Towards Differentially Private Inference on Network Data

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Chapter 1: Introduction

1.1 Motivation

Networks are ubiquitous both as subjects of scientific study and as fixtures of everyday life. From social networks like Facebook to webs of financial transactions, mobile phone call records, and communications via email, rich network data constitutes a popular subject for statistical inquiry in a broad range of disciplines. However, due to the interconnected nature of the data, protecting the privacy of participants in a network while conducting statistical analysis can be difficult. A number of recent examples highlight the challenges of keeping network data private:

The Cambridge Analytica Case [RCC18]: In late March 2018, revelations emerged that a political consulting firm, Cambridge Analytica, had harvested over 50 million user profiles off Facebook allowing them to build psychological profiles of a vast portion of the American electorate. Only $270,000$ users actually consented to give Cambridge Analytica access to their profile information via an online survey. However, by leveraging users' friend networks, it was possible for Cambridge Analytica to violate the privacy of a much larger number of people.

"Gaydar" [JM09]: Consider an individual on Facebook who does not publicly disclose their sexual orientation, presumably because they wish this data to be kept private. By analyzing the proportion of this user's friends who publicly reveal being gay, it is possible to learn with high accuracy whether this user is gay or straight. In effect, then, a person's relationships along with publicly available data about their acquaintances, friends, and coworkers implicitly disclose private information about the person.

The Structure of Intimate Relationships [BK14]: Even with no profile information made public, it is possible to identify the romantic partner of a Facebook user with high accuracy using only the structure of their friend network – an intimate relationship is highly likely between individuals in the network who have many mutual friends, but whose friends have few mutual friends, a phenomenon known as "dispersion." Therefore, suppressing identities of people in the network does not suffice to protect privacy, as the links within the network alone may reveal potentially sensitive, private information like a person's romantic partner.

These examples highlight the fundamental difficulty of analyzing relational data while respecting the privacy of data holders – network structure discloses an extensive amount of ostensibly private information about both the identity of participants and the nature of their relationships. Further, distinctive substructures of a network can make it relatively easy to reconstruct a naively anonymized network given auxiliary information: a number

of proposed attacks on networks have demonstrated that an attacker with relatively little additional information may (with high probability) identify participants in any unlabeled network ([BDK11],[NS09]). Thus, statistical analysis of network data, while popular, is also problematic from a privacy standpoint. In response to this issue, a growing body of work seeks to answer the question: is it possible to protect the privacy of individuals included in a network dataset while enabling researchers to conduct useful analysis of network structure?

1.1.1 Differential Privacy

A promising direction for answering this question involves employing a rigorous and meaningful concept of privacy first proposed for analysis of tabular data: differential privacy [DMNS06]. At a high level, differential privacy promises that the participation of any single individual in a dataset will not noticeably alter the results of an analysis of the dataset. The differential privacy guarantee holds even if an adversary is equipped with arbitrary auxiliary information about the participants in a dataset. Indeed, differential privacy promises that were an adversary to know the data of every other individual in a dataset, she still could not discover the private information of the final unknown participant in the dataset.

Differential privacy provides a quantifiable notion of privacy, as it is operationalized by two small, non-negative parameters ϵ and δ. The parameter ϵ captures the amount of privacy leaked by the analysis. As ϵ decreases, it becomes more difficult for an adversary to discern the private information of an individual in the dataset. The parameter δ specifies the probability of a potentially catastrophic privacy leakage. When $\delta = 0$, we speak of ϵ-differential privacy. For $\delta > 0$, we provide (ϵ, δ)-differential privacy, which promises that an algorithm is ϵ-differentially private with probability $1 - \delta$, but permits an arbitrarily bad privacy leak with probability δ. We may be comfortable with this relaxed concept of privacy if δ is vanishingly small (one in a million, for instance), so that the chance of a privacy leak is low. The quantifiability of differential privacy allows for rigorous study of the trade-off between privacy and utility in data analysis.

Differential privacy is a property of an algorithm – it promises privacy by process. Only randomized algorithms meet the definition of differential privacy, suggesting that the algorithm must introduce noise in some way, either by perturbing the inputted data, the steps taken by the algorithm, or the output. One simple way to answer queries on a dataset under differential privacy constraints is by adding statistical noise drawn from an appropriately scaled Laplace distribution to the output of the query. This mechanism, called the Laplace mechanism, gives a general way of providing differential privacy, but requires one to compute the appropriate scale of Laplace noise to protect privacy for a specific query. In this work, we will employ the Laplace mechanism to provide differential privacy.

Protecting the privacy of individuals included in a network dataset requires identifying what aspects of the network should be considered private. A network abstractly represents relationships between various entities – the entities are referred to as *nodes* in the network

(potentially with *labels* specifying nodal attributes) and the links between entities are referred to as *edges*. As the examples of "Gaydar" and romantic relationships on Facebook suggest, even if only node labels are considered private data, treating labels alone as private may not in itself preserve privacy, as the edge structure can reveal sensitive information about the labels. Furthermore, the goal may be to explicitly protect the relationships in a network, not just identities of participants. For instance, in a network of romantic partnerships the identity of participants may be public information, while the relationships could be sensitive. Thus, meaningful notions of privacy should seek to also protect the privacy of edges in a network.

In protecting the privacy of edges in a network there is ambiguity as to what granularity of privacy to provide. Differential privacy specifies that the "participation" of any single individual in a dataset should not alter the result of an analysis significantly. We could take "participation" to mean the inclusion of a single edge in the network and guarantee *edge-level privacy* by protecting the privacy of any sensitive relationship in the network. Alternatively, we could protect the inclusion of a node and all edges incident to that node in the network, providing a much stronger privacy guarantee known as *node-level privacy*. While node-level privacy offers a strictly stronger privacy guarantee than edge-level privacy, there may be cases where we are only concerned with protecting any single relationship in a network, not all of an individual's relationships. Further, node-level privacy may have an extensive cost in utility. Therefore, we consider both the notions of edge-level privacy and node-level privacy in our analysis.

1.1.2 Inference on Network Data

The goal of this work is to enable differentially private statistical inference for network data using a general class of models known as exponential random graph models or ERGMs. In contrast to simply computing statistics of a network – like degree distributions or clustering coefficients – a statistical model posits an explicit probability distribution over the space of possible networks, allowing researchers to study the distinctive structural properties of an instantiated network and the processes that gave rise to such structure. ERGMs are among the most commonly employed statistical models of network data, having been applied to a broad range of problems, including analysis of corporate management structures at Enron [UHH13], the demographics of high school friendships [GKM09], interactions between proteins in the human body [RAS10], and networks of neurons in the brain as people age [SDC+16].

ERGMs are described fully by a vector of "sufficient statistics" computed on the network, which are generally aggregations of small substructures of the network like the number of edges or the number of triangles (groups of three connected nodes.) The ERGM associates a different parameter with each sufficient statistic, allowing researchers to understand the relative importance of different substructures in a network. Then, the goal of inference over an ERGM is to estimate parameters of the model that make sufficient statistics of the real-world network likely in the modeled probability distribution.

There is a growing body of research on differentially private release of various statistics

of networks, such as degree distributions ([HLMJ09], [DLL16]), clustering coefficients [WWZX12], and counts of small subgraphs like triangles [KRSY14] among many others. In comparison, there has been relatively little study of differentially private *inference* over network data. Karwa and Slavkokvic propose a differentially private inference method for a specific class of ERGM known as the β-model [KS16]. The β-model uses only the degree distribution as a sufficient statistic. While Karwa and Slavkokvic provide an elegant mathematical formulation of differentially private inference on this model, the β-model is used relatively infrequently in actual analyses, as it cannot capture many structures of interest in network data. For general ERGMs there exist two proposed methods. Lu and Miklau [LM14] give an (ϵ, δ)-differentially private inference method based on adding Laplace noise to sufficient statistics of an ERGM, while Karwa et. al [KKS17] propose an ϵ-differentially private method based on flipping edges of the underlying network inputted to inference. Both of these approaches work only for edge-level privacy with node labels taken to be public. Additionally, they permit accurate inference only for large privacy budgets (with ϵ taken to be greater than 3 or δ taken to be 0.5,) while in practice we want smaller privacy budgets with ϵ less than 1 and δ taken to be very small (on the order of one in a million, for instance) [NSW+17]. In short, existing approaches to differentially private inference on network data only enable useful analysis for a weak privacy guarantee, namely, for settings where we use relatively large privacy budgets in the edge-privacy model with publicly known node labels.

1.2 Contributions

Motivated by the goal of performing useful inference on network data while providing meaningful privacy guarantees, we propose a new framework for differentially private inference on ERGMs. We prove the privacy of our methods and then empirically evaluate their performance relative to alternative approaches and to non-private inference. There are three primary features of our proposed methods that move us towards the goal of practical differentially private inference for network data using ERGMs:

- We enable accurate inference under edge-level privacy at *smaller, more realistic, privacy budgets* than current methods.
- Unlike previously proposed methods, we permit differentially private inference under edge-level privacy with *private node labels*, rather than treating labels as public.
- We suggest the first (to our knowledge) method for differentially private inference under the stronger notion of *node-level privacy.*

Our approach takes advantage of the recently proposed machinery of "restricted sensitivity" ([BBDS13], [KNRS13]) to perturb network data much less dramatically than existing methods for guaranteeing differential privacy. Restricted sensitivity exploits the observation that many real world networks are sparse: individuals in the network tend to have relatively few relationships compared to the size of the network. For instance, Facebook has over 2 billion users, but users have no more than a few thousand friends. While restricted sensitivity was initially proposed for edge-level and node-level differentially

private release of statistics of networks, it has not been utilized for statistical inference over networks. We propose employing restricted sensitivity to elegantly leverage the sparsity common in real-world network data to perform useful private inference. In particular, we use the framework of restricted sensitivity to add statistical noise to computed sufficient statistics and then use these statistics to perform inference. Our inference method takes into account the noise introduced to sufficient statistics by the privacy mechanism to infer valid parameter estimates.

To evaluate the performance of our methods, we conduct extensive experimentation on both synthetic network data and a real high school friendship network. Compared to prior work in differentially private inference on network data, which tests on networks of under 150 nodes, we run inference experiments on larger networks of $200 - 300$ nodes. This is useful as current analyses using ERGMs increasingly look at larger networks, both because larger network datasets are becoming available and because greater computational power now enables inference on large networks. Further, by virtue of their size, larger networks are easier to keep private than smaller networks: when there are few participants, any one participant has a strong impact on an analysis, compared to a network with many participants. Thus, our experiments use networks large enough to guarantee meaningful levels of differential privacy while permitting useful inference.

Our experiments on synthetic networks offer evidence that for small privacy budgets of $\epsilon = 1$ or 2, our method estimates parameters more accurately than existing methods. Then, we show in a case study on high school friend network data that our method allows researchers to accurately estimate parameters and standard errors for a budget of $\epsilon = 2$, while existing approaches fail even for a larger budget of $\epsilon = 3$. This suggests that our approach enables researchers to reach reliable conclusions about the structure of real-world network data under substantive privacy constraints. Finally, we demonstrate the viability of our proposed methods for inference in the node-level privacy model by evaluating the noise addition of our proposed methods under a variety of assumptions. In summary, our experimental results suggest that our proposed restricted sensitivity-based methods allow for accurate inference under strong privacy guarantees in many settings where current methods do not, moving us closer to the goal of useful differentially private statistical modeling of network data.

1.3 A Road Map

The remainder of this thesis is structured as follows:

In **Chapter 2** we introduce the mathematical formulation of ERGMs. We define sufficient statistics of these models commonly used to capture network structure and then describe a standard non-private Bayesian inference method known as the Exchange Algorithm on which our private inference method is based.

In **Chapter 3** we give detailed background on differential privacy. We specify general mechanisms that meet the definition of differential privacy and useful properties of differential privacy and then describe the machinery of restricted sensitivity.

In **Chapter 4** we propose new methods for differentially private inference on ERGMs. In particular, we prove bounds on the restricted sensitivity of the common sufficient statistics of ERGMs introduced in Chapter 2 under both edge-level and node-level privacy, so that using mechanisms introduced in Chapter 4, we can provably protect differential privacy by adding statistical noise to sufficient statistics. We introduce a modified version of the Exchange Algorithm for Bayesian inference on ERGMs, which takes into account the noise of the privacy mechanism, thereby performing valid inference over the private posterior. Finally, we compare our method to existing work, giving high-level motivation for why we expect restricted sensitivity to outperform current methods.

In **Chapter 5** we empirically evaluate the performance of our proposed methods against current work in both the edge-level and node-level privacy models. We look at both the level of noise addition to sufficient statistics and the accuracy of parameter estimation for 3 synthetic network models. In addition, we test inference on a friend network of high school students.

In **Chapter 6** we conclude with suggestions for future work.

Chapter 2: Statistical Modeling of Networks

An increasingly popular approach in quantitative analysis of networks is to fit statistical models to realized network data. Many of these models have generative interpretations, allowing researchers to understand the relative importance of multiple endogenous processes to the resulting structure of the network. The advantage of such an approach is best illustrated in contrast to computing statistics – like degree distributions or clustering coefficients – to describe the network structure, without an explicit model of the network. While such metrics are useful in summarizing the structural properties of a given network, they cannot tease out the underlying processes that may give rise to such structures.

For example, one of the distinguishing characteristics of many real-world social networks is the tendency to have more triangles (sets of three connected nodes) than would be expected by drawing random edges of a graph [GKM09]. There are a number of different processes in the formation of a friend network that could give rise to this outcome. One potential explanation is the notion of "triangle closure," or the tendency for people to become friends with friends-of-friends, since they are easier to meet. Another subtly different explanation is that triangles arise out of "assortative matching," the propensity for people with the same attributes to become friends with one another, leading to clustering in the network. Finally, a high number of triangles in a social network could arise for reasons of "sociality," the presence of only a few highly social individuals in the network, who are mutual friends to many people. In order to consider what global or local processes best explain particular structures of a network, a statistical model of network data posits a probability distribution over the space of possible networks. The goal of inference is to tune parameters of the distribution, such that the realized network is likely to be observed under the probability distribution.

A simple example of such a model is the Erdös-Rényi Random Graph Model, known as the $G(n, p)$ model, which proposes that edges are drawn independently with probability p between any two nodes of a network with n nodes. While this model has been studied in great depth by graph theorists, it does not capture many important features of real world networks, like the tendency for clustering or the power-law distribution of degrees. In order to model such structures in networks, a more general class of random graph models are Exponential Random Graph Models.

In this chapter, we give the mathematical foundations for exponential random graph models. In Section 2.1, we introduce the formulation of the general class of models. In Section 2.2, we provide definitions and intuition for some of the most commonly used sufficient statistics in ERGMS. We introduce both statistics that capture substructure of the network like triangles or stars and statistics that account for labeled nodes in the network. Lastly, in Section 2.3 we describe a standard method used for Bayesian inference on ERGMs called the Exchange Algorithm.

2.1 Exponential Random Graph Models (ERGMs)

Formally, a graph $G = (V, E)$ is defined by a set of nodes (or vertices) V, with $|V| = n$ and edges E, representing the presence or absence of relationships between nodes. We will use the "adjacency matrix" representation of a graph, which we denote x, where $x_{ij} = 1$ if an edge exists between nodes i and j and $x_{ij} = 0$ otherwise. The models we consider are defined over undirected graphs, where all the edges are bidirectional, and the adjacency matrix is therefore symmetric. We refer to the number of edges adjacent to node i as the *degree* of node i so $d_i = \sum_{j=1}^{n} x_{ij}$. Then, the *degree distribution* is $D = (D_0, ..., D_{n-1})$ where $D_k = |\{i \in V : d_i = k\}|$.

Definition 2.1 (Exponential Random Graph [WP96])**.** A probability distribution over graphs of n vertices belongs to the family of *exponential random graph models* (henceforth referred to as ERGMs) if it takes the form:

$$\Pr(x|\theta) = \exp\left\{\theta^T u(x) - \psi(\theta)\right\}$$

where θ is a vector of parameters of the model, $u(x)$ is a vector of sufficient statistics computed on graph x, and $\psi(\theta)$ is a normalization constant needed to ensure a valid probability distribution so:

$$\psi(\theta) = \log \sum_{x'} \exp\left\{\theta^T u(x')\right\}$$

ERGMs describe a broad class of random graphs, with varying conditional dependence relationships between edges. For instance, the $G(n, p)$ graph can be viewed as an ERGM:

Example 2.1 ($G(n, p)$ graphs)**.** We can represent the Erdös-Rényi Random Graph ($G(n, p)$) model as an ERGM, by taking

$$u(x) = |E|, \quad \theta = \log \frac{p}{1-p}$$

$$\psi(\theta) = -\binom{n}{2} \log(1-p) = -\binom{n}{2} \log \frac{e^{-\theta}}{1+e^{-\theta}}$$

Then,

$$\begin{aligned}
\Pr(x|\theta) &= \exp\left\{|E| \log \frac{p}{1-p} + \binom{n}{2} \log(1-p)\right\} \\
&= p^{|E|}(1-p)^{\binom{n}{2}-|E|} \\
&= \prod_{i<j} p^{x_{ij}}(1-p)^{1-x_{ij}}
\end{aligned}$$

so each possible edge is included independently with probability p as specified by the Erdös-Rényi Model.

In order to model more complex structures in a network, researchers have proposed higher order sufficient statistics of ERGMs that imply more general conditional independence

assumptions than the Erdös-Rényi Model. For instance, "Markov" graphs, allow the probabilities of any two possible edges in a graph to be conditionally dependent if the edges share a common endpoint. This dependency allows for node level effects on edge formation. In fact, Markov dependencies are captured by ERGMs of the following form:

Example 2.2 (Markov graphs [FS86])**.** Any undirected *Markov graph* has probability distribution:

$$\Pr(x|\theta,\tau) = \exp\left\{\sum_{k=1}^{n-1}\theta_k S_k(x) + \tau T(x) - \psi(\theta,\tau)\right\}$$

where the sufficient statistics are

$$\begin{aligned}
&\text{number of edges:} && S_1(x) = \sum_{1\le i<j\le n} x_{ij} = |E| \\
&\text{number of } k\text{-stars } (k\ge 2): && S_k(x) = \sum_{i=1}^{n-1}\binom{i}{k} D_i(x) \\
&\text{number of triangles:} && T(x) = \sum_{1\le h<i<j<\le n} x_{hi}x_{ij}x_{hj}
\end{aligned}$$

and the parameters are $\{\theta_k\}_{k=1}^n$ and τ.[1]

2.2 Sufficient Statistics of ERGMs

In practice, due to its simplicity, the $G(n,p)$ model is used only as a starting point in inference over real-world data, while the full Markov graph model is infrequently used as it suffers from poor statistical properties. In particular, the Markov graph model is degenerate for many parameter configurations, representing only distributions that put all of their probability mass on either nearly-complete graphs (graphs with all edges present) or on $G(n,p)$ graphs [Jon99]. In response to these problems of degeneracy with Markov graphs, more robust "alternating" sufficient statistics are generally used in ERGMs to capture structural properties of networks. We will first provide definitions of these statistics and then expand on the mathematical motivation behind them.

