress the log CPIS portfolio share against the log market caps of the origin and destination and report results in column (2) of Table 2.1. The theory would predict a coefficient of 1 for the destination market cap, and close to zero for the origin. Column (2) shows that the coefficient for the destination country's market cap is 0.832 (0.039), while that for the origin is statistically indistinguishable from zero. In terms of fit, the size model performs slightly worse than the network portfolio ($R^2 = 0.41$) despite having one more free parameter.

Another alternative theory is that portfolio shares are determined by informational or physical frictions, which I proxy using the geographical distance between the origin and destination country. Column (3) shows that distance by itself does not explain bilateral portfolios ($R^2 = 0.01$), despite being a significant predictor. The poor fit indicates that geographical distance may not be the "right" measure for frictions in asset trade. Column (4) shows the combination of market caps and distance. This model actually does better than the network model ($R^2 = 0.53$). Controlling for market caps, distance now has a large negative coefficient (-0.8).

In the next section, I use a larger set of variables often included in gravity equations in the trade literature to proxy for familiarity and/or frictions, and see whether the network portfolio retains its explanatory power after controlling for these factors.

2.2.3 Network portfolio in a gravity model for assets

Several authors have suggested estimating a gravity equation for assets that is similar to the popular gravity model in the international trade literature.[4] Portes and Rey (2005) empirically

[4]For a review of the gravity model in trade, see Anderson (2011).

	Dependent variable: CPIS portfolio share			
	(1)	(2)	(3)	(4)
Network portfolio share	1.179			
	(0.054)			
Market cap, origin		-0.074		0.0658
		(0.036)		(0.034)
Market cap, destination		0.832		1.0186
		(0.039)		(0.037)
Distance			-0.229	-0.8020
			(0.074)	(0.057)
Constant	0.012	-1.866	-1.446	1.073
	(0.098)	(0.033)	(0.262)	(0.203)
Adj. R-squared	0.45	0.40	0.01	0.53
Number of covariates	2	3	2	4
AIC	1655	1717	2082	1551
Number of observations	720	720	720	720

Table 2.1: *Regression results for three models: network, size, and distance*

Column (1) reports the regression result of data portfolio share (CPIS) on network equity portfolio share as predicted by the theory (construction detailed in section 2.2). The model is compared against two alternative simple models: "size" and "distance", reported in corresponding columns (2) and (3). Numerical variables are in log form, so coefficients are interpreted as elasticities. Heteroskedasticity-consistent standard errors are given in parentheses.

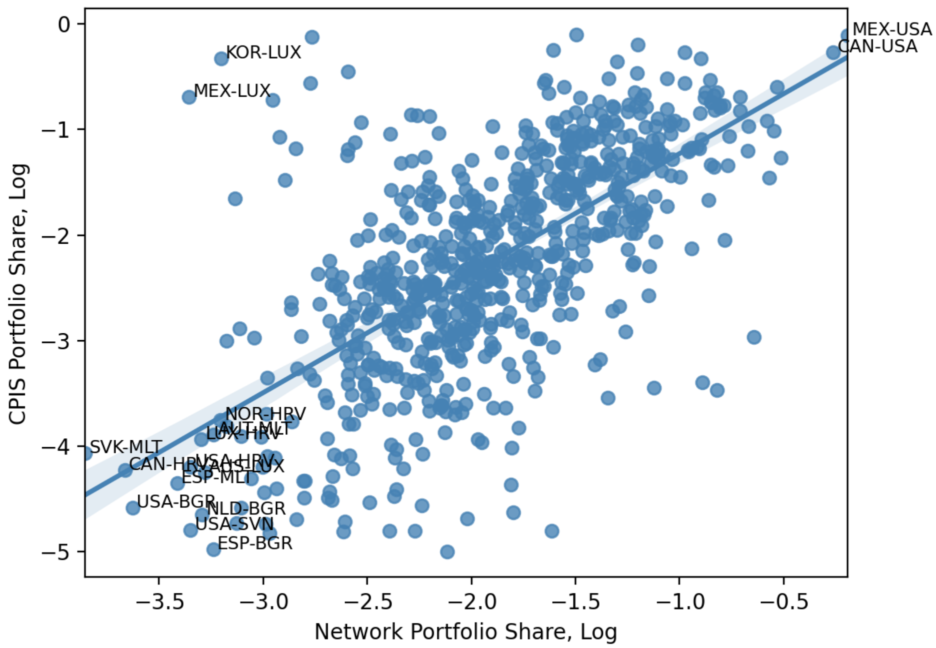

Figure 2.1: *Bilateral portfolio equity shares: network portfolio versus CPIS data*

This figure plots the CPIS portfolio equity shares against the theoretical network shares predicted by the model. Theoretical shares were calculated using WIOD data on international input-output linkages, as detailed in section 2.2. Equity share here is the total equity investment in a particular foreign destination divided by total foreign equity investment by a given origin country. CPIS data has been "restated" to account for indirect investments through tax havens and financial centers using the procedure by Coppola *et al.* (2020). Some points in the scatter plot has been labeled in the format "origin-destination." The adjusted $\boldsymbol{R}$-squared of the corresponding regression including the constant is 0.45, and the regression slope is 1.19 (standard error = 0.05).

estimated a gravity model for equity trade flows[5] of the type:

$$\log\left(\text{Equity flow}_{ij}\right) = \beta_1 m_i + \beta_2 m_j + \beta_3 \tau_{ij} + u_{ij}$$

where m_i is a measure of size (e.g. log of market cap), and τ_{ij} a measure of transaction cost *in the asset market*. While popular, the theoretical underpinning of a gravity model for asset is less clear. In addition, as noted by Chaney (2018), while the effect of size can be explained more easily, it is puzzling why distance has a strong effect on trade flows. Chaney (2018) provided the first theory of the role of distance in gravity equation using a network model of contact acquisition. However, theoretical papers providing foundation for gravity variables in general are scarce.

In terms of gravity models for assets, Martin and Rey (2004) provided a model in which Arrow-Debreu assets are endogenously created and traded internationally. When consumptions in different states are imperfectly substitutable, a gravity model arises for asset trade in the same way it does for good trade. Okawa and van Wincoop (2012) pointed out that the Arrow-Debreu assets traded in Martin and Rey (2004) do not resemble the assets we observe in the data, because international equities often have positive payoffs in the same state of the world. Okawa and van Wincoop (2012) argued that the general structure of the covariance makes the estimating equation for bilateral equity unlikely to have a gravity form.

Despite the lack of theoretical support, the gravity model does have explanatory power for international equity holdings. In Table 2.2, I compare the network portfolio presented in this paper with the gravity model. Column (1) reports the baseline regression result in Table 2.1. Column (2) reports a gravity model that includes market caps, distance, and proxies of trade frictions such as contiguity, common language, historical dependence (colony). I also include two dummies for the US as an origin or destination due to its special role in international finance, as well as a measure of the current capital stock (data from the Penn World Table). The inclusion of the capital stock is due to the classical Lucas argument that capital should flow to where its marginal product is highest (i.e. countries where capital is scarce).

In terms of goodness-of-fit, the gravity model ($R^2 = 0.57$) explains the the data better than the

[5] As opposed to explaining asset holdings, a stock concept.

network-only model ($R^2 = 0.45$), at the cost of parsimony. Market caps have similar coefficients to the size-only model (column (2) of Table 2.1), i.e. a coefficient near 1 for the market cap of the destination and zero for that of the origin. Other significant variables include distance, contiguity, joint EU membership, US dummies, and whether the origin was a colony of the destination country. The capital stock of the destination country enters with a negative coefficient, suggesting evidence for the North-South investment motive (higher weights on capital-scarce destinations).

Column (3) tests whether the network portfolio has explanatory power above and beyond what can be explained by a gravity model. We see that controlling for gravity variables, the network portfolio share remains a significant predictor. The coefficient on network portfolio share is 0.846, with a standard error of 0.103. The p-value for the hypothesis that the coefficient on network portfolio share is 1 is 0.166, implying we cannot reject the null hypothesis that it is 1 at 5% level. A partial regression of network portfolio shares on CPIS portfolio shares, residualizing out the gravity variables, has an R^2 of 0.096.

The partial regression plot is shown in Figure 2.2, with residualized network portfolio on the x-axis and the data portfolio on the y-axis. This plot shows that the network portfolio can predict the high investment for country pairs such as Czech Republic → Slovakia, or the low investments from Hungary to Greece, that cannot be explained by gravity factors alone.

As before, the obvious outliers are the high investment shares into Luxembourg (from Korea, Mexico, Turkey) that cannot be explained through the production structure. Similarly, we see that the model over-predicts investment into China, but likely this is due to the CPIS data set underestimates Chinese liabilities due to Chinese firms' issuance through infrastructures abroad (Coppola *et al.*, 2020). Recall that the reallocation matrix is not available for all countries. As a result, the current restatement of the CPIS data set is only a partial fix, and part of the "residency vs. nationality" problem remains.

It is also worth noting that the model taken literally implies that once the estimated network portfolio share is held fixed, the other explanatory variables should have coefficients of zero. This hypothesis is rejected for some variables individually (particularly for the market cap of the destination country), and for all non-network portfolio variables jointly (p-value = 0.000). The significance of gravity variables after controlling for the network portfolio share can be

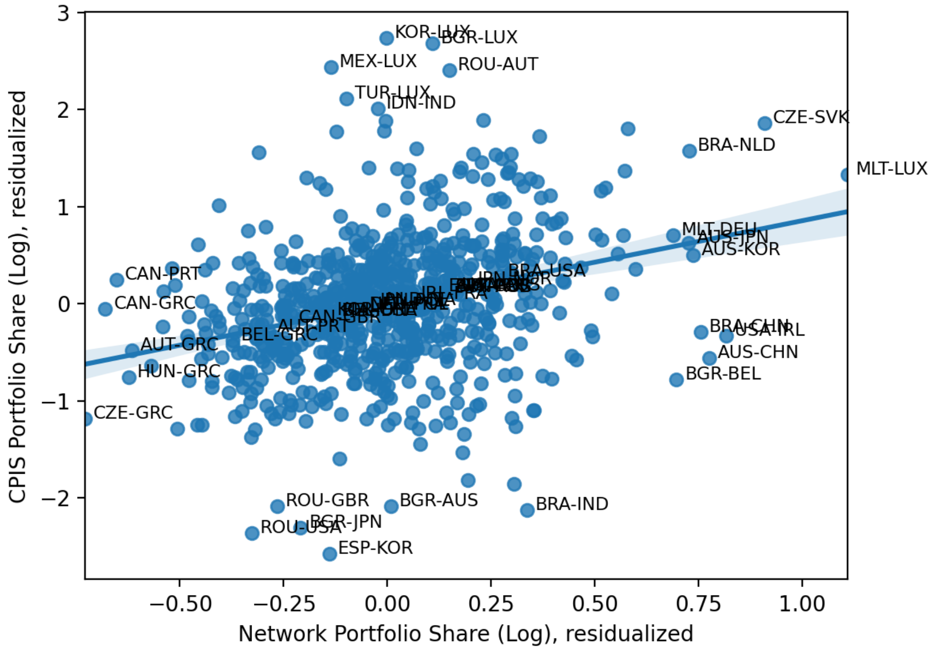

Figure 2.2: *Partial regression plot: network portfolio versus CPIS data*

This figure is the partial regression plot of CPIS portfolio equity shares against the theoretical network shares predicted by the model. Both variables were residualized by gravity variables (market caps, common language, contiguity, etc.). Notable points in the scatter plot have been labeled in the format "origin-destination." The partial $\boldsymbol{R}$-squared associated with this graph is 0.096.

due to possible measurement errors for the network portfolio shares, which can come from the measurement errors in the input - output data. Another reason is that the gravity variables proxy for information frictions that are not present in the model.

2.2.4 Effect of distance on asset holdings

Distance is a significant predictor of bilateral asset holdings, as shown in column (2) of Table 2.2. The coefficient (elasticity) of log distance is -0.45, which is slightly smaller than previous estimates.[6] However, presuming other covariates have proxied well for informational cost, it

[6] Portes and Rey (2005) estimates a coefficient in the range of $(-0.9, -0.5)$ for distance, depending on specifications.

	Dependent variable: CPIS portfolio share		
	(1)	(2)	(3)
Network portfolio share	1.179		0.846
	(0.054)		(0.103)
Market cap, origin		0.0995	0.124
		(0.062)	(0.057)
Market cap, destination		1.2124	0.917
		(0.073)	(0.081)
Distance		-0.4548	0.142
		(0.089)	(0.107)
Contiguity		0.3465	0.177
		(0.099)	(0.094)
Joint EU membership		0.3325	0.335
		(0.069)	(0.064)
Common language		0.0720	0.0398
		(0.062)	(0.060)
Origin = US		0.5255	0.4411
		(0.109)	(0.104)
Destination = US		0.2684	0.4775
		(0.104)	(0.101)
Colony of origin ever		0.2069	0.1822
		(0.265)	(0.246)
Colony of destination ever		0.6623	0.3507
		(0.296)	(0.331)
Capital stock, origin		-0.1504	-0.2885
		(0.079)	(0.073)
Capital stock, destination		-0.3776	-0.8100
		(0.112)	(0.114)
Constant	0.012	3.1239	6.175
	(0.098)	(0.858)	(0.844)
Adj. R-squared	0.45	0.57	0.61
Number of covariates	2	13	14
AIC	1655	1494	1422
Number of observations	720	720	720

Table 2.2: *Regression result of network model versus gravity model*

Column (1) reports the baseline regression result of data portfolio share (CPIS) on network equity portfolio share (detailed in section 2.2). Column (2) regresses CPIS equity share against covariates often included in a gravity equation. Column (3) adds the network portfolio to the gravity model to show that network portfolio has explanatory power beyond the gravity factors. Numerical variables are all included in logs, so coefficients are interpreted as elasticities. Heteroskedasticity-consistent standard errors are given in parentheses. The p-value for the hypothesis that the coefficient of network portfolio share equals 1 is 0.166.

remains puzzling why physical distance matters at all for trade in assets.

Adding the network portfolio to the gravity regression resolves the distance puzzle. Column (3) of Table 2.2 shows that the effect of distance becomes statistically insignificant when network portfolio is added. Not only distance, gravity variables that proxy non-geographical distances such as contiguity or colonial history also no longer have an effect on bilateral asset holding (EU membership and common language still have a significant effect).

In summary, the empirical result in Table 2.2 shows that the effect of distance on international trade in assets is purely through its effect on trade in goods. Holding constant the trade structure, distance by itself does not have an effect on asset holdings.

2.2.5 Role of indirect linkages

In this section, I test for the role of indirect trade linkages in determining optimal equity portfolios by controlling for the "direct-trade-only portfolio." The optimal equity holding in the model is a nonlinear function of the trade matrices (Ω, Ξ). In particular, performing a Taylor expansion on the optimal equity holding matrix Λ gives:

$$\Lambda = \theta M [\mathbf{I} - \alpha M]^{-1} = \theta M + \theta \alpha M^2 + \text{higher order terms}$$

The matrix of international Domar weights is in turn given by:

$$M = (1-\gamma)\left[\mathbf{I} - \gamma \Omega'\right]^{-1} \Xi' = (1-\gamma)\Xi' + (1-\gamma)\gamma\Omega'\Xi' + \text{higher order terms}$$

So, to the first-order, optimal equity holding Λ is simply proportional to the transpose of the consumption matrix Ξ.[7] This portfolio has the portfolio decision rule $\Lambda_{ij} = \Xi'_{ij} = \Xi_{ji}$, so that the portfolio share of j in i's portfolio is given by:

$$\text{Direct-trade-only portfolio}_{ij} = \frac{\Xi_{ji} P^S_j}{\sum_{k \neq i} \Xi_{ki} P^S_k},$$

where again P^S denotes the steady state stock market cap.

For completeness, I also run a regression controlling for a "model-free" variable of bilateral

[7] This first-order term is the exact result, not approximation, in a model without intermediate inputs ($\gamma = 0$).

trade linkage. Let trade be the sum of imports and exports. The trade share between origin country i and destination j is given by:[8]

$$\text{Trade share}_{ij} = \frac{\text{Trade}_{ij}}{\sum_k \text{Trade}_{ik}}. \tag{2.4}$$

This measure of trade share also proxies for "familiarity," i.e. the hypothesis that a country's investors tend to be more familiar to their large trade partners.

The result is presented in Table 2.3. Column (1) shows again the result of the baseline regression (2.3). Column (2) shows the performance of the model where only direct trade is concerned. This portfolio explains the data less well ($R^2 = 0.36$ versus the network model's $R^2 = 0.45$). Column (4) shows that the network portfolio is still a significant predictor after controlling for the direct-trade-only portfolio. The coefficient of the direct-trade-only portfolio, however, turns negative due to the relatively high correlation between the two measures. Column (3) of Table (2.3) shows that a model-free model of trade share explains the data relatively well ($R^2 = 0.42$, with a trade-share elasticity of 1.03). However, this measure ceases to be a significant predictor when the network portfolio is added to the regression, as shown in column (5).

In sum, this section shows that indirect trade linkages, not just direct trades, matter in explaining international equity portfolios. This result highlights the need to account for intermediate inputs and their trade networks in theoretical and empirical works.

2.2.6 Explaining Home bias

I now turn to evaluate the model's predictive power on equity home bias.

I measure equity home bias (EHB) in the data as the share of country equity portfolio made up by domestic assets:

$$\text{Data EHB}_i = \frac{\text{Domestic equity investment}_i}{\text{domestic equity investment}_i + \text{external equity investment}_i}$$

The external equity component of a country's portfolio is calculated simply by summing its portfolio investments across destinations using CPIS data. However, CPIS does not include the

[8] This is similar to the variable "trade intensity" often used in the trade literature, where the denominator of (2.4) is instead the sum of the two countries' GDPs.

	Dependent variable: CPIS portfolio share				
	(1)	(2)	(3)	(4)	(5)
Network portfolio share	1.179			1.7579	1.1933
	(0.054)			(0.163)	(0.207)
Direct-trade-only portfolio		0.9038		-0.5245	
		(0.054)		(0.145)	
Trade share			1.0321		-0.0133
			(0.053)		(0.189)
Constant	0.012	-0.4019	-0.1976	0.0511	0.0127
	(0.098)	(0.102)	(0.097)	(0.095)	(0.098)
Adj. R-squared	0.45	0.36	0.42	0.47	0.45
Number of covariates	2	2	2	3	3
AIC	1655	1767	1700	1640	1657
Number of observations	720	720	720	720	720

Table 2.3: *Explanatory power of network portfolio after controlling for direct trade measures*

Column “Network” reports the regression result of data portfolio share (CPIS) on network equity portfolio share as predicted by the theory. This portfolio is constructed in section 2.2 using WIOD data. All numerical variables are in log form. The variable “direct-trade-only portfolio” is the portfolio constructed by ignoring intermediate input-output linkages and looking at direct imports only. The variable “trade share” is calculated by bilateral trade between two countries divided by total trade of the origin country.

value of domestic equities held by domestic investors. I calculate this object as the total equity market cap less equity held by foreigners:

$$\text{Domestic equity investment}_i = \text{Market cap}_i - \text{Equities held by foreigners}_i.$$

I smooth out the time series of equity home bias by calculating a 5-year moving-average for each year in the sample. As the theory put forth in this paper focuses more on longer-run changes, smoothing out the time series helps focus more on the trend of the data and less of transitory movements.

