Dynamics and topology of absolute period foliations of strata of holomorphic 1-forms

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presented by **Karl Winsor**

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Signature ____________________________

Typed name: Professor Curtis McMullen

Signature ____________________________

Typed name: Professor Laura DeMarco

Signature ____________________________

Typed name: Professor Dawei Chen (Boston College)

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Karl Winsor

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Abstract

Let S_g be a closed oriented surface of genus g, and let $\Omega\mathcal{M}_g(\kappa)$ be a stratum of the moduli space of holomorphic 1-forms of genus g. In this thesis, we study dynamical and topological properties of the absolute period foliation of $\Omega\mathcal{M}_g(\kappa)$. We show that in most cases, the absolute period foliation is ergodic on the area-1 locus, and we give an explicit full measure set of dense leaves. These dynamical results are obtained as an application of a topological result on the connectedness of the space of holomorphic 1-forms in $\Omega\mathcal{M}_g(\kappa)$ representing a given cohomology class in $H^1(S_g;\mathbb{C})$. As another application, we give a new proof that in most cases, the monodromy representation of $\pi_1(\Omega\mathcal{M}_g(\kappa))$ on absolute homology surjects onto $\mathrm{Sp}(2g,\mathbb{Z})$.

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1. Introduction

Let $\mathcal{M}_g$ be the moduli space of closed Riemann surfaces of genus $g \geq 2$. Let $\Omega\mathcal{M}_g \to \mathcal{M}_g$ be the bundle of pairs (X, ω) with $X \in \mathcal{M}_g$ and ω a nonzero holomorphic 1-form on X. In general, ω has $2g-2$ zeros counted with multiplicity. The space $\Omega\mathcal{M}_g$ is a union of strata $\Omega\mathcal{M}_g(\kappa)$, indexed by partitions $\kappa = \{m_1, \ldots, m_n\}$ of $2g-2 = \sum m_j$, and consisting of holomorphic 1-forms with n distinct zeros of multiplicities $m_1, \ldots, m_n$. Also associated to (X, ω) is its group of absolute periods, the subgroup of $\mathbb{C}$ obtained by integrating ω over closed loops in X. These two invariants give rise to the absolute period foliation $\mathcal{A}(\kappa)$ of $\Omega\mathcal{M}_g(\kappa)$. A leaf of $\mathcal{A}(\kappa)$ is navigated by varying a holomorphic 1-form in $\Omega\mathcal{M}_g(\kappa)$ without changing its absolute periods or its number of zeros.

In this thesis, we study the dynamics of the absolute period foliation of $\Omega\mathcal{M}_g(\kappa)$. We are interested in the distribution of leaves of $\mathcal{A}(\kappa)$ within $\Omega\mathcal{M}_g(\kappa)$. Our main dynamical results will describe the measurable subsets of $\Omega\mathcal{M}_g(\kappa)$ that are unions of leaves of $\mathcal{A}(\kappa)$, and will give a criterion for the closure of a leaf of $\mathcal{A}(\kappa)$ in $\Omega\mathcal{M}_g(\kappa)$ to be as big as possible. These results will follow from a sufficient condition for two holomorphic 1-forms to lie on the same leaf of $\mathcal{A}(\kappa)$. Our results suggest that a version of Ratner's theorems for unipotent flows may hold in this setting, and we raise some open questions along these lines.

Absolute periods. Let S_g be a closed oriented surface of genus g. The absolute periods of a cohomology class $\phi \in H^1(S_g; \mathbb{C})$ are defined by

$$\mathrm{Per}(\phi) = \{\phi(c) \ : \ c \in H_1(S_g; \mathbb{Z})\} \subset \mathbb{C}.$$

When $\mathrm{Per}(\phi)$ has rank $2g$, the algebraic intersection form on $H_1(S_g; \mathbb{Z})$ induces a unimodular symplectic form on $\mathrm{Per}(\phi)$. For $(X, \omega) \in \Omega\mathcal{M}_g$, the holomorphic 1-form ω determines a cohomology class $[\omega] \in H^1(X; \mathbb{C})$, and we define $\mathrm{Per}(\omega) = \mathrm{Per}([\omega])$. For $\kappa = \{m_1, \ldots, m_n\}$ a partition of $2g-2$, let $|\kappa| = n$. The foliation $\mathcal{A}(\kappa)$ is a holomorphic foliation of $\Omega\mathcal{M}_g(\kappa)$

whose leaves have complex dimension $|\kappa| - 1$. Two holomorphic 1-forms are on the same leaf of $\mathcal{A}(\kappa)$ precisely when they are joined by a path (X_t, ω_t) in $\Omega\mathcal{M}_g(\kappa)$ such that $\mathrm{Per}(\omega_t)$ is constant.

Measurable dynamics. Let $\Omega_1\mathcal{M}_g(\kappa)$ be the area-1 locus in $\Omega\mathcal{M}_g(\kappa)$, that is, the subset of holomorphic 1-forms (X, ω) with $\int_X |\omega|^2 = 1$. Our main result on measurable dynamics is the following.

Theorem 1.1. The absolute period foliation of $\Omega_1\mathcal{M}_g(\kappa)$ is ergodic, provided $|\kappa| > 1$ and $\Omega\mathcal{M}_g(\kappa)$ is connected.

Here, ergodicity means that any measurable union of leaves has either zero Lebesgue measure or full Lebesgue measure. Regarding the hypotheses, we remark that most strata are connected. Specifically, by [KZ] a stratum $\Omega\mathcal{M}_g(\kappa)$ is connected if and only if there is $m_j \in \kappa$ that is odd and not equal to $g - 1$, or $g = 2$. We exclude the case $|\kappa| = 1$, since in that case leaves of $\mathcal{A}(\kappa)$ are points. Theorem 1.1 was previously known for the principal stratum $\Omega\mathcal{M}_g(1, \ldots, 1)$ [CDF, Theorem 1.5], [Ham, Theorem 1], [McM5, Proposition 2.6].

Topological dynamics. A free abelian group $\Lambda \subset \mathbb{C}$ of rank r is *algebraically generic* if it has the following two properties.

(1) For any $z_1, z_2 \in \Lambda$, if $\mathbb{R}z_1 = \mathbb{R}z_2$ then $\mathbb{Q}z_1 = \mathbb{Q}z_2$.
(2) For any number field $K \subset \mathbb{R}$, we have $K \cdot \Lambda \cong K^r$ as a K-vector space.

A holomorphic 1-form $(X, \omega) \in \Omega\mathcal{M}_g$ is *algebraically generic* if $\mathrm{Per}(\omega)$ has rank $2g$ and $\mathrm{Per}(\omega)$ is algebraically generic. The set of algebraically generic holomorphic 1-forms in $\Omega_1\mathcal{M}_g(\kappa)$ is a union of leaves of $\mathcal{A}(\kappa)$, and its complement is contained in a countable union of suborbifolds of positive codimension. In particular, the complement has measure zero. We emphasize that for a holomorphic 1-form, being algebraically generic is an explicit condition on its group of absolute periods. Our main result on topological dynamics shows that any leaf of $\mathcal{A}(\kappa)$ in $\Omega_1\mathcal{M}_g(\kappa)$ is either dense in $\Omega_1\mathcal{M}_g(\kappa)$ or contained in an explicit countable union of suborbifolds of positive codimension. In particular, this result provides an explicit full measure set of dense leaves.

Theorem 1.2. Let $(X,\omega) \in \Omega_1\mathcal{M}_g(\kappa)$ be algebraically generic. The leaf of $\mathcal{A}(\kappa)$ through (X,ω) is dense in $\Omega_1\mathcal{M}_g(\kappa)$, provided $|\kappa| > 1$ and $\Omega\mathcal{M}_g(\kappa)$ is connected.

Examples of dense leaves were previously given in $\Omega_1\mathcal{M}_g(1,\ldots,1)$ [CDF, Theorem 1.5], a certain connected component of $\Omega_1\mathcal{M}_g(g-1,g-1)$ [HW, Theorem 1], and $\Omega_1\mathcal{M}_3(2,1,1)$ [Ygo1, Theorem C].

Connectedness. Theorems 1.1 and 1.2 are in fact consequences of a closely related connectedness result for spaces of holomorphic 1-forms with the same absolute periods.

Theorem 1.3. Let $(X,\omega),(Y,\eta) \in \Omega\mathcal{M}_g(\kappa)$ be algebraically generic. If $\mathrm{Per}(\omega) = \mathrm{Per}(\eta)$ as symplectic modules, then there is a path in $\Omega\mathcal{M}_g(\kappa)$ from (X,ω) to (Y,η) along which the absolute periods are constant, provided $|\kappa| > 1$ and $\Omega\mathcal{M}_g(\kappa)$ is connected.

Theorems 1.1 and 1.2 can be deduced from Theorem 1.3 using a transfer principle from [CDF], by applying Moore's ergodicity theorem and Ratner's orbit closure theorem to the action of $\mathrm{Sp}(2g,\mathbb{Z})$ on $\mathrm{Sp}(2g,\mathbb{R})/\mathrm{Sp}(2g-2,\mathbb{R})$. Theorem 1.3 was previously known for the principal stratum [CDF, Theorem 1.2].

Connected components of strata. We are also able to establish Theorems 1.1, 1.2, and 1.3 for some connected components of disconnected strata.

Theorem 1.4. For $g \geq 4$ even, Theorems 1.1, 1.2, and 1.3 hold for the nonhyperelliptic connected component of $\Omega\mathcal{M}_g(g-1,g-1)$.

However, Theorem 1.3 cannot be extended to any other connected components of strata. By [KZ], the connected components of strata that are not addressed by Theorems 1.3 and 1.4 are the connected components of $\Omega\mathcal{M}_g(\kappa)$ when every $m_j \in \kappa$ is even, and the hyperelliptic component of $\Omega\mathcal{M}_g(g-1, g-1)$ when $g \geq 4$ is even.

Theorem 1.5. Let $\mathcal{C}$ be a connected component of a stratum $\Omega\mathcal{M}_g(\kappa)$ with every $m_j \in \kappa$ even, or the hyperelliptic component of $\Omega\mathcal{M}_g(\kappa)$ with $\kappa = \{g-1, g-1\}$ and $g \geq 4$ even, and fix an algebraically generic $(X, \omega) \in \mathcal{C}$. There exists $(Y, \eta) \in \mathcal{C}$, such that $\mathrm{Per}(\omega) = \mathrm{Per}(\eta)$ as symplectic modules, that is not in the leaf of $\mathcal{A}(\kappa)$ through (X, ω).

Here, the obstruction comes from the failure of the monodromy representation of the fundamental group of $\mathcal{C}$ on absolute homology $\pi_1(\mathcal{C}) \to \mathrm{Sp}(2g, \mathbb{Z})$ to be surjective. In [Gut], the image of this homomorphism is explicitly described for all connected components of strata. In particular, when $|\kappa| > 1$, this homomorphism is surjective if and only if some $m_j \in \kappa$ is odd and $\mathcal{C}$ is nonhyperelliptic, or $g = 2$. These are precisely the connected components of strata to which Theorem 1.3 and Theorem 1.4 apply. As a complement, we will use Theorem 1.3 and Theorem 1.4 to give a new proof of the surjectivity of $\pi_1(\mathcal{C}) \to \mathrm{Sp}(2g, \mathbb{Z})$ in these cases.

Open questions. Theorem 1.2 gives hope for a complete classification of closures of leaves of $\mathcal{A}(\kappa)$ in $\Omega_1\mathcal{M}_g(\kappa)$. Here, we raise some open questions that suggest a possible classification, in the spirit of Ratner's theorems for unipotent flows on homogeneous spaces [Rat].

For context, we recall that Ratner's orbit closure theorem in homogeneous dynamics places strong restrictions on the closures of orbits of 1-parameter unipotent subgroups. Briefly, if Γ is a lattice in a connected Lie group G, and if U is a 1-parameter unipotent subgroup of G, then the closures of orbits of U in G/Γ all come from intermediate closed subgroups $U \subset H \subset G$. A striking version of Ratner's orbit closure theorem was established in [EMM, Theorem 2.1] and [Fil2, Theorem 1.1] for the action of $\mathrm{GL}^+(2, \mathbb{R})$ on a stratum $\Omega\mathcal{M}_g(\kappa)$, where it was proven that $\mathrm{GL}^+(2,\mathbb{R})$-orbit closures are locally linear subvarieties of $\Omega\mathcal{M}_g(\kappa)$. Surprisingly, by [CSW, Theorem 1.3], there does not seem to be a version of Ratner's orbit closure theorem where one replaces $\mathrm{GL}^+(2,\mathbb{R})$ with its upper-triangular unipotent subgroup.

Our results suggest that closures of leaves of $\mathcal{A}(\kappa)$ enjoy rigidity properties similar to $\mathrm{GL}^+(2,\mathbb{R})$-orbit closures. Let L be the leaf of $\mathcal{A}(\kappa)$ through $(X,\omega) \in \Omega_1\mathcal{M}_g(\kappa)$, and let $\overline{L}$ be its closure in $\Omega_1\mathcal{M}_g(\kappa)$.

Question 1.6. Is $\overline{L}$ always a properly immersed real-analytic suborbifold of $\Omega_1\mathcal{M}_g(\kappa)$?

Question 1.7. If $\mathrm{Per}(\omega)$ is dense in $\mathbb{C}$ and $\overline{L} \neq L$, is it the case that $\overline{L} = \overline{\mathrm{SL}(2,\mathbb{R}) \cdot L}$?

Question 1.7 addresses the three known obstructions to the density of L in its connected component in $\Omega_1\mathcal{M}_g(\kappa)$. First, $\mathrm{Per}(\omega)$ might be contained in a proper closed subgroup of $\mathbb{C}$. Since $\mathrm{Per}(\omega)$ contains a lattice in $\mathbb{C}$, up to the action of $\mathrm{GL}^+(2,\mathbb{R})$ the only possible subgroups are $\mathbb{R} + i\mathbb{Z}$ and $\mathbb{Z} + i\mathbb{Z}$. Second, L might lie in a proper closed $\mathrm{SL}(2,\mathbb{R})$-invariant subset of its connected component in $\Omega_1\mathcal{M}_g(\kappa)$. This occurs, for instance, in loci of double covers of quadratic differentials when $|\kappa| = 2$. Third, L might be closed and consist of branched covers of holomorphic 1-forms of lower genus. An answer to the following question is likely needed for a complete classification of closures of leaves of $\mathcal{A}(\kappa)$.

Question 1.8. What are the closed $\mathrm{SL}(2,\mathbb{R})$-invariant subsets of $\Omega_1\mathcal{M}_g(\kappa)$ that are saturated for $\mathcal{A}(\kappa)$?

Here, a subset of $\Omega\mathcal{M}_g(\kappa)$ is saturated for $\mathcal{A}(\kappa)$ if it is a union of leaves of $\mathcal{A}(\kappa)$. Even in the stratum $\Omega\mathcal{M}_2(1,1)$, Question 1.8 is surprisingly subtle. In [Cal] and [McM1], it was discovered that loci of eigenforms for real multiplication provide examples of closed $\mathrm{SL}(2,\mathbb{R})$-invariant subsets of $\Omega\mathcal{M}_2(1,1)$ that are saturated for $\mathcal{A}(1,1)$, and in [McM3], it is shown that these loci of eigenforms also arise as closures of leaves of $\mathcal{A}(1,1)$.

Methods. We outline the proof of Theorem 1.3 here. As mentioned previously, Theorems 1.1 and 1.2 can be deduced from Theorem 1.3. Our proof consists of two inductive arguments.

The first inductive argument addresses the case of strata of holomorphic 1-forms with exactly 2 distinct zeros, and we induct on genus. The base case of genus 2 involves only the stratum $\Omega\mathcal{M}_2(1,1)$, for which Theorem 1.3 is already known [CDF]. For the inductive step, we analyze the interaction of the absolute period foliation with connected sums with a torus. Choose a flat torus $T = (\mathbb{C}/\Lambda, dz)$ and a closed geodesic $\alpha \subset T$. Given a holomorphic 1-form $(X, \omega) \in \Omega\mathcal{M}_g(m_1, m_2)$, we can slit T along α, slit (X, ω) along a parallel segment of the same length from a zero of order m_1 to a point that is not a zero of ω, and reglue opposite sides to obtain a new holomorphic 1-form $(X', \omega') \in \Omega\mathcal{M}_{g+1}(m_1 + 2, m_2)$. The connected sum construction is well-defined provided (X, ω) does not have a saddle connection that is parallel to α and whose length is less than or equal to that of α. Now let L be the leaf of $\mathcal{A}(m_1, m_2)$ through (X, ω), and let L' be the leaf of $\mathcal{A}(m_1 + 2, m_2)$ through (X', ω'). One of our main observations is that if α is not parallel to an absolute period of (X, ω), then this connected sum construction is defined on a path-connected subset of L whose complement in L is a countable union of line segments. The rough idea of our inductive argument is then to "forget" the torus T and deform the complementary holomorphic 1-form (X, ω) along the leaf L while avoiding the line segments where the connected sum construction is not defined. Making this precise requires studying the absolute period foliation of a certain finite cover of $\Omega\mathcal{M}_g(m_1, m_2)$. Next, we study how a single holomorphic 1-form in $\Omega\mathcal{M}_{g+1}(m_1 + 2, m_2)$ can be presented as a connected sum in multiple ways. Our argument proceeds by taking two holomorphic 1-forms satisfying the hypotheses of Theorem 1.3 and using multiple presentations as connected sums to deform them along their respective leaves of $\mathcal{A}(m_1 + 2, m_2)$ until they "agree" on a torus as above. The inductive hypothesis then allows us to conclude that they lie on the same leaf of $\mathcal{A}(m_1 + 2, m_2)$. In general, we show that if Theorem 1.3 holds for a connected component $\mathcal{C}$ of $\Omega\mathcal{M}_g(m_1, m_2)$, and if $\mathcal{C}'$ is a connected component of $\Omega\mathcal{M}_{g+1}(m_1+2, m_2)$ that contains connected sums of holomorphic 1-forms in $\mathcal{C}$ with a torus as above, then Theorem 1.3 also holds for $\mathcal{C}'$.

Our second inductive argument is easier and addresses the general case, and we induct on the number of zeros. The base case is the case of 2 zeros, discussed above. For the inductive step, we use the surgery of splitting a zero. Given $(X, \omega) \in \Omega\mathcal{M}_g(\kappa)$ with a zero Z of order $m \geq 2$, and $1 \leq j < m$, there is a local surgery which splits Z into a pair of zeros of orders $m-j$ and j, respectively. This surgery does not change the absolute periods of (X, ω). Let $\kappa' = (\kappa \setminus \{m\}) \cup \{m-j, j\}$. We show that if Theorem 1.3 holds for a connected component $\mathcal{C}$ of $\Omega\mathcal{M}_g(\kappa)$, and if $\mathcal{C}'$ is a connected component of $\Omega\mathcal{M}_g(\kappa')$ that contains holomorphic 1-forms arising from splitting a zero on a holomorphic 1-form in $\Omega\mathcal{M}_g(\kappa)$, then Theorem 1.3 also holds for $\mathcal{C}'$.

Every connected stratum $\Omega\mathcal{M}_g(\kappa)$ with $|\kappa| > 1$ can be accessed from $\Omega\mathcal{M}_2(1,1)$ by iteratively forming a connected sum with a torus and then iteratively splitting a zero. For $g \geq 4$ even, the nonhyperelliptic connected component of $\Omega\mathcal{M}_g(g-1, g-1)$ can also be accessed in this way. The inductive steps in our method apply to all nonhyperelliptic connected components of strata $\Omega\mathcal{M}_g(\kappa)$ with $|\kappa| > 1$. To establish Theorems 1.1 and 1.2 in these cases, we would need one additional base case, namely, the stratum $\Omega\mathcal{M}_3(2,2)$.

Notes and references. The particular case of the dynamics of the absolute period foliation of $\Omega\mathcal{M}_g$ are studied in [CDF], [Ham], and [McM5]. For $g = 2$ and $g = 3$, the fact that any principally polarized abelian variety of dimension g is the Jacobian of a stable curve is exploited in [McM5] to prove ergodicity on $\Omega_1\mathcal{M}_g$, and this idea is pushed further in [CDF] to obtain a classification of leaf closures. The approach in [CDF] uses the classification in [Kap] of cohomology classes in $H^1(S_g; \mathbb{C})$ that can be represented by a holomorphic 1-form, along with an inductive argument involving isoperiodic degenerations to the boundary of moduli space, in order to classify leaf closures and to prove ergodicity on $\Omega_1\mathcal{M}_g$ for all

$g \geq 2$. An independent and simpler proof of ergodicity on $\Omega_1\mathcal{M}_g$ for $g \geq 2$ is given in [Ham], also using induction and degenerations. All of these results apply to the principal stratum $\Omega\mathcal{M}_g(1,\ldots,1)$ as well. We remark that the boundary of moduli space does not play a role in our proofs, and so we obtain a new proof of ergodicity on $\Omega_1\mathcal{M}_g$ for $g \geq 3$. The methods in [CDF], [Ham], and [McM5] do not seem to be easily adaptable to non-principal strata, due to our limited understanding of the Schottky locus and a lack of available base cases for induction.