2.2.1 Alternating Sufficient Statistics

Definition 2.2 (Alternating k-star statistic [SPRH06])**.** The *alternating k-star* statistic on graph x with weighting parameter $\lambda \ge 1$ is defined as

$$\begin{aligned}
u_\lambda^{(s)}(x) &= S_2 - \frac{S_3}{\lambda} + \frac{S_4}{\lambda^2} - \cdots + (-1)^{n-2}\frac{S_{n-1}}{\lambda^{n-3}} \\
&= \sum_{k=2}^{n-1}\frac{S_k}{\lambda^{k-2}}
\end{aligned}$$

[1]Note that setting $\theta_2 = ... = \theta_k = \tau = 0$ in the Markov model, we recover the $G(n,p)$ model, which is an instance of a Markov graph since any two edges are conditionally independent in the $G(n,p)$ model.

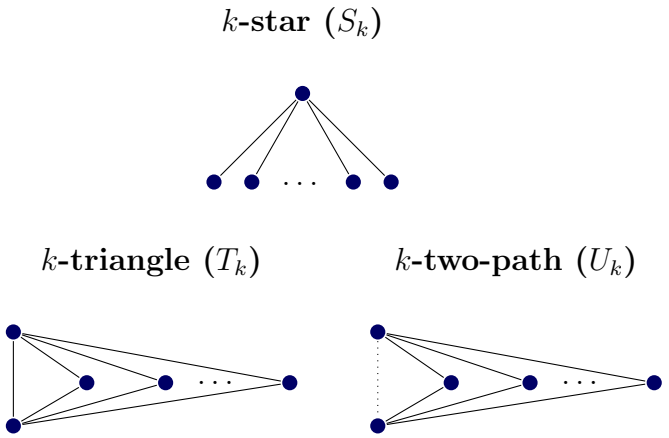


Figure 2.1: Subgraphs used in sufficient statistics of ERGMs.

We introduce the notion of "shared partners" of two nodes – the number of common neighbors that two nodes share – which give a clean way to count k-triangles and k-two-paths.

Definition 2.3 (Shared partners)**.** We denote the *shared partner count* of nodes i and j:

$$P_{ij}(x) = \sum_{\ell \in V} x_{i\ell} x_{j\ell} \tag{2.1}$$

We define *k-triangles* analogously to k-stars, so that a k-triangle consists of k triangles that all share an edge. We can count the total number of k-triangles in a graph using the number of shared partners:

$$T_k(x) = \sum_{1 \le i < j \le n} x_{ij} \binom{P_{ij}}{k} \quad \text{for } (k \ge 2), \quad \text{and } T_1 = \frac{1}{3} \sum_{1 \le i < j \le n} x_{ij} P_{ij} \tag{2.2}$$

where T_1 has an extra factor of $\frac{1}{3}$ in front because of the symmetry of a 1-triangle for all three edges included in the triangle.

Definition 2.4 (Alternating k-triangle statistic [SPRH06])**.** The *alternating k-triangle* statistic on graph x with weighting parameter $\gamma \ge 1$ is defined as

$$\begin{aligned} u_\gamma^{(t)}(x) &= 3T_1 - \frac{T_2}{\gamma} + \frac{T_3}{\gamma^2} - \cdots + (-1)^{n-3} \frac{T_{n-2}}{\gamma^{n-3}} \\ &= 3T_1 + \sum_{k=2}^{n-2} \left(\frac{-1}{\gamma} \right)^{k-1} T_k \end{aligned}$$

with a factor of 3 in front of T_1 so that each k-triangle is counted once for each edge that is a base of the k-triangle.

We define an *independent k-two-path* as a pair of nodes (possibly connected or unconnected) with k paths of length 2 connecting them. We can think of a k-two-path as a precondition for a k-triangle, since every k-triangle must contain an independent k-two-path. In terms

of shared partners, independent k-two-paths can be represented as:

$$U_k(x) = \sum_{1 \le i < j \le n} \binom{P_{ij}}{k} \text{ for } k \neq 2 \quad \text{and } U_2(x) = \frac{1}{2} \sum_{1 \le i < j \le n} \binom{P_{ij}}{2} \tag{2.3}$$

where U_2 is preceded by a factor of $\frac{1}{2}$, because a k-two-path with $k = 2$ is a 4-cycle and hence is symmetric with respect to the two pairs of non-adjacent nodes making up the cycle.

Definition 2.5 (Alternating k-two-path statistic [SPRH06])**.** The *alternating k-two-path* statistic on graph x with weighting parameter $\gamma \geq 1$ is defined as

$$\begin{aligned} u_\gamma^{(p)}(x) &= U_1 - \frac{2U_2}{\gamma} + \frac{U_3}{\gamma^2} - \cdots + (-1)^{n-3} \frac{U_{n-2}}{\gamma^{n-3}} \\ &= U_1 - \frac{2U_2}{\gamma} + \sum_{k=3}^{n-2} \left(\frac{-1}{\gamma} \right)^{k-1} U_k \end{aligned}$$

Now, having defined the "alternating" sufficient statistics, the proposed model has the form

$$\Pr(x|\theta) = \exp\left\{ \theta_1 E(x) + \theta_2 u_\lambda^{(s)}(x) + \theta_3 u_\gamma^{(t)}(x) + \theta_4 u_\gamma^{(p)}(x) - \psi(\theta) \right\} \tag{2.4}$$

where $E(x)$ is the number of edges in graph x, the alternating k-two-path and k-triangle statistics generally use the same weighting parameter γ. In practice, a subset of the sufficient statistics can be used in the model, depending on what properties of a graph are pertinent to model for a given network.

The overarching motivation behind introducing "alternating" sufficient statistics of the ERGMs is that these statistics are robust to addition or removal of an edge adjacent to an individual node, alleviating degeneracies in the Markov graph model. For instance, consider adding an edge to a high degree node with degree k. This new edges contributes one $(k+1)$-star, $\binom{k}{k-1}$ k-stars, $\binom{k}{k-2}$ $(k-1)$-stars and so on. Therefore, the total number of additional stars in the graph resulting from adding this edge is $\sum_{i=0}^{k} \binom{k}{i} = 2^k$. For Markov graphs including all stars with arbitrary associated parameters, this could lead to a large increase (or decrease) in the likelihood of the graph making the model degenerate as it places almost all of its probability on either near-complete or near-empty graphs. However, by imposing constraints on the parameters θ_k, namely by alternating the signs of the k-star statistics, the additional $(k-1)$-stars and k-stars balance each-other out. The same general reasoning applies to the use of alternating statistics for k-triangles and k-two-paths – alternation prevents the probability distribution from putting all of its mass on graphs with many high degree nodes, preventing degeneracy of the model.

This interpretation of alternating statics as limiting the sensitivity of the likelihood to addition or removal of edges to high degree nodes can be understood by looking at an alternative representation of the statistics in terms of the degree distribution and the number of shared partners for nodes. Below, we present these equivalent representations of the statistics, which will also be helpful in proofs of privacy in Chapter 4.

Alternating k-star

Note that using the relationship between k-stars and degrees given in Example 2.2 along with the binomial theorem we can rewrite the *alternating k-star* statistic as:

$$\begin{aligned} u_\lambda^{(s)}(x) &= \sum_{i=1}^{n-1} D_i(x) \sum_{k=2}^{n-1} \left(\frac{-1}{\lambda}\right)^{k-2} \binom{i}{k} \\ &= \lambda^2 \sum_{i=0}^{n-1} \left(\frac{\lambda-1}{\lambda}\right)^i D_i + 2\lambda|E| - n\lambda^2 \end{aligned} \tag{2.5}$$

The alternating k-star statistic is thus made up of the number of edges as well as a linear combination of the degree sequence where lower degree nodes are up-weighted exponentially compared to higher degree nodes, reflecting the tendency towards a power law degree distribution. Since a term representing the number of edges in the network is generally included along with this statistic, the model is mathematically equivalent to a model using a geometrically weighted average of the degree sequence. Sociologically, the coefficient of the k-star statistic can thus be interpreted as the propensity for high degree nodes in the network. If the coefficient of the statistic is positive, then networks with a few high degree "hubs" are observed, while if it is negative, high degree nodes are discouraged and the network consists of mostly low-degree nodes [SPRH06].

Alternating k-triangle

Similarly, for the alternating k-triangle statistic, we can gain insight by rewriting in terms of the number of shared partners for pairs of nodes. By using this representation of k-triangles from Equation (2.2) along with the binomial theorem, we can rewrite the *alternating k-triangle* statistic as:

$$\begin{aligned} u_\gamma^{(t)}(x) &= \sum_{1\le i<j\le n} x_{ij} \sum_{k=1}^{n-2} \left(\frac{-1}{\gamma}\right)^{k-1} \binom{P_{ij}}{k} \\ &= \gamma \sum_{1\le i<j\le n} x_{ij} \left(1 - \left(\frac{\gamma-1}{\gamma}\right)^{P_{ij}}\right) \\ &= \gamma|E| - \gamma \sum_{1\le i<j\le n} x_{ij} \left(\frac{\gamma-1}{\gamma}\right)^{P_{ij}} \end{aligned} \tag{2.6}$$

Note, then, that any edge that does not participate in a triangle (so $x_{ij} = 1$ but $P_{ij} = 0$) does not contribute to the the alternating k-triangle statistic. On the other hand, as we add additional shared partners to an edge, the second term in (2.6) falls exponentially so the statistic increases, but by less for higher order k-triangles than for lower-order triangles. Sociologically, this term can be interpreted as the importance of triangle closure in the generation of the graph [GKM09]. In contrast to directly including the number of triangles in the graph, the alternating k-triangles statistic is more stable, preventing the model degeneracies discussed above.

Alternating k-two-path

Using the representation of k-two-paths in terms of shared partners from Equation (2.3) and the binomial theorem, we can rewrite the *alternating k-two-path* statistic as:

$$\begin{aligned} u_\gamma^{(p)}(x) &= \sum_{1\le i<j\le n} \sum_{k=1}^{n-2} \left(\frac{-1}{\gamma}\right)^{k-1} \binom{P_{ij}}{k} \\ &= \gamma \sum_{1\le i<j\le n} \left(1 - \left(\frac{\gamma-1}{\gamma}\right)^{P_{ij}}\right) \\ &= \gamma\binom{n}{2} - \gamma \sum_{1\le i<j\le n} \left(\frac{\gamma-1}{\gamma}\right)^{P_{ij}} \end{aligned} \tag{2.7}$$

Thus, the alternating k-two-path has an interpretation similar to that of the alternating k-triangle. As shared partners are added for any two nodes, the second term of the statistic increases, but the increase falls exponentially with additional partners. This term is generally only included in conjunction with the k-triangle statistic to try to separate out the effects of two-paths forming between unconnected nodes and mutual connections forming between already connected nodes.

2.2.2 Sufficient Statistics for Labeled Nodes

The alternating statistics over k-stars, k-triangles and k-two-paths capture structural properties of network data. Frequently, however, there are labels associated with nodes in the network, which are important to model. ERGMs can take into consideration both the structure of the network and the labels associated with nodes, by including sufficient statistics based off of the labels, allowing researchers to capture properties like homophily, the tendency for similar actors to build relationships with one another within a network. Generally, labels are taken to be fixed and exogenous to the edges, so that attributes of the nodes may affect the formation of the network, whereas relationships in the network are not thought of as impacting attributes. This is generally a reasonable assumption, as labels often represent the identity of an individual, containing characteristics like gender, race, or age. As there are many potential ways to incorporate labeled data into an ERGM, we will focus here on three of the most commonly used statistics for discrete nodal attributes, "homophily", "popularity" and "mixing."[2]

In particular, letting z_i be a discrete attribute of node i (gender, for instance) we introduce the following sufficient statistics to represent different processes of social selection [LKR12]:

[2]These have fairly straightforward analogues for continuous nodal attributes, but we focus on the discrete case, as this is applicable to the dataset analyzed.

Table 2.1: Common sufficient statistics for discrete nodal attributes.

Parameter	*Statistic*
Homophily (Uniform)	$\sum_{i<j} x_{ij}\mathbb{I}(z_i = z_j)$
Homophily (Differential)	$\sum_{i<j} x_{ij}\mathbb{I}(z_i = z_j = a)$
Popularity	$\sum_{i<j} x_{ij}\left(\mathbb{I}(z_i = a) + \mathbb{I}(z_j = a)\right)$
Mixing	$\sum_{i<j} x_{ij}\mathbb{I}(z_i = a)\mathbb{I}(z_j = b)$

Uniform homophily captures the tendency for nodes with the same attribute to share an edge, while differential homophily captures this phenomenon for a specific attribute, which may be useful if, for instance, we thought that men and women have different propensities to become friends with people of the same gender. The popularity parameter is fairly self-explanatory as it measures the number of edges that have nodes with a given attribute as an endpoint and can be thought of as the overall sociability of a group with a specific attribute. Finally, the mixing parameter represents the number of edges between nodes with two different, specific attributes. Including such nodal attribute statistics in conjunction with the alternating sufficient statistics discussed in Section 2.2 allows for specification of ERGMs that separate out social selection effects like homophily from structural effects like triangle closure, making ERGMs a powerful modeling tool.

2.3 Bayesian Inference on ERGMs

Having provided an ERGM specification that captures the characteristics of interest in a network, the goal of inference is to find parameters θ that describe the realized data well. In the framework of maximum likelihood estimation, this means finding a θ that maximizes the probability of drawing observed network x_{obs} from the distribution $p(X|\theta)$. In the Bayesian paradigm, an analyst specifies a prior distribution over θ and then wishes to compute a posterior distribution of θ given the observed network. Bayesian inference is more general than maximum likelihood inference in the sense that if an analyst chooses a flat prior on θ (a uniform prior over the parameter space) and takes the maximum of the posterior as a point estimate, then Bayesian inference reduces to maximum likelihood inference.

In general, exact inference is not feasible for ERGMs due to the presence of the intractable normalizing constant $\psi(\theta)$ in the likelihood (Definition 2.1), which is a sum over the space of possible graphs on n nodes of size $2^{\binom{n}{2}}$. Therefore, a number of approximate MCMC approaches have been proposed to perform inference. In this work, we focus on Bayesian inference over ERGMs, because it constitutes the state-of-the-art in non-private inference methods and has been shown to be more stable than MCMC-MLE approaches

[CF11]. Additionally, the noise from differentially private mechanisms can be incorporated quite naturally into the Bayesian framework. The non-private Bayesian inference method proposed by Caimo and Friel is based on the Exchange Algorithm[MGM12] and is fairly simple to describe:

Algorithm 1 Non-Private Bayesian Inference for ERGMs (Exchange Algorithm) [CF11]

Input: ERGM distribution $\pi(X|\theta)$, prior $p(\theta)$, observed graph x_{obs}, number of burn-in draws r, symmetric proposal distribution $h(\cdot|\theta)$.
Output: sequence of draws $\theta^{(r)}, ...\theta^{(T)}$ from posterior distribution $p(\theta|x_{obs})$.

For $t = 1, ..., T$:

1. Draw parameter vector $\theta^* \sim h(\cdot|\theta^{(t-1)})$
2. Sample graph $x^* \sim \pi(\cdot|\theta^*)$
3. Accept the proposed move with probability $\min\{1, \alpha\}$. If the move is accepted, set $\theta^{(t)} = \theta^*$. Otherwise, set $\theta^{(t)} = \theta^{(t-1)}$

where

$$\alpha = \frac{p(\theta^*)}{p(\theta^{(t-1)})} \exp\left\{\left(\theta^* - \theta^{(t-1)}\right)^T \left(u(x_{obs}) - u(x^*)\right)\right\}$$

The algorithm can be justified by considering sampling from an augmented distribution with two auxiliary variables x^*, θ^*:

$$p(x^*, \theta^*, \theta|x_{obs}) \propto \pi(x_{obs}|\theta)p(\theta)h(\theta^*|\theta)\pi(x^*|\theta^*)$$

where π refers to the ERGM probability distribution. Marginalizing out θ^* and x from the augmented distribution gives the posterior distribution $p(\theta|x_{obs})$ of interest. Steps 1 and 2 are Gibbs updates of θ^* and x^*, while step 3 can be justified as the appropriate Metropolis-Hastings acceptance ratio:

$$\begin{aligned}\alpha &= \frac{\pi(x_{obs}|\theta^*)p(\theta^*)h(\theta^{(t-1)}|\theta^*)\pi(x^*|\theta^{(t-1)})}{\pi(x_{obs}|\theta^{(t-1)})p(\theta)h(\theta^*|\theta^{(t-1)})\pi(x^*|\theta^*)} \\ &= \frac{p(\theta^*)}{p(\theta)} \frac{\pi(x_{obs}|\theta^*)\pi(x^*|\theta^{(t-1)})}{\pi(x_{obs}|\theta^{(t-1)})\pi(x^*|\theta^*)}\end{aligned}$$

where we drop the h transition probabilities by symmetry and the intractable normalizing constants for θ^* and $\theta^{(t-1)}$ cancel, allowing easy computation of α. Thus, by standard MCMC theory, the draws $\theta^{(t)}$ come asymptotically from the desired posterior distribution.

In practice, Caimo and Friel advocate the use of a population-MCMC variant of their basic algorithm, in which multiple Markov chains are run in parallel, with the state space defined over the θ's of these multiple chains, as this population MCMC approach tends to converge faster and lead to less temporal dependence in draws from the Markov chain. We use this method, known as Parallel Adaptive Direction Sampling, in our private inference methods and explain it in detail in Appendix A along with a Metropolis-Hastings sampler to simulate networks from an ERGM with specified parameters.

Chapter 3: Differential Privacy for Networks

We employ the framework of differential privacy to protect individuals' personal data while analyzing networks. First, in Section 3.1 we provide the basic definitions of differential privacy and mechanisms that meet this definition. We also explain specific challenges that arise in applying these general definitions to network data. Then, in Section 3.2 we detail the machinery of "restricted sensitivity" which we propose to use for differentially private inference over ERGMs.

3.1 Basics of Differential Privacy

Definitions

Let $\mathcal{D}$ denote the space of all possible datasets. Then:

Definition 3.1. Two datasets $x, x' \in \mathcal{D}$ are *adjacent*, written as $x \sim x'$, if they differ in the record of one individual. For tabular data, this means that the datasets differ in a single row.

Definition 3.2. The *distance* between two datasets $x, x' \in \mathcal{D}$, denoted $d(x, x')$ is the minimum length of the sequence of datasets beginning with x and ending with x' such that every two consecutive datasets on the path are adjacent. So, two datasets are clearly adjacent, or neighboring, if $d(x, x') = 1$.