Furthermore, some entries in CPIS corresponding to investments into tax havens actually represent investment into another country's equities, not the tax havens themselves. As a result, some market cap measure is less than the amount of equities held by foreigners, leading to a negative domestic equity investment value. Given the lack of data with higher quality, these countries will be dropped from the analysis.[9]

Figure 2.3 plots the equity home bias in the data for five major economies: US, Japan, France, Germany, and Brazil. We can see that there is a large heterogeneity in levels of equity home bias. France and Germany tend to be more diversified than larger economies like US or Japan. We can also see that the level of asset home bias has decreased across the board (with the exception of Brazil). In fact, in my sample, the average home equity bias has declined from 0.88 in 1997 to 0.77 in 2013.

I will perform two exercises. First, I compare the *level* of equity home bias in the data with the theory prediction. Again, let Λ be the optimal equity portfolio given in Proposition 1 and $\boldsymbol{P}_S^i$ the stead-state stock price (market cap). The theoretical equity home bias is given by:

$$\text{Theory EHB} = \frac{\Lambda_{ii} P_S^i}{\sum_j \Lambda_{ij} P_S^j},$$

The regression in level is:

$$\text{Data EHB}_i = \alpha + \beta \cdot \text{Theory EHB}_i + \varepsilon_i, \tag{2.5}$$

where i denotes country.

[9]The dropped countries are Cyprus, Bulgaria, Malta, Ireland, Luxembourg, Cayman Islands, and Portugal.

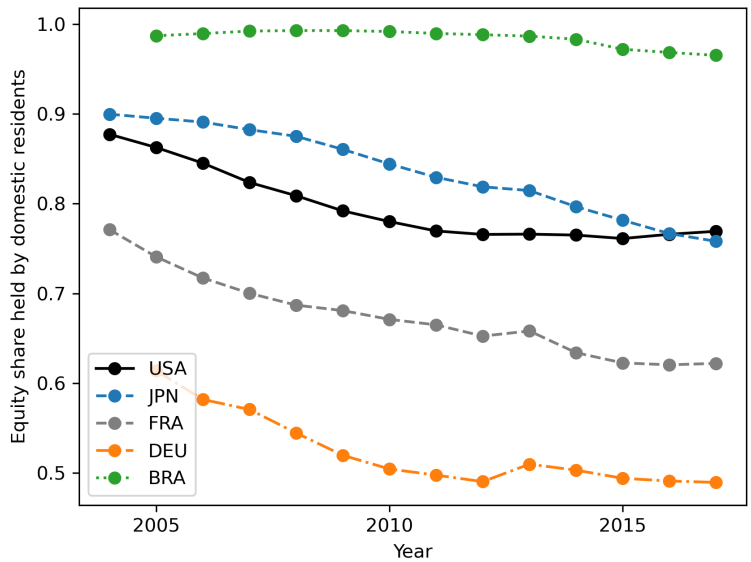

Figure 2.3: *Declining asset home bias in the data*

Second, I test whether changes in the global trade structures, reflected in the network portfolio, can explain the decline in asset home bias across countries. To this end, I run the regression:

$$\Delta \text{Data EHB}_{2005-2014} = \alpha + \beta \cdot \Delta \text{Theory EHB}_{2005\text{-}2014} + \varepsilon_i$$

Changes are taken from 2005 to 2014, the beginning and end year of the smoothed home bias time series.

Table 2.4 reports the results of regressing data home bias on the theory's prediction.

The first column of Table 2.4 reports the cross-country regression in *level* using data of the year 2010. The theory equity home bias level is a significant predictor of the data, with adjusted $R^2 = 0.14$. The level regression (2.5) is run for each year of the samples, and the result for the year with median R^2 (2010) is reported. The adjusted R^2 and β for other years is plotted in figure 2.5. The theory EHB is a significant predictor for most years in the sample.

Figure 2.4 visualizes the fit of the model. The model predicts a US EHB of 77%, very close to the level 0.8 observed in the data. Many countries lie near the 45-degree line. Larger, less open economies (Japan, Brazil, Australia) have a higher Domar weight for home market and are predicted to have a higher EHB level than more open economies such as Belgium or Austria.

	Dependent variable: Equity home bias, data	
	Level, 2010	Changes, 2005-2014
Equity home bias, theory	0.815	0.623
	(0.314)	(0.250)
Constant	0.281	-0.024
	(0.201)	0.007
Adj. R-squared	0.14	0.19
Number of observations	24	17

Table 2.4: *Regression results for equity home bias, theory vs. data*

This table reports the regression result of data equity home bias versus that predicted by the network model. Data from CPIS and WIOD, with author's calculations. Column "Level, 2010" refers to the regression in level (2010 is the year with median R-squared), and column "Changes, 2015-2014" reports the corresponding regression in log changes.

This prediction is borne out in the data. The model under-predicts EHB for a group of emerging markets (e.g. Mexico, Turkey, Korea, Romania, India), reflecting that there are relevant frictions not accounted for by the model.

Turning focus now to the regression in changes, we see in the second column of Table 2.4 that the model can also predict the changes in EHB that have occurred in the data. The R^2 in that regression is 0.19, and the theory EHB predictor is significant at the 5% level. Figure 2.6 shows the fit of this regression. Countries that have reduced their demand exposure to domestic market faster (e.g. Belgium, Japan) are predicted to have a larger decline in EHB, as opposed to countries that have not opened up their trade structure (e.g. Indonesia).

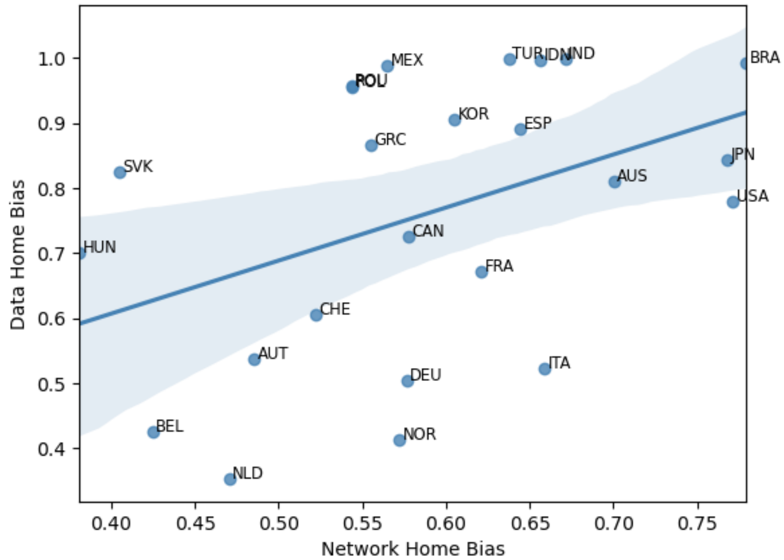

Figure 2.4: *Model fit of asset home bias in level.*

This scatterplot visualizes the fit of the *level* of equity home bias predicted by the network model versus that observed in the data. This plot uses the data of the year with the median R^2 in the level regression (year 2010). Data: CPIS, WIOD, and author's calculations.

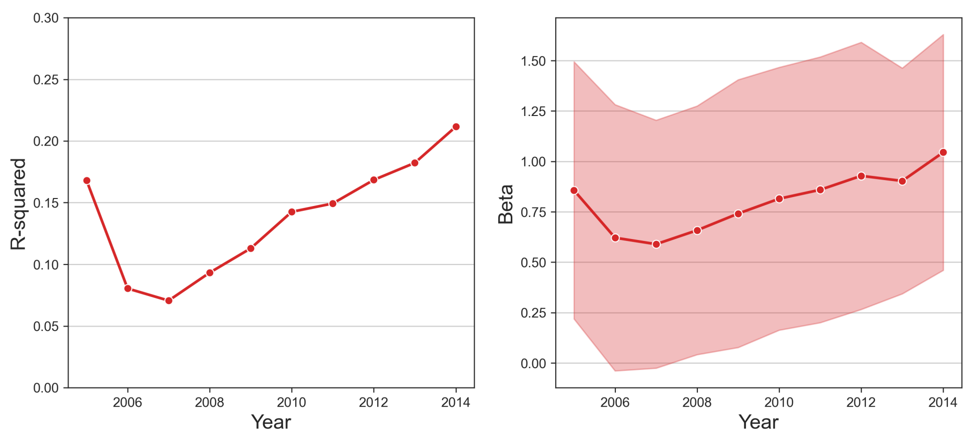

Figure 2.5: R^2 *and* β *of Equity Home Bias level regression for multiple years*

This figure plots the R^2 (left panel) and β (right panel) of the regression of the *level* of equity home bias predicted by the network model versus that observed in the data for each year in the sample. The shaded area in the right panel represents 95% confidence intervals. Data: CPIS, WIOD, and author's calculations.

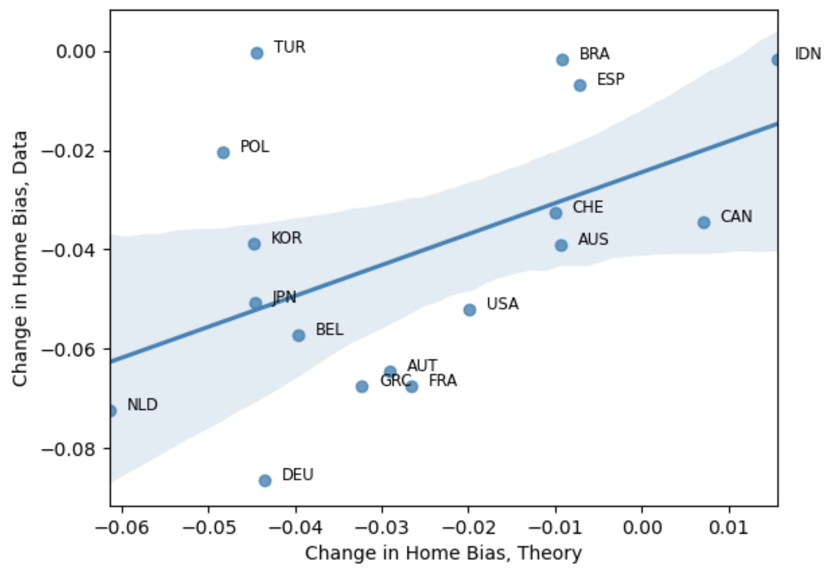

Figure 2.6: *Model fit of asset home bias in changes (2005-2014).*

This scatterplot visualizes the fit of the changes in equity home bias as predicted by the theory versus that observed in the data. Data: CPIS, WIOD, and author's calculations. Norway and Italy are obvious outliers and have been removed from the analysis.

Chapter 3

A Behavioral-Attention Phillips Curve: Theory and Evidences from Inflation Surveys[1]

3.1 Introduction

The existence of a Phillips Curve, a classic relationship between inflation and output gap, has been questioned in recent years. In particular, it has become near consensus that the Phillips Curve slope has been declining since the early 1980s. Consequently, the Phillips Curve does not seem to perform well out-of-sample: a constant-slope Phillips Curve estimated using US data before the Great Recession predicts counterfactually large disinflation after the 2008-2009 Financial Crisis. Essentially, unemployment after 2008 was extremely high, yet inflation does not fall nearly as much, a phenomenon called the "Missing Disinflation puzzle" by Coibion and Gorodnichenko (2015).[2] Another puzzling phenomenon of inflation dynamics in recent years is

[1] I want to extend my gratitude to Xavier Gabaix, Ken Rogoff, Gita Gopinath, Greg Mankiw for their valuable guidance on this project. I also want to thank Michael Woodford, Andrew Caplin, Robert Barro, Masao Fukui, Maria Voronina, Taehoon Kim, seminar participants at Harvard Macro Lunch and the 2019 Sloan-Nomis Conference for useful comments and discussions.

[2] Talking about the Missing Inflation Puzzle, the former Federal Reserve chairwoman Janet Yellen said:

> "Now, I recognize and it's important that inflation has been running under our 2 percent objective for a number of years, and that is a concern, particularly if it were to translate into lower inflation expectations. [...] This year, the shortfall of inflation from 2 percent, when none of [low energy price

the anchoring of inflation (or inflation expectations), first popularized by Bernanke (2007) and Blanchard *et al.* (2015).

This paper proposes propose a tractable theory of endogenous attention and uncertainty that can jointly explain the two puzzles. In particular, I derive a Behavioral Attention Phillips Curve (BAPC) whose slopes on the output gap and inflation expectations decline when inflation is less uncertain:

$$\pi_t = \omega_t^d \pi_t^d + \left(1 - \omega_t^d\right) \beta \mathbb{E}_t[\pi_{t+1}] + \kappa_t x_t \tag{3.1}$$

where ω_t^d is the weight on default inflation and κ_t the output gap slope.

Define inflation uncertainty as $\sigma_{\pi,t}^2 = \mathbb{E}_t\left[\left(\pi_t - \pi_t^d\right)^2\right]$, the variance of inflation deviation from the firms' default level. I show that as inflation uncertainty declines, the output gap slope κ_t is lower (Phillips Curve flattening) while β_t^b rises, i.e. firms lean more heavily on their default inflation (well-anchored expectation). Gabaix (2016) developed a tractable New Keynesian Phillips Curve with constant, exogenous attention. The contribution of this paper is to allow for endogenous attention and uncertainty, as well as using the device to study the empirical changes of the Phillips Curve.

The intuition is relatively straightforward. Firms try to price as close as possible to the optimal rational level that a frictionless firms would set. Knowing this rational price requires information about current demand and economic activity (output gap), as well as past, current, and future inflations (if prices are sticky). In the real world, these variables are not free information, and paying attention to them is costly. Misperceiving these variables cause firms to set a sub-optimal price that is different from the rational price. With costly attention, the firms have to weigh the benefit of setting the correct price versus having to pay the cost of attention.

The benefit of attention is lower when inflation uncertainty is low. Therefore, during times of low inflation uncertainty, firms pay little attention to monetary shocks and aggregate variables. As a result, firms that can change their price react less to shocks. The dampened price response

> or cheaper import price due to dollar appreciation] is operative, is more of a mystery, and I will not say that the Committee clearly understands what the causes are of that." - Janet Yellen, September 2017 FOMC meeting press conference.

It is fair to say that understanding the cause of low inflation has been a central question faced by Central Banks (CB) in recent years.

flattens the Phillips Curve, because there is relatively less inflation response and more output response to nominal shocks. Similarly, inflation becomes more anchored with low uncertainty because costly attention motivates firms to rely more on "rules-of-thumb" such as the 2% inflation target.

In Section 3.4, I provide empirical supports for the Behavioral-Attention Phillips Curve (BAPC). To implement the BAPC empirically, I show that the output gap slope κ depends linearly on inflation uncertainty when the cost of attention takes the linear or quadratic form. This allows me to run a regression of the following form:

$$\pi_t = \alpha + (\kappa_0 + \kappa_1 \sigma^2_{\pi,t})\text{Output Gap}_t + (\xi_0 + \xi_1 \sigma^2_{\pi,t})\pi_{t-1} + \varepsilon_t \tag{3.2}$$

While we cannot directly observe the attention level of the economy, inflation uncertainty is more directly observable, and I will draw proxies for $\sigma^2_{\pi,t}$ from inflation surveys.

It is worth noting that the traditional PC is regression (3.2) with the restrictions $\kappa_1 = \beta_0 = \beta_1 = 0$. Using US inflation dynamics since 1979 (when inflation surveys start being available), I show that both κ_1 and ξ_1 are significant and positive, supporting the theory. The BAPC also performs better in an out-of sample exercise than a traditional Phillips Curve, as shown in Figure 3.1.

The construction of inflation uncertainty measure $\sigma^2_{\pi,t}$ here is paramount to our statistical test. For this purpose, I constructed five different measures of inflation uncertainty. Among which, a main measure is the variance of inflation forecasts among households in the Michigan Survey of Consumers (MSC). This is a disagreement-based measure of inflation uncertainty. However, since disagreement may not necessarily imply uncertainty, I construct another subjective uncertainty-based measure using the probability bin forecasts from the Survey of Professional Forecasters. From the probability distribution forecasts of a cross-section of professional forecasters, I construct an entropy statistic which indicates how uncertain forecasters are on average at a given quarter. I will show that the empirical findings in support of the BAPC are robust to the choice of inflation uncertainty.

Theoretically, my paper departs from previous literature in two main ways. First, firms in my framework only watch variables that are directly important to them, which are endogenous state

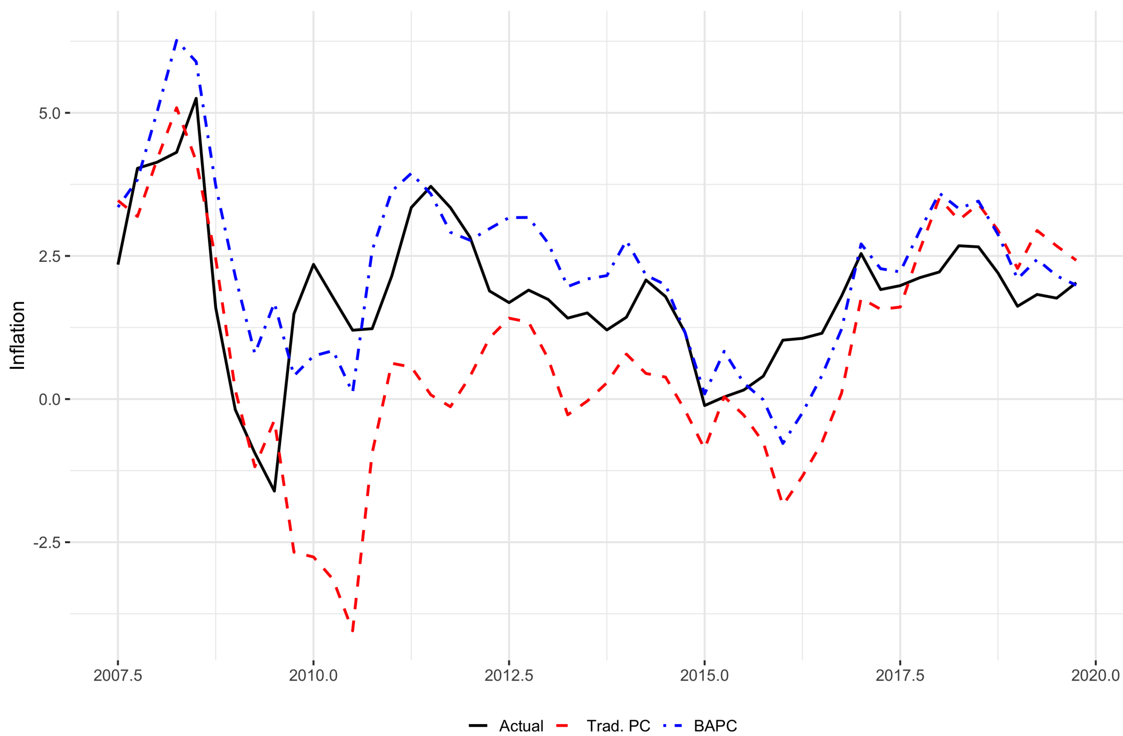

Figure 3.1: *Out-of-sample performance of the Behavioral-Attention Phillips Curve (BAPC) versus the traditional Phillips Curve*

Both curves were fit using data from 1970-2007Q3, and used to predict inflation out of sample for the period 2007Q3-now. The traditional Phillips Curve predicts a large deflation in 2010, which did not occur (the Missing Deflation Puzzle). The BAPC, on the other hand, did not make that prediction. The root mean square error, used to gauge out-of-sample performance of model, is roughly 2% for the traditional model, and only 1% for BAPC.

variables such as output and price, instead of the underlying structural shocks (e.g. monetary policy shocks). Second, I assume agents exhibit behavioral inattention: they take actions thinking the simplified economy (due to imperfect attention) is the actual truth.