Much less is known about the dynamics of the absolute period foliation of non-principal strata. In [HW], it is shown that the Arnoux-Yoccoz surfaces of genus $g \geq 3$ give examples of dense leaves in a fixed-area locus in a certain connected component of $\Omega\mathcal{M}_g(g-1, g-1)$. Additional examples of dense leaves in $\Omega_1\mathcal{M}_3(2,1,1)$ arising from Prym loci are given in [Ygo1]. In [Win1], it is shown that there exist dense relative period geodesics in the area-1 locus of each connected component of every non-minimal stratum. In [McM3] and [Ygo2], it is shown that leaves of the absolute period foliation of eigenform loci in $\Omega\mathcal{M}_2(1,1)$, and more generally of rank 1 affine invariant manifolds, are either closed or dense in the area-1 locus. Jon Chaika and Barak Weiss have informed us of work in progress in which they prove that real Rel flows are ergodic on the area-1 locus of each connected component of a stratum of holomorphic 1-forms with multiple zeros, conditional on a generalization of the results in [EM].

The surgeries we consider are special cases of the surgery of bubbling a handle in [KZ] and the figure-eight construction in [EMZ]. These surgeries play an important role in the classification of connected components of strata in [KZ], and in the computation of Siegel-Veech constants for strata in [EMZ]. Detailed studies of presentations of holomorphic 1-forms in genus 2 as connected sums are carried out in [McM2] and [CM]. In [McM2], connected sums are used to classify all $\mathrm{SL}(2,\mathbb{R})$-orbit closures and invariant measures in $\Omega_1\mathcal{M}_2(1,1)$, and in [CM], connected sums are used to exhibit minimal non-uniquely ergodic straight-line flows on every non-Veech surface in genus 2.

Our proof of Theorem 1.1 only relies on the genus 2 case in [McM5] as a base case, which uses Moore's ergodicity theorem [Zim], and on the ergodicity of the $\mathrm{SL}(2,\mathbb{R})$-action on the area-1 locus of connected components of strata [Mas], [Vee1], [Vee2]. Our proofs of Theorems 1.2 and 1.3 rely on the genus 2 case in [CDF] as a base case, and on the explicit full measure sets of dense $\mathrm{GL}^+(2,\mathbb{R})$-orbits in strata given in [Wri], which in turn relies on the rigidity results for $\mathrm{GL}^+(2,\mathbb{R})$-orbit closures in strata in [EMM]. By [Wri] and [Fil1], Theorems 1.2 and 1.3 still hold if one only considers totally real number fields of degree at most g in the definition of "algebraically generic."

The intrinsic geometry of leaves of the absolute period foliation of $\Omega\mathcal{M}_g$ and of strata are studied in [BSW], [McM4], [McM5], and [MW]. Completeness results for the natural metric on leaves are given in these papers. In [McM5], it is shown that the metric completion of a typical leaf in $\Omega\mathcal{M}_2$ is a Riemann surface biholomorphic to the upper half-plane. In contrast, examples of infinite-genus leaves in certain strata of holomorphic 1-forms with exactly 2 zeros are given in [Win3]. The geometry of leaves of $\mathcal{A}(1,1)$ is studied in [EMS] in order to count periodic billiard trajectories in a square with a barrier, and in [Dur] to make progress toward classifying square-tiled surfaces in $\Omega\mathcal{M}_2(1,1)$.

Outline. In Chapter 2, we provide background material on holomorphic 1-forms and describe the surgeries used in our proofs. In Chapter 3, we discuss the absolute period foliation of a stratum of holomorphic 1-forms and establish the key connectivity lemma for our inductive arguments. In Chapter 4, we study presentations of holomorphic 1-forms as connected sums in multiple ways. In Chapter 5, we prove our main theorems. Most of the material in this thesis is contained in [Win2].

2. Splitting zeros and connected sums

This chapter is based on Section 2 of [Win2]. We recall relevant material on strata of holomorphic 1-forms and the $\mathrm{GL}^+(2,\mathbb{R})$-action on strata. We then discuss the surgeries of splitting zeros and forming a connected sum with a torus. For additional background material, we refer to [FM] and [Zor].

2.1. **Holomorphic 1-forms.** We denote by (X,ω) a closed Riemann surface X of genus $g \geq 2$ equipped with a holomorphic 1-form ω. We always assume $\omega \neq 0$. The zero set $Z(\omega)$ is finite and nonempty, and the orders of the zeros form a partition of $2g-2$. Integration of ω on $X \setminus Z(\omega)$ gives an atlas of charts to the complex plane $\mathbb{C}$, whose transition maps are translations. Geometric structures on $\mathbb{C}$ that are invariant under translations can be pulled back to $X \setminus Z(\omega)$ using this atlas. In particular, the Euclidean metric on $\mathbb{C}$ determines a singular flat metric $|\omega|$ on X with a cone point with angle $2\pi(k+1)$ at a zero of order k.

In our figures, we will present holomorphic 1-forms as finite disjoint unions of polygons in $\mathbb{C}$, possibly with slits, with pairs of edges identified by translations in $\mathbb{C}$. In most cases, the edge identifications will be implicit from the requirement that identified edges must be parallel and of the same length.

A *saddle connection* on (X,ω) is an oriented geodesic segment γ with endpoints in $Z(\omega)$ and otherwise disjoint from $Z(\omega)$. The *holonomy* of γ is the nonzero complex number $\int_\gamma \omega$. A closed geodesic in $X \setminus Z(\omega)$ is contained in a maximal connected open subset of $X \setminus Z(\omega)$ foliated by parallel closed geodesics. Such an open subset C is called a *cylinder*. The boundary of C consists of a finite union of parallel saddle connections. Each homotopy class of paths in X with endpoints in $Z(\omega)$ has a unique *geodesic representative* of minimal length in the metric $|\omega|$, consisting of finitely many saddle connections such that each angle formed by two consecutive saddle connections is at least π. Let

$$\mathrm{Per}(\omega) = \left\{ \int_c \omega \ : \ c \in H_1(X;\mathbb{Z}) \right\}$$

be the subgroup of $\mathbb{C}$ of *absolute periods* of ω. Let

$$\Gamma(\omega) = \left\{ \int_\gamma \omega \ : \ \gamma \text{ is a saddle connection} \right\}$$

be the subset of $\mathbb{C}$ of holonomies of saddle connections. The subset $\Gamma(\omega)$ is discrete. In particular, for any $B > 0$, there are only finitely many saddle connections on (X,ω) of length at most B. Let $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, and let

$$\Delta(\omega) = \mathbb{C}^* \setminus \{tz \ : \ t \in [1,\infty), \ z \in \Gamma(\omega)\}.$$

Note that since $\Gamma(\omega)$ is discrete, $\Delta(\omega)$ is open.

2.2. **Strata.** Let S_g be a closed oriented surface of genus $g \geq 2$. The Teichmüller space $\mathcal{T}_g$ of marked Riemann surfaces $f : S_g \to X$ of genus g is a complex manifold of dimension $3g-3$. The mapping class group Mod_g acts properly discontinuously on $\mathcal{T}_g$ by biholomorphisms. The moduli space of Riemann surfaces of genus g is the complex orbifold $\mathcal{M}_g = \mathcal{T}_g / \mathrm{Mod}_g$. The action of Mod_g on $\mathcal{T}_g$ induces an action on the bundle $\Omega\mathcal{T}_g \to \mathcal{T}_g$ of nonzero holomorphic 1-forms on marked Riemann surfaces. The *moduli space of holomorphic* 1*-forms of genus* g is the complex orbifold $\Omega\mathcal{M}_g = \Omega\mathcal{T}_g / \mathrm{Mod}_g$. The space $\Omega\mathcal{T}_g$ decomposes into strata $\Omega\mathcal{T}_g(\kappa)$ indexed by partitions $\kappa = \{m_1, \ldots, m_n\}$ of $2g-2$. The stratum $\Omega\mathcal{T}_g(\kappa)$ consists of holomorphic 1-forms on marked Riemann surfaces with exactly n distinct zeros of orders $m_1, \ldots, m_n$. The action of Mod_g preserves each stratum, and the space $\Omega\mathcal{M}_g$ decomposes into *strata* $\Omega\mathcal{M}_g(\kappa) = \Omega\mathcal{T}_g(\kappa) / \mathrm{Mod}_g$ which are complex suborbifolds of $\Omega\mathcal{M}_g$.

Fix $(X_0, \omega_0) \in \Omega\mathcal{T}_g(\kappa)$. There is a neighborhood $\mathcal{U} \subset \Omega\mathcal{T}_g(\kappa)$ of (X_0, ω_0), and a natural isomorphism $H^1(X, Z(\omega); \mathbb{C}) \cong H^1(X_0, Z(\omega_0); \mathbb{C})$ for any $(X, \omega) \in \mathcal{U}$, provided by the Gauss-Manin connection on the bundle of relative cohomology groups over $\Omega\mathcal{T}_g(\kappa)$. *Period coordinates* on $\mathcal{U}$ are defined using these isomorphisms by

$$\mathcal{U} \to H^1(X_0, Z(\omega_0); \mathbb{C}), \quad (X, \omega) \mapsto [\omega],$$

and this map is a biholomorphism from an open subset of $\Omega\mathcal{T}_g(\kappa)$ to an open subset of a complex vector space of dimension $2g + |\kappa| - 1$. Given a choice of basis $c_1, \ldots, c_{2g+|\kappa|-1}$ for $H_1(X_0, Z(\omega_0); \mathbb{Z})$, we get a map

$$\mathcal{U} \mapsto \mathbb{C}^{2g+|\kappa|-1}, \quad (X, \omega) \mapsto \left(\int_{c_1} \omega, \ldots, \int_{c_{2g+|\kappa|-1}} \omega \right).$$

The components $\int_{c_j} \omega$ are the *period coordinates of* (X, ω). Transition maps between period coordinate charts are integral linear maps that preserve $H^1(X_0, Z(\omega_0); \mathbb{Z})$. Period coordinates give $\Omega\mathcal{M}_g(\kappa)$ the structure of an affine orbifold.

The *area* of (X, ω) is the area of X with respect to the metric $|\omega|$, and is given by

$$\text{Area}(X, \omega) = \frac{i}{2} \int_X \omega \wedge \overline{\omega} = \sum_{j=1}^{g} \text{Im} \left(\int_{a_j} \overline{\omega} \int_{b_j} \omega \right)$$

where $\{a_j, b_j\}_{j=1}^g$ is a symplectic basis for $H_1(X; \mathbb{Z})$. The area of (X, ω) is an invariant of the absolute cohomology class $[\omega] \in H^1(X; \mathbb{C})$. Let

$$\Omega_1\mathcal{M}_g(\kappa) = \{(X, \omega) \in \Omega\mathcal{M}_g(\kappa) \ : \ \text{Area}(X, \omega) = 1\}$$

be the area-1 locus in $\Omega\mathcal{M}_g(\kappa)$. The area-1 locus $\Omega_1\mathcal{M}_g(\kappa)$ is a real-analytic orbifold and has a canonical Lebesgue measure class.

Let $\mathrm{GL}^+(2,\mathbb{R})$ be the group of linear automorphisms of $\mathbb{R}^2$ with positive determinant. Let $\mathrm{SL}(2,\mathbb{R})$ be the subgroup of matrices with determinant 1. The standard $\mathbb{R}$-linear action of $\mathrm{GL}^+(2,\mathbb{R})$ on $\mathbb{C} \cong \mathbb{R}^2$ induces an action on $\Omega\mathcal{M}_g$ by postcomposition with an atlas of charts on $X \setminus Z(\omega)$ as above. The action of $\mathrm{GL}^+(2,\mathbb{R})$ preserves each stratum $\Omega\mathcal{M}_g(\kappa)$, and the action of $\mathrm{SL}(2,\mathbb{R})$ preserves $\Omega_1\mathcal{M}_g(\kappa)$. The action of $\mathrm{SL}(2,\mathbb{R})$ is *ergodic* on each connected component of $\Omega_1\mathcal{M}_g(\kappa)$ with respect to the Lebesgue measure class [Mas], [Vee1], [Vee2], meaning that any measurable $\mathrm{SL}(2,\mathbb{R})$-invariant subset of $\Omega_1\mathcal{M}_g(\kappa)$ has either zero measure or full measure.

Most strata in $\Omega\mathcal{M}_g$ are connected. However, in general, strata can have up to 3 connected components, which are classified by hyperellipticity and the parity of a spin structure. We recall their classification from [KZ].

Let $\kappa = \{m_1, \ldots, m_n\}$ be a partition of $2g-2$ with all m_j even, and fix $(X,\omega) \in \Omega\mathcal{M}_g(\kappa)$. The *index* of a smooth oriented closed loop $\gamma \subset X \setminus Z(\omega)$ is the degree of the associated Gauss map $\gamma \to S^1$, that is, $1/2\pi$ times the total change in angle of a tangent vector travelling once around γ. We denote the index of γ by $\mathrm{ind}(\gamma)$. Let $\{\alpha_j, \beta_j\}_{j=1}^g$ be a collection of smooth oriented closed loops in $X \setminus Z(\omega)$ representing a symplectic basis for $H_1(X;\mathbb{Z})$. The *parity of the spin structure* $\phi(\omega)$ is defined by

$$\phi(\omega) = \sum_{j=1}^{g} (\mathrm{ind}(\alpha_j)+1)(\mathrm{ind}(\beta_j)+1) \mod 2.$$

It is a fact that $\phi(\omega)$ is independent of the choice of symplectic basis of $H_1(X;\mathbb{Z})$ and the choice of representatives for the symplectic basis. Moreover, $\phi(\omega)$ is an invariant of the connected component of $(X,\omega) \in \Omega\mathcal{M}_g(\kappa)$. A connected component $\mathcal{C} \subset \Omega\mathcal{M}_g(\kappa)$ is *even* or *odd* according to whether $\phi(\omega) = 0$ or $\phi(\omega) = 1$ for $(X,\omega) \in \mathcal{C}$.

If $\mathcal{C} \subset \Omega\mathcal{M}_g(2g-2)$ consists of holomorphic 1-forms on hyperelliptic curves, or if $\mathcal{C} \subset \Omega\mathcal{M}_g(g-1, g-1)$ consists of holomorphic 1-forms on hyperelliptic curves whose hyperelliptic involution exchanges the two zeros, then $\mathcal{C}$ is *hyperelliptic*. A connected component which is not hyperelliptic is *nonhyperelliptic*.

Theorem 2.1. ([KZ], Theorems 1-2 and Corollary 5) For $g \geq 4$, the connected components of $\Omega\mathcal{M}_g(\kappa)$ are as follows.

(1) If $\kappa = \{2g-2\}$ or $\kappa = \{g-1, g-1\}$, then $\Omega\mathcal{M}_g(\kappa)$ has a unique hyperelliptic connected component.

(2) If all $m_j \in \kappa$ are even, then $\Omega\mathcal{M}_g(\kappa)$ has exactly two nonhyperelliptic connected components: one even connected component and one odd connected component.

(3) If some $m_j \in \kappa$ is odd, then $\Omega\mathcal{M}_g(\kappa)$ has a unique nonhyperelliptic connected component.

For $g \leq 3$, the stratum $\Omega\mathcal{M}_g(\kappa)$ is connected unless $\kappa = \{4\}$ or $\kappa = \{2, 2\}$, in which case $\Omega\mathcal{M}_g(\kappa)$ has exactly two connected components: one odd connected component, and one hyperelliptic connected component which is also an even connected component.

Corollary 2.2. A stratum $\Omega\mathcal{M}_g(\kappa)$ is connected if and only if there is $m_j \in \kappa$ that is odd and not equal to $g-1$, or $g = 2$.

Let κ be a partition of $2g-2$, and choose $m \in \kappa$. It will be convenient for us to work with a finite cover of a stratum

$$p : \widetilde{\Omega}\mathcal{M}_g(\kappa; m) \to \Omega\mathcal{M}_g(\kappa) \tag{1}$$

consisting of holomorphic 1-forms in $\Omega\mathcal{M}_g(\kappa)$ equipped with a distinguished rightward horizontal direction θ at a zero Z of order m. We denote elements of $\widetilde{\Omega}\mathcal{M}_g(\kappa; m)$ by (X, ω, θ), and we refer to θ as a *prong*. The degree of p is $(m+1)N_m$, where N_m is the number of times m appears in κ. An automorphism of (X, ω, θ) is required to fix the distinguished zero Z and the prong θ, so (X, ω, θ) has no nontrivial automorphisms.

A subset $\mathcal{K} \subset \Omega\mathcal{M}_g(\kappa)$ is compact if and only if $\mathcal{K}$ is closed and there exists $\varepsilon > 0$ such that every saddle connection on every holomorphic 1-form in $\mathcal{K}$ has length at least ε. The analogous statement holds for $\widetilde{\Omega\mathcal{M}}_g(\kappa; m)$.

Let $\mathcal{U} \subset \widetilde{\Omega\mathcal{M}}_g(\kappa; m)$ be a contractible open subset whose closure is compact. For each homotopy class γ of paths on $(X, \omega, \theta) \in \mathcal{U}$ with endpoints in $Z(\omega)$, there is a well-defined continuous *length function*

$$\ell_\gamma : \mathcal{U} \to \mathbb{R}_{>0}$$

whose value at (Y, η, θ') is the length of the geodesic representative of the corresponding homotopy class of paths on (Y, η, θ'). Since there are only finitely many saddle connections on (X, ω, θ) with length at most B, there are only finitely many homotopy classes γ such that $\ell_\gamma(X, \omega, \theta) \leq B$.

Lemma 2.3. For any $B > 0$, there are only finitely many homotopy classes γ as above such that $\inf_{\mathcal{U}} \ell_\gamma < B$.

Proof. Suppose $\inf_{\mathcal{U}} \ell_\gamma < B$. Since the closure of $\mathcal{U}$ is compact, there is $0 < \varepsilon < B$ such that every saddle connection on every holomorphic 1-form in $\mathcal{U}$ has length at least ε. Fix $(X, \omega, \theta) \in \mathcal{U}$. There is a neighborhood $\mathcal{V} \subset \mathcal{U}$ of (X, ω, θ) such that for all $(Y, \eta, \theta') \in \mathcal{V}$, every saddle connection γ' on (X, ω, θ) of length at most B persists as a saddle connection on (Y, η, θ') and satisfies $|\ell_{\gamma'}(X, \omega, \theta) - \ell_{\gamma'}(Y, \eta, \theta')| < \varepsilon/2$. On (X, ω, θ), the geodesic representative of γ is a finite union of saddle connections $\gamma_1, \ldots, \gamma_j$ whose lengths lie in the interval $[\varepsilon, B]$. For each γ_k, and for any $(Y, \eta, \theta') \in \mathcal{V}$, we have

$$\sup_{\mathcal{V}} \ell_{\gamma_k} \leq \ell_{\gamma_k}(Y, \eta, \theta') + \varepsilon \leq 2\ell_{\gamma_k}(Y, \eta, \theta').$$

Therefore, if $\inf_{\mathcal{V}} \ell_\gamma < B$, then $\sup_{\mathcal{V}} \ell_\gamma < 2B$. Since the closure of $\mathcal{U}$ is compact, there is a finite covering $\mathcal{U} = \bigcup_{k=1}^{N} \mathcal{V}_k$ by open subsets as above. For each $1 \leq k \leq N$, fix

$(X_k, \omega_k, \theta_k) \in \mathcal{V}_k$. Since $\inf_{\mathcal{V}_k} \ell_\gamma < B$ for some $1 \leq k \leq N$, we have $\ell_\gamma(X_k, \omega_k, \theta_k) < 2B$ for some $1 \leq k \leq N$, and thus there are only finitely many possibilities for γ. $\square$

Fix $z \in \mathbb{C}^*$, let $I = [0, z] = \{tz \ : \ t \in [0, 1]\}$, and let $\Omega\mathcal{M}_g(\kappa; I)$ be the set of holomorphic 1-forms in $\Omega\mathcal{M}_g(\kappa)$ with a saddle connection whose holonomy is in I.

Lemma 2.4. The subset $\Omega\mathcal{M}_g(\kappa; I) \subset \Omega\mathcal{M}_g(\kappa)$ is closed.