Definition 3.3 (ϵ-differential privacy [DMNS06])**.** Let $\mathcal{A}$ be an algorithm over datasets in $\mathcal{D}$. Then $\mathcal{A}$ is *ϵ-differentially private* if for all $S \subseteq \text{Range}(\mathcal{A})$ and for every pair of neighboring datasets $x, x' \in \mathcal{D}$,

$$\Pr[\mathcal{A}(x) \in S] \leq e^{\epsilon} \Pr[\mathcal{A}(x') \in S]$$

Intuitively, differential privacy promises that the participation of any individual in a dataset does not significantly change the outcome of an analysis run on the dataset, limiting the potential harm (or benefit) to a data provider due to the inclusion of her data. Smaller values of ϵ correspond to stronger guarantees of privacy where $\epsilon = 0$ suggests that the algorithm does not learn anything from the data and therefore the algorithm is useless. Additionally, it is clear that no non-trivial deterministic algorithm satisfies ϵ-differential privacy for any value of ϵ. If $\mathcal{A}$ is deterministic and its output differs on at least two datasets, then there must be neighboring datasets such that the probability of a specific output is 0 on one dataset and 1 on the other, preventing the ratio between probabilities on this response from being bounded as required. Therefore, mechanisms that provide differential privacy will have to introduce some randomness, or noise, into their answers.

We can relax the definition of ϵ-differential privacy to allow for a small probability of potentially catastrophic privacy leakage:

Definition 3.4 ((ϵ, δ)-differential privacy [DMNS06])**.** $\mathcal{A}$ is (ϵ, δ)-*differentially private* if for all $S \subseteq \text{Range}(\mathcal{A})$ and for every pair of neighboring datasets $x, x' \in \mathcal{D}$,

$$\Pr[\mathcal{A}(x) \in S] \leq e^{\epsilon} \Pr[\mathcal{A}(x') \in S] + \delta$$

It is immediate from the definition that (ϵ, δ)-differential privacy is equivalent to ϵ-differential privacy when $\delta = 0$. For $\delta > 0$, however, (ϵ, δ) guarantees that the mechanism is ϵ-differentially private with probability $1 - \delta$, but makes no promises about the privacy loss that occurs with probability δ. Therefore, if δ is on the order of $\frac{1}{n}$ where n is the size of the dataset, it is possible to satisfy (ϵ, δ)-DP by releasing a row of the data. Further, a mechanism that sometimes releases the entire dataset still satisfies (ϵ, δ)-DP.

Example 3.1. An algorithm that selects at random one record in the dataset and exactly releases this record is $(\epsilon, \frac{1}{n})$-differentially private for any value of ϵ.

Example 3.2. An algorithm that releases the entire dataset with probability δ and a constant value with probability $1 - \delta$ is (ϵ, δ)-differentially private for any value of ϵ.

As these examples demonstrate, (ϵ, δ)-differential privacy only provides meaningful privacy for values of δ much smaller than $\frac{1}{n}$. In particular, ϵ should be taken to be "cryptographically small" (e.g. take $\delta = \frac{1}{1{,}000{,}000}$ for networks over a few 100 nodes.)

Properties

One of the desirable properties of differential privacy is its immunity to *post-processing* – armed with the output of a differentially private mechanism, an analyst cannot degrade privacy any further without additional information about the private dataset. In the context of inference over ERGMs, this property suggests that after computing sufficient statistics of a model in a differentially private manner, inference using these sufficient statistics can be thought of as a post-processing step that does not further degrade privacy. Formally:

Property 1 (Post-processing [DMNS06])**.** If $\mathcal{A}$ is an (ϵ, δ)-differentially private algorithm, then for an arbitrary mapping f, $f \circ \mathcal{A}$ is also (ϵ, δ)-differentially private.

A second useful property of differential privacy is that multiple differentially private algorithms compose, so applying many differentially private algorithms to the same dataset still provides privacy, albeit with higher privacy loss. This allows for basic DP algorithms to be used as building blocks in more complicated algorithms and in particular to split a privacy budget across multiple private computations on the data. Specifically, basic composition states that the privacy loss incurred by running multiple DP algorithms on a dataset grows linearly:

Property 2 (Basic Composition [DMNS06])**.** Let $\mathcal{A}_i$ be an (ϵ_i, δ_i)-differentially private algorithm for $i \in [k]$. Then, the algorithm releasing the result of running all k algorithms on the dataset $\mathcal{A}_{[k]}(x) = (\mathcal{A}_1(x), ..., A_k(x))$ is $\left(\sum_{i=1}^k \epsilon_i, \sum_{i=1}^k \delta_i\right)$-DP.

Mechanisms

We now describe two simple mechanisms that satisfy differential privacy. First, we describe the Laplace Mechanism, which answers queries on a dataset in a differentially private manner by adding Laplace noise to queries. Then, we introduce Randomized Response, which provides privacy by randomly perturbing the underlying dataset.

Laplace Mechanism

We define a query to be a function mapping the dataset to a vector of real numbers, $f : \mathcal{D} \to \mathbb{R}^m$. Then the local sensitivity of a query on a dataset x is the maximum ℓ_1-norm of the difference in the query over neighbors of dataset x.

Definition 3.5 (Local sensitivity)**.** The *local sensitivity* of a query f on a dataset x is

$$LS_f(x) = \max_{x' \sim x} ||f(x) - f(x')||_1$$

The global sensitivity is the worst-case local sensitivity over all possible datasets:

Definition 3.6 (Global sensitivity)**.** The *global sensitivity* of a query f is

$$GS_f = \max_{x \in \mathcal{D}} LS_f(x)$$

A basic result in differential privacy is that adding Laplace noise scaled to the global sensitivity provides differential privacy:

Theorem 3.1 (Laplace mechanism [DMNS06])**.** *Let f be a query on dataset x with global sensitivity GS_f and let* Lap *denote the zero-mean Laplace distribution*[1]*. Then, the Laplace mechanism $\mathcal{A}_L$ that outputs*

$$\mathcal{A}_L(x, f, \epsilon) = f(x) + (Y_1, ..., Y_m)$$

where $Y_i \overset{i.i.d.}{\sim}$ Lap $\left(\frac{GS_f}{\epsilon}\right)$ is ϵ-differentially private.

Note that the Laplace mechanism scales noise to the *global sensitivity*. While it is tempting to calibrate noise to local sensitivity, this does not protect privacy, because the noise level may disclose information about the underlying dataset. However, we can add noise scaled to a smooth upper bound on the local sensitivity, namely a function S that is larger

[1]The Laplace distribution centered at 0 with scale parameter b has probability density function $p(x|b) = \frac{1}{2b}e^{-|x|/b}$ and the variance of the distribution is $\sigma^2 = 2b^2$.

than the local sensitivity for all datasets and for which $\ln(S(\cdot))$ is not too sensitive. The smoothness is parameterized by β, where β depends on ϵ and δ:

Theorem 3.2 (Calibrating Noise to β-Smooth Upper Bound on Local Sensitivity [NRS07])**.** *A β-smooth upper bound on the local sensitivity of query f is a function $S_{f,\beta}$ that satisfies:*

(i) $S_{f,\beta}(x) \geq LS_f(x) \quad \forall x \in \mathcal{D}$

(ii) $S_{f,\beta}(x) \leq \exp\{-\beta d(x, x')\} S_{f,\beta}(x') \quad \forall x, x' \in \mathcal{D}$

It is possible to satisfy (ϵ, δ)-differential privacy by adding Laplace noise scaled to $\frac{2S_{f,\beta}(x)}{\epsilon}$ where $\beta = -\frac{\epsilon}{2\ln(\delta)}$ and $S_{f,\beta}$ is a β-smooth upper bound on $LS_f(x)$. It is possible to satisfy ϵ-differential privacy by adding Cauchy noise[2] scaled to $\sqrt{2}S_{f,\beta}(x)$ where $\beta = \epsilon/\sqrt{2}$.

Then, global sensitivity trivially satisfies the definition of a β-smooth upper bound on local sensitivity, but it is a very conservative bound. The smallest function S to satisfy the definition of a β-smooth upper bound is known as the *smooth sensitivity*:

Definition 3.7 (Smooth Sensitivity [NRS07])**.** For query f and dataset x, define the *local sensitivity at distance of t* to be

$$LS_f^{(t)}(x) = \max_{\substack{x' \in \mathcal{D}: \\ d(x,x') \leq t|}} LS_f(x')$$

Then the *smooth sensitivity* is

$$S_{f,\beta}^*(D) = \max_t e^{-t\beta} LS^{(t)}(D)$$

The smooth sensitivity is the smallest β-smooth upper bound on the local sensitivity in the sense that for any other β-smooth upper bound S, $S_{f,\beta}^*(D) \leq S(D)$ for all datasets D. Thus, if we can compute the smooth sensitivity efficiently, then we can potentially add much less noise by calibrating to smooth rather than global sensitivity.

Randomized Response

In contrast to the Laplace Mechanism, which perturbs the output of a query on a dataset, randomized response perturbs the underlying dataset by randomly introducing spurious data. A typical version of randomized response over binary data proceeds as follows:

For each bit in a dataset consisting of $\{0, 1\}$ values:

1. Flip a biased coin with probability p_1 of heads.
2. If tails, then record the bit truthfully.

[2]The Cauchy distribution with median 0 and scale parameter b has probability density function $p(x|b) = 1/(b\pi(1 + (x/b)^2))$. Roughly, the Cauchy distribution can provide ϵ-DP because it has fatter tails than the Laplace distribution.

3. If heads, then flip a second biased coin with probability p_2 of heads and record 1 if heads, 0 if tails.

A benefit of randomized response is that it can be employed while collecting data, by using the coin-flipping procedure to collect responses in a study. The method provides plausible deniability for respondents, so it may incentivize participation in surveys for sensitive information. It is easy to verify that taking $p_1 = 2p$ and $p_2 = \frac{1}{2}$ yields the following simpler description:

Theorem 3.3 (Binary Randomized Response [War65],[KKS17])**.** *Let $\mathcal{D} = \{0,1\}^n$ so $x \in \mathcal{D}$ consists of binary data. Then, randomized response flips each bit of x with probability $p \in (0, \frac{1}{2})$ and releases the resulting noisy bits. This process provides ϵ-differential privacy taking $p \geq \frac{1}{e^\epsilon + 1}$.*

3.1.1 Edge-Level vs. Node-Level Adjacency

We now turn to the question of how to define "adjacency" for graphs, as opposed to tabular data. We will define graphs abstractly in terms of vertex sets and edge sets, rather than as adjacency matrices in this section, as it makes the definitions easier to specify and more intuitive. There are two reasonable and widely used definitions of adjacency, which provide privacy at very different granularities and thus may be appropriate in different circumstances:

Definition 3.8 (Edge-level adjacency)**.** We define two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ to be *edge-adjacent* if they have the same vertex set ($V_1 = V_2$) and they differ in only one edge ($|E_1 \triangle E_2| = 1$).

Differential privacy with respect to edge-adjacency protects the privacy of individual relationships between nodes. Thus, edge-level privacy could protect a Facebook friendship with a controversial political leader. However, privacy at the edge-level could not promise to prevent an adversary from discerning whether an individual has mostly Republican or Democratic friends on Facebook. Such concerns motivate a stronger definition of neighboring graphs:

Definition 3.9 (Node-level adjacency)**.** We define two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ to be *node-adjacent* if $G_1 - v_i = G_2 - v_i$ for some vertex v_i, where $G - v_i$ means deleting edges adjacent to node v_i.

An additional consideration in defining adjacent graphs is how to account for labeled nodes. In the node-level case, labels are protected since removing a vertex and replacing it with a different vertex suggests changing the labeling on that vertex. For edge-level privacy, labels could be taken to be either public or private information. There may be cases where the only sensitive information is the edges in the graph, not the identities of nodes (for instance, in a public social network, where people's identities may be readily searchable online, while their friendships are kept private.) However, in many settings, it seems preferable to protect the labels in addition to the relationships. Thus, letting

there be some labeling function associated with a network that specifies a vector of nodal attributes for each node $\ell : V \to \mathbb{R}^m$, we define edge-level adjacency for labeled networks as follows:

Definition 3.10 (Edge-level adjacency with private labels)**.** We define two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with labeling functions ℓ_1 and ℓ_2 to be *edge-adjacent with private labels* if they have the same vertex set ($V_1 = V_2$) and either they differ in only one edge ($|E_1 \triangle E_2| = 1$) or differ in one label ($\ell_1(v) \neq \ell_2(v)$ for exactly one vertex v.)

3.2 Restricted Sensitivity

Node-level privacy constitutes a strictly stronger guarantee than edge-level privacy, but it is often much more difficult to perform accurate analysis under node-level privacy. For instance, consider computing the degree distribution on an n-node graph. The global sensitivity under edge-level adjacency is only 2, since the degree of two nodes will change by 1 due to the addition or removal of an edge. However, under node-level adjacency, removing or adding all edges to a node of degree $n-1$ would affect n entries of the degree distribution, so the global sensitivity is n and naive application of the Laplace mechanism would completely destroy the counts of the degree distribution. Furthermore, even under edge-level adjacency many statistics computed on networks have high global sensitivity. For instance, the count of triangles in a graph (which is used in the alternating k-triangle sufficient statistic in ERGMs) has global sensitivity $O(n)$ in the edge-level case, since a single edge could be the base of a triangle with each other node in the graph.

The high global sensitivity of many graph statistics is particularly problematic for sparse graphs, where the noise completely overwhelms the true statistics. This is especially troubling, because sparsity is a characteristic of many real world networks. For instance, Facebook has billions of users, but users tend to have on the order of 1000 friends or fewer. We can formalize the hypothesis that a graph is sparse by considering the *degree* of the graph, the maximum degree of any of its nodes. If we hypothesize that all the graphs under consideration have limited degree, then the global sensitivity might be much lower over these limited-degree graphs than over all graphs on n nodes. For example, considering the space of graphs with degree of at most $k << n$, the triangle count would have a much lower global sensitivity of $O(k)$ rather than $O(n)$ over the space of all graphs on n nodes.

If we were certain that the graphs under consideration always had limited degree, we could scale noise to the sensitivity over limited degree graphs. However, our hypothesis might be false, so adding noise assuming that the graph has limited degree would not protect privacy for an arbitrary graph. Therefore, it is necessary to first project the graph into the space of limited degree graphs. If the limited degree hypothesis is true then the projection will not alter the graph at all, so the analysis is accurate up to the distortion of the noise-adding procedure. We formally define the *limited degree hypothesis*:

Definition 3.11 (Limited Degree Hypothesis)**.** Let $\mathcal{G}_n$ be the space of graphs on n nodes. Then, a graph $G \in \mathcal{G}_n$ satisfies the *limited degree hypothesis* if it belongs to the class $\mathcal{H}_k$

where $\mathcal{H}_k$ is the set of graphs:

$$\mathcal{H}_k = \{G = (V, E) \in \mathcal{G}_n : \deg(v) \leq k, \forall v \in V\}$$

Then, the restricted sensitivity is the global sensitivity of the query restricted to limited degree graphs:

Definition 3.12 (Restricted sensitivity [BBDS13])**.** For a given notion of adjacency (either edge or node), we define the *restricted sensitivity* of query f over hypothesis $\mathcal{H}_k \in \mathcal{G}_n$ as

$$RS_f(\mathcal{H}) = \max_{\substack{G,G' \in \mathcal{H}_k: \\ G \sim G'}} ||f(G - f(G')||_1$$

To protect privacy over arbitrary graphs, while calibrating noise to the restricted sensitivity rather than the global sensitivity, we require a projection $\mu : \mathcal{G} \to \mathcal{H}_k$. We can define the sensitivity of the projection in terms of how much it changes the distance by a multiplicative factor between any two adjacent graphs. In particular:

Definition 3.13 (Local sensitivity of projection μ [KNRS13])**.** Define the local sensitivity of projection $\mu : \mathcal{G}_n \to \mathcal{H}_k$ on graph $G \in \mathcal{G}_n$ to be:

$$LS_\mu(G) = \max_{G' \sim G} d(\mu(G), \mu(G'))$$

Then, the global sensitivity and smooth sensitivity can be defined as before. Now, if we can find a projection μ, where it is possible to bound the global sensitivity by a small constant, so $\forall G \in \mathcal{G}_n : LS_\mu(G) \leq c$, then for any two neighboring graphs the effect of first projecting a graph to $\mathcal{H}_k$ before answering a query only increases global sensitivity by a multiplicative factor of c:

Lemma 3.1 (Global Sensitivity on Composed Functions)**.** *For projection $\mu : \mathcal{G}_n \to \mathcal{H}_k$ and query $f : \mathcal{G}_n \to \mathbb{R}^m$, define $f_{\mathcal{H}_k} = f \circ \mu$ to be the query applied to the projection. Then $GS_{f_{\mathcal{H}_k}} \leq GS_\mu \cdot RS_f(\mathcal{H}_k)$.*

In particular, this suggests that if we find a projection to $\mathcal{H}_k$ with low global sensitivity c, then using ϵ-differentially private mechanisms like the Laplace mechanism that calibrate noise to $c \cdot RS_f(\mathcal{H})$ can give significant accuracy gains over global sensitivity. Blocki et al. give such a projection for the edge-adjacency model with $GS_\mu = 3$ that is also efficient (linear in the number of edges in the graph.) We give the details of this projection in Appendix B.

In the node level-adjacency model an efficient projection with low global sensitivity is not known [KNRS13]. However, it can be shown that if we use the smooth sensitivity of μ, then multiplying this β-smooth upper bound by the restricted sensitivity of f gives a β-smooth bound on the local sensitivity of the composition $f_{\mathcal{H}_k}$ as above:

Lemma 3.2 (β-Smooth Bound on Composed Functions)**.** *Let $S_\mu(G)$ be a β-smooth upper bound on the local sensitivity of μ on graph $G \in \mathcal{G}_n$. Then $S_{f_{\mathcal{H}_k}} = S_\mu(G) \cdot RS_f(\mathcal{H}_k)$ is a β-smooth bound on the local sensitivity of $f_{\mathcal{H}_k} = f \circ \mu$.*

We detail two possible projections for the node-adjacency model in in Appendix B ([KNRS13], [BBDS13]). The first, which we refer to as μ_{trunc} simply removes nodes of high degree and the other, μ_{LP} solves a linear program. It is possible to give β-smooth upper bounds on the local sensitivity for each of these projections. Roughly speaking, the benefits of node truncation are that it is more efficient than the LP and has low smooth sensitivity when there are few nodes with degree close to the cutoff k, which is often applicable since degree distributions frequently follow a power law. However, the smooth sensitivity of μ_{trunc} could be high for graphs in $\mathcal{H}_k$ if the graph does in fact have many nodes with degree close to the cutoff k. On the other hand, the smooth sensitivity of μ_{LP} is always relatively low when the hypothesis $\mathcal{H}_k$ is true, but the LP is not strictly a projection in that it is guaranteed to project graphs in $\mathcal{H}_k$ to themselves, but its image is $\mathcal{H}_{2k}$ not $\mathcal{H}_k$. Therefore, we must calibrate noise to the restricted sensitivity over $\mathcal{H}_{2k}$ when using the LP. Thus, we expect μ_{trunc} to have low smooth sensitivity in practice and to give a lower bound on restricted sensitivity than μ_{LP} (since node truncation requires calibrating noise to restricted sensitivity over $\mathcal{H}_k$ rather than $\mathcal{H}_{2k}$.) For these reasons, in addition to the greater efficiency of μ_{trunc}, we propose using node truncation as a smooth projection for the node-adjacency model.