The first departure is important to separate between two concepts: volatility and uncertainty. Uncertainty about a variable (say, inflation) is the product of the volatility of underlying structural shock and the pass-through from the shock to the variable. For example, suppose that inflation depends solely on monetary shock ε_t^{ν}, which is normally distributed with mean zero and variance σ_{ν}^2. Let $\phi_{\pi\nu,t}\varepsilon_t^{\nu}$ denotes the pass-through of monetary shocks into inflation $\pi_t = \phi_{\pi\nu}\varepsilon_t^{\nu}$, then inflation uncertainty is given by

$$\text{var}(\pi_t) = \phi_{\pi\nu}^2 \sigma_{\nu}^2 \tag{3.3}$$

The pass-through, of course, depends on how steep the Phillips Curve currently is. If the economy is relatively money-neutral (i.e. Phillips Curve is steep), then monetary shock fully passes through to inflation while having no effect on output. When the Phillips Curve is flat, on the other hand, monetary shock affects inflation very little.

Thus, even when monetary policy is volatile (high σ_{ν}^2), but the Phillips Curve is currently flat ($\phi_{\pi\nu,t}$ is low), then the product $\phi_{\pi\nu,t}^2 \sigma_{\nu}^2$ is also low. A monetary shock by central bank would not increase uncertainty about inflation as much. This confirms to firms that they should not pay attention to inflation, which lead to small price-resetting, and reinforcing the flattened Phillips Curve. The feedback loop is demonstrated by Figure (3.2).

This is important for us to understand why inflation has been low in recent years. One might ask why is the Phillips Curve flat after the financial crisis when Central Banks around the world have been very active in implementing their monetary policies (like Quantitative Easing programs). The model in this paper will say that because inflation uncertainty is already low up until the crisis, attention to prices are low, which imply that monetary stimulus would only have an effect on output and not on inflation, reinforcing the low inflation uncertainty.

Given the complementarity between attention and optimal pricing, I show in section 3.3.3 that multiple equilibria arise with medium volatility. I propose a novel policy paradox for the Central Bank: to raise inflation in a quiet, low-volatility period, monetary policy needs to be sufficiently volatile to retain firms' attention and a meaningful Phillips Curve (volatility of Taylor rule). Yet,

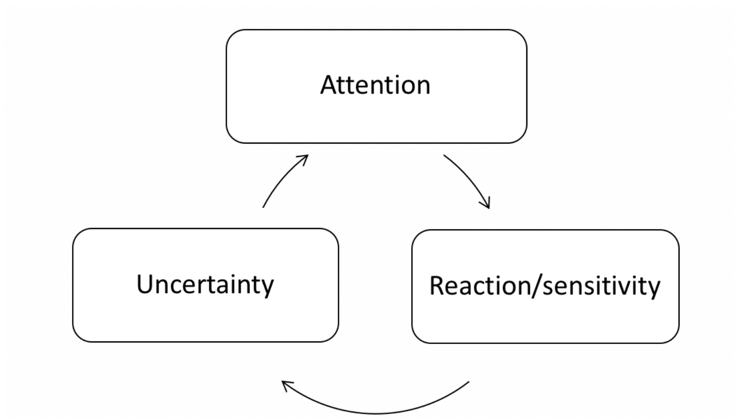

Figure 3.2: *Endogenous relationship between uncertainty, attention, and equilibrium sensitivity/reaction*

at the same time, the Central Bank needs to sufficiently aggressive in targeting inflation to retain determinacy (slope of Taylor rule). These two objectives are hard to accomplish at the same time if firms care only about inflation and not the Taylor rule.

Section 2 reviews the literature. In section 3.3, I provide the general set up and derive and BAPC for both the flexible prices and rigid prices cases. Section 3.3.3 studies policy implication and the paradox for Central Bank. Section 3.4 provides empirical evidence for the BAPC. Section 3.5 concludes.

3.2 Literature Review

This paper relates to a large literature that tries to estimate the Phillips Curve empirically. Gordon (2011) provides a good survey of the Phillips Curve, and Mavroeidis *et al.* (2014) gives a comprehensive survey for empirical estimation of the Phillips Curve using various estimation methods (single-equation, full system with GMM-VAR, etc.). The resulting estimates of the slope coefficient are often wide-ranging and very dependent on the data periods chosen for estimation. Coibion and Gorodnichenko (2015) uses the Michigan Consumer Survey as inflation expectation in the Phillips Curve, but still find it generally hard to fit the Phillips Curve over a long time period. The non-robustness of these estimates likely stem from omitting variables, such as attention and uncertainty. This was an original motivation for this paper, which aims to make the Phillips Curve robust by not having omitted variable.

My theoretical model adds to this literature by providing a framework that carries many of the new micro data stylized facts on the behavior of forming expectation. The agents in this paper only observe partially the variables, thus having the delayed information property that is mentioned in Mankiw and Reis (2002). As a result, the mean ex-post forecast error of our sparse agents is predictable by mean forecast revisions, a recently established stylized fact (Coibion (2016)) paper). Furthermore, the nowcast prediction error is serially correlated, a feature present in our framework.

To my best knowledge, this is one of the few papers that derived a time-varying Phillips Curve. The first paper is Vavra (2014), which is based on the menu-cost and time-varying volatility of idiosyncratic shock. My paper, in contrast, would focus on the uncertainty about aggregate endogenous variables. I believe the first approach would dictate that the Phillips Curve be steep after the financial crisis due to rising volatility of idiosyncratic shock; but the PC remained flat during this period.

The noisy-information approach pioneered by Lucas (1973), with continuing work by Woodford (2001), Angeletos and Lian (2018), also has the feature that the Phillips Curve tends to be steep whenever monetary shock is volatile. However, this strand of the literature did not derive a Phillips Curve with time-varying slope that one can test using the data as the BAPC. Secondly, this literature would also imply that the Phillips Curve should be steep after the Great Recession. I discuss this point further in Section (3.3).

Finally, this paper benefits from the tractable formulation of behavioral inattention pioneered by Gabaix (2014), Gabaix (2016). Gabaix (2016) focuses on a different set of questions than this paper (fiscal policy, determinacy, forward guidance puzzle, etc.), and thus has a constant-slope (New Keynesian) Phillips Curve. I add to this literature by using behavioral inattention to endogenize time-varying attention, which allows us to speak on issues such as the flattening of the Phillips Curve, time-varying effect of monetary policy, so on.

I first present the main working model in Section 3.3. There I detail how I treat attention and uncertainty. In section 3.3, I also present the main theoretical results, which include the BAPC and a theorem about multiple equilibria. In Section 3.4, I provide empirical supports for the BAPC. Section 3.5 concludes.

3.3 Model

The model follows the standard New Keynesian structure where firms choose optimal prices to maximize profit in the presence of nominal rigidity à la Calvo. The additional key ingredient here is that paying attention to the aggregate price and output is costly; therefore, firms do not always have up-to-date information about these variables. The firms have to first choose an optimal attention level before optimizing conditional on the information obtained under that attention level. The attention problem is formulated similar to Gabaix (2014) and Gabaix (2016). The key difference is that the second moments of this economy are endogenous and time-varying, which will generate time-varying attention levels and time-varying slopes for the Phillips Curve.

3.3.1 Environment

Consumers

There is a representative household who supplies L units of labor, consumes a CES basket of differentiated goods C, and maximizes their lifetime utility:

$$U_t = \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \left(\frac{C_t^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} - \frac{L_t^{1+\varphi}}{1+\varphi} \right). \tag{3.4}$$

There is a continuum of firms, denoted by $\omega \in [0,1]$, that produce differentiated products. The aggregate consumption good is given by:

$$C_t = \left[\int_0^1 C_t(\omega)^{1-\frac{1}{\varepsilon}} d\omega \right]^{\frac{\varepsilon}{\varepsilon-1}}.$$

The household can save in a risk-free nominal bond which costs Q_t and pays 1 dollar per unit at date $t+1$. Denote by B_t the bond position, W_t the aggregate wage, Π_t the firm profits, and T_t the lump-sum transfers from the government. The household's budget constraint is given by:

$$\int_0^1 P_t(\omega) C_t(\omega) d\omega + Q_t B_t \leq B_{t-1} + W_t L_t + T_t + \Pi_t, \quad t \geq 0 \tag{3.5}$$

As usual, the household faces a No-Ponzi condition $\lim_{T\to\infty} \mathbb{E}_t \exp\left(-\int_t^T r(\tau) d\tau\right) B_T \leq 0$.

Firms

Firm $\omega \in [0,1]$ has a decreasing-to-scale (DRS) production technology:

$$Y_t(\omega) = A_t L_t(\omega)^{1-\alpha}. \tag{3.6}$$

The (log) aggregate TFP follows an AR(1) structure: $a_t = \rho_a a_{t-1} + \varepsilon_t^a$, where $\varepsilon_t^a \sim \mathcal{N}\left(0, \sigma_a^2\right)$. The only factor of production is labor, which is rented from a competitive market at wage W_t. The firm receives a payroll subsidy τ_w, which the government uses to remove equilibrium inefficiency arising from monopolistic market power.

I assume nominal rigidities à la Calvo: in each period, only a fraction θ of firms can reset its price. Firms face a demand curve $Y(\omega) = (P(\omega)/P)^{-\varepsilon} Y$ when the aggregate price and output are P and Y, respectively. When attention is costless, the firm chooses an optimal price P_t^{rat} to maximize its expected stream of profit until the next reset opportunity:

$$P_t^{rat} \equiv \arg\max_{P^*} V(P^*) = \sum_{s=0}^{\infty} \theta^s \mathbb{E}_t \left\{ \Theta_{t,t+s} \left(P^* Y_{t+s|t}(P^*) - \Phi_{t+s}\left(Y_{t+s|t}(P^*) \right) \right) \right\} \tag{3.7}$$

$$s.t. \quad Y_{t+s|t}(P^*) = Y_{t+s}\left(P^*/P_{t+s}\right)^{-\varepsilon} \tag{3.8}$$

where $\Theta_{t,t+s} \equiv \beta^s (C_{t+s}/C_t)^{-\frac{1}{\sigma}}$ is the Stochastic Discount Factor and $\Phi_t(Y) = (1-\tau_w) W_t (Y/A_t)^{\frac{1}{1-\alpha}}$ is the date-t cost function of a firm that produces output Y.

I define the natural output y_t^n as the level of output that would prevail when there is neither nominal friction nor costly attention. Let $x_t \equiv y_t - y_t^n$ denote the output gap, defined as the deviation of aggregate output from the natural output. Up to a first-order approximation around the perfect foresight zero inflation steady state, Appendix C.1 shows that the (log) optimal rational reset price is given by:

$$p_t^{rat} = \left(1 - \beta\theta\right) \sum_{s=0}^{\infty} \left(\beta\theta\right)^s \mathbb{E}_t \left[p_{t+s} + \lambda \psi_x x_{t+s} \right],$$

where $\lambda \equiv \frac{1-\alpha}{1-\alpha+\alpha\varepsilon}$ and $\psi_x = \frac{\sigma^{-1}(1-\alpha)+\varphi+\alpha}{1-\alpha}$.

Intuitively, firms, facing nominal rigidities, want to set their prices at a markup over their expected average of today and future marginal costs. Forming expectations about these marginal costs requires forming expectations about the future aggregate price levels and output gaps

$\{p_{t+s}, x_{t+s}\}_{s\geq 0}$.

Suppose the firms find it costly to pay attention to current and future variables. They instead rely on a perceived, simplified version of the economy to decide on their reset prices. Concretely, let $m_t^X(\omega)$ be the attention level that firm ω pays to variable X. Conditional on a vector of attention $\mathbf{m}(\omega)$, the firm replaces the rational expectation $\mathbb{E}_t[X_{t+s}]$ by an alternative behavioral expectation $\mathbb{E}_t^{BR}[X_{t+s}]$:

$$\mathbb{E}_{\omega,t}^{BR}[X_{t+s}] = \begin{cases} m_t^X(\omega)X_{t+s} + \left(1 - m_t^X(\omega)\right)X_t^d(\omega), & s = 0 \\ \mathbb{E}_t\left[\mathbb{E}_{\omega,t+1}^{BR}[X_{t+s}]\right], & s \geq 1. \end{cases} \tag{3.9}$$

There are two parts to the specification in equation (3.9). The first part specifies that in the same period, the behavioral agent can misperceive the true values of contemporary variables. In particular, they perceive X as a weighted average of the truth (weight m^X) and a default value X^d (weight $1 - m^X$). This default value could be the steady state level of X, the last available value of X, or even ad-hoc rules of thumb. When the firm pays full attention ($m^X = 1$), it shares the same expectations as the rational agents (those with costless attention): $\mathbb{E}_t^{BR}[X_t] = X_t$. When the firm pays zero attention ($m^X = 0$), it instead relies on the default value: $\mathbb{E}_t^{BR}[X_t] = X_t^d(\omega)$.

The second part specifies how behavioral agents perceive the future. Here, I assume that $\mathbb{E}_t^{BR}[X_{t+s}] = \mathbb{E}_t\left[\mathbb{E}_{t+1}^{BR}[X_{t+s}]\right]$, i.e. they are rational about their future beliefs. This assumption implies that if firms find it optimal to choose a high level of attention in the future (for example, if uncertainty becomes very high), then they correctly anticipate that today.

The behavioral firms set their prices to:

$$p_t^{BR}(\mathbf{m}(\omega)) = \left(1 - \beta\theta\right)\sum_{s=0}^{\infty}\left(\beta\theta\right)^s \mathbb{E}_t^{BR}\left[p_{t+s} + \lambda\psi_x x_{t+s} | \mathbf{m}(\omega)\right]. \tag{3.10}$$

From here onwards, I assume that the firms have a default level of inflation π_t^d (could be a 2% rule of thumb) and a default output gap of zero: $x_t^d = 0$.

Attention choice

If a firm chooses an attention vector $\mathbf{m}(\omega)$, it sets a price $p^{BR}(\mathbf{m}(\omega))$ that deviates from the rational reset price p^{rat} (unless $\mathbf{m}(\omega) = \mathbf{1}$). This deviation incurs a loss of current and future profits for the firm equal to $V(p^{rat}) - V(p^{BR}(\mathbf{m}(\omega))) > 0$ (since p^{rat} is the value that maximizes V, defined in equation (3.7). To a second order of approximation, this is equal to:

$$V(p^{rat}) - V(p^{BR}(\mathbf{m}(\omega))) = \sum_{i,j} \mathbb{E}_t \left[-\frac{\partial^2 V}{\partial P^2} P^2 \frac{\partial p^{BR}}{\partial m_i} \frac{\partial p^{BR}}{\partial m_j} \right] (1 - m_i)(1 - m_j)$$

where the summation is taken over the set of variables that firm ω pays attention to (in this model, the price level and the output gap). All derivatives are evaluated at $\mathbf{m} = 1$ and $P = P^{rat}$. Since the attention level chosen today does not affect future variables, the impact of attention choices today on the behavioral reset price:

$$\frac{\partial p_t^{BR}}{\partial m_t^p} = p_t - p_t^d, \quad \frac{\partial p_t^{BR}}{\partial m_t^x} = \lambda \psi_x x_t.$$

To avoid making the attention choice problem even harder to solve than the original problem, I make three simplifying assumptions

1. The firms do not consider the cross-terms, i.e. they think that the coefficients in front of $(1 - m_i)(1 - m_j)$ as zeros.

2. The firms replace the term $\left[-\frac{\partial^2 V}{\partial P^2} P^2 \right]_{P = P_t^{rat}}$, which is the average benefit of attention, by its steady state value, denote by Λ.

Denote by $\sigma_{p,t}^2 = \mathbb{E}_t \left[(p_t - p_t^d(\omega))^2 \right]$ the inflation uncertainty of firm ω, and $\sigma_{x,t}^2 = \mathbb{E}_t \left[x_t^2 \right]$ its uncertainty about the output gap. With the simplifying assumptions, we obtain a simple version of the firm's attention choice problem:

$$\min_{m_x, m_p} \Lambda \left[\sigma_{p,t}^2 (1 - m_t^p)^2 + \lambda^2 \psi_x^2 \sigma_{x,t}^2 \left(1 - m_t^x \right)^2 \right] + \chi_p m_t^p + \chi_x m_t^x.$$

This is a straightforward cost-benefit analysis that firms can do: they weigh the cost of inattention (potential profit loss due to mispricing) against the cost of attention. The parametersχ_p and χ_x parameterize the cost per unit of attention for inflation and the output gap.

The optimal attention of firm ω is given by:

$$m_t^p(\omega) = \begin{cases} 1 - \frac{\chi_p}{2\Lambda \sigma_{p,t}^2}, & \sigma_{p,t}^2(\omega) > \frac{\chi_p}{2\Lambda} \\ 0, & \sigma_{p,t}^2(\omega) \leq \frac{\chi_p}{2\Lambda}. \end{cases}, \qquad m_t^x(\omega) = \begin{cases} 1 - \frac{\chi_p}{2\Lambda \lambda^2 \psi_x^2 \sigma_{x,t}^2}, & \sigma_{x,t}^2(\omega) > \frac{\chi_x}{2\Lambda \lambda^2 \psi_x^2} \\ 0, & \sigma_{p,t}^2(\omega) \leq \frac{\chi_x}{2\Lambda \lambda^2 \psi_x^2}. \end{cases}. \tag{3.11}$$

In general, the optimal attention to a variable increases with uncertainty. As uncertainty approaches infinity, the level of attention approaches one. When uncertainty reaches a certain lower bound, attention becomes zero.

Monetary policy

To complete the model, I specify the monetary policy block. The short-term interest rate $i_t \equiv -\log Q_t$ is set by a central bank that follows a Taylor rule:

$$i_t = \rho + \phi_x x_t + \phi_\pi(\pi_t - \overline{\pi}_t^{CB}) + \nu_t \tag{3.12}$$

where $\overline{\pi}_t$ is the Central Bank's inflation target. The random component of the Taylor rule has an AR(1) structure:

$$\nu_t = \rho_\nu \nu_{t-1} + \varepsilon_t^\nu,$$

where $\varepsilon_t^\nu \sim \mathcal{N}\left(0, \sigma_\nu^2\right)$ is referred to as monetary policy shocks.

3.3.2 The Behavioral-Attention Phillips Curve (BAPC)

I now state the the Phillips Curve that arises in this model and give intuition.

Proposition 5. *The Behavioral-Attention Phillips Curve is given by:*

$$\pi_t = \omega_t^d \pi_t^d + (1 - \omega_t^d)\beta \mathbb{E}_t[\pi_{t+1}] + \kappa_t x_t, \tag{3.13}$$

The weight on default inflation is

$$\omega_t^d = \frac{(1-\theta)\left(1-\beta\theta\right)(1-m_t^p)}{\theta + (1-\theta)\left(1-\beta\theta\right)(1-m_t^p)},$$

and the output-gap slope is given by:

$$\kappa_t = \frac{(1-\theta)\left(1-\beta\theta\right)\lambda \upsilon m_t^x}{\theta + (1-\theta)\left(1-\beta\theta\right)(1-m_t^p)} \in [0, \kappa_{max}]$$

where $\lambda \equiv \frac{1-\alpha}{1-\alpha+\alpha\varepsilon}$ *and* $\upsilon \equiv \frac{\sigma^{-1}(1-\alpha)+\varphi+\alpha}{1-\alpha}$.