Proof. Fix $(X, \omega) \in \Omega\mathcal{M}_g(\kappa) \setminus \Omega\mathcal{M}_g(\kappa; I)$ and $(X, \omega, \theta) \in p^{-1}(X, \omega)$ with p the stratum cover in (1). By Lemma 2.3, there is a neighborhood $\mathcal{U}$ of (X, ω, θ) such that there are only finitely many homotopy classes $\gamma_1, \ldots, \gamma_j$ of paths on X with endpoints in $Z(\omega)$ satisfying $\inf_{\mathcal{U}} \ell_{\gamma_k} \leq |z|$. For each γ_k, either $\ell_{\gamma_k}(X, \omega, \theta) > |z|$, or the geodesic representative of γ_k on (X, ω) contains a saddle connection δ_k with $\int_{\delta_k} \omega \notin \mathbb{R}z$. Both of these properties of γ_k persist on a neighborhood $\mathcal{U}_k \subset \mathcal{U}$ of (X, ω, θ). Since p is open, $p(\bigcap_{k=1}^{j} \mathcal{U}_k)$ is a neighborhood of (X, ω) disjoint from $\Omega\mathcal{M}_g(\kappa; I)$. $\square$

2.3. **Domains of surgeries.** Since the fundamental group of $\mathrm{GL}^+(2, \mathbb{R})$ is isomorphic to $\mathbb{Z}$, there is a degree $m+1$ connected covering of topological groups

$$\zeta : \widetilde{\mathrm{GL}}^+(2, \mathbb{R}) \to \mathrm{GL}^+(2, \mathbb{R})$$

which is unique up to isomorphism. There is a unique continuous action of $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ on $\widetilde{\Omega}\mathcal{M}_g(\kappa; m)$ such that p is ζ-equivariant. There is also a degree $m+1$ connected covering of topological groups

$$\sigma : \widetilde{\mathbb{C}}^* \to \mathbb{C}^*$$

which is unique up to isomorphism. We have polar coordinates $\widetilde{\mathbb{C}}^* \cong \mathbb{R}_{>0} \times \mathbb{R}/2\pi(m+1)$ and $\mathbb{C}^* \cong \mathbb{R}_{>0} \times \mathbb{R}/2\pi$ in which the identity elements correspond to $(1, 0)$. In these coordinates, σ is given by reduction mod 2π in the angular coordinate. There is a unique continuous action of $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ on $\widetilde{\mathbb{C}}^*$ such that σ is ζ-equivariant. Let

$$S(\omega) \to \Delta(\omega)$$

be the degree $m+1$ covering consisting of oriented geodesic segments γ starting at the distinguished zero Z such that $\int_\gamma \omega \in \Delta(\omega)$. Let $\mathbb{H} = \{z \in \mathbb{C} \, : \, \mathrm{Im}(z) > 0\}$, and let

$$T(\omega) \subset S(\omega) \times \mathbb{C}^*$$

be the subset of pairs (γ, w) such that $w / \int_\gamma \omega \in \mathbb{H}$. Let

$$\mathcal{S}(\kappa; m) \to \widetilde{\Omega}\mathcal{M}_g(\kappa; m)$$

be the bundle of pairs $((X, \omega, \theta), \gamma)$, where $(X, \omega, \theta) \in \widetilde{\Omega}\mathcal{M}_g(\kappa; m)$ and $\gamma \in S(\omega)$, and let

$$\mathcal{T}(\kappa; m) \to \widetilde{\Omega}\mathcal{M}_g(\kappa; m)$$

be the bundle of pairs $((X, \omega, \theta), (\gamma, w))$, where $(X, \omega, \theta) \in \widetilde{\Omega}\mathcal{M}_g(\kappa; m)$ and $(\gamma, w) \in T(\omega)$. The actions of $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ on $\widetilde{\Omega}\mathcal{M}_g(\kappa; m)$ and $\widetilde{\mathbb{C}}^*$ induce actions on the bundles $\mathcal{S}(\kappa; m)$ and $\mathcal{T}(\kappa; m)$.

For $(X, \omega, \theta) \in \widetilde{\Omega}\mathcal{M}_g(\kappa; m)$, we have a natural inclusion $S(\omega) \hookrightarrow \widetilde{\mathbb{C}}^*$ determined by requiring that the image of $\gamma \in S(\omega)$ projects to $\int_\gamma \omega \in \Delta(\omega)$ and that the image of a segment in the direction of the prong θ lies in $\mathbb{R}_{>0} \times \{0\}$ in polar coordinates. Similarly, we have a natural inclusion $T(\omega) \hookrightarrow \widetilde{\mathbb{C}}^* \times \mathbb{C}^*$. We will implicitly regard elements of $S(\omega)$ and $T(\omega)$ as elements of $\widetilde{\mathbb{C}}^*$ and $\widetilde{\mathbb{C}}^* \times \mathbb{C}^*$, respectively, using these inclusions. We then obtain $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$-equivariant inclusions

$$\mathcal{S}(\kappa; m) \hookrightarrow \widetilde{\Omega}\mathcal{M}_g(\kappa; m) \times \widetilde{\mathbb{C}}^*$$

and

$$\mathcal{T}(\kappa; m) \hookrightarrow \widetilde{\Omega}\mathcal{M}_g(\kappa; m) \times \widetilde{\mathbb{C}}^* \times \mathbb{C}^*$$

which respect the projections to $\widetilde{\Omega}\mathcal{M}_g(\kappa;m)$.

2.4. **Splitting a zero.** Suppose $m \geq 1$, and fix $1 \leq j \leq m$. Given $((X,\omega,\theta),\gamma) \in \mathcal{S}(\kappa;m)$, let $I = [0,\sigma(\gamma)]$ be the oriented segment in $\mathbb{C}$ from 0 to $\sigma(\gamma)$, and let

$$\gamma_1, \dots, \gamma_{j+1} : I \to X$$

be the isometric embeddings that preserve the direction of I, such that $\gamma_k(0)$ is the distinguished zero Z, and such that the counterclockwise angle around Z from γ to $\gamma_k(I)$ is $2\pi(k-1)$. Since $\int_{\gamma_k(I)} \omega \in \Delta(\omega)$, the segments $\gamma_k(I)$ are disjoint from $Z(\omega)$ and from each other except at their common starting point. Slit X along $\gamma_1(I) \cup \dots \cup \gamma_{j+1}(I)$ to obtain a surface with boundary X_0, and let $\gamma_k^+ : I \to X_0$ and $\gamma_k^- : I \to X_0$ be the left and right edges of the slit coming from γ_k, respectively. Glue $\gamma_k^+(z)$ to $\gamma_{k+1}^-(z)$ for $1 \leq k \leq j$, and glue $\gamma_{j+1}^+(z)$ to $\gamma_1^-(z)$. The complex structure and the holomorphic 1-form on the interior of X_0 extend over the slits to give a holomorphic 1-form (X',ω'). If $j < m$, then $|Z(\omega')| = |Z(\omega)| + 1$ and the distinguished zero Z is split into two zeros joined by a single saddle connection γ' such that

$$\int_{\gamma'} \omega' = \int_{\gamma} \omega.$$

The order of ω' at the starting point of γ' is $m - j$, and the order of ω' at the ending point of γ' is j. If $j < m$, let

$$\kappa' = (\kappa \setminus \{m\}) \cup \{m-j, j\},$$

and if $j = m$, let $\kappa' = \kappa$. Then κ' is the partition of $2g-2$ given by the orders of the zeros of ω'. We regard (X',ω') as an element of $\Omega\mathcal{M}_g(\kappa')$, and we say that (X',ω') arises from (X,ω) by *splitting a zero*. See Figure 1 for an example. The above surgery defines a *zero splitting map*

$$\Phi = \Phi(\kappa;m,j) : \mathcal{S}(\kappa;m) \to \Omega\mathcal{M}_g(\kappa')$$

which is a ζ-equivariant local covering of orbifolds. The zero splitting map preserves the area of the underlying holomorphic 1-form. Let

$$\mathcal{S}_1(\kappa; m) = \{((X, \omega, \theta), \gamma) \in \mathcal{S}(\kappa; m) \ : \ \mathrm{Area}(X, \omega) = 1\}$$

be the area-1 locus in $\mathcal{S}(\kappa; m)$. We can restrict Φ to get a map

$$\Phi_1 = \Phi_1(\kappa; m, j) : \mathcal{S}_1(\kappa; m) \to \Omega_1\mathcal{M}_g(\kappa')$$

which we also refer to as a zero splitting map. We can restrict ζ to get a degree $m+1$ connected covering of topological groups

$$\widetilde{\mathrm{SL}}(2, \mathbb{R}) \to \mathrm{SL}(2, \mathbb{R}).$$

The subset

$$\mathcal{S}_1(\kappa; m) \subset \widetilde{\Omega}_1\mathcal{M}_g(\kappa; m) \times \widetilde{\mathbb{C}}^*$$

is an $\widetilde{\mathrm{SL}}(2, \mathbb{R})$-invariant open subset of full measure with respect to the Lebesgue measure class on the product. The image of Φ_1 is nonempty, open, and $\mathrm{SL}(2, \mathbb{R})$-invariant. Since $\mathrm{SL}(2, \mathbb{R})$ acts ergodically on each connected component of $\Omega_1\mathcal{M}_g(\kappa')$, the image of Φ_1 is a full measure subset of a union of connected components of $\Omega_1\mathcal{M}_g(\kappa')$.

2.5. **Connected sums with a torus.** Given $((X, \omega, \theta), (\gamma, w)) \in \mathcal{T}(\kappa; m)$, let $I = [0, \sigma(\gamma)]$ be the oriented segment in $\mathbb{C}$ from 0 to $\sigma(\gamma)$. The pair (γ, w) determines a flat torus

$$T = (\mathbb{C}/(\mathbb{Z}\sigma(\gamma) + \mathbb{Z}w), dz).$$

Let $\gamma_1 : I \to X$ be the isometric embedding that preserves the direction of I and satisfies $\gamma_1(I) = \gamma$. Let $\gamma_2 : I \to T$ be the projection of I, which gives a closed geodesic in T. Slit X along $\gamma_1(I)$ to obtain a surface with boundary X_0, and let $\gamma_1^+ : I \to X_0$ and $\gamma_1^- : I \to X_0$ be

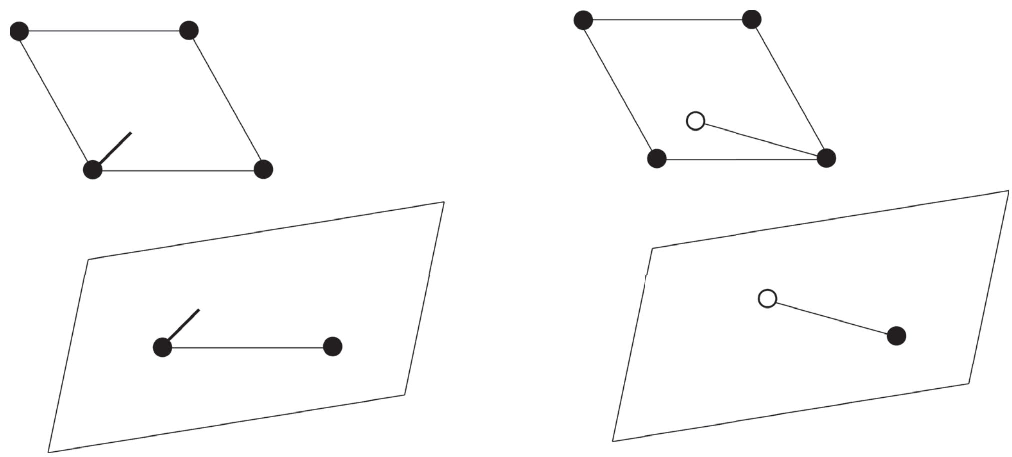

FIGURE 1. A holomorphic 1-form in $\Omega\mathcal{M}_2(1,1)$ (right) that arises from a holomorphic 1-form in $\Omega\mathcal{M}_2(2)$ (left) by splitting a zero. The two segments being slit are shown in bold.

the left and right edges of the slit coming from γ_1, respectively. Slit T along $\gamma_2(I)$ to obtain a cylinder with boundary T_0, and let $\gamma_2^+ : I \to T_0$ and $\gamma_2^- : I \to T_0$ be the left and right edges of the slit coming from γ_2, respectively. Glue $\gamma_1^+(z)$ to $\gamma_2^-(z)$, and glue $\gamma_2^+(z)$ to $\gamma_1^-(z)$. The result is a holomorphic 1-form (X', ω') with a pair of homologous saddle connections $\gamma^\pm$ forming a figure-eight on X' and arising from $\gamma_1^\pm(I) \subset X_0$. The order of ω' at the zero Z' arising from the distinguished zero Z is $m+2$. The counterclockwise angle around Z' from the end of γ^- to the end of γ^+ is 2π. Let

$$\kappa' = (\kappa \setminus \{m\}) \cup \{m+2\}$$

be the partition of $2g$ given by the orders of the zeros of ω'. We regard (X', ω') as an element of $\Omega\mathcal{M}_{g+1}(\kappa')$, and we say that (X', ω') arises from (X, ω) by a *connected sum with a torus*. A pair of homologous saddle connections that presents (X', ω') as a connected sum with a torus is a *splitting* of (X', ω'). See Figure 2 for an example. The above surgery defines a *connected sum map*

$$\Psi = \Psi(\kappa; m) : \mathcal{T}(\kappa; m) \to \Omega\mathcal{M}_{g+1}(\kappa')$$

which is a ζ-equivariant local covering of orbifolds. A connected sum of a holomorphic 1-form of area $0 < a < 1$ with a flat torus of area $1 - a$ has area 1. Letting

$$\mathcal{T}_1(\kappa; m) = \left\{ ((X, \omega, \theta), (\gamma, w)) \in \mathcal{T}(\kappa; m) \ : \ \mathrm{Area}(X, \omega) = \mathrm{Im}\left(\overline{\sigma(\gamma)} w\right) = 1 \right\} \times (0, 1),$$

we have an inclusion

$$\mathcal{T}_1(\kappa; m) \hookrightarrow \mathcal{T}(\kappa; m), \quad ((X, \omega, \theta), (\gamma, w), a) \mapsto (a^{1/2}(X, \omega, \theta), (1-a)^{1/2}(\gamma, w)),$$

and we regard $\mathcal{T}_1(\kappa; m)$ as the area-1 locus in $\mathcal{T}(\kappa; m)$. We then have a map

$$\Psi_1 = \Psi_1(\kappa; m) : \mathcal{T}_1(\kappa; m) \to \Omega_1 \mathcal{M}_{g+1}(\kappa')$$

which we also refer to as a connected sum map. We have an identification

$$\widetilde{\mathrm{SL}}(2, \mathbb{R}) \cong \left\{ (\gamma, w) \in \widetilde{\mathbb{C}}^* \times \mathbb{C}^* \ : \ \mathrm{Im}(\overline{\sigma(\gamma)} w) = 1 \right\}$$

given by sending $\widetilde{M} \in \widetilde{\mathrm{SL}}(2, \mathbb{R})$ to the image of $((1, 0), (1, 0)) \in \widetilde{\mathbb{C}}^* \times \mathbb{C}^*$ in polar coordinates under the diagonal action of $\widetilde{M}$. The subset

$$\mathcal{T}_1(\kappa; m) \subset \widetilde{\Omega}_1 \mathcal{M}_g(\kappa; m) \times \widetilde{\mathrm{SL}}(2, \mathbb{R}) \times (0, 1)$$

is an $\widetilde{\mathrm{SL}}(2, \mathbb{R})$-invariant open subset of full measure with respect to the Lebesgue measure class on the product. Here, $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ acts trivially on the third factor $(0, 1)$. The image of Ψ_1 is nonempty, open, and $\mathrm{SL}(2, \mathbb{R})$-invariant. Since $\mathrm{SL}(2, \mathbb{R})$ acts ergodically on each connected component of $\Omega_1 \mathcal{M}_{g+1}(\kappa')$, the image of Ψ_1 is a full measure subset of a union of connected components of $\Omega_1 \mathcal{M}_{g+1}(\kappa')$.

For our inductive arguments, we will need to understand the relationship between the connected components of (X', ω') and (X, ω) in their respective strata, when (X', ω') arises

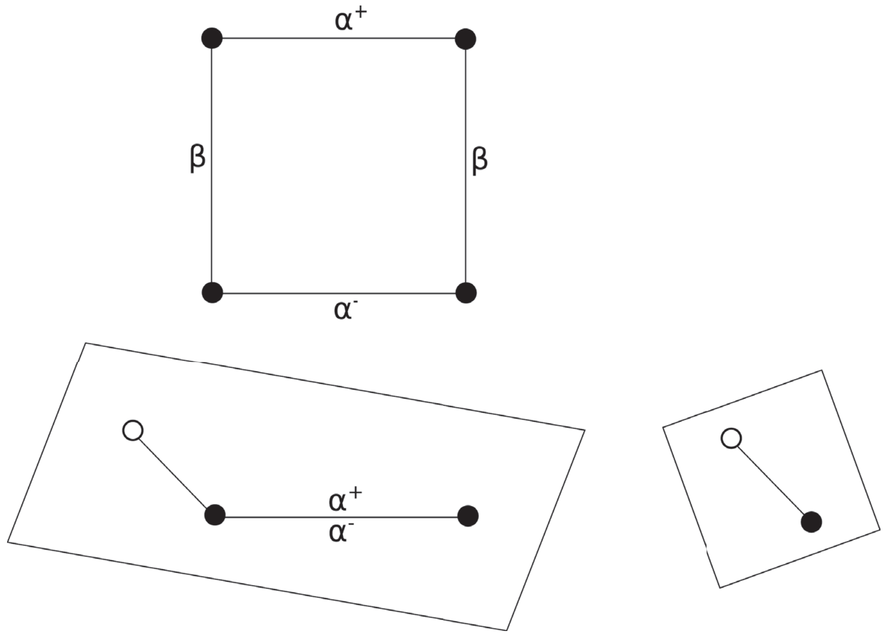

FIGURE 2. A holomorphic 1-form $(X, \omega) \in \Omega\mathcal{M}_3(3,1)$ arising from a holomorphic 1-form in $\Omega\mathcal{M}_2(1,1)$ by a connected sum with a torus. The pair of homologous saddle connections $\alpha^{\pm}$ is a splitting of (X, ω).

from (X, ω) by splitting a zero or by a connected sum with a torus. We only address the cases relevant to our proofs. We refer to [EMZ] and [KZ] for more general results.

Lemma 2.5. Let $\Omega\mathcal{M}_g(\kappa')$ be a connected stratum with $|\kappa'| \geq 3$. There is a connected stratum $\Omega\mathcal{M}_g(\kappa)$ with $|\kappa| = |\kappa'| - 1$ such that $\Omega\mathcal{M}_g(\kappa')$ contains holomorphic 1-forms that arise from holomorphic 1-forms in $\Omega\mathcal{M}_g(\kappa)$ by splitting a zero.

Proof. Note that $g \geq 3$ since $|\kappa'| \geq 3$. By Corollary 2.2, there is $m' \in \kappa'$ such that m' is odd and not equal to $g-1$. Choose $m_1, m_2 \in \kappa' \setminus \{m'\}$, let $m = m_1 + m_2$, and let $\kappa = (\kappa' \setminus \{m_1, m_2\}) \cup \{m\}$. We have $|\kappa| = |\kappa'| - 1$, and $\Omega\mathcal{M}_g(\kappa)$ is connected by Corollary 2.2 since $m' \in \kappa$. By splitting a zero of order m into two zeros of orders m_1 and m_2, respectively, we obtain holomorphic 1-forms in $\Omega\mathcal{M}_g(\kappa')$ that arise from holomorphic 1-forms in $\Omega\mathcal{M}_g(\kappa)$ by splitting a zero. □

Lemma 2.6. Let $\Omega\mathcal{M}_{g+1}(\kappa')$ be a stratum with $g+1 \geq 3$ such that $|\kappa'| = 2$ and the elements of κ' are odd. There is a connected stratum $\Omega\mathcal{M}_g(\kappa)$ with $|\kappa| = 2$ such that the nonhyperelliptic connected component of $\Omega\mathcal{M}_{g+1}(\kappa')$ contains holomorphic 1-forms that arise from holomorphic 1-forms in $\Omega\mathcal{M}_g(\kappa)$ by a connected sum with a torus.

Proof. If $g+1 \geq 4$, then there is $m' \in \kappa'$ such that $m' \geq 3$ and such that the elements of $\kappa = (\kappa' \setminus \{m'\}) \cup \{m'-2\}$ are odd and distinct. If $g+1 = 3$, then $\kappa' = \{3, 1\}$ and we let $m' = 3$. Let $m = m' - 2 \in \kappa$. In either case, $\Omega\mathcal{M}_g(\kappa)$ is connected by Corollary 2.2, and by applying the connected sum construction at a zero of order m, we obtain holomorphic 1-forms in $\Omega\mathcal{M}_{g+1}(\kappa')$ that arise from holomorphic 1-forms in $\Omega\mathcal{M}_g(\kappa)$ by a connected sum with a torus.