Then, taking advantage of restricted sensitivity over $\mathcal{H}_k$ and the appropriate projections, we can perform inference over ERGMs while adding relatively low noise to sufficient statistics that have high global sensitivity. Our primary focus in proving privacy will be bounding the restricted sensitivity of the queries of interest over $\mathcal{H}_k$ in order to take advantage of this machinery of restricted sensitivity.

Chapter 4: Private Inference on ERGMs

In this chapter, we propose new methods for differentially private inference over ERGMs with the alternating and nodal attribute sufficient statistics defined in Section 2.2. We propose perturbing the sufficient statistics, taking advantage of restricted sensitivity to limit the amount of noise needed to protect privacy. Then, since the restricted sensitivity is public, it is possible to perform Bayesian inference while accounting for the level of noise addition. So we propose a principled inference method that converges asymptotically (in the number of iterations of the MCMC procedure) to the posterior distribution incorporating the randomness of the privacy mechanism. Incorporating the noise of the privacy mechanism in inference has been shown to lead to more reliable results in many cases (see [FGWC16], [KS16], [KKS17], [LM14] for instance.)

Our primary contribution is the proposal to use the machinery of restricted sensitivity in adding noise to sufficient statistics. The advantages of employing restricted sensitivity for inference over ERGMs are threefold:

- Calibrating noise to restricted sensitivity enables *lower noise in the edge-adjacency model* than current methods, permitting accurate inference at smaller privacy budgets.
- Restricted sensitivity permits private release of sufficient statistics under edge-level privacy with *private labels*, whereas prior work has treated labels as public.
- By using restricted sensitivity, we suggest the first (to our knowledge) method that performs differentially private inference under the *node-adjacency model*, a strictly stronger notion of privacy than the edge-adjacency model.

Restricted sensitivity relies on the hypothesis that the graph under analysis is sparse, namely that its max degree node has degree less than k. There are a number of reasons to believe that the limited degree hypothesis $\mathcal{H}_k$ is a reasonable assumption when modeling real social network data with ERGMs. First, previous empirical analyses of ERGMs have demonstrated that for reasonable parameter values, the distribution tends to put low probability mass on high-degree graphs [SPRH06]. Thus, given that we assume that an observed network is roughly drawn from the probability distribution specified by an ERGM, we believe with high probability that the graph has relatively low degree. Second, many real-world social networks are fairly sparse and have bounded degree. Therefore, an analyst is likely to believe that a network dataset under analysis represents a sparse graph and could reasonably choose a degree cutoff based on similar public datasets or domain knowledge.

4.1 Releasing Private Sufficient Statistics

In this section, we bound the restricted sensitivity under $\mathcal{H}_k$ of a number of the most commonly used sufficient statistics in ERGMs. As the following summary shows, in the edge-adjacency case, restricted sensitivity is much lower than global sensitivity for alt-k-triangle and alt-k-two path, assuming $k << n$. In the node-level case, adding noise scaled to the global sensitivity overwhelms the computed statistics in most cases, motivating the need for restricted sensitivity. For labeled networks, the global sensitivity is very low if labels are considered public and only edges are taken to be private. However, if labels are private, then the restricted sensitivity is much lower than the global sensitivity.

Table 4.1: Restricted Sensitivity on $\mathcal{H}_k$ for Common Structural Statistics

	Edge-Level		Node-Level	
	$RS_f(\mathcal{H}_k)$	GS_f	$RS_f(\mathcal{H}_k)$	GS_f
Edges	1	1	k	$n-1$
Alt k-star ($u_\lambda^{(s)}$)	2λ	2λ	$3\lambda k$	$O(n)$
Alt k-triangle ($u_\gamma^{(t)}$)	$2(k-1)+\gamma$	$O(n)$	$k^2+(\gamma-1)k$	$O(n^2)$
Alt k-two-path ($u_\gamma^{(p)}$)	$2(k-1)$	$O(n)$	k^2	$O(n^2)$

Table 4.2: Restricted Sensitivity on $\mathcal{H}_k$ for Common Statistics of Labeled Networks

	Public Labels	Private Labels	
	GS_f	$RS_f(\mathcal{H}_k)$	GS_f
Homophily	1	k	$n-1$
Popularity	2	$2k$	$2n$
Mixing	1	k	$n-1$

Below, we derive the restricted sensitivity of the alternating sufficient statistics of an ERGM under edge level privacy and node level privacy respectively. The "weighting parameters" of the alternating statistics γ and λ are generally set to be small constants between roughly 1 and 5 (most empirical work seems to find that values between 1 and 2 suffice) so the choice of this parameter has a fairly minor effect on the level of noise.

4.1.1 Edge Level Privacy

For the alternating k-star statistic under edge-level privacy, restricted sensitivity does not give any advantage over using global sensitivity, as the global sensitivity of this statistic is quite low:

Claim 4.1.1 (Global sensitivity of alternating k-star under edge-level privacy)**.** *The global sensitivity of the alternating k-star statistic is less than 2λ.*

Proof. We use the alternative formulation of the statistic given in Equation (2.5):

$$u_\lambda^{(s)}(x) = \lambda^2 \sum_{i=0}^{n-1} \left(\frac{\lambda-1}{\lambda}\right)^i D_i + 2\lambda|E| - n\lambda^2$$

Then, consider adjacent graphs x, x' differing in one edge where x has the additional edge. Then, the first term of the alternating k-statistic is larger for x' than for x and by at most 2λ and at least 0, while the second term is larger for x than for x' by 2λ. Hence, the difference between the alternating k-star statistic computed on x and x' is at most $|2\lambda - 0| = 2\lambda$. □

Claim 4.1.2 (Restricted sensitivity of alternating k-triangle under edge-level privacy)**.** *The restricted sensitivity of the alternating k-triangle statistic under $\mathcal{H}_k$ is less than $2(k-1)+\gamma$.*

Proof. Consider two adjacent graphs $x, x' \in \mathcal{H}_k$ differing in exactly one edge, so that $x_{ij} = 1$ and $x'_{ij} = 0$. Now, note that for nodes i and j, the number of shared partners is the same in x and x' since all edges are the same except for the edge between i and j. Then, let $P_{ij} = P'_{ij} = m \leq k-1$ by the limited degree hypothesis. Note that there are $2m$ edges for which $P'_e = P_e - 1$, since there are two other edges in each triangle. Then, recalling the definition of the alternating k-triangle statistic in terms of the shared partners of i and j given in Equation (2.6):

$$u_\gamma^{(t)}(x) = \gamma|E| - \gamma \sum_{1\leq i<j\leq n} x_{ij} \left(\frac{\gamma-1}{\gamma}\right)^{P_{ij}}$$

we have that

$$\begin{aligned}
|u_\gamma^{(t)}(x) - u_\gamma^{(t)}(x')| &= \left|\gamma - \gamma\left(\frac{\gamma-1}{\gamma}\right)^m + \gamma\sum_{e=1}^{2m}\left[\left(\frac{\gamma-1}{\gamma}\right)^{P_e-1} - \left(\frac{\gamma-1}{\gamma}\right)^{P_e}\right]\right| \\
&= \left|\gamma - \gamma\left(\frac{\gamma-1}{\gamma}\right)^m + \sum_{e=1}^{2m}\left(\frac{\gamma-1}{\gamma}\right)^{P_e-1}\right| \\
&\leq 2m + \gamma \\
&\leq 2(k-1) + \gamma
\end{aligned}$$

□

Note the usefulness of restricted sensitivity here, in contrast to global sensitivity. The global sensitivity of this statistic is $O(n)$, since in the worst case there could be a graph with an $(n-1)$-triangle where removing the base of the triangle leads to the removal of $O(n)$ triangles. However, if we restrict degrees, we add much less noise.

Claim 4.1.3 (Restricted sensitivity of alternating k-two-path under edge-level privacy)**.** *The restricted sensitivity of the alternating k-two-path statistic under $\mathcal{H}_k$ is less than $2(k-1)$.*

Proof. The proof will proceed in roughly the same way as for k-triangles. Define x and x' in the same way and recall the definition of the alternating k-two-path statistic in terms of shared partners as given in Equation (2.7):

$$u_\gamma^{(p)}(x) = \gamma\binom{n}{2} - \gamma \sum_{1\le i<j\le n} \left(\frac{\gamma-1}{\gamma}\right)^{P_{ij}}$$

Then, the change between the statistic on x and x' is equal to

$$|u_\gamma^{(p)}(x) - u_\gamma^{(p)}(x')| = \sum_{e=1}^{2m} \left(\frac{\gamma-1}{\gamma}\right)^{P_e-1} \le 2m \le 2(k-1)$$

□

4.1.2 Node Level Privacy

Claim 4.1.4 (Restricted sensitivity of alternating k-star under node-level privacy)**.** *The restricted sensitivity with hypothesis $\mathcal{H}_k$ of alternating k-star under node-level differential privacy is less than $3\lambda k$.*

Proof. We will again use the formulation of the alternating k-star statistic in terms of degree distribution from Equation (2.5). Now, consider two graphs $x, x' \in \mathcal{H}_k$ differing in one node i of degree $m \le k$, with all of its incident edges removed in x'. Then, the degree of node i is m in x and 0 in x', while the degrees of m other nodes are 1 lower in x' than in x, so:

$$\begin{aligned}
|u_\lambda^{(s)}(x) - u_\lambda^{(s)}(x')| &= \left| 2\lambda m + \lambda^2 \left(\left(\frac{\lambda-1}{\lambda}\right)^m - 1 \right) + \sum_{j:x_{ij}=1} \lambda \left(\frac{\lambda-1}{\lambda}\right)^{d_j-1} \right| \\
&\le \left| 3\lambda m + \lambda^2 \left(\left(\frac{\lambda-1}{\lambda}\right)^m - 1 \right) \right|
\end{aligned}$$

and note that $0 \le \left(\frac{\lambda-1}{\lambda}\right)^m \le 1$ and that $|\lambda^2| \le 3\lambda m$ for reasonable choices of k and λ (since generally we choose $1 < \lambda < 5$, so in order to have the λ^2 term dominate the $3\lambda k$ term we would have to restrict k to 1, which would not be interesting or realistic). Thus, because $m \le k$, the sensitivity is bounded by $3\lambda k$. □

Claim 4.1.5 (Restricted sensitivity of alternating k-triangle under node-level privacy)**.** *The restricted sensitivity with hypothesis $\mathcal{H}_k$ of the alternating k-triangle statistic under node-level differential privacy is less than $k^2 + (\gamma-1)k$.*

Proof. Consider two adjacent graphs $x, x' \in \mathcal{H}_k$ differing in one node i of degree m. Now, since each of the m edges incident to node i is removed this changes m edges $x_{ij} = 1$ to

$x'_{ij} = 0$, so $E(x) - E(x') = m$ and for each of these m edges

$$x_{ij}\left(\frac{\gamma-1}{\gamma}\right)^{P_{ij}} - x'_{ij}\left(\frac{\gamma-1}{\gamma}\right)^{P'_{ij}} = \left(\frac{\gamma-1}{\gamma}\right)^{P_{ij}}$$

so the direct effect of removing the x_{ij} is that $u_\gamma^{(t)}(x') - u_\gamma^{(t)}(x') \leq m\gamma - 0$ (ignoring the effect on the shared partners of edges not adjacent to i.)

Now, we consider edges e such that the endpoints of e have i as a shared partner. Note that there are $\binom{m}{2} = m^2 - m$ such edges, because we can choose any 2 edges of i and the endpoints of these edges have i as a shared partner. Now, each of these edges still exists in x'_{ij} but has its number of shared partners decrease by 1. Then, we have

$$\begin{aligned}
|u_\gamma^{(t)}(x) - u_\gamma^{(t)}(x')| &= \left|\gamma m - \gamma \sum_{j:x_{ij}=1}\left(\frac{\gamma-1}{\gamma}\right)^{P_{ij}} + \sum_{e=1}^{m^2-m}\left(\frac{\gamma-1}{\gamma}\right)^{P_e-1}\right| \\
&\leq |\gamma m + (m^2 - m)| \\
&\leq k^2 + (\gamma - 1)k
\end{aligned}$$

□

Claim 4.1.6 (Restricted sensitivity of alternating k-two-path under node-level privacy)**.** *The restricted sensitivity with hypothesis $\mathcal{H}_k$ of the alternating k-two-path statistic under node-level differential privacy is less than k^2.*

Proof. As for k-triangles, consider two adjacent graphs $x, x' \in \mathcal{H}_k$ differing in node i of degree m. Then, the removal of these m edges impacts the shared partners of m^2 edges, the m incident to i and the $\binom{m}{2} = m^2 - m$ that have i as a shared partner and the decrease in shared partners for each of these edges can change the statistic by at most 1 so the overall change is at most $m^2 \leq k^2$. □

4.1.3 Private Labels

If labels are considered public, then the global sensitivity of the sufficient statistics using discrete attributes of nodes given in Table 2.1 have low global sensitivity in the edge-adjacency model, since they are effectively counts of edges for nodes with certain attributes, and adjacent graphs have only a single edge changed and all labels kept the same. However, if labels are considered private then the change in a single label could change the count for all edges incident to that node, leading to very high global sensitivity of $O(n)$. By using restricted sensitivity, we can bound the sensitivity to be $O(k)$. Additionally, note that differential homphily and popularity are vectors of queries, with dimension the size of the number of attributes under consideration. However, these queries are structurally disjoint as a change in one attribute can only change the counts of two entries of the vector, making it easy to bound the ℓ_1-sensitivity of the entire vector. We give the proof for homophily, while the proofs for popularity and mixing follow from the same argument:

Claim 4.1.7 (Restricted sensitivity of homophily with private labels.)**.** *Both differential and uniform homophily have ℓ_1-restricted sensitivity k.*

Proof. Recall that for attributes $a_1, ..., a_m$ differential homophily is given by

$$\left(\sum_{i<j} x_{ij} \mathbb{I}(z_i = z_j = a_1), ..., \sum_{i<j} x_{ij} \mathbb{I}(z_i = z_j = a_m) \right)$$

Then, changing nodal attribute z_i from a to b changes the endpoint of at most k edges. If both endpoints of an edge had endpoint a, then this reduces the count of entry a in the vector by 1, while if the endpoints of the edge were a and b to start with, this increases the count in entry b by 1. These cases are disjoint so the largest ℓ_1 difference in the vector is k. For uniform homophily, it is easy to see that changing one label could change at most k edges and each edge is counted only once in uniform homophily, so the global sensitivity is 1. □

4.2 Inference Using Noisy Sufficient Statistics

Now, by projecting a network into $\mathcal{H}_k$ using the projections specified in Appendix B and then applying the Laplace mechanism (3.1), we can release the sufficient statistics of the ERGM in a differentially private manner by calibrating the noise of the Laplace mechanism to the restricted sensitivity. We could now release these sufficient statistics to analysts who wish to study the network, since the likelihood of the ERGM depends on the data only through the sufficient statistics. Using noisy statistics directly for standard inference techniques has been shown to lead to biased estimates, however, as the sufficient statistics may not even be graphical. Therefore, in the framework of Bayesian inference we want to compute the posterior over both the observed network and the privacy mechanism. In particular, letting $\tilde{y}$ be the "noisy network" defined by the application of our privacy mechanism to the true network we wish to compute the posterior:

$$p(\theta|\tilde{y}) \propto p(\tilde{y}|\theta)p(\theta) = \sum_x p(\tilde{y}|x)p(x|\theta)p(\theta) \tag{4.1}$$

where $p(\tilde{y}|x)$ is the privacy distribution defined by our mechanism, $p(x|\theta)$ is the ERGM distribution and $p(\theta)$ is the prior on θ which is specified by the analyst. Then, along the lines of [LM14], it is simple to modify the Exchange Algorithm for non-private inference to draw from the posterior that incorporates the privacy distribution, as shown in Algorithm 2.

As in the non-private case, Algorithm 2 draws samples from the true posterior of interest as $T \to \infty$ by MCMC theory. Steps 1 and 2 can be justified as Gibbs updating steps as in the non-private case, while steps 3 and 4 are component-wise Metropolis-Hastings updates, where we update the variables separately rather than in a block, because this tends to lead to higher acceptance ratios and thus faster convergence [GL06], [LM14]. Intuitively, x^* can be thought of as our best guess of the true underlying network. Then, θ^* is replaced in step 3 if it explains this best guess of the network well, while x^* is updated if the

Algorithm 2 Bayesian Inference for ERGMs with Differentially Private Network Data

Input: ERGM distribution $\pi(X|\theta)$, prior $p(\theta)$, noisy network $\tilde{y}$, privacy distribution $\pi_p(\tilde{y}|y)$, number of burn-in draws r, symmetric proposal distribution $h(\cdot|\theta)$.
Output: sequence of draws $\theta^{(r)}, ...\theta^{(T)}$ from posterior distribution $p(\theta|\tilde{y})$.

For $t = 1, ..., T$:

1. Draw parameter vector $\theta^* \sim h(\cdot|\theta^{(t-1)})$
2. Sample graph $x^* \sim \pi(\cdot|\theta^*)$
3. Replace $\theta^{(t-1)}$ with θ^* with probability $\min\{1, \alpha_1\}$.
4. Replace $x^{(t-1)}$ with x^* with probability $\min\{1, \alpha_2\}$.

where

$$\alpha_1 = \frac{p(\theta^*)}{p(\theta^{(t-1)})} \exp\left\{\left(\theta^* - \theta^{(t-1)}\right)^T \left(u(x^{(t-1)}) - u(x^*)\right)\right\}$$
$$\alpha_2 = \frac{\pi_p(\tilde{y}|x^*)}{\pi_p(\tilde{y}|x^{(t-1)})} \exp\left\{\left(\theta^* - \theta^{(t-1)}\right)^T \left(u(x^{(t-1)}) - u(x^*)\right)\right\}$$

new network is likely to be the true network over the noise of the privacy mechanism. Additionally, we propose using the population MCMC version of the exchange algorithm, as this leads to better convergence in practice and still converges to the correct posterior.