Proof. See Appendix C.2. □

The Behavioral-Attention Phillips Curve in equation (3.13) nests the traditional Phillips Curve (see for example, Galí (2008)) as the special case with full attention to both price and the output gap ($m^p = m^x = 1$). When firms pay full attention to prices, inflation expectation relies only on forward-looking rational expectation of future inflation $\beta\mathbb{E}_t\pi_{t+1}$, and put a weight $\omega^d = 0$ on default inflation π_t^d. As firms become inattentive to prices, they rely more on the default value to optimize on attention cost.

The output gap slope κ increases with both types of attention. To zoom in on the effect of attention choices, and not nominal frictions, on the output gap slope κ, let us first consider the special case of Proposition 5 where prices are entirely flexible ($\theta = 0$).

Proposition 6. *If prices are flexible ($\theta = 0$), the BAPC is given by:*

$$\pi_t = \pi_t^d + \kappa_t x_t, \quad \textit{where } \kappa_t = \frac{\lambda\upsilon m_t^x}{1 - m_t^p}.$$

Proof. The flexible-price BAPC is simply the limit of equation (3.13) when $\theta \to 0$. □

Intuitively, if prices are entirely flexible and firms have perfect information, then monetary policy shocks only cause price changes while output remains at the natural level. However, if firms are inattentive to the aggregate price movement ($m_t^p < 1$), they want to raise their individual prices by less than the aggregate level. In equilibrium, when all firms increase prices by less than the frictionless benchmark, the real wage and output increase above the natural level. Thus, there is a positive relationship between inflation and the output gap. The more inattentive firms are, the stronger is the output response relative to inflation, leading to a more flattened Phillips Curve.

The attention to the output gap m^x matters for the output-gap slope κ because it governs how firms perceive conditions in the labor market. When the output gap is positive, inattentive firms fail to notice that their marginal costs have increased as a result. Once again, this leads to a price increase that is lower than the frictionless benchmark, a larger output increase, and a more flattened Phillips Curve.

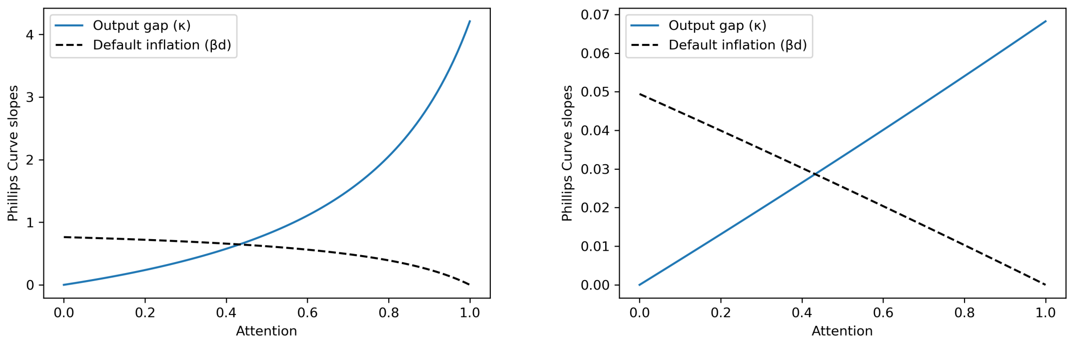

Figure 3.3: *Effects of attention on Phillips Curve slope (left: $\theta = 0.2$, right: $\theta = 0.8$)*

3.3.3 Volatility and Equilibrium Uniqueness

Why has inflation stayed so low in recent years despite a large, constant monetary stimulus by the Fed and other central banks? The model presented in this paper implies that because inflation uncertainty is already low up until the crisis, firms do not pay attention to prices, and this implies that monetary expansions would only result in changes in output rather than prices. The lack of movement in inflation in turn reinforces low inflation uncertainty and low attention.

In discussing the roles of monetary policy, one must be specific about what monetary policies are. Here, the Central Bank has two margins: the monetary volatility $\sigma^2_{v,t}$, i.e. the variance of the deviation from the Taylor rule, and the coefficients ϕ_π, ϕ_x of the Taylor rule itself.

Let me first focus on the effects of changing $\sigma^2_{v,t}$. Suppose that prices are entirely flexible, and that attention output is costless ($\chi_x = m^x = 0$), so that we can focus on attention to inflation. I now provide a theorem about the the dependence of the number of equilibrium on the volatility of monetary shock (exogenous volatility). The idea is that if the monetary policy volatility is sufficiently high, the only optimal choice is to pay full attention, and the equilibrium with high attention is the unique equilibrium. Conversely, when exogenous volatility is sufficiently low, the only optimal choice is to pay zero attention, and the equilibrium is again uniquely determined. However, if the central bank is sufficiently non-aggressive to inflation, then there could be two equilibria, one associated with a low attention economy with a flattened Phillips Curve, and the other equilibrium has high attention and a steep Phillips Curve.

The existence of multiple equilibria depends crucially on how aggressive the Central Bank is in

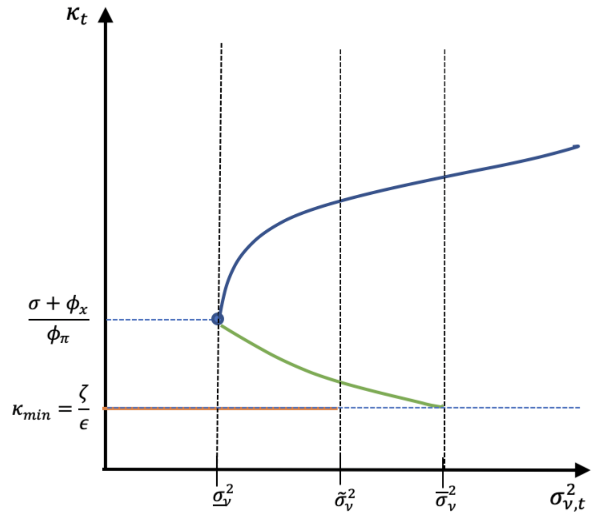

Figure 3.4: *Multiple equilibria can occur at low enough inflation volatility.*

This graph plots the possible values of κ slope (colored lines) under different levels of monetary volatility $\sigma_{\nu,t}^2$ on the x-axis.

targeting inflation versus output. For the baseline, let us first consider the case of a non-aggressive Central Bank, defined by the condition:

$$\frac{\sigma^{-1} + \phi_x}{\phi_\pi} > \lambda \upsilon. \tag{3.14}$$

The number of equilibrium is characterized in Proposition 7, and a visualization is provided in Figure 3.4.

Proposition 7. *[Multiple equilibria] Consider the case of flexible prices, costless attention to output, and a non-aggressive Central Bank (defined by condition (3.14)). Then, there exist cutoff thresholds $\underline{\sigma}_\nu^2 \leq \overline{\sigma}_\nu^2 \leq \tilde{\sigma}_\nu^2$ for monetary shock volatility such that:*

- *If $\sigma_\nu^2 < \underline{\sigma}_\nu^2$: There is a unique attention-equilibrium with zero attention, $m_t^\pi = 0$. The Phillips Curve attains minimum slope: $\kappa_t = \lambda \upsilon$.*
- *If $\sigma_\nu^2 \in [\underline{\sigma}_\nu^2, \overline{\sigma}_\nu^2]$: there is a zero-attention equilibrium and at least one more equilibrium with positive attention and a steep Phillips Curve $\kappa > \lambda \upsilon$.*
- *If $\sigma_\nu^2 > \overline{\sigma}_\nu^2$: the zero-attention equilibrium ceases to exist. There remains at least one more*

positive-attention equilibrium.

- *If $\sigma_v^2 > \tilde{\sigma}_v^2$: there is a unique equilibrium with positive attention and a steep Phillips Curve.*

Proof. See Appendix C.3. □

This result is visualized in Figure 3.4. This results highlight how the strategic complementarity in optimal attention by firms could be an important force to counter any change in monetary policy.

When inflation volatility is already very high, even when pass-through is low, firms face high inflation uncertainty and would like to pay full attention. The only equilibrium that can happen when inflation is very volatile is one with full attention. The United States in the late 1970s can be thought to be to the right end of Figure 3.4, when inflation uncertainty was high, firms are highly attentive to monetary variables, and Phillips Curve slope is very steep. By contrast, the US economy after the year 2000 can be thought of as the leftmost region of Figure 3.4, where the only equilibrium is the low-attention equilibrium.

Multiple equilibria occur when inflation volatility gets sufficiently low, but not too low (the region $\sigma_{v,t}^2 \in [\underline{\sigma}_v^2, \tilde{\sigma}_v^2]$). In this region, given the same inflation volatility, if firms are very attentive to monetary shocks, pass through will be high, the Phillips Curve will be steep, which justifies the high level of attention. Conversely, if firms are very inattentive already, then pass through is low, and the Phillips Curve is flat. In such a scenario, even if the Central Bank becomes more active in monetary policy, that does not necessary translate into higher inflation uncertainty and firms' attention.

The assumption of a non-aggressive Central Bank was crucial to obtain the result in Proposition 7. Consider now the case of $\frac{\sigma+\phi_x}{\phi_\pi} > \lambda \upsilon$. At one extreme, if the Central Bank is extremely aggressive about inflation ($\phi_\pi \to \infty$), then attention does not matter at all, since inflation is always fixed at the Central Bank's target. However, once the Central Bank begins to care more about output relative to inflation, there is more room for inflation to be determined by the behaviors of firms. Thus, a way to remove the dependence of the economy on attention is to be aggressive about inflation. This result is made clear in Proposition 8 and visualized by Figure 3.5.

Proposition 8. *To attain unique equilibrium, the Central Bank needs to be sufficiently aggressive against inflation. A necessary condition is:*

$$\frac{\phi_\pi}{\sigma^{-1} + \phi_x} \geq \kappa_{min} = \lambda \upsilon.$$

.

The results presented in this section show that the Central Bank faces a "policy paradox" after the Global Financial Crisis. In particular, raising inflation in a period of low inflation uncertainty and low attention requires the Central Bank to simultaneously do two things:

1. Commit strongly to keeping inflation at a target rather than targeting output (ϕ_π/ϕ_x must be high) to remove multiplicity of equilibria.

2. Raise the volatility of monetary shock, which raises attention (in a determined way.)

This, however, can be hard to do in practice. While ϕ_π and σ_ν^2 are two separate objects in theory, it may not be perceived separately by the market. A central bank which has a significant random component of its policy (high σ_ν^2) may be perceived as non-committal to an inflation target. Vice versa, a central bank which commits strongly to a target (high ϕ_π) reduces inflation uncertainty and may reduce attention. To resolve this paradox, further nontraditional policies aiming to increase the attention level of the economy is needed.

3.4 Empirical Evidences for BAPC from Inflation Surveys

3.4.1 Testing for BAPC

Specification

I now test the empirical relevance of the benchmark BAPC given in Proposition (6)

$$\pi_t = \pi_t^d + \kappa_t x_t. \tag{3.15}$$

where κ_t is time-varying and depends on inflation uncertainty that prevails in the economy.

The traditional Phillips Curve with backward-looking expectation is nested in equation (3.15). It is the case when default inflation is formed as a function of past inflations and the output-gap

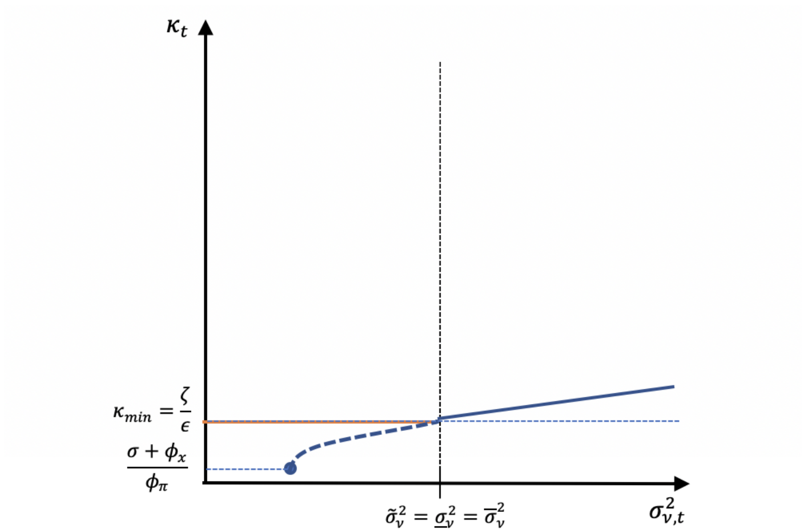

Figure 3.5: *If the Central Bank is aggressive enough about inflation, multiple equilibria can be eliminated.*

slope is constant: $\kappa_t = \overline{\kappa}$. Coibion and Gorodnichenko (2015) points out that this curve does not fit well the US inflation dynamics both in sample and out-of-sample. In the lens of this paper, the traditional Phillips Curve is mis-specified because (i) κ_t is time-varying and (ii) what constitutes default inflation π_t^d has changed. I now consider these two objects in turn.

The output gap slope

In the benchmark model, when there is only inattention to inflation ($m_t^x = 1$), the slope on output gap is given by:

$$\kappa_t = \frac{\lambda v}{1 - m_t^p}$$

The empirical challenge is that level of inattention to inflation $\left(1 - m_t^p\right)$ is not directly observable. To overcome this challenge, I use the positive relationship between optimal attention and inflation uncertainty predicated by the theory to replace attention in equation (3.15) by inflation uncertainty, which is arguably more quantifiable.

In particular, the optimal level of inattention is inversely related to inflation uncertainty, as shown in equation (3.11):

$$1 - m_t^\pi = \min\left\{1, \frac{\chi_p}{2\Lambda\sigma_{p,t}^2}\right\}.$$

The Phillips Curve slope κ_t can then be written as a piecewise linear function of inflation uncertainty:

$$\kappa_t = \frac{\lambda v}{1-m_t^p} = \lambda v \max\left\{1, \frac{2\Lambda}{\chi_p}\sigma_{p,t}^2\right\}.$$

Empirically, I parametrize κ as:

$$\kappa_t = \kappa_0 + \kappa_1 \sigma_{p,t}^2.$$

The theory predicts that $\kappa_0 = 0$ and $\kappa_1 = \frac{2\lambda v \Lambda}{\chi_p} > 0$. The slope κ_1 is inversely proportional to the cost-to-benefit ratio of inattention. A higher cost-to-benefit ratio of inattention implies a weaker relationship between the output gap slope and inflation uncertainty, as firms are hesitant to change their attention.

Choosing the appropriate measure of uncertainty about inflation is important for my test. In the baseline regression, I measure inflation uncertainty by the variance of one-year-ahead inflation forecasts made by consumers in the Michigan Surveys of Consumers data set. As robustness check, I will consider a range of alternative proxies such as the entropy from probabilistic forecasts, rolling sample variance of inflation, etc. The baseline result is robust across those measures.

Default inflation

The empirical section requires us to take a stand on what is default inflation. I suppose that default inflation is a weighted average between a constant $\overline{\pi}^{CB}$ (e.g. the 2% inflation target of the central bank) and the backward-looking inflation π_{t-1}:

$$\pi_t^d = \Big((1-\xi_t)\cdot\overline{\pi}^{CB} + \xi_t \pi_{t-1}\Big).$$

The weight ξ_t is linked to the persistence of inflation dynamics. If $\xi_t = 1$, $\pi_t^d = \pi_{t-1}$, and inflation will be extremely persistent, corresponding to the *accelerationist view* of inflation. Contrarily, if $\xi_t = 0$, then the default inflation is simply top-of-mind number such as the Central Bank's 2% inflation target. In this case, inflation expectation is said to be well-anchored.

I parameterize this slope ξ_t to also be linear in inflation uncertainty:

$$\xi_t = \xi_0 + \xi_1 \sigma_{\pi,t}^2.$$

The motivation for this specification is as following. Suppose that monetary policy shocks are persistent, so that past inflation carries meaningful information about current inflation. A natural default would then be past inflation. In the baseline model, at any given date t, the firms are given past values such as π_{t-1} for free. In the real world, however, past inflation is not free information, and it is costly to track that variable as well. We can hypothesize that only when there is high uncertainty $\sigma^2_{\pi,t}$ that the firms find it worth paying the attention cost and know π_{t-1}, otherwise they use a rule-of-thumb to predict inflation (such as the 2% inflation target).

We can re-arrange to write the default inflation as:

$$\pi^d_t = \overline{\pi}^{CB} + \left(\xi_0 + \xi_1 \sigma^2_{\pi,t}\right)\left(\pi_{t-1} - \overline{\pi}^{CB}\right)$$

Regression specification

For reasons given in previous sections, I run three regressions nested in the following general form:

$$\pi_t = \alpha + (\kappa_0 + \kappa_1 \sigma^2_{\pi,t}) x_t + (\xi_0 + \xi_1 \sigma^2_{\pi,t})\left(\pi_{t-1} - \overline{\pi}^{CB}\right) + u_t \tag{3.16}$$

The first regression is the simple traditional Phillips Curve with a constant slope and only backward-inflation expectation. This is the benchmark regression in Coibion and Gorodnichenko (2015), and also used as the benchmark here. It is nested in specification (3.16) by setting $\kappa_1 = \xi_1 = 0$, and $\xi_0 = 1$:

$$\pi_t - \pi_{t-1} = \alpha + \kappa_0 x_t + u_t. \tag{3.17}$$

In the result section, I will label this model as "Trad. PC" (traditional Phillips Curve).

The second model allows for time-varying slope on the output gap, but not past inflation (so $\xi_0 = 1$, and $\xi_1 = 0$). I run this specification to isolate the explanatory power of a changing output slope and not be confounded with the effect of inflation anchoring. This model will be referred to as "BAPC" (the Behavioral Attention Phillips Curve) in the result.

Finally, I run the third model, termed "BAPC Full", which allows for both a changing output-gap slope and inflation-expectation slope.

Finally, I report a separate set of results that includes oil inflation to account for supply shocks. If supply shocks such as oil shocks are fully visible to firms, then they are uncorrelated with the output gap (but they affect the natural output), but still affecting the inflation level. I show that the baseline result is robust to whether one includes oil inflation or not.

Data

In the baseline model, I use the year-over-year CPI inflation (including food and energy) at quarterly frequency as a measure of inflation. To proxy for output gap x_t, I use the (negative) unemployment gap measure constructed by the CBO.[3]

The final piece is measure of inflation uncertainty $\sigma^2_{\pi,t}$. For the baseline regression, I use the sample variance of households' inflation forecasts from the Michigan Survey of Consumers dataset. This measure is plotted in Figure ??. The original survey question asks households to forecast inflation for 1-year ahead, and I have lagged the series by 4-quarter to match the forecast uncertainty with the quarter corresponding to that forecast. This measure shows that inflation uncertainty was highest during the 1973-1975 recession (the "stagflation" period) and the Volcker period (1979-1982). After 1982, inflation uncertainty has declined, except for temporary surges in the 1990s, due to an oil price shock and the ensuing recession; early 2000s (dot-com bubble); and 2009-2010 (Great Recession). It is worth noting that even though inflation uncertainty rose during the Great Recession, the level of uncertainty pales compared to mid-1990s, when the inflation level was relatively low and stable.

