What Drives Currency Revaluation Decisions?

Acknowledgements

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Contents

Part I

Introduction

An abundance of literature considers whether a switch in the exchange rate regime is predictable when there is depreciative pressure on the currency. The literature is relatively scarce when it comes to studying the change of exchange rate regimes for countries whose currencies are under appreciative pressure. Inspired by Switzerland's currency crisis in January 2015, the paper examines the strategy that a forward-looking policy maker would take under appreciative speculative pressure on their currency. Different from existing literature, my model captures the interaction between the market and a forward-looking policy maker that is aware of the output cost following the unpegging of their currency.

This model sheds light on the timing and nature of Switzerland's unpegging decision on January 15, 2015 and whether they are consistent with a rational, optimizing policy maker. It also explains why Denmark, whose currency is also pegged to Euro, did not experience the speculative pressure that Switzerland went through during the same period of time. Looking forward, a modified version of this model can also be applied to understand the exchange rate dynamics of China and Saudi Arabia.

My model captures the short-term output shock following the abandonment of the exchange rate peg, which is caused by the normal rigidities in prices. Since prices do not adjust fully in the short term, an appreciation in the exchange rate results in a hike in the price of the exports, making the manufacturers less competitive. The cost vanishes in the long run as firms adjust their prices.

Each period the policy maker compares the cost of abandoning the peg to that of maintaining the peg. Observing the policy maker's objective function, the traders in the market would keep purchasing domestic currency until the policy maker is indifferent between pegging and unpegging. Otherwise there would be an arbitrage opportunity for the speculators. Since the policy maker is kept indifferent in each period, it is essentially pursuing a randomized strategy in equilibrium. So instead of a sudden speculative attack, there is a prolonged period of increased probability for the policy maker to abandon the peg accompanied by increased speculation in the market.

My core model contributes to the current literature in the following ways: first, it bridges the gap in the theoretical literature on exchange rate where currency revaluation models are scarce; second, I introduce a short-term output shock caused by the change in exchange rate after the unpegging occurs, which is also a relatively underexplored area. This output shock element helps the model better capture the reality in a currency crisis. For example, in Switzerland's case, after the unpegging happened, Swiss GDP suffered tremendously both due to the soaring prices of output and also the deflationary spiral caused by the public's delay in purchasing imports expecting that Swiss Franc would become more expensive in the future; third, the model is set in continuous time framework rather than discrete time, different from most other existing models. This change enables me to capture how speculators adjust to a policy maker's decision to unpeg its currency when there is a short-term output cost. If the model were not continuous, the speculators, whose actions are governed by Uncovered Interest Rate Parity, would not adjust fully to compensate for the short-term output cost. (See Proposition 1 of the model).

The policy maker's objective function includes the cost from output loss and also the cost of holding excessive reserves. The cost of reserves mainly originates from the following two sources. First, reserves are low in return and so they generate opportunity cost. Second, it is empirically documented that countries hold disproportionately high level of reserves in the currency they are pegged to. This behavior increases their exposure to the risk of the particular country they are pegged to. For example, Switzerland's holding of foreign reserves is equivalent to 77% of its GDP at the end of 2014, among which close to 40% was denominated in Euro. As a result, for a country like Switzerland, the previous two concerns about high reserve level have led to another political economy cost of holding excessive reserves; that is, the so-called gold referendum. Swiss People's Party initiated the gold referendum to prevent further accumulation of Swiss foreign reserves. The initiative proposes that Swiss National Bank hold at least 20% of their foreign reserves in gold and stop further selling gold. This proposal put the high level of Swiss foreign reserves under spotlight and reflected the general public's concern about it. In November 2014, a voting conducted by all Swiss citizens rejected the proposal by 78%. Although it was rejected in the end, the referendum still put Switzerland's high reserve level under spotlight and the fear of a second gold referendum constitutes yet a third source of concern about high reserve level.

The basic intuition for the paper is as follows. There is a point $\underline{T}$ before which the policy maker would not abandon the fixed exchange rate regime because the reserve level is not high enough. And there is a period $\bar{T}$ where the policy maker would abandon the peg for sure due to excessive accumulation of reserves. Before $\underline{T}$ reserves are low enough so that if speculators attacked and drove reserves up to its upper reserve threshold, forcing a move to a floating exchange rate, they would increase the domestic money supply to a point where the floating rate would exceed the fixed rate, which is a loss to the speculators; hence they do not attack. The domestic interest rate is therefore equal to the world rate.

After $\underline{T}$ both speculators and the policy maker realize that $\bar{T}$ is approaching, the reserve level is high enough that the unpegging would be imminent. Each period, the government evaluates the cost of remaining the peg versus that of abandoning the peg. The no arbitrage condition in the market ensures that the speculators always purchase enough domestic currency (increase Central Bank's holding of foreign reserves) so that the policy maker is indifferent between the two choices. The increased amount of domestic currency in the hands of the speculators drives down the domestic interest rate and forward exchange rate premium while driving up the shadow floating exchange rate. Therefore the opportunity cost of holding foreign currency is decreasing, and the benefit to successful speculation is declining. But the probability that speculation will be successful is increasing rapidly, ensuring that the expected return from holding domestic currency remains constant at zero. However, in the event of a move to floating exchange rates, while there is no sudden increase in Central Bank reserves, the ex-post profits from holding domestic currency can be substantial, since the shadow floating exchange rate is below the fixed rate until $\bar{T}$.

The model also intuits how the government's probability for abandoning the peg changes with different parameters. Increased output growth results in increased market expectation of the unpegging, which in turn leads to higher reserves because speculators would purchase more domestic currency to earn profit when it appreciates. The more the output shock is correlated with the magnitude of change in exchange rate upon unpegging, the less the market expects the policy maker to unpeg, which in turn results in less accumulation of reserves during the speculation period. This is intuitive because the greater output shock increases with exchange rate jump upon unpegging, the more cost that the policy maker incurs, the less likely they would abandon the peg.

Part II

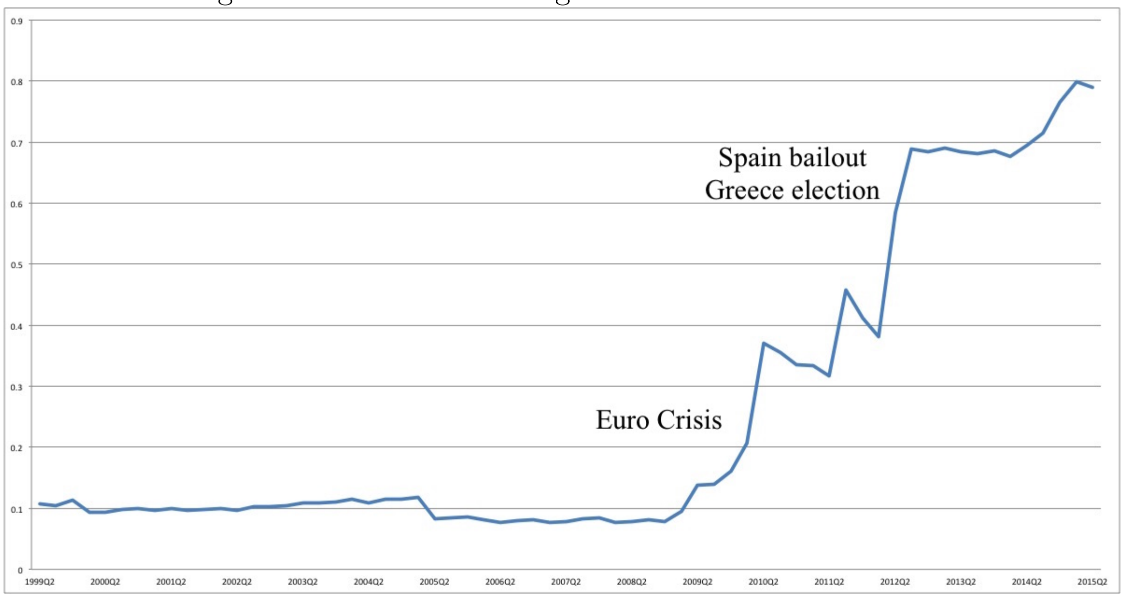
Empirical Motivation

In this section, I motivate my model with the evolution of Switzerland's currency peg to Euro and show how the market's expectation for Swiss unpegging moves in correlation with its foreign reserves level. Switzerland is a small rich country with real GDP growth whose currency is constantly under appreciative pressure due to its reputation as a safe haven. To prevent large capital inflow, Switzerland's currency peg to Euro was put in place in September 2011. Ever since then, Switzerland's foreign reserves has experienced several big jumps since the Central Bank needed to continuously purchase foreign reserves in order to maintain the peg under appreciative pressure. March 2013 marks a turning point where the high level of Swiss foreign reserves started to draw public attention through a proposal filed by Swiss People's Party—the proposal received more than 100,000 signatures, which means there would be a referendum held to determine whether Swiss National Bank (SNB) would hold twenty percent of its reserve in gold. The main concerns about holding excessive reserves, as explained in the previous section is first, the low return on reserve assets and second, the heightened risk exposure of holding large portions of reserve in the currency that the peg is in, which is the case for Switzerland. This event makes one ponder the existence of a threshold of the level of foreign reserves a country holds, upon reaching which the extra amount would impose a cost on the Central Bank. As the level of excessive reserves increased, the market sentiment leaned towards betting against the peg. This increased speculation against the Switzerland peg can be seen in the Euro-Swiss 25-Delta Risk Reversal Data, which reflects the market's sentiment towards the future movement of the currency, as will be explained in detail in the Empitics Section (Section 5). A decrease in the Euro-Swiss Franc risk reversal data represents the market's expectation that the currency would appreciate and vice versa. The following graphs show the accumulation of Swiss foreign reserves and the downward trend in the risk reversal data after December 19, 2014, signaling increased speculation against the Swiss peg.

Following the Swiss unpeg on January 15, 2015, Swiss GDP experienced a large output shock. From 2014 Q4 to 2015 Q1, the GDP growth is -0.3%, while the average quarterly GDP growth from 2013 Q3 to 2014 Q4 is 0.53%.

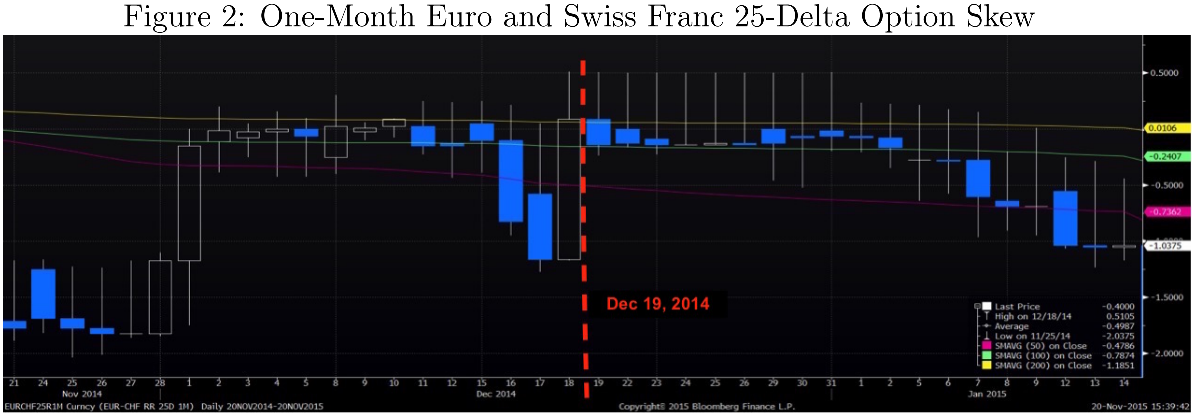
Although I can only show correlations rather than causations between risk reversal and reserve level as well as between unpegging and output shock, these relationships call for a theoretical model to uncover an underlying mechanism where increased reserve level potentially causes increased speculation against the peg. The model that

Figure 1: Switzerland foreign reserves as a Share of GDP



Source: World Economic Outlook and SNB

Figure 2: One-Month Euro and Swiss Franc 25-Delta Option Skew



Source: Bloomberg

I introduce below captures a small rich economy whose currency is pegged and under appreciative pressure. The Central Bank incurs a cost from accumulating excessive reserves. The speculators in the market, after observing the reserve accumulation and knowing the fact that after a certain point, foreign reserves would be too costly for the Central Bank to further accumulate, increasingly bet against the peg by purchasing domestic currency. The Central Bank incurs a short-term output cost as well as an additional cost every period after it abandons the currency peg. Each period, the Central Bank decides between pegging and unpegging after a cost-benefit analysis. The speculators would purchase domestic currency (and thus increasing the Central Bank's holding of foreign reserves), until the Central Bank is indifferent between pegging and unpegging. Thus in my model, predictable attacks are not possible because if an attack is expected, the Central Bank would choose to abandon the peg earlier to avoid the cost from the speculative attack.

Part III

Literature Review

The seminal work in modeling exchange rate peg and the government's decision to abandon it is Krugman's 1979 paper, "A Model of Balance-of-Payments Crises." In Krugman's model, a balance of payments crisis is generated by a monetary authority who prioritizes a policy of domestic credit expansion over the fixed exchange rate regime. Foreign exchange reserves inevitably run out and the fixed rate has to be abandoned. Krugman shows that the crisis culminates with sudden discrete loss of reserves in a "speculative attack". Modified first-generation models include Flood, Garber and Kramer (1996). Their model adds a bond-based risk premium to the spread between domestic and foreign-currency intrest rate, thus modifying the Uncovered Intersest Parity in the standard model. The speculative attack is sterilized.

Growing domestic credit puts depreciative pressure on the currency. Their model imposes the additional condition to prevent domestic-currency interest rate from jumping when there is a predictable attack. It shows that sterilization is compatible with a fixed exchange rate. First generation models show that an attack need not be caused by a large shock. However, in real-life crisis uncertainly is a crucial element, since traders do not know the timing of an attack. One of the papers that capture the uncertainty element is Flood and Marion (1996). Their model captures a stochastic environment with full sterilization and a time-varying stochastic risk premium. They show that currency crises can either be the outcome of inconsistent policies, or some self-fulfilling prophecies about exchange-market risk fixing the fundamentals.

The second category of models originates from the Obstfeld (1986) model. It differs from the Krugman model in that the second-generation models do not depend on a policymaker with noncompatible and thus unsustainable policy goals. Instead it is the private sectors' expectations of a loosening of monetary policy after a collapse of the fixed rate regime that causes the speculative attack. The expectation of a devaluation makes it rational for private sector agents to join an attack when one occurs. An attack exhausts reserves and forces the authorities to abandon the fixed rate. If the authorities do in fact then loosen monetary policy the exchange rate depreciates and the expectations of speculators are fulfilled. Thus a rational expectations equilibrium can exist with a speculative attack even when the initial policy stance is sustainable. The main takeaway from Obstfeld is that strategic complementarities generate multiple equilibria, so that speculative attacks can occur irrespective of fundamentals.

Morris and Shin (1998) presents the idea that strategic uncertainty, which is the uncertainty about the actions and beliefs of others, can restore a unique equilibrium that is a function of the underlying fundamentals, while retaining the "coordination" element of models with multiple equilibria. As such, it can help bridge the gap between the first and second generation models of speculative attacks. One example where extreme beliefs change the market is the 1992-93 currency crisis in Europe. After each new announcement by the Bundesbank, market participants needed to consider how others in the market make of the official announcement. And thus a crisis can be caused by market participants' expectation of others' beliefs. Another explanation for the cause of a speculative attack is information cascades, which is presented in Banerjee (1992) and Bikhchandani, Hirshleifer and Welch (1992). Their models rely on what others' actual actions are rather than the lack of common knowledge about the state of fundamentals, as in Morris and Shin (1998). These models predict that traders would follow what others are doing rather than using their own information. Another type of models such as Calvo and Mendoza (2000) consider a global market with many identical investors forming decisions simultaneously rather than the sequential decision-making framework in Banerjee (1992). They show that small rumors can induce herding behavior, which could move the economy from the no-attack to the attack equilibrium.

Subsequent papers have studied the impact of the level of reserves on government's decision to abandon the peg. The Krugman-Flood-Garber (KFG) model of balance of payment crises considers currency pegs that are under depreciative pressure. It assumes the existence of a lower threshold for the amount of foreign reserves a Cantral Bank can hold, upon reaching which the fixed exchange rate must be abandoned. Rebelo, Sergio, and Carlos A. Végh (2008) modifies this assumption by constructing a model to explain why many countries abandon the peg with still a considerable amount of international reserves in the central bank's vault. In their model, an unexpected increase in government spending makes the fixed exchange rate regime unstable. The model predicts that when there are no exit costs, it is optimal for the government to abandon the exchange rate peg immediately. When there are exit costs, the optimal abandonment time decreases with the size of the fiscal shock. For large fiscal shocks, it is optimal for the government to abandon the peg immediately.

Another branch of papers consider the role private information plays in the timing of the currency crisis. Broner (2002) incorporates private information into the investors' decision. The model shows that disaggregated information delays the currency crises, causing discrete devaluations. Regarding policy recommendations, the model demonstrates that high interest rates can delay (or possibly avoid) the abandonment of the peg. In a subsequent paper, Broner (2008) shows that when all consumers are not perfectly informed about the level of fundamentals, the lack of perfect information could delay the attack on the currency past the point where the shadow rate equals the peg, which leads to unpredictable and discrete devaluations. On a similar note, Minguez-Afonso (2006) incorporates into the model the uncertainty about the willingness of a Central Bank to defend the peg. Their model predicts a unique equilibrium where the exchange rate is abandoned and the lack of common knowledge will lead to a discrete devaluation of the local currency at the unpegging.

Coordination between speculators is also a topic of interest. For example, Chamley (2003) presents a model in a regime where exchange rate is pegged within a band. Speculators decide whether their mass is large enough for a successful attack by observing the exchange rate within the band.

Economists have also modeled forward-looking policy makers with an objective function. Ozkana and Sutherland (1998) models a currency crisis triggered by an optimizing policymaker who wants to loosen monetary policy and boost aggregate demand. Agents in the foreign exchange market know the policymaker's objective function. And interest differentials signal these agents' expectations regarding a regime change. Subsequently, the resulting rise in interest rates affects the policymaker's decision to switch regime. In their model, there exists a rational expectations equilibrium where the fixed rate is abandoned due to adverse demand shocks. Multiple equilibria could also arise leading to potential self-fulfilling crises.

Pastine (2002) also models a forward-looking government that chooses a critical level of fundamentals to abandon the fixed exchange rate regime and accepting a speculative attack. The policy maker introduces uncertainty into the decision of the speculators in order to make it difficult for speculators to determine when it would change the exchange rate policy. The model is a single good, small open economy that is governed by purchasing power parity, uncovered interest rate parity, money demand, money supply and domestic credit growth function. The model also assigns an objective function to the policy maker—it prefers more reserves to less, and fixed exchange rates over floating exchange rates. There is also a cost associated with abandoning the peg. Pastine uses backward induction to calculate the level of reserve, output, shadow exchange rate, and the government's probability of abandoning the peg. The model finds that before a certain period $\underline{T}$ the government would not abandon the peg but after $\underline{T}$ both the traders and the government realize that the period when they have to abandon the peg based on fundamentals approaches, and therefore there is increased speculation that the peg will be abandoned.

My paper extends Pastine's model in an economy with exchange rate revaluation pressure. I also incorporate a short-term output cost that is proportional to the change in exchange rate after the currency crisis. Under a continuous time framework, my model predicts that predictable speculative attacks are not possible because the optimizing policy-maker can avoid it through a randomization strategy.

Part IV

Model

1 Model with a Myopic Policy Maker

1.1 Basic Setup with a Myopic policy maker

This model considers a single-good, small open economy with real output growth, adapted from Pastine (2002) to capture an economy experiencing appreciative pressure in their currency. All equations are in log forms. In this model, I introduce an output shock that is proportional to the change in exchange rate that lasts exactly one period after the unpegging happens. The period length is Δ, $\Delta < 1$. The reason I introduce continuous time is that between the time when the peg is abandoned and the time when the exchange rate no longer affects output, the interest rate would adjust in continuous time to ensure the no-arbitrage condition (Uncovered Interest Rate Parity). In other words, if we only consider discrete time where the period length is one, then the model fails to capture the exchange rate dynamics between the time T and $T+1$. Purchasing Power Parity (Equation (1)) and money supply (Equation (4)) are not affected by the switch from discrete to continuous time. Uncovered Interest Parity, on the other hand, would adjust according to the period length. In the Cagan model money demand function, i_t stands for $log(1+i_t)$, which is the change in interest rate, hence would also vary with period length. Output is assumed to grow linearly, so it also varies with period length.

Purchasing Power Parity implies

$$p_t = e_t + p_t^* \tag{1}$$

at all times t, where p_t, p_t^* are the logs of domestic- and foreign-currency price of the consumption basket respectively and we assume here that $p_t^* = 0$. e_t is the log of nominal exchange rate (foreign in terms of home). Uncovered Interest Rate Parity holds for each period with length Δ,

$$\Delta i_t = \Delta i^* + E_t e_{t+\Delta} - e_t \tag{2}$$

where $i_t = log(1 + i'_t)$, i' being the nominal interest rate on domestic securities and $i^* = log\left(1 + i_t^{*'}\right)$, with $i_t^{*'}$ being the nominal interest rate on foreign securities, which is assumed to be constant. It is an approximation in logs of Uncovered Interest Rate Parity, which is $1 + i_{t+1} = (1 + i^*_{t+1})E_t(\frac{\xi_{t+1}}{\xi_t})$. Money demand is given by

$$m_t - p_t = -\eta \Delta i_t + \phi y_t \tag{3}$$

where m_t is log of nominal money balances held at the end of period t; y_t is the output at the end of period. The real money demand function above is an adaption of the Cagan Model. The Money Supply consists of the book value of of Central Bank foreign currency reserves r_t and domestic credit d_t. In order to simplify the model, we assume that money supply consists solely of foreign reserves (i.e. $d_t = 0$). In one of the extensions of the paper, I show that all propositions hold with the addition of d_t in the model.

$$m_t = r_t \tag{4}$$

Output grows at a constant rate of $\Delta\theta$ each period with period length Δ without change in exchange rate regimes. The economy experiences an output shock after the unpegging happens, which lasts exactly one period. Assume T is period where the exchange rate unpegging happens.

$$y_{t+\Delta} = \begin{cases} y_t + \Delta\left[\theta + \alpha\left(\tilde{e}_{t+\Delta} - e_t\right)\right] & t + \Delta \leqslant T + 1 \\ y_t + \Delta\theta & t > T + 1 \; or \; t \leqslant T \end{cases} \tag{5}$$

Exchange rate appreciation causes a continuous linear fall in output (and vice versa, depreciation causes a continuous linear increase in output) in the period after the peg is abandoned. The output shock is caused by short-term price rigidity, while in the long term, price would adjust accordingly so there should be no long-term cost, which is why the cost lasts exactly one period after the unpegging.