Note that α_1 does not depend on the choice of privacy mechanism, while α_2 is simple to compute under the addition of Laplace noise. In particular, if we add Laplace noise scaled to L (for instance, $L = 3 \cdot RS_f(\mathcal{H}_k)$ in the edge-adjacency case) to the sufficient statistics of the network then:

$$\begin{aligned} \log \frac{\pi_p(\tilde{y}|x^*)}{\pi_p(\tilde{y}|x^{(t-1)})} &= \log\left(Lap\left(\tilde{y} - x^*|L\right)\right) - \log\left(Lap\left(\tilde{y} - x^{(t-1)}|L\right)\right) \\ &= \frac{|\tilde{y} - x^{(t-1)}|}{L} - \frac{|\tilde{y} - x^*|}{L} \end{aligned} \tag{4.2}$$

In the edge-adjacency model, the restricted sensitivity is public, so we can easily compute the ratio in *eq.* (4.2) using $L = 3RS_f(\mathcal{H}_k)$ where the factor of 3 comes from the global sensitivity of the projection. In the node-level case, we compute smooth sensitivity of the projection which cannot be publicly released, but we can release the value of the smooth sensitivity assuming the graph belongs to $\mathcal{H}_k$ as explained in *Appendix B*, so inference is still valid when the analyst's assumptions about the degree of the graph are met.

In general, we assume that the number of nodes in a graph is known and public. For the case where labels are public, sampling a graph in step 2 is straightforward as we sample a graph from the space of all possible graphs with the n labeled nodes of the original

graph. However, if labels are private, then these labels must be privatized as well. This is straightforward to do by releasing a noisy histogram of the labels, which has global sensitivity of 1 and therefore is high accuracy assuming that there are a limited number of types of labels [DMNS06]. Then, the noisy histogram can be used as the node-set over which graphs are sampled in step 2 of the algorithm.

The Full Workflow

Putting together bounds on restricted sensitivity and the inferential procedure, the workflow for differentially private inference looks as follows. Given hypothesis $\mathcal{H}_k$, privacy budget ϵ and network data y:

1. Split privacy budget between sufficient statistics under consideration.
2. Project y to $\mathcal{H}_k$ using smooth projections μ specified for edge and node level privacy respectively in Appendix B.
3. Compute and release restricted sensitivity of sufficient statistics.
4. If labels are considered private, then release noisy histogram of node labels.
5. Draw noise scaled to restricted sensitivity. Add this noise to sufficient statistics and release these noisy sufficient statistics.
6. Using the noisy sufficient statistics from step 4 and the restricted sensitivity levels from step 3, perform inference using Algorithm 2.

Then, privacy follows by applying composition in step 1, restricted sensitivity with the Laplace Mechanism in steps 2-5 and post-processing in step 6. Post-processing is particularly useful here, because MCMC methods frequently require tuning of the inference, whereby we run the inferential procedure multiple times and run diagnostics to make sure it converges (for instance, by checking that every 100 samples from the posterior are not highly correlated.) By post-processing, we can run step 6 an arbitrary number of times to tune the inferential procedure because differential privacy is provided by steps 1 to 5.

4.3 Related Work

Our work builds on two proposed methods, both of which only consider the edge-adjacency privacy model with labels taken to be public. The method most closely related to our work is that of Lu and Miklau [LM14] who also suggest adding noise to sufficient statistics and then performing Bayesian inference. In order to avoid adding noise scaled to the high global sensitivity of these statistics, they calibrate noise to a private bound on the local sensitivity of the network. In particular, they use an approach suggested in [KRSY14] to compute a differentially private over-estimate of the local sensitivity and scale noise to this estimate of local sensitivity. This provides (ϵ, δ)-differential privacy and introduces significantly less noise than calibrating to the global sensitivity. In particular

the expected scale of Laplace noise added when calibrating to private local sensitivity is $\frac{2LS_f(G)}{\epsilon} + \frac{4GS_{LS_f}\ln(1/\delta)}{\epsilon^2}$ whereas we add noise scaled to $3RS_f(\mathcal{H}_k)/\epsilon$ to sufficient statistics[1]. For reasonable choice of k, the restricted sensitivity is close to the local sensitivity on graph G, while the second term in the private local sensitivity can be quite large for small privacy budgets in ϵ and δ. Miklau and Lu test their approach with $\delta = 0.1$, which is an unreasonable choice of this parameter in practice, since a method that released the entire dataset one tenth of the time would satisfy (ϵ, δ)-DP with this parameter (Example 3.2). We find through a battery of tests that our proposed method (which takes $\delta = 0$) adds much less noise than the private local sensitivity approach (which we test with $\delta = 10^{-6}$ in order to give a privacy guarantee that is more comparable to pure ϵ-DP) especially for small privacy budgets in ϵ. This difference in the magnitude of noise makes a significant difference in the accuracy of inference, as our method can perform accurate inference for realistic, small privacy budgets.

The primary drawback of our approach is that it requires a reasonable estimate of the the maximum degree of the network, while the private bounding of local sensitivity can calibrate noise to the sparsity of the graph without assumptions on the part of the analyst. However, given the availability of many public network datasets and domain knowledge, it is often feasible for an analyst to give a reasonable assumption on the degree of a private network. Further, we show empirically that even if the max degree assumption is a fairly large over-estimate or under-estimate of the true degree of the network, restricted sensitivity tends to introduce less error to sufficient statistics than private local sensitivity.

Additionally, the local sensitivity approach does not apply well in the node-level privacy model as local sensitivity of a statistic for a network in this model can be much higher than restricted sensitivity [BBDS13]. In particular, under node-level adjacency, any graph is neighbors with a graph in which a node has all edges removed or all edges added. Thus local sensitivity for the statistics used in ERGMs tends to be scaled to a function of n. In contrast, due to the projection of the restricted sensitivity framework we only consider neighbors to be other graphs with degree less than k, so noise is scaled to a function of k.

Another approach suggested by Karwa et. al. [KKS17] uses randomized response on edges of the network, where the network is thought of as a binary dataset of 0s and 1s indicating the presence or absence of an edge between any two nodes. Then, they employ maximum likelihood estimation on this perturbed network taking into account the flipping of edges.[2] The main benefit of this method is that it potentially permits greater flexibility as the perturbed network can be released for public use and researchers can use any sufficient statistics they like. Our method requires a commitment to use a specific set of sufficient statistics, although we bound restricted sensitivity for a broad range of commonly used sufficient statistics in ERGMs. The primary drawback of the randomized response approach is that for small privacy budgets, it leads to extensive distortion of the underlying network. For instance, taking $\epsilon = 1$ suggests a probability of flipping each edge of around 25%, which for a sparse network in which tends to completely overwhelm network structure. Thus, Karwa et al. only demonstrate the utility of their method for ϵ taken to be 3.5 or larger.

[1] In private local sensitivity, GS_{LS_f} is 2 for alternating k-triangle and k-two-path statistics [LM14].

[2] We test randomized response using Bayesian inference, as it is straightforward to account for this perturbation in the Bayesian inferential framework.

Chapter 5: Empirical Evaluation of Private Inference

It is typical in the literature on inference for ERGMs to evaluate performance empirically through experiments ([HH06], [LM14], [HG10]). This is because there are no general theoretical results on the utility of non-private inference. In this chapter, we conduct a series of experiments, both on synthetic graphs drawn from a variety of ERGMs and on a high school friend network dataset collected by sociologists. For the edge-level privacy model, we compare the accuracy in releasing differentially private sufficient statistics using our proposed method of calibrating noise to restricted sensitivity against the private local sensitivity. Next, we compare the quality of parameter estimation using our proposed inference method with restricted sensitivity, private bounding of local sensitivity and randomized response used to perturb the network data. Finally, we evaluate the noise addition of our method under node-level privacy. We give a brief summary of our main findings in this chapter in Section 5.4.

5.1 Experimental Setup

Data

We test our proposed methods on networks drawn from three different ERGM specifications using the alternating structural sufficient statistics introduced in Section 2.2, so the probability distribution has the form:

$$\Pr(x|\theta) \propto \exp\left\{\theta_1 E(x) + \theta_2 u_\lambda^{(s)}(x) + \theta_3 u_\gamma^{(t)}(x) + \theta_4 u_\gamma^{(p)}(x)\right\}$$

with the following parameters for each model:

Model	θ_1 (Edges)	θ_2 (k-Stars)	θ_3 (k-Triangles)	θ_4 (k-Two-Paths)
1	-4.6	0.0	1.0	0.0
2	-4.6	0.0	2.0	-0.1
3	-4.6	2.0	2.0	-0.5

Table 5.1: Parameters of synthetic networks

We fix the edge parameter at -4.6, because in the absence of any other sufficient statistics this corresponds to a $G(n,p)$ model with $p = 1\%$, leading to sparse networks. Then, the other parameters are chosen based on typical values from analyses of real network data, with Model 1 constituting a simple model introducing only the alternating k-

triangle parameter in addition to edges and Model 3 representing the most complex model, including all four structural parameters. In Model 2, we incorporate a two-path parameter in addition to the triangle parameter, because the small negative parameter encourages sparsity while still leading to a large number of triangles in the network. In all three models we use a positive parameter for triangles, because triangulation is one of the most distinctive structures of networks captured by ERGMs that other stochastic models of networks frequently have difficulty capturing. Additionally, triangle count is one of the most difficult statistics to compute privately as it has high sensitivity relative to the scale of the statistic.

We simulate networks using the sampling method detailed in Appendix A.1 for graphs on n nodes with n ranging from 100 to 1000 in order to study networks with a range of sizes. Current standard inference methods for ERGMs are generally computationally feasible only for networks of around 1000 nodes at most, hence our choice of this upper limit on the size of the networks.[1] For each n and each model we draw 50 networks. Looking at the structure of the simulated networks in Figure 5.1 indicates that the 3 models put most of their probability mass on distinctive networks. On average, networks drawn from Model 1 have edge density (proportion of possible edges present) of 6%, networks from Model 2 have average density of 1%, and networks from Model 3 average 0.5%. Further, the max degree and triangle counts of the networks vary highly between the three models.

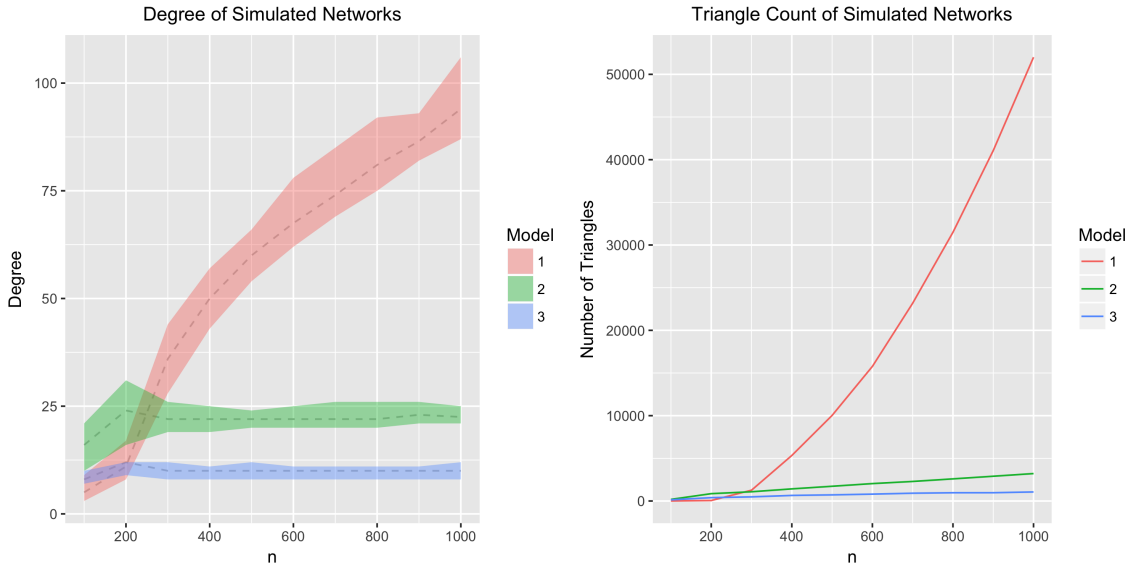


Figure 5.1: Degree of simulated networks (with shading indicating max and min over the 50 samples and dashed line showing median degree) along with the mean triangle count for simulated networks.

Networks drawn from Model 1 have the highest degree and triangle count, while networks simulated from Model 2 have approximately three times as many triangles as networks from Model 3. Additionally, all three models capture more complex structure in networks than the basic $G(n, p)$ model with $p = 1\%$, for which simulated networks tend to have only one tenth the number of triangles as Model 3 despite having higher edge density.

[1]Indeed, current work ([LM14], [KKS17]) only evaluates differentially private inference on networks of under 150 nodes, which are too small to perform accurate private inference on under strong privacy constraints.

In addition to the simulated networks, we test our methods on a dataset known as the "Faux Mesa High School Network", which is a publicly available social network released by sociologists who studied high schools in the Southwestern United States. These researchers surveyed the entire student body of a high school and formulated a social network with 205 nodes corresponding to students (labeled with attributes like Race and Sex) and 203 undirected edges representing reported mutual friendships between students [HHB+08]. The publicly available network on which we test inference was generated by researchers by fitting an ERGM to the dataset and then releasing a network sampled from this distribution. This suggests that privacy was a concern for the underlying network data. Therefore, the synthetic network maintains the interesting structural properties of the underlying dataset. Performing inference over this dataset allows us to evaluate whether researchers would reach similar conclusions using our proposed differentially private inference methods as using standard non-private inference.

Implementation

Below, we give a number of important implementation details that apply to all conducted tests:

- Code is written in the `R` programming language, with the procedure for sampling networks used for inference built on top of the packages `ergm` [HHB+08] and `Bergm` [CF14], which support non-private inference over ERGMs.
- In order to explore a broad parameter space and compare a variety of approaches, the experiments ran in parallel on Harvard's Odyssey computing cluster. To speed up individual experiments (which was particularly important during initial debugging), the inference step exploited thread level parallelism on 32-core machines.
- Unless otherwise specified, we give overall privacy budget (in terms of ϵ and δ), which is evenly split between sufficient statistics needed for inference, employing the composition property of differential privacy. Throughout, when we guarantee the relaxed notion of (ϵ, δ)-DP we set the privacy budget of δ to 10^{-6} in order to give a meaningful privacy guarantee akin to that of pure ϵ-differential privacy.
- For all parameters, we specify an "uninformative" prior of $\mathcal{N}(0, 50)$ in inference. Since parameter estimates are generally small compared to the variance of the prior, this prior has little effect on the posterior parameter estimates. It may be possible to improve performance by incorporating simple prior information, like the expected signs of parameters, about which researchers often have beliefs, which may be an interesting notion to investigate in future work.
- To obtain point estimates of fitted parameters, we take the mean of the posterior distribution over parameters, which is standard practice for Bayesian inference on ERGMs [CF11].

5.2 Edge-Adjacency Model

Under edge-level privacy constraints, we compare our proposed method against two alternative methods, the private bound on local sensitivity approach suggested by Lu and Miklau [LM14] and the randomized response method of Karwa et al. [KKS17], both detailed in Section 4.3. Since the private local sensitivity method also adds noise to sufficient statistics and then performs inference just as our restricted sensitivity-based method does, we first compare the noise addition step between these methods. We find that restricted sensitivity allows for much lower noise addition, measured in terms of root-mean-square error, especially at small values of ϵ.

Then, we evaluate how difference in perturbation of the network data impacts quality of inference. We compare the performance of differentially private inference between the three methods by looking at how close parameter estimates are to the "ground-truth" parameters. For the three synthetic networks, we take ground-truth to be the true parameters of the ERGMs from which we drew synthetic data (specified in Table 5.1) while for the Mesa high school data we learn parameters non-privately and take these to be a best guess of true parameters.

5.2.1 Noise Addition Comparison

Setup

For noise addition comparisons, we test the error introduced by perturbing sufficient statistics for the three synthetic network models and all values of n. We compare noise addition under restricted sensitivity with various max degree cutoffs and private local sensitivity. For each of the 50 networks of size n drawn from a model, we draw noise 50 times for each noise addition method, resulting in 2500 simulated noise draws for each n. We compute the four sufficient statistics (edges, alt-k-star, alt-k-triangle, and alt-k-two-path) with privacy levels per-statistic of $\epsilon = 0.025, 0.125$, and 0.25 (so by the composition property of differential privacy, overall privacy budgets using the 4 statistics $0.1, 0.5$, and 1 respectively) and display results below for the largest per-statistic budget of $\epsilon = 0.25$. For the private bounding of local sensitivity, which can only guarantee (ϵ, δ)-DP, we use an overall budget of $\delta = 10^{-6}$ so that the privacy guarantee is comparable to that of pure ϵ-differential privacy.

Further, we test varying degree cutoffs k of the restricted degree hypothesis. Lower (more aggressive) setting of k allows for less Laplace noise to be added since the restricted sensitivity is smaller. However, if k is lower than the true degree of the network, then the projection to the space of networks of degree k requires removing edges from the network, introducing bias into the released sufficient statistics. Since mean square error is variance plus squared bias, using RMSE captures both the error from projection bias and from Laplace noise addition. Specifically, we test three choices of k:

1. Take k equal to the minimum degree over the 50 networks drawn from a given model, which may allow for low Laplace noise, but at the expense of potentially high bias

induced by removing many edges.

2. Take k to be the median degree over the 50 networks, allowing relatively low scale of Laplace noise, while also introducing limited bias since the edge-level projection requires removing edges from nodes that have degree higher than k of which we expect there to be relatively few.

3. Take k to be a conservative estimate of 1.5 times the maximum degree of the 50 networks drawn from a given model, which guarantees that we never under-estimate the degree of a network.

In the plots that follow, we refer to these settings of k as 'min', 'median' and 'conservative' respectively.

Results

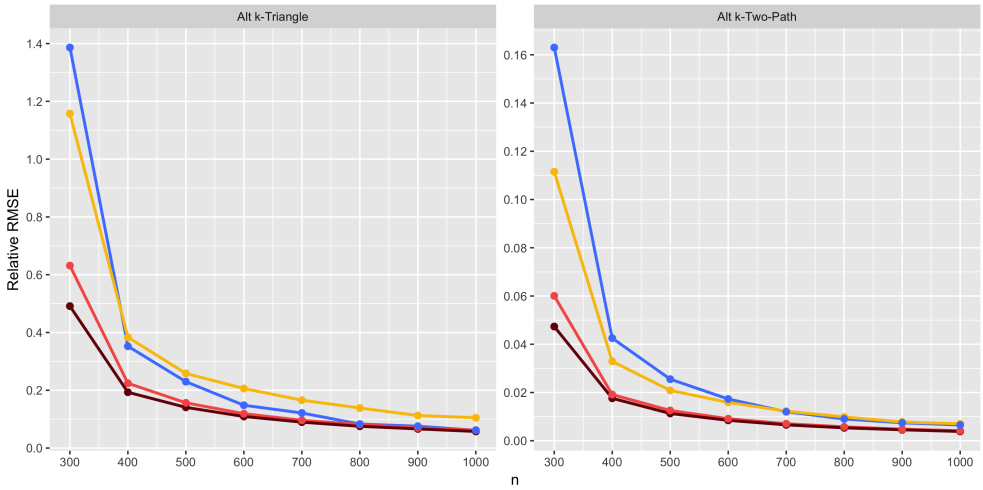
As comparison of noise in Figure 5.2 demonstrates, for $\epsilon = 0.25$ per statistic, our proposal of scaling noise to restricted sensitivity introduces much lower error into sufficient statistics than the private bounding of local sensitivity described in Section 4.3. The difference in noise addition is larger for the sparser networks drawn from Models 2 and 3, than for networks drawn from Model 1 because the degree of these networks is lower, giving a restricted sensitivity bound closer to the local sensitivity of the statistics. Further, the scale of noise added in the private bound on local sensitivity (as discussed in 4.3) is $O\left(\frac{LS}{\epsilon} + \frac{\log(1/\delta)}{\epsilon^2}\right)$, whereas restricted sensitivity adds noise scaled to $O\left(\frac{RS}{\epsilon}\right)$. Therefore, as ϵ gets smaller, the $\frac{1}{\epsilon^2}$ term grows faster than the $\frac{1}{\epsilon}$ term leading to even higher noise addition for private bounding of local sensitivity. Consistent with this observation, for tests run with smaller ϵ of 0.025 and 0.125-per statistic, we observed that restricted sensitivity outperforms private local sensitivity in the amount of noise added by an even larger margin. This suggests that for smaller overall privacy budgets than $\epsilon = 1$ or for cases where we introduce more sufficient statistics (for instance, when we want terms incorporating nodal attributes) so that we must split the privacy budget between additional terms, restricted sensitivity will dominate private local sensitivity by an even larger factor.