It is worth noting that this measure is survey-based and may represent disagreement between consumers rather than within-consumer uncertainty. In general, a statistician cannot distinguish from the surveys whether (i) responders have very certain point forecasts that disagree or (ii) responders are uncertain in the same way, and when asked, pick randomized responses from the same distribution. The theoretical and statistical models specified in this paper work well if it is the latter case for the Michigan Surveys of Consumers. To be more prudent in assessing the first case, I also consider within-forecaster uncertainty measured from the probabilistic forecasts in the Survey of Professional Forecasters. Details are given in section 3.4.4, and we will see that the baseline results is robust when using other measures of inflation uncertainty.

[3]Since mean unemployment appears very stable, one can also use the unemployment rate as a proxy for economic slackness. See Fernald *et al.* (2017) for a discussion.

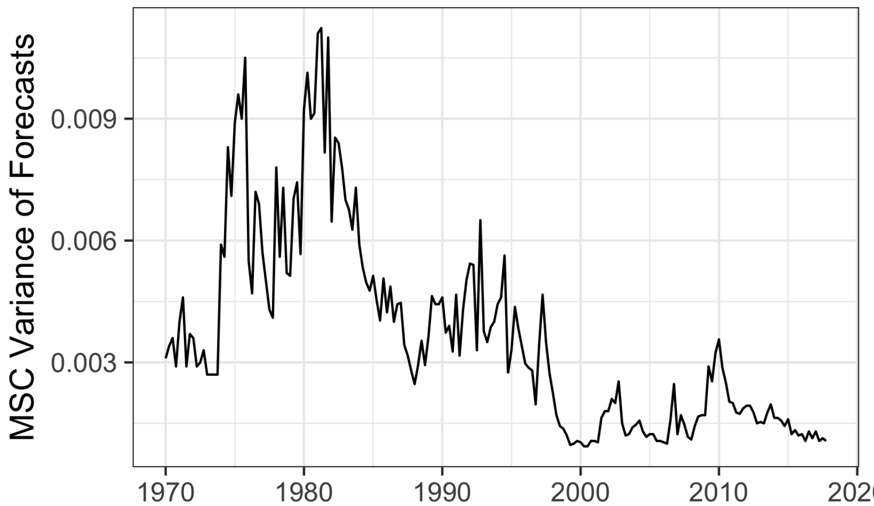

Figure 3.6: *Variance of inflation forecasts from the Michigan Survey of Consumers*

3.4.2 Results

Table 3.1 presents the estimates for the three candidate models: traditional PC, BAPC (with only uncertainty-dependent output-gap slope κ), and Full BAPC (with uncertainty-dependent output-gap slope $\kappa$and persistence ξ).

The output-gap slope is time-varying and increases with uncertainty

Column (1) of Table 3.1 shows that the traditional backward-looking PC estimates a significant and positive output gap slope $\kappa_0 = 0.632$. However, in column (2), we see that this term becomes statistically insignificant once the uncertainty-adjusted output gap term is added. The coefficient κ_1 is significant and positive, indicating that the PC κ-slope flattens as uncertainty declines.

In other words, the output gap or unemployment gap per se is not linked to inflation; only when inflation uncertainty is high do we see this relationship. Given that the BAPC nest the traditional PC, the finding that κ_1 is significant rejects the null hypothesis of a traditional Phillips Curve. Furthermore, the insignificance of κ_0 suggests that the data favors a linear attention cost function overs a quadratic one.

The effect of uncertainty is both significant and large. The full evolution of the κ-slope and 95% confidence interval band are plotted in figure 3.7. The estimated values of κ_0 and κ_1 imply

$\pi_t = \alpha + (\kappa_0 + \kappa_1 \sigma^2_{\pi,t})x_t + (\xi_0 + \xi_1 \sigma^2_{\pi,t})\left(\pi_{t-1} - \overline{\pi}^{CB}\right) + u_t$			
	Trad. PC	BAPC	Full BAPC
κ_0	0.632	0.023	-0.027
	(0.244)	(0.236)	(0.229)
κ_1		0.015	0.018
		(0.003)	(0.007)
ξ_0	1	1	0.407
			(0.288)
ξ_1	0	0	0.007
			(0.003)
α	2.359	2.450	2.776
	(0.233)	(0.231)	(0.285)
RMSE	2.52	2.11	1.70
N	196	196	196
Adjusted R^2	0.178	0.264	0.351

Table 3.1: *Estimation Results for Specification* (3.16)

This table reports estimation result for the Traditional Phillips Curve (Traditional PC), the baseline Behavioral-Attention Phillips Curve (BAPC), and the Full Behavioral-Attention Phillips Curve (Full BAPC) using US data from 1970Q1-2018Q2. Inflation is year-over-year CPI inflation. Newey-West standard errors are reported in parentheses. The RMSE row reports the root mean squared error statistic when specification (3.16) is estimated using data up to 2007Q3 and fitted out-of-sample for data after 2007Q3.

a very steep historical PC output-gap slope of near 1 at the peak of uncertainty in 1981Q2. However, that slope has essentially flattened to zero at the beginning of the 2000s.

The backward-inflation slope is time-varying and increases with uncertainty

Not only does the output-gap slope, κ, changes with uncertainty, the data also favors an uncertainty-dependent slope of lagged inflation. Table 3.1 shows that the lower bound for persistence is $\xi_0 = 0.407$; however, this number is statistically not different from zero. The implication is that in an environment without uncertainty, the effect of past inflation on current inflation largely disappears. This accords well with the well-anchored inflation expectation phenomenon in the recent years (Bernanke (2007), Blanchard *et al.* (2015)).

Importantly, I find that the degree of anchoring increases with uncertainty. This can be seen in the Full BAPC column of Table 3.1: the coefficient ξ_1 is positive and significantly different from zero.

Figure 3.8 plots the estimate and 95% confidence band of the ξ-slope over time. Recall that $\xi = 1$ suggests that inflation expectation is entirely backward-looking (past-anchoring),

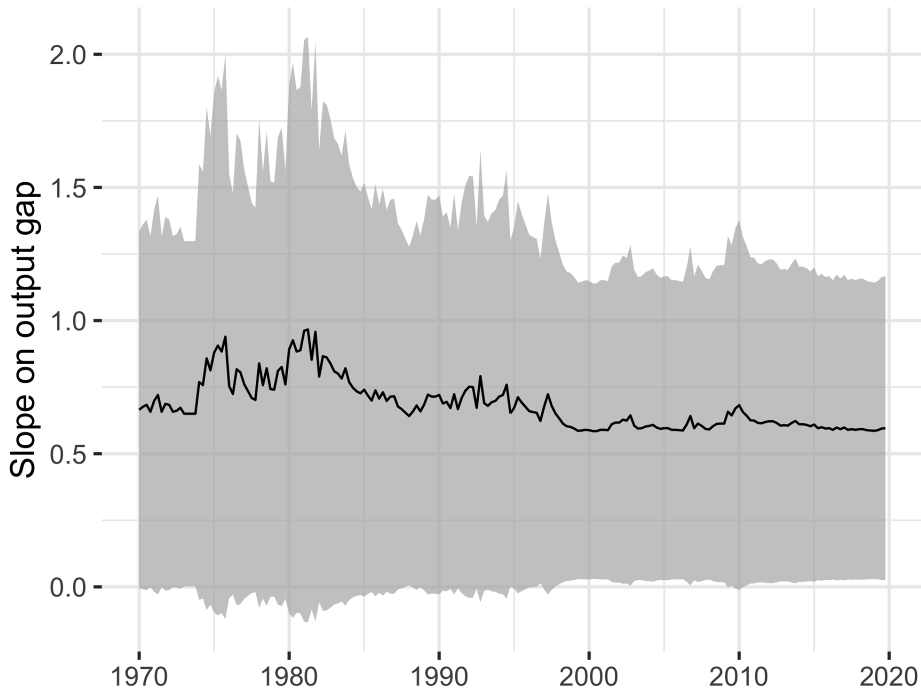

Figure 3.7: *The κ-slope implied by the estimated Full BAPC*

while $\xi = 0$ suggests perfect inflation anchoring at a fixed inflation target (target-anchoring). Estimates of Full BAPC from Table 3.1 imply that inflation dynamics was better described by the "accelerationist theory" in the late 70s and early 80s when the coefficient on past inflation is approximately 1. After year 2000, however, while backward expectation still plays a role, the decline in uncertainty has reduced that role significantly, with coefficient approximately 0.5.

	Trad. PC	BAPC	Full BAPC	Trad. PC	BAPC	Full BAPC
RMSE (%)	2.48	2.03	1.76	1.96	1.36	1.11
Oil included	No	No	No	Yes	Yes	Yes

Table 3.2: *Root Mean Square Error (RMSE) of Traditional PC versus BAPC*

This table reports the out-of-sample test result for the Traditional Phillips Curve (Traditional PC) versus the baseline Behavioral-Attention Phillips Curve (BAPC) and the Full Behavioral-Attention Phillips Curve (Full BAPC). The models, specified in specification 3.16, were estimated using US data from 1970Q1-2007Q3 and predicted out-of-sample using data from 2007Q3 to 2019Q4. Inflation is year-over-year CPI inflation. The first three columns report results when energy shocks are not included, while the last three columns include oil shocks in the regression.

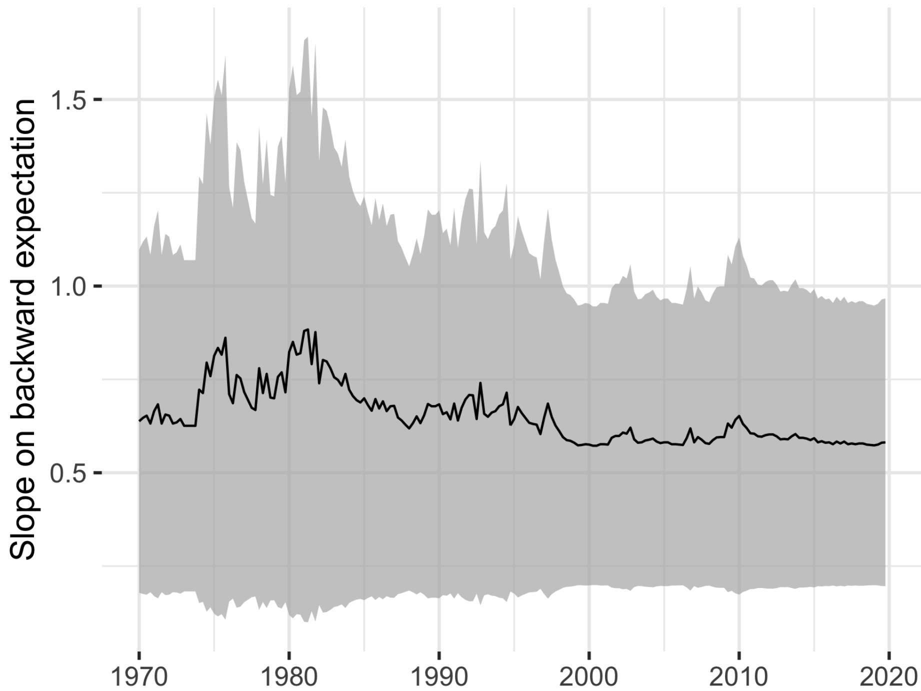

Figure 3.8: *The ξ-slope implied by the estimated Full BAPC*

3.4.3 Out-of-sample test: BAPC "resolves" the Missing Disinflation Puzzle

In the aftermath of the Great Recession, unemployment rose substantially and remained high for over 5 years. A constant-sloped Phillips Curve predicts that such slack in the economy should be associated with a large, sustained disinflation. However, after a brief decline in 2009Q2, inflation large rebounded and stayed in the positive domain since, defying the traditional PC's prediction. Statistically, estimating the traditional Phillips Curve using data prior to the 2008-2009 crisis and uses that model to predict inflation dynamics post-crisis generates large errors. Coibion and Gorodnichenko (2015) called this phenomenon the *Missing Disinflation Puzzle*.

I re-did the same out-of-sample exercise as in Coibion and Gorodnichenko (2015), now extending to include data up to 2019, and show that the BAPC "resolves" this puzzle in the sense that the Root Mean Squared Errors (RMSE) of the BAPC models are significantly lower than the traditional PC.

The RMSE of various models are reported in Table 3.2, and a graphical illustration of how

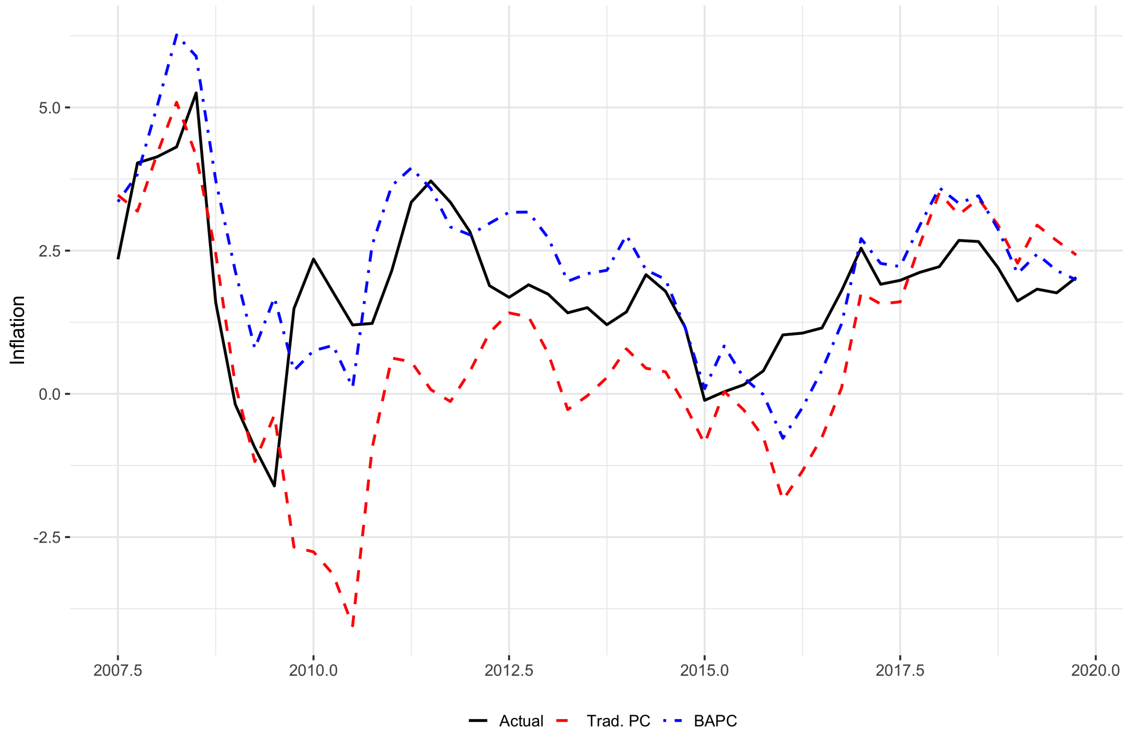

Figure 3.9: *Out-of-sample Performance of the Traditional PC versus BAPC*

This figure reports the out-of-sample fit for the Traditional Phillips Curve (Traditional PC) versus the the Full Behavioral-Attention Phillips Curve (Full BAPC) when oil shocks are included. The models, specified in specification 3.16, were estimated using US data from 1970Q1-2007Q3 and predicted out-of-sample using data from 2007Q3 to 2019Q4. Inflation is year-over-year CPI inflation.

the BAPC outperforms the traditional PC out-of-sample is presented in Figure 3.9.

On average, the traditional PC makes a 2.48% error (in absolute inflation level) each quarter, while the Full BAPC without oil shocks makes only 1.76% prediction error, a reduction of 0.72% in level. Including oil shocks, the traditional PC still makes significant error (RMSE = 1.96%), while the full version of BAPC makes only half the error (RMSE = 1.11%).

Intuitively, the traditional PC, having to fit a steep PC in the 1980s, estimates a high constant κ, and this high estimate produces the Missing Disinflation Puzzle as described. Both the behavioral curves, however, recognize that the PC was steep in the 1980s due to high inflation uncertainty $\sigma^2_{\pi,t}$ rather than a high constant κ, and correctly lower the PC $\kappa-$slope to rectify for the relatively low inflation uncertainty in recent years.

Figure 3.9 shows clearly the difference between the two models: while the traditional PC predicts large disinflation after the 2008 crisis and throughout the recovery, the BAPC fits inflation dynamics remarkably well.

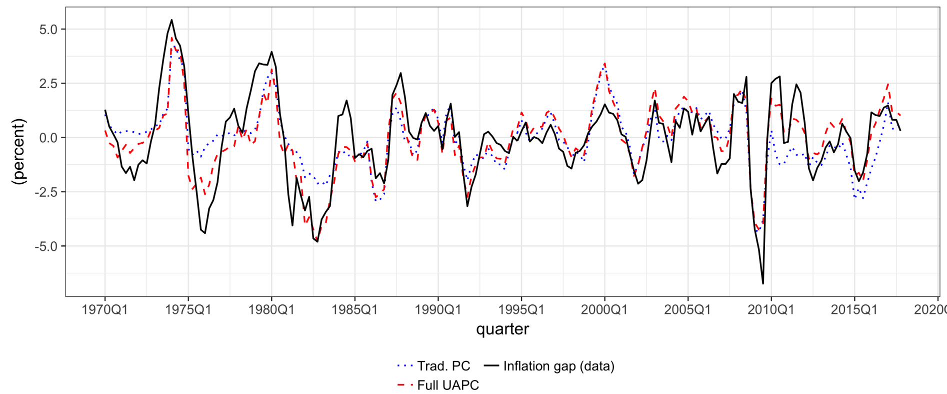

Figure 3.10: *In-sample Fit of the Trad. PC, Rest. BAPC, and Full BAPC in 1970Q1-2018Q2*

The two behavioral PCs also fit better in sample. Let us look at the flipside of the "Missing Disinflation" in recent years: inflation dynamics in the 1970s and early 1980s. Figure 3.10 plots the in-sample fit of the three models embedded in specification (3.16) for 1970Q1 to 1990Q1 (estimated using full sample). Due to the high inflation in this period, I plot on the y-axis "inflation surprise", $\pi_t - \pi_{t-1}$, rather than π_t itself, though one should note that the regression is the same.

We can see that the traditional backward-looking PC does a good job explaining the "stagflation" in 1973-1975, when both inflation and unemployment were high. What it could not account for, however, is the rapid fall of inflation from over 10% at the peak of recession in 1974 to just above 5% two years later. The unemployment gap was sizable in 1975-1977, and a correctly estimated PC should observe that such slackness should predict rapidly falling price. However, since the traditional Phillips Curve only has a constant slope, it is "flattened" to fit inflation dynamics in the later years, thus predicting only a modest slowdown of inflation. Meanwhile, the two BAPC curve notes the high inflation uncertainty during this period, predicts both a steep slope and a rapid fall in inflation growth that fits the data well. The same phenomenon repeats after the Volcker recession 1980-1983, when the traditional PC once again fails to see the rapidly declining inflation, while the behavioral curves do not.