Lastly, suppose (X, ω) is in the hyperelliptic connected component of $\Omega\mathcal{M}_g(g-1, g-1)$. Recall the hyperelliptic involution exchanges the zeros of ω. Moreover, since we can increase the height of a cylinder on (X, ω) arbitrarily while remaining in the same stratum component, the hyperelliptic involution preserves every cylinder on (X, ω) (Lemma 2.1 in [Lin]). Therefore, (X, ω) does not have a cylinder bounded by a pair of saddle connections that form a figure-eight at a zero of ω. In particular, (X, ω) does not arise from another holomorphic 1-form by a connected sum with a torus. □

3. The absolute period foliation and surgeries

This chapter is based on Section 3 in [Win2]. We review the absolute period foliation of a stratum of holomorphic 1-forms. We then study the absolute period foliation of the finite covers of strata from Chapter 2, and we study the interaction between the absolute period foliation and the surgeries from Chapter 2. In the case of strata of holomorphic 1-forms with exactly two zeros, we establish a key lemma about the connectedness of the intersection of a leaf with the locus of holomorphic 1-forms with no saddle connections whose holonomy lies in a given interval.

In the literature, the absolute period foliation is also referred to as the isoperiodic foliation, the Rel foliation, and the kernel foliation. For related discussions and further background, we refer to [BSW], [CDF], [McM4], [McM5], [Zor].

3.1. **The period map.** Let S_g be a closed oriented surface of genus $g \geq 2$. For $X \in \mathcal{M}_g$, a *marking* of $H^1(X;\mathbb{C})$ is a symplectic isomorphism $m : H^1(S_g;\mathbb{C}) \to H^1(X;\mathbb{C})$ that sends $H^1(S_g;\mathbb{Z})$ to $H^1(X;\mathbb{Z})$. Let $\mathcal{S}_g \to \mathcal{M}_g$ be the *Torelli cover* of moduli space, whose points (X, m) are closed Riemann surfaces of genus g equipped with a marking of $H^1(X;\mathbb{C})$. Let $\Omega\mathcal{S}_g \to \mathcal{S}_g$ be the associated bundle of nonzero holomorphic 1-forms. The space $\Omega\mathcal{S}_g$ decomposes into strata $\Omega\mathcal{S}_g(\kappa)$ indexed by partitions $\kappa = \{m_1, \dots, m_n\}$ of $2g-2$. The *period map*

$$\mathrm{Per}_g : \Omega\mathcal{S}_g \to H^1(S_g;\mathbb{C}), \quad (X, \omega, m) \mapsto m^{-1}([\omega])$$

sends a holomorphic 1-form to the associated cohomology class on S_g. The period map is a holomorphic submersion, and the connected components of nonempty fibers of Per_g are leaves of a holomorphic foliation of $\Omega\mathcal{S}_g$. This foliation descends to a foliation $\mathcal{A}$ of $\Omega\mathcal{M}_g$, called the *absolute period foliation of* $\Omega\mathcal{M}_g$. The restriction of Per_g to a stratum $\Omega\mathcal{S}_g(\kappa)$ is also a holomorphic submersion, and we similarly obtain a foliation $\mathcal{A}(\kappa)$ of $\Omega\mathcal{M}_g(\kappa)$, called the *absolute period foliation of* $\Omega\mathcal{M}_g(\kappa)$. Leaves of $\mathcal{A}(\kappa)$ are immersed complex suborbifolds of dimension $|\kappa| - 1$.

3.2. **Geometry of leaves.** Let $\Omega\mathcal{M}_g(\kappa)$ be a stratum with $|\kappa| > 1$. Fix $(X_0, \omega_0) \in \Omega\mathcal{M}_g(\kappa)$, and let L be the leaf of $\mathcal{A}(\kappa)$ through (X_0, ω_0). Let $v = (1, \dots, 1) \in \mathbb{C}^{|\kappa|}$, let $\mathbb{X} = \mathbb{C}^{|\kappa|}/\mathbb{C}v$, and let $G = \mathbb{C}^{|\kappa|}/\mathbb{C}v \rtimes \mathrm{Sym}(|\kappa|)$, where the symmetric group $\mathrm{Sym}(|\kappa|)$ acts by permuting components. Choose an open disk $U \subset L$ containing (X_0, ω_0), a labelling $Z_1, \dots, Z_{|\kappa|}$ of $Z(\omega)$, a point $x \in X_0$, and paths γ_j from x to Z_j. The *relative period map*

$$\rho : U \to \mathbb{X}, \quad (X, \omega) \mapsto \left(\int_{\gamma_1} \omega, \dots, \int_{\gamma_{|\kappa|}} \omega \right) \tag{2}$$

provides local coordinates on U and is independent of the choice of x. Different choices of labellings and paths may permute the components of ρ and may translate the components of ρ by absolute periods, which are constant on L. Thus, L has a $(G, \mathbb{X})$-structure, and in particular a canonical locally Euclidean metric. In general, this metric is incomplete, since the holonomy of a saddle connection with distinct endpoints may approach 0 along a path in L of finite length. For all $M \in \mathrm{GL}^+(2, \mathbb{R})$, the action of $\mathrm{GL}^+(2, \mathbb{R})$ on $\Omega\mathcal{M}_g(\kappa)$ induces a homeomorphism $L \to M \cdot L$ to another leaf of $\mathcal{A}(\kappa)$, and this homeomorphism is affine in the coordinates provided by relative period maps.

3.3. **Lifting to finite covers.** Choose $m \in \kappa$, and let $p : \widetilde{\Omega}\mathcal{M}_g(\kappa; m) \to \Omega\mathcal{M}_g(\kappa)$ be the stratum cover in (1). The foliation $\mathcal{A}(\kappa)$ lifts to a foliation $\mathcal{A}(\kappa; m)$ of $\widetilde{\Omega}\mathcal{M}_g(\kappa; m)$ which we call the *absolute period foliation of* $\widetilde{\Omega}\mathcal{M}_g(\kappa; m)$. The action of $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ on $\widetilde{\Omega}\mathcal{M}_g(\kappa; m)$ induces affine homeomorphisms between leaves of $\mathcal{A}(\kappa; m)$.

Given $\ell \in \kappa \setminus \{m\}$, $1 \leq j \leq \min\{\ell+1, m+1\}$, $\kappa_1 = (a_1, \ldots, a_j)$ an ordered partition of $m+1$ with j parts, and $\kappa_2 = (b_1, \ldots, b_j)$ an ordered partition of $\ell+1$ with j parts, we define

$$\widetilde{A}(\kappa, \kappa_1, \kappa_2) \subset \widetilde{\Omega}\mathcal{M}_g(\kappa; m)$$

to be the subset of (X, ω, θ) with a collection of j homologous saddle connections $\gamma_1, \ldots, \gamma_j$ from the distinguished zero Z to a different zero Z' of order ℓ, cyclically ordered in counterclockwise order around their common starting point Z, and with the following properties.

(1) If γ_k has length $\varepsilon > 0$ for $1 \leq k \leq j$, then every other saddle connection on (X, ω, θ) has length at least 3ε.

(2) Let $X_1, \ldots, X_j$ be the connected components of $X \setminus (\gamma_1 \cup \cdots \cup \gamma_j)$, where X_k is bounded by $\gamma_k \cup \gamma_{k+1}$, indices taken moduli j. The cone angle around Z inside X_k is $2\pi a_k$, and the cone angle around Z' inside X_k is $2\pi b_k$.

Since homologous saddle connections have the same holonomy, they have the same length. We also define

$$A(\kappa, \kappa_1, \kappa_2) = p(\widetilde{A}(\kappa, \kappa_1, \kappa_2)).$$

A collection of saddle connections as above persists on an open neighborhood, so $\widetilde{A}(\kappa, \kappa_1, \kappa_2)$ and $A(\kappa, \kappa_1, \kappa_2)$ are open subsets of $\widetilde{\Omega}\mathcal{M}_g(\kappa; m)$ and $\Omega\mathcal{M}_g(\kappa)$, respectively.

The question of which configurations of homologous saddle connections can occur on a holomorphic 1-form in a given connected component of $\Omega\mathcal{M}_g(\kappa)$ was studied in detail in [EMZ]. As a consequence of some special cases of their results, we have the following.

Lemma 3.1. Let $\Omega\mathcal{M}_g(\kappa)$ be a stratum with $|\kappa| > 1$, and fix $m \in \kappa$.

(1) For all $\ell \in \kappa \setminus \{m\}$, $A(\kappa, (m+1), (\ell+1))$ intersects each connected component of $\Omega\mathcal{M}_g(\kappa)$.

(2) If some $m_j \in \kappa$ is odd, then for all $\ell \in \kappa \setminus \{m\}$, $A(\kappa, (m, 1), (\ell, 1))$ intersects each nonhyperelliptic connected component of $\Omega\mathcal{M}_g(\kappa)$.

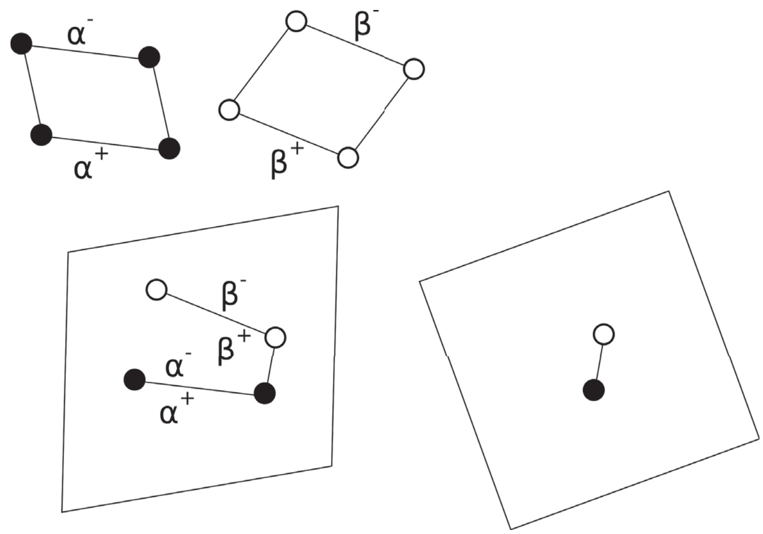

FIGURE 3. A holomorphic 1-form in the intersection of $A(\{3,3\},(3,1),(3,1))$ with the nonhyperelliptic connected component of $\Omega\mathcal{M}_4(3,3)$.

(3) If all $m_j \in \kappa$ are even, then for all $\ell \in \kappa\backslash\{m\}$, $A(\kappa,(m-1,1,1),(\ell-1,1,1))$ intersects each nonhyperelliptic connected component of $\Omega\mathcal{M}_g(\kappa)$.

Proof. Each of statements (1), (2), and (3) in Lemma 3.1 follows from Lemmas 9.1, 10.2, and 10.3 in [EMZ]. Statement (1) is part of the case of these lemmas where $p = 1$ in the notation of [EMZ]. Statement (2) is part of the case where $p = 2$. Statement (3) is part of the case where $p = 3$. □

See Figure 1 (right) for an illustration of Case 1, where the saddle connection arises from the slits on the left. See Figure 3 for an illustration of Case 2.

The next lemma says that leaves of $\mathcal{A}(\kappa)$ typically lift to leaves of $\mathcal{A}(\kappa;m)$, as opposed to a disjoint union of leaves.

Lemma 3.2. Let $\Omega\mathcal{M}_g(\kappa)$ be a stratum with $|\kappa| > 1$. Fix $m \in \kappa$, and let $p : \widetilde{\Omega}\mathcal{M}_g(\kappa;m) \to \Omega\mathcal{M}_g(\kappa)$ be the stratum cover in (1). There is an open $\mathrm{GL}^+(2,\mathbb{R})$-invariant subset $A \subset \Omega\mathcal{M}_g(\kappa)$ that intersects each connected component of $\Omega\mathcal{M}_g(\kappa)$, such that if L is a leaf of $\mathcal{A}(\kappa)$ that intersects A, then $p^{-1}(L)$ is a leaf of $\mathcal{A}(\kappa;m)$.

Proof. Fix $\ell \in \kappa \setminus \{m\}$, $1 \leq j \leq \min\{\ell+1, m+1\}$, $\kappa_1 = (a_1, \ldots, a_j)$ an ordered partition of $m+1$, and $\kappa_2 = (b_1, \ldots, b_j)$ an ordered partition of $\ell+1$. Suppose that $A(\kappa, \kappa_1, \kappa_2)$ is nonempty. Fix $(X, \omega) \in A(\kappa, \kappa_1, \kappa_2)$, fix $(X, \omega, \theta) \in p^{-1}(X, \omega)$, and let $\gamma_1, \ldots, \gamma_j$ be saddle connections as in the definition of $A(\kappa, \kappa_1, \kappa_2)$. Let L be the leaf of $\mathcal{A}(\kappa)$ through (X, ω), and let $\widetilde{L}$ be the leaf of $\mathcal{A}(\kappa; m)$ through (X, ω, θ).

By slitting X along $\gamma_1 \cup \cdots \cup \gamma_j$ and gluing the left side of γ_k to the right side of γ_{k+1}, indices taken modulo j, we obtain a finite collection of holomorphic 1-forms

$$(X_1, \omega_1), \ldots, (X_j, \omega_j).$$

Each (X_k, ω_k) has an oriented geodesic segment δ_k from a point Z_k to a point Z_k' coming from the slits. The order of ω_k at Z_k is $a_k - 1$, and the order of ω_k at Z_k' is $b_k - 1$. Each saddle connection on (X_k, ω_k) has length at least 3ε, except δ_k when $a_k, b_k > 1$. Let $\delta_{k,1}, \ldots, \delta_{k,b_k}$ be the oriented straight segments starting at Z_k' such that

$$\int_{\delta_{k,r}} \omega_k = -\int_{\delta_k} \omega_k$$

cyclically ordered in counterclockwise order around Z_k. We may assume that $\delta_{k,1}$ is δ_k with the opposite orientation. Slit X_k along $\delta_{k,1} \cup \cdots \cup \delta_{k,b_k}$ and glue the left side of $\delta_{k,r}$ to the right side of $\delta_{k,r+1}$, indices taken modulo b_k, to obtain a holomorphic 1-form (X_k', ω_k'). The order of ω_k' at Z_k is $a_k + b_k - 2$, and the order of ω_k' at Z_k' is 0. Moreover, (X_k', ω_k') has no saddle connections of length less than 2ε.

We can reverse the process above to recover (X, ω) from the (X_k', ω_k'). More generally, for each $1 \leq k \leq j$, choose a collection of oriented segments $\delta_{k,1}', \ldots, \delta_{k,b_k}'$ on X_k' starting at Z_k such that

$$\int_{\delta_{k,r}'} \omega_k' = \int_{\delta_k} \omega_k$$

and such that the counterclockwise angle around Z_k from $\delta_{k,1}'$ to $\delta_{k,r}'$ is $2\pi(r-1)$ for $1 \leq r \leq b_k$. Slit X_k' along $\delta_{k,1}' \cup \cdots \cup \delta_{k,b_k}'$ and glue the left side of $\delta_{k,r}'$ to the right side of $\delta_{k,r+1}'$, indices taken modulo b_k. The resulting holomorphic 1-form has a distinguished oriented geodesic segment δ_k coming from $\delta'_{k,1}$ and $\delta'_{k,r}$. Next slit along the segments δ_k, $1 \leq k \leq j$, and glue the left side of δ_k to the right side of δ_{k+1}, indices taken modulo j, to obtain a holomorphic 1-form.

The oriented geodesic segments of length ε on (X'_k, ω'_k) starting at Z_k are parametrized by $\mathbb{R}/2\pi(a_k + b_k - 1)$. By rotating the chosen segment $\delta'_{k,1}$ counterclockwise in the construction above, we obtain a family of holomorphic 1-forms $s_k(t)$ such that $s_k(0) = (X_k, \omega_k)$ and

$$\int_{\delta_k} s_k(t) = e^{it} \int_{\delta_k} s_k(0).$$

Moreover, since $s_k(t)$ is obtained from $s_k(0)$ by only modifying a contractible neighborhood of δ_k, the absolute periods do not change. Thus, by slitting $s_k(t)$ along δ_k for $1 \leq k \leq j$ and gluing the left side of δ_k to the right side of δ_{k+1} as above, we obtain a path

$$s : \mathbb{R} \to \widetilde{L}$$

such that $s(0) = (X, \omega, \theta)$ such that

$$\int_{\gamma_k} s(t) = e^{it} \int_{\gamma_k} s(0)$$

for $t \in \mathbb{R}$. Informally, $s(t)$ is obtained from $s(0)$ by rotating each saddle connection γ_k around its starting point counterclockwise through an angle t.

Rotating these saddle connections counterclockwise through an angle $2\pi(a_k + b_k - 1)$ does not change (X_k, ω_k), that is, $s_k(t + 2\pi(a_k + b_k - 1)) = s_k(t)$. Therefore, letting

$$N(\kappa_1, \kappa_2) = \mathrm{lcm}_{1 \leq k \leq j}(a_k + b_k - 1),$$

we have

$$p(s(t)) = p(s(t + 2\pi N(\kappa_1, \kappa_2)))$$

for all $t \in \mathbb{R}$. Letting $c(t)$ be the counterclockwise angle around Z from the prong on $s(t)$ to the saddle connection γ_1 on $s(t)$, we have

$$c(t) = c(0) + t$$

for all $t \in \mathbb{R}$. For $n \in \mathbb{Z}$, let θ_n be the prong on (X, ω, θ) such that the counterclockwise angle from θ to θ_n is $2\pi n$. Then we have

$$s(2\pi N(\kappa_1, \kappa_2)) = (X, \omega, \theta_{-N(\kappa_1,\kappa_2)}) \in \widetilde{L}.$$

The cone angle around Z is $2\pi(m+1)$, meaning $(X, \omega, \theta_{m+1}) = (X, \omega, \theta)$. Since the action of $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ on $\widetilde{\Omega}\mathcal{M}_g(\kappa; m)$ respects leaves of $\mathcal{A}(\kappa; m)$, and since p is ζ-equivariant, we similarly have $(X, \omega, \theta_{-N(\kappa_1,\kappa_2)}) \in \widetilde{L}$ whenever L intersects $\mathrm{GL}^+(2, \mathbb{R}) \cdot A(\kappa, \kappa_1, \kappa_2)$. Therefore,

$$(X, \omega, \theta_{n \gcd(m+1, N(\kappa_1,\kappa_2))}) \in \widetilde{L}$$

for all $n \in \mathbb{Z}$ whenever L intersects $\mathrm{GL}^+(2, \mathbb{R}) \cdot A(\kappa, \kappa_1, \kappa_2)$.

Now let $\mathcal{C}$ be a connected component of $\Omega\mathcal{M}_g(\kappa)$. Since $A(\kappa, \kappa_1, \kappa_2) \cap \mathcal{C}$ is open, it has positive measure whenever it is nonempty. By Lemma 3.1, there is a nonempty subset of the form $A(\kappa, \kappa_1, \kappa_2) \cap \mathcal{C}$. Let $A_{\mathcal{C}}$ be the intersection of the finitely many nonempty subsets of the form $\mathrm{GL}^+(2, \mathbb{R}) \cdot (A(\kappa, \kappa_1, \kappa_2) \cap \mathcal{C})$. By ergodicity of the $\mathrm{GL}^+(2, \mathbb{R})$-action on $\mathcal{C}$, we have that $A_{\mathcal{C}}$ is nonempty. Moreover, $A_{\mathcal{C}}$ is open and $\mathrm{GL}^+(2, \mathbb{R})$-invariant. We claim that

$$\gcd\left(\{m+1\} \cup \{N(\kappa_1, \kappa_2) \ : \ A_{\mathcal{C}} \subset A(\kappa, \kappa_1, \kappa_2)\}\right) = 1. \tag{3}$$

We verify this claim in 3 cases.

Case 1: Suppose that $\ell = m$. By Lemma 3.1,

$$A_{\mathcal{C}} \subset A(\kappa, (m+1), (m+1)).$$

Since $N((m+1),(m+1)) = 2m+1$, the gcd in (3) divides

$$\gcd(m+1, 2m+1) = 1.$$

Additionally, in this case, by applying the zero splitting construction from Chapter 2 to $s(\pi(2m+1))$ with $j = m$ using the segments $\gamma'_1, \ldots, \gamma'_{m+1}$ starting at the distinguished zero Z and satisfying $\int_{\gamma'_k} \omega = -\int_{\gamma_1} \omega$, and then applying the zero splitting construction again with $j = m$ using the segments $\gamma''_1, \ldots, \gamma''_{m+1}$ starting at Z' and satisfying $\int_{\gamma''_k} \omega = -\int_{\gamma_1} \omega$, we obtain an element of $p^{-1}(X, \omega)$ where the prong is at a different zero of order m.