Comparing the RMSE of statistics to the true values, we observe that restricted sensitivity may permit inference in settings where the noise added by private local sensitivity would overwhelm the true statistic value. For instance, looking at the lower left subplot in Figure 5.2, for networks with 300 nodes drawn from Model 3, the RMSE of the k-triangle statistic is over three times the value of the true statistic when noise is scaled to private local sensitivity, while restricted sensitivity adds noise under half the true statistic value. Further, we observe that as networks grow larger the error relative to the size of the network decreases, since the degree of the network remains low, while the magnitude of the sufficient statistics increases, allowing for lower noise addition relative to the magnitude of the statistics.

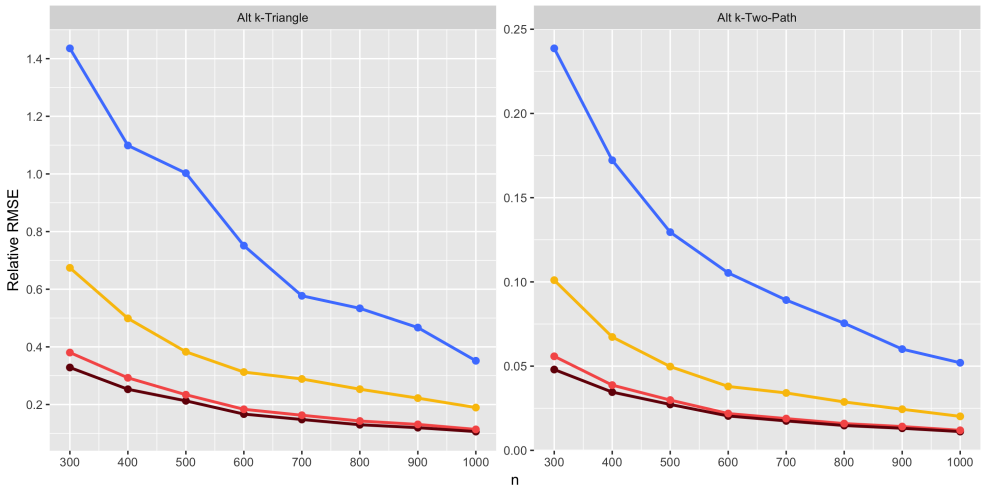
Comparison of the different thresholds for the degree cutoff k suggests that even a conservative cutoff outperforms private local sensitivity. However, choosing a more aggressive cutoff leads to much lower error, indicating that the lowered variance induced by Laplace noise addition has a larger impact on error than the bias introduced by

Figure 5.2: Comparison of Relative RMSE of edge-DP sufficient statistics released at a privacy level of $\epsilon = 0.25$-per statistic. Laplace noise is scaled to either a private bound on local sensitivity or to restricted sensitivity with 3 settings of degree cutoff k.

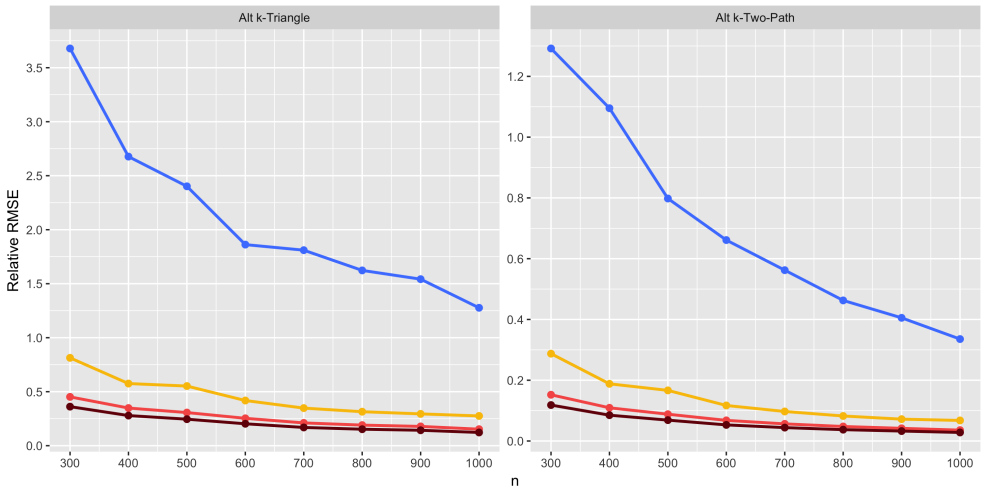
(a) Model 1



(b) Model 2



(c) Model 3



removing edges from high-degree nodes in networks of degree larger than k. In fact, for the alternating k-triangle and the alternating k-two-path statistics, using the min k cutoff, such that the degree of all simulated networks is beneath the cutoff, slightly outperforms restricted median. However, this aggressive setting of the cutoff introduced more bias into the edges and alt k-star terms. While the median cutoff leads to the removal of less than 0.05% of edges in the network for all n, the min cutoff leads to the removal of around 0.3% of edges. Additional testing with a highly aggressive cutoff of 0.75(min degree) led to the removal of $7 - 8\%$ of edges, making the RMSE of released private statistics higher for this setting than for the conservative cutoff. The ability to use cutoffs slightly below the actual degree of networks, while introducing low bias to computed statistics, most likely arises because ERGMs tend to capture power laws of degree distributions, such that there are relatively few nodes with degree close to the max degree. Therefore, choosing a cutoff below the max degree leads to the removal of only a few edges incident to these few nodes, skewing the structure of the network very little. Our comparison indicates that setting k equal to the median of all max degrees for simulated networks constitutes a good choice of k, so we use this setting of k in our inference tests.

Overall, our tests suggest that for reasonable choices of privacy budget ϵ, restricted sensitivity introduces much lower error to sufficient statistics than privately bounding local sensitivity. Further, aggressively setting the degree cutoff k to be beneath the max degree of actual networks can decrease the overall error in privatizing network statistics.

5.2.2 Inference Comparison on Synthetic Networks

Next, we test how the differing approaches to perturbing the network data impact the quality of inference. We find that for small privacy budgets our proposed method outperforms randomized response and for either small privacy budgets or the use of many sufficient statistics our method outperforms private bounding of local sensitivity.

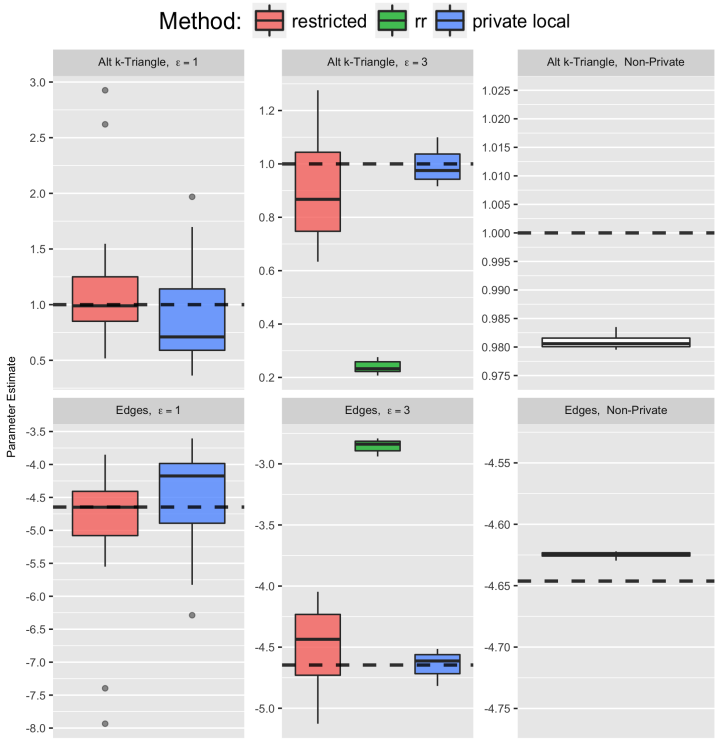
Setup

We test the performance of private inference on networks drawn from of each of the three models with overall privacy budgets of $\epsilon = 1$ and $\epsilon = 3$. Specifically, for $N = 25$ tests of inference, we randomly choose one of the 50 networks on 300 nodes drawn from a model and run inference on this network to learn the parameters of the model from which the network was drawn. This implies that Model 1 has the largest privacy budget per-statistics, since we split the fixed privacy budget over only the 2 statistics for Model 1, 3 statistics for Model 2, and 4 statistics for Model 3. We test inference on networks of 300 nodes because private inference is expected to have higher utility on larger networks (as the amount of added noise is lower compared to the value of true statistics as shown in Figure 5.2) but running a relatively large number of tests on networks with more than 300 nodes was computationally infeasible within the scope of this work. For the small privacy budget of $\epsilon = 1$, the inference method using randomized response fails to converge (which is unsurprising, because this corresponds to a probability of over 25% of flipping each edge in the network, so the posterior distribution spreads probability mass over a very large range of possible networks.) Therefore, for $\epsilon = 1$ we only report results for private

local sensitivity and restricted sensitivity, while for $\epsilon = 3$ (corresponding to a flipping probability of 5.6% in randomized response), we report results for all three methods. We display the distribution of parameter estimates using boxplots (which denote min, max, quantiles and median parameter estimates) against the ground truth parameters shown as dashed black lines. Additionally, we display the results of running non-private inference 25 times, which represents the best possible performance we could expect from differentially private inference. Non-private inference tends to be have low variance in parameter estimates across tests, as we fit networks using the models from which synthetic networks were drawn, thereby controlling for inference error caused by mis-specification of the model.

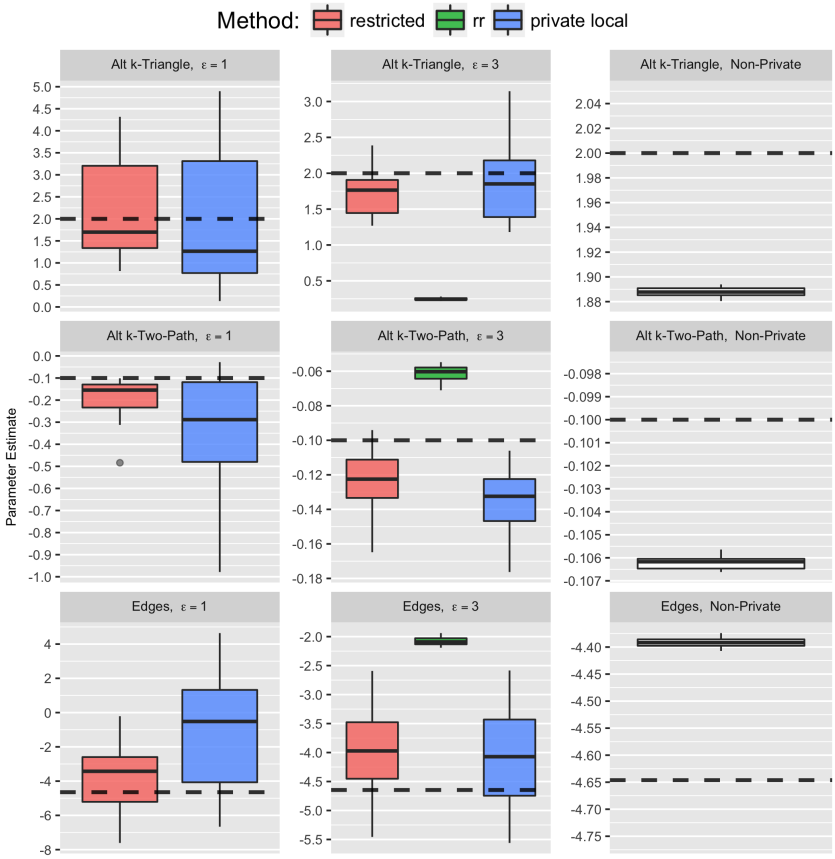
Results

Figure 5.3: Differentially private parameter estimates for 300-node synthetic networks drawn from *Model 1*. The dashed black line denotes ground-truth parameter values. We abbreviate randomized response as "rr".



Model 1 uses only two sufficient statistics, allowing for high values of ϵ per statistic, since the privacy budget is split between few statistics. Additionally, only the k-triangle statistic has high sensitivity, as edges have a global sensitivity of 1 (Table 4.1). For the privacy budget of $\epsilon = 1$, restricted sensitivity slightly outperforms private local sensitivity in the accuracy of parameter estimation as shown in Figure 5.3. This is because restricted sensitivity has added slightly less noise to the alternating k-triangle statistic than private local sensitivity: restricted introduces a relative RMSE of 0.3 to the statistic, while private local introduces a relative RMSE of 0.35. Our inference tests demonstrate that using noisy sufficient statistics with error at this level still allows for highly accurate parameter estimates. Thus, extrapolating to larger networks where even less noise can be added relative to the magnitude of sufficient statistics, we expect to be able to accurately

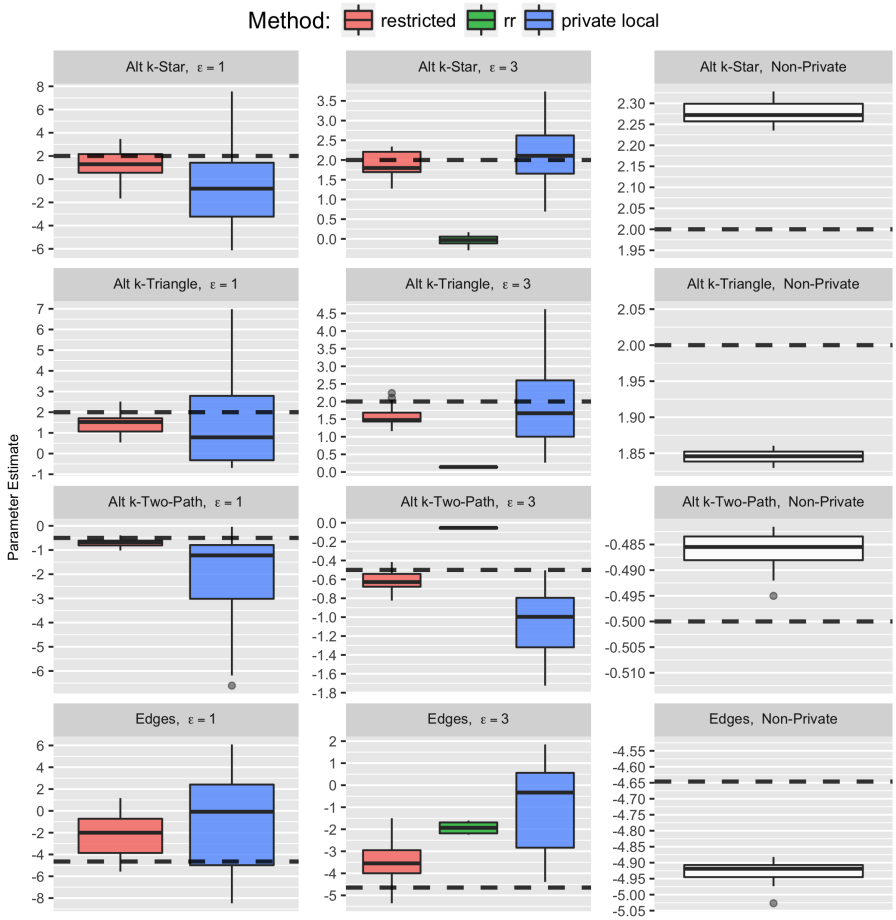
Figure 5.4: Parameter estimates for 300-node synthetic networks drawn from *Model 2*.



estimate parameters with smaller privacy budgets. In contrast to adding noise to sufficient statistics, randomized response performs poorly even at the privacy budget of $\epsilon = 3$, for which each edge is flipped with probability 4.7%. This perturbation of edges heavily distorts network structure, leading the network to resemble a denser $G(n, p)$ network, as randomized response underestimates the k-triangle parameter and overestimates the edge parameter.

Model 2 introduces a third sufficient statistic, requiring the privacy budget to be split between 3 statistics instead of 2 as in Model 1. This leads private local sensitivity to perform poorly at $\epsilon = 1$, as adding noise to the alt-k-triangle statistic overwhelms the value of the statistic, which leads parameter estimates to vary wildly as shown in Figure 5.4. For $\epsilon = 1$, restricted sensitivity gives better estimates of the edge and alt-k-two-path parameters than private local does. Using restricted sensitivity, we consistently estimate the alternating k-triangle statistic to be much larger than 0 and the alternating k-two-path statistic to be slightly negative, suggesting that researchers could accurately discern the importance of these network structures using differentially private inference. Also, we observe that even non-private inference produces biased parameter estimates compared to ground-truth, which points to underlying stochasticity in parameter estimation inherent in the specification of the synthetic network model. Relative to the error of non-private inference, private inference using restricted sensitivity performs approximately two times worse in estimating parameters at $\epsilon = 3$. Finally, we observe for Model 2 as for Model 1 that randomized response lead to parameter estimates that describe a $G(n, p)$ graph, with

Figure 5.5: Parameter estimates for 300-node synthetic networks drawn from *Model 3*.



two-path and triangle parameters close to 0, indicating the loss of important substructure in the perturbed network when applying randomized response.

As we see in Figure 5.5, for Model 3, which uses all 4 sufficient statistics, private local sensitivity does not enable useful inference even for $\epsilon = 3$, as the average bias of parameter estimates is over 2 times the value of the parameters. However, restricted sensitivity gives reasonable parameter estimates that would permit a researcher to understand the relative importance of the various alternating sufficient statistics in describing network structure. In contrast, private local sensitivity gives highly biased estimates of the two-path and edges parameters and has much higher variance for alt k-star and alt k-triangle, confirming that as more parameters are added, so that the privacy budget needs to be split between multiple statistics, restricted sensitivity increasingly outperforms private local sensitivity.