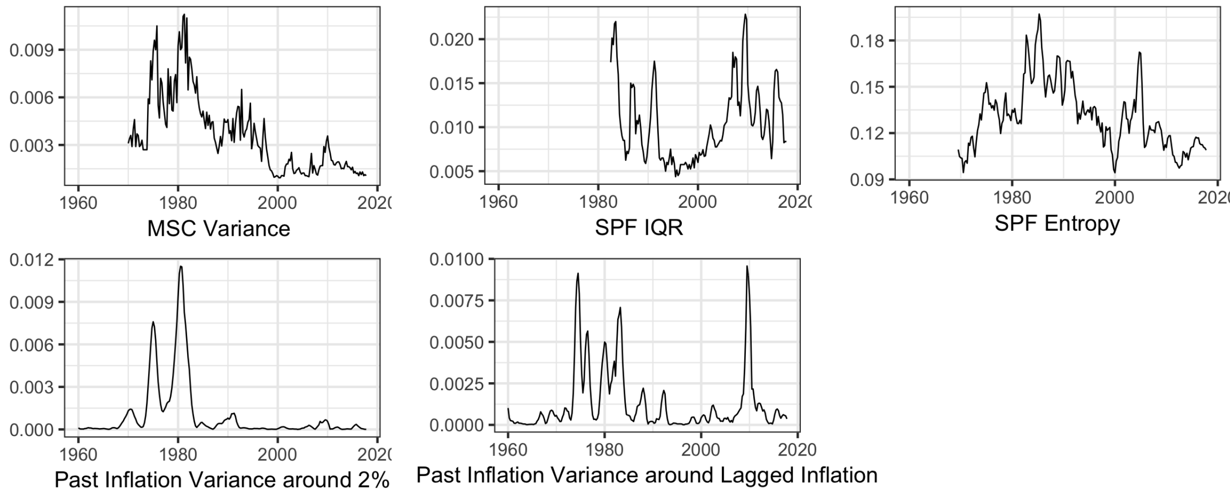

Figure 3.11: *Measures of Inflation Uncertainty*

3.4.4 Other Measures of Inflation Uncertainty

Constructing other measures of $\sigma^2_{\pi,t}$

Having a good measure of inflation uncertainty is central to my empirical analysis. I here present five alternative measures of inflation uncertainty, plotted in Figure (3.11), and see how they affect the baseline's result.

The first approach to measure inflation uncertainty is the dispersion of individual inflation forecasts from micro survey datasets. Within this approach, I have two time series: the between-forecasters forecast variance from Michigan Survey of Consumers, and the interquartile range (IQR) of forecasts of the Survey of Professional Forecasters. These two series differ both in the subjects being surveyed (MSC surveys consumers, while SPF surveys professional forecasters) and the treatment of tails. The MSC variance measure is more subject to large, highly disagreed forecasts, while the SPF IQR is more robusts to outliers. From Figure (3.11), we can see that the main difference between these two measures are that consumer disagreement seems more volatile over time than the professional forecasters. Consumer inflation uncertainty was much higher in the 1980s than in the 2000s, while professional forecasters disagree almost equally at the height of the Volcker years and the Great Recession.

Whether disagreement proxies for uncertainty or not remains a subject of debate. Forecasters can disagree because they are uncertain; they can also disagree if they are extremely certain about their own posterior belief. To tease out true *within-subject* uncertainty, I look next to the *probabilistic forecasts*. In particular, the SPF survey asks forecasters not only for a point estimate, but also for a probability measure over a set of inflation scenario. The survey then reports the average probability for each inflation scenario (averaging over subjects). The inflation scenarios take the form: "How many percent chance do you think year-over-year inflation this quarter will be <3% / 3-4% / 4-5% / ?" etc. Let $p_t^i(s)$ be the probability that forecaster i puts on inflation event s in quarter t. I then have data of $\overline{p}_t(s) \equiv \frac{1}{n}\sum_{i=1}^{n} p_t^i(s)$. I then construct an *entropy statistic* H to summarize the average within-subject uncertainty each quarter:

$$H_t \equiv -\sum_s \overline{p}_t(s) \cdot \ln(\overline{p}_t(s))$$

This measure is plotted on the top-right corner in Figure (3.11). Interestingly, the SPF Entropy measure correlates well with the MSC Forecast Variance measure (with a correlation of 0.65), while the SPF IQR measure does not (correlation 0.16). This suggests that while special forecasters do not seem to disagree as much as the consumers, *internally* they are almost as uncertain as consumer.

Finally, I have two other forecasts that are not survey-based. I construct in turn the moving-average square prediction error for someone who always sets $\pi_t^e = 2\%$, i.e. a inflation-target believer, and an individual who always sets $\pi_t^e = \pi_{t-1}$, i.e. a past-inflation anchorer. The formula is $\sigma_{\pi,t}^2 = \frac{1}{h}\sum_{s=t-h}^{h-1}(\pi_s - \pi_s^e)^2$, with horizon h sets at 8 quarters. I label the measures "2%-Dev. Var" and "Back-Dev. Var" respectively, and plot them on the second row of figure (3.11). We can see that a inflation-target believer has very high uncertainty only in the 1970s and early 1980s, while the past-inflation anchorer had high uncertainty in the late 1970s, early 1980s, and during the Great Recession.

Robustness check with other Measures

Table (3.4) provides the estimates for our main regression but using the five measures of inflation uncertainty constructed above. Newey West standard errors are reported in parentheses.

We can see that across the measures, κ_1 is highly significant and with the same sign in

	MSC Var	SPF Entropy	SPF IQR	2%-Dev. Var	Back-Dev. Var
MSC Var	1.000	0.650	0.161	0.637	0.314
SPF Entropy	0.650	1.000	0.044	0.382	0.126
SPF IQR	0.161	0.044	1.000	0.443	0.263
2%-Dev. Var	0.637	0.382	0.443	1.000	0.327
Back-Dev. Var	0.314	0.126	0.263	0.327	1.000

Table 3.3: *Correlation between Measures of Inflation Uncertainty*

almost all columns (with the exception of the last). This is strong evidence for the robustness of the BAPC, indicating that the slope of the output gap does indeed rise with uncertainty. The evidence for $\xi_1 > 0$ (less anchor when inattentive) is more mixed, but also significant in two of the cases.

Table 3.4: *Robustness Check: Using Other Measures of Inflation Uncertainty*

	$\pi_t = \alpha + (\kappa_0 + \kappa_1 \sigma^2_{\pi,t}) x_t + (\xi_0 + \xi_1 \sigma^2_{\pi,t}) \left(\pi_{t-1} - \overline{\pi}^{CB}_t\right) + \varepsilon_t$				
	MSC Var	SPF Entropy	SPF IQR	2%-Dev. Var	Back-Dev. Var
κ_0	-0.027	-2.10	-0.164	0.336	0.980
	(0.229)	(1.01)	(0.193)	(0.180)	(0.314)
κ_1	0.018	22.55	0.296	0.013	-0.258
	(0.007)	(8.98)	(0.141)	(0.005)	(0.103)
ξ_0	0.408	-0.95	0.489	0.943	-0.066
	(0.288)	(0.895)	(0.200)	(0.246)	(0.232)
ξ_1	0.007	13.76	-0.049	0	-0.062
	(0.003)	(6.87)	(0.138)	(0.002)	(0.161)
α	2.78	2.67	2.56	2.50	2.53
	(0.285)	(0.242)	(0.128)	(0.26)	(0.295)
N	196	190	144	196	196
Adjusted R^2	0.350	0.34	0.54	0.28	0.31

3.5 Conclusion

This paper has made two main contribution.

Firstly, I derived a new Phillips Curve with a slope that increases with inflation uncertainty in the economy using a theory of behavioral inattention. This curve, which I named the Behavioral-

Attention Phillips Curve, is useful to explain the gradual flattening of the Phillips Curve over time, as well as to answer the question of why inflation has stayed low in recent years.

The second contribution is using various measures of inflation uncertainty with various approaches, disagreement-based and subjective uncertainty-based, I have provided support for the BAPC and shown that the relationship is robust across varied specifications and measures of inflation uncertainty.

For future research, the presented NKBAPC can be structurally estimated to obtain estimates for some behavioral parameters, such as the cost and benefit of attention. Estimates for these parameters are still rare and it would be nice to obtain them. Another direction is to test the BAPC using a cross-section of countries. The challenge, however, is the lack of reliable inflation uncertainty measure, even for OECD countries. Country-level surveys that are similar to the Michigan Consumers Survey or the Survey of Professional Forecasters are unavailable in most countries. With the hope of better availability of data in the future, one can test the external validity of this theory in a wider range of countries.

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Appendix A

Appendix to Chapter 1

A.1 The Social Planner problem

We first compute the set of efficient allocations as the optimal allocation by a benevolent Social Planner. Given a vector of Pareto weights $\chi = (\chi_1, \chi_2, ..., \chi_N)$, with $\sum_{i=1}^{N} \chi_i = 1$, the Social Planner maximizes

$$\max \quad \sum_{i=1}^{N} \chi_i \left(\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u \left(C_t^i \right) \right)$$

by choosing for each country i and time t their consumption C_t^i, investment I_t^i (equivalently, next period capital K_{t+1}^i), and intermediate goods used in the production of final and intermediate sector $\left(G_{jt}^i, X_{jt}^i \right)_{j=1}^{N}$.

The SP faces the resource constraints:

$$C_t^i + \frac{1}{Z_{It}^i} \left(K_{t+1}^i - (1-\delta) K_t^i \right) = \left[\sum_{j=1}^{N} \xi_{ij}^{1/\varepsilon} \left(G_{jt}^i \right)^{1-\frac{1}{\varepsilon}} \right]^{\frac{\varepsilon}{\varepsilon-1}} \equiv G_t^i \qquad \forall i \in \mathcal{N}, \tag{A.1}$$

$$\sum_{j=1}^{N} (G_{it}^j + X_{it}^j) = Z_t^i \left[\left(K_t^i \right)^{\alpha} \left(L_t^i \right)^{\theta} \right]^{1-\gamma_i} \left(X_t^i \right)^{\gamma_i} \equiv Y_t^i \tag{A.2}$$

$$X_t^i = \left[\sum_{j=1}^{N} \omega_{ij}^{\frac{1}{\varepsilon}} \left(X_{jt}^i \right)^{1-\frac{1}{\varepsilon}} \right]^{\frac{\varepsilon}{\varepsilon-1}} \tag{A.3}$$

Let $\left(\exp\left(\mu_t^i\right), \exp\left(\nu_t^i\right), \exp\left(\varsigma_t^i\right) \right)_{i \in \mathcal{N}, t \geq 0}$ be the Lagrange multipliers corresponding to the resource constraints for final goods (A.1) (μ), intermediate goods (A.2) (ν), and the intermediate bundle (A.3) (ς) respectively.

The optimal conditions for intermediate inputs are:

$$g^i_{jt} - g^i_t = \ln \xi_{ij} - \varepsilon(\nu^j_t - \mu^i_t) \tag{A.4}$$

$$x^i_{jt} - x^i_t = \ln \omega_{ij} - \varepsilon\left(\nu^j_t - \varsigma^i_t\right), \quad \text{where } \varsigma^i_t + x^i_t = \ln \gamma_i + \nu^i_t + y^i_t. \tag{A.5}$$

When the elasticity of substitution is 1, these conditions simplify to:

$$\nu^j_t + g^i_{jt} = \ln \xi_{ij} + \mu^i_t + g^i_t, \quad \nu^j_t + x^i_{jt} = \ln\left(\gamma_i \omega_{ij}\right) + \nu^i_t + y^i_t. \tag{A.6}$$

The allocation of consumption between countries is given by:[1]

$$\ln \chi_i - \frac{1}{\sigma} c^i_t = \mu^i_t \tag{A.7}$$

Dynamically, the SP optimally chooses investment level for each country that satisfies the Euler equations:

$$\mathbb{E}_t \left[\beta \exp\left(\mu^i_{t+1} - \mu^i_t\right)\left((1-\delta)\frac{Z^i_{It}}{Z^i_{It+1}} + r^i_{t+1} Z^i_{It}\right)\right] = 1 \tag{A.8}$$

where the return on capital is $r^i_{t+1} = (1-\gamma_i)\alpha \exp\left(\nu^i_{t+1} - \mu^i_{t+1}\right) \frac{Y^i_{t+1}}{K^i_{t+1}}$.

The Social Planner solution satisfies the resource constraints (A.1) - (A.3), optimality input choices (A.4), (A.5), optimal consumption choice (A.7), and the Euler equation (A.8).

A.2 Proof of Proposition 1

The competitive equilibrium is characterized by the following equations.

[1]Condition (A.7) is essentially the Backus-Smith condition. When utility is log ($\sigma = 1$), it reduces to $\mu^i_t C^i_t = \chi_i$ constant, which means relative *nominal* consumption is the same across all states and times (note that μ^i_t's are the shadow prices of consumption). Generally, dividing equation (A.7) between two countries i and j and take logs:

$$\ln \chi_i - \ln \chi_j - \frac{1}{\sigma}\left(c^i_t - c^j_t\right) = \mu^i_t - \mu^j_t$$

$$c^i_t - c^j_t = -\sigma\left(\mu^i_t - \mu^j_t\right) + \sigma\left(\ln \chi_i - \ln \chi_j\right)$$

$$\left(\mu^i_t + c^i_t\right) - \left(\mu^j_t + c^j_t\right) = (1-\sigma)\left(\mu^i_t - \mu^j_t\right) + \sigma\left(\ln \chi_i - \ln \chi_j\right)$$

which is the Backus Smith condition.

Production

With Cobb-Douglas production, intermediate-producing firms spend a fixed share of revenue on individual inputs. The optimal conditions (in logs) are:

$$w_t^i + l_t^i = \ln\left((1-\gamma_i)\theta\right) + p_t^i + y_t^i \tag{A.9}$$

$$p_{Xt}^i + x_t^i = \ln\gamma_i + p_t^i + y_t^i \tag{A.10}$$

$$\ln r_t^i + k_t^i = \ln\left(\left(1-\gamma_i\right)\alpha\right) + p_t^i + y_t^i. \tag{A.11}$$

The optimal choices of intermediate inputs satisfy:

$$g_{jt}^i - g_t^i = \ln\xi_{ij} - \varepsilon p_{jt}^i \tag{A.12}$$

$$x_{jt}^i - x_t^i = \ln\omega_{ij} - \varepsilon\left(p_{jt}^i - p_{Xt}^i\right). \tag{A.13}$$

Capital accumulation

Optimal capital accumulation in the capital sector:

$$\mathbb{E}\left[\Theta_{t,t+1}^i\left((1-\delta)P_{Kt+1}^i + r_{t+1}^i/P_{Kt}^i\right)\right] = 1, \tag{A.14}$$

where $P_{Kt}^i = 1/Z_{It}^i$ is the price of capital.

Consumer - investor's problem

The consumer in country i solves:

$$\max_{\left(C_t^i,\Lambda_{j,t+1}^i\right)_{t=0}^{\infty}} \quad \mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t u(C_t^i)\right]$$

subject to budget constraint:

$$C_t^i + \sum_{j=1}^N \mathscr{E}_{it}^j\Lambda_{j.t+1}^i P_{St}^j + \sum_{j=1}^N \mathscr{E}_{it}^j P_{Bt}^j B_{j,t+1}^i = W_t^i\overline{L^i} + \sum_{j=1}^N \mathscr{E}_{it}^j\Lambda_{jt}^i\left(\Pi_t^j + P_{St}^j\right) + \sum_{j=1}^N \mathscr{E}_{it}^j B_{jt}^i\left(1 + P_{Bt}^j\right). \tag{A.15}$$

Let $\exp\left(\kappa_t^i\right)$ be the Lagrange multipliers on the budget constraints, the consumers' optimality conditions are:

$$\kappa_t^i = -\frac{1}{\sigma} c_t^i \tag{A.16}$$

$$\mathbb{E}_t\left[\Theta_{t,t+1}^i \frac{\mathcal{E}_{i,t+1}^j}{\mathcal{E}_{it}^j} \frac{P_{St+1}^j + \Pi_{t+1}^j}{P_{St}^j}\right] = 1 \tag{A.17}$$

$$\mathbb{E}_t\left[\Theta_{t,t+1}^i \frac{\mathcal{E}_{i,t+1}^j}{\mathcal{E}_{it}^j} \frac{P_{Bt+1}^j + 1}{P_{Bt}^j}\right] = 1 \tag{A.18}$$

with $\Theta_{t,t+1}^i \equiv \beta \exp\left(\kappa_{t+1}^i - \kappa_t^i\right)$.

These optimal conditions, together with the budget constraint (1.8) and the market clearing conditions fully characterize the competitive equilibrium.

The set of optimal conditions here are very similar to those of the SP. By choosing $\kappa_t^i = \mu_t^i - \ln \chi_i$ to equate the shadow prices of consumption, the decentralized equation (A.16) becomes equivalent to equation (A.7) from the SP problem.

The real exchange rate $\mathcal{E}_{jt}^i$ is the relative price of consumption between country i and j, thus can be set to the SP counterpart: $e_{jt}^i \equiv \ln \mathcal{E}_{jt}^i = \mu_t^i - \mu_t^j$. The price of the intermediate good i in local currency is simply $p_t^i = \nu_t^i - \mu_t^i$, which makes equations (A.4) and (A.5) equivalent to (A.12), (A.10) respectively.

The SDF is $\Theta_{t,t+1}^i \equiv \beta u'(C_{t+1}^i)/u'(C_t^i) = \beta \exp\left(\mu_{t+1}^i - \mu_t^i\right)$ and the equity return in country i is $r_t^i = (1-\gamma)\alpha P_t^i Y_t^i / K_t^i = \alpha \frac{\nu_t^i Y_t^i}{\mu_t^i K_t^i}$. Thus, the decentralized optimal investment condition (A.14) is equivalent to the SP condition (A.8).

The return to investment by the capital sector is equal to the return to investment in domestic equity by households. To see this, set $P_{St}^i = P_{Kt}^i K_{t+1}^i = K_{t+1}^i / Z_{It}^i$ makes the household's Euler equation (A.17) equivalent to (A.14) (which is equivalent to (A.8)). The bond prices are simply given by $P_{Bt}^i = \sum_{s=0}^{\infty} \mathbb{E}_t\left[\Theta_{t,t+s}^i\right]$.

It remains to check that under these prices, there exists a portfolio (Λ, B) and a particular Social Planner allocation (corresponding to specific Pareto weights) that satisfy the Competitive Equilibrium budget constraint. We guess-and-verify that we can decentralize using a constant portfolio. The budget constraint (1.8) reduces to the following static constraint:

$$\widetilde{C}_t^i = \widetilde{L}_t^i + \sum_j \Lambda_j^i \widetilde{\Pi}_t^j + \sum_j \mathscr{E}_t^j B_j^i.$$

Put in matrix form and simplify:

$$\begin{aligned}
\widetilde{C}_t &= \widetilde{L}_t + \Lambda \widetilde{\Pi}_t + B\mathscr{E}_t \\
&= (1-\gamma_i)\theta \widetilde{Y}_t + \Lambda \left[(1-\gamma_i)\alpha \widetilde{Y}_t - \widetilde{I}_t \right] + B\mathscr{E}_t \\
&= [\theta \mathbf{I} + \alpha \Lambda](I - \Gamma)\widetilde{Y}_t - \Lambda \widetilde{I}_t + B\mathscr{E}_t,
\end{aligned}$$

where $\Gamma \equiv diag(\gamma_1, ..., \gamma_N)$.