1.2 Timing

To fix ideas I will adopt the following timing specification: at the beginning of each period t where the period length is Δ, the economy undergoes a shock to its output, which is $\Delta\theta$ in normal times and $\Delta\left[\theta+\alpha\left(\tilde{e}_{t+\Delta}-e_t\right)\right]$ after the peg is abandoned.

1.3 Solving for the Speculators Problem in the Myopic Policy Maker Case

1.3.1 Equilibrium

First I rule out a case where speculators believe that the fixed regime will never be abandoned, in which case $E_t(e_{t+\Delta})=\bar{e}$ for all t. In this case, output growth becomes

$$\Delta y_t = y_{t-\Delta}+\Delta\theta \tag{6}$$

And from Equations 1 - 4 we have

$$r_t=\bar{e}-\eta\Delta i^{*}+\phi y_t \tag{7}$$

since (2) implies that $\Delta i_t=\Delta i^{*}$. But the log reserve needs to grow at a rate of $\phi\Delta\theta$. Hence given the upper threshold on the reserves that the Central Bank can take, it is not rational for them to think that the bank will never abandon the exchange rate regime. In fact, the fixed regime will be abandoned at a state $\tilde{T}$ defined by the following equations. Setting money supply equal money demand,

$$\bar{r}=\bar{e}-\eta\Delta i^{*}+\phi y_{\tilde{T}} \tag{8}$$

Note that the highest reserve level can go is $\bar{r}$.

$$\bar{e}-\eta\Delta i^{*}+\phi y_{\tilde{T}}=\bar{r} \tag{9}$$

Rearranging the terms yields,

$$y_{\tilde{T}} = \frac{\bar{r} + \eta\Delta i^{*} - \bar{e}}{\phi} \tag{10}$$

The intuition for the previous equation is that the more sensitive money demand is to output growth (which translates to a higher ϕ), the faster speculative attack will happen. This is intuitive because in the myopic policy maker case, the peg has to be abandoned when the reserve threshold is hit. So if money demand grows faster since it is more sensitive to output growth, then the time it would take to hit the threshold would be shorter.

To solve the speculators' optimization problem, it is necessary to determine the exchange rate that will prevail after the fixed rate regime is abandoned. Define the log of the shadow floating exchange rate, $\tilde{e}_t$, as the exchange rate that would prevail if the exchange rate were floating at t. In order for the exchange rate to be floating, in the myopic policy maker case, it must be that the reserves rose to $\bar{r}$ at one point, and after that point all foreign currency transactions will take place in private markets so the money supply will simply be $\bar{r}$. Solving for $\tilde{e}_t$ using Equations (1)—(5):

Here we are asking the question of what would the shadow exchange rate be if the peg is abandoned in period t and speculators only look out to period Δ beyond the current period (rather than 1 period ahead as in discrete time models). First set money demand to equal money supply

$$\bar{r} = \tilde{e}_t - \eta\Delta i_t + \phi y_t \tag{11}$$

Defining $\bar{T}$ as the date of the speculative attack in this traditional version of the model. In the period right after the unpegging, by Equation (5), $y_{\bar{T}+\Delta} = y_{\bar{T}} + \Delta\theta + \Delta\alpha \left[E_{\bar{T}}\left(\tilde{e}_{\bar{T}+\Delta}\right) - \tilde{e}_{\bar{T}}\right]$

$$E_T\left(\tilde{e}_{\bar{T}+\Delta}\right) - \tilde{e}_{\bar{T}} = -\Delta\phi\left[\theta + \alpha\left(E_{\bar{T}}\left(\tilde{e}_{\bar{T}+\Delta}\right) - \tilde{e}_{\bar{T}}\right)\right] \tag{12}$$

Solving for $E_t\left(\tilde{e}_{T+\Delta}\right) - \tilde{e}_T$ yields

$$E_t\left(\tilde{e}_{\bar{T}+\Delta}\right) - \tilde{e}_{\bar{T}} = -\frac{\Delta\phi\theta}{1+\Delta\phi\alpha} \tag{13}$$

Notice that the greater output growth (θ) is, the greater shadow exchange rate would change from period to period. This is because shadow exchange rate essentially reflects the demand and supply of currency. The demand of currency is positively affected by economic growth, hence exchange rate would appreciate faster if economic growth is faster. On the other hand, since output in the short term is negatively affected by exchange rate by a size of α, and exchange rate is in turn dependent on output, the change in exchange rate is negatively correlated with α. Let $\kappa = \frac{\Delta\theta}{1+\Delta\phi\alpha}$, then shadow exchange rate is

$$\tilde{e}_{\bar{T}} = \bar{r} + \eta\Delta i^* - \eta\phi\kappa - \phi y_{\bar{T}} \tag{14}$$

After period $\bar{T}$, the growth rate goes back $\Delta\theta$ every period. In other words, $y_{t+\Delta} = y_t + \Delta\theta$ for $t > \bar{T}+1$. Solving for the shadow exchange rate[1] in the same way yields

$$\tilde{e}_t = \bar{r} + \eta\Delta i^* - \eta\Delta\phi\theta - \phi y_t, \qquad t \geqslant \bar{T}+1 \tag{15}$$

If the shadow floating exchange rate is more than the fixed exchange rate then there is clearly no incentive for speculators to engage in a speculative attack. An attack would cause a breakdown of the exchange rate regime and speculators would make a loss. Speculators will therefore wait until the post-attack exchange rate equals the fixed rate. At that point if the breakdown is delayed there will be profits to be made by speculating in foreign currency. In this model, competition for speculative profits ensures that a speculative attack occurs the instant they become available. Therefore the attack must occur at the state where $\tilde{e}_{\bar{T}} = \bar{e}$ so that there is no in-

[1]Derivation available in Appendix A on page 55.

stantaneous jump in the exchange rate. The reason that we use $\tilde{e}_{\bar{T}}$ instead of $\tilde{e}_t$ for $t > \bar{T} + 1$ is that when the speculators make the decision when to attack, they consider the instantaneous profit they can make rather than the shadow exchange rate in future periods.

Setting $\tilde{e}_{\bar{T}} = \bar{e}$ in Equation (14) yields,

$$\bar{e} = \bar{r} + \eta \Delta i^* - \eta \phi \kappa - \phi y_{\bar{T}} \tag{16}$$

$$y_{\bar{T}} = \frac{\bar{r} + \eta \Delta i^* - \bar{e}}{\phi} - \eta \kappa \tag{17}$$

Together with Equation (17) and (10), we have

$$y_{\bar{T}} - y_{\tilde{T}} = -\eta \kappa \; < 0, \;\; \kappa = \frac{\Delta \theta}{1 + \Delta \phi \alpha} \tag{18}$$

Since output increases with time each period, the speculative attack happens before the reserve reaches its upper threshold. Initially reserves follow Equation (8), increasing with the growth of real output. At $\bar{T}$, the shadow floating exchange rate increases to the fixed rate and a sudden speculative attack would increase the reserves by $\eta \phi \kappa$, and forcing the reserve level to reach its upper threshold, which forces the Central Bank to abandon the peg.

2 Model with an Optimizing Policy Maker

2.1 Timing

As in the myopic policy maker case, to fix ideas I will adopt the following timing specification: although this is a continuous model, again it is helpful to specify the sequence of action. The economy experiences economic growth ($\Delta\theta$ when it is not the period after the peg is abandoned and $\Delta\theta - \frac{\alpha \Delta \phi \theta}{1+\Delta \phi \alpha} = \Delta\theta - \alpha\phi\kappa$ when it is the period after the peg is abandoned) which affects money supply. Given economic growth and the money demand, the speculators choose how much domestic currency to purchase based on their belief about the likelihood of the peg being abandoned. In the myopic policy maker case, the peg is immediately pegged or unpegged depending on whether the reserve level reaches $\bar{r}$. In the forward looking policy maker case, however, the policy maker can decide whether to abandon the peg after speculators take action.

2.2 The Policy Maker's Problem

Each period, the Central Bank incurs a cost τ for each period when it abandons the exchange rate. It also immediately experiences a short-term cost to the output after the abandonment. However, holding on to the peg is costly with the accumulation of excessive foreign reserves. Disutility increases with the level of foreign reserves due to two reasons as explained in the Introduction Section. First, the opportunity cost of holding reserves could be high because foreign reserves are usually held in highly liquid form that generates low rates of return and second, countries tend to hold foreign reserves in the currency that they are pegged to. Although one can argue that countries have the choice of diversifying their reserve portfolio, sometimes they still accumulate large amounts of reserve in a single currency in order to influence exchange rate more easily. These concerns about excessive foreign reserves have also resulted in other political economic considerations. One example of the political economy concern in Switzerland is the chance of a gold referendum that once passed, would mandate the Central Bank to hold 20% of their reserves in gold. In fear of excessive reserve building, especially in Euro, the Central Bank is willing to trade with the cost of unpegging.

Since the foreign reserves is central to my analysis, I elaborate more on the functions and opportunity cost of reserves to explain why they might be a concern in the central bank's objective function. Foreign reserves typically serve the following three purposes. First, foreign reserves serve as a tool to affect exchange rates. Second, reserves could be used to calm disorderly markets. For example, when speculators are exclusively betting one-sided on the exchange rate (either appreciation or depreciation), the government would be able to make the market believe that there is still a two-sided risk to the movement of the exchange rate. Third, foreign reserves could be insurance against liquidity losses and disruptions to capital market access. Although foreign reserves is an important tool of the Central Bank's monetary policy, holding excessive foreign reserves generates opportunity cost. This is mainly due to the low rate of returns to holding funds in currency and asset portfolios, which are the common forms of foreign reserves in most industrialized countries. Countries usually invest reserves in highly liquid assets such as foreign government securities. Due to their high liquidity, these assets are able to insure a country against a loss of access to capital markets but meanwhile, they generally yield low rates of returns.

Besides the opportunity cost to holding foreign reserves, they can also be costly for political economy reasons such as the threat of a gold referendum that could potentially mandate the government to hold 20% of their reserves in gold. The gold referendum was organized by Swiss People's Party, members of which argues that the gold measures are necessary in face of excessive foreign reserves especially the proportion denominated in Euro. The actual gold referendum proposal of Switzerland consists of two parts—the Swiss Central Bank is required to hold a certain proportion of their reserves in the form of gold and second, they are not allowed to sell that proportion of gold. The drawbacks to this proposal is significant to economic policies. First, the illiquidity of gold makes it hard for policy makers to respond to financial crises using monetary policies. The second part of the proposal which prevents the Central Bank from selling the gold would be troublesome when the Central Bank wants to shrink its balance sheet. To hold onto the gold, the Central Bank might have to sell other assets. Members of Swiss Central Bank have all been vocal about the negative implications of the gold referendum on future Swiss national policies. Swiss central banker Thomas Jordan says initiative would have hindered the effectiveness of monetary policy, calling it both "unnecessary and dangerous" because there is no link between price stability and the share of gold in the CB balance sheet.

The Central Bank's cost objective function consists not only of cost from excessive foreign reserves but also that from the gap between current output and the ideal output level. From the basic setup of the model, the economy experiences a short-term output shock following the unpegging, whose size is proportional to the size of fluctuation in exchange rate.

Each period, the government has an ideal output level y^*, which is constant across periods. Let $y_{a,b}$ denote the output in period b when the peg is abandoned in period a. Output in this case depends on the period where the peg is abandoned because the output growth is $y_t = y_{t-\Delta} + \Delta\left[\theta + \alpha\left(e_t - e_{t-\Delta}\right)\right]$. So the earlier the peg is abandoned, the earlier the output will start incurring cost from the appreciation of exchange rate but also the earlier the output shock terminates because the output shock lasts exactly one period. The Central Bank incurs a utility cost $y^* - y_{t,k}$ whenever the output level fails to reach y^*. At the beginning of each period after the output growth is realized, the government compares the cost between unpegging this period and unpegging in the future.

$$V_t = min\left\{\frac{\Delta\left(r_t+\tau\right)}{1-\delta^{\Delta}} + E_t\Delta\sum_{k\Delta=t}^{\infty}\delta^{k\Delta-t}\left(y^* - y_{t,k\Delta}\right)\ ,\ \Delta r_t + \Delta\left(y^* - y_t\right) + \delta^{\Delta}E_t(V_{t+\Delta})\right\}, \tag{19}$$

Each period, the policy maker compares the cost of abandoning the fixed exchange rate system with the current and expected future cost of maintaining it. Note that the output only starts changing right after time t. Hence the output when the peg is abandoned at t or $t+\Delta$ are both y_t.

This specification implies that if the policy maker is expecting a speculative attack, it will find it optimal to abandon the fixed exchange rate regime in the period before the attack; otherwise more reserve would only cause more pain in the future. That is, if in the coming period the expected increase in reserves is large, the policy maker will move to a floating exchange rate regime to avoid the attack. Notice that $E_t(V_{t+\Delta})$ includes the information on the expected level of future reserves. Therefore, for a given value of $V_{t+\Delta}$, low current period reserves make abandoning the fixed exchange rate system more attractive.

Since the policy maker is pursuing a fixed exchange rate policy, it must find it optimal to do so. This places a lower bound on the value of the parameter τ, the policy maker's preference for fixed exchange rates. In other words, the policy maker must find the cost of abandoning the peg next period smaller than the cost of abandoning the peg in this period, when there is no speculative attack. Mathematically, it means the following

$$\begin{aligned} \frac{\Delta(r_t+\tau)}{1-\delta^\Delta} + E_t\sum_{k\Delta=t}^{\infty}\delta^{k\Delta-t}\Delta\left(y^*-y_{t,t+k\Delta}\right) \quad &\geqslant \quad \Delta r_t + \Delta\left(y^*-y_t\right) \\ &+\delta^\Delta\left[\frac{\Delta\left(r_{t+\Delta}+\tau\right)}{1-\delta^\Delta} + E_t\sum_{k\Delta=t+\Delta}^{\infty}\Delta\delta^{(k-1)\Delta-t}\left(y^*-y_{t+\Delta,t+k\Delta}\right)\right] \end{aligned} \tag{20}$$

where the left-hand side is the cost of abandoning the fixed rate this period, and the right hand-side is abandoning the peg next period.

Simplifying Inequality (20) yields the following[2]:

$$\frac{\delta^\Delta\Delta}{1-\delta^\Delta}\left(r_t-r_{t+\Delta}\right)+\Delta\tau \geqslant E_t\sum_{k=t}^{\infty}\delta^{k\Delta-t}\Delta\left(y_{t,t+k\Delta}-y_{t+\Delta,t+k\Delta}\right)$$

I then examine the evolutionof output and calculate $E_t\sum_{k=t}^{\infty}\delta^{k\Delta-t}\Delta\left(y_{t,t+k\Delta}-y_{t+\Delta,t+k\Delta}\right)$.

I will explain how the output is calculated in terms of y_t. Pick N such that $N\Delta=1$, and by assumption of the model that output shock lasts exactly one period, when period length is Δ the exchange rate would affect output for N periods (with length Δ) after the peg is abandoned. By Equation (13), the growth rate of exchange rate is $E_t\left(\tilde{e}_{t+\Delta}-e_t\right)=-\alpha\phi\kappa$

[2]Derivation available in Appendix A on page 57

$$
\begin{aligned}
E_t y_{t,t+\Delta} &= y_t + \Delta \left[\theta + \alpha E_t \left(\tilde{e}_{t,t+\Delta} - \tilde{e}_{t,t}\right)\right] \\
&= y_t + \Delta \left(\theta - \alpha\phi\kappa\right)
\end{aligned}
$$

Right after time $t+1$, output again grows by $\Delta\theta$ each period with length Δ.

$$
\begin{aligned}
E_t y_{t,t+2\Delta} &= E_t y_{t,t+\Delta} + \Delta \left(\theta - \alpha\phi\kappa\right) \\
&= y_t + 2\Delta\theta - 2\Delta\alpha\phi\kappa \\
&\vdots \\
E_t y_{t,t+N\Delta} &= y_t + N\Delta\theta - N\Delta\alpha\phi\kappa
\end{aligned}
$$

The same calculation applies to the output path when the peg is abandoned $t+\Delta$. This time the effect of exchange rate on output stops at $t+(N+1)\Delta$ because it also starts one period (with length Δ) late. After period $t+(N+1)\Delta$, the differences between the two output paths goes away. The following is the different paths that output will follow when the peg is abandoned at t and $t+\Delta$ respectively.

Hence adding in the discount rate and calculate $E_t \sum_{k=t}^{\infty} \delta^{k\Delta-t}\Delta\left(y_{t,t+k\Delta} - y_{t+\Delta,t+k\Delta}\right)$:[3]

$$
E_t \sum_{k=t}^{\infty} \delta^{k\Delta-t}\Delta\left(y_{t,t+k\Delta} - y_{t+\Delta,t+k\Delta}\right) = -\frac{\delta^{\Delta}\left(1-\delta^{N\Delta}\right)}{1-\delta^{\Delta}}\Delta\alpha\phi\kappa \tag{21}
$$

Equation (20) thus turns into

[3]Derivation available in Appendix A on page 58.

Table 1: Evolution of Output

Time	t	$t+\Delta$	$t+2\Delta$
Peg abandoned at t			
output	y_t	$y_{t,t+\Delta}$	$y_{t,t+2\Delta}$
output in terms of previous period	y_t	$y_t+\Delta\left[\theta+\alpha\left(\tilde{e}_{t+\Delta}-e_t\right)\right]$	$y_{t,t+\Delta}+\Delta\left[\theta+\alpha\left(\tilde{e}_{t,t+2\Delta}-e_{t,t+\Delta}\right)\right]$
output in terms of y_t	y_t	$y_t+\Delta\left(\theta-\alpha\phi\kappa\right)$	$y_t+2\Delta\theta-2\Delta\alpha\phi\kappa$
Peg Abandoned at $t+1$			
output	y_t	$y_{t+\Delta,t+\Delta}$	$y_{t+\Delta,t+2\Delta}$
output in terms of previous period	y_t	$y_t+\Delta\theta$	$y_t+\Delta\theta-\Delta\alpha\phi\kappa$
output in terms of y_t	y_t	$y_t+\Delta\theta$	$y_t+2\Delta\theta-\Delta\alpha\phi\kappa$
$y_{t,t+n\Delta}-y_{t+\Delta,t+n\Delta}$	0	$-\Delta\alpha\phi\kappa$	$-\Delta\alpha\phi\kappa$

Time	t	$t+3\Delta$	... until time $t+1$
Peg abandoned at t			
output	y_t	$y_{t,t+3\Delta}$	$y_{t,t+N\Delta}$
output in terms of previous period	y_t	$y_{t,t+2\Delta}$ $+\Delta\left[\theta+\alpha\left(\tilde{e}_{t,t+3\Delta}-e_{t,t+2\Delta}\right)\right]$	$y_{t,t+(N-1)\Delta}+\Delta\left(\theta-\alpha\phi\kappa\right)$
output in terms of y_t	y_t	$y_t+3\Delta\theta-3\Delta\alpha\phi\kappa$	$y_t+N\Delta\theta-N\Delta\alpha\phi\kappa$
Peg Abandoned at $t+1$			
output	y_t	$y_{t+\Delta,t+3\Delta}$	$y_{t+\Delta,t+N\Delta}$
output in terms of previous period	y_t	$y_{t+\Delta,t+2\Delta}+\Delta\theta$	$y_{t+\Delta,t+(N-1)\Delta}+\Delta\left(\theta-\alpha\phi\kappa\right)$
output in terms of y_t	y_t	$y_t+3\Delta\theta-2\Delta\alpha\phi\kappa$	$y_t+N\Delta\theta-(N-1)\Delta\alpha\phi\kappa$
$y_{t,t+n\Delta}-y_{t+\Delta,t+n\Delta}$	0	$-\Delta\alpha\phi\kappa$	$-\Delta\alpha\phi\kappa$

Time	t	$t+1+\Delta$	$t+1+2\Delta$
Peg abandoned at t			
output	y_t	$y_{t,t+1+\Delta}$	$y_{t,t+1+2\Delta}$
output in terms of previous period	y_t	$y_{t,t+1}+\Delta\theta$	$y_{t,t+1+\Delta}+\Delta\theta$
output in terms of y_t	y_t	$y_t+(N+1)\Delta\theta$ $-N\Delta\alpha\phi\kappa$	$y_t+(N+2)\Delta\theta-N\Delta\alpha\phi\kappa$
Peg Abandoned at $t+1$			
output	y_t	$y_{t+\Delta,t+1+\Delta}$	$y_{t+\Delta,t+1+2\Delta}$
output in terms of previous period	y_t	$y_{t+\Delta,t+1}+\Delta\left(\theta-\alpha\phi\kappa\right)$	$y_{t,t+1+\Delta}+\Delta\theta$
output in terms of y_t	y_t	$y_t+(N+1)\Delta\theta$ $-N\Delta\alpha\phi\kappa$	$y_t+(N+2)\Delta\theta-N\Delta\alpha\phi\kappa$
$y_{t,t+n\Delta}-y_{t+\Delta,t+n\Delta}$	0	0	0

$^{*}\kappa=\frac{\Delta\theta}{1+\Delta\phi\alpha}$

**N is such that $N\Delta=1$.

$$\begin{aligned}
\frac{\delta^{\Delta}\Delta}{1-\delta^{\Delta}}\left(r_t - r_{t+\Delta}\right) + \Delta\tau &\geqslant E_t \sum_{k=t}^{\infty} \delta^{k\Delta - t}\Delta\left(y_{t,t+k\Delta} - y_{t+\Delta,t+k\Delta}\right) \\
\frac{1-\delta^{\Delta}}{\delta^{\Delta}}\tau &\geqslant \left(r_{t+\Delta} - r_t\right) - \left(1-\delta^{N\Delta}\right)\alpha\phi\kappa
\end{aligned}$$

Here we are calculating a lower bound for τ when there is no speculative attack and the change in reserve level for times with no speculative attack is given by Equation (7). It implies that $r_{t+\Delta} - r_t = \phi\left(y_{t+\Delta} - y_t\right) = \Delta\theta$. And so

$$r_{t+\Delta} - r_t = \phi\Delta\theta \tag{22}$$

Hence from Equation (23) and the previous equation, given a period length Δ, the lower bound for τ is becomes

$$\frac{1-\delta^{\Delta}}{\delta^{\Delta}}\tau \geqslant \phi\Delta\theta - \left(1-\delta^{N\Delta}\right)\alpha\phi\kappa \tag{23}$$

where $\kappa = \frac{\Delta\theta}{1+\Delta\alpha\phi}$. This is **Assumption 1** in our model.

2.3 The Speculators' Problem

The solution to the speculators' problem is described by Uncovered Interest Parity, Equation (2). However, speculators will be aware that the policy maker may decide to allow the exchange rate to float before reserves reaches $\bar{r}$. If it decides to do this the floating exchange rate will no longer be described by (14).