5.2.3 Inference Comparison on Mesa High School Data

To understand how differential privacy impacts analysis of network data in practice, we provide a case study of differentially private inference on a high school friend network where nodes are labeled by race and sex. In line with demographers analyses of high school friend networks in [GKM09], we study various processes of social network formation

including homophily by race and sex and triangulation. Because the network is relatively small with 203 nodes, we are not able to provide useful inference while taking labels as private, for this case it may be reasonable to assume that Race and Sex of nodes are public information and treat labels as public.

Setup

We apply the three private inference approaches – restricted sensitivity, private local sensitivity, and randomized response – to the Mesa High School Friend Network. We run 25 tests of inference for each method and $\epsilon \in \{1, 3\}$. For the restricted degree cutoff we take $k = 15$, whereas the real network has max degree 13. We choose degree 15, as a realistic estimate of the max degree of the network based on similar published datasets collected by sociologists at high schools [HHB+08], as students administered surveys tend to not list more than 15 friends. Based on models used to study high school friendship networks in the published literature on the subject [GKM09] as well as the models used for non-private inference on this particular network [HHB+08] we fit the a model to the data incorporating 'Sex' and 'Race' labels on nodes. Race can take on one of five values – Black, Hispanic, Native American, White, and Other – while sex is either Male or Female. We use the k-triangle statistic as well as measures of overall homophily by race, homophily among men, and homophily among women. We perform non-private inference using 25 tests and find that it gives consistent parameter estimates across test runs, suggesting that our model is well-specified. Thus, we take mean parameter estimates across all 25 tests as ground truth parameters.

Results

Compared to both adding noise scaled to private bound local sensitivity and randomized response, our proposed method using restricted sensitivity allows for accurate parameter estimation at lower privacy budgets. As we show in Table 5.2, restricted sensitivity with $\epsilon = 1$ gives estimates of edges and alternating k-triangle that are unbiased overall, although with fairly high error. For $\epsilon = 2$, restricted sensitivity gives consistently accurate parameter estimates for all parameters except Male homophily. However, the non-private estimate for this parameter in 5.3 is not statistically significant, so it is unsurprising that perturbing the statistics leads to differing estimates of this parameter. In contrast, private local sensitivity even with overall privacy budget of $\epsilon = 3$ is highly unstable returning parameter estimates with wildly varying values as evidenced by the high MSE and MAE of the parameter estimate. This is due to the splitting of the privacy budget between parameters, which leads to high noise addition to the k-triangle statistic when calibrating noise to private local sensitivity. Randomized response with $\epsilon = 3$ overwhelms interesting structure of the network, leading all parameter estimates to be close to 0 except for edges.

As shown in Table 5.3, for private inference with $\epsilon = 1$ there is a large increase in standard errors, demonstrating that inference over the private posterior distribution captures the uncertainty in true sufficient statistics introduced by the privacy mechanism. This increased uncertainty means that parameter estimates are no longer significant for $\epsilon = 1$. However, for $\epsilon = 2$, homophily by race, homophily among women, the edges parameter and the k-triangles parameter all remain statistically significant and close to the original parameters. This suggests that, in the typical case, a researcher performing

Table 5.2: Average differentially private parameter estimates, mean-square error, and bias for 25 differentially private inference test runs on Mesa High School Friend Network.

		Restricted ($\epsilon = 1$)			Restricted ($\epsilon = 2$)		
Parameter	True	Avg.	MSE	MAE	Avg.	MSE	MAE
Edges	-5.86	-6.30	1.36	0.84	-5.81	0.30	0.41
Homophily (Female)	0.59	0.72	0.48	0.52	0.39	0.28	0.37
Homophily (Male)	0.26	-0.22	3.20	1.01	-0.28	3.89	0.74
Homophily (Race)	0.44	0.32	2.31	0.85	0.59	0.34	0.32
Alt k-Triangle	1.39	1.56	1.96	0.93	1.43	0.44	0.50

		Private Local ($\epsilon = 3$)			Rand. Response ($\epsilon = 3$)		
Parameter	True	Avg.	MSE	MAE	Avg.	MSE	MAE
Edges	-5.86	-6.13	28.56	3.75	-3.00	8.18	2.86
Homophily (Female)	0.59	-2.36	31.91	4.20	0.15	0.20	0.44
Homophily (Male)	0.26	-1.86	35.53	4.00	0.02	0.06	0.23
Homophily (Race)	0.44	-3.16	48.89	5.27	0.09	0.12	0.35
Alt k-Triangle	1.39	-3.25	50.75	4.93	0.06	1.76	1.33

differentially private using the restricted sensitivity approach with $\epsilon = 2$ would reach the same conclusions about network structure as under non-private inference – that the network is sparse, that triangulation, or tendency for friends-of-friends to become friends, constitutes an important process in network formation, and that homophily has a significant impact on formation of friendship between people of the same race and between women, but not for men.

Table 5.3: Typical parameter estimates and standard errors for differentially private inference on Mesa Data."Typical" estimates denote the test run with median overall error in parameter estimates (normalized by magnitude of the parameters.)

	Non-Private		Restricted ($\epsilon = 1$)		Restricted ($\epsilon = 2$)	
Parameter	Estimate	SE	Estimate	SE	Estimate.	SE
Edges	-5.86	0.15	-5.38	0.49	-5.83	0.20
Homophily (Female)	0.59	0.18	0.04	0.38	0.62	0.16
Homophily (Male)	0.26	0.23	-0.63	0.48	0.25	0.17
Homophily (Race)	0.44	0.17	0.41	0.50	0.52	0.17
Alt k-Triangle	1.39	0.06	1.38	1.02	1.85	0.19

Overall, the results from this case study suggest that, in contrast to current methods, using restricted sensitivity to perturb sufficient statistics of a network enables researchers to reach meaningful conclusions about the processes giving rise to network structure for reasonable privacy budgets. For similar, even larger, sparse friend networks – which are common in the sociology and demography literature – it should be feasible to perform useful inference with even smaller privacy budgets as the magnitude of sufficient statistics will be larger compared to the max degree of the network.

5.3 Node-Adjacency Model

Finally, we assess the level of noise addition in releasing node-differentially private sufficient statistics of ERGMs. We test the error introduced by calibrating noise to restricted sensitivity for a range of different degree cutoffs and for the two different projections detailed in Appendix B. Comparing the level of noise addition to that under edge-level privacy suggests that for large, sparse networks it is feasible to perform differentially private inference on network data under node-level privacy constraints.

Setup

We evaluate the error introduced to structural sufficient statistics for the three synthetic network models with networks of up to 1000 nodes. To guarantee ϵ-differential privacy we project the networks to the space of limited degree graphs and then add Cauchy noise to sufficient statistics, scaled to the restricted sensitivity specified in Table 4.1 times a β-smooth upper bound on the projection.[2] Because the Cauchy distribution has fat tails, summary statistics based on the mean and variance are not meaningful measures of noise addition. Therefore, we use the median absolute error normalized by the scale of the statistics to measure the level of distortion introduced to sufficient statistics by the privacy mechanism. We test two possible projections to the space of limited degree networks, one based on a linear program proposed in [BBDS13] and the other based on removing edges incident to nodes of maximum degree proposed in [KNRS13], both of which are described in detail in Appendix B. For the linear program we use the GLPK package to find solutions. We hypothesize that the node truncation approach will work better than the LP, because for max degree k, the LP requires adding noise scaled to the restricted sensitivity on $\mathcal{H}_{2k}$, while node truncation scales noise to restricted sensitivity on $\mathcal{H}_k$. We test 4 different degree cutoffs: the min, median, max and a conservative estimate of 1.5 times the max degree over samples on n nodes.

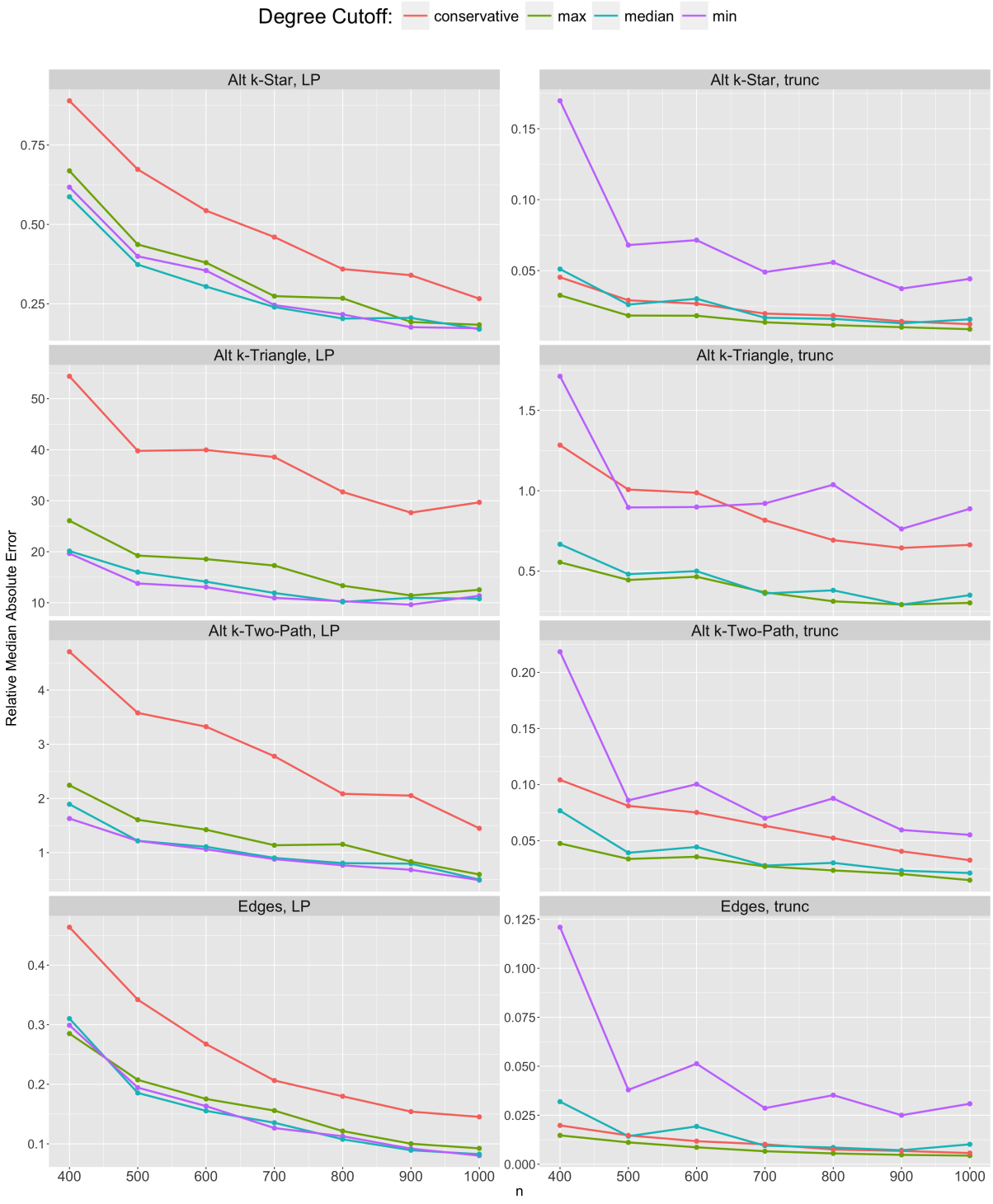
Results

We display results for Model 1 in Figure 5.6 and results for Models 2 and 3 in Appendix C. Using the node truncation projection, it is possible to perturb the sufficient statistics relatively little. In general, star and edge statistics can be released with high accuracy, as the restricted sensitivity is scaled to the degree cutoff k, while for triangles and two-paths it is scaled to k^2. The k-two-path statistic has low relative error, as the statistic value tends to be high for networks with many triangles, so noise is unlikely to overwhelm the statistic. However, the magnitude of the k-triangle statistic tends to be much lower than the k-two-path statistic, hence the difficulty in releasing the k-triangle statistic under privacy constraints.

For all three models, we show that it is possible using the node truncation projection on large networks to add noise well below a relative error of 50% the size of the alternating

[2]While we also tested the addition of Laplace noise which gives an (ϵ, δ)-differentially private algorithm, we found that for small δ, the scale of Laplace noise was almost a factor of 10 higher than Cauchy, as the factors of δ led to very high smooth sensitivity of the projections. Since the Cauchy distribution gives pure ϵ-DP and gives much better accuracy in the typical case, we present results using Cauchy noise.

Figure 5.6: Comparison of Relative Median Absolute Error of node-DP sufficient statistics released at a privacy level of $\epsilon = 0.5$-per statistic for networks from *Model 1*. Cauchy noise is scaled to the restricted sensitivity times a β-smooth bound on the local sensitivity of projection. Projection using an LP is shown on the left and using node truncation is shown on the right



k-triangle statistic. For instance, looking at the second subplot from the top on the right side of Figure 5.6, we can see that for for Model 1, we introduce error of under 25% of the magnitude of the k-triangle statistic for networks above 700 nodes (with either max or median cutoff for k.) As we demonstrated for edge-level inference on 300-node networks drawn from Model 1, introducing error of 30% of the magnitude of the k-triangle statistic enabled highly accurate parameter estimation, so extrapolating to the larger networks, it may be possible to perform accurate inference under node-level privacy.

However, the node-projection is much more sensitive to choice of degree cutoff than projection for the edge-level privacy model. In particular, for edge-level privacy, the projection has fixed global sensitivity of 3 so changing the degree cutoff has no effect on noise addition, only on the value of sufficient statistics in the projected network. In contrast, for the node-level projections, the β-smooth bound on local sensitivity of the projection is affected by choice of cutoff. Relative to the LPs projection, the node-truncation projection tends to perform well for conservative choices of the degree cutoff, as choosing the cutoff close to the max degree can lead to high smooth sensitivity (which is particularly true for Models 2 and 3). For the LP-projection, the opposite is true: aggressive choice of the degree cutoff leads to better performance. This occurs because the LP-projection is closer to optimal smoothness, so the smooth sensitivity of the projection is not much worse for under-estimates of the degree than for over-estimates. However, using the LP requires calibrating noise to the restricted sensitivity on $\mathcal{H}_{2k}$ so this noise addition dominates the noise added, especially for k-triangle and two-path where restricted sensitivity is $O(k^2)$ meaning that the restricted sensitivity is worse by a factor of 4 for the LP as compared to node truncation. We observe that noise addition using the LP tends to be even worse than this factor, however, suggesting that for the sparse networks under consideration, node truncation has low smooth sensitivity. Additionally, the node truncation projection is much more computationally efficient than the LP in practice, as for large networks of more than 700 nodes it tends to run about 100 times faster, although some speed-up may be possible using more efficient LP solvers like CPLEX. However, this computational gap is inherent to the two projections as the node truncation projection runs in time $O(n)$ while the LP projection requires solving a linear program with $O(n^2)$ variables. In short, the node truncation projection seems better suited to the setting of differentially private inference over sparse network data than the linear program projection.

In summary, restricted sensitivity using Cauchy noise and the node truncation projection allows for low noise addition for k-star, edge, and k-two-path sufficient statistics in the strong node-level privacy model. For the k-triangle statistic, the error on large networks with conservative estimates of the degree cutoff is comparable to that of edge-level privacy for small networks, suggesting that even for this high sensitivity statistic it is feasible to perform inference for large, sparse networks under the node-level privacy model.

5.4 Summary of Findings

The chapter concludes with a summary of experimental findings:

- Calibrating noise to restricted sensitivity leads to substantially lower noise addition than private bounding of local sensitivity for sparse networks with privacy budgets of $\epsilon = 0.25$ or less per-sufficient statistic (Figure 5.2).
- Inference using restricted sensitivity gives more accurate parameter estimates than using private local sensitivity on 300-node synthetic networks for small privacy budgets of $\epsilon = 1$ or for estimation of 3 or more parameters (Figures 5.4 and 5.5).
- Randomized response does not allow for useful inference on synthetic networks of 300 nodes even for large privacy budgets of $\epsilon = 3$. In particular, it overwhelms interesting features of network structure, causing estimated parameters to resemble those of $G(n, p)$ graphs (Figures5.3,5.4,5.5).
- Restricted sensitivity enables accurate private parameter estimation on real network data, while existing methods do not. In particular, for restricted sensitivity calibrated to $\epsilon = 2$ we can estimate parameters on a 205-node high school friend network with low error, while maintaining statistical significance of parameters as compared to non-private inference (Table 5.3.) In contrast, private bounding of local sensitivity and randomized response both give highly inaccurate parameter estimates for $\epsilon = 3$ on this dataset (Table 5.2.)
- In the node-level privacy model, relatively low levels of noise can be added to networks of 700 or more nodes using a node truncation projection with a conservative estimate of the maximum degree of a network (Figure 5.6 and Appendix C.) Extrapolating from the performance of edge-level private inference to the node-level case, suggests that noise addition to sufficient statistics may be low enough for large networks to permit node-level private inference.

Chapter 6: Conclusions

We have presented a novel framework for differentially private inference on network data using ERGMs with a variety of sufficient statistics. By taking advantage of the sparsity of many real world networks, our approach allows for inference in the edge-level privacy model with smaller privacy budgets than previous work. Further, our work suggests a path towards feasible differentially private inference under the stronger notion of node-level privacy.

Our empirical testing demonstrates the utility of our methods for practical inference. In particular, we showed for a variety of synthetic networks that our method outperforms current methods for meaningful privacy budgets of $\epsilon = 1$ and for realistic numbers of sufficient statistics. We demonstrate through a differentially private analysis on a dataset consisting of friendships among high school students that researchers could practically apply our methods for inference on real network data and reach similar conclusions as under non-private inference. Finally, we present evidence that for large networks our approach makes it feasible to protect node-level privacy while estimating parameters accurately. Thus, our proposed approach moves us toward the goal of providing meaningful privacy guarantees to participants in network datasets while also allowing researchers to perform accurate inference on the dataset.

We conclude with suggestions for future work. Running tests for node differentially-private inference and for inference with private labels on large networks of over 700 nodes would be useful follow-ups on our experiments. Our proposed framework could also be extended in a number of ways:

- **Drawing synthetic networks from privately fitted ERGMs.** Currently, there exist ad-hoc methods for releasing "anonymized" data to researchers for statistical analysis. A common approach taken by statistical agencies is to fit an ERGM to a sensitive network dataset and then release a network drawn from the fitted model for use by researchers [HHB+08]. Since our proposed methods allow for accurate parameter estimation, it may be possible to simulate networks from ERGMs using differentially private parameter estimates to generate rigorously privatized networks. Doing so requires understanding how variation in parameter estimates impacts the probability distribution defined by these parameters. A measurement of distance between distributions like KL-divergence could be helpful in understanding this difference. However, it is only known how to approximate KL-divergence between ERGMs for maximum-likelihood estimates of parameters [HG10], not Bayesian posterior means. Establishing a useful framework for understanding the difference in ERGM probability distributions defined by parameters inferred using Bayesian inference may be a natural starting point.