Case 2: Some part of κ is odd. By Case 1, we may assume that $\mathcal{C}$ is nonhyperelliptic. Note that κ contains at least two odd parts, so we may assume that ℓ is odd. By Lemma 3.1,

$$A_{\mathcal{C}} \subset A(m, (m+1), (\ell+1)), \quad A_{\mathcal{C}} \subset A(m, (m,1), (\ell,1)).$$

Since ℓ is odd and

$$N((m+1),(\ell+1)) = m+\ell+1, \quad N((m,1),(\ell,1)) = m+\ell-1,$$

the gcd in (3) divides

$$\gcd(m+1, m+\ell+1, m+\ell-1) = \gcd(m+1, \ell, 2) = 1.$$

Case 3: All parts of κ are even. By Case 1, we may assume that $\mathcal{C}$ is nonhyperelliptic. By Lemma 3.1,

$$A_{\mathcal{C}} \subset A(m, (m+1), (\ell+1)), \quad A_{\mathcal{C}} \subset A(m, (m-1,1,1), (\ell-1,1,1)).$$

Since $m+1$ is odd and

$$N((m-1,1,1),(\ell-1,1,1)) = m+\ell-3,$$

the gcd in (3) divides

$$\gcd(m+1, m+\ell+1, m+\ell-3) = \gcd(m+1, \ell, 4) = 1.$$

To conclude, let

$$A = \bigcup_{\mathcal{C}} A_{\mathcal{C}}.$$

We have that A is open, $\mathrm{GL}^+(2,\mathbb{R})$-invariant, and intersects every connected component of $\Omega\mathcal{M}_g(\kappa)$. By the claim and the surgery in Case 1, for any $(X,\omega) \in A$ and $(X,\omega,\theta) \in p^{-1}(X,\omega)$, the leaf of $\mathcal{A}(\kappa;m)$ through (X,ω,θ) contains $p^{-1}(X,\omega)$. □

When $|\kappa| > 1$, Lemma 3.2 implies that the preimage under p of a connected component of $\Omega\mathcal{M}_g(\kappa)$ is a connected component of $\widetilde{\Omega}\mathcal{M}_g(\kappa;m)$. The same holds when $|\kappa| = 1$, since in that case the orbit of (X,ω,θ) under the rotation subgroup of $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ contains $p^{-1}(X,\omega)$.

3.4. **Splitting zeros along leaves.** The foliation $\mathcal{A}(\kappa;m)$ lifts to a foliation $\mathcal{F}_{\mathcal{S}}$ of $\mathcal{S}_1(\kappa;m)$. The leaf of $\mathcal{F}_{\mathcal{S}}$ through $((X,\omega,\theta),\gamma)$ consists of the elements of $\mathcal{S}_1(\kappa;m)$ that can be reached from $((X,\omega,\theta),\gamma)$ by a path in $\mathcal{S}_1(\kappa;m)$ along which the absolute periods are constant. The segment γ may vary along the leaf.

Lemma 3.3. Let $L_{\mathcal{S}}$ be the leaf of $\mathcal{F}_{\mathcal{S}}$ through $((X,\omega,\theta),\gamma)$. Then $((X',\omega',\theta'),\gamma') \in L_{\mathcal{S}}$ if and only if (X',ω',θ') is in the leaf of $\mathcal{A}(\kappa;m)$ through (X,ω,θ) and $\gamma' \in S(\omega')$.

Proof. Let $\widetilde{L}$ be the leaf of $\mathcal{A}(\kappa;m)$ through (X,ω,θ), and fix $(X',\omega',\theta') \in \widetilde{L}$. Let $s : [0,1] \to \widetilde{L}$ be a path such that $s(0) = (X,\omega,\theta)$ and $s(1) = (X',\omega',\theta')$. Let $(X_t,\omega_t,\theta_t) = s(t)$. By compactness, there is $\varepsilon > 0$ such that for all $t \in [0,1]$, every saddle connection on $s(t)$ has length at least ε. Since $S(\omega)$ is path-connected, there is a path $s_1 : [0,1] \to L_{\mathcal{S}}$ from $((X,\omega,\theta),\gamma)$ to $((X,\omega,\theta),\gamma_1)$ such that γ_1 has length less than ε. Using the natural inclusions $S(\omega_t) \hookrightarrow \widetilde{\mathbb{C}}^*$, we obtain a well-defined path $\widetilde{s} : [0,1] \to L_{\mathcal{S}}$ given by $\widetilde{s}(t) = (s(t),\gamma_1)$. Then since $S(\omega')$ is path-connected, we have $((X', \omega', \theta'), \gamma') \in L_{\mathcal{S}}$. The other containment is clear by definition of $\mathcal{F}_{\mathcal{S}}$. □

Fix $1 \le j < m$, let $\kappa' = (\kappa \setminus \{m\}) \cup \{m-j, j\}$, and consider the associated zero splitting map

$$\Phi_1 : \mathcal{S}_1(\kappa; m) \to \Omega_1\mathcal{M}_g(\kappa').$$

Splitting a zero is a local surgery which only modifies a holomorphic 1-form in a contractible neighborhood of one of its zeros, so it does not change the absolute periods. Therefore, Φ_1 sends leaves of $\mathcal{F}_{\mathcal{S}}$ into leaves of $\mathcal{A}(\kappa')$. If $E \subset \Omega_1\mathcal{M}_g(\kappa')$ is saturated for $\mathcal{A}(\kappa')$, then $\Phi_1^{-1}(E)$ is saturated for $\mathcal{F}_{\mathcal{S}}$.

3.5. **Geodesics on leaves.** Next, let $\Omega\mathcal{M}_g(\kappa)$ be a stratum with $|\kappa| = 2$. In this case, a leaf L of $\mathcal{A}(\kappa)$ is a Riemann surface equipped with a canonical quadratic differential q. To describe q, fix $(X_0, \omega_0) \in L$, and let γ be a saddle connection on (X_0, ω_0) with distinct endpoints. Let Z_1 and Z_2 be the starting point and ending point, respectively, of γ. The map

$$(X, \omega) \mapsto \int_\gamma \omega \in \mathbb{C}$$

provides a local coordinate on L near (X_0, ω_0), and we have $q = dr^2$. For any $z \in \mathbb{C}^*$, there is a locally defined geodesic with respect to $|q|$ through (X_0, ω_0), given by

$$s : (-\varepsilon, \varepsilon) \to L, \quad s(t) = (X_t, \omega_t),$$

such that $\frac{d}{dt}\int_\gamma \omega_t = z$. The maximal domain of definition of s is not necessarily $\mathbb{R}$. However, the only obstruction is the existence of a saddle connection on (X_0, ω_0) with distinct endpoints and with holonomy in $\mathbb{R}z$.

Corollary 3.4. ([BSW], Corollary 6.2) The maximal domain of definition of s contains $t_0 \in \mathbb{R}$ if and only if (X_0, ω_0) does not have a saddle connection from Z_2 to Z_1 with holonomy in $\{tt_0 z \ : \ t \in [0,1]\}$.

A more general version of Corollary 3.4 is proven in [BSW], which applies to any stratum $\Omega\mathcal{M}_g(\kappa)$ with $|\kappa| > 1$. Note that [BSW] work with strata with labelled singularities. See also [McM4], [MW].

Lemma 3.5. Let $\Omega\mathcal{M}_g(\kappa)$ be a stratum with $|\kappa| = 2$. Fix $(X, \omega) \in \Omega\mathcal{M}_g(\kappa)$, let L be the leaf of $\mathcal{A}(\kappa)$ through (X, ω), and let q be the canonical quadratic differential on L. Fix $z \in \mathbb{C}^*$ such that

$$z \notin \bigcup_{z_0 \in \mathrm{Per}(\omega)} \mathbb{R} z_0.$$

Let $I = [0, z] = \{tz \ : \ t \in [0,1]\}$, and let

$$L(I) = \{(Y, \eta) \in L \ : \ \Gamma(\eta) \cap I \neq \emptyset\}.$$

The subspace $L(I) \subset L$ is closed, and is a countable union of embedded isolated parallel line segments with respect to q. Moreover, the complement $L \setminus L(I)$ is path-connected.

Proof. Fix $(X_0, \omega_0) \in L(I)$, and let γ be a saddle connection on (X_0, ω_0) with holonomy in I. Since $(X_0, \omega_0) \in L$, we have $\mathrm{Per}(\omega_0) = \mathrm{Per}(\omega)$. Then since $I \subset \mathbb{R}z$ and

$$\mathbb{R}z \cap \left(\bigcup_{z_0 \in \mathrm{Per}(\omega_0)} \mathbb{R} z_0 \right) = \{0\},$$

the saddle connection γ must have distinct endpoints. Moreover, any other saddle connection on (X_0, ω_0) with holonomy in $\mathbb{R}_{>0} z$ must have the same starting point and ending point as γ and must have the same holonomy as γ. Let $\gamma_1, \ldots, \gamma_j$ be this finite collection of saddle connections. By Corollary 3.4, there is a geodesic ray with respect to $|q|$ through (X_0, ω_0), given by

$$s : \mathbb{R}_{>0} \to L, \quad s(t) = (X_t, \omega_t),$$

such that for all $t > 0$ and $k = 1, \ldots, j$,

$$\int_{\gamma_k} \omega_t = tz.$$

In particular, s is injective and $s^{-1}(L(I)) = (0, 1]$. The period coordinates of $s(1)$ lie in $\mathbb{Q} \cdot \mathrm{Per}(\omega) + \mathbb{Q}z$, so there are only countably many possibilities for $s(1)$. We have shown that with respect to q, the subspace $L(I) \subset L$ is a countable union of embedded parallel line segments.

By Lemma 2.4, the subset $\Omega\mathcal{M}_g(\kappa; I)$ of holomorphic 1-forms in $\Omega\mathcal{M}_g(\kappa)$ such that $\Gamma(\omega) \cap I \neq \emptyset$ is closed. We have

$$L(I) = L \cap \Omega\mathcal{M}_g(\kappa; I)$$

so $L(I)$ is closed in the subspace topology on L, and since L is immersed, $L(I)$ is closed in the intrinsic topology on L.

Fix $0 < \varepsilon < |z|$, and let $s : \mathbb{R}_{>0} \to L$ be a geodesic ray as above, so $\ell = s((0, 1])$ is a maximal line segment in $L(I)$. Let $\ell_\varepsilon = s([\varepsilon/|z|, 1])$. Let γ be a homotopy class of paths on $s(1)$ with endpoints in the zero set. Parallel transport along ℓ_ε gives a homotopy class of paths $\gamma(t)$ on $(X_t, \omega_t) = s(t)$ for all $t \in [\varepsilon/|z|, 1]$. By compactness and Lemma 2.3, there are only finitely many homotopy classes $\gamma'_1, \ldots, \gamma'_n$ on $s(1)$ such that for some $t \in [\varepsilon/|z|, 1]$, the length of the geodesic representative on (X_t, ω_t) is at most $2|z|$. The Euclidean distance in $\mathbb{C}$ from $\int_{\gamma'_k(t)} \omega_t$ to $\mathbb{R}z$ is constant along ℓ_ε, so there is $\delta > 0$ such that for all $t \in [\varepsilon/|z|, 1]$ and $k = 1, \ldots, n$, the distance from $\int_{\gamma'_k(t)} \omega_t$ to $\mathbb{R}z$ is at least δ. Along a path in L starting at (X_t, ω_t), the change in $\int_{\gamma'_k(t)} \omega_t$ has absolute value at most the $|q|$-length of the path. Therefore, letting $d_q : L \times L \to \mathbb{R}_{\geq 0}$ be the distance on L induced by $|q|$, we have

$$d_q(\ell_\varepsilon, L \setminus \ell) > \delta.$$

We have shown that the maximal line segments in $L(I)$ are isolated from each other.

Choose a path $\varphi : [0,1] \to L$ such that $\varphi(0) \notin L(I)$ and $\varphi(1) \notin L(I)$. By applying a homotopy rel endpoints to φ, we may assume that with respect to q, the path φ is piecewise-linear with finitely many pieces, that the endpoints of each piece do not lie in $L(I)$, and that each piece is not parallel to the line segments in $L(I)$. By compactness, there is $0 < \varepsilon < |z|$ such that for all $t \in [0,1]$, each saddle connection on $\varphi(t)$ has length at least ε. Since $L(I) \subset L$ is closed and the line segments in $L(I)$ are isolated from each other, $\varphi([0,1]) \cap L(I)$ is compact and discrete, and therefore finite. Let

$$0 < t_1 < \cdots < t_n < 1$$

be the finite set of times such that $\varphi(t_j) \in L(I)$, let $s_j : \mathbb{R}_{>0} \to L$ be the geodesic ray as above through $\varphi(t_j)$, let $\ell_j = s_j((0,1])$, and let $\ell_{j,\varepsilon} = s_j([\varepsilon/|z|, 1])$. Fix $\varepsilon' > 0$ such that for $1 \le j \le n$,

$$d_q(\ell_{j,\varepsilon}, L \setminus \ell_j) > \varepsilon'$$

and the embedding $s_j : [\varepsilon/|z|, 1] \to L$ extends to an embedding of the $(\varepsilon'/|z|)$-neighborhood of $[\varepsilon/|z|, 1]$ in $\mathbb{C}$ with respect to the Euclidean metric, whose image is the ε'-neighborhood of $\ell_{j,\varepsilon}$ in L with respect to $|q|$. Fix $\delta' > 0$ such that for $1 \le j \le n$ and $t \in (t_j - \delta', t_j + \delta')$, we have

$$d_q(\varphi(t), \ell_j) < \varepsilon'.$$

For each j, we can apply a homotopy rel endpoints to the restriction $\varphi|_{[t_j-\delta', t_j+\delta']}$ to arrange that the image of $\varphi|_{[t_j-\delta', t_j+\delta']}$ is contained in the ε'-neighborhood of $\ell_{j,\varepsilon}$ and disjoint from ℓ_j. This gives us a path $[0,1] \to L \setminus L(I)$ with the same starting point $\varphi(0)$ and the same ending point $\varphi(1)$, thus $L \setminus L(I)$ is path-connected. $\square$

3.6. **Connected sums along leaves.** Lemma 3.5 also holds with $\widetilde{\Omega}\mathcal{M}_g(\kappa; m)$ in place of $\Omega\mathcal{M}_g(\kappa)$, and the proof is the same. The foliation $\mathcal{A}(\kappa; m)$ lifts to a foliation $\mathcal{F}_{\mathcal{T}}$ of $\mathcal{T}_1(\kappa; m)$.

The leaf of $\mathcal{F}_{\mathcal{T}}$ through $((X,\omega,\theta),(\gamma,w),a)$ consists of the elements of $\mathcal{T}_1(\kappa;m)$ that can be reached from $((X,\omega,\theta),(\gamma,w),a)$ by a path in $\mathcal{T}_1(\kappa;m)$ along which the absolute periods and $((\gamma,w),a)$ are constant.

Corollary 3.6. Let $\Omega\mathcal{M}_g(\kappa)$ be a stratum with $|\kappa|=2$. Fix $m\in\kappa$, let $p:\widetilde{\Omega}\mathcal{M}_g(\kappa;m)\to\Omega\mathcal{M}_g(\kappa)$ be the associated stratum cover, and consider the full measure subset of $\mathcal{T}_1(\kappa;m)$ given by

$$\mathcal{T}_{\mathrm{conn}}(\kappa;m)=\left\{((X,\omega,\theta),(\gamma,w),a)\in\mathcal{T}_1(\kappa;m)\;:\;\int_\gamma\omega\notin\bigcup_{z\in\mathrm{Per}(\omega)}\mathbb{R}z\right\}.$$

For $((X,\omega,\theta),(\gamma,w),a)\in\mathcal{T}_{\mathrm{conn}}(\kappa;m)$, letting $\widetilde{L}$ be the leaf of $\mathcal{A}(\kappa;m)$ through (X,ω,θ), $L_{\mathcal{T}}$ the leaf of $\mathcal{F}_{\mathcal{T}}$ through $((X,\omega,\theta),(\gamma,w),a)$, and $I=[0,\int_\gamma\omega]$, we have

$$L_{\mathcal{T}}=(\widetilde{L}\setminus\widetilde{L}(I))\times\{(\gamma,w)\}\times\{a\}.$$

Let $\kappa'=(\kappa\setminus\{m\})\cup\{m+2\}$, and consider the associated connected sum map

$$\Psi_1:\mathcal{T}_1(\kappa;m)\to\Omega_1\mathcal{M}_{g+1}(\kappa').$$

The connected sum map Ψ_1 sends leaves of $\mathcal{F}_{\mathcal{T}}$ into leaves of $\mathcal{A}(\kappa')$. If $E\subset\Omega_1\mathcal{M}_{g+1}(\kappa')$ is saturated for $\mathcal{A}(\kappa')$, then $\Psi_1^{-1}(E)$ is saturated for $\mathcal{F}_{\mathcal{T}}$. Moreover, if E is measurable, then up to a set of measure zero, $\Psi_1^{-1}(E)$ is saturated for the foliation of $\widetilde{\Omega}\mathcal{M}_g(\kappa;m)\times\widetilde{\mathrm{SL}}(2,\mathbb{R})\times(0,1)$ whose leaves have the form $\widetilde{L}\times\{(\gamma,w)\}\times\{a\}$ with $\widetilde{L}$ a leaf of $\mathcal{A}(\kappa;m)$.

4. Pairs of splittings

This chapter is based on Section 4 in [Win2]. We give a criterion for presenting a holomorphic 1-form as a connected sum with a torus in multiple ways. Our constructions are similar in spirit to the connected sum constructions for holomorphic 1-forms in $\Omega\mathcal{M}_2(2)$ studied in [CM] and [McM2].

4.1. **Splittings.** Recall from Chapter 2 that a splitting of (X, ω) is a pair of homologous saddle connections $\alpha^\pm$ on (X, ω) that form a figure-eight at a zero Z of ω, such that

(1) the counterclockwise angle around Z from the end of α^- to the end of α^+ is 2π;

(2) one of the connected components of $X \setminus (\alpha^+ \cup \alpha^-)$ is a cylinder.

Slitting (X, ω) along $\alpha^\pm$ and regluing the sides of the slits gives a holomorphic 1-form of genus $g-1$ and a flat torus. In this way, the pair $\alpha^\pm$ gives a presentation of (X, ω) as a connected sum with a torus.

We borrow some notation from [CM]. Given $z, w \in \mathbb{C}$, the cross-product $z \times w$ is the signed area of the parallelogram spanned by z and w, that is,

$$z \times w = \operatorname{Im}(\overline{z} w).$$

Lemma 4.1. Suppose that $\alpha^\pm$ is a splitting of (X, ω), and suppose that (X, ω) has an embedded open parallelogram P bounded by $\alpha^\pm$ and another pair of homologous saddle connections $\gamma_0^\pm$. Let C be the cylinder given by one of the connected components of $X \setminus (\alpha^+ \cup \alpha^-)$, and choose a saddle connection $\beta \subset C \cup Z(\omega)$. Let

$$z = \int_{\alpha^\pm} \omega, \quad w = \int_\beta \omega, \quad z' = \int_{\gamma_0^\pm} \omega,$$

and suppose that

$$z \times w > 0, \quad z \times z' > 0, \quad z' \times w > 0, \quad (z+w) \times (z'+w) > 0. \tag{4}$$

Then $P \cup \overline{C}$ contains another splitting $\gamma^\pm$ of (X, ω) with the same starting point and ending point as $\alpha^\pm$, and there is a saddle connection $\delta \subset C' \cup Z(\omega)$, where C' is the cylinder given by one of the connected components of $X \setminus (\gamma^+ \cup \gamma^-)$, such that

$$[\gamma^\pm] = -[\gamma_0^\pm] - [\beta], \quad [\delta] = [\alpha^\pm] + [\beta]$$

in $H_1(X; \mathbb{Z})$.