To describe the shadow floating exchange rate, let $\tilde{e}_{T,t}$ denote the exchange rate that would prevail at t if the exchange rate was first floated in T. So the first subscript gives the date the fixed exchange rate was abandoned and the second subscript refers to the current date. Notice that the interpretation of that shadow floating exchange rate in the optimizing model is slightly different than it is in the myopic model. In the myopic model $\tilde{e}_t$ is the exchange rate that will prevail if the reserve level reaches $\bar{r}$, which is the only binding criteria used to judge whether the policy maker would abandon the fixed exchange rate regime. In the optimizing model $\tilde{e}_{T,t}$ is the exchange rate that will prevail at t if the policy maker abandons the fixed exchange rate at date T. The difference arises because here the policy maker may choose to abandon the fixed exchange rate without a speculative attack. In the latter case, after policy maker abandons the peg at time T, the reserve level will remain at r_T, since all later transactions take place in private markets.

Since all foreign currency transactions in the floating rate period take place in private markets, the log of money supply will be r_T. Here we denote q_t as the probability for the Central Bank to abandon the fixed exchange rate. We set $E_t(e_{t+\Delta} - e_t) = q_t(E_t e_{t+\Delta} - e_t)$. So when the speculators expect the policy maker to abandon the peg this period ($q_t = 1$), $E_t(e_{t+\Delta} - e_t) = E_t e_{t+\Delta} - e_t$ and the expected change of exchange rate would be zero if the speculators believe the policy maker does not abandon the peg this period. Solving for the shadow exchange rate from Equation (1)—(5) similar to the myopic case.

$$r_T = \tilde{e}_{T,t} - \eta\,[\Delta i^* + E_t \tilde{e}_{T,t+\Delta} - \tilde{e}_{T,t}] + \phi y_t \tag{24}$$

The exchange rate dynamics for the period right after the abandonment is different from the exchange rate dynamics one period after. Between T and $T+1$, output is linearly affected by change in exchange rate. After $T+1$, it is not affected by exchange rate. In order to capture the nuanced dynamics of this one-period output shock, at time T, I assume the speculators will be concerned with the interest rate change between time t and $t+\Delta$, where $\Delta < 1$. At time T, if the speculators only consider the exchange rate in one period ahead, then they lose the interest rate dynamics between T and $T+1$ caused by this one period shock to output. And that is why a continuous time framework is necessary.

First I calculate the shadow exchange rate between T and $T+1$. Here the shadow exchange rate $\tilde{e}_{T,T+n\Delta}$, where $n \in [0, N]$ (where $N\Delta = 1$), denotes the exchange rate that would prevail at time $T+\Delta$ when the peg is abandoned at T and the speculators look out to period Δ ahead.

$$r_T = \tilde{e}_{T,T+n\Delta} - \eta\Delta i_{T+n\Delta} + \phi y_{T+n\Delta} \tag{25}$$

We can derive that [4]

$$E_t\left(\tilde{e}_{T,T+(n+1)\Delta}\right) - \tilde{e}_{T,T+n\Delta} = -\frac{\Delta\phi\theta}{1+\Delta\phi\alpha} \tag{26}$$

Let $\kappa = \frac{\Delta\theta}{1+\Delta\phi\alpha}$, then the change in expected shadow exchange rate

$$\tilde{e}_{T,T+n\Delta} = r_T + \eta\Delta i^* - \eta\phi\kappa - \phi y_{T,T+n\Delta} \qquad n < N \tag{27}$$

where $N\Delta = 1$. Otherwise when $t \geqslant T+1$, output growth returns to be $\Delta\theta$ per period.

$$\tilde{e}_{T,t} = r_T + \eta\Delta i^* - \eta\phi\Delta\theta - \phi y_t, \quad t \geqslant T+1 \tag{28}$$

If speculators were not holding very much foreign currency at the time of the abandonment, then the money supply will be relatively high, resulting in a high path for the floating exchange rate. Thus if reserves are high at the time of the move to a floating exchange rate then the exchange rate itself will be relatively high as well. In fact, if speculators did not expect a change in the fixed exchange rate, reserves would be given by Equation (7), which is obtained by setting $E_t(e_{t+1}) = \bar{e}$ for all t.

$$\bar{e} = r_t + \eta\Delta i^* - \phi y_t \tag{29}$$

If the speculators don't expect the government to abandon the fixed exchange rate

[4] Derivation available in Appendix A on page 59.

($q_t = 0$), the shadow exchange rate would be given by setting the period where the government abandons the peg, T to be t in the expression for shadow exchange rate (Equation (27))

$$\tilde{e}_{t,t} = r_t + \eta \Delta i^* - \eta \phi \kappa - \phi y_t \tag{30}$$

which is because in calculating the shadow rate, we assume that there is no speculative bubble.

Replacing $r_t + \eta \Delta i^* - \phi y_t$ with $\bar{e}$ by Equation (29), we find that the shadow floating exchange rate in this case would be strictly greater than the fixed rate,

$$\{\tilde{e}_{t,t} |\, E(q_t) = 0\} = \bar{e} - \eta \phi \kappa \tag{31}$$

This means that if the speculators does not expect the government to unpeg while the government does unpeg, then the

Proposition 1. In equilibrium, high probabilities of abandonment will result in high levels of reserves.[5]

Intuitively, higher probabilities of abandonment would cause speculators to hold more domestic currency (with the expectation that it would appreciate once the regime becomes floating).

Proof

Speculators are still governed by Uncovered Interest Parity. If we are in the fixed regime in the current period, UIP becomes

$$\Delta i_t = \Delta i^* + q_t (\tilde{e}_{t,t+\Delta} - \bar{e}) \tag{32}$$

where $\tilde{e}_{t,t+\Delta}$ denotes the shadow exchange rate in the next period assuming the peg is abandoned in this period. Since $\Delta < 1$, we use the first equation for shadow

[5] Derivation available in Appendix A on page 61.

exchange rate (Equation (28)), $E_t\tilde{e}_{t,t+\Delta} = r_t + \eta\Delta i^* - \eta\Delta\phi\kappa - \phi y_{t+\Delta}$. Combining it with Equations (1), (3), (4), (5), we get

$$r_t = \bar{e} - \eta\Delta i^* + \phi y_t + \frac{q_t}{1+\eta q_t}\eta\phi\Delta\theta\left[\frac{1+\eta}{1+\phi\alpha\Delta}\right] \tag{33}$$

The derivative of r_t with respect to q_t is

$$\frac{\partial r_t}{\partial q_t} = \left[\frac{1}{1+\eta q_t} - \frac{q_t\eta}{(1+\eta q_t)^2}\right]\left[\frac{1+\eta}{1+\alpha\phi\Delta}\right]\eta\phi\Delta\theta > 0 \tag{34}$$

Since $q_t, \eta < 1$ and $\frac{1}{1+\eta q_t} > \frac{1}{(1+\eta q_t)^2}$, we have $\frac{\partial r_t}{\partial q_t} > 0$, which completes the proof of **Proposition 1**. □

Proposition 2. If $q < 1$ then $\tilde{e}_{t,t} < \bar{e}$. If $q_t = 1$, then $\tilde{e}_{t,t} = \bar{e}$.[6]

Intuitively, if speculators are certain that the exchange rate will be abandoned they will continue to buy domestic currency as long as the shadow floating exchange rate is smaller than the fixed rate. These domestic currency purchases increase the money supply and hence decrease the shadow floating exchange rate until it equals the fixed rate. Put another way, the shadow floating exchange rate will always be at most as high as the fixed rate since otherwise speculators would find it profitable to buy foreign currency, thereby decreasing the money supply and lowering the shadow floating exchange rate.

Proof

Setting $q_t = 1$ in Equation (33), we get

$$r_t = \bar{e} - \eta\Delta i^* + \phi y_t + \frac{\Delta\eta\phi\theta}{1+\phi\alpha\Delta}$$

And hence $\bar{e} = r_t + \eta i^* - \phi y_t - \eta\phi\kappa$. The shadow exchange rate at the time when the peg is first abandoned is given in (27). Setting $T = t$ and set $n = 0$, we get

[6] Derivation available in Appendix A on page 63.

$\tilde{e}_{t,t} = \bar{e}$ when $q_t = 1$. To prove the first part, note that the shadow rate is decreasing in the level of reserves and the level of reserves is increasing in q_t by Proposition 1. Hence given that $\tilde{e}_{t,t} = \bar{e}$ when $q_t = 1$, we have that $\tilde{e}_{t,t} < \bar{e}$ when $q_t < 1$. □

2.4 Equilibrium

In the standard model where the policy maker remains passive, speculators attack the fixed exchange rate as soon as the shadow floating exchange rate decreases to the fixed rate. However, this attack imposes losses on the Central Bank, making it attractive to abandon the fixed rate just before the attack. By abandoning the fixed exchange rate regime one period early the policy maker would avoid the speculative attack. One might presume that this would be an equilibrium, since in the traditional model with a myopic policy maker, the shadow floating exchange rate does not rise to the fixed rate until one period later. However, this is not the case. If the policy maker chooses to abandon the fixed exchange rate when reserves are still below $\bar{r}$, then the shadow floating exchange rate will be correspondingly lower and speculators will find it profitable to plan an attack in the beginning of that period.

Proposition 3. If $q_{t-\Delta} = 0$ and $q_t = 1$ for any $t \leqslant \bar{T}$, where $\bar{T}$ is the period when the speculative attack takes place in the myopic case, then $r_t - r_{t-\Delta} = \eta\phi\kappa + \phi\Delta\theta$.[7]

Intuitively, since reserves increase by $\phi\Delta\theta$ each period due to real output growth, if speculators know for sure that the government would abandon the fixed regime in the next period, they would immediately plan an attack of size $\eta\phi\kappa$. Note that the size of the speculative attack is the same as the size of the attack in the myopic case and it does not depend on the date of the abandonment. In the myopic model the size of the attack does not depend on the critical level of reserves $\bar{r}$, but the date of the speculative attack is uniquely determined by $\bar{r}$. Thus in the myopic model there is a

[7] Derivation available in Appendix A on page 64.

one-to-one correspondence between $\bar{r}$ and the date of abandonment, given either, one can deduce the other. In Proposition 3 the specification of the date of abandonment ($q_{t-1} = 0$ and $q_t = 1$) is therefore tantamount to picking a different critical level of reserves in the myopic model.

Proof

Trivial application of Equation (33).

Proposition 4. In equilibrium $q_t < 1$ for all $t < \bar{T}$.[8]

Intuitively, if the policy maker plans to abandon the fixed exchange rate, speculators will try to take advantage of this by purchasing domestic currency, resulting in high Central Bank foreign reserves. So when the time comes for the policy maker to actually implement the switch to the floating rate, it will find that reserves are already quite high and that the damage from speculators has already been done. The additional damage that they could do if the policy maker waited one more period is relatively small. So it is in the policy maker's best interest to continue maintaining the fixed exchange rate. Therefore switching exchange rate regimes at time $t < \bar{T}$ with certainty cannot be part of an equilibrium strategy. This implies that in equilibrium the policy maker will not be able to preemptively abandon the fixed rate with certainty at t in order to avoid a speculative attack in $t + \Delta$.

Proof

Consider the case where $q_t = 1$ while $t < \bar{T}$ for the sake of contradiction. This assumption means that the Central Bank must find it optimal to abandon the fixed exchange rate regime, which means it must find the cost of abandoning the peg this period would be smaller than the cost of abandoning the peg next period. Their objective function (19) implies that

[8] Derivation available in Appendix A on page 65.

$$\frac{\Delta\left(r_t+\tau\right)}{1-\delta^{\Delta}}+E_t\sum_{k\Delta=t}^{\infty}\delta^{k\Delta-t}\Delta\left(y^*-y_{t,t+k\Delta}\right) \leqslant \Delta r_t+\Delta\left(y^*-y_t\right)+\delta^{\Delta}\frac{\Delta\left(E_t r_{t+\Delta}+\tau\right)}{1-\delta^{\Delta}}$$
$$+\delta^{\Delta}E_t\sum_{k\Delta=t+\Delta}^{\infty}\Delta\delta^{(k-1)\Delta-t}\left(y^*-y_{t+\Delta,t+k\Delta}\right)$$

where the left-hand side is the cost of abandoning the fixed rate this period, and the right hand-side is abandoning the peg next period. The previous inequality can be simplified to

$$\frac{1-\delta^{\Delta}}{\delta^{\Delta}}\tau \leqslant \left(E_t r_{t+\Delta}-r_t\right)-\left(1-\delta^{N\Delta}\right)\alpha\phi\kappa \tag{35}$$

Now consider the speculators' optimal choice. Since $q_t=1$, Proposition 2 implies that they purchase domestic currency to a point where $\tilde{e}_{t,t}=\bar{e}$. From Equation (33), we know the reserve level is

$$r_t=\bar{e}-\eta\Delta i^*+\phi y_t+\eta\phi\kappa \tag{36}$$

The speculative attack in $t+\Delta$ would only continue if the shadow rate is at least as low as the fixed rate, $\tilde{e}_{t+\Delta,t+\Delta}\leqslant\bar{e}$, an inequality since $q_{t+\Delta}$ might be smaller than or equal to 1. This implies, by the fact that the level of foreign reserves increases in shadow exchange rate. And we are given that when $\tilde{e}_{t+\Delta,t+\Delta}=\bar{e}$, $E_t r_{t+\Delta}=\bar{e}-\eta i^*+\eta\phi\kappa+\phi y_{t+\Delta}$. So when $\tilde{e}_{t+\Delta,t+\Delta}\leqslant\bar{e}$,

$$E_t r_{t+\Delta}\leqslant\bar{e}-\eta\Delta i^*+\eta\phi\kappa+\phi E_t y_{t+\Delta} \tag{37}$$

Hence (37) -(36) yields

$$E_t r_{t+\Delta}-r_t\leqslant\phi(y_{t+\Delta}-y_t) \tag{38}$$

In our derivation of $E_t r_{t+\Delta}$, we are assuming that the peg is not abandoned until time $t+\Delta$, hence the output $y_{t,t+\Delta}$ is not affected by exchange rate yet. Therefore

$$\begin{aligned} E_t r_{t+\Delta} - r_t &\leqslant \phi(y_{t+\Delta} - y_t) && (39) \\ &= \phi\Delta\theta && (40) \end{aligned}$$

Substitute this into Equation (35),

$$\phi\Delta\theta - \left(1-\delta^{N\Delta}\right)\alpha\phi\kappa \geqslant \frac{1-\delta^{\Delta}}{\delta^{\Delta}}\tau \tag{41}$$

But by **Assumption 1** in Equation (23), $\phi\Delta\theta - \left(1-\delta^{N\Delta}\right)\alpha\phi\kappa \leqslant \frac{1-\delta^{\Delta}}{\delta^{\Delta}}\tau$. $\Rightarrow\Leftarrow$

This completes the proof of Proposition 4 that the Central Bank will not pursue $q_t = 1$ in any period before $\bar{T}$. $\square$

The argument given in the myopic case implies that $q_t = 1$ for all $t \geqslant \bar{T}$. Therefore an implication of Proposition 4 is that the only remaining potential pure-strategy for the policy maker involves setting $q_t = 0$ for all $t < \bar{T}$ and $q_t = 1$ for all $t \geqslant \bar{T}$. Consider in the next proposition what is necessary for this to be an equilibrium.

Proposition 5. If $t < \bar{T}$ and $q_{t+1} > 0$, then $q_t = 0$ can only be part of an equilibrium strategy if $(E_t r_{t+\Delta} - r_t) \leqslant \frac{1-\delta^{\Delta}}{\delta^{\Delta}}\tau + \left(1-\delta^{N\Delta}\right)\alpha\phi\kappa$. As $\Delta \to 0$, in continuous time framework, such equilibrium doesn't exist.

Intuitively, this means that passively waiting for an attack can only be optimal when the increase in reserves, and thus disutility, is small relative to the policy maker's preference for the exchange rate peg.

Proof

For the choice of $q_t = 0$ (not abandoning the peg in period t) to be optimal for the policy maker, it must find that abandoning the peg is more costly than continuing with the fixed regime. In other words,

$$
\begin{aligned}
\frac{\Delta\left(r_t+\tau\right)}{1-\delta^{\Delta}}+E_t \sum_{k\Delta=t}^{\infty} \delta^{k\Delta-t}\Delta\left(y^*-y_{t,t+k\Delta}\right) \;\geqslant\; & \Delta r_t+\Delta\left(y^*-y_t\right)+\delta^{\Delta}\frac{E_t r_{t+\Delta}+\tau}{1-\delta^{\Delta}} \\
& +\delta^{\Delta} E_t \sum_{k\Delta=t+\Delta}^{\infty} \delta^{(k-1)\Delta-t}\left(y^*-y_{t+\Delta,t+k\Delta}\right)
\end{aligned}
$$

By Equation (21), $\sum_{k=t}^{\infty} \delta^{k-t}\left(y^*-y_{t+1,k}\right)-\sum_{k=t}^{\infty} \delta^{k-t}\left(y^*-y_{t,k}\right)=-\frac{\delta^{\Delta}\left(1-\delta^{N\Delta}\right)}{1-\delta^{\Delta}}\Delta\alpha\phi\kappa$. Hence the Inequality can be simplified to

$$
\frac{1-\delta^{\Delta}}{\delta^{\Delta}}\tau \;\geqslant\; \left(E_t r_{t+\Delta}-r_t\right)-\left(1-\delta^{N\Delta}\right)\alpha\phi\kappa \tag{42}
$$

which reduces to

$$
\left(E_t r_{t+\Delta}-r_t\right) \;\leqslant\; \tfrac{1-\delta^{\Delta}}{\delta^{\Delta}}\tau \;+\left(1-\delta^{N\Delta}\right)\alpha\phi\kappa \tag{43}
$$

Put in words, maintaining the fixed exchange rate will be optimal as long as in the next period reserves will not fall by more than $\frac{1-\delta^{\Delta}}{\delta^{\Delta}}\tau+\left(1-\delta^{N\Delta}\right)\alpha\phi\kappa$. As $\Delta\to 0$, $\frac{1-\delta^{\Delta}}{\delta^{\Delta}}\tau+\left(1-\delta^{N\Delta}\right)\alpha\phi\kappa\to 0$, which means that in continuous time framework, $q_t=0$ is not an equilibrium for $t<\bar{T}$. $\square$

Proposition 6. With short period lengths no pure-strategy sub-game-perfect Nash equilibrium exists for $t<\bar{T}$.

Proof

This is the result of Proposition 4 and 5.

3 Backward Induction

Intuitively, if the policy maker can predict an imminent speculative attack then it will wish to abandon the fixed exchange rate regime just before the attack. Likewise, if speculators can predict this preemptive abandonment of the fixed exchange rate regime, then they will exploit this knowledge by buying foreign currency just before the abandonment. Thus in order to avoid a speculative attack the policy maker must introduce uncertainty into the decisions of speculators. It cannot follow a predictable pure strategy, since such a strategy would result in a speculative attack.

The subgame-perfect Nash equilibrium can be constructed by backward induction. The argument made in the myopic case implies that in any equilibrium $q_{\bar{T}} = 1$. Given this, it is possible to examine the policy maker's optimal strategy in $\bar{T} - \Delta$. And from this it is possible to examine the policy maker's optimal strategy in $\bar{T} - 2\Delta$, and so on.

Consider a time $t < \bar{T}$ where $q_{t+1} > 0$. That is, in the coming period there will be a positive probability that the policy maker will abandon the fixed exchange rate. The policy maker must find this optimal in period $t + 1$, which implies that it either strictly prefers to abandon the fixed rate regime in $t + 1$ or it is indifferent between abandoning and maintaining the peg. In either case from the their objective function, Equation (19), the maximized expected present value of its cost function is given by $\frac{\Delta(r_{t+1}+\tau)}{1-\delta^{\Delta}} + E_t \sum_{k\Delta=t}^{\infty} \delta^{k\Delta-t} \Delta \left(y^* - y_{t,t+k\Delta}\right)$.

From Proposition 4, $q_t = 1$ when $t < \bar{T}$ is not possible. Suppose for a moment that $q_t \in (0, 1)$, which means that the policy maker is indifferent between maintaining and abandoning the fixed exchange rate regime. From Equation (19), hence

$$\frac{1-\delta^{\Delta}}{\delta^{\Delta}}\tau \quad = \quad (r_{t+\Delta} - r_t) - \left(1 - \delta^{N\Delta}\right)\alpha\phi\kappa$$

and the Central Bank log reserve value must equal to

$$r_t = r_{t+\Delta} - \left[\frac{1-\delta^{\Delta}}{\delta^{\Delta}} \tau + \left(1 - \delta^{N\Delta}\right) \alpha\phi\kappa \right] \tag{44}$$

Working backwards from time $\bar{T}$, $r_{\bar{T}-\Delta} = r_{\bar{T}} - \left[\frac{1-\delta^{\Delta}}{\delta^{\Delta}} \tau + \left(1 - \delta^{N\Delta}\right) \alpha\phi\kappa \right]$. Since $r_{\bar{T}} = \bar{r}$, we have

$$r_{\bar{T}-\Delta} = \bar{r} - \left[\frac{1-\delta^{\Delta}}{\delta^{\Delta}} \tau + \left(1 - \delta^{N\Delta}\right) \alpha\phi\kappa \right] \tag{45}$$

Then working backward from $\bar{T} - \Delta$, it is possible to establish $\underline{\mathrm{T}}$, the earliest state where abandonment of the fixed exchange rate can be an equilibrium outcome. Iterating (44) yields[9]

$$r_t = \bar{r} - \left(\frac{\bar{T} - t}{\Delta} \right) \left[\frac{1-\delta^{\Delta}}{\delta^{\Delta}} \tau + \left(1 - \delta^{N\Delta}\right) \alpha\phi\kappa \right] \tag{46}$$

Equation (46) describes the equilibrium level of reserves as long as it yields reserves that are greater than those given by Equation (7) because that point and earlier $q_t = 0$ is an equilibrium and so reserves will follow Equation (7). After $\underline{\mathrm{T}}$ reserves increase quickly, but continuously, until they reach their upper bound at $\bar{T}$.

During $[\underline{\mathrm{T}}, \bar{T}]$, the Central Bank plays a randomized strategy. Since we will be interested in expressing shadow exchange rate $\tilde{e}_{t,t}$ with $y_{\bar{T}}$ and $\bar{r}$, and the expression for shadow exchange rate $\tilde{e}_{t,t}$ involves $y_{t,t}$, as shown in Equation (27). Hence here I derive $y_{t,t}$ in terms of $y_{\bar{T}}$. According to the output table (where $N\Delta = 1$), $y_{t,t} = y_{t,t+\Delta} - N\Delta\theta - N\Delta\alpha\phi\kappa$.[10]

$$y_{t,t} = y_{\bar{T}} - \left(\bar{T} - t\right) \theta - \alpha\phi\kappa \tag{47}$$

Substituting (47) into the shadow exchange rate in Equation (27), solving for

[9] Derivation available in Appendix A on page 67.
[10] Derivation available in Appendix A on page 69.