Written in terms of deviation from the steady state, and denote $\widetilde{B} \equiv B\overline{\mathscr{E}}$ to be the dollar bond position in steady state, the condition is given by:

$$\widehat{\widetilde{C}}_t = \left(1-\gamma\right)[\theta \mathbf{I} + \alpha \Lambda] \widehat{\widetilde{Y}}_t - \Lambda \widehat{\widetilde{I}}_t + \widetilde{B} \widehat{e}_t. \tag{A.19}$$

Next, we linearize the market clearing conditions. Written in nominal quantities, the market clearing conditions are given by:

$$\widetilde{Y}_t^i = \sum_{j=1}^{N} (\widetilde{G}_{it}^j + \widetilde{X}_{it}^j), \quad i = 1, 2, .., N$$

where again $\widetilde{Y}_t^i \equiv \mathscr{E}_t^i P_t^i Y_t^i$, $\widetilde{G}_{it}^j \equiv \mathscr{E}_t^i P_t^i G_{it}^j$, and $\widetilde{X}_{it}^j = \mathscr{E}_t^i P_t^i X_{it}^j$. Denote $\widehat{X} \equiv X - \overline{X}$ to be the deviation in level from the steady state, and lowercase variables to be the log of uppercase ones: $x \equiv \ln X$. For small changes, $\widehat{X} = \overline{X}\widehat{x}$. The market clearing conditions can be log linearized as following:

$$\widehat{\widetilde{Y}}_t^i = \sum_{j=1}^{N} \left(\widehat{\widetilde{G}}_{it}^j + \widehat{\widetilde{X}}_{it}^j \right) = \sum_{j=1}^{N} \left(\overline{\widetilde{G}}_i^j \widehat{\widetilde{g}}_{it}^j + \overline{\widetilde{X}}_i^j \widehat{\widetilde{x}}_{it}^j \right) \tag{A.20}$$

The optimal conditions for input choices are:

$$\begin{aligned}
\widetilde{g}_{it}^j &= \widetilde{g}_t^j + \ln \xi_{ji} + (1-\varepsilon)\left(p_t^i + e_t^i - e_t^j\right), \\
\widetilde{x}_{it}^j &= \widetilde{x}_t^j + \ln \omega_{ji} + (1-\varepsilon)\left(p_t^i + e_t^i - p_{Xt}^j - e_t^j\right)
\end{aligned}$$

Substitute into the market clearing condition (A.20) and simplify, we have:

$$\begin{aligned}\widehat{\widetilde{Y}}_t^i &= \sum_{j=1}^N \left(\overline{\widetilde{G}}_i^j \widehat{\widetilde{g}}_t^j + \overline{\widetilde{X}}_i^j \widehat{\widetilde{x}}_t^j\right) + (1-\varepsilon)\overline{\widetilde{Y}}^i \left(\widehat{p}_t^i + \widehat{e}_t^i\right) \\ &\quad - (1-\varepsilon)\sum_{j=1}^N \overline{\widetilde{G}}_i^j \widehat{e}_t^j - (1-\varepsilon)\sum_{j=1}^N \overline{\widetilde{X}}_i^j \left(\widehat{p}_{Xt}^j + \widehat{e}_t^j\right).\end{aligned}$$

Since $\overline{\widetilde{G}}_i^j = \Xi_{ji}\overline{\widetilde{G}}^j$, $\overline{\widetilde{X}}_i^j = \gamma\overline{\Omega}_{ji}\overline{\widetilde{Y}}^j$, $\widehat{\widetilde{x}}_t^j = \widehat{\widetilde{y}}_t^j$, we can simplify the equation above further:

$$\begin{aligned}\widehat{\widetilde{Y}}_t^i &= \sum_{j=1}^N \left(\overline{\Xi}_{ji}\widehat{\widetilde{G}}_t^j + \gamma\overline{\Omega}_{ji}\widehat{\widetilde{Y}}_t^j\right) + (1-\varepsilon)\overline{\widetilde{Y}}^i \left(\widehat{p}_t^i + \widehat{e}_t^i\right) \\ &\quad - (1-\varepsilon)\sum_{j=1}^N \overline{\Xi}_{ji}\overline{\widetilde{G}}^j \widehat{e}_t^j - (1-\varepsilon)\sum_{j=1}^N \gamma\overline{\Omega}_{ji}\overline{\widetilde{Y}}^j \left(\widehat{p}_{Xt}^j + \widehat{e}_t^j\right).\end{aligned}$$

Put into matrix form:

$$\begin{aligned}\widehat{\widetilde{Y}}_t &= \overline{\Xi}'\widehat{\widetilde{G}}_t + \gamma\overline{\Omega}'\widehat{\widetilde{Y}}_t + (1-\varepsilon)\Phi_Y\left(\widehat{p}_t + \widehat{e}_t\right) \\ &\quad - (1-\varepsilon)\overline{\Xi}'\Phi_G\widehat{e}_t - (1-\varepsilon)\gamma\overline{\Omega}'\Phi_Y\left(\widehat{p}_{Xt} + \widehat{e}_t\right).\end{aligned} \tag{A.21}$$

Log linearizing the price indices for intermediate and final consumption bundle:

$$\begin{aligned}\mathscr{E}_t^i &= \left[\sum_{j=1}^N \Xi_{ij}\left(\mathscr{E}_t^j P_t^j\right)^{1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}} \\ \mathscr{E}_t^i P_{Xt}^i &= \left[\sum_{j=1}^N \Omega_{ij}\left(\mathscr{E}_t^j P_t^j\right)^{1-\varepsilon}\right]\end{aligned}$$

we have:

$$\widehat{e}_t = \overline{\Xi}(\widehat{p}_t + \widehat{e}_t)$$

$$\widehat{p}_{Xt} + \widehat{e}_t = \overline{\Omega}(\widehat{p}_t + \widehat{e}_t).$$

Substitute into (A.21), we have:

$$(1-\gamma)\widehat{\widetilde{Y}}_t = \overline{M}\widehat{\widetilde{G}}_t - \overline{M}_P\left(\widehat{p}_t + \widehat{e}_t\right) \tag{A.22}$$

where $\overline{M} \equiv (1-\gamma)\left[\mathbf{I} - \gamma\overline{\Omega}'\right]^{-1}\overline{\Xi}'$ is the steady state International Domar Weights, and

$$\overline{M}_P \equiv (\varepsilon-1)(1-\gamma)\left[\mathbf{I} - \gamma\overline{\Omega}'\right]^{-1}\left(\overline{\Xi}'\Phi_G\overline{\Xi} + \gamma\overline{\Omega}'\Phi_Y\overline{\Omega} - \Phi_Y\right)$$

is a matrix that describes the expenditure-switching effect.[2]

Substitute the linearized market clearing condition (A.22) into the budget constraint (A.19), we obtain:

$$\begin{aligned}\widehat{\widetilde{C}}_t &= (1-\gamma)[\theta\mathbf{I}+\alpha\Lambda]\widehat{\widetilde{Y}}_t - \Lambda\widehat{\widetilde{I}}_t + \widetilde{B}\widehat{e}_t \\ &= [\theta\mathbf{I}+\alpha\Lambda]\overline{M}\widehat{\widetilde{G}}_t - [\theta\mathbf{I}+\alpha\Lambda]\overline{M}_P(\widehat{p}_t+\widehat{e}_t) - \Lambda\widehat{\widetilde{I}}_t + \widetilde{B}\widehat{e}_t\end{aligned}$$

Use the identity $\widehat{\widetilde{G}}_t = \widehat{\widetilde{C}}_t + \widehat{\widetilde{I}}_t$ and re-arrange:

$$\begin{aligned}\left\{\mathbf{I}-[\theta\mathbf{I}+\alpha\Lambda]\overline{M}\right\}\widehat{\widetilde{C}}_t &= \left\{[\theta\mathbf{I}+\alpha\Lambda]\overline{M}-\Lambda\right\}\widehat{\widetilde{I}}_t \\ &\quad -[\theta\mathbf{I}+\alpha\Lambda]\overline{M}_P(\widehat{p}_t+\widehat{e}_t)+\widetilde{B}\widehat{e}_t \qquad \text{(A.23)}\end{aligned}$$

The equity portfolio $\Lambda = \theta\overline{M}\left[\mathbf{I}-\alpha\overline{M}\right]^{-1}$ (as in (1.17)) ensures that the first term on the RHS of (A.23) is zero. In other words, the equity portfolio hedges completely against demand risk arising from fluctuating investment. The second term on the RHS of (A.23) can be made zero by having a bond portfolio:

$$\begin{aligned}\widetilde{B}^{TOT}\widehat{e}_t - [\theta\mathbf{I}+\alpha\Lambda]\overline{M}_P(\widehat{p}_t+\widehat{e}_t) = 0 \\ \left[\widetilde{B}^{TOT}\Xi - [\theta\mathbf{I}+\alpha\Lambda]\overline{M}_P\right](\widehat{p}_t+\widehat{e}_t) = 0\end{aligned}$$

Setting $\widetilde{B}^{TOT}\Xi = [\theta\mathbf{I}+\alpha\Lambda]\overline{M}_P$ makes this term zero for all values of TOT $\widehat{p}+\widehat{e}$. In other words, this TOT-hedging portfolio hedges completely income risks arising from TOT fluctuations.

Finally, we are left with choosing the bond portfolio $\widetilde{B}^{RER}$ that hedges consumption risk:

$$\left\{\mathbf{I}-[\theta\mathbf{I}+\alpha\Lambda]\overline{M}\right\}\widehat{\widetilde{C}}_t = \widetilde{B}^{RER}\widehat{e}_t.$$

Note that $[\theta\mathbf{I}+\alpha\Lambda]\overline{M} = \Lambda$ from the choice of optimal equity. This simplifies to:

$$\widetilde{B}^{RER}\widehat{e}_t = (\mathbf{I}-\Lambda)\widehat{\widetilde{C}}_t$$

[2]Note that:

$$\left(\Xi'\Phi_G\Xi + \gamma\overline{\Omega}'\Phi_Y\overline{\Omega} - \Phi_Y\right)1 = \Xi'\overline{\widetilde{G}} + \gamma\overline{\Omega}'\overline{\widetilde{Y}} - \overline{\widetilde{Y}} = 0$$

from the market clearing condition. This implies that $M_P 1 = 0$, which has the intuitive meaning that a proportional change in prices does not affect output, since the relative price has not changed.

To this end, we guess-and-verify that complete markets are achieved. This implies that consumption satisfies:

$$\widehat{\widetilde{c}}_t = \widehat{\widetilde{c}}_t \mathbf{1} + (1-\sigma)\widehat{e}_t.$$

This is the multi-country version of the Backus-Smith condition. Using $\widehat{\widetilde{C}}_t = \Phi_C \widehat{\widetilde{c}}_t$, where Φ_C is a diagonal matrix containing steady state consumption, we have:

$$\begin{aligned}
\widetilde{B}^{RER}\widehat{e}_t &= (\mathbf{I}-\Lambda)\Phi_C\left(\widehat{\widetilde{c}}_t \mathbf{1} + (1-\sigma)\widehat{e}_t\right) \\
&= \widehat{\widetilde{c}}_t^1 (\mathbf{I}-\Lambda)\overline{\widetilde{C}} + (1-\sigma)(\mathbf{I}-\Lambda)\Phi_C \widehat{e}_t.
\end{aligned}$$

If we choose the bond portfolio $\widetilde{B}^{RER} = (1-\sigma)(\mathbf{I}-\Lambda)\Phi_C$, and $\overline{\widetilde{C}}$ that satisfies $(\mathbf{I}-\Lambda)\overline{\widetilde{C}} = 0$, then the budget constraint is completely satisfied.

A.2.1 Proof of Proposition 2

To focus on the equity portfolio, let us assume $\sigma = \varepsilon = 1$, but the proof follows more generally.

The equity portfolio is chosen to satisfy the household budget constraint

$$\widetilde{C}_t = \widetilde{L}_t + \Lambda \widetilde{\Pi}_t$$

where Λ has dimension $N \times S$ ($S = |\mathcal{K}|$ is the number of world industries), $\widetilde{\Pi} = \left(1-\gamma\right)\alpha \widetilde{Y} - \widetilde{I}$ has dimension $S \times 1$. The labor income is found as the fraction $(1-\gamma)\theta$ of domestic industries' sales:

$$\widetilde{L}_t = \left(1-\gamma\right)\theta E \widetilde{Y}_t$$

where $E_{N\times S}$ denotes the modified identity matrix (1.25). Expanding, we have:

$$\begin{aligned}
\widetilde{C}_t &= \left(1-\gamma\right)\theta E \widetilde{Y}_t + \Lambda\left(1-\gamma\right)\alpha \widetilde{Y}_t - \Lambda \widetilde{I}_t \\
&= [\theta E + \alpha\Lambda](1-\gamma)\widetilde{Y}_t - \Lambda \widetilde{I}_t
\end{aligned}$$

We have:

$$\widetilde{Y}_t^i = \sum_{j=1}^{S} \gamma \Omega_{ji} \widetilde{Y}_t^j + \sum_{j=1}^{N} \Xi_{ji} \widetilde{G}_t^j$$

$$\widetilde{Y}_t = \left[\mathbf{I} - \gamma \Omega'\right]^{-1} \Xi' \widetilde{G}_t$$

$$\left(1-\gamma\right) \widetilde{Y}_t = M \widetilde{G}_t,$$

with $M = \left(1-\gamma\right)\left[\mathbf{I} - \gamma \Omega'\right]^{-1} \Xi'$ is the IDWs matrix. M has dimension $S \times N$. Substitute into the budget constraint:

$$\widetilde{C}_t = [\theta E + \alpha \Lambda] M \left(\widetilde{C}_t + E \widetilde{I}_t\right) - \Lambda \widetilde{I}_t$$

$$[\mathbf{I} - [\theta E + \alpha \Lambda] M] \widetilde{C}_t = [[\theta E + \alpha \Lambda] M E - \Lambda] \widetilde{I}_t$$

The optimal equity portfolio satisfies:

$$\Lambda = \theta E M E + \alpha \Lambda M E \Rightarrow \Lambda = \theta E M E \left[\mathbf{I} - \alpha M E\right]^{-1}.$$

A.2.2 Proof of Proposition 3

To focus on the equity portfolio, let us assume $\sigma = \varepsilon = 1$, but the proof follows more generally.

Labor income now comes from two sources, traded and nontraded sector:

$$\widetilde{L} = \left(1-\gamma\right) \theta \widetilde{Y} + \iota \widetilde{G}.$$

(Since labor is the only factor of production in the nontraded sector, they get the entire revenue of that sector, which is $\iota \widetilde{G}$.)

Consider the static budget constraint:

$$\begin{aligned}
\widetilde{C} &= \widetilde{L} + \Lambda \widetilde{I} \\
&= \left(1-\gamma\right) \theta \widetilde{Y} + \iota \widetilde{G} + \Lambda \left[\left(1-\gamma\right) \alpha \widetilde{Y} - \widetilde{I}\right] \\
&= [\theta \mathbf{I} + \alpha \Lambda] \left(1-\gamma\right) \widetilde{Y} - \Lambda \widetilde{I} + \iota \widetilde{I} + \iota \widetilde{C} \\
(1-\iota) \widetilde{C} &= [\theta \mathbf{I} + \alpha \Lambda] \left(1-\gamma\right) \widetilde{Y} - \Lambda \widetilde{I} + \iota \widetilde{I}
\end{aligned}$$

The market clearing condition with nontraded sector is:

$$\left(1-\gamma\right)\widetilde{Y} = (1-\iota)M\widetilde{G}$$

since the traded sector only gets fraction $1-\iota$ of final expenditure. Substitute into the budget constraint, we have:

$$(1-\iota)\widetilde{C} = (1-\iota)[\theta\mathbf{I}+\alpha\Lambda]M\widetilde{G} - \Lambda\widetilde{I} + \iota\widetilde{I}$$

$$(1-\iota)[\mathbf{I}-[\theta\mathbf{I}+\alpha\Lambda]M]\widetilde{C} = [(1-\iota)[\theta\mathbf{I}+\alpha\Lambda]M - \Lambda + \iota\mathbf{I}]\widetilde{I}$$

The optimal equity portfolio solves:

$$\Lambda = \iota\mathbf{I} + (1-\iota)[\theta\mathbf{I}+\alpha\Lambda]M$$

$$\Lambda = [\iota\mathbf{I} + (1-\iota)\theta M][\mathbf{I}-(1-\iota)\alpha M]^{-1}.$$

A.2.3 Proof of Proposition 4

Given complete markets, I work with the Social Planner problem (detailed in Appendix A.1). Let μ_t and ν_t be the (log) Lagrange multiplier on the resource constraint for final and intermediate good, respectively.

Guess, and later verify, that the investment-GDP ratio is constant:

$$\frac{\widetilde{I}_t}{\widetilde{Y}_t} = \frac{\widetilde{K}_{t+1}}{\widetilde{Y}_t} = \zeta,$$

for some ζ to be determined. Recall that $\widetilde{I}_t = e^{\mu_t}I_t$, $\widetilde{K}_{t+1} = e^{\mu_t}K_{t+1}$, and $\widetilde{Y}_t = e^{\nu_t}Y_t$ are nominal amounts. The market clearing conditions are given by:

$$\widetilde{Y}_t = M\left(\widetilde{C}_t + \widetilde{I}_t\right),$$

where $M = \left(1-\gamma\right)\left[I-\gamma\Omega'\right]^{-1}\Xi'$ is the matrix of IDWs as defined in Section 1.4. When utility is log and production function is Cobb-Douglas, $\widetilde{C}_t = \chi$, a vector of constant (χ is the Pareto weights, normalized). Since $\widetilde{I}_t = \zeta\widetilde{Y}_t$, this implies that these two variables are constants as well:

$$\widetilde{Y}_t = M\chi + \zeta M\widetilde{Y}_t \Rightarrow \widetilde{Y}_t = [\mathbf{I}-\zeta M]^{-1}M\chi.$$

The Euler equation when $\delta = 1$ is:

$$\mathbb{E}_t \left[\exp\left(-\rho + \Delta\mu^i_{t+1} + \ln r^i_{t+1} \right) \right] = 1$$

where $\ln r^i_{t+1} \equiv \ln\left[(1-\gamma)\alpha\right] + \widetilde{y}^i_{t+1} - k^i_{t+1} - \mu^i_{t+1}$ and $\rho = -\ln\beta$. The constant ζ can be solved out from the Euler equation in the steady state:

$$-\rho + \ln\left[(1-\gamma)\alpha\right] - \ln\zeta = 0 \Rightarrow \zeta = \beta\left(1-\gamma\right)\alpha.$$

The return on capital is then given by:

$$\ln r^i_{t+1} = \ln\left[(1-\gamma)\alpha\right] - \ln\zeta - \Delta\mu^i_{t+1} = \rho - \Delta\ln\mu^i_{t+1}.$$

Thus, the Euler equation is satisfied with a constant investment-GDP ratio.

Finally, cost minimization in the intermediate sector implies:

$$v_t = -z_t + \left(1-\gamma\right)\alpha\left(\mu_t + \ln r_t\right) + \gamma\Omega v_t + h$$

with $h = \left(1-\gamma\right)\alpha\ln\left[(1-\gamma)\alpha\right] + \left(1-\gamma\right)\theta\widetilde{y} + \sum_j \gamma\omega_{ij}\ln\left[\gamma\omega_{ij}\right]$. Re-arranging, we have:

$$v_t = \left[\mathbf{I} - \gamma\Omega\right]^{-1}\left[-z_t + \left(1-\gamma\right)\alpha\mu_{t-1} + h + \left(1-\gamma\right)\alpha\rho\right]$$

Multiply both sides by Ξ, we have:

$$\mu_t = \frac{M'}{1-\gamma}\left[-z_t + \left(1-\gamma\right)\alpha\mu_{t-1} + h_2\right]$$

with $h_2 = h + (1-\gamma)\alpha\rho$. Let $\overline{\mu}$ be the solution to $\overline{\mu} = -\left[\mathbf{I} - \alpha M'\right]^{-1} h_2$, we have:

$$\mu_t - \overline{\mu} = -\frac{M'}{1-\gamma} z_t + \alpha M'\left(\mu_{t-1} - \overline{\mu}\right)$$

Take the first difference, we have:

$$\Delta\mu_t = -\frac{M'}{1-\gamma}\Delta z_t + \alpha M'\Delta\mu_{t-1}.$$

When TFP follows a random walk, $\Delta z = \varepsilon \sim \mathcal{N}(0,\Sigma)$ is independent of the second term $\alpha M'\Delta\mu_{t-1}$.