Proof. For $M \in \mathrm{GL}^+(2, \mathbb{R})$, the cross-products $z \times w \in \mathbb{R}$ and $Mz \times Mw \in \mathbb{R}$ have the same sign. There is an affine homeomorphism $(X, \omega) \to M(X, \omega)$ that sends zeros to zeros, and sends a saddle connection on (X, ω) with holonomy z_0 to a saddle connection on $M(X, \omega)$ with holonomy Mz_0. A pair of homologous saddle connections is a splitting of (X, ω) if and only if the corresponding pair on $M(X, \omega)$ is a splitting of $M(X, \omega)$. Thus, it is enough to show that Lemma 4.1 holds for $M(X, \omega)$. Since $z \times w > 0$, by applying an appropriate element of $\mathrm{GL}^+(2, \mathbb{R})$ to (X, ω), we may assume that

$$z = 1, \quad w = i.$$

We regard C as a unit square with its vertical sides glued together to form β. The bottom side of C is α^-, and the top side of C is α^+. The inequalities in (4) imply there is a straight segment $\gamma^- \subset P \cup \overline{C}$ from the top-left corner of P to the bottom-left corner of C that crosses α^+, and a straight segment $\gamma^+ \subset P \cup \overline{C}$ from the top-right corner of C to the bottom-right corner of P that crosses α^-. The segments $\gamma^\pm$ are a pair of homologous saddle connections with the same starting point and ending point Z as $\alpha^\pm$, and $\gamma^\pm$ bound a cylinder $C' \subset P \cup \overline{C}$. Since the counterclockwise angle around Z from the end of α^- to the end of α^+ is 2π, the counterclockwise angle around Z from the end of γ^- to the end of γ^+ is also 2π. Moreover, the cylinder C' is one of the connected components of $X \setminus (\gamma^+ \cup \gamma^-)$. Thus, $\gamma^\pm$ is another splitting of (X, ω). Let δ be the straight segment from the bottom-left corner of C to the top-right corner of C. Then δ is a saddle connection contained in $C' \cup Z(\omega)$. All of the

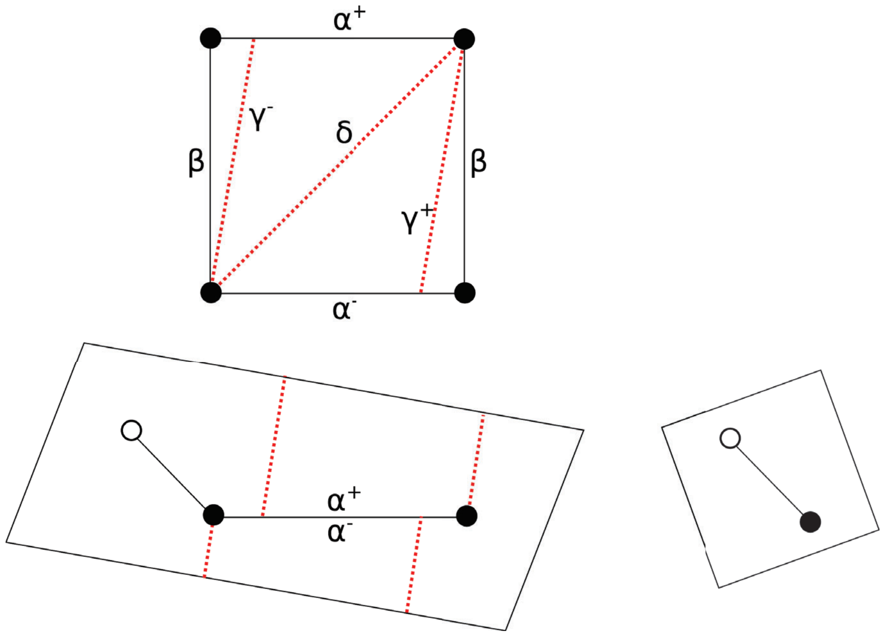

FIGURE 4. A holomorphic 1-form in $\Omega\mathcal{M}_3(3,1)$ with a splitting $\alpha^\pm$ and a second splitting $\gamma^\pm$ as in Lemma 4.1.

saddle connections $\alpha^\pm, \gamma_0^\pm, \gamma^\pm, \beta, \delta$ have the same starting and ending point, and therefore represent elements of $H_1(X;\mathbb{Z})$. The relations $[\gamma^\pm] = -[\gamma_0^\pm] - [\beta]$ and $[\delta] = [\alpha^\pm] + [\beta]$ are clear. □

See Figure 4 for an illustration of Lemma 4.1.

4.2. **Related splittings.** Let

$$\mathcal{T}_{(0,1)} = \left\{(z,w) \in \mathbb{C}^2 \;:\; 0 < z \times w < 1\right\}$$

and let $\sim$ be an equivalence relation on $\mathcal{T}_{(0,1)}$ that satisfies

$$(z,w) \sim (z, nz + w) \tag{5}$$

for all $(z,w) \in \mathcal{T}_{(0,1)}$ and $n \in \mathbb{Z}$, and that satisfies

$$(z,w) \sim (-z' - w, z + w) \tag{6}$$

for all $(z,w) \in \mathcal{T}_{(0,1)}$ and $z' \in \mathbb{C}$ such that

$$0 < z \times z' < 1 - z \times w, \quad 0 < z' \times w < z \times (z' + w). \tag{7}$$

A splitting $\alpha^\pm$ of a holomorphic 1-form (X,ω) of area 1 determines an element of $\mathcal{T}_{(0,1)}$ as follows. Let C be the cylinder given by one of the connected components of $X \setminus (\alpha^+ \cup \alpha^-)$, and let $\beta \subset C \cup Z(\omega)$ be a saddle connection. Let $z = \int_{\alpha^\pm} \omega$ and let $w = \int_\beta \omega$. Reversing the orientation of β if necessary, we may assume that $z \times w > 0$. Then $z \times w$ is the area of C with respect to $|\omega|$, so $z \times w < 1$ and we have $(z,w) \in \mathcal{T}_{(0,1)}$. By changing the choice of β, we can obtain $(z, nz + w) \in \mathcal{T}_{(0,1)}$ for all $n \in \mathbb{Z}$.

Lemma 4.1 provides a way of constructing holomorphic 1-forms with a pair of splittings with associated pairs $(z,w) \in \mathcal{T}_{(0,1)}$ and $(-z' - w, z + w) \in \mathcal{T}_{(0,1)}$, respectively, whenever z, w, z' satisfy (7).

Lemma 4.2. Let $\mathcal{C}$ be a nonhyperelliptic connected component of a stratum $\Omega\mathcal{M}_g(m_1, m_2)$ with $m_1 \geq 3$ odd. Fix $(z,w) \in \mathcal{T}_{(0,1)}$ and $z' \in \mathbb{C}$ satisfying (7). There exists $(X,\omega) \in \mathcal{C}_1$ with a pair of splittings $\alpha^\pm$ and $\gamma^\pm$ that all start and end at the same zero of order m_1, and there are saddle connections $\beta \subset C \cup Z(\omega)$ and $\delta \subset C' \cup Z(\omega)$, where C and C' are the cylinders given by a connected component of $X \setminus (\alpha^+ \cup \alpha^-)$ and $X \setminus (\gamma^+ \cup \gamma^-)$, respectively, such that

$$z = \int_{\alpha^\pm} \omega, \quad w = \int_\beta \omega, \quad -z' - z = \int_{\gamma^\pm} \omega, \quad z + w = \int_\delta \omega.$$

Proof. Let T_0 be the flat torus $\mathbb{C}/(\mathbb{Z}z + \mathbb{Z}w)$. Choose $w' \in \mathbb{C}$ such that

$$0 < z \times z' < z' \times w' < 1 - z \times w \tag{8}$$

and such that $z \notin \mathbb{Q}z' + \mathbb{Q}w'$. Let T_1 be the flat torus $\mathbb{C}/(\mathbb{Z}z' + \mathbb{Z}w')$. Let T_2 be a flat torus with area less than $1 - z \times w - z' \times w'$. The segment $[0, z] \subset \mathbb{C}$ projects to a closed geodesic $\alpha_0 \subset T_0$, and projects to an embedded geodesic segment $\alpha \subset T_1$. The segments $[0, z'], [z, z + z'] \subset \mathbb{C}$ project to a pair of closed geodesics $\gamma_0^\pm \subset T_1$ passing through the endpoints of α and otherwise disjoint from α. The inequalities in (8) imply that $\gamma_0^\pm$ and the two sides of α bound an embedded open parallelogram $P \subset T_1$. For $j = 1, 2$, choose short embedded segments $s_j \subset T_j$ with the same length and in the same direction, such that s_1 starts at the starting point of α and is otherwise disjoint from $\overline{P}$. Slit T_j along s_j, and let s_j^+ and s_j^- be the left and right sides of the slit coming from s_j, respectively. Glue s_1^+ to s_2^-, and glue s_1^- to s_2^+. The result is a holomorphic 1-form $(X_0, \omega_0) \in \Omega\mathcal{M}_2(1, 1)$, given by a connected sum of two flat tori along a pair of homologous saddle connections $s^\pm$.

Let $\alpha_1, \ldots, \alpha_{(m_1-3)/2}$ be a collection of short embedded segments in T_1, starting at the starting point of α and otherwise disjoint from each other and from $\alpha \cup s^+ \cup s^- \cup \overline{P}$. Let $\alpha'_1, \ldots, \alpha'_{(m_2-1)/2}$ be a collection of short embedded segments in T_2, starting at the other zero of ω_0 and otherwise disjoint from each other and from $s^+ \cup s^-$. Slit (X_0, ω_0) along α, slit T_0 along α_0, and glue opposite sides to get a holomorphic 1-form in $\Omega\mathcal{M}_3(3, 1)$ with a splitting $\alpha^\pm$ bounding a cylinder C. Then, iterate this procedure using the segments $\alpha_1, \ldots, \alpha_{(m_1-3)/2}$ and $\alpha'_1, \ldots, \alpha'_{(m_2-1)/2}$ and using flat tori with appropriate areas to get a holomorphic 1-form $(X, \omega) \in \Omega_1\mathcal{M}_g(m_1, m_2)$. As in the proof of Lemma 2.6, (X, ω) cannot lie in a hyperelliptic connected component, therefore $(X, \omega) \in \mathcal{C}_1$.

On (X, ω), we have $\int_{\alpha^\pm} \omega = z$. Let $\beta \subset C \cup Z(\omega)$ be a saddle connection such that $\int_\beta \omega = w$. The saddle connections $\alpha^\pm, \gamma_0^\pm$ and the parallelogram P on (X, ω) satisfy the hypotheses of Lemma 4.1. Letting $\gamma^\pm$ be a splitting of (X, ω) and $C' \subset P \cup \overline{C}$ a cylinder given by a connected component of $X \setminus (\gamma^+ \cup \gamma^-)$, and letting $\delta \subset C' \cup Z(\omega)$ be a saddle connection as in Lemma 4.1, we are done. □

Lemma 4.3. For all $(z, w) \in \mathcal{T}_{(0,1)}$ and $(z', w') \in \mathcal{T}_{(0,1)}$, we have $(z, w) \sim (z', w')$.

Proof. Since $\mathcal{T}_{(0,1)}$ is connected, it is enough to show that every equivalence class for $\sim$ is open. For $(z,w),(z',w') \in \mathcal{T}_{(0,1)}$ and $M \in \mathrm{GL}^+(2,\mathbb{R})$ with $0 < \det(M) \leq 1$, we have $(z,w) \sim (z',w')$ if and only if $M(z,w) \sim M(z',w')$. For any $(z,w) \in \mathcal{T}_{(0,1)}$, either $(z,w) = M(1/2, i/2)$ for some $M \in \mathrm{GL}^+(2,\mathbb{R})$ with $0 < \det(M) \leq 1$, or $M(z,w) = (1/2, i/2)$ for some $M \in \mathrm{GL}^+(2,\mathbb{R})$ with $0 < \det(M) < 1$. Thus, it is enough to show that the equivalence class of $(1/2, i/2)$ contains a neighborhood of $(1/2, i/2)$.

By (7), for any $(z_0, w_0) \in \mathcal{T}_{(0,1)}$ and $z, z' \in \mathbb{C}$, if

$$0 < z_0 \times z < 1 - z_0 \times w_0, \quad 0 < z \times w_0 < z_0 \times (z + w_0)$$

and

$$0 < (-z-w_0) \times z' < 1-(-z-w_0) \times (z_0+w_0), \quad 0 < z' \times (z_0+w_0) < (-z-w_0) \times (z'+z_0+w_0),$$

then

$$(z_0, w_0) \sim (-z' - z_0 - w_0, -z + z_0). \tag{9}$$

Fix $\theta \in \mathbb{R}/2\pi$, let $z = \frac{1}{2}\left(1 - e^{i(\theta+\pi/2)}\right)$, and let $z' = -\frac{1}{2}\left(1 + i + e^{i\theta}\right)$. We have

$$0 < \frac{1}{2} \times z < 1 - \frac{1}{2} \times \frac{i}{2}$$

if and only if $\theta \in (\pi/2, 3\pi/2)$, and

$$0 < z \times \frac{i}{2} < \frac{1}{2} \times \left(z + \frac{i}{2}\right)$$

if and only if $\theta \in (3\pi/4, 7\pi/4)$ and $\theta \neq 3\pi/2$. Then by (6)-(7),

$$\left(\frac{1}{2}, \frac{i}{2}\right) \sim \left(-z - \frac{i}{2}, \frac{1}{2} + \frac{i}{2}\right)$$

for all $\theta \in (3\pi/4, 3\pi/2)$. Next, we have

$$0 < \left(-z - \frac{i}{2}\right) \times z' < 1 - \left(-z - \frac{i}{2}\right) \times \left(\frac{1}{2} + \frac{i}{2}\right)$$

if and only if $\theta \in (-\pi/6, 7\pi/6)$, and

$$0 < z' \times \left(\frac{1}{2} + \frac{i}{2}\right) < \left(-z - \frac{i}{2}\right) \times \left(z' + \frac{1}{2} + \frac{i}{2}\right)$$

if and only if $\theta \in (\pi/4, 5\pi/4)$. Then by (6)-(7),

$$\left(-z - \frac{i}{2}, \frac{1}{2} + \frac{i}{2}\right) \sim \left(\frac{1}{2}e^{i\theta}, \frac{i}{2}e^{i\theta}\right)$$

for all $\theta \in (\pi/4, 7\pi/6)$.

Thus, for all

$$\theta \in \left(\frac{3\pi}{4}, \frac{7\pi}{6}\right),$$

we have

$$\left(\frac{1}{2}, \frac{i}{2}\right) \sim \left(\frac{1}{2}e^{i\theta}, \frac{i}{2}e^{i\theta}\right).$$

Then since (7) is an open condition, the equivalence class of $(1/2, i/2)$ contains a neighborhood of $(-1/2, -i/2)$. Since $(z, w) \sim (z', w')$ if and only if $(-z, -w) \sim (-z', -w')$, the equivalence class of $(-1/2, -i/2)$ contains a neighborhood of $(1/2, i/2)$. We conclude that the equivalence class of $(1/2, i/2)$ contains a neighborhood of $(1/2, i/2)$, as desired. $\square$

Let $\Omega\mathcal{M}_g(\kappa)$ be a stratum. Fix $m \in \kappa$, let $\kappa' = (\kappa \setminus \{m\}) \cup \{m+2\}$, and let

$$\Psi_1 : \mathcal{T}_1(\kappa; m) \to \Omega_1\mathcal{M}_{g+1}(\kappa')$$

be the associated connected sum map. Recall that

$$\mathcal{T}_1(\kappa; m) \subset \widetilde{\Omega}_1\mathcal{M}_g(\kappa; m) \times \widetilde{\mathrm{SL}}(2, \mathbb{R}) \times (0, 1)$$

is an open subset of full measure, and that the diagonal action of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ on $\widetilde{\mathbb{C}}^* \times \mathbb{C}$ gives an identification

$$\widetilde{\mathrm{SL}}(2,\mathbb{R}) \cong \left\{(\gamma, w) \in \widetilde{\mathbb{C}}^* \times \mathbb{C} \; : \; \sigma(\gamma) \times w = 1\right\}.$$

We have a map

$$\widetilde{\mathrm{SL}}(2,\mathbb{R}) \times (0,1) \to \mathcal{T}_{(0,1)}$$

which sends $(\widetilde{M}, a)$ to $(1-a)^{1/2}(\sigma(\gamma), w)$, where $(\gamma, w) \in \widetilde{\mathbb{C}}^* \times \mathbb{C}$ corresponds to $\widetilde{M}$ under the identification above. By composing with the projection $\mathcal{T}_1(\kappa; m) \to \widetilde{\mathrm{SL}}(2,\mathbb{R}) \times (0,1)$, we obtain a map

$$\sigma_{\mathcal{T}} : \mathcal{T}_1(\kappa; m) \to \mathcal{T}_{(0,1)}.$$

Given $(X, \omega) \in \Omega_1\mathcal{M}_{g+1}(\kappa')$, an element of $\sigma_{\mathcal{T}}(\Psi_1^{-1}(X,\omega))$ is a pair of complex numbers recording the holonomy of the saddle connections in a splitting $\alpha^{\pm}$, and the holonomy of a saddle connection crossing the cylinder given by a connected component of $X \setminus (\alpha^+ \cup \alpha^-)$.

Lemma 4.4. Let $\mathcal{C}$ be a nonhyperelliptic connected component of a stratum $\Omega\mathcal{M}_g(\kappa)$ with $\kappa = \{m_1, m_2\}$ and m_1 odd, or let $\mathcal{C} = \Omega\mathcal{M}_g(\kappa)$ with $\kappa = \{1,1\}$. Fix $m \in \kappa$, let $p : \widetilde{\Omega}\mathcal{M}_g(\kappa; m) \to \Omega\mathcal{M}_g(\kappa)$ be the stratum cover in (1), and let $\widetilde{\mathcal{C}} = p^{-1}(\mathcal{C})$. Let

$$\mathcal{C}_1(\kappa; m) = \mathcal{T}_1(\kappa; m) \cap \left(\widetilde{\mathcal{C}}_1 \times \widetilde{\mathrm{SL}}(2,\mathbb{R}) \times (0,1)\right),$$

let $\kappa' = (\kappa \setminus \{m\}) \cup \{m+2\}$, and consider the restrictions

$$\Psi_1 : \mathcal{C}_1(\kappa; m) \to \mathcal{C}_1', \quad \sigma_{\mathcal{T}} : \mathcal{C}_1(\kappa; m) \to \mathcal{T}_{(0,1)},$$

where $\mathcal{C}_1'$ is a connected component of $\Omega\mathcal{M}_{g+1}(\kappa')$. If $F \subset \mathcal{C}_1(\kappa; m)$ is a nonempty subset of the form

$$F = \Psi_1^{-1}(F_1) = \sigma_{\mathcal{T}}^{-1}(F_2),$$

then $F = \mathcal{C}_1(\kappa; m)$.

Proof. Fix $(X, \omega) \in F_1$, and fix $(z, w) \in \sigma_{\mathcal{T}}(\Psi_1^{-1}(X, \omega)) \subset F_2$. By Lemma 4.2, for all $z' \in \mathbb{C}$ satisfying (7), there exists

$$(Y, \eta) \in \Psi_1(\sigma_{\mathcal{T}}^{-1}(z, w)) \subset F_1$$

such that

$$(-z' - w, z + w) \in \sigma_{\mathcal{T}}(\Psi_1^{-1}(Y, \eta)) \subset F_2.$$

Also, as discussed above Lemma 4.2, we have $(z, nz + w) \in \sigma_{\mathcal{T}}(\Psi_1^{-1}(X, \omega))$ for all $n \in \mathbb{Z}$. Then by definition of $\sim$, the equivalence class of (z, w) for $\sim$ is contained in F_2. Then by Lemma 4.3, $F_2 = \mathcal{T}_{(0,1)}$ and thus $F = \mathcal{T}_1(\kappa; m)$. $\square$

Remark 4.5. Lemma 4.4 is significantly simpler to prove when $g + 1 \geq 4$. In this case, for any $(z, w) \in \mathcal{T}_{(0,1)}$ and $(z', w') \in \mathcal{T}_{(0,1)}$ satisfying

$$z \times w + z' \times w' < 1,$$

there is $(X, \omega) \in \Omega_1\mathcal{M}_{g+1}(\kappa')$ with a pair of splittings whose associated cylinders are disjoint, realizing

$$(z, w) \in \sigma_{\mathcal{T}}(\Psi_1^{-1}(X, \omega)), \quad (z', w') \in \sigma_{\mathcal{T}}(\Psi_1^{-1}(X, \omega)).$$

See Figure 5 for an example with $\kappa' = \{5, 1\}$.

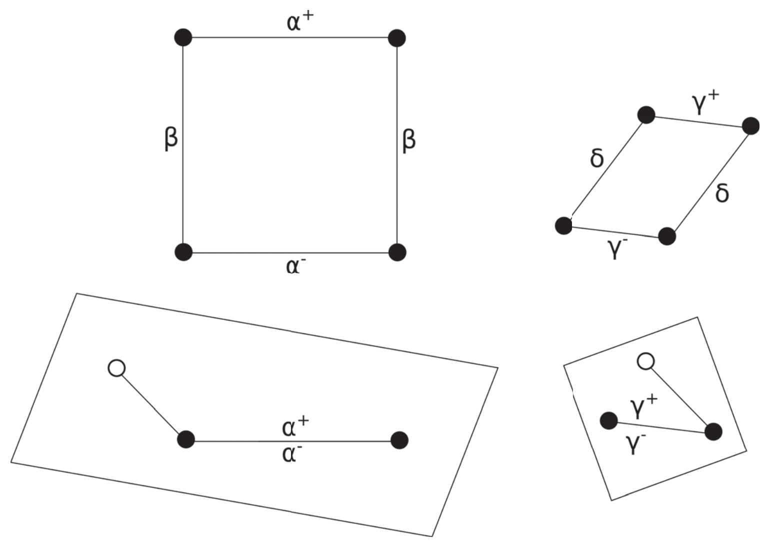

FIGURE 5. A holomorphic 1-form in $\Omega\mathcal{M}_4(5,1)$ with a pair of splittings $\alpha^\pm$ and $\gamma^\pm$, whose associated cylinders are disjoint.

5. Connected spaces of isoperiodic forms

This chapter is based partly on Section 7 of [Win2]. In this chapter, we prove our results on connected components of spaces of holomorphic 1-forms in a stratum representing a given cohomology class. We then deduce our results on the ergodicity of the absolute period foliation, dense leaves of the absolute period foliation, and the monodromy representation of the fundamental group of a stratum component on absolute homology.