$\tilde{e}_{t,t}$,[11]

$$\tilde{e}_{t,t} = (\bar{r} + \eta\Delta i^* - \eta\phi\kappa - \phi y_{\bar{T}}) - (\bar{T} - t)\left\{\left[\frac{1-\delta^\Delta}{\delta^\Delta}\tau + \left(1 - \delta^{N\Delta}\right)\alpha\phi\kappa\right]/\Delta - \phi\theta - \left(\frac{\phi}{\bar{T} - t}\right)\alpha\phi\kappa\right\}, \ \kappa = \frac{\theta}{1 + \alpha\phi} \tag{48}$$

According to Equation (16), $\bar{e} = \bar{r} + \eta i^* - \eta\phi\kappa - \phi y_{\bar{T}}$, hence

$$\tilde{e}_{t,t} = \bar{e} - (\bar{T} - t)\left\{\left[\frac{1-\delta^\Delta}{\delta^\Delta}\tau + \left(1 - \delta^{N\Delta}\right)\alpha\phi\kappa\right]/\Delta - \phi\theta - \left(\frac{\phi}{\bar{T} - t}\right)\alpha\phi\kappa\right\}, \ \kappa = \frac{\Delta\theta}{1 + \Delta\alpha\theta} \tag{49}$$

To examine the behavior of the interest rate in equilibrium before the unpegging happens, note that (1), (3) and (4) yield that,[12]

$$i_t = \frac{\bar{e} + \phi y_t - r_t}{\Delta\eta} \tag{50}$$

and during $t \in [\underline{\mathrm{T}}, \bar{T}]$, reserves are given by Equation (46) and output before unpegging is given by $y_t = y_{\bar{T}} - \left(\bar{T} - t\right)\theta$. Plug in $y_{\bar{T}}$ from Equation (17) into the Equation above, $\phi y_{\bar{T}} = \bar{r} + \eta\Delta i^* - \bar{e} - \eta\phi\kappa$, we have

$$i_t = i^* - \phi\kappa/\Delta + \left(\bar{T} - t\right)\left[\frac{\left(\frac{1-\delta^\Delta}{\delta^\Delta}\tau + \left(1 - \delta^{N\Delta}\right)\alpha\phi\kappa\right)/\Delta - \phi\theta}{\Delta\eta}\right] \tag{51}$$

From the definition of $\underline{\mathrm{T}}$ the interest rate is equal to the world rate at $\underline{\mathrm{T}}$. This is because $\underline{\mathrm{T}}$ is defined as the period where the reserve level at which the speculators don't expect a change in exchange regime (and thus $i_{\underline{\mathrm{T}}} = i^*$) equals the reserve level from backward induction as in Equation (46). Since i_t is derived from the period where the reserve level at which speculators don't expect a change in exchange rate, $i_{\underline{\mathrm{T}}} = i^*$. It then decreases linearly until it reaches $i^* - \phi\kappa/\Delta$ at $\bar{T}$. Intuitively it makes sense for i_t to decrease because by UIP, $i_t = i^* + E_t\left(\tilde{e}_{t+1} - e_t\right)$ and after $\underline{\mathrm{T}}$, there is chance that the government would abandon the peg, and thus the exchange rate next period experiences appreciative pressure. Hence the term $E_t\left(\tilde{e}_{t+1} - e_t\right)$ is negative and therefore $i_t < i^*$ for $t > \underline{\mathrm{T}}$.

[11] Derivation available in Appendix A on page 78.
[12] Derivation available in Appendix A on page 70.

The one remaining endogenous variable is q_t, the probability that the policy maker abandons the fixed exchange regime. This can be derived by noting that $\tilde{e}_{t,t+\Delta} - \tilde{e}_{t,t} = \phi(y_t - y_{t+\Delta})$. From Uncovered Interest Rate Parity (32), $i = i^* + q_t(\tilde{e}_{t,t+\Delta} - \bar{e})$ yields,[13]

$$q_t = \frac{i^* - i_t}{(\bar{T} - t)\left\{\left[\frac{1-\delta^\Delta}{\delta^\Delta}\tau + (1 - \delta^{N\Delta})\,\alpha\phi\kappa\right]/\Delta - \phi\theta - \left(\frac{\phi}{\bar{T}-t}\right)\alpha\phi\kappa\right\} + \phi\kappa} \tag{52}$$

At $\underline{\mathrm{T}}$, $i_t = i^*$ and then i_t decreases linearly afterwards. So Equation (52) says that the probability of abandoning the fixed exchange rate is zero at $\underline{\mathrm{T}}$. It then increases, at an increasing rate, until it reaches 1 at $\bar{T}$.

3.1 Interpretation

Before the point $\underline{\mathrm{T}}$, the policy maker would not abandon the fixed exchange rate regime because the reserve level is not high enough, while at $\bar{T}$ the policy maker have to abandon the peg due to excessive accumulation of reserves. Before $\underline{\mathrm{T}}$ reserves are low enough so that if speculators attacked and drove reserves up to its upper reserve threshold, forcing a move to a floating exchange rate, they would increase the domestic money supply to a point where the floating rate would exceed the fixed rate, which is a loss to the speculators; hence they do not attack. The domestic interest rate is therefore equal to the world rate.

After $\underline{\mathrm{T}}$ both speculators and the policy maker realize that $\bar{T}$ is approaching, the reserve level is high enough that the unpegging would be imminent. Each period, the government evaluates the cost of remaining the peg versus that of abandoning the peg. The no arbitrage condition in the market ensures that the speculators always purchase enough domestic currency (increase Central Bank's holding of foreign reserves) so that the policy maker is indifferent between the two choices. The increased amount of domestic currency in the hands of the speculators drives down the domestic

[13] Derivation available in Appendix A on page 71.

interest rate and forward exchange rate premium while driving up the shadow floating exchange rate. Therefore the opportunity cost of holding foreign currency is decreasing, and the benefit to successful speculation is declining. But the probability that speculation will be successful is increasing rapidly, ensuring that the expected return from holding domestic currency remains constant at zero. However, in the event of a move to floating exchange rates, while there is no sudden increase in Central Bank reserves, the ex-post profits from holding domestic currency can be substantial, since the shadow floating exchange rate is below the fixed rate until $\bar{T}$.

The model also intuits how the government's probability for abandoning the peg changes with different parameters. Increased output growth results in increased market expectation of the unpegging, which in turn leads to higher reserves because speculators would purchase more domestic currency to earn profit when it appreciates. The more the output shock is correlated with the magnitude of change in exchange rate upon unpegging, the less the market expects the policy maker to unpeg, which in turn results in less accumulation of reserves during the speculation period. This is intuitive because the greater output shock increases with exchange rate jump upon unpegging, the more cost that the policy maker incurs, the less likely they would abandon the peg.

4 Extensions of the Model

In the appendix, I include several variations of my model. Appendix B presents a discrete-time model where the output grows stochastically each period. The model is still governed by purchasing power parity, uncovered interest rate parity, money demand, money supply and domestic credit growth function. In this model, the economy does not experience a short-term shock upon the onset of a currency crisis. Each period the policy maker compares the cost of abandoning the peg to that of maintaining it. The policy maker plays a randomized strategy. If it had a preferred strategy, the speculators would find it profitable to purchase more domestic currency to the extent that no more profit can be earned even if the policy maker decides to unpeg the fixed exchange rate regime. This purchase of domestic currency increases the Central Banks' holding of foreign reserves, which is a cost to the government as governed by its objective function. However, since the speculators would purchase domestic currency (and increase foreign reserves) to the level where the government is indifferent between pegging or unpegging, the loss of increase in reserve (in other words, the cost of a speculative attack), is already incurred. So the government would be indifferent between pegging and unpegging.

Appendix C introduces a stochastic shock to the cost of holding foreign reserves in the policy maker's objective cost function. This stochastic element aims to capture events that happen abroad but influence the cost of holding reserves. For example, Denmark experienced a speculative attack following Switzerland's unpegging because traders lost confidence in safe haven countries and noticed that Denmark also accumulated a notable amount of reserves. This spillover effect can either be captured in a stochastic shock to the money demand function or a stochastic shock to the cost of holding excessive reserves. The model shows that a big enough shock could force the policy maker to unpeg is currency.

5 Empirics

This section aims to quantitatively capture the correlation between Switzerland's reserve level and the market's sentiment towards its unpegging risk during 2011 and 2015. Recall that my model predicts, there exists a point $\underline{T}$, after which the speculation against the peg increases more dramatically with reserve level compared to before $\underline{T}$. The reason is that after $\underline{T}$, the Central Bank's foreign reserves holding is so costly that unpegging is imminent. In Switzerland's case, $\underline{T}$ is March 2013, when the gold referendum proposal was approved with 100,000 signatures from the Swiss citizens. According to my model, before March 2013 we should expect a positive but less significant correlation between foreign reserves level and market speculation against the peg. After March 2013, the positive correlation between the two should be much more significant. I use 25-Delta Risk Reversal data from Bloomberg as a proxy for market's sentiment towards the unpegging of Swiss Franc. At the end of the section, I apply the parameters derived in Switzerland's case to Denmark whose currency is also pegged to Euro. I argue that since Denmark's data best fit with the set of parameters that capture Switzerland's reserve dynamics before $\underline{T}$ (i.e. March 2013), in the context of my model, Denmark is likely to still be in the period before $\underline{T}$ and that is potentially why Denmark did not experience speculation when Switzerland did. However, readers should note that all the empirics in this paper are correlations rather than causations. My goal in the empirics section is not to prove my model. Rather it is to show my model could be a potential explanation for the dynamics in the foreign exchange markets of Switzerland and Denmark.

In order to understand the 25-Delta Option Skew, I first clarify a couple definitions including that of Delta, implied volatility and risk reversal. First, Delta is the partial derivative of the value of the option with respect to the value of the underlying asset. In other words, Delta measures the degree to which an option is exposed to shifts in the price of the underlying asset (i.e. stock) or commodity (i.e. futures contract).

Implied volatility is derived from an option's price and shows what the market "implies" about the stock's volatility in the future. It acts as a critical surrogate for option value—specifically, the higher the implied volatility, the higher the option premium.

For a given maturity, the 25-Delta risk reversal is the implied volatility of the 25-Delta call less the implied volatility of the 25-Delta put. The 25-Delta put is the put whose strike has been chosen such that the Delta is -25%. Hence the risk reversal being positive means the underlying asset would have big volatile upward movement and small and less volatile downward movement. The reasoning is the following—risk reversal being positive means the implied volatility of a call option is more than the implied volatility of a put option. And since larger implied volatility of a call option means the option premium for a call option is higher, which means the demand for the option is higher. Further, the reason for the increased call option is that market thinks the underlying asset would have big volatile upward movement and small and less volatile downward movement.

One-Month 25-Delta EURCHF Risk Reversal reflects the difference between the implied volatility on the 25-delta One-Month Euro call and the implied volatility of the One-Month Euro put against Swiss Franc. A positive risk reversal means the market expects the underlying asset (Swiss Franc per Euro) to have big volatile upward movement and small and less volatile downward movement, which indicates the depreciation of Swiss Franc. A negative 25-Delta risk reversal would mean that the market expects the underlying asset (Swiss Franc per Euro) to have small upward movement and big and more volatile downward movement. In terms of currency, more volatile downward movement means the appreciation of Swiss Franc.

Almost all major events that potentially impact the price of a currency are reflected in its risk reversal market. For example, the following graph shows the Euro-Swiss One-Month Risk Reversal data between 2007 and 2016. In September 2011 after the peg was established, the market clearly expected Swiss Franc to depreciate and in December 2014, the gold referendum again is reflected as a decrease in risk reversal, signaling that the market expected an increase in probability for Switzerland to unpeg. Then comes the Swiss unpegging on January 15, 2015 that caused Swiss Franc to appreciate 30% within 13 minutes.

5.1 Switzeralnd Case Study

My theoretical model predicts that before $\underline{\mathrm{T}}$, which is March 2013 for Switzerland, there would be a negative correlation between Foreign Reserve-GDP ratio and Euro-Swiss Franc One-Month 25-Delta Risk Reversal data. After March 2013, I expect a

Figure 3: Three-Month Euro and Swiss Franc 25-Delta Option Skew

Source: Bloomberg

more significant negative correlation between the two.

5.1.1 Data

The data I use for foreign reserves is the monthly foreign currency reserves published by SNB. Nominal GDP data come from World Economic Outlook, monthly CPI data from Bloomberg (indicator name: SZCPIYOY:IND) and Euro-Swiss Franc One-Month Risk Reversal data from Bloomberg (indicator name: EURCHF25R1M). Since reserve data are only available monthly, the frequency of my regression data is monthly. Risk reversal data are available every three days, which are averaged on a monthly basis before entering into regression. Nominal GDP data is available annually, and thus to calculate the Foreign Reserve-GDP ratio, I use the same GDP for every month within the year.

5.1.2 Results

Once we restrict out attention to post-March 2013 data, a one percent increase in foreign-reserve-GDP ratio is associated with a -20.4 vol decrease in risk reversal, indicating that increase in reserve is correlated with an increased speculation about the appreciation of Swiss Franc. Although this regression with limited observations only shows the correlation between foreign-reserve-GDP ratio and risk reversal, it

Table 2: Correlation between EUR-CHF Risk Reversal and Swiss Reserve Ratio post March 2013

OLS Regression
Dependent variable: Monthly Averaged EURCHF Risk Reversal

Reserve-GDP Ratio	-20.43***
	(5.996)
GDP Growth	4.195
	(6.376)
CPI	-1.103*
	(0.536)
Constant	14.68***
	(4.155)
Observations	22
R-squared	0.446

Standard errors in parentheses
*** p<0.01, ** p<0.05, * p<0.1

qualitatively agrees with the predictions of my theoretical model.

When we restrict our sample to September 2011 and March 2013, I find statistically significant correlation and negative correlation between reserve-GDP ratio and risk reversal, although the magnitude of the correlation is much smaller than the in the post-March, 2013 data (See Table 3). To be specific, a one percent increase in reserve-GDP ratio is correlated with a -4.0 vol drop in the EURCHF risk reversal (compared to -20.3 for post 2013 March regression). These two sets of regression results confirm the model's prediction that before $\underline{T}$, the market speculation against the peg would not be as strongly correlated with the increase in reserve level than after $\underline{T}$, where $\underline{T}$ represents a time after which the unpegging is imminent.

5.2 Denmark Case Study

Denmark's currency is also pegged to Euro and is often under appreciative pressure. As I will show, Denmark could be an application of my model whose reserve level is still at the pre-$\underline{T}$ level. Overall, Denmark's Foreign Reserve-GDP ratio is at a much lower position compared to Switzerland. Even at the highest point, their Foreign

Table 3: Correlation between EUR-CHF Risk Reversal and Swiss Reserve-GDP Ratio

OLS Regression
Dependent variable: Monthly Averaged EURCHF Risk Reversal

Reserve-GDP Ratio	-3.954**
	(1.782)
Nominal GDP Growth	-10.81
	(11.55)
CPI	4.392***
	(0.824)
Constant	5.207***
	(1.307)
Observations	17
R-squared	0.688

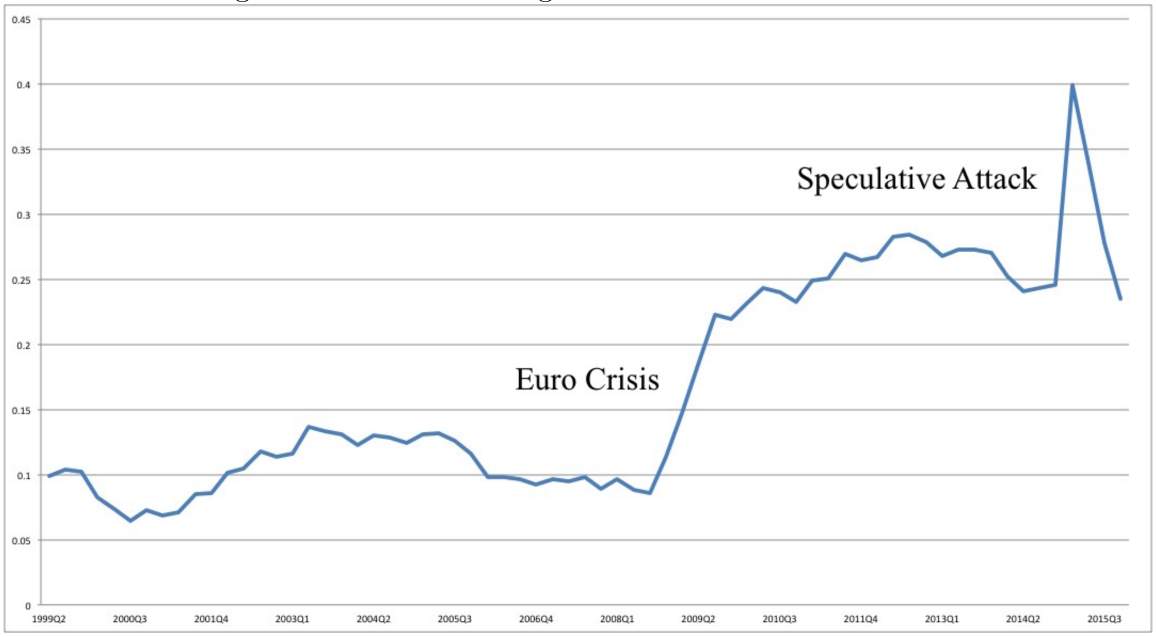
Standard errors in parentheses
*** $p<0.01$, ** $p<0.05$, * $p<0.1$

Reserve-GDP ratio is only 40%, while Switzerland has reached 80% by the end of 2014. The data I use for Denmark foreign reserves is the foreign exchange reserve published by Denmark NationalBank (See Figure 4).

Given the relatively low foreign-reserve-GDP ratio in Denmark compared to Switzerland, my model would predict that Denmark still lies in the pre-$\underline{T}$ range where the risk of unpegging is low. To check this, I plug the two sets of coefficients derived from the two regressions in the Switzerland case (pre-March,2013 and post-March,2013) to Denmark data and find that the pre-March, 2013 data fit Denmar data much better. To be specific, the correlation between the predicted risk reversal values and the actual values using pre-$\underline{T}$ coefficients from Switzerland case is 0.4873, whereas the correlation between the two using the post-$\underline{T}$ coefficients -0.4387. Again although all the statistical analysis presented in this section is only correlations rather than causations, my model at least agrees with the empirics qualitatively.

Next I aim to explain the difference in Denmark and Switzerland's foreign reserves level in the context of my model. In the model, the increase in reserve level when there is no speculative attack depends on GDP growth rate (see Equation (22)). Turning

Figure 4: Denmark foreign reserves as a Share of GDP



Source: World Economic Outlook and Denmarks NationalBank

to the statistics of Switzerland and Denmark, note that from 2012 to 2013, when the reserve difference experienced a major jump, the difference in their real GDP growth also did experience a big jump (1.32% in 2012 and 2.33% in 2013 compared to 0.01% in 2011). The gap between the reserve levels of the two countries also widened in 2010 when the difference in real GDP is 1.52%. From 2010 to 2015, the correlation between the difference in the two countries' real GDP and the difference in reserve-GDP ratio is 0.37. Readers should note that these correlations do not prove the correctness of my model. Rather it verifies again that my model manages to capture this aspect of the correlation between foreign reserve and real GDP growth.

6 Conclusion

The paper explores the problem of whether countries that chronically resist revaluations are ever vulnerable to speculative attacks the way countries that resist devaluations are. It sheds light on the timing and nature of Switzerland's unpegging decision on January 15, 2015 and whether they are consistent with a rational, optimizing policy maker. It could also provide one explanation as to why Denmark, whose currency is also pegged to Euro, did not experience the speculative pressure that Switzerland went through during the same period of time. Looking forward, a modified version of this model can also be applied to understand the exchange rate dynamics of China and Saudi Arabia.

My model captures a forward-looking policy maker that evaluates the costs of abandoning versus maintaining the peg every period. Excessive reserves impose a cost for the Central Bank due to their low return and the potential risk they expose the country to when a large portion of the foreign reserves is in a single currency. My model captures the short-term output shock following the change in exchange rate regime, which is caused by the normal rigidities in prices. Observing the policy maker's objective function, the traders in the market would keep purchasing domestic currency until the policy maker is indifferent between pegging and unpegging. Otherwise there would be an arbitrage opportunity for the speculators. Since the policy maker is indifferent in each period, it is essentially pursuing a randomized strategy in equilibrium. So instead of a sudden speculative attack, there is a prolonged period of increased probability for the policy maker to abandon the peg accompanied by increased speculation in the market.

The basic intuition for the paper is as follows. There is a point $\underline{\mathrm{T}}$ before which the policy maker would not abandon the fixed exchange rate regime because the reserve level is not high enough. And there is a period $\bar{T}$ where the policy maker would abandon the peg for sure due to excessive accumulation of reserves. Before $\underline{\mathrm{T}}$ reserves are low enough so that if speculators attacked and drove reserves up to its upper reserve threshold, forcing a move to a floating exchange rate, they would increase the domestic money supply to a point where the floating rate would exceed the fixed rate, which is a loss to the speculators; hence they do not attack. The domestic interest rate is therefore equal to the world rate. After $\underline{\mathrm{T}}$ both speculators and the policy maker realize that $\bar{T}$ is approaching, the reserve level is high enough that the unpegging would be imminent. The increased amount of domestic currency in the hands of the speculators drives down the domestic interest rate and forward exchange rate premium while driving up the shadow floating exchange rate. Therefore the opportunity cost of holding foreign currency is decreasing, and the benefit to successful speculation is declining. But the probability that speculation will be successful is increasing rapidly, ensuring that the expected return from holding domestic currency remains constant at zero. However, in the event of a move to floating exchange rates, while there is no sudden increase in Central Bank reserves, the ex-post profits from holding domestic currency can be substantial, since the shadow floating exchange rate is below the fixed rate until $\bar{T}$.

The model also intuits how the government's probability for abandoning the peg changes with different parameters. Increased output growth results in increased market expectation of the unpegging, which in turn leads to higher reserves because speculators would purchase more domestic currency to earn profit when it appreciates. The more the output shock is correlated with the magnitude of change in exchange rate upon unpegging, the less the market expects the policy maker to unpeg, which in turn results in less accumulation of reserves during the speculation period. This is intuitive because the greater output shock increases with exchange rate jump upon unpegging, the more cost that the policy maker incurs, the less likely they would abandon the peg.

I then confirm the predictions of the model using data from Switzerland and Denmark. For Switzerland, I consider March 2013 to be the point $\underline{T}$ in the model because the gold referendum proposal reflected the public's concern about the high foreign reserves level. Running regressions for pre-March,2013 and post-March,2013 data shows that my model is able to qualitatively capture the relationship between risk reversal data and reserve level. To be specific, in the pre-March, 2013 period, the speculation against the Swiss peg increased as the reserve-GDP ratio increased. But the magnitude is much smaller than the post-March, 2013 period.