We can calculate $\text{cov}(\Delta\mu)$ as:

$$\text{cov}(\Delta\mu_t) = \frac{M'}{1-\gamma}\Sigma\frac{M}{1-\gamma} + \alpha^2 M'\text{cov}(\Delta\mu_{t-1})M.$$

Let $\Sigma_\mu = \text{cov}(\Delta\mu_t)$. Let vec denote the vectorization operator (stacking matrix into a column) and $\otimes$denote the Kronecker product. We have:

$$vec(\Sigma_\mu) = \frac{1}{(1-\gamma)^2}M'\otimes M' vec(\Sigma) + \alpha^2 M'\otimes M' vec(\Sigma_\mu)$$
$$vec(\Sigma_\mu) = \frac{1}{(1-\gamma)^2}\left[I - \alpha^2 M'\otimes M'\right]^{-1} M'\otimes M' vec(\Sigma).$$

Given that $c+\mu$ is a constant in this setting, $vec(\Sigma_c) = vec(\Sigma_\mu)$ as well. Since $r_{t+1} = \rho - \Delta\mu_{t+1}$, the return to capital also has the same covariance matrix as μ.

Finally, the vector of exchange rate in the decentralized economy is given by:

$$e = A\mu,$$

with

$$A = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & -1 & 0 & \dots & 0 \\ 1 & 0 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & -1 \end{bmatrix}.$$

Thus:

$$\Sigma_e = A\Sigma_\mu A'$$
$$vec(\Sigma_e) = (A\otimes A)vec(\Sigma_\mu).$$

Appendix B

Appendix to Chapter 2

Appendix C

Appendix to Chapter 3

C.1 Optimal reset price of a rational firm

Solving the rational firm's optimization problem, the optimal reset price is a markup μ above the firm's average of today and future expected marginal costs:

$$p_t^{rat} = \mu + \left(1-\beta\theta\right)\sum_{s=0}^{\infty}\left(\beta\theta\right)^s \mathbb{E}_t\left[\psi_{t+s}(p_t^{rat})\right] \tag{C.1}$$

where $\mu = \log\frac{\varepsilon}{\varepsilon-1}$ is the (log) frictionless markup, and $\psi_{t+s}(p) = \log\Phi'\left(Y_{t+s|t}(e^p)\right)$ is the (log) marginal cost at date $t+s$ of a firm whose price is last reset on date t.

The cost function of a firm with producing output $Y(\omega)$ at date $t+s$ $\Phi = (1-\tau_w)W_{t+s}\left(\frac{Y(\omega)}{A_{t+s}}\right)^{1/(1-\alpha)}$. The log marginal cost is:

$$\begin{aligned}\psi = \log\Phi'(Y(\omega)) &= \log\left[\left(\frac{1-\tau_w}{1-\alpha}\right)W_{t+s}Y(\omega)^{\frac{\alpha}{1-\alpha}}A_{t+s}^{\frac{-1}{1-\alpha}}\right]\\ &= \log\left(\frac{1-\tau_w}{1-\alpha}\right) + w_{t+s} + \frac{\alpha}{1-\alpha}y(\omega) - \frac{1}{1-\alpha}a_{t+s}.\end{aligned}$$

For a firm with price $p(\omega)$ at date $t+s$, its log marginal cost is:

$$\psi = \log\left(\frac{1-\tau_w}{1-\alpha}\right) + w_{t+s} + \frac{\alpha}{1-\alpha}(y_{t+s} - \varepsilon(p(\omega) - p_{t+s})) - \frac{1}{1-\alpha}a_{t+s}.$$

From the household's consumption-leisure choice:

$$\begin{aligned} w_t - p_t &= \sigma^{-1} y_t + \varphi l_t \\ &= \sigma^{-1} y_t + \varphi \frac{y_t - a_t}{1-\alpha} \\ &= \frac{\sigma^{-1}(1-\alpha)+\varphi}{1-\alpha} y_t - \frac{\varphi}{1-\alpha} a_t. \end{aligned}$$

Substitute into the expression for marginal cost, we have:

$$\psi = \log\left(\frac{1-\tau_w}{1-\alpha}\right) + p_{t+s} + \frac{\sigma^{-1}(1-\alpha)+\varphi+\alpha}{1-\alpha} y_{t+s} - \frac{\alpha\varepsilon}{1-\alpha}(p(\omega) - p_{t+s})) - \frac{1+\varphi}{1-\alpha} a_{t+s}. \tag{C.2}$$

Apply equation (C.2) for the frictionless economy, where $p(\omega) = p_{t+s} = \psi + \mu$, we have:

$$0 = \log\left(\frac{1-\tau_w}{1-\alpha}\right) + \mu + \frac{\sigma^{-1}(1-\alpha)+\varphi+\alpha}{1-\alpha} y^n_{t+s} - \frac{1+\varphi}{1-\alpha} a_{t+s}.$$

Choose $\log(1-\tau_w) = -\mu$ (i.e. $\tau_w = 1/\varepsilon$) to be the subsidy that removes inefficiency arising from monopolistic competition, we have an expression for the natural output y^n:

$$0 = \log\frac{1}{1-\alpha} + \frac{\sigma^{-1}(1-\alpha)+\varphi+\alpha}{1-\alpha} y^n_{t+s} - \frac{1+\varphi}{1-\alpha} a_{t+s}. \tag{C.3}$$

Substitute (C.3) into (C.2), and define the output gap $x_t = y_t - y^n_t$:

$$\psi_{t+s}(p(\omega)) = p(\omega) - \mu - \lambda^{-1}(p(\omega) - p_{t+s}) + \upsilon x_{t+s}, \tag{C.4}$$

with $\lambda \equiv \frac{1-\alpha}{1-\alpha+\alpha\varepsilon}$ and $\upsilon \equiv \frac{\sigma^{-1}(1-\alpha)+\varphi+\alpha}{1-\alpha}$.

Substitute into equation (C.1) and rearrange, we have the desired relationship:

$$p^{rat}_t = (1-\beta\theta) \sum_{s=0}^{\infty} (\beta\theta)^s \, \mathbb{E}_t [p_{t+s} + \lambda\upsilon x_{t+s}].$$

C.2 Proof for Proposition 5

The optimal reset price of a behavioral firm is:

$$\begin{aligned} p^{BR}_t &= (1-\beta\theta) \sum_{s=0}^{\infty} (\beta\theta)^s \, \mathbb{E}^{BR}_t [p_{t+s} + \lambda\upsilon x_{t+s}] \\ &= (1-\beta\theta)\mathbb{E}^{BR}_t [p_t + \lambda\upsilon x_t] + \beta\theta \mathbb{E}^{BR}_t \left\{ (1-\beta\theta) \sum_{s=0}^{\infty} (\beta\theta)^s (p_{t+1+s} + \lambda\upsilon x_{t+1+s}) \right\} \end{aligned} \tag{C.5}$$

Use the assumption that behavioral expectations for future periods satisfy $\mathbb{E}_t^{BR}[X_{t+s}] = \mathbb{E}_t\left[\mathbb{E}_{t+1}^{BR} X_{t+s}\right]$ $(s \geq 1)$, we can rewrite equation C.5 recursively:

$$p_t^{BR} = (1-\beta\theta)\mathbb{E}_t^{BR}[p_t + \lambda \upsilon x_t] + \beta\theta\mathbb{E}_t\left[p_{t+1}^{BR}\right]. \tag{C.6}$$

Subtract p_{t-1} from both sides of equation (C.6), we have:

$$p_t^{BR} - p_{t-1} = \left(1-\beta\theta\right)\left(\mathbb{E}_t^{BR} p_t - p_{t-1} + \lambda\upsilon\mathbb{E}_t^{BR} x_t\right) + \beta\theta\mathbb{E}_t\left[p_{t+1}^{BR} - p_{t-1}\right] \tag{C.7}$$

In equilibrium, a fraction $1-\theta$ of firms can reset prices and choose p_t^{BR}, while the remaining firms are stuck at their old prices. Therefore, inflation is given by:

$$\pi_t = p_t - p_{t-1} = (1-\theta)(p_t^{BR} - p_{t-1}). \tag{C.8}$$

As a result, the term on LHS of (C.7) equals $\pi_t/(1-\theta)$. Furthermore, the RHS term $\mathbb{E}_t^{BR} p_t - p_{t-1}$, which indicates inflation as perceived by the behavioral agents, can be written as a weighted average of actual inflation and the default inflation:

$$\mathbb{E}_t^{BR} p_t - p_{t-1} = m_t^p(p_t - p_{t-1}) + (1-m_t^p)(p_t^d - p_{t-1}) = m_t^p \pi_t + (1-m_t^p)\pi_t^d. \tag{C.9}$$

Substitute equations (C.8) and (C.9) into equation (C.7), we have:

$$\frac{\pi_t}{1-\theta} = (1-\beta\theta)\left(m_t^p \pi_t + (1-m_t^p)\pi_t^d + \lambda\upsilon m_t^x x_t\right) + \beta\theta\mathbb{E}_t\left[\frac{\pi_{t+1}}{1-\theta} + \pi_t\right]$$

Collect terms and rearrange, we get the BAPC:

$$\pi_t = \omega_t^d \pi_t^d + \left(1-\omega_t^d\right)\beta\mathbb{E}_t[\pi_{t+1}] + \kappa_t x_t,$$

where the weight on default inflation is given by:

$$\omega_t^d = \frac{(1-\theta)\left(1-\beta\theta\right)(1-m_t^p)}{\theta + (1-\theta)\left(1-\beta\theta\right)(1-m_t^p)} \in [0,1]$$

and the output-gap slope is given by:

$$\kappa_t = \frac{(1-\theta)\left(1-\beta\theta\right)\lambda\upsilon m_t^x}{\theta + (1-\theta)\left(1-\beta\theta\right)(1-m_t^p)} \in [0, \kappa_{max}]$$

where $\kappa_{max} = \frac{(1-\theta)(1-\beta\theta)}{\theta+(1-\theta)(1-\beta\theta)}\lambda\upsilon$. (Recall that $\lambda \equiv \frac{1-\alpha}{1-\alpha+\alpha\varepsilon}$ and $\upsilon \equiv \frac{\sigma^{-1}(1-\alpha)+\varphi+\alpha}{1-\alpha}$.)

C.3 Proof of Proposition 7

Suppose it is only costly to think about prices, i.e. $\chi_y = 0$ and $m_t^y = 1$. The flexible-price BAPC is:

$$\pi_t - \pi_t^d = \kappa_t x_t, \quad \text{where } \kappa_t = \frac{\lambda \upsilon}{1 - m_t^p}. \tag{C.10}$$

Recall that $\lambda \equiv \frac{1-\alpha}{1-\alpha+\alpha\varepsilon}$ and $\upsilon \equiv \frac{\sigma^{-1}(1-\alpha)+\varphi+\alpha}{1-\alpha}$. From the firm optimal attention problem, inattention to prices is given by equation (3.11):

$$1 - m_t^p = \min\left\{1, \frac{\chi_p}{2\Lambda\sigma_{p,t}^2}\right\}.$$

Thus, the output gap slope can be rewritten as a weakly increasing function of inflation uncertainty:

$$\kappa_t = \begin{cases} \lambda\upsilon, & \text{if } \sigma_{p,t}^2 \leq \frac{\chi_p}{2\Lambda}, \\ \lambda\upsilon \frac{2\Lambda}{\chi_p}\sigma_{p,t}^2, & \text{if } \sigma_{p,t}^2 > \frac{\chi_p}{2\Lambda}. \end{cases} \tag{C.11}$$

Let γ_π (resp. γ_x) denote the pass through from monetary policy shocks to inflation (resp. output gap) in equilibrium. We can write:

$$\pi_t = \pi_t^d + \gamma_{\pi t}\varepsilon_t^\nu$$
$$x_t = \gamma_{xt}\varepsilon_t^\nu.$$

Differentiate the flexible-price BAPC with respect to ε_t^ν, we have:

$$\gamma_{\pi t} = \kappa_t \gamma_{xt}.$$

Turn to the demand block of the model, the consumer Euler equation is:

$$c_t = \mathbb{E}_t c_{t+1} - \sigma\left(i_t - \mathbb{E}_t\pi_{t+1} - \rho\right).$$

Substitute $c_t = y_t$, $x_t = y_t - y_t^n$, we get:

$$x_t = \mathbb{E}_t x_{t+1} - \sigma\left(i_t - r_t^n - \mathbb{E}_t\pi_{t+1}\right), \quad r_t^n = \rho + \frac{1}{\sigma}\mathbb{E}_t\Delta y_{t+1}^n.$$

In the flexible price case, $\mathbb{E}_t x_{t+1} = \mathbb{E}_t\varepsilon_{t+1}^\nu = 0$. Given the Taylor rule $i_t = r_t^n + \phi_\pi\pi_t + \phi_x x_t + \nu_t$, we get:

$$x_t = -\sigma\left(\phi_\pi \pi_t + \phi_x x_t + \nu_t - \mathbb{E}_t \pi_{t+1}\right).$$

We can solve for inflation:

$$\pi_t = -\frac{\sigma^{-1} + \phi_x}{\phi_\pi} x_t + \nu_t + \frac{1}{\phi_\pi}\mathbb{E}_t \pi_{t+1}$$

Solve forward:

$$\pi_t = -\frac{\sigma^{-1} + \phi_x}{\phi_\pi} x_t + \sum_{s=0}^{\infty} \frac{1}{\phi_\pi^s}\mathbb{E}_t \nu_{t+s} = -\frac{\sigma^{-1} + \phi_x}{\phi_\pi} x_t + \left[\sum_{s=0}^{\infty}\left(\frac{\rho_\nu}{\phi_\pi}\right)^s\right]\nu_t$$

$$\pi_t = -\frac{\sigma^{-1} + \phi_x}{\phi_\pi} x_t + \frac{\phi_\pi}{\phi_\pi - \rho_\nu}\nu_t. \tag{C.12}$$

Differentiate equation (C.12) with respect to ε_t^ν, we have:

$$\gamma_{\pi t} = -\frac{\sigma^{-1} + \phi_x}{\phi_\pi}\gamma_{xt} + \frac{\phi_\pi}{\phi_\pi - \rho_\nu} = -\frac{\sigma^{-1} + \phi_x}{\phi_\pi}\frac{\gamma_{\pi t}}{\kappa_t} + \frac{\phi_\pi}{\phi_\pi - \rho_\nu}.$$

This implies that the pass through from monetary policy to inflation is:

$$\gamma_{\pi t} = \frac{-\phi_\pi}{\left(\phi_\pi - \rho_\nu\right)\left(1 + \frac{\sigma^{-1} + \phi_x}{\phi_\pi}\frac{1}{\kappa_t}\right)} = \frac{-a}{1 + b/\kappa_t}$$

with $a \equiv \phi_\pi / \left(\phi_\pi - \rho_\nu\right)$ and $b \equiv \left(\sigma^{-1} + \phi_x\right)/\phi_\pi$. Inflation uncertainty is then given by:

$$\sigma_{p,t}^2 = \mathbb{E}_t\left[\left(\pi_t - \pi_t^d\right)^2\right] = \mathbb{E}_t\left[\gamma_{\pi t}^2\left(\varepsilon_t^\nu\right)^2\right] = \gamma_{\pi t}^2\sigma_\nu^2.$$

Take square root on both sides, we have:

$$\sigma_{p,t} = |\gamma_{\pi t}|\sigma_\nu^2 = \frac{a}{1 + b/\kappa_t}\sigma_\nu. \tag{C.13}$$

Equations (C.11) and (C.13) together form a system to solve simultaneously for the output gap slope κ and inflation uncertainty $\sigma_{p,t}^2$, for a given level of exogenous volatility σ_ν^2. Combine the two equations, we get a single equation for the output gap slope:

$$\kappa_t = \lambda v \max\left\{1, \frac{a^2}{(1 + b/\kappa_t)^2 \frac{\chi_p}{2\Lambda}}\sigma_{\nu,t}^2\right\}.$$

The obvious solution for κ is the smallest root, $\kappa = \lambda v$. This root requires that the second in the max operator to be smaller than 1. This occurs when $\sigma^2_{\nu,t}$ is sufficiently small:

$$\sigma^2_{\nu,t} \leq \overline{\sigma}^2_\nu = \frac{(1+b/(\lambda v))^2}{a^2}\frac{\chi_p}{2\Lambda}.$$

We now look at the case where $\kappa = \lambda v \frac{\sigma^2_{p,t}}{\chi_p/2\Lambda} > \lambda v$. The output gap slope solves:

$$\kappa_t = 2\frac{\Lambda\lambda v}{\chi_p}\frac{a^2}{(1+b/\kappa_t)^2}\sigma^2_{\nu,t}.$$

Re-arrange and collect terms:

$$\kappa^2 + 2\left(b - \frac{\Lambda\lambda v a^2}{\chi_p}\sigma^2_\nu\right)\kappa + b^2 = 0$$

This quadratic equation has at least one root when exogenous volatility exceeds a certain threshold:

$$\sigma^2_\nu \geq \underline{\sigma}^2_\nu \equiv \frac{2b}{\Lambda\lambda v a^2/\chi_p}.$$

Whenever $\sigma^2_\nu > \underline{\sigma}^2_\nu$, the quadratic equation has two positive roots $\kappa_1 > \kappa_2 > 0$, given by:

$$\kappa_1 = c\sigma^2_\nu - b + \sqrt{\left(c\sigma^2_\nu - b\right)^2 - b^2}$$
$$\kappa_2 = c\sigma^2_\nu - b - \sqrt{\left(c\sigma^2_\nu - b\right)^2 - b^2}$$

where $c \equiv \frac{\Lambda\lambda v a^2}{\chi_p}$ is a constant.

It can be shown that $\underline{\sigma}^2_\nu$ is always below $\overline{\sigma}^2_\nu$:

$$\overline{\sigma}^2_\nu \geq \underline{\sigma}^2_\nu \Leftrightarrow \frac{\chi_p(1+b/(\lambda v))^2}{2\Lambda a^2} \geq \frac{2b\chi_p}{\Lambda\lambda v a^2} \Leftrightarrow \left(1+\frac{b}{\lambda v}\right)^2 \geq \frac{4b}{\lambda v} \Leftrightarrow \left(1-\frac{b}{\lambda v}\right)^2 \geq 0.$$

We can also show that κ_1 increases with σ^2_ν, while κ_2 decreases with σ^2_ν. We can also show that at $\sigma^2_\nu = \underline{\sigma}^2_\nu$, the quadratic equation has a unique root equal to:

$$\kappa_1 = \kappa_2 = c\underline{\sigma}^2_\nu - b = b = \frac{\sigma^{-1}+\phi_x}{\phi_\pi}.$$