5.1. Cohomology classes represented by holomorphic 1-forms.

Let S_g be a closed oriented surface of genus $g \geq 2$. Let

$$\langle \alpha, \beta \rangle = \frac{i}{2} \int_{S_g} \alpha \wedge \overline{\beta}$$

be the intersection form on $H^1(S_g;\mathbb{C})$. A cohomology class $\phi \in H^1(S_g;\mathbb{C})$ is *positive* if $\langle \phi, \phi \rangle > 0$, *elliptic of degree* $d > 0$ if $\mathrm{Per}(\phi)$ is a lattice in $\mathbb{C}$ and the associated homotopy class of maps $S_g \to \mathbb{C}/\mathrm{Per}(\phi)$ has degree d, and *algebraically generic* if $\mathrm{Per}(\phi)$ has rank $2g$ and $\mathrm{Per}(\phi)$ is algebraically generic. The *moduli space of holomorphic* 1*-forms representing*

ϕ is defined by

$$\mathcal{M}(\phi) = \big\{(X, \omega) \in \Omega\mathcal{M}_g \; : \; [\omega] = m(\phi) \text{ for some marking } m \text{ of } H^1(X; \mathbb{C})\big\}$$

and we define

$$\mathcal{M}(\phi; \kappa) = \mathcal{M}(\phi) \cap \Omega\mathcal{M}_g(\kappa).$$

By Haupt's theorem [Hau], the space $\mathcal{M}(\phi)$ is nonempty if and only if ϕ is positive and ϕ is not elliptic of degree 1. Haupt's theorem was rediscovered in [Kap] using tools from homogeneous dynamics, and another proof is given in [CDF]. A generalization of Haupt's theorem to strata was proven in [BJJP], [Fil3].

5.2. **Related periods.** A *polarized module* is a free abelian group $\Lambda \subset \mathbb{C}$ of rank $2g$ equipped with a unimodular symplectic form $\Lambda \times \Lambda \to \mathbb{Z}$, $(a, b) \mapsto a \cdot b$, such that

$$\sum_{j=1}^{g} a_j \times b_j > 0$$

where $\{a_j, b_j\}_{j=1}^{g}$ is a symplectic basis for Λ and $\times$ is the cross-product on $\mathbb{C}$ as in Chapter 4. If $\phi \in H^1(S_g; \mathbb{C})$ is positive and $\mathrm{Per}(\phi)$ has rank $2g$, then $\mathrm{Per}(\phi)$ is a polarized module with the symplectic form inherited from the algebraic intersection form on $H_1(S_g; \mathbb{Z})$. Let

$$\Lambda_{(0,1)} = \{(a, b) \in \Lambda \times \Lambda \; : \; a \cdot b = 1, \; 0 < a \times b < 1\}$$

and let $\sim_\Lambda$ be an equivalence relation on $\Lambda_{(0,1)}$ that satisfies

$$(a, b) \sim_\Lambda (a, na + b) \tag{10}$$

for all $n \in \mathbb{Z}$, and

$$(a, b) \sim_\Lambda (-c - b, a + b) \tag{11}$$

for all $c \in \{a, b\}^\perp$ such that

$$0 < a \times c < 1 - a \times b, \quad 0 < c \times b < a \times (b + c). \tag{12}$$

Lemma 5.1. Suppose $g \geq 3$. Let $\Lambda \subset \mathbb{C}$ be a polarized module of rank $2g$, such that for any $z_1, z_2 \in \Lambda$, if $\mathbb{R}z_1 = \mathbb{R}z_2$ then $\mathbb{Q}z_1 = \mathbb{Q}z_2$. For all $(a, b) \in \Lambda_{(0,1)}$ and $(c, d) \in \Lambda_{(0,1)}$, we have $(a, b) \sim_\Lambda (c, d)$.

Proof. Let $V \subset \Lambda$ be a submodule of rank 2, and fix $z \in \Lambda$ such that for all nonzero $n \in \mathbb{Z}$, $nz \notin V$. Since V is a lattice in $\mathbb{C}$ and $z \notin \bigcup_{v \in V} \mathbb{R}v$, the submodule $V + \mathbb{Z}z$ is dense in $\mathbb{C}$. Therefore, every submodule of Λ of rank at least 3 is dense in $\mathbb{C}$. For any $a \in \Lambda$ and $b_0 \in \Lambda$ such that $a \cdot b_0 = 1$,

$$\{b \in \Lambda \ : \ a \cdot b = 1\} = b_0 + a^\perp$$

is a coset of a submodule of rank $2g - 1 \geq 5$, and is therefore dense in $\mathbb{C}$. The submodule $\{a, b_0\}^\perp$ has rank $2g - 2 \geq 4$, and is therefore dense in $\mathbb{C}$. Then since

$$\mathcal{T}_{(0,1)} = \left\{(z, w) \in \mathbb{C}^2 \ : \ 0 < z \times w < 1\right\}$$

is an open subset of $\mathbb{C}^2$, we have that $\Lambda_{(0,1)}$ is dense in $\mathcal{T}_{(0,1)}$. Since (5) applies to all elements of $\mathcal{T}_{(0,1)}$, and since (7) is an open condition, by Lemma 4.3 the equivalence classes for $\sim_\Lambda$ are dense in $\mathcal{T}_{(0,1)}$. Thus, it is enough to show that $(a, b) \sim_\Lambda (c, d)$ for all $(a, b) \in \Lambda_{(0,1)}$ and $(c, d) \in \Lambda_{(0,1)}$ sufficiently close to $(1/2, i/2)$.

Fix $\varepsilon > 0$ small, and fix $(a, b) \in \Lambda_{(0,1)}$ such that

$$\left|a - \frac{1}{2}\right| < \varepsilon, \quad \left|b - \frac{i}{2}\right| < \varepsilon.$$

The proof of Lemma 4.3 up through (9) gives us that

$$(a, b) \sim_\Lambda (-a_2 - a - b, -a_1 + a)$$

for all $a_1 \in \{a, b\}^\perp$ and $a_2 \in \{-a_1 - b, a + b\}^\perp$ such that

$$|a_1 - (a + b)| < 4\varepsilon, \quad |a_2 + b| < 4\varepsilon.$$

Since $\{b' \in b^\perp \ : \ a \cdot b' = 1\}$ is a coset of a submodule of rank $2g - 2 \geq 4$, there exists $b' \in b^\perp$ such that

$$(a, b') \in \Lambda_{(0,1)}, \quad \left| b' - \frac{i}{2} \right| < \varepsilon.$$

Since the submodule $\{a, b, b'\}^\perp$ has rank at least $2g - 3 \geq 3$, and since $|b - b'| < 2\varepsilon$, there exists $a_1 \in \{a, b, b'\}^\perp$ such that

$$|a_1 - (a + b)| < 2\varepsilon, \quad |a_1 - (a + b')| < 2\varepsilon.$$

The relation

$$(-a_1 - b) + (a + b) = (-a_1 - b') + (a + b')$$

implies that the submodule

$$\{-a_1 - b, a + b, -a_1 - b', a + b'\}^\perp$$

has rank at least $2g - 3 \geq 3$. Since $b' - b \in \{-a_1 - b, a + b\}^\perp$, we have

$$(b' - b) \cap \{-a_1 - b, a + b, -a_1 - b', a + b'\}^\perp = \{-a_1 - b, a + b\}^\perp \cap \left((b' - b) + \{-a_1 - b', a + b'\}^\perp \right).$$

Then there exists

$$a_2 \in \{-a_1 - b, a + b\}^\perp \cap \left((b' - b) + \{-a_1 - b', a + b'\}^\perp \right)$$

such that $|a_2 + b| < 2\varepsilon$, and then

$$a_2' = a_2 + b - b' \in \{-a_1 - b', a + b'\}^\perp$$

satisfies $|a_2' + b| < 4\varepsilon$. Thus,

$$(a,b) \sim_\Lambda (-a_2 - a - b, -a_1 + a) = (-a_2' - a - b', -a_1 + a) \sim_\Lambda (a, b').$$

Now fix $b'' \in \Lambda$ such that $(a, b'') \in \Lambda_{(0,1)}$ and $|b'' - i/2| < \varepsilon$. The subset

$$\{b' \in \{b, b''\}^\perp \ : \ a \cdot b' = 1\}$$

is a coset of a submodule of rank $2g - 3 \geq 3$, so there exists $b' \in \{b, b''\}$ such that

$$(a, b') \in \Lambda_{(0,1)}, \quad \left| b' - \frac{i}{2} \right| < \varepsilon.$$

By the previous paragraph,

$$(a,b) \sim_\Lambda (a, b') \sim_\Lambda (a, b'').$$

Next, since $\{a' \in a^\perp \ : \ a' \cdot b = 1\}$ is a coset of a submodule of rank $2g - 2 \geq 4$, there exists $a' \in a^\perp$ such that

$$(a', b) \in \Lambda_{(0,1)}, \quad \left| a' - \frac{1}{2} \right| < \varepsilon.$$

Since the submodule $\{a, a', b\}^\perp$ has rank at least $2g - 3 \geq 3$, and since

$$\{a, b\}^\perp \cap ((a - a') + \{a', b\}^\perp) = (a - a') + \{a, a', b\}^\perp,$$

there exists

$$a_1 \in \{a, b\}^\perp \cap ((a - a') + \{a', b\}^\perp)$$

such that $|a_1 - (a + b)| < 2\varepsilon$. Then

$$a_1' = a_1 + a' - a \in \{a', b\}^\perp$$

satisfies $|a_1' - (a + b)| < 4\varepsilon$. The relation

$$(-a_1 - b) + (a + b) = -a_1 + a = -a_1' + a' = (-a_1' - b) + (a' + b)$$

implies the submodule

$$\{-a_1 - b, a + b, -a_1' - b, a' + b\}^\perp$$

has rank at least $2g - 3 \geq 3$, and we have

$$\{-a_1 - b, a + b\}^\perp \cap ((a' - a) + \{-a_1' - b, a' + b\}^\perp) = (a' - a) + \{-a_1 - b, a + b, -a_1' - b, a' + b\}^\perp,$$

so there exists

$$a_2 \in \{-a_1 - b, a + b\}^\perp \cap ((a' - a) + \{-a_1' - b, a' + b\}^\perp)$$

such that $|a_2 + b| < 2\varepsilon$. Then

$$a_2' = a_2 + a - a' \in \{-a_1' - b, a' + b\}^\perp$$

satisfies $|a_2' + b| < 4\varepsilon$. Thus,

$$(a, b) \sim_\Lambda (-a_2 - a - b, -a_1 + a) = (-a_2' - a' - b, -a_1' + a') \sim_\Lambda (a', b).$$

Now fix $a'' \in \Lambda$ such that $(a'', b) \in \Lambda_{(0,1)}$ and $|a'' - 1/2| < \varepsilon$. The subset

$$\{a' \in \{a, a''\}^\perp \ : \ a' \cdot b = 1\}$$

is a coset of a submodule of rank $2g - 3 \geq 3$, so there exists $a' \in \{a, a''\}$ such that

$$(a', b) \in \Lambda_{(0,1)}, \quad \left| a' - \frac{1}{2} \right| < \varepsilon.$$

By the previous paragraph,

$$(a, b) \sim_\Lambda (a', b) \sim_\Lambda (a'', b).$$

To conclude, fix $(c, d) \in \Lambda_{(0,1)}$ such that $|c - 1/2| < \varepsilon$ and $|d - i/2| < \varepsilon$. There exists $b' \in \Lambda$ such that

$$a \cdot b' = c \cdot b' = 1, \quad \left| b' - \frac{i}{2} \right| < \varepsilon,$$

and by the above,

$$(a, b) \sim_\Lambda (a, b') \sim_\Lambda (c, b') \sim_\Lambda (c, d).$$

□

5.3. **Ergodicity, density, and connectedness.**

Lemma 5.2. Suppose $(X', \omega') \in \Omega\mathcal{M}_{g+1}(\kappa')$ arises from $(X, \omega) \in \Omega\mathcal{M}_g(\kappa)$ by a connected sum with a torus, and let γ be the associated segment in (X, ω). If (X', ω') is algebraically generic, then (X, ω) is algebraically generic and

$$\int_\gamma \omega \notin \bigcup_{z \in \mathrm{Per}(\omega)} \mathbb{R} z.$$

Proof. We have an injection on homology

$$f : H_1(X; \mathbb{C}) \hookrightarrow H_1(X'; \mathbb{C})$$

such that $\int_c \omega = \int_{f(c)} \omega'$. Since (X', ω') is algebraically generic, the subgroup

$$\mathrm{Per}(\omega) = \left\{ \int_c \omega \ : \ c \in f(H_1(X; \mathbb{Z})) \right\} \subset \mathrm{Per}(\omega')$$

satisfies property (1), and satisfies property (2) with g in place of $g + 1$, so (X, ω) is algebraically generic. Let $\gamma^\pm$ be the given splitting of (X', ω'), and let $c' = [\gamma^\pm] \in H_1(X'; \mathbb{Z})$. Since $c' \notin f(H_1(X; \mathbb{Z}))$, and since (X', ω') is algebraically generic, for all nonzero $c \in H_1(X; \mathbb{Z})$ we have $\int_c \omega \notin \mathbb{R} \int_{f(c')} \omega'$. Therefore, $\int_\gamma \omega \notin \bigcup_{z \in \mathrm{Per}(\omega)} \mathbb{R} z$. □

Lemma 5.3. Let $(X_0, \omega_0) \in \Omega_1\mathcal{M}_g(\kappa)$ be algebraically generic, and let L be the leaf of $\mathcal{A}(\kappa)$ through (X_0, ω_0). For a dense subset of $(X, \omega) \in L$, the $\mathrm{SL}(2, \mathbb{R})$-orbit of (X, ω) is dense in its connected component in $\Omega_1\mathcal{M}_g(\kappa)$.

Proof. Choose a basis $\{a_j, b_j\}_{j=1}^g$ for $H_1(X_0; \mathbb{Z})$, and extend to a basis for $H_1(X_0, Z(\omega_0); \mathbb{Z})$ by adding relative cycles $c_1, \ldots, c_{m-1}$ represented by paths $\gamma_1, \ldots, \gamma_{m-1}$ that all start at the same zero of ω_0. For any number field $K \subset \mathbb{R}$, the absolute periods $\int_{a_1} \omega_0, \ldots, \int_{b_g} \omega_0$ are linearly independent over K. The map

$$(X, \omega) \mapsto \left(\int_{c_1} \omega, \ldots, \int_{c_{m-1}} \omega \right)$$

provides local coordinates on a neighborhood of (X_0, ω_0) in L, so there are nearby holomorphic 1-forms $(X, \omega) \in L$ such that for any number field $K \subset \mathbb{R}$, the period coordinates $\int_{a_1} \omega, \ldots, \int_{b_g} \omega, \int_{c_1} \omega, \ldots, \int_{c_{m-1}} \omega$ of (X, ω) are linearly independent over K. Then by Corollary 1.3 in [Wri], the $\mathrm{SL}(2, \mathbb{R})$-orbit of (X, ω) is dense in its connected component in $\Omega_1\mathcal{M}_g(\kappa)$. □

Theorem 5.4. Let $\Omega\mathcal{M}_g(\kappa)$ be a connected stratum with $|\kappa| = 2$. If $\phi \in H^1(S_g; \mathbb{C})$ is algebraically generic, then $\mathcal{M}(\phi; \kappa)$ is connected.

Proof. We induct on genus. The base case of genus 2 is part of Theorem 2.3 in [CDF].

Fix $g \geq 2$, and suppose the theorem is true in genera at most g. Let $\Omega\mathcal{M}_{g+1}(\kappa')$ be a connected stratum with $|\kappa'| = 2$. By Lemma 2.6, there is a connected stratum $\Omega\mathcal{M}_g(\kappa)$ with $|\kappa| = 2$ and a connected sum map

$$\Psi_1 : \mathcal{T}_1(\kappa; m) \to \Omega_1\mathcal{M}_{g+1}(\kappa').$$

Recall that the image of Ψ_1 is nonempty, open, and $\mathrm{SL}(2, \mathbb{R})$-invariant, and therefore dense.

We may assume that $\mathcal{M}(\phi;\kappa')$ is nonempty and that $\mathcal{M}(\phi;\kappa') \subset \Omega_1\mathcal{M}_{g+1}(\kappa')$. Fix $(X_1,\omega_1),(X_2,\omega_2) \in \mathcal{M}(\phi;\kappa')$. Since ϕ is algebraically generic, (X_1,ω_1) and (X_2,ω_2) are algebraically generic. By Lemma 5.3, by replacing (X_1,ω_1) and (X_2,ω_2) with nearby elements of their respective connected components in $\mathcal{M}(\phi;\kappa')$, we may assume that the $\mathrm{SL}(2,\mathbb{R})$-orbits of (X_1,ω_1) and (X_2,ω_2) are dense in $\Omega_1\mathcal{M}_{g+1}(\kappa')$. Then we can write

$$(X_1,\omega_1) = \Psi_1((X_1',\omega_1',\theta_1),(\gamma_1,w_1),a_1), \quad (X_2,\omega_2) = \Psi_1((X_2',\omega_2',\theta_2),(\gamma_2,w_2),a_2).$$

By Lemma 5.2, (X_1',ω_1') and (X_2',ω_2') are algebraically generic, so there are markings $m_j : H^1(S_g;\mathbb{C}) \to H^1(X_j';\mathbb{C})$ and algebraically generic cohomology classes $\phi_j \in H^1(S_g;\mathbb{C})$ such that $m_j(\phi_j) = [\omega_j']$. Suppose

$$\sigma_{\mathcal{T}}((\gamma_1,w_1),a_1) = \sigma_{\mathcal{T}}((\gamma_2,w_2),a_2).$$

Then there is a symplectic automorphism of $H^1(S_g;\mathbb{C})$ that preserves $H^1(S_g;\mathbb{Z})$ and sends ϕ_1 to ϕ_2, so

$$\mathcal{M}(\phi_1;\kappa) = \mathcal{M}(\phi_2;\kappa).$$

By the inductive hypothesis, (X_1',ω_1') and (X_2',ω_2') lie on the same leaf of $\mathcal{A}(\kappa)$, and by Lemma 5.3 and Lemma 3.2, $(X_1',\omega_1',\theta_1)$ and $(X_2',\omega_2',\theta_2)$ lie on the same leaf of $\mathcal{A}(\kappa;m)$. Then by Corollary 3.6, $((X_1',\omega_1',\theta_1),(\gamma_1,w_1),a_1)$ and $((X_2',\omega_2',\theta_2),(\gamma_2,w_2),a_2)$ lie on the same leaf of $\mathcal{F}_{\mathcal{T}}$. Since Ψ_1 maps leaves of $\mathcal{F}_{\mathcal{T}}$ into leaves of $\mathcal{A}(\kappa')$, we have that (X_1,ω_1) and (X_2,ω_2) lie on the same leaf of $\mathcal{A}(\kappa')$.

Since ϕ is algebraically generic, by the inductive hypothesis and Lemma 4.1, the connected component of (X_1,ω_1) in $\mathcal{M}(\phi;\kappa')$ contains elements of $\Psi(\sigma_{\mathcal{T}}^{-1}(z,w))$ for any $(z,w) \in \Lambda_{(0,1)}$ in the equivalence class of $\sigma_{\mathcal{T}}((\gamma_1,w_1),a_1)$ for $\sim_\Lambda$. By Lemma 5.1, this equivalence class is all of $\Lambda_{(0,1)}$, so (X_1,ω_1) and (X_2,ω_2) lie in the same connected component of $\mathcal{M}(\phi;\kappa')$. Thus, $\mathcal{M}(\phi;\kappa')$ is connected. $\square$

Theorem 5.5. Let $\Omega\mathcal{M}_g(\kappa)$ be a connected stratum with $|\kappa| > 1$, and suppose that $m \geq 2$ for some $m \in \kappa$. Fix $1 \leq j < m$, and let $\kappa' = (\kappa \setminus \{m\}) \cup \{m-j, j\}$. If $\phi \in H^1(S_g; \mathbb{C})$ is algebraically generic and $\mathcal{M}(\phi; \kappa)$ is connected, then $\mathcal{M}(\phi; \kappa')$ is connected.