Plugging in the two sets of coefficients (pre-March, 2013 and post-March, 2013) derived in the Switzerland's case to Denmark data, I find that Denmark data are a better match with the set of parameters that belong to pre-March, 2013 period. This illustratively shows that Denmark might still be in the pre-$\underline{T}$ period in the context of my model.

This paper is only a first step towards understanding the dynamics of the change in exchange rate regime that is under appreciative pressure. Moving forward, an interesting extension to the model would be the introduction of risk premium to capture scenarios such as spillover effects. One example is that Denmark experienced appreciative speculation attack against its peg following Switzerland's unpegging decision, which caused speculators to lose confidence in safe haven currencies. A shock to the risk premium would be able to model this loss of confidence. Other directions to pursue includes further research on the existence of the upper reserve threshold, $\bar{r}$. The literature would also benefit from further exploration of Central Banks' reserve portfolio choice; specifically, for a country like Switzerland, why does it choose to hold a large portion of foreign reserves in Euro instead of diversifying its portfolio.

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Appendix A: Derivations

1. Section 1.3, Model with a Myopic Policy Maker, Solving for shadow exchange rate

   Here we are asking the question of what would the shadow exchange rate be if the peg is abandoned in period t and speculators only look out to period Δ beyond the current period (rather than 1 period ahead as in discrete time models). First set money demand to equal money supply

$$\bar{r} = \tilde{e}_t - \eta \Delta i_t + \phi y_t \tag{53}$$

   Substitute in the logs of PPP and UIP, then we get

$$\bar{r} = \tilde{e}_t - \eta \left[\Delta i^* + E_t \tilde{e}_{t+\Delta} - \tilde{e}_t\right] + \phi y_t \tag{54}$$

   or after rearranging terms,

$$(1+\eta)\tilde{e}_t = \bar{r} + \eta \Delta i^* + \eta E_t \tilde{e}_{t+\Delta} - \phi y_t \tag{55}$$

   We also know that $m^s = \bar{r}$ and so

$$\dot{m}_t = 0 \tag{56}$$

   Taking the time derivative of money demand function, $m^d = e_t - \eta\left[i^* + \dot{e}_t\right] + \phi y_t$ yields,

$$\dot{m}_t = \dot{e}_t - E_t \ddot{e}_t + \phi \dot{y}_t \tag{57}$$

If we assume $E_t\ddot{e}_t = 0$, then $\dot{e}_t = -\phi\dot{y}_t$. The left hand side equal 0. Hence

$$E_t\left(\tilde{e}_{t+\Delta}\right) = \tilde{e}_t - \phi\left(y_{t+\Delta} - y_t\right) \tag{58}$$

Defining $\bar{T}$ as the date of the speculative attack in this traditional version of the model. In the period right after the unpegging, by Equation (5), $y_{\bar{T}+\Delta} = y_{\bar{T}} + \Delta\theta + \Delta\alpha\left[E_{\bar{T}}\left(\tilde{e}_{\bar{T}+\Delta}\right) - \tilde{e}_{\bar{T}}\right]$

$$E_T\left(\tilde{e}_{\bar{T}+\Delta}\right) - \tilde{e}_{\bar{T}} = -\Delta\phi\left[\theta + \alpha\left(E_{\bar{T}}\left(\tilde{e}_{\bar{T}+\Delta}\right) - \tilde{e}_{\bar{T}}\right)\right] \tag{59}$$

Solving for $E_t\left(\tilde{e}_{T+\Delta}\right) - \tilde{e_T}$ yields

$$E_t\left(\tilde{e}_{\bar{T}+\Delta}\right) - \tilde{e}_{\bar{T}} = -\frac{\Delta\phi\theta}{1+\Delta\phi\alpha} \tag{60}$$

Let $\kappa = \frac{\Delta\theta}{1+\Delta\phi\alpha}$, then shadow exchange rate is

$$\tilde{e}_{\bar{T}} = \bar{r} + \eta\Delta i^* - \eta\phi\kappa - \phi y_{\bar{T}} \tag{61}$$

After period $\bar{T}$, the growth rate goes back $\Delta\theta$ every period. In other words, $y_{t+\Delta} = y_t + \Delta\theta$ for $t > \bar{T}+1$. Solving for the shadow exchange rate in the same way by setting money supply ($\bar{r}$) equaling money demand, we get the same Equation (54) and Equation (58). Following Equation (58),

$$E_t\left(\tilde{e}_{t+\Delta}\right) - \tilde{e}_t = -\Delta\phi\theta, \qquad t \geqslant \bar{T}+1 \tag{62}$$

Plug Equation (62) back to Equation (54), we get

$$\bar{r} = \tilde{e}_t - \eta\left[\Delta i^* - \Delta\phi\theta\right] + \phi y_t, \qquad t \geqslant \bar{T}+1 \tag{63}$$

$$\tilde{e}_t = \bar{r} + \eta\Delta i^* - \eta\Delta\phi\theta - \phi y_t, \qquad t \geqslant \bar{T} + 1 \tag{64}$$

2. Section 2.2, the policy maker's problem, deriving the lower bound on τ.

   Since the policy maker is pursuing a fixed exchange rate policy, it must find it optimal to do so. This places a lower bound on the value of the parameter τ, the policy maker's preference for fixed exchange rates. In other words, the policy maker must find the cost of abandoning the peg next period smaller than the cost of abandoning the peg in this period, when there is no speculative attack. Mathematically, it means the following

$$\begin{aligned} \frac{\Delta\left(r_t + \tau\right)}{1 - \delta^{\Delta}} + E_t \sum_{k\Delta = t}^{\infty} \delta^{k\Delta - t} \Delta\left(y^* - y_{t,t+k\Delta}\right) \quad &\geqslant \quad \Delta r_t + \Delta\left(y^* - y_t\right) \\ &+\delta^{\Delta}\left[\frac{\Delta\left(r_{t+\Delta} + \tau\right)}{1 - \delta^{\Delta}} + E_t \sum_{k\Delta = t+\Delta}^{\infty} \Delta\delta^{(k-1)\Delta - t}\left(y^* - y_{t+\Delta,t+k\Delta}\right)\right] \end{aligned} \tag{65}$$

   where the left-hand side is the cost of abandoning the fixed rate this period, and the right hand-side is abandoning the peg next period.

   Simplifying Inequality (65) as follows:

$$
\begin{aligned}
\frac{\Delta\left(r_t+\tau\right)}{1-\delta^{\Delta}}+E_t\sum_{k=t}^{\infty}\delta^{k\Delta-t}\Delta\left(y^*-y_{t,t+k\Delta}\right) &\geqslant \Delta r_t+\Delta\left(y^*-y_t\right)+\delta^{\Delta}\frac{\Delta\left(r_{t+\Delta}+\tau\right)}{1-\delta^{\Delta}} \\
&\quad +\delta^{\Delta}E_t\Delta\sum_{k=t+\Delta}^{\infty}\delta^{(k-1)\Delta-t}\Delta\left(y^*-y_{t+\Delta,t+k\Delta}\right) \\
\frac{\Delta r_t}{1-\delta^{\Delta}}-\Delta r_t-\delta^{\Delta}\frac{\Delta r_{t+\Delta}}{1-\delta^{\Delta}}+\frac{\Delta\tau}{1-\delta^{\Delta}}-\delta^{\Delta}\frac{\tau}{1-\delta^{\Delta}} &\geqslant \Delta\left(y^*-y_t\right)-E_t\sum_{k=t}^{\infty}\delta^{k\Delta-t}\Delta\left(y^*-y_{t,t+k\Delta}\right) \\
&\quad +\delta^{\Delta}\left[E_t\sum_{k=t+\Delta}^{\infty}\delta^{(k-1)\Delta-t}\Delta\left(y^*-y_{t+\Delta,t+k\Delta}\right)\right] \\
\frac{\delta^{\Delta}\Delta\left(r_t-r_{t+\Delta}\right)}{1-\delta^{\Delta}}+\frac{1-\delta^{\Delta}}{1-\delta^{\Delta}}\Delta\tau &\geqslant \Delta\left(y^*-y_t\right)-E_t\sum_{k=t}^{\infty}\delta^{k\Delta-t}\left(y^*-y_{t,t+k\Delta}\right) \\
&\quad +\delta^{\Delta}\left[E_t\sum_{k=t+\Delta}^{\infty}\delta^{(k-1)\Delta-t}\Delta\left(y^*-y_{t+\Delta,t+k\Delta}\right)\right] \\
\frac{\delta^{\Delta}\Delta\left(r_t-r_{t+\Delta}\right)}{1-\delta^{\Delta}}+\frac{1-\delta^{\Delta}}{1-\delta^{\Delta}}\Delta\tau &\geqslant \Delta\left(y^*-y_t\right)-E_t\sum_{k=t}^{\infty}\delta^{k\Delta-t}\Delta\left(y^*-y_{t,t+k\Delta}\right) \\
&\quad +\delta^{\Delta}\left[E_t\sum_{k=t+\Delta}^{\infty}\delta^{(k-1)\Delta-t}\Delta\left(y^*-y_{t+\Delta,t+k\Delta}\right)\right] \\
\frac{\delta^{\Delta}\Delta}{1-\delta^{\Delta}}\left(r_t-r_{t+\Delta}\right)+\Delta\tau &\geqslant E_t\sum_{k=t}^{\infty}\delta^{k\Delta-t}\Delta\left[\left(y^*-y_{t+\Delta,t+k\Delta}\right)-\left(y^*-y_{t,t+k\Delta}\right)\right] \\
\frac{\delta^{\Delta}\Delta}{1-\delta^{\Delta}}\left(r_t-r_{t+\Delta}\right)+\Delta\tau &\geqslant E_t\sum_{k=t}^{\infty}\delta^{k\Delta-t}\Delta\left(y_{t,t+k\Delta}-y_{t+\Delta,t+k\Delta}\right)
\end{aligned}
$$

3. Section 2.2, The Policy-Maker's Problem, adding in the discount rate and calculate $E_t\sum_{k=t}^{\infty}\delta^{k\Delta-t}\Delta\left(y_{t,t+k\Delta}-y_{t+\Delta,t+k\Delta}\right)$

$$
\begin{aligned}
E_t\sum_{k=t}^{\infty}\delta^{k\Delta-t}\Delta\left(y_{t,t+k\Delta}-y_{t+\Delta,t+k\Delta}\right) &= \left(y_t-y_t\right)+\delta^{\Delta}E_t\left(y_{t,t+\Delta}\text{-}y_{t+\Delta,t+\Delta}\right)+\delta^{2\Delta}E_t\left(y_{t+\Delta,t+2\Delta}\text{-}y_{t,t+2\Delta}\right)+\cdots \\
&= 0-\delta^{\Delta}\Delta\alpha\phi\kappa-\delta^{2\Delta}\Delta\alpha\phi\kappa-\delta^{3\Delta}\Delta\alpha\phi\kappa \\
&= -\left(\delta^{\Delta}+\delta^{2\Delta}+\delta^{3\Delta}+\cdots\delta^{N\Delta}\right)\Delta\alpha\phi\kappa \qquad (66) \\
&= -\frac{\delta^{\Delta}\left(1-\delta^{N\Delta}\right)}{1-\delta^{\Delta}}\Delta\alpha\phi\kappa \qquad (67)
\end{aligned}
$$

Equation (65) thus turns into

$$
\begin{aligned}
\frac{\delta^{\Delta}\Delta}{1-\delta^{\Delta}}\left(r_t - r_{t+\Delta}\right) + \Delta\tau &\geqslant E_t \sum_{k=t}^{\infty} \delta^{k\Delta - t}\Delta\left(y_{t,t+k\Delta} - y_{t+\Delta,t+k\Delta}\right) \\
\frac{\delta^{\Delta}}{1-\delta^{\Delta}}\left(r_t - r_{t+\Delta}\right) + \tau &\geqslant -\frac{\delta^{\Delta}\left(1-\delta^{N\Delta}\right)}{1-\delta^{\Delta}}\alpha\phi\kappa \\
\tau &\geqslant \left(r_{t+\Delta} - r_t\right) - \frac{\delta^{\Delta}\left(1-\delta^{N\Delta}\right)}{1-\delta^{\Delta}}\alpha\phi\kappa \\
\frac{1-\delta^{\Delta}}{\delta^{\Delta}}\tau &\geqslant \left(r_{t+\Delta} - r_t\right) - \left(1-\delta^{N\Delta}\right)\alpha\phi\kappa
\end{aligned} \tag{68}
$$

4. Shadow Exchange Rate with Optimizing Policy Maker

First I calculate the shadow exchange rate between T and $T+1$. Here the shadow exchange rate $\tilde{e}_{T,T+n\Delta}$, where $n \in [0, N]$ (where $N\Delta = 1$), denotes the exchange rate that would prevail at time $T + \Delta$ when the peg is abandoned at T and the speculators look out to period Δ ahead.

$$
r_T = \tilde{e}_{T,T+n\Delta} - \eta\Delta i_{T+n\Delta} + \phi y_{T+n\Delta} \tag{69}
$$

Substitute in the logs of PPP and UIP, then we get

$$
r_T = \tilde{e}_{T,T+n\Delta} - \eta\left[\Delta i^* + E_{T+n\Delta}\tilde{e}_{T,T+(n+1)\Delta} - \tilde{e}_{T,T+n\Delta}\right] + \phi y_{T+n\Delta} \tag{70}
$$

or after rearranging terms,

$$
(1+\eta)\tilde{e}_{T,T+n\Delta} = r_T + \eta i^* + \eta E_{T+n\Delta}\tilde{e}_{T,T+(n+1)\Delta} - \phi y_{T+n\Delta} \tag{71}
$$

We also know that $m^s = \bar{r}$ and so

$$
\dot{m_T} = 0 \tag{72}
$$

Taking the time derivative of money demand function, $m^d = e_T - \eta\left[\Delta i^* +\right.$

$\dot{e}_{T+n\Delta}] + \phi y_{T+n\Delta}$ yields,

$$\dot{m}_T = \dot{e}_{T+n\Delta} - E_t \ddot{e}_{T+n\Delta} + \phi \dot{y}_{T+n\Delta} \tag{73}$$

If we assume $E_T \ddot{e_{T,T}} = 0$, then $\dot{e}_{T,T} = -\phi \dot{y}_T$. The left hand side equal 0. Hence

$$E_T\left(\tilde{e}_{T,T+n\Delta}\right) = \tilde{e}_{T,T} - \phi\left(y_{T+n\Delta} - y_T\right) \tag{74}$$

By Equation (5), $y = y_T + \theta + \alpha\left[E_T\left(\tilde{e}_{T+1}\right) - \tilde{e}_T\right]$

$$E_T\left(\tilde{e}_{T,T+(n+1)\Delta}\right) - \tilde{e}_{T,T+n\Delta} = -\Delta\phi\left[\theta + \alpha\left(E_T\left(\tilde{e}_{T,T+(n+1)\Delta}\right) - \tilde{e}_{T,T+n\Delta}\right)\right] \quad (n+1)\,\Delta < 1 \tag{75}$$

Solving for $E_t\left(\tilde{e}_{T+1}\right) - \tilde{e_T}$ yields

$$E_t\left(\tilde{e}_{T,T+(n+1)\Delta}\right) - \tilde{e}_{T,T+n\Delta} = -\frac{\Delta\phi\theta}{1+\Delta\phi\alpha} \tag{76}$$

Let $\kappa = \frac{\Delta\theta}{1+\Delta\phi\alpha}$, then the change in expected shadow exchange rate

$$\tilde{e}_{T,T+n\Delta} = r_T + \eta\Delta i^* - \eta\phi\kappa - \phi y_{T,T+n\Delta} \qquad n < N \tag{77}$$

where $N\Delta = 1$. Otherwise when $t \geqslant T+1$, output growth returns to be $\Delta\theta$ per period. Again setting money supply equal money demand,

$$r_T = \tilde{e}_{T,t} - \eta\left[i^* + E_t\tilde{e}_{T,t+\Delta} - \tilde{e}_{T,t}\right] + \phi y_t \tag{78}$$

Assuming that $E_t \ddot{e_{T,t}} = 0$,

$$E_T\left(\tilde{e}_{T,t+\Delta}\right) = \tilde{e}_{T,t} - \phi\left(y_{t+\Delta} - y_t\right) \tag{79}$$

After $T+1$, $(y_{t+1} - y_t) = \phi\Delta\theta$. Hence

$$r_T = \tilde{e}_{T,t} - \eta\left[\Delta i^* - \phi\Delta\theta\right] + \phi y_t \tag{80}$$

$$\tilde{e}_{T,t} = r_T + \eta\Delta i^* - \eta\phi\Delta\theta - \phi y_t, \quad t \geqslant T+1 \tag{81}$$

5. Derivation of Proposition 1

Speculators are still governed by Uncovered Interest Parity. If we are in the fixed regime in the current period, UIP becomes

$$\Delta i_t = \Delta i^* + q_t(\tilde{e}_{t,t+\Delta} - \bar{e}) \tag{82}$$

where $\tilde{e}_{t,t+\Delta}$ denotes the shadow exchange rate in the next period assuming the peg is abandoned in this period. Since $\Delta < 1$, we use the first equation for shadow exchange rate (Equation (81)), $E_t\tilde{e}_{t,t+\Delta} = r_t + \eta\Delta i^* - \eta\Delta\phi\kappa - \phi y_{t+\Delta}$. Combining it with Equations (1), (3), (4), (5), we get

$$r_t = \bar{e} - \eta\left[\Delta i^* + q_t(\tilde{e}_{t,t+\Delta} - \bar{e})\right] + \phi y_t \tag{83}$$

$$r_t = \bar{e} - \eta\left[\Delta i^* + q_t(r_t + \eta\Delta i^* - \eta\Delta\phi\kappa - \phi y_{t+\Delta} - \bar{e})\right] + \phi y_t \tag{84}$$

$$r_t = \bar{e} - \eta\Delta i^* - \eta q_t(r_t + \eta\Delta i^* - \eta\phi\Delta\kappa - \phi y_{t+\Delta} - \bar{e}) + \phi y_t \tag{85}$$

Rearranging the terms above, we have

$$(1+\eta q_t)\, r_t = (1+\eta q_t)\,\bar{e} - (1+\eta q_t)\,\eta\Delta i^* + \eta^2 q_t\phi\kappa + \eta q_t\phi y_{t+\Delta} + \phi y_t \tag{86}$$

The expected output in this case would be

$$E_t y_{t+\Delta} = y_t + \Delta \left[\theta + \alpha \left(E_t \tilde{e}_{t,t+\Delta} - e_t\right)\right] \tag{87}$$

As calculated in Equation (126), $E_t \left(\tilde{e}_{t,t+\Delta}\right) - \tilde{e}_{t,t} = -\frac{\Delta\phi\theta}{1+\Delta\phi\alpha} = -\Delta\phi\kappa.$

$$E_t y_{t+\Delta} = y_t + \Delta \left(\theta - \alpha\Delta\phi\kappa\right) \tag{88}$$

Equation (86) thus becomes

$$(1+\eta q_t) r_t = (1+\eta q_t) \bar{e} - (1+\eta q_t) \eta i^* + \eta^2 q_t \phi\kappa + \eta q_t \phi \left(y_t + \Delta\left(\theta - \alpha\Delta\phi\kappa\right)\right) + \phi y_t \tag{89}$$

Plug in $\kappa = \frac{\Delta\theta}{1+\Delta\phi\alpha}$,

$$\begin{aligned}
(1+\eta q_t) r_t &= (1+\eta q_t) \bar{e} - (1+\eta q_t) \eta i^* + (1+\eta q_t) \phi y_t + \Delta\eta q_t \phi\theta \left(\frac{\eta}{1+\Delta\phi\alpha} + 1 - \frac{\Delta\phi\alpha}{1+\phi\alpha\Delta}\right) \\
(1+\eta q_t) r_t &= (1+\eta q_t) \bar{e} - (1+\eta q_t) \eta i^* + (1+\eta q_t) \phi y_t + \Delta\eta q_t \phi\theta \left(\frac{\eta}{1+\Delta\phi\alpha} + 1 - \frac{\Delta\phi\alpha}{1+\phi\alpha\Delta}\right)
\end{aligned}$$

Solving for r_t yields

$$r_t = \bar{e} - \eta\Delta i^* + \phi y_t + \frac{q_t}{1+\eta q_t} \eta\phi\Delta\theta \left[\frac{1+\eta}{1+\phi\alpha\Delta}\right] \tag{90}$$

The derivative of r_t with respect to q_t is

$$\frac{\partial r_t}{\partial q_t} = \left[\frac{1}{1+\eta q_t} - \frac{q_t \eta}{(1+\eta q_t)^2}\right] \left[\frac{1+\eta}{1+\alpha\phi\Delta}\right] \eta\phi\Delta\theta > 0 \tag{91}$$

Since $q_t, \eta < 1$ and $\frac{1}{1+\eta q_t} > \frac{1}{(1+\eta q_t)^2}$, we have $\frac{\partial r_t}{\partial q_t} > 0$, which completes the proof of **Proposition 1**. $\square$

6. Proof of Proposition 2

Proposition 2. If $q < 1$ then $\tilde{e}_{t,t} < \bar{e}$. If $q_t = 1$, then $\tilde{e}_{t,t} = \bar{e}$.

Intuitively, if speculators are certain that the exchange rate will be abandoned they will continue to buy domestic currency as long as the shadow floating exchange rate is smaller than the fixed rate. These domestic currency purchases increase the money supply and hence decrease the shadow floating exchange rate until it equals the fixed rate. Put another way, the shadow floating exchange rate will always be at most as high as the fixed rate since otherwise speculators would find it profitable to buy foreign currency, thereby decreasing the money supply and lowering the shadow floating exchange rate.

Proof

Setting $q_t = 1$ in Equation (92), we get

$$\begin{aligned} r_t &= \bar{e} - \eta\Delta i^* + \phi y_t + \frac{\eta\phi}{1+\eta}\left[\frac{1+\eta}{1+\phi\alpha\Delta}\right]\eta\phi\Delta\theta && (92) \\ &= \bar{e} - \eta\Delta i^* + \phi y_t + \frac{\Delta\eta\phi\theta}{1+\phi\alpha\Delta} && (93) \end{aligned}$$

$$\begin{aligned} \bar{e} &= r_t + \eta i^* - \phi y_t - \frac{\Delta\eta\phi\theta}{1+\phi\alpha\Delta} && (94) \\ \bar{e} &= r_t + \eta i^* - \phi y_t - \eta\phi\kappa && (95) \end{aligned}$$

The shadow exchange rate at the time when the peg is first abandoned is given in (96),

$$\tilde{e}_{T,T+n\Delta} = r_T + \eta\Delta i^* - \eta\phi\kappa - \phi y_{T,T+n\Delta} \qquad (96)$$

Here we are interested in the shadow rate at time t when the peg is abandoned at time t. Hence set $T = t$ and set $n = 0$, we get

$$\tilde{e}_{t,t} = r_t + \eta\Delta i^* - \eta\phi\kappa - \phi y_t \tag{97}$$

Hence $\bar{e} = \tilde{e}_{t,t}$ when $q_t = 1$.