- **Adapting more scalable inference methods for differentially private inference.** Recently, more efficient inference methods for ERGMs have been proposed, which can perform inference on networks of well over 1000 nodes in reasonable time [BFM17]. A simple approach (since noise addition is very low for large networks) would be to use these inference methods directly on noisy sufficient statistics without accounting for noise introduced by the privacy mechanism at all. This could be an interesting starting point for testing the usefulness of restricted sensitivity on larger networks. Ideally, more scalable methods could be adapted to incorporate the privacy mechanism and thus infer parameters of the private posterior distribution as our method does.

- **Privacy over directed networks.** The sufficient statistics described in our work can be defined analogously for networks with directed edges, which describe many real world datasets [LKR12]. Proving differential privacy requires bounding the restricted sensitivity of these new sufficient statistics. It may be possible to do so using a similar approach to ours, namely by defining triangle and two-path statistics in terms of an analog to shared partners on directed networks.

- **Comparison of Maximum Likelihood Estimation and Bayesian Inference** In this thesis we considered Bayesian inference for ERGMs both because it constitutes the state-of-the art in inference on ERGMs and because this allowed for simple and intuitive incorporation of the noise of the privacy mechanism into the inferential procedure. However, it is straightforward to incorporate the noise of our restricted sensitivity mechanism into the maximum likelihood estimation method originally proposed by Karwa et al. [KKS17] for inference using networks perturbed by randomized response. Since maximum likelihood estimation constitutes the other most common inference procedure in practice, it may be useful to develop both maximum likelihood and Bayesian approaches for differentially private inference. Further, comparing the choice of inference procedure on noisy data while fixing the noise addition procedure could provide insight into the trade-offs between these two inferential frameworks under differential privacy constraints.

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Appendix A: MCMC Methods for Bayesian Inference over ERGMs

A.1 Simulating Networks from an ERGM

First, we describe a simple MCMC method for simulating networks from an ERGM given parameters of the model. This method is used both to generate synthetic graphs in our experiments and to draw samples needed for inference.

Algorithm 3 Metropolis-Hastings Sampler for ERGMs

Input: parameter vector θ, initial graph $x^{(0)}$, number of iterations T
Output: sequence of graphs $x^{(1)}, ..., x^{(T)}$ such that $x^{(T)} \sim p(X|\theta)$ as $T \to \infty$

For $t = 1, ..., T$:

1. Select nodes i and j at random
2. Propose graph x^* which is the same as $x^{(t-1)}$ except that we "toggle" the edge between i and j so $x^*_{ij} = 1 - x^{(t-1)}_{ij}$
3. Accept the proposed move with probability $\min\left\{1, \frac{p(x^*|\theta)}{p(x^{(t-1)}|\theta)}\right\}$. If the move is accepted set $x^{(t)} = x^*$. Otherwise, set $x^{(t)} = x^{(t-1)}$

The acceptance ratio (assuming all pairs of nodes are chosen with equal probability) is just $\exp\{\theta^T(u(x^*) - u(x^{(t-1)}))$. As the difference in sufficient statistics between two graphs differing in an edge (known as the "change statistic") is typically a simple function of the nodes participating in that edge, this ratio is easy to compute (for instance, for the edges sufficient statistic, it is always just 1 if adding an edge and -1 if removing).

If an ERGM specification puts most of its probability mass on relatively sparse graphs, the sampler that proposes all pairs of nodes with equal probability in step 1 will reject the addition of an edge in most steps, leading to slow convergence. Therefore, Tie-No-Tie (TNT) sampling is generally used in step 1, where we first select either the set of edges or the set of non-edges with equal probability and then pairs of nodes are selected uniformly at random from within the chosen set, biasing step 1 to consider removing edges more frequently than adding (and accounting for the non-uniform proposal distribution in the acceptance ratio). Therefore, throughout this thesis we use TNT sampling to efficiently draw samples from ERGMs. [LKR12]

A.2 Population MCMC Version of the Exchange Algorithm

The basic exchange algorithm for Bayesian inference over ERGMs can be easily modified to take advantage of population MCMC methods, which tend to converge faster, since using various chains reduces temporal dependency between time-steps in the Markov Chain. In particular, Caimo and Friel propose using parallel ADS, which maintains a collection of H chains that interact with one another.

Algorithm 4 Non-Private Bayesian Inference for ERGMs (Parallel ADS) [CF11]

Input: ERGM distribution $\pi(X|\theta)$, prior $p(\theta)$, observed graph x_{obs}, number of chains to use H, tuning parameter γ.
Output: sequence of draws $(\theta_1^{(r)}, ...\theta_1^{(T)}), ..., (\theta_H^{(r)}, ...\theta_H^{(T)})$ from posterior distributions $p(\theta_h|x_{obs})$.

For $t = 1, ..., T$:

For each chain $h = 1, ..., H$:

1. Select at random two different chains h_1 and h_2 from $\{1, ..., H\}\backslash h$
2. Propose $\theta_h^* = \theta_h^{(t-1)} + \gamma\left(\theta_{h_1}^{(t-1)} - \theta_{h_2}^{(t-1)}\right) + \epsilon$
   where ϵ is random noise drawn from a symmetric distribution, such as a Normal.
3. Sample graph $x_h^* \sim \pi(\cdot|\theta_h^*)$
4. Accept the proposed move with probability $\min\{1, \alpha\}$. If the move is accepted, set $\theta_h^{(t)} = \theta_h^*$. Otherwise, set $\theta_h^{(t)} = \theta_h^{(t-1)}$

where

$$\alpha = \frac{p(\theta_h^*)}{p(\theta_h^{(t-1)})} \exp\left\{\left(\theta_h^* - \theta_h^{(t-1)}\right)^T (u(x_{obs}) - u(x_h^*))\right\}$$

The MH acceptance ratio reamins the same as in the single-site update, because the proposal distribution is still symmetric – making the reverse jump from θ_h^* to $\theta_h^{(t-1)}$ simply requires reversing ϵ and the order of h_1 and h_2. The tuning parameter γ controls the amount of interaction between chains and is generally taken to be in the range 0.5 and 1 (in this case we take $\gamma = 0.5$ throughout.) Additionally, the number of chains to use can be tuned in inference, but we choose to use 3 chains throughout as this seems to lead to fast convergence.

Appendix B: Smooth Projections to $\mathcal{H}_k$

B.1 Edge-Adjacency Model

Blocki et al. give an efficient projection to $\mathcal{H}_k$ in the edge-adjacency model with $GS_\mu = 3$ [BBDS13]:

Algorithm 5 3-smooth Projection to $\mathcal{H}_k$ for Edge-Adjacency Model

Input: graph G, cutoff k
Output: graph $\mu(G)$ with max degree k

1. Fix a canonical ordering over all possible edges in a graph on n vertices. Let $e_1^v ... e_t^v$ denote the edges incident to vertex v in this canonical ordering.
2. Delete edge $e = (u, v)$ if and only if:
   (i) $e = e_j^v$ for $j > k$, or
   (ii) $e = e_j^u$ for $j > k$

Intuitively, we keep only the first k edges in the canonical ordering for any node with degree above k. It is clear, then, that this algorithm results in a graph of max degree k and that any graph where all nodes have degree less than k are unchanged. The global sensitivity follows fairly straightforwardly. Consider two graphs G_1 and G_2 that are neighbors differing on a single edge $e = (x, y)$ where wlog G_1 contains e. Then, for every vertex that is not x or y, exactly the same set of edges is deleted, since e does not appear in any other nodes' canonical ordering. If e is deleted, then $\mu(G_1) = \mu(G_2)$. However, if e is not deleted than there may be at most one edge incident to x and one edge incident to y that were deleted from $\mu(G_1)$ but not $\mu(G_2)$, so the neighboring graphs differ in 3 edges. In practice, since this algorithm deletes edges from high degree nodes, it may not bias results too extensively to aggressively estimate k for a graph, as this will only mark edges for deletion on a few nodes that are above the cutoff. However, choosing a cutoff that is too low may remove many edges from many high degree nodes, which will bias not only the number of edges, but also many other sub-graph counts like triangles k-stars, which we explore in our experimental results.

B.2 Node-Adjacency Model

Naive Truncation

The naive truncation projection $\mu_{trunc} : \mathcal{G}_n \to \mathcal{H}_k$ proposed by Kasiviswanathan et. al. simply removes all nodes from the graph with degree above the cutoff k [KNRS13]. It is clear, then, that μ_{trunc} maps any graph in $\mathcal{H}_k$ to itself and that its image is $\mathcal{H}_k$. Moreover, μ_{trunc} is quite efficient, requiring $O(n + \binom{n}{2})$ time. It is also fairly simple to characterize the smooth sensitivity of μ_{trunc}. First, note that the local sensitivity of μ_{trunc} on graph G is $1 + D_k(G) + D_{k+1}(G)$ where D_i is the number of nodes of degree i in graph G since rewiring one node in the graph may affect whether all nodes of degree k or $k+1$ are truncated by μ_{trunc}. We can characterize the smooth sensitivity as follows:

Proposition B.1 (Smooth Sensitivity of μ_{trunc} [KNRS13])**.** Given graph G and hypothesis $\mathcal{H}_k$, let $N_t(G)$ denote the number of nodes with degrees in the range $[k-t, k+t+1]$ and let $C_t = 1 + t + N_t(G)$. Then:

1. The local sensitivity of μ_{trunc} is $C_0(G)$.
2. The local sensitivity at distance t of μ_{trunc} is $C_{t-1}(G)$.
3. The β-smooth sensitivity of μ_{trunc} is $\max_{t \geq 0} e^{-\beta t} C_t(G)$.
4. If $N_{\ln n/\beta}(G) \leq \ell$, so there are at most ℓ nodes in G with degree in range $k \pm (\ln n/\beta)$, then
$$S^*_{\mu,\beta}(G) \leq 1 + \ell + \frac{1}{\beta}$$

Thus, we can compute β-smooth sensitivity efficiently using either part 3 or 4 of the above proposition. Notice that even if a graph is in $\mathcal{H}_k$, it may have high smooth sensitivity if it has many nodes with degree close to the cutoff k. However, part 4 gives a guideline for choosing a conservative cutoff k. In particular, by choosing k to be $\ln n/\beta$ above what is thought to be the max degree of the graph, then the smooth sensitivity would simply be 1. This is not an unreasonable quantity to add to the cutoff, if the cutoff is itself $O(\log n)$, which is often the case. Further, degree distributions are often thought to fall exponentially, so that it is unlikely that there are very many nodes with degree near the cutoff, especially if a conservative cutoff is chosen, suggesting that ℓ might be quite low, even for cutoffs close to the believed cutoff k.

LP-Based Projection

Blocki et al. propose a projection using linear programming. Their method satisfies a relaxed definition of a projection, where $\mu_{LP} : \mathcal{G}_n \to \mathcal{H}_{2k}$ and $\forall G \in \mathcal{H}_k$, $\mu(G) = G$, (but graphs in $\mathcal{H}_k$ are not necessarily mapped to themselves). Because the image is $\mathcal{H}_{2k}$, their method requires calibrating the restricted sensitivity to $\mathcal{H}_{2k}$. However, in contrast to

naive truncation, their approach guarantees that graphs in $\mathcal{H}_k$ always have low smooth sensitivity, because their algorithm outputs an estimator of the distance between a graph and its projection, used to compute a β-smooth upper bound, where the distance estimator is always 0 for graphs in $\mathcal{H}_k$.

The algorithm is also less efficient than naive truncation as it requires solving a linear program with $n + \binom{n}{2}$ decision variables: a variable x_u per node u representing whether node u should be removed from the projected graph or not and a variable $w_{u,v}$ per edge (u, v) representing whether the edge from u to v remains in the projected graph:

Algorithm 6 Projection and 4-Smooth Distance Estimator to $\mathcal{H}_{2k}$ for Node-Adjacency Model [BBDS13]

Input: graph G, cutoff k
Output: graph $\mu_{LP}(G)$ with max degree $2k$, 4-smooth estimate of distance from graph to its projection $\hat{d}(G)$

1. Solve the following LP to get fractional solution $(\bar{x}^*, \bar{w}^*)$. Let there be n decision variables x_u, one for each vertex, and $\binom{n}{2}$ decision variables $w_{u,v}$ one for each potential edge. Additionally, let $a_{uv} = 1$ if the edge $\{u, v\}$ is in G and 0 otherwise. Then, solve the following LP:

$$\min \sum_{v \in V} x_v \quad s.t.$$

$$(1) \quad \forall v, x_v \geq 0$$
$$(2) \quad \forall u, v, w_{u,v} \geq 0$$
$$(3) \quad \forall u, v, a_{uv} \geq w_{uv} \geq a_{uv} - x_u - x_v$$
$$(4) \quad \forall u, \sum_{v \neq u} w_{u,v} \leq k$$

2. Let $\mu_{LP}(G)$ be the graph resulting from removing every edge in G for which either endpoint has weight greater than $\frac{1}{4}$, so either $x_u^* > \frac{1}{4}$ or $x_v^* > \frac{1}{4}$ for edge (u, v).

3. Define distance estimator to be $\hat{d}(G) = 4 \sum_u x_u^*$.

It is clear that if $G \in \mathcal{H}_k$, then the algorithm will return a distance estimator of 0 and $\mu_{LP}(G) = G$, since we can take all x_v to be equal to 0, $w_{uv} = a_{uv}$ so that conditions 1 to 3 of the LP are met and condition 4 is met because all vertices have degree less than k. Using the distance estimator gives a β-smooth upper bound on the local sensitivity of μ_{LP}:

Proposition B.2 (Smooth Sensitivity of μ_{LP} [BBDS13])**.** The smooth sensitivity of μ_{LP} can be bounded by

$$S_{\mu,\beta}(G) \leq \exp\left\{\tfrac{\beta}{4}\hat{d}(G)\right\} \cdot g\left(\tfrac{\beta}{4}\right)$$

where

$$g(x) = \begin{cases} \frac{2}{x} e^{-1+\frac{5}{2}x}, & 0 \leq x \leq \frac{2}{5} \\ 5, & x > \frac{2}{5} \end{cases}$$

so $S_{\mu,\beta}(G) RS_f(\mathcal{H}_{2k})$ is a β-smooth upper bound on the local sensitivity of $f \circ \mu_{LP}$ on graph G.

Comparing the two proposed methods, it is preferable to use naive truncation in cases where we believe $k \geq \ln n/\beta$, because then setting the cutoff to be $\hat{k} = k + \ln n/\beta$, we expect smooth sensitivity of μ_{trunc} to be below $1 + \frac{1}{\beta}$ and the restricted sensitivity will be lower than $RS_f(\mathcal{H}_{2k})$. In general, since we believe the graphs under consideration to have very few high degree nodes close to the cutoff, we expect naive truncation to perform quite well, since the smooth sensitivity should be relatively low for the graphs considered, while considering restricted sensitivity on $\mathcal{H}_{2k}$ may introduce more noise.

Appendix C: Additional Empirical Results

Figure C.1: Comparison of Relative Median Absolute Error of node-DP sufficient statistics released with $\epsilon = 0.5$-per statistic and Cauchy noise addition for networks from *Model 2*.

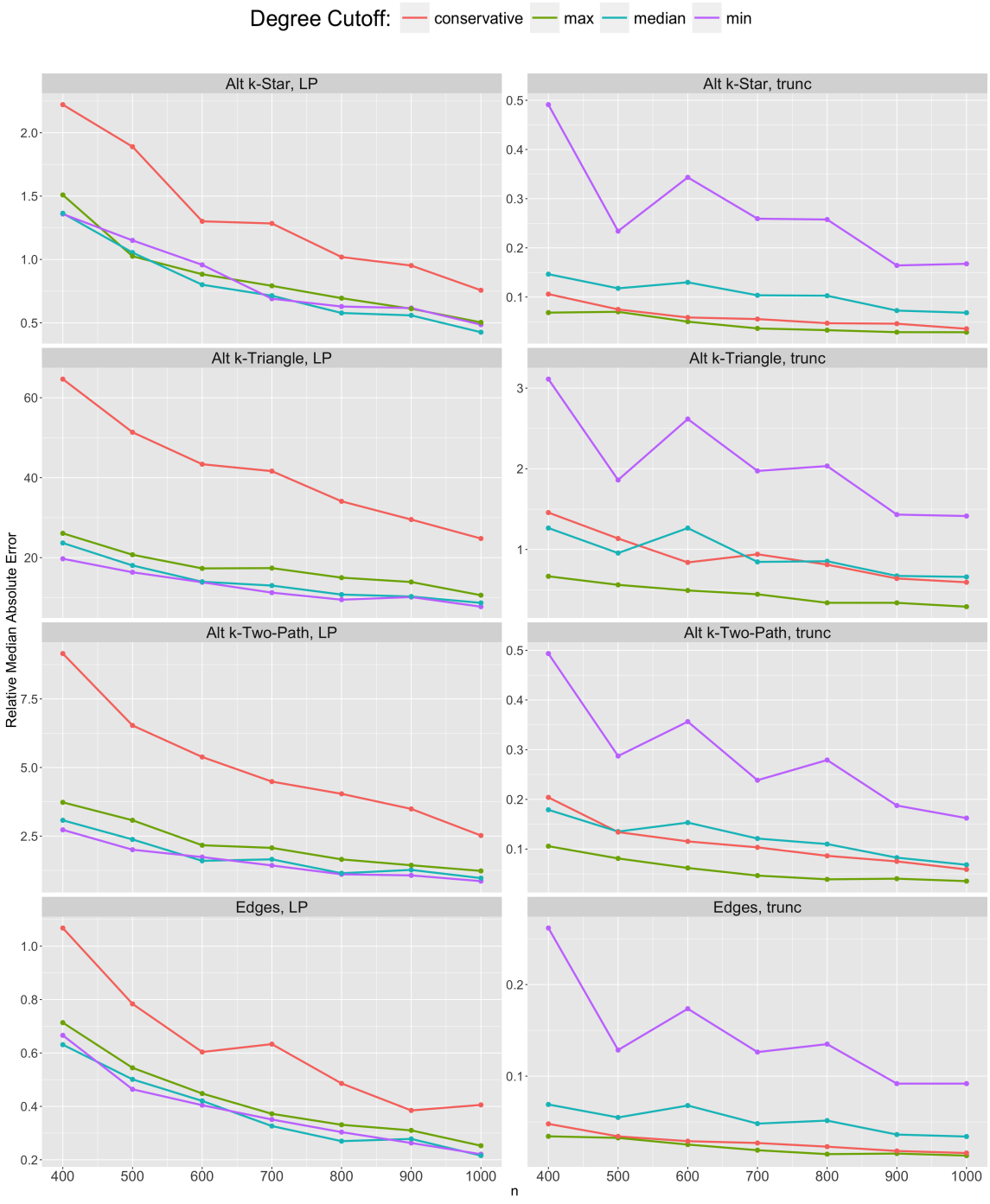

Figure C.2: Comparison of Relative Median Absolute Error of node-DP sufficient statistics released with $\epsilon = 0.5$-per statistic and Cauchy noise addition for networks from *Model 3*.