Proof. By Corollary 2.2, since $\Omega\mathcal{M}_g(\kappa)$ is connected, $\Omega\mathcal{M}_g(\kappa')$ is connected. We may assume that $\mathcal{M}(\phi; \kappa')$ is nonempty. Fix $(X_1, \omega_1), (X_2, \omega_2) \in \mathcal{M}(\phi; \kappa')$. By Lemma 5.3, by replacing (X_1, ω_1) and (X_2, ω_2) with nearby elements of their respective connected components in $\mathcal{M}(\phi; \kappa')$, we may assume that the $\mathrm{GL}^+(2, \mathbb{R})$-orbits of (X_1, ω_1) and (X_2, ω_2) are dense in $\Omega\mathcal{M}_g(\kappa')$. Since the image of the zero splitting map

$$\Phi : \mathcal{S}(\kappa; m) \to \Omega\mathcal{M}_g(\kappa')$$

is open and dense, and since splitting zeros does not change the absolute periods, we can write

$$(X_1, \omega_1) = \Phi((X_1', \omega_1', \theta_1), \gamma_1), \quad (X_2, \omega_2) = \Phi((X_2', \omega_2', \theta_2), \gamma_2)$$

with $(X_1', \omega_1'), (X_2', \omega_2') \in \mathcal{M}(\phi; \kappa)$. By assumption, $\mathcal{M}(\phi; \kappa)$ is connected, so (X_1', ω_1') and (X_2', ω_2') lie on the same leaf of $\mathcal{A}(\kappa)$. Then by Lemma 3.2, $(X_1', \omega_1', \theta_1)$ and $(X_2', \omega_2', \theta_2)$ lie on the same leaf of $\mathcal{A}(\kappa; m)$, and by Lemma 3.3, $((X_1', \omega_1', \theta_1), \gamma_1)$ and $((X_2', \omega_2', \theta_2), \gamma_2)$ lie on the same leaf of $\mathcal{F}_{\mathcal{S}}$. Since Φ maps leaves of $\mathcal{F}_{\mathcal{S}}$ into leaves of $\mathcal{A}(\kappa')$, we have that (X_1, ω_1) and (X_2, ω_2) lie on the same leaf of $\mathcal{A}(\kappa')$. Thus, $\mathcal{M}(\phi; \kappa')$ is connected. □

We now complete the proof of our main connectedness result.

Proof. (of Theorem 1.3) Induct on $|\kappa|$, using Theorem 5.4 for the base case $|\kappa| = 2$, and using Lemma 2.5 and Theorem 5.5 for the inductive step. □

Similar inductive steps can be used to prove Theorem 1.4, by forming connected sums with a torus using holomorphic 1-forms in $\Omega\mathcal{M}_{g-1}(g-1, g-3)$ for g even.

Theorem 1.3 can be used to prove Theorems 1.1 and 1.2, using the transfer principle from [CDF] and results from homogeneous dynamics, which we briefly explain. Let

$$\langle \alpha, \beta \rangle = \frac{i}{2} \int_{S_g} \alpha \wedge \overline{\beta}$$

be the intersection form on $H^1(S_g; \mathbb{C})$. For $\phi \in H^1(S_g; \mathbb{C})$, let $V(\phi) \subset H^1(S_g; \mathbb{R})$ be the span of $\mathrm{Re}(\phi)$ and $\mathrm{Im}(\phi)$. The symplectic automorphism group $\mathrm{Sp}(H^1(S_g; \mathbb{R}))$ acts transitively on the set of $\phi \in H^1(S_g; \mathbb{C})$ such that $\langle \phi, \phi \rangle = 1$ by acting on the real and imaginary parts of ϕ simultaneously, and the stabilizer of ϕ is $\mathrm{Sp}(V(\phi)^\perp)$. Let

$$\Pi : \Omega\mathcal{S}_g(\kappa) \to \Omega\mathcal{M}_g(\kappa)$$

be the Torelli cover of a stratum, whose points are holomorphic 1-forms $(X, \omega) \in \Omega\mathcal{M}_g(\kappa)$ equipped with a marking of $H^1(X; \mathbb{C})$, and consider the restriction of the period map

$$\mathrm{Per}_g : \Omega\mathcal{S}_g(\kappa) \to H^1(S_g; \mathbb{C}).$$

Since Per_g is a holomorphic submersion on $\Omega\mathcal{S}_g(\kappa)$, the image of Per_g is open. Moreover, the image of Per_g is invariant under the action of $\mathrm{Sp}(H^1(S_g; \mathbb{Z}))$. The set

$$G_g = \left\{ \phi \in H^1(S_g; \mathbb{C}) \ : \ \langle \phi, \phi \rangle = 1 \text{ and } \phi \text{ is algebraically generic} \right\}$$

is $\mathrm{Sp}(H^1(S_g; \mathbb{Z}))$-invariant, and is contained in the image of Per_g by Proposition 3.10 in [CDF]. Since $\mathrm{Sp}(H^1(S_g; \mathbb{R})) \cong \mathrm{Sp}(2g, \mathbb{R})$ and $\mathrm{Sp}(V(\phi)^\perp) \cong \mathrm{Sp}(2g-2, \mathbb{R})$, we can identify G_g with an $\mathrm{Sp}(2g; \mathbb{Z})$-invariant full measure subset of $\mathrm{Sp}(2g, \mathbb{R}) / \mathrm{Sp}(2g-2, \mathbb{R})$. The set

$$G(\kappa) = \{ (X, \omega) \in \Omega_1\mathcal{M}_g(\kappa) \ : \ (X, \omega) \text{ is algebraically generic} \}$$

is saturated for $\mathcal{A}(\kappa)$ and has full measure.

Now suppose that $|\kappa| > 1$ and that $\Omega\mathcal{M}_g(\kappa)$ is connected. Theorem 1.3 implies that $\mathrm{Per}_g^{-1}(\phi)$ is connected for $\phi \in G_g$, and this provides a bijection $A \mapsto \mathrm{Per}_g(\Pi^{-1}(A))$ between subsets of $G(\kappa)$ that are saturated for $\mathcal{A}(\kappa)$ and subsets of G_g that are invariant under the action of $\mathrm{Sp}(H^1(S_g;\mathbb{Z}))$. Positive measure subsets correspond to positive measure subsets, and dense subsets correspond to dense subsets. Theorem 1.1 then follows from Moore's ergodicity theorem [Zim], and Theorem 1.2 follows from Ratner's orbit closure theorem [Rat], applied to the action of $\mathrm{Sp}(2g,\mathbb{Z})$ on $\mathrm{Sp}(2g,\mathbb{R})/\mathrm{Sp}(2g-2,\mathbb{R})$.

5.4. **Monodromy.** Let $\mathcal{C}$ be a connected component of a stratum $\Omega\mathcal{M}_g(\kappa)$ with $|\kappa| > 1$. We will relate the image of the monodromy representation $\pi_1(\mathcal{C}) \to \mathrm{Sp}(2g,\mathbb{Z})$ to the question of whether holomorphic 1-forms in $\mathcal{C}$ with the same absolute periods lie on the same leaf of $\mathcal{A}(\kappa)$.

To illustrate the idea, we first consider part of the case where $\kappa = \{g-1, g-1\}$. Choose $z_j, w_j \in \mathbb{C}$, $j = 1, \dots, g$, such that the subgroup $\Lambda \subset \mathbb{C}$ generated by $z_1, w_1, \dots, z_g, w_g$ has rank $2g$ and is algebraically generic, and such that

$$z_j \times w_j > 0, \quad j = 1, \dots, g, \tag{13}$$

$$(z_1 - w_2) \times w_1 > 0, \quad (z_2 - w_1) \times w_2 > 0. \tag{14}$$

For instance, one may choose each z_j close to 1 and each w_j close to i. Consider the flat tori $T_j = (\mathbb{C}/(\mathbb{Z}z_j + \mathbb{Z}w_j), dz)$, $j = 1, \dots, g$. Choose very short segments $s_j \in T_j$ that are parallel and of the same length. Slit T_j along s_j, and glue the left side of s_j to the right side of s_{j+1}, indices taken modulo g. The result is a holomorphic 1-form (X, ω). Let $\mathcal{C}$ be the connected component of (X, ω) in $\Omega\mathcal{M}_g(\kappa)$. If $g = 2$, then $\mathcal{C} = \Omega\mathcal{M}_2(1,1)$. If $g \geq 4$ is even, then $\mathcal{C}$ is the nonhyperelliptic component of $\Omega\mathcal{M}_g(g-1, g-1)$, and if $g \geq 3$ is odd, then $\mathcal{C}$ is the odd component of $\Omega\mathcal{M}_g(g-1, g-1)$. This construction provides a symplectic basis $\{a_j, b_j\}_{j=1}^g$

for $H_1(X;\mathbb{Z})$ such that

$$\int_{a_j} \omega = z_j, \quad \int_{b_j} \omega = w_j, \quad j = 1, \ldots, g,$$

where a_j, b_j are represented by closed geodesics in T_j. Let $(Y, \eta) \in \mathcal{C}$ be another holomorphic 1-form arising from this construction, such that $\mathrm{Per}(\omega) = \mathrm{Per}(\eta)$ as symplectic modules. Let $\{a'_j, b'_j\}_{j=1}^g$ be the associated symplectic basis of $H_1(Y;\mathbb{Z})$, and write

$$\int_{a'_j} \eta = z'_j, \quad \int_{b'_j} \eta = w'_j, \quad j = 1, \ldots, g.$$

There is a path $\rho : [0,1] \to \mathbb{C}^{2g}$, $\rho(t) = (z_1(t), w_1(t), \ldots, z_g(t), w_g(t))$, from $(z_1, w_1, \ldots, z_g, w_g)$ to $(z'_1, w'_1, \ldots, z'_g, w'_g)$ such that $\rho(t)$ satisfies (13) for all t. This path induces a path $\gamma_1 : [0,1] \to \mathcal{C}$ from (X, ω) to (Y, η), given by replacing T_j with $(\mathbb{C}/(\mathbb{Z}z_j(t) + \mathbb{Z}w_j(t)), dz)$ in the construction above and keeping the slits very short. Parallel transport along γ_1 sends a_j to a'_j and b_j to b'_j. On the other hand, if Theorem 1.3 holds for $\mathcal{C}$, then there is another path $\gamma_2 : [0,1] \to \mathcal{C}$ from (Y, η) to (X, ω) along which the absolute periods are constant. Parallel transport along γ_2 sends a'_j to the homology class a''_j satisfying $\int_{a''_j} \omega = z'_j$ and sends b'_j to the homology class b''_j satisfying $\int_{b''_j} \omega = w'_j$. Concatenating γ_1 and γ_2 gives a loop $\gamma : [0,1] \to \mathcal{C}$ based at (X, ω). Using the basis $\{a_j, b_j\}_{j=1}^g$ to identify $\mathrm{Per}(\omega) \cong \mathbb{Z}^{2g}$, the associated monodromy matrix $A \in \mathrm{Sp}(2g, \mathbb{Z})$ is determined by the requirement that

$$\int_{Aa_j} \omega = z'_j, \quad \int_{Ab_j} \omega = w'_j, \quad j = 1, \ldots, g.$$

We recall a convenient generating set for $\mathrm{Sp}(2g, \mathbb{Z})$ from [FM, Section 6.1], using slightly different terminology.

- Shear S: $S(a_1) = a_1 + b_1$.
- Rotation R: $R(a_1) = b_1$, $R(b_1) = -a_1$.
- Factor mix M: $M(a_1) = a_1 - b_2$, $M(a_2) = a_2 - b_1$.

- Factor swaps W_j, $1 \leq j \leq g-1$: $W_j(a_j) = a_{j+1}$, $W_j(b_j) = b_{j+1}$, $W_j(a_{j+1}) = a_j$, $W_j(b_{j+1}) = b_j$.

In each case, each element of $\{a_j, b_j\}_{j=1}^{g}$ not mentioned is fixed by the above generator. By shearing T_1 in the construction above, we can realize the shear S in the image of $\pi_1(\mathcal{C}) \to \mathrm{Sp}(2g, \mathbb{Z})$. Using a suitable path in $\mathbb{C}^2$ from (z_1, w_1) to $(w_1, -z_1)$, we can realize the rotation R. (We have not used the assumption that Theorem 1.3 holds for $\mathcal{C}$ yet.) Since (14) holds, using a suitable path in $\mathbb{C}^4$ from (z_1, w_1, z_2, w_2) to $(z_1 - w_2, w_1, z_2 - w_1, w_2)$, and assuming Theorem 1.3 holds for $\mathcal{C}$, we can realize the factor mix M. Lastly, for $1 \leq j \leq g-1$, using a suitable path in $\mathbb{C}^4$ from $(z_j, w_j, z_{j+1}, w_{j+1})$ to $(z_{j+1}, w_{j+1}, z_j, w_j)$, and assuming Theorem 1.3 holds for $\mathcal{C}$, we can realize the factor swap W_j. We conclude that if Theorem 1.3 holds for $\mathcal{C}$, then the monodromy representation $\pi_1(\mathcal{C}) \to \mathrm{Sp}(2g, \mathbb{Z})$ is surjective.

Next, we indicate how to generalize this construction. By the classification of connected components of $\Omega\mathcal{M}_g(\kappa)$ in Theorem 2.1, and Lemma 2.5 in this thesis, and Lemma 10.1 in [EMZ], it is enough to address the case of strata of holomorphic 1-forms with exactly 2 distinct zeros. Write $\kappa = \{m_1, m_2\}$ with $m_1 \geq m_2$. Since $m_1 + m_2 = 2g - 2$ is even, $m_1 - m_2$ is even. By the previous paragraph, we may assume $m_1 > m_2$. As before, choose $z_1, w_1, \ldots, z_g, w_g \in \mathbb{C}$ generating a subgroup $\Lambda \subset \mathbb{C}$ of rank $2g$ such that Λ is algebraically generic, and satisfying (13) and (14). Assume that each z_j is very close to 1 and that each w_j is very close to i. Since Λ is algebraically generic, we can additionally assume that the arguments of the complex numbers z_j satisfy $0 < \arg(z_1) < \cdots < \arg(z_g) < \varepsilon$ for some small $\varepsilon > 0$. Let $T_j = (\mathbb{C}/(\mathbb{Z}z_j + \mathbb{Z}w_j), dz)$ for $j = 1, \ldots, g$. Choose very short vertical segments $s_j \in T_j$, $j = 1, \ldots, m_2 + 1$, that are parallel and of the same length. Slit T_j along s_j, and glue the left side of s_j to the right side of s_{j+1}, indices taken modulo $m_2 + 1$, to obtain a holomorphic 1-form $(X_0, \omega_0) \in \Omega\mathcal{M}_{m_2+1}(m_2, m_2)$. Let $p \in T_{m_2+1}$ be the top endpoint of the vertical slit coming from s_{m_2+1}. By our assumptions, there are embedded geodesic segments

$r_{m_2+2}, \ldots, r_g$ in T_{m_2+1} starting at p, disjoint except at p, and satisfying

$$\int_{r_j} \omega_0 = z_j, \quad j = m_2 + 2, \ldots, g.$$

For $j = m_2 + 2, \ldots, g$, slit T_j along a closed geodesic α_j satisfying $\int_{\alpha_j} dz = z_j$, slit T_{m_2+1} along r_j, and reglue opposite sides. The result is a holomorphic 1-form $(X, \omega) \in \Omega\mathcal{M}_g(\kappa)$, obtained from (X_0, ω_0) by iteratively applying the connected sum with a torus construction from Chapter 2. Let $\mathcal{C}$ be the connected component of (X, ω) in $\Omega\mathcal{M}_g(\kappa)$. If m_1, m_2 are odd, then by Corollary 2.2, $\mathcal{C} = \Omega\mathcal{M}_g(\kappa)$, and if m_1, m_2 are even, then by Lemma 11 in [KZ], $\mathcal{C}$ is the odd component of $\Omega\mathcal{M}_g(\kappa)$. Since the only slit on T_1 comes from the very short segment s_1, we can realize the shear S and the rotation R in the image of $\pi_1(\mathcal{C}) \to \mathrm{Sp}(2g, \mathbb{Z})$ in the same way as before. If $m_2 \geq 2$, then the only slit on T_2 comes from the very short segment s_2, and we can realize the factor mix M in the same way as before. If $m_2 = 1$, then we can realize M by first shrinking the length of the slits $r_{m_2+2}, \ldots, r_g$ (keeping their directions and their starting point p fixed), then deforming T_1, T_2 using a suitable path in $\mathbb{C}^4$ from (z_1, w_1, z_2, w_2) to $(z_1 - w_2, w_1, z_2 - w_1, w_2)$, then stretching the slits $r_{m_2+2}, \ldots, r_g$ to their original lengths, and then returning to (X, ω) along a path in a leaf of $\mathcal{A}(\kappa)$. The factor swaps $W_1, \ldots, W_{m_2}$ can be realized in the same way as before. The factor swap W_{m_2+1} can be realized by shrinking the slits $r_{m_2+2}, \ldots, r_g$, deforming T_{m_2+1} using a very short path in $\mathbb{C}^2$ from (z_{m_2+1}, w_{m_2+1}) to (z_{m_2+2}, w_{m_2+2}), rotating r_{m_2+2} so that its new direction is parallel to z_{m_2+1}, enlarging the slits and deforming T_{m_2+2} using a very short path in $\mathbb{C}^2$ from (z_{m_2+1}, w_{m_2+2}) to (z_{m_2+1}, w_{m_2+1}), and then returning to (X, ω) along a path in a leaf of $\mathcal{A}(\kappa)$. For $j = m_2 + 2, \ldots, g - 1$, the factor swap W_j can be realized as follows. First, shrink the slits r_j, r_{j+1}. Rotate the slits r_j and r_{j+1} counterclockwise around p a small amount so that the new direction of r_j is the old direction of r_{j+1}. By our discussion of the domain of definition of the surgery of forming a connected sum with a torus in Chapter 2, we can continue rotating the slit r_{j+1} counterclockwise around p until this slit is in between

r_j and r_{j-1}, and so that the new direction of r_{j+1} is the old direction of r_j. (Rotate w_{j+1} counterclockwise as well in the process.) Enlarge the slits r_j and r_{j+1}, and deform w_j and w_{j+1} slightly, so that the slit r_j bounds the torus $(\mathbb{C}/(\mathbb{Z}z_{j+1}, \mathbb{Z}w_{j+1}), dz)$ and the slit r_{j+1} bounds the torus $(\mathbb{C}/(\mathbb{Z}z_j, \mathbb{Z}w_j), dz)$. Lastly, return to (X, ω) along a path in a leaf of $\mathcal{A}(\kappa)$. Next, to produce holomorphic 1-forms in the hyperelliptic component of $\Omega\mathcal{M}_g(g-1, g-1)$ for $g \geq 3$, let $z_1, w_1, \ldots, z_g, w_g$ and $T_1, \ldots, T_g$ be as before. Slit T_1 and T_3 along very short horizontal segments s_1, s_3 of the same length and reglue opposite sides. Let s_3' be the segment in T_3 with the same endpoints as s_3 such that $\int_{s_3'} dz$ is close to 1, and let s_4 be a segment on T_4 that is parallel to s_3' and has the same length. Slit T_3 and T_4 along s_3' and s_4, and reglue opposite sides. Let s_4' be the short segment in T_4 with the same endpoints as s_4, and let s_5 be a segment in T_5 that is parallel to s_4' and has the same length. Slit T_4 and T_5 along s_4' and s_5, and reglue opposite sides. Continue this process through T_{g-1} and T_g, and then continue by gluing T_g and T_2, to obtain a holomorphic 1-form (X, ω) in the hyperelliptic component of $\Omega\mathcal{M}_g(g-1, g-1)$. The tori T_1 and T_2 each only have one very short slit, and the argument in this case is exactly the same as in the previous paragraph. Lastly, to construct holomorphic 1-forms in the even component of $\Omega\mathcal{M}_g(\kappa)$ when m_1 and m_2 are even with $g \geq 4$, start with (X_0, ω_0) in the hyperelliptic component of $\Omega\mathcal{M}_3(2, 2)$ in the previous construction. Let Z_1, Z_2 be the zeros of ω_0. There are embedded segments $r_4, \ldots, r_g$ in T_3 such that $r_4, \ldots, r_{2+m_1/2}$ start at Z_1, $r_{3+m_1/2}, \ldots, g$ start at Z_2, and $\int_{r_j} \omega_0 = z_j$ for $j = 4, \ldots, g$, and the segments $r_4, \ldots, r_g$ are disjoint except at their starting points. Slit T_3 along the r_j, and slit T_j along a closed geodesic α_j satisfying $\int_{\alpha_j} dz = z_j$, and reglue opposite sides, to obtain a holomorphic 1-form (X, ω) in the even component of $\Omega\mathcal{M}_g(\kappa)$. The tori T_1 and T_2 each only have one very short slit, and the argument in this case is the same as the case at the beginning of this paragraph.

Thus, if Theorem 1.3 holds for $\mathcal{C}$, then the homomorphism $\pi_1(\mathcal{C}) \to \mathrm{Sp}(2g, \mathbb{Z})$ is surjective. Since we have proven that Theorem 1.3 holds when $|\kappa| > 1$ and $\Omega\mathcal{M}_g(\kappa)$ is connected, and when $g \geq 4$ is even and $\mathcal{C}$ is the nonhyperelliptic component of $\Omega\mathcal{M}_g(g-1, g-1)$, we recover the result of [Gut] that $\pi_1(\mathcal{C}) \to \mathrm{Sp}(2g, \mathbb{Z})$ is surjective in these cases. On the other hand, by [Gut], in all other cases $\pi_1(\mathcal{C}) \to \mathrm{Sp}(2g, \mathbb{Z})$ is not surjective, so there must be $(Y, \eta) \in \mathcal{C}$ with $\mathrm{Per}(\omega) = \mathrm{Per}(\eta)$ as symplectic modules but not on the same leaf of $\mathcal{A}(\kappa)$. This establishes Theorem 1.5.

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