To prove the first part, note that the shadow rate is decreasing in the level of reserves and the level of reserves is increasing in q_t by Proposition 1. Hence given that $\tilde{e}_{t,t} = \bar{e}$ when $q_t = 1$, we have that $\tilde{e}_{t,t} < \bar{e}$ when $q_t < 1$. □

7. Proof of Proposition 3

Proposition 3. If $q_{t-\Delta} = 0$ and $q_t = 1$ for any $t \leqslant \bar{T}$, where $\bar{T}$ is the period when the speculative attack takes place in the myopic case, then $r_t - r_{t-\Delta} = \eta\phi\kappa + \phi\Delta\theta$.

Proof

Setting $q_t = 1$ and $q_{t-\Delta} = 0$ in Equation (92), $r_t = \bar{e} - \eta\Delta i^* + \phi y_t + \frac{q_t}{1+\eta q_t}\eta\phi\Delta\theta\left[\frac{1+\eta}{1+\phi\alpha\Delta}\right]$. we have

$$r_{t-\Delta} = \bar{e} - \eta\Delta i^* + \phi y_{t-\Delta} \tag{98}$$

$$r_t = \bar{e} - \eta\Delta i^* + \phi y_t + \eta\phi\kappa \tag{99}$$

Taking the difference,

$$\begin{aligned} r_t - r_{t-\Delta} &= (\bar{e} - \eta\Delta i^* + \phi y_t + \eta\phi\kappa) - (\bar{e} - \eta\Delta i^* + \phi y_{t-\Delta}) \\ &= \eta\phi\kappa + \phi\left(y_t - y_{t-\Delta}\right) \end{aligned}$$

Since the peg is only abandoned at time t, the exchange rate before time t remains at $\bar{e}$. Hence $\phi\left(y_t - y_{t-\Delta}\right) = \phi\Delta\theta$.

$$r_t - r_{t-\Delta} \quad = \quad \eta\phi\kappa + \phi\Delta\theta$$

$\square$

8. Proof of Proposition 4

Proposition 4. In equilibrium $q_t < 1$ for all $t < \bar{T}$.

Proof

Consider the case where $q_t = 1$ while $t < \bar{T}$ for the sake of contradiction. This assumption means that the Central Bank must find it optimal to abandon the fixed exchange rate regime, which means it must find the cost of abandoning the peg this period would be smaller than the cost of abandoning the peg next period. Their objective function implies that

$$\begin{aligned}\frac{\Delta\left(r_t + \tau\right)}{1-\delta^{\Delta}} + E_t \sum_{k\Delta=t}^{\infty} \delta^{k\Delta - t}\Delta\left(y^* - y_{t,t+k\Delta}\right) \quad &\leqslant \quad \Delta r_t + \Delta\left(y^* - y_t\right) + \delta^{\Delta}\frac{\Delta\left(E_t r_{t+\Delta} + \tau\right)}{1-\delta^{\Delta}} \\ &+\delta^{\Delta} E_t \sum_{k\Delta=t+\Delta}^{\infty} \Delta\delta^{(k-1)\Delta - t}\left(y^* - y_{t+\Delta,t+k\Delta}\right)\end{aligned}$$

where the left-hand side is the cost of abandoning the fixed rate this period, and the right hand-side is abandoning the peg next period. By Equation (66), $\sum_{k=t}^{\infty}\delta^{k-t}\left(y^* - y_{t+1,k}\right) - \sum_{k=t}^{\infty}\delta^{k-t}\left(y^* - y_{t,k}\right) = -\frac{\delta^{\Delta}\left(1-\delta^{N\Delta}\right)}{1-\delta^{\Delta}}\Delta\alpha\phi\kappa$. So the previous inequality can be simplified to

$$\frac{1-\delta^{\Delta}}{\delta^{\Delta}}\tau \quad \leqslant \quad \left(E_t r_{t+\Delta} - r_t\right) - \left(1-\delta^{N\Delta}\right)\alpha\phi\kappa \tag{100}$$

Now consider the speculators' optimal choice. Since $q_t = 1$, Proposition 2 implies that they purchase domestic currency to a point where $\tilde{e}_{t,t} = \bar{e}$. From Equation (92), we know the reserve level is

$$r_t = \bar{e} - \eta \Delta i^* + \phi y_t + \eta \phi \kappa \tag{101}$$

The speculative attack in $t+\Delta$ would only continue if the shadow rate is at least as low as the fixed rate, $\tilde{e}_{t+\Delta,t+\Delta} \leqslant \bar{e}$, an inequality since $q_{t+\Delta}$ might be smaller than or equal to 1. This implies, by the fact that the level of foreign reserves increases in shadow exchange rate. And we are given that when $\tilde{e}_{t+\Delta,t+\Delta} = \bar{e}$, $E_t r_{t+\Delta} = \bar{e} - \ \eta \, i^* \ + \eta \phi \kappa + \ \phi y_{t+\Delta}$. So when $\tilde{e}_{t+\Delta,t+\Delta} \leqslant \bar{e}$,

$$E_t r_{t+\Delta} \leqslant \bar{e} - \eta \Delta i^* + \eta \phi \kappa + \phi E_t y_{t+\Delta} \tag{102}$$

Hence (102) -(101) yields

$$E_t r_{t+\Delta} - r_t \leqslant \phi (y_{t+\Delta} - y_t) \tag{103}$$

In our derivation of $E_t r_{t+\Delta}$, we are assuming that the peg is not abandoned until time $t+\Delta$, hence the output $y_{t,t+\Delta}$ is not affected by exchange rate yet. Therefore

$$\begin{aligned} E_t r_{t+\Delta} - r_t &\leqslant \phi (y_{t+\Delta} - y_t) && (104) \\ &= \phi \Delta \theta && (105) \end{aligned}$$

Substitute this into Equation (100),

$$\phi\Delta\theta - \left(1 - \delta^{N\Delta}\right)\alpha\phi\kappa \geqslant \frac{1-\delta^{\Delta}}{\delta^{\Delta}}\tau \tag{106}$$

But by **Assumption 1** in Equation (23), $\phi\Delta\theta - \left(1 - \delta^{N\Delta}\right)\alpha\phi\kappa \leqslant \frac{1-\delta^{\Delta}}{\delta^{\Delta}}\tau$. $\Rightarrow\Leftarrow$

This completes the proof of Proposition 4 that the Central Bank will not pursue $q_t = 1$ in any period before $\bar{T}$. $\square$

9. Section 3, Backward induction to obtain r_t in terms of $\bar{r}$

The subgame-perfect Nash equilibrium can be constructed by backward induction. The argument made in the myopic case implies that in any equilibrium $q_{\bar{T}} = 1$. Given this, it is possible to examine the policy maker's optimal strategy in $\bar{T} - \Delta$. And from this it is possible to examine the policy maker's optimal strategy in $\bar{T} - 2\Delta$, and so on.

Consider a time $t < \bar{T}$ where $q_{t+1} > 0$. That is, in the coming period there will be a positive probability that the policy maker will abandon the fixed exchange rate. The policy maker must find this optimal in period $t + 1$, which implies that it either strictly prefers to abandon the fixed rate regime in $t + 1$ or it is indifferent between abandoning and maintaining the peg. In either case from the their objective function, Equation (19), the maximized expected present value of its cost function is given by $\frac{\Delta(r_{t+1}+\tau)}{1-\delta^{\Delta}} + E_t \sum_{k\Delta=t}^{\infty} \delta^{k\Delta-t}\Delta\left(y^* - y_{t,t+k\Delta}\right)$.

From Proposition 4, $q_t = 1$ when $t < \bar{T}$ is not possible. Suppose for a moment that $q_t \in (0, 1)$, which means that the policy maker is indifferent between maintaining and abandoning the fixed exchange rate regime. From Equation (19), hence

$$
\begin{aligned}
\frac{\Delta\left(r_t+\tau\right)}{1-\delta^{\Delta}}+E_t \sum_{k\Delta=t}^{\infty} \delta^{k\Delta-t} \Delta\left(y^*-y_{t,t+k\Delta}\right) &= \Delta r_t+\Delta\left(y^*-y_t\right) \\
&\quad +\delta^{\Delta}\left[\frac{\Delta\left(r_{t+\Delta}+\tau\right)}{1-\delta^{\Delta}}+E_t \Delta \sum_{k\Delta=t+\Delta}^{\infty} \delta^{(k-1)\Delta-t}\left(y^*-y_{t+\Delta,t+k\Delta}\right)\right] \\
\frac{\delta^{\Delta}\Delta}{1-\delta^{\Delta}}\left(r_t-r_{t+\Delta}\right)+\Delta\tau &= E_t \sum_{k=t}^{\infty} \delta^{k\Delta-t} \Delta\left(y_{t,t+k\Delta}-y_{t+\Delta,t+k\Delta}\right)
\end{aligned}
$$

According to Equation (66), $E_t \sum_{k=t}^{\infty} \delta^{k\Delta-t} \Delta\left(y_{t,t+k\Delta}-y_{t+\Delta,t+k\Delta}\right)=\frac{\delta^{\Delta}\left(1-\delta^{N\Delta}\right)}{1-\delta^{\Delta}} \Delta\alpha\phi\kappa$. Hence the Equation above reduces to

$$
\frac{1-\delta^{\Delta}}{\delta^{\Delta}}\tau = \left(r_{t+\Delta}-r_t\right)-\left(1-\delta^{N\Delta}\right)\alpha\phi\kappa
$$

and the Central Bank log reserve value must equal to

$$
r_t=r_{t+\Delta}-\left[\frac{1-\delta^{\Delta}}{\delta^{\Delta}}\tau+\left(1-\delta^{N\Delta}\right)\alpha\phi\kappa\right] \tag{107}
$$

Working backwards from time $\bar{T}$, $r_{\bar{T}-\Delta}=r_{\bar{T}}-\left[\frac{1-\delta^{\Delta}}{\delta^{\Delta}}\tau+\left(1-\delta^{N\Delta}\right)\alpha\phi\kappa\right]$. Since $r_{\bar{T}}=\bar{r}$, we have

$$
r_{\bar{T}-\Delta}=\bar{r}-\left[\frac{1-\delta^{\Delta}}{\delta^{\Delta}}\tau+\left(1-\delta^{N\Delta}\right)\alpha\phi\kappa\right] \tag{108}
$$

Then working backward from $\bar{T}-\Delta$, it is possible to establish $\underline{\mathrm{T}}$, the earliest state where abandonment of the fixed exchange rate can be an equilibrium outcome. Iterating (109) yields

$$
\begin{aligned}
r_{\bar{T}-2\Delta} &= \bar{r} - 2\left[\frac{1-\delta^{\Delta}}{\delta^{\Delta}}\tau + \left(1-\delta^{N\Delta}\right)\alpha\phi\kappa\right] && (109)\\
&\vdots && (110)\\
r_t &= \bar{r} - \left(\frac{\bar{T}-t}{\Delta}\right)\left[\frac{1-\delta^{\Delta}}{\delta^{\Delta}}\tau + \left(1-\delta^{N\Delta}\right)\alpha\phi\kappa\right] && (111)
\end{aligned}
$$

10. Deriving $y_{t,t}$ in terms of $y_{\bar{T}}$

During $[\underline{T}, \bar{T}]$, the Central Bank plays a randomized strategy. Since we will be interested in expressing shadow exchange rate $\tilde{e}_{t,t}$ with $y_{\bar{T}}$ and $\bar{r}$, and the expression for shadow exchange rate $\tilde{e}_{t,t}$ involves $y_{t,t}$, as shown in Equation (96). Hence here I derive $y_{t,t}$ in terms of $y_{\bar{T}}$. According to the output table (where $N\Delta = 1$), $y_{t,t} = y_{t,t+\Delta} - N\Delta\theta - N\Delta\alpha\phi\kappa$.

$$
\begin{aligned}
y_{t,t} &= y_{t,t+\Delta} - N\Delta\theta - N\Delta\alpha\phi\kappa && (112)\\
y_{t,t} &= y_{\bar{T}} - \left(\frac{\bar{T}-t-1}{\Delta}\Delta\theta\right) - N\Delta\theta - N\Delta\alpha\phi\kappa && (113)\\
&= y_{\bar{T}} - \left(\bar{T}-t-1\right)\theta - \theta - \alpha\phi\kappa && (114)\\
&= y_{\bar{T}} - \left(\bar{T}-t\right)\theta - \alpha\phi\kappa && (115)
\end{aligned}
$$

11. Deriving $\tilde{e}_{t,t}$ in terms of $\bar{r}$ and $y_{\bar{T}}$.

Substituting (115) into the shadow exchange rate in Equation (96), solving for $\tilde{e}_{t,t}$,

$$
\begin{aligned}
\tilde{e}_{t,t} &= r_t + \eta\Delta i^* - \eta\phi\kappa - \phi y_{t,t} \qquad n < N && (116)\\
&= \bar{r} - \left(\frac{\bar{T}-t}{\Delta}\right)\left[\frac{1-\delta^{\Delta}}{\delta^{\Delta}}\tau + \left(1-\delta^{N\Delta}\right)\alpha\phi\kappa\right] + \eta\Delta i^* - \eta\phi\kappa - \phi\left[y_{\bar{T}} - \left(\bar{T}-t\right)\theta - \alpha\phi\kappa\right] && (117)
\end{aligned}
$$

Rearranging the terms yield,

$$\tilde{e}_{t,t} = (\bar{r} + \eta\Delta i^* - \eta\phi\kappa - \phi y_{\bar{T}}) - (\bar{T} - t)\left\{\left[\frac{1-\delta^{\Delta}}{\delta^{\Delta}}\tau + \left(1 - \delta^{N\Delta}\right)\alpha\phi\kappa\right]/\Delta - \phi\theta - \left(\frac{\phi}{\bar{T}-t}\right)\alpha\phi\kappa\right\},\ \kappa = \frac{\theta}{1+\alpha\phi} \tag{118}$$

12. Backward Induction, interest rate

To examine the behavior of the interest rate in equilibrium before the unpegging happens, note that (1), (3) and (4) yield that,

$$i_t = \frac{\bar{e} + \phi y_t - r_t}{\Delta\eta} \tag{119}$$

and during $t \in [\underline{T}, \bar{T}]$, reserves are given by Equation (109) and output before unpegging is given by $y_t = y_{\bar{T}} - \left(\bar{T} - t\right)\theta$. So the previous equation can be expressed as

$$i_t = \frac{\bar{e} + \phi y_{\bar{T}} - \bar{r} + (\bar{T} - t)\left[\left(\frac{1-\delta^{\Delta}}{\delta^{\Delta}}\tau + \left(1 - \delta^{N\Delta}\right)\alpha\phi\kappa\right)/\Delta - \phi\theta\right]}{\Delta\eta} \tag{120}$$

Plug in $y_{\bar{T}}$ from Equation (17) into the Equation above, $\phi y_{\bar{T}} = \bar{r} + \eta\Delta i^* - \bar{e} - \eta\phi\kappa$, we have

$$i_t = \frac{\bar{e} + \bar{r} + \eta\Delta i^* - \bar{e} - \eta\phi\kappa - \bar{r} + (\bar{T} - t)\left[\left(\frac{1-\delta^{\Delta}}{\delta^{\Delta}}\tau + \left(1 - \delta^{N\Delta}\right)\alpha\phi\kappa\right)/\Delta - \phi\theta\right]}{\Delta\eta} \tag{121}$$

this can be reduced to

$$i_t = i^* - \phi\kappa/\Delta + \left(\bar{T} - t\right)\left[\frac{\left(\frac{1-\delta^{\Delta}}{\delta^{\Delta}}\tau + \left(1 - \delta^{N\Delta}\right)\alpha\phi\kappa\right)/\Delta - \phi\theta}{\Delta\eta}\right] \tag{122}$$

13. Backward Induction, Deriving q_t

The one remaining endogenous variable is q_t, the probability that the policy maker abandons the fixed exchange regime. This can be derived in the following way. Note that from Equation 96,

$$\tilde{e}_{t,t+\Delta} = r_t + \eta i^* - \eta\phi\kappa - \phi y_{t+\Delta}, \;\; \kappa = \frac{\Delta\theta}{1+\Delta\alpha\phi} \tag{123}$$

$$\tilde{e}_{t,t} = r_t + \eta i^* - \eta\phi\kappa - \phi y_t \tag{124}$$

And so

$$\tilde{e}_{t,t+\Delta} - \tilde{e}_{t,t} = \phi(y_t - y_{t+\Delta}) \tag{125}$$

Here because we are looking at $\tilde{e}_{t,t+\Delta}$ and $\tilde{e}_{t,t}$, we are assuming that the peg is abandoned in period t. $y_t - y_{t+\Delta} = \theta + \alpha(E_t e_{t,t+\Delta} - e_{t,t})$. Same as the calculation of Equation (126), we know that after the peg is abandoned, the evolution of exchange rate follows

$$E_t\left(\tilde{e}_{t,t+\Delta}\right) - \tilde{e}_{t,t} = -\frac{\Delta\phi\theta}{1+\Delta\phi\alpha} \tag{126}$$

Therefore we can calculate $\tilde{e}_{t,t+\Delta}$ from Equation (49), again set $\kappa = \frac{\Delta\theta}{1+\Delta\phi\alpha}$

$$\tilde{e}_{t,t+\Delta} = \tilde{e}_{t,t} - \phi\kappa = \bar{e} - (\bar{T} - t)\left\{\left[\frac{1-\delta^{\Delta}}{\delta^{\Delta}}\tau + \left(1-\delta^{N\Delta}\right)\alpha\phi\kappa\right]/\Delta - \phi\theta - \left(\frac{\phi}{\bar{T}-t}\right)\alpha\phi\kappa\right\} - \phi\kappa \tag{127}$$

Plug Equation (127) into Uncovered Interest Rate Parity (82), $i = i^* + q_t(\tilde{e}_{t,t+\Delta} - \bar{e})$ yields,

$$i_t = i^* + q_t \left\{ \bar{e} - (\bar{T} - t) \left\{ \left[\frac{1-\delta^{\Delta}}{\delta^{\Delta}} \tau + \left(1 - \delta^{N\Delta}\right) \alpha\phi\kappa \right] / \Delta - \phi\theta - \left(\frac{\phi}{\bar{T} - t} \right) \alpha\phi\kappa \right\} - \phi\kappa - \bar{e} \right\} \quad (128)$$

$$i_t = i^* + q_t \left\{ -(\bar{T} - t) \left\{ \left[\frac{1-\delta^{\Delta}}{\delta^{\Delta}} \tau + \left(1 - \delta^{N\Delta}\right) \alpha\phi\kappa \right] / \Delta - \phi\theta - \left(\frac{\phi}{\bar{T} - t} \right) \alpha\phi\kappa \right\} - \phi\kappa \right\} \quad (129)$$

Rearrange the terms, we have

$$q_t = \frac{i_t - i^*}{-(\bar{T} - t) \left\{ \left[\frac{1-\delta^{\Delta}}{\delta^{\Delta}} \tau + \left(1 - \delta^{N\Delta}\right) \alpha\phi\kappa \right] / \Delta - \phi\theta - \left(\frac{\phi}{\bar{T} - t} \right) \alpha\phi\kappa \right\} - \phi\kappa} \quad (130)$$

$$q_t = \frac{i^* - i_t}{(\bar{T} - t) \left\{ \left[\frac{1-\delta^{\Delta}}{\delta^{\Delta}} \tau + \left(1 - \delta^{N\Delta}\right) \alpha\phi\kappa \right] / \Delta - \phi\theta - \left(\frac{\phi}{\bar{T} - t} \right) \alpha\phi\kappa \right\} + \phi\kappa} \quad (131)$$

At $\underline{T}$, $i_t = i^*$ and then i_t decreases linearly afterwards. So Equation (130) says that the probability of abandoning the fixed exchange rate is zero at $\underline{T}$. It then increases, at an increasing rate, until it reaches 1 at $\bar{T}$.

Appendix B: Basic Model

Basic Setup with a Myopic policy maker

Small open economy with real output growth. I build the model around five equations.

$$p_t = e_t + p_t^* \tag{132}$$

$$i_t = i^* + E_t e_{t+1} - e_t \tag{133}$$

$$m_t - p_t = -\eta i_t + \phi y_t \tag{134}$$

$$m_t = r_t \tag{135}$$

$$y_t = y_{t-1} + \theta \tag{136}$$

where p_t, p_t^* are the logs of domestic- and foreign-currency price of the consumption basket and we assume here that $p_t^* = 0$; $i_t = log(1 + i_t')$, where i' is nominal interest rate; e_t is the log of nominal exchange rate (foreign in terms of home); m_t is log of nominal money balances held at the end of period t; y_t is the output at the end of period; r_t is the log of foreign reserves at period t. Equation 132 is Purchasing Power Parity and we assume without loss of generality that $p_t^* = 0$; Equation 167 is an approximation in logs of Uncovered Interest Rate Parity, which is $1 + i_{t+1} = (1 + i_{t+1}^*)E_t(\frac{\xi_{t+1}}{\xi_t})$; Equation 134 is the real money demand function, which is an adaption of the Cagan Model; Equation135 is money supply—we assume for simplicity that there is no domestic credit but the case with domestic credit is derived in an extension to this paper. I find that all major implications hold as in this case. Hence money supply consists only of foreign reserves, although the sterilization case will also be discussed in the extension of the paper; Equation 168 says the growth of log of output is θ for each period.

Solving for the Speculators Problem in the Myopic Policy Maker Case

Timing

For now consider the traditional case where the Central Bank abandons the fixed exchange rate system if, and only if, reserves reaches its upper bound, $\bar{R}$. Once the reserve level reaches $\bar{R}$, the Central Bank must leave the foreign exchange market forever and the exchange rate will float freely. There are at least three reasonable specifications of the timing in this market which could be adopted. The speculators could choose their levels of speculative balances first followed by the policy maker's move, the policy maker could move first followed by speculators, or the players could make their decisions simultaneously.

Equilibrium

First I rule out a case where speculators believe that the fixed regime will never be abandoned, in which case $E_t(e_{t+1}) = \bar{e}$ for all t. In this case, from Equations 1 - 4 we have

$$r_t = \bar{e} - \eta i^* + \phi y_t \tag{137}$$

But the log reserve needs to grow at a rate of $\phi\theta$, given the upper threshold on the reserves that the Central Bank can take, it is not rational for them to think that the bank will never abandon the exchange rate regime.

In fact, the fixed regime will be abandoned at a state $\tilde{T}$ defined by the following equations. Setting money supply equal money demand,

$$\bar{r} = \bar{e} - \eta i^* + \phi y_{\tilde{T}} \tag{138}$$

Note that the highest reserve level can go is $\bar{r}$.

$$\bar{e} - \eta i^* + \phi y_{\tilde{T}} = \bar{r} \tag{139}$$

Rearranging the terms yields,

$$y_{\tilde{T}} = \frac{\bar{r} + \eta i^* - \bar{e}}{\phi} \tag{140}$$

To solve the speculators' optimization problem, it is necessary to determine the exchange rate that will prevail after the fixed rate regime is abandoned. Define the shadow floating exchange rate, $\tilde{\xi}_t$, as the exchange rate that would prevail if the exchange rate were floating at t. Let $\tilde{e}$ be the log of this shadow rate. In order for the exchange rate to be floating, it must be that the reserves rose to $\bar{r}$ at one point, and after that point all foreign currency transactions will take place in private markets so the money supply will simply be $\bar{r}$. Solving for $\tilde{e}$ using Equations (1)—(5):

$$\tilde{e}_t = \bar{r} + \eta i^* - \eta\phi\theta - \phi y_t \tag{141}$$

Defining $\bar{T}$ as the date of the speculative attack in this traditional version of the model. Setting $\tilde{e}_{\bar{T}} = \bar{e}$ in Equation 141 yields,

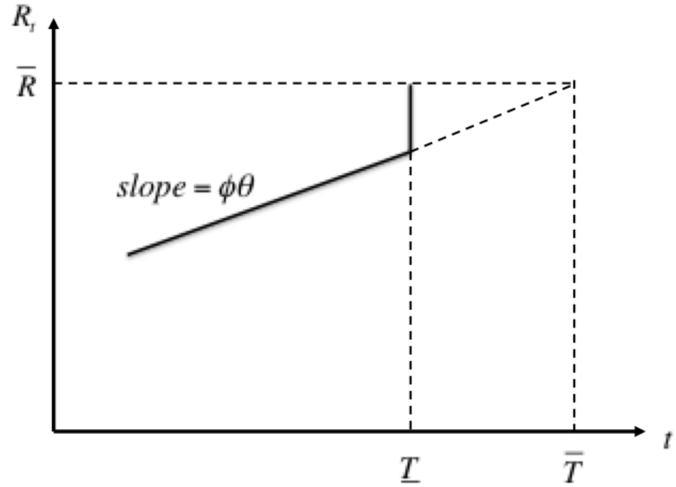
$$y_{\bar{T}} = \frac{\bar{r} + \eta i^* - \bar{e}}{\phi} - \eta\theta \tag{142}$$

Together with Equation 142 and 140, we have

$$y_{\bar{T}} - y_{\tilde{T}} = -\eta\theta < 0 \tag{143}$$

Hence the speculative attack happens before the reserve reaches its upper thresh-

Figure 5: Time Evolution of Reserve Accumulation



old. Fig. 1 shows the path of reserves. Initially reserves follow Equation 138 , increasing with the growth of real output. At $\bar{T}$, the shadow floating exchange rate increases to the fixed rate and a sudden speculative attack would increase the reserves by $\eta\phi\theta$, and forcing the reserve level to reach its upper threshold, which forces the Central Bank to abandon the peg.

Fig.1 Reserve level with a myopic policy maker.

Model with an Optimizing Policy Maker

The Policy Maker's Problem

A similar objective function as the one presented in my core model is used as the policy maker's objective function. Here the objective function is simplified to only include reserve level in order to simplify the model and highlight the impact of reserves.

$$V_t = min \left\{ \frac{r_t + \tau}{1 - \delta} \, , \; r_t + \delta E_t(V_{t+1}) \right\} \tag{144}$$

The policy maker compares the cost of abandoning the fixed exchange rate system with the current and expected future cost of maintaining it.

Since the policy maker is pursuing a fixed exchange rate policy, it must find it optimal to do so. This places a lower bound on the value of the parameter τ, the policy maker's preference for fixed exchange rates. In other words, the policy maker must find the cost of abandoning the peg next period smaller than the cost of abandoning the peg in this period, when there is no speculative attack. Mathematically,

$$\frac{r_t + \tau}{1-\delta} \geqslant r_t + \delta\left(\frac{r_{t+1}+\tau}{1-\delta}\right) \tag{145}$$

where the left-hand side is the cost of abandoning the fixed rate this period, and the right hand-side is abandoning the peg next period. The above equation reduces to

$$\delta(r_{t+1} - r_t) \leqslant (1-\delta)\tau \tag{146}$$

The reserve level when there's no speculative attack is given by Equation 137. Notice that $y_{t+1} - y_t = \theta$; so $r_{t+1} - r_t = \phi\theta$.

We get **Assumption 1**

$$(1-\delta)\tau \geqslant \delta\phi\theta \tag{147}$$

The Speculators' Problem

The solution to the speculators' problem is still described by Uncovered Interest Parity, Equation 167. However, speculators will be aware that the policy maker may decide to allow the exchange rate to float before reserves reaches $\bar{r}$. If it decides to do this the floating exchange rate will no longer be described by 141.

Since all foreign currency transactions in the floating rate period take place in private markets, the log of money supply will be r_T. Here we denote q_t as the probability for the Central Bank to abandon the fixed exchange rate. We set $E_t(e_{t+1} - e_t) = q_t(e_{t+1} - e_t)$ in Equation **??**. So $E_t(e_{t+1} - e_t) = e_{t+1} - e_t$ when the speculators expect the policy maker to abandon the peg this period ($q_t = 1$) and the expected change of exchange rate would be zero if the speculators believe the policy maker does not abandon the peg this period. Solving for the shadow exchange rate from Equation (1)—(5) similar to the myopic case.

$$r_T = \tilde{e}_{T,t} - \eta\,[i^* + E_t\tilde{e}_{t+1} - \tilde{e}_t] + \phi y_t \tag{148}$$

$$\tilde{e}_{T,t} = r_T + \eta i^* - \eta\phi\theta - \phi y_t \tag{149}$$

If speculators were not holding very much foreign currency at the time of the abandonment, then the money supply will be relatively high, resulting in a high path for the floating exchange rate. Thus if reserves are high at the time of the move to a floating exchange rate then the exchange rate itself will be relatively high as well. In fact, if speculators did not expect a change in the fixed exchange rate, reserves would be given by Equation 138

$$\bar{e} = r_t + \eta i^* - \phi y_t \tag{150}$$

If the speculators don't expect the government to abandon the fixed exchange rate ($q_t = 0$), the shadow exchange rate would be given by the following, which is essentially setting $T = t$ in Equation 149 because in calculating the shadow rate, we assume that there is no speculative bubble. So the shadow rate when there is no speculative bubble (i.e. speculators do not expect the government to abandon the peg) would be

$$\tilde{e}_{t,t} = r_t + \eta i^* - \eta\phi\theta - \phi y_t \tag{151}$$

Plugging in the expression for $\bar{e}$ from Equation 150, we find that the shadow floating exchange rate in this case would be strictly greater than the fixed rate,

$$\{\tilde{e}_{t,t}|\, E(q_t) = 0\} = \bar{e} - \eta\phi\theta \tag{152}$$

Proposition 1. In equilibrium, high probabilities of abandonment will result in high levels of reserves.

Intuitively, higher probabilities of abandonment would cause speculators to hold more domestic currency (with the expectation that it would appreciate once the regime becomes floating).

Proof

The speculators' problem is still defined by Uncovered Interest Parity. In the fixed regime, the speculators find out the level of interest rate given the probability that the policy maker would abandon the peg this period(q_t). It can be written as

$$i_t = i^* + q_t(\tilde{e}_{t,t+1} - \bar{e}) \tag{153}$$

where $\tilde{e}_{t,t+1}$ denotes the shadow exchange rate in the next period assuming the peg is abandoned in this period. By Equation 149, $\tilde{e}_{t,t+1} = r_t + \eta i^* - \eta\phi\theta_t - \phi y_{t+1}$. Combining it with Equations 132, 134, 135, 168,

$$r_t = \bar{e} -\ \eta\, i^* \ +\ \phi y_t + \frac{q_t}{1+\eta q_t}(1+\eta)\eta\phi\theta \tag{154}$$

The derivative of r_t with respect to q_t is

$$\frac{\partial r_t}{\partial q_t} = \left[\frac{1}{1+\eta q_t} - \frac{q_t\eta}{(1+\eta q_t)^2}\right](1+\eta)\eta\phi\theta \tag{155}$$

Since $q_t, \eta < 1$ and $\frac{1}{1+\eta q_t} > \frac{1}{(1+\eta q_t)^2}$, we have $\frac{\partial r_t}{\partial q_t} > 0$, which completes the proof of Proposition 1. $\square$

Proposition 2. If $q < 1$ then $\tilde{e}_{t,t} < \bar{e}$. If $q_t = 1$, then $\tilde{e}_{t,t} = \bar{e}$.

Intuitively, if speculators are certain that the exchange rate will be abandoned they will continue to buy domestic currency as long as the shadow floating exchange rate is smaller than the fixed rate. These domestic currency purchases increase the money supply and hence decrease the shadow floating exchange rate until it equals the fixed rate. Put another way, the shadow floating exchange rate will always be at most as high as the fixed rate since otherwise speculators would find it profitable to buy foreign currency, thereby decreasing the money supply and lowering the shadow floating exchange rate.

Proof

Setting $q_t = 1$ in Equation 154, we get

$$\bar{e} = r_t + \eta i^* - \eta\phi\theta - \phi y_t \tag{156}$$

Compare the equation above to the shadow exchange rate in 149, when we set $T = t$ and get

$$\tilde{e}_{t,t} = r_t + \eta i^* - \eta\phi\theta - \phi y_t \tag{157}$$

We get $\tilde{e}_{t,t} = \bar{e}$, which proves the second part of Proposition 2. To prove the first part, note that the shadow rate is decreasing in the level of reserves and the level of reserves is increasing in q_t by Proposition 1. Hence given that $\tilde{e}_{t,t} = \bar{e}$ when $q_t = 1$, we have that $\tilde{e}_{t,t} < \bar{e}$ when $q_t < 1$. □

Equilibrium

Proposition 3. If $q_{t-1} = 0$ and $q_t = 1$ for any $t \leqslant \bar{T}$, where $\bar{T}$ is the period when the speculative attack takes place in the myopic case, then $r_t - r_{t-1} = \phi\theta\eta + \phi\theta$.

Intuitively, since reserves increase by $\phi\theta$ each period due to real output growth, if speculators know for sure that the government would abandon the fixed regime in the next period, they would immediately plan an attack of size $\phi\theta\eta$. Note that the size of the speculative attack is the same as the size of the attack in the myopic case and it does not depend on the date of the abandonment. In the myopic model the size of the attack does not depend on the critical level of reserves $\bar{r}$, but the date of the speculative attack is uniquely determined by $\bar{r}$. Thus in the myopic model there is a one-to-one correspondence between $\bar{r}$ and the date of abandonment, given either, one can deduce the other. In Proposition 3 the specification of the date of abandonment ($q_{t-1} = 0$ and $q_t = 1$) is therefore tantamount to picking a different critical level of reserves in the myopic model.

Proof

Setting $q_t = 1$ and $q_{t-1} = 0$ in Equation 154, we have

$$r_{t-1} = \bar{e} - \eta i^* + \phi y_{t-1} \tag{158}$$

$$r_t = \bar{e} - \eta i^* + \eta\phi\theta + \phi y_t \tag{159}$$

Taking the difference,

$$r_t - r_{t-1} = \phi\theta\eta + \phi\theta \tag{160}$$

□

Proposition 4. In equilibrium $q_t < 1$ for all $t < \bar{T}$.

Intuitively, if the policy maker plans to abandon the fixed exchange rate, speculators will try to take advantage of this by purchasing domestic currency, resulting in high Central Bank foreign reserves. So when the time comes for the policy maker to actually implement the switch to the floating rate, it will find that reserves are already quite high and that the damage from speculators has already been done. The additional damage that they could do if the policy maker waited one more period is relatively small. So it is in the policy maker's best interest to continue maintaining the fixed exchange rate. Therefore switching exchange rate regimes at time $t < \bar{T}$ with certainty cannot be part of an equilibrium strategy. This implies that in equilibrium the policy maker will not be able to preemptively abandon the fixed rate with certainty at t in order to avoid a speculative attack in $t + 1$.

Proof

Consider the case where $q_t = 1$ while $t < \bar{T}$ for the sake of contradiction. This assumption means that the Central Bank must find it optimal to abandon the fixed exchange rate regime, which means it must find the cost of abandoning the peg this period would be smaller than the cost of abandoning the peg next period. Their objective function 169 implies that

$$\frac{r_t + \tau}{1 - \delta} \leqslant r_t + \delta \left(\frac{r_{t+1} + \tau}{1 - \delta} \right) \tag{161}$$

where the left-hand side is the cost of abandoning the fixed rate this period, and the right hand-side is abandoning the peg next period. The above equation reduces to

$$r_{t+1} - r_t \geqslant \frac{(1 - \delta)\tau}{\delta} \tag{162}$$

Now consider the speculators' optimal choice. Since $q_t = 1$, Proposition 2 implies that they purchase domestic currency to a point where $\tilde{e}_{t,t} = \bar{e}$. From Equation 154, we know the reserve level is

$$r_t = \bar{e} - \eta i^* + \eta \phi \theta + \phi y_t \tag{163}$$

The speculative attack in $t + 1$ would only continue if the shadow rate is at least as low as the fixed rate, $\tilde{e}_{t+1,t+1} \leqslant \bar{e}$, an inequality since q_{t+1} might be smaller than or equal to 1. This implies, by the fact that the level of foreign reserves increases in shadow exchange rate. And we are given that when $\tilde{e}_{t+1,t+1} = \bar{e}$, $r_{t+1} = \bar{e} - \eta i^* + \eta\phi\theta + \phi y_{t+1}$. So when $\tilde{e}_{t+1,t+1} \leqslant \bar{e}$,

$$r_{t+1} \leqslant \bar{e} - \eta i^* + \eta\phi\theta + \phi y_{t+1} \tag{164}$$

Hence 163 - 164 yields

$$r_{t+1} - r_t \leqslant \phi(y_{t+1} - y_t) = \phi\theta \tag{165}$$

In equilibrium, both 162 and 165 hold, which implies that $(1-\delta)\tau \leqslant \delta(r_{t+1} - r_t) \leqslant \delta\phi\theta$. This contradicts with our assumption in Equation 147. Therefore $q_t = 1$ can't be an equilibrium. □

The argument given in the myopic case implies that $q_t = 1$ for all $t \geqslant \bar{T}$. Therefore an implication of Proposition 4 is that the only remaining potential pure-strategy for the policy maker involves setting $q_t = 0$ for all $t < \bar{T}$ and $q_t = 1$ for all $t \geqslant \bar{T}$. Consider in the next proposition what is necessary for this to be an equilibrium.

Proposition 5. If $t < \bar{T}$ and $q_{t+1} > 0$, then $q_t = 0$ can only be part of an equilibrium strategy if $r_{t+1} - r_t \leqslant \frac{1-\delta}{\delta}\tau$.

Intuitively, this means that passively waiting for an attack can only be optimal when the increase in reserves, and thus disutility, is small relative to the policy maker's preference for the exchange rate peg.

Proof

For the choice of $q_t = 0$ (not abandoning the peg in period t) to be optimal for the policy maker, it must be that the cost of abandoning the peg is bigger than the cost of continuing with the fixed regime. In other words,

$$r_{t+1} - r_t \leqslant \frac{(1-\delta)}{\delta}\tau \tag{166}$$

Put in words, maintaining the fixed exchange rate will be optimal as long as in the next period reserves will not fall by more than $\frac{(1-\delta)}{\delta}\tau$, which completes the proof of Proposition 5. □

Proposition 6. With short period lengths no pure-strategy sub-game-perfect Nash equilibrium exists.

Proof

From Proposition 4 and the argument made in the myopic case, the only potential pure-strategy remaining for the policy maker involves $q_{\bar{T}-1} = 0$ and $q_{\bar{T}} = 1$. By Proposition 3 this would imply that there would be a speculative attack in $\bar{T}$, $r_{\bar{T}} - r_{\bar{T}-1} = \phi\theta\eta + \phi\theta$, the size of the speculative attack being $\phi\theta\eta$. So from Proposition 5 if $\phi\theta\eta + \phi\theta < \frac{(1-\delta)}{\delta}\tau$, this will be an equilibrium. If the expected attack is very small relative to the policy maker's preference for fixed exchange rates, then it can be optimal to deliberately accept the attack rather than to give up even one period of the fixed exchange rate regime. In this case $q_{\bar{T}-1} = 0$ can be part of an equilibrium strategy. However, we should note that this equilibrium is an artifact of the period length. If we consider short periods, this equilibrium cannot exist, as derived below.

To examine the possibility that the model might be able to generate Krugman style speculative attacks with a frequently optimizing policy maker, the analysis is extended to allow for an arbitrary period length, n. Low values of n imply frequent decision making on the part of the policy maker and the preceding analysis is a special case where $n = 1$. As $n \to 0$ the model approaches continuous time.

Equations 132, 134 and 135 are not dependent on the period length and can be retained without modification, noting only that it in the money demand Eq.134 refers to the interest return over calendar time 1. Therefore the per-period interest rate is given by ni_t, so Equation 167 becomes,

$$ni_t = ni^* + E_t e_{t+1} - e_t \tag{167}$$

and 168 is

$$y_t = y_{t-1} + n\theta \tag{168}$$

From these equations the shadow floating exchange rate in the traditional model can be derived using the same method to show that 141 still holds. The time of the speculative attack in the traditional model $\bar{T}$ is given by $\tilde{e_t} = \bar{e}$ which implies that reserves increase by at $\eta\phi\theta$ at $\bar{T}$. So the size of the speculative attack in the traditional model is not dependent on the period length. This is unsurprising since the model generates attacks in continuous time as well.

Now moving to the problem of the policy maker, the policy maker's per-period discount rate is δ^n. Its per-period cost is proportional to the period length: nr_t for fixed exchange rates and $n(r_t + \tau)$ with floating rates. Therefore the policy maker's maximization problem is

$$V_t = min \ \left\{ \frac{n(r_t + \tau)}{1 - \delta^n} \ , \ nr_t + \delta^n E_t(V_{t+1}) \right\} \tag{169}$$

We are interested here in whether it is optimal for the policy maker to set $q_{\bar{T}-1} = 0$ knowing that there will be a speculative attack in the next period. If this is so, then from the maximization of 169,

$$\frac{n(r_{\bar{T}-1} + \tau)}{1 - \delta^n} \geqslant nr_{\bar{T}-1} + \frac{n\delta^n(r_{\bar{T}} + \tau)}{1 - \delta^n} \tag{170}$$

which reduces to

$$r_{\bar{T}} - r_{\bar{T}-1} \leqslant \tau(1 - \delta^n)/\delta^n \tag{171}$$

Since reserves increase by $n\phi\theta$ each period just due to real economic growth and since there will be a speculative attack boosting reserves by $\eta\phi\theta$ at $\bar{T}$, $r_{\bar{T}} - r_{\bar{T}-1} = n\phi\theta + \eta\phi\theta$. Therefore Equation 171 requires

$$\eta\phi\theta \leqslant \tau(1-\delta^n)/\delta^n - n\phi\theta \tag{172}$$

This condition must hold for it to be optimal for the policy maker to remain passive in the face of an expected speculative attack. In the limit as $n \to 0$ the right-hand side goes to zero and the inequality cannot hold for any set of parameters. Therefore with short periods there cannot be an equilibrium where the policy maker remains passive in the face of a predictable speculative attack.

The size and cost of the speculative attack is not dependent on the period length, but the opportunity cost of abandoning the fixed exchange rate system one period earlier is. With short enough periods the opportunity cost is negligible. Therefore if we consider short periods it will not be optimal for the policy maker to remain passive in the face of a predictable speculative attack. Hence $q_{\bar{T}-1}$ will not be zero in equilibrium. Thus from Proposition 4, $q_{\bar{T}-1} \in (0,1)$ which completes the proof of Proposition 6. □

Sub-game-Perfect Nash Equilibrium

Intuitively, if the policy maker can predict an imminent speculative attack then it will wish to abandon the fixed exchange rate regime just before the attack. Likewise, if speculators can predict this preemptive abandonment of the fixed exchange rate regime, then they will exploit this knowledge by buying foreign currency just before the abandonment. Thus in order to avoid a speculative attack the policy maker must introduce uncertainty into the decisions of speculators. It cannot follow a predictable pure strategy, since such a strategy would result in a speculative attack.

The subgame-perfect Nash equilibrium can be constructed by backward induction.

The argument made in the myopic case implies that in any equilibrium $q_{\bar{T}} = 1$. Given this, it is possible to examine the policy maker's optimal strategy in $\bar{T}-1$. And from this it is possible to examine the policy maker's optimal strategy in $\bar{T}-2$, and so on.

Consider a time $t < \bar{T}$ where $q_{t+1} > 0$. That is, in the coming period there will be a positive probability that the policy maker will abandon the fixed exchange rate. The policy maker must find this optimal in period $t+1$, which implies that it either strictly prefers to abandon the fixed rate regime in $t+1$ or it is indifferent between abandoning and maintaining the peg. In either case from the their objective function, Equation 169, the maximized expected present value of its cost function is given by $V_{t+1} = \frac{r_{t+1}+\tau}{1-\delta}$.

From Proposition 4, $q_t = 1$ when $t < \bar{T}$ is not possible. Suppose for a moment that $q_t \in (0,1)$, which means that the policy maker is indifferent between maintaining and abandoning the fixed exchange rate regime. From Equation 169, hence

$$\frac{r_t + \tau}{1-\delta} = r_t + \delta\left(\frac{r_{t+1}+\tau}{1-\delta}\right) \tag{173}$$

and the Central Bank log reserve value must equal to

$$r_t = r_{t+1} - (1-\delta)\tau/\delta \tag{174}$$

On the other hand, if $q_t = 0$, reserves are given by Equation 138 because the speculators do not expect the peg to be abandoned this period. This can be part of an equilibrium only if it implies $r_t \geqslant r_{t+1} - \frac{(1-\delta)}{\delta}\tau$ by Proposition 5.

Thus, working backward from $\bar{T}-1$, it is possible to establish $\underline{\mathrm{T}}$, the earliest state where abandonment of the fixed exchange rate can be an equilibrium outcome. Iterating 174 yields

$$r_t = \bar{r} - (\bar{T} - t)(1-\delta)\tau/\delta \tag{175}$$

Equation 175 describes the equilibrium level of reserves as long as it yields reserves that are greater than those given by Equation 138 because that point and earlier $q_t = 0$ is an equilibrium and so reserves will follow Equation 138. After $\underline{T}$ reserves increase quickly, but continuously, until they reach their upper bound at $\bar{T}$.

Consider the problem of an individual speculator. Suppose that, at the moment, reserves are low and that in the next period they will be relatively high. The speculator therefore realizes that the policy maker will prefer to abandon the fixed rate regime this period rather than permit such a large increase in reserves. So he will purchase extra domestic currency in this period. This means that the increase in foreign currency reserves will not be as dramatic in the coming period. Therefore, the policy maker will no longer be quite as eager to abandon the fixed rate in the next period.

Speculators will purchase domestic currency as long as the policy maker prefers to abandon the peg. When it is indifferent between abandoning and maintaining the fixed rate, speculators will realize that if they purchase additional domestic currency, the jump in reserves between the current period and the next will be small enough so that the policy maker will prefer not to abandon. Since speculation increases the money supply, it decreases domestic interest rate below the foreign rate. Hence purchasing domestic currency entails an opportunity cost and they will not purchase additional domestic currency once the threshold is reached.

So optimizing behavior on the part of speculators ensure that reserves are at a level where the policy maker either strictly prefers maintaining the fixed exchange rate one more period or is indifferent between abandoning and maintaining the fixed rate. In other words, optimizing speculators will always ensure that the policy maker does not strictly prefer to abandon the fixed exchange rate regime. If it did so, speculators could make profits by purchasing additional domestic currency, making abandonment less attractive to the policy maker. Since sudden attacks on foreign currency reserves make preemptive abandonment of the fixed exchange rate attractive, in equilibrium there can be no such predictable attacks.

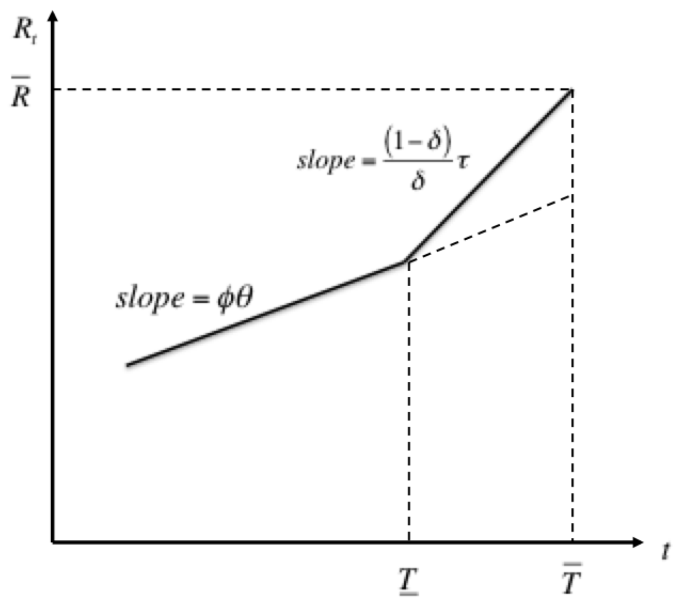


Fig.2 Reserve level with a smart policy maker.

In the range $t < \underline{\mathrm{T}}$ the policy maker prefers to continue maintaining the fixed exchange rate so reserves increase one for every ϕ percent growth in output ($\dot{r} = \phi\theta$ in continuous time). In the period $t \in [\underline{\mathrm{T}}, \bar{T})$, speculators ensure that the policy maker is indifferent between abandoning an maintaining the fixed rate regime, which implies that reserves are increasing at a greater rate.

From this information on the path of reserves it is straightforward to determine the behavior of other variables. Firstly the behavior of all the endogenous variables will be derived and then the intuition for the results will be discussed. During the period $t < \underline{\mathrm{T}}$ there is no chance that the fixed exchange rate will be abandoned ($q_t = 0$). This implies that $i_t = i^*$ and that the shadow floating exchange rate is given by Equation 152. After $\underline{\mathrm{T}}$ reserves are given by Equation 175 and output growth can always be written as $y_t = y_{\bar{T}} - (\bar{T} - t)\theta$. The shadow floating exchange rate can be found by substituting these into 149, $\tilde{e}_{t,t} = r_t + \eta i^* - \eta\phi\theta - \phi y_t$.

$$\tilde{e}_{t,t} = [\bar{r} + \eta i^* - \eta\phi\theta - \phi y_{\bar{T}}] - (\bar{T} - t)\left[\frac{1-\delta}{\delta}\tau - \phi\theta\right] \tag{176}$$

The first bracketed term in the previous equation is equal to the fixed exchange rate, according to Equation 142. So the shadow floating exchange rate is simply

$$\tilde{e}_{t,t} = \bar{e} - (\bar{T} - t)\left[\frac{1-\delta}{\delta}\tau - \phi\theta\right] \tag{177}$$

We also show that $\tilde{e}_{\underline{T},\underline{T}} = \bar{e} - \eta\phi\theta$. So the shadow floating exchange rate is constant and below the fixed rate before $\underline{T}$. Then it increases linearly, reaching the fixed exchange rate at $\bar{T}$.

To examine the behavior of the interest rate in equilibrium before the unpegging, note that 132, 134 and 135 yield that,

$$i_t = \frac{\bar{e} + \phi y_t - r_t}{\eta} \tag{178}$$

and during $t \in [\underline{T}, \bar{T}]$, reserves are given by Equation 175. So the previous equation can be expressed as

$$i_t = \frac{\bar{e} + \phi y_{\bar{T}} - \bar{r} + (\bar{T} - t)[(1-\delta)\tau/\delta - \phi\theta]}{\eta} \tag{179}$$

Plug in $y_{\bar{T}}$ into the Equation above, $\phi y_{\bar{T}} = \bar{r} + \eta i^* - \bar{e} - \eta\phi\theta$

$$i_t = \frac{\bar{e} + \bar{r} + \eta i^* - \bar{e} - \eta\phi\theta - \bar{r} + (\bar{T} - t)[(1-\delta)\tau/\delta - \phi\theta]}{\eta} \tag{180}$$

this can be reduced to

$$i_t = i^* - \phi\theta + (\bar{T} - t)\frac{(1-\delta)\tau/\delta - \phi\theta}{\eta} \tag{181}$$

From the definition of $\underline{T}$ the interest rate is equal to the world rate at $\underline{T}$. It then increases linearly until it reaches $i^* - \phi\theta$ at $\bar{T}$.

The one remaining endogenous variable is q_t, the probability that the policy maker abandons the fixed exchange regime.

$$q_t = \frac{i^* - i_t}{(\bar{T} - t)\left[\frac{1-\delta}{\delta}\tau\text{-}\phi\theta\right] + \phi\theta} \tag{182}$$

At $\underline{\mathrm{T}}$, $i_t = i^*$ and then $i^* > i_t$ afterwards. So Equation **??** says that the probability of abandoning the fixed exchange rate is zero at $\underline{\mathrm{T}}$. It then increases, at an increasing rate, until it reaches 1 at $\bar{T}$.

It is also worth noting that if we made the additional assumption of risk neutrality and perfectly competitive forward markets, the log of the one period ahead forward exchange rate, F_t, would be equal to the expected exchange rate in the next period

$$f_t = q_t\tilde{e}_{t,t+1} + (1 - q_t)\bar{e} \tag{183}$$

Combining this with Equation 153 yields,

$$f_t = \bar{e} + i - i^* \tag{184}$$

Let FP_t denote the one period ahead forward exchange rate premium, defined as

$$FP_t = \frac{F_t - \bar{S}}{\bar{S}} \tag{185}$$

and fp_t as the log of FP_t. Hence

$$fp_t = f_t - \bar{e} = i - i^* \tag{186}$$

So the forward exchange rate premium is equal to the interest rate differential, starting at zero at $\underline{\mathrm{T}}$ and dropping linearly to $-\phi\theta$ at $\bar{T}$.

Appendix C: Extension of Basic Model with Stochastic Utility Function

The main purpose of deriving this extension with stochastic utility is to show that with either a big enough shock to holding reserves or the expectation of a sudden increase in the reserve level could initiate an unexpected unpegging. The shock to reserves is represented in a stochastic element in the government's objective function that is distributed exponentially with an unconditional probability density function.

Model with an Optimizing Policy Maker

The Policy Maker's Problem

The basic setup is the same as before,

$$p_t = e_t + p_t^* \tag{187}$$

$$i = i^* + E_t e_{t+1} - e_t \tag{188}$$

$$m_t - p_t = -\eta i + \phi y_t \tag{189}$$

$$m_t = r_t \tag{190}$$

$$\dot{y}_t = \theta \tag{191}$$

except now we add a stochastic element to the Central Bank utility function,

$$V_t = min\left\{\frac{r_t + \epsilon_t + \tau}{1-\delta}\ ,\ r_t + \epsilon_t + \delta E_t(V_{t+1})\right\} \tag{192}$$

where ϵ_t represents a random disturbance and is distributed exponentially with an unconditional probability density function. It is a positive disturbance because as time goes on, the Central Bank would accumulate more and more reserve due to economic growth, which is worse for the Central Bank.

$$f(\epsilon_t) = \begin{cases} \lambda exp(-\lambda\epsilon_t) & \epsilon_t > 0 \\ 0 & \epsilon_t \leqslant 0 \end{cases} \tag{193}$$

Note that $E_t(\epsilon_{t+1}) = 1/\lambda$.

Here we also want to find a lower bound for τ, the benefit to holding fixed exchange rate. Previously, we assumed that without a speculative attack, the cost of abandoning the peg this period would be bigger than the cost of abandoning next period and thus derived that $(1-\delta)\tau \geqslant \delta\phi\theta$. However, here we have a disturbance term ϵ_t every period. Suppose there is a lower level of ϵ called $\epsilon^L > 0$ such that for any $\epsilon > \epsilon^L$, if there is no speculative attack, the Central Bank would find it optimal to remain fixed, i.e.,

$$\frac{r_t + \epsilon^L + \tau}{1-\delta} \geqslant r_t + \epsilon^L + \delta\left(\frac{r_{t+1} + \epsilon_{t+1} + \tau}{1-\delta}\right) \tag{194}$$

which reduces to

$$\frac{(1-\delta)\tau}{\delta} \geqslant \frac{1}{\lambda} - \epsilon^L + r_{t+1} - r_t \tag{195}$$

The reserve level when there's no speculative attack is given in the myopic case in the original model,

$$r_t = \bar{e} - \eta i^* + \phi y_t \tag{196}$$

So $r_{t+1} - r_t = \phi\theta$. Hence Equation 197 becomes our **Assumption I**

$$\frac{(1-\delta)\tau}{\delta} \geqslant \frac{1}{\lambda} - \epsilon^L + \phi\theta \tag{197}$$

$\tilde{e}_{T,t}$ is still the shadow exchange rate that would prevail at t if the peg is abandoned at T. q_t is the probability that the Central Bank would abandon the peg in period t.

The Speculators' Problem

Same as before, the shadow exchange rate is

$$\tilde{e}_{T,t} = r_T + \eta i^* - \eta\phi\theta - \phi y_t \tag{198}$$

If speculators were not holding very much foreign currency at the time of the abandonment, then the money supply will be relatively high, resulting in a high path for the floating exchange rate. Thus if reserves are high at the time of the move to a floating exchange rate then the exchange rate itself will be relatively high as well. In fact, if speculators did not expect a change in the fixed exchange rate, which means $E_t(e_{t+1}) = \bar{e}$, so by equating money supply and demand, reserves would be given by $r_t = \bar{e} - \eta i^* + \phi y_t$.

Propositions 1-3 still hold as in the original model because the speculators' problem governed by the UIP doesn't change. To recap, we have **Proposition 1**: In equilibrium, high probabilities of abandonment will result in high levels of reserves. And the reserve level given q_t is given by

$$r_t = \bar{e} - \eta\, i^*_{t+1} + \frac{\eta q_t \phi(1+\eta)}{1+\eta q_t}\theta + \phi y_t \tag{199}$$

Proposition 2: If $q < 1$ then $\tilde{e}_{t,t} < \bar{e}$. If $q_t = 1$, then $\tilde{e}_{t,t} = \bar{e}$. Hence when $q_t = 1$,

$\tilde{e}_{t,t} = \bar{e}$; **Proposition 3**: If $q_{t-1} = 0$ and $q_t = 1$ for any $t \leqslant \bar{T}$, then $r_t - r_{t-1} = \phi\theta\eta$.

Starting from **Proposition 4** is where the current extension would differ from the original one.

Proposition 4. This proposition examines whether there is a period $t < \bar{T}$ where the Central Bank would strictly prefer to abandon the peg, i.e., $q_t = 1$ and if so, under what conditions.

Let's let $q_t = 1$ and see what happens. Since the Central Bank strictly prefers to abandon the peg, we must have

$$\frac{r_t + \epsilon_t + \tau}{1-\delta} \leqslant r_t + \epsilon_t + \delta\left(\frac{r_{t+1} + \epsilon_{t+1} + \tau}{1-\delta}\right) \tag{200}$$

The above equation reduces to

$$E_t(\epsilon_{t+1}) - \epsilon_t + r_{t+1} - r_t \geqslant \frac{(1-\delta)\tau}{\delta} \tag{201}$$

$E_t(\epsilon_{t+1}) = 1/\lambda$, so

$$\frac{(1-\delta)\tau}{\delta} \leqslant \frac{1}{\lambda} - \epsilon_t + r_{t+1} - r_t \tag{202}$$

Now consider the speculators' optimal choice. Since $q_t = 1$, **Proposition 2** implies that they purchase domestic currency to a point where $\tilde{e}_{t,t} = \bar{e}$. From Equation 199, we know

$$r_t = \bar{e} - \eta\, i^* + \eta\phi\theta + \phi y_t \tag{203}$$

The speculative attack in $t+1$ would only continue if the shadow rate is at least as low as the fixed rate, $\tilde{e}_{t+1,t+1} \leqslant \bar{e}$, an inequality since q_{t+1} might be smaller than or equal to 1. And we are given that when $\tilde{e}_{t+1,t+1} = \bar{e}$, $r_{t+1} = \bar{e} - \eta\, i^* + \eta\phi\theta + \phi y_{t+1}$.

From Equation 199the $r_{t+1} = \tilde{e}_{t+1,t+1} - \eta i^* + \eta\phi\theta + \phi y_t$. r_{t+1} is increasing in $\tilde{e}_{t+1,t+1}$.

$$r_{t+1} \leqslant \bar{e} - \eta i^* + \eta\phi\theta + \phi y_{t+1} \tag{204}$$

Hence 203 - 204 yields

$$r_{t+1} - r_t \leqslant \phi(y_{t+1} - y_t) = \phi\theta \tag{205}$$

In equilibrium, both 201 and 205 hold, which implies that $\frac{(1-\delta)}{\delta}\tau \leqslant r_{t+1} - r_t + \frac{1}{\lambda} - \epsilon_t \leqslant \phi\theta + \frac{1}{\lambda} - \epsilon_t$.

To investigate whether there is a pure Nash equilibrium strategy, we need to examine two cases:

Case I : If $\epsilon_t \leqslant \epsilon^L$, then $\phi\theta + \frac{1}{\lambda} - \epsilon_t \geqslant \phi\theta + \frac{1}{\lambda} - \epsilon^L$. $q_t = 1$ could happen for some $t < \bar{T}$ when

$$\phi\theta + \frac{1}{\lambda} - \epsilon^L \leqslant \frac{(1-\delta)\tau}{\delta} \leqslant \phi\theta + \frac{1}{\lambda} - \epsilon_t \tag{206}$$

Case II: If $\epsilon_t > \epsilon^L$, then $\phi\theta + \frac{1}{\lambda} - \epsilon_t < \phi\theta + \frac{1}{\lambda} - \epsilon^L$. By **Assumption I**, $\frac{(1-\delta)}{\delta}\tau \geqslant \phi\theta + \frac{1}{\lambda} - \epsilon^L$, but here we require that $\frac{(1-\delta)}{\delta}\tau \leqslant \phi\theta + \frac{1}{\lambda} - \epsilon_t$, which is a contradiction. Hence q_t would not equal one in any period when $\epsilon_t > \epsilon^L$.

In Case I where $\epsilon_t \leqslant \epsilon^L$, the disturbance this period is small enough, which means the Central Bank would expect the future disturbance $E_t(\epsilon_{t+1})$ to be greater than ϵ_t, so the expected pain caused by ϵ_{t+1} outweighs the benefit of keeping the peg this period (which is captured by $\phi\theta + \frac{1}{\lambda} - \epsilon^L \leqslant \frac{(1-\delta)\tau}{\delta} \leqslant \phi\theta + \frac{1}{\lambda} - \epsilon_t$), the Central Bank would adopt a pure strategy of abandoning the peg.

On the other hand, in Case II where $\epsilon_t > \epsilon^L$, the magnitude of the disturbance this period is already high compared to the expected future disturbance ($E_t(\epsilon_{t+1})$). If the Central Bank were to abandon the peg, it would incur τ in addition to a high

ϵ_t. Hence the Central Bank wouldn't strictly prefer to abandon the peg this period.

Notice that if $\epsilon^L = 0$, then since $\epsilon_t > 0$ for any t, only Case II exists, i.e., the Central Bank would never strictly prefer to abandon the peg until $\bar{T}$, which is the date when $\bar{r}$ is reached. Intuitively speaking, this is because the government's preference for fixed exchange rate is high enough (τ has a higher lower bound) that however low the disturbance is this period, although the disturbance can be higher next period, the Central Bank would not prefer to abandon the fixed rate definitely in any period.

□

Here we consider the other possible pure strategy game for the Central Bank, i.e., if $q_t = 0$ for any $t < \bar{T}$ in the discrete time case.

Proposition 5. If $t < \bar{T}$ and $q_{t+1} > 0$, then $q_t = 0$ can only be part of an equilibrium strategy if $r_{t+1} - r_t \leqslant \frac{1-\delta}{\delta}\tau - \frac{1}{\lambda}$. In continuous time, this would only be an equilibrium if $\epsilon_{\bar{T}-1} \geqslant \eta\phi\theta + \frac{1}{\lambda}$.

Intuitively speaking, passively waiting for an attack can only be optimal when the increase in reserves is small relative to the policy maker's preference for the exchange rate peg.

Proof

For the Central Bank to choose $q_t = 0$,

$$\frac{r_t + \epsilon_t + \tau}{1-\delta} \geqslant r_t + \epsilon_t + \delta\left(\frac{r_{t+1} + \epsilon_{t+1} + \tau}{1-\delta}\right) \tag{207}$$

which reduces to

$$r_{t+1} - r_t \leqslant \frac{(1-\delta)\tau}{\delta} - \frac{1}{\lambda} + \epsilon_t \tag{208}$$

Since $\epsilon_t > 0$, if $r_{t+1} - r_t \leqslant \frac{(1-\delta)\tau}{\delta} - \frac{1}{\lambda}$, then the Central Bank would choose to remain passive, even knowing that there would be a speculative attack in the next period.

In continuous time, Equations 187, 189 and 190 are not dependent on the period length and can be retained without modification, noting only that it in the money demand Eq.189 refers to the interest return over calendar time 1. Therefore the per-period interest rate is given by ni_t, so Eq. 188 becomes,

$$ni_{t+1} = ni^* + E_t e_{t+1} - e_t \tag{209}$$

and 191 is

$$y_t = y_{t-1} + n\theta \tag{210}$$

From these equations the shadow floating exchange rate in the traditional model can be derived using the same method to show that Equation 198 still holds. The time of the speculative attack in the traditional model $\bar{T}$ is given by $\tilde{e_t} = \bar{e}$, which implies that reserves increase by at $\eta\phi\theta$ at $\bar{T}$. So the size of the speculative attack in the traditional model is not dependent on the period length. This is unsurprising since the model generates attacks in continuous time as well.

Now moving to the problem of the policy maker, the policy maker's per-period discount rate is δ^n. Its per-period cost is proportional to the period length: $n(r_t + \epsilon_t)$ for fixed exchange rates and $n(r_t + \epsilon_t + \tau)$ with floating rates. Therefore the policy maker's maximization problem is

$$V_t = min\left\{ \frac{n(r_t + \epsilon_t + \tau)}{1 - \delta^n} \;,\; n(r_t + \epsilon_t) + \delta^n E_t(V_{t+1}) \right\} \tag{211}$$

We are interested here in whether it is optimal for the policy maker to set $q_{\bar{T}-1} = 0$ knowing that there will be a speculative attack in the next period. If this is so, then from the maximization of 211,

$$\frac{n(r_{\bar{T}-1}+\epsilon_{\bar{T}-1}+\tau)}{1-\delta^n} \geqslant n(r_{\bar{T}-1}+\epsilon_{\bar{T}-1}) + \frac{n\delta^n(r_{\bar{T}}+\epsilon_{\bar{T}}+\tau)}{1-\delta^n} \tag{212}$$

which reduces to

$$\frac{1}{\lambda} - \epsilon_{\bar{T}-1} + r_{\bar{T}} - r_{\bar{T}-1} \leqslant \tau(1-\delta^n)/\delta^n \tag{213}$$

Since reserves increase by $n\phi\theta$ each period just due to real economic growth and since there will be a speculative attack boosting reserves by $\eta\phi\theta$ at $\bar{T}$, $r_{\bar{T}} - r_{\bar{T}-1} = n\phi\theta + \eta\phi\theta$. Therefore Equation 213 requires

$$\eta\phi\theta \leqslant \tau(1-\delta^n)/\delta^n - n\phi\theta + \epsilon_{\bar{T}-1} - \frac{1}{\lambda} \tag{214}$$

This condition must hold for it to be optimal for the policy maker to remain passive in the face of an expected speculative attack. In the limit as $n \to 0$

$$\epsilon_{\bar{T}-1} \geqslant \eta\phi\theta + \frac{1}{\lambda} \tag{215}$$

This is saying that when $\epsilon_{\bar{T}-1}$ is big enough ($\epsilon_{\bar{T}-1} \geqslant \eta\phi\theta + \frac{1}{\lambda}$), then it is sensible for the Central Bank to remain passive knowing there would be a speculative attack next period. Intuitively speaking, the left hand side of Equation 213 captures how much more painful it would be to remain passive this period and wait until next period to abandon the fixed regime. When $\epsilon_{\bar{T}-1} \geqslant \eta\phi\theta + \frac{1}{\lambda}$, it is already painful this period, so rather than incurring an additional cost τ to abandon the peg this period, the Central Bank will just wait until next period (there is the discount rate δ that makes the Central Bank want to procrastinate) since it won't be much worse.

□

Subgame-Perfect Nash Equilibrium

In our model, there are three cases that could happen before r reaches $\bar{r}$, which is the upper threshold for reserves:

Case I

If in any period $t < \bar{T}$, $\epsilon_t \leqslant \epsilon^L$, then as discussed in **Proposition 4**, the Central Bank would abandon the peg in that period ($q_t = 1$).

Case II

If in any period $t < \bar{T}$, $\epsilon_{\bar{T}-1} \geqslant \eta\phi\theta + \frac{1}{\lambda}$, then the Central Bank would remain passive ($q_t = 0$) even when knowing the peg will be abandoned next period.

Case III

If for all periods $t < \bar{T}$, $\epsilon^L \leqslant \epsilon_t \leqslant \eta\phi\theta + \frac{1}{\lambda}$, then the Central Bank would adopt play a mixed-strategy game as in our original model.