Observational Constraints on Dissipative Dark Matter

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ABSTRACT

Models for dissipative, disk-forming dark matter are explored in the context of Double Disk Dark Matter. The basics of the model are reviewed. Observational constraints are placed on the parameter space of these models. The first constraint comes from the kinematics of local Milky Way stars. It is argued that most constraints in the literature only apply to the dark matter halo but not to a dark disk. Moreover it is shown that the constraints that do apply to a dark disk are mitigated by non-equilibrium features in the tracer star populations. Constraints are also placed from the distribution of Milky Way interstellar gas. Here, it is shown that a disk of dark matter may be needed to counter the effect of the gas’s magnetic pressure. The possible relationship between a dark disk and the periodicity in the crater record on Earth is also revisited and a dark disk scenario is found to be strongly favored. Direct detection prospects are also explored. The density enhancement in the dark disk and the lower relative velocity are found to play a key role. The exclusion limits on the $X$ particles from nuclear recoil are worked out, as well as ranges of sensitivity for the lighter $C$ particles.
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THIS THESIS IS DEDICATED TO MY WIFE SARAH.
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HaShem, my heart is not haughty, nor mine eyes lofty; neither do I exercise myself in things too great, or in things too wonderful for me.

David, Psalms 131

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Introduction

Dark Matter is 85% of matter in the universe and we still do not know what it is made of. Its presence drove the growth of galaxies from small fluctuations to the vast tapestry we see today. Although we can measure its gravitational interactions, no other interactions with known matter have ever been observed. First proposed in 1922 by Jacobus Cornelius Kapteyn by looking at the dynamics of Milky Way stars (Kapteyn 1922), dark matter has been repeatedly
confirmed from multiple viewpoints, including the galaxy rotation curves (Rubin et al. 1980) and gravitational lensing (Clowe et al. 2004; Markevitch et al. 2004). Dark matter, present in large halos around visible galaxies, is also crucial to seeding galaxy formation in the standard cosmology. These galaxy formation models require that dark matter be cold (Blumenthal et al. 1984). The lensing observations imply additionally that it is relatively collisionless (Clowe et al. 2004; Markevitch et al. 2004). Pauli exclusion constrains the number of fermions in a given volume. Therefore, if dark matter is fermionic, the mass distribution of dwarf galaxies implies a dark matter mass greater than about a keV (Boyarsky et al. 2009b). The galaxy formation history inferred from the Lyman-alpha forest gives similar constraints (Boyarsky et al. 2009a). A popular candidate for dark matter is the WIMP (Weakly Interacting Massive Particle). The reason is that in order to obtain the correct abundance of dark matter today, a self-annihilation cross-section of $\langle \sigma v \rangle \simeq 3 \times 10^{-26} \text{cm}^3\text{s}^{-1}$ would be needed, which coincides with the cross-section for a weakly interacting particle with mass roughly 100 GeV. This is known as the “WIMP miracle” (Jungman et al. 1996).

From a scientific methods point of view, Occam’s razor might seem to prefer a model for dark matter comprised of a single particle type. On the other hand, our experience in the visible matter sector suggests that a dark sector made of a single particle may be statistically “too-good-to-be-true”. An important question is therefore what the possible constraints are on subsectors of dark matter with interacting dynamics. One such model was proposed by Fan et al. (2013), called Double-Disk Dark Matter. In this model, a subsector of dark matter containing a heavy dark matter particle X as well as a lighter particle C, both oppositely charged under a dark $U(1)$. 

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The presence of the lighter particle C allows for bremsstrahlung cooling in this sector so that the sector loses energy to radiation, eventually cooling and, by angular momentum conservation, forming a disk of dark matter. This disk would presumably continue to cool until the X and C bind to form atoms. The model thus predicts a disk of dark matter with a temperature of the order the binding energy of XC atom.

Were such a dark disk to exist, it would be of interest to measure its mass distribution. In particular, we would like to know what observations can tell us about the presence of such a disk, or whether they rule it out, and specifically what would the mass and thickness of the disk be. In this thesis, I describe several studies performed over the course of my PhD describing the observational limits on a dark disk and these dissipative dark matter models. We show that not only are these models viable, but some observations may even seem to favor a disk. We also describe what direct detection experiments can potentially tell us about these models.

Chapter 3 will deal with the limits from local Milky Way Kinematics. Chapter 4 will deal with a similar constraint from the distribution of local interstellar gas. Chapter 5 will describe the potential relationship between a disk of dark matter and the crater record on Earth. Chapter 6 will describe the direct detection prospects.
This section will discuss dissipative dark matter models, specifically in the context of Double-Disk Dark matter. Fan et al. (2013) proposed the Double-Disk Dark Matter (DDDM) model. In this paper, they pointed out that the constraints on dark matter interactions only apply to the majority of dark matter, but that it is possible for a sub-sector to have interactions that are dissipative.
Specifically, dark interactions lead to halos that are spherical. Lensing studies of MS 2137-23, for example, show an ellipticity that constrains the cross-section to be $\sigma_x/m_x < 10^{-25.5} \text{cm}^2 \text{GeV}^{-1}$ (Miralda-Escudé 2002). However, as pointed out by Fan et al. (2013), these bounds can be evaded for a small subsector of dark matter while being satisfied for the other 90% or so of dark matter.

2.1 Temperature of the Dark Sector

Fan et al. assumed a dark $U(1)_D$ gauge field along with a heavy fermion $x$ and light fermion $C$ (for “coolant”) with opposite charges $q_x = +1$ and $q_c = -1$ under $U(1)_D$. They argued that the $C$ particles should contribute to the number of relativistic degrees of freedom at the time of BBN. Assuming that the dark and visible sector decoupled before DDDM, we can use the fact that

\begin{align*}
    s_{\text{vis}}(t) &= \frac{2\pi^2}{45} g_{ss,\text{vis}} T_{\text{vis}}^3(t) \\
    s_D(t) &= \frac{2\pi^2}{45} g_{ss,D} T_D^3(t)
\end{align*}

are both separately conserved to infer

\begin{align*}
    g_{ss,\text{vis}}(t) T_{\text{vis}}^3(t) &= g_{ss,\text{vis}}(t_{\text{dec}}) T_{\text{dec}}^3 \\
    g_{ss,D}(t) T_D^3(t) &= g_{ss,D}(t_{\text{dec}}) T_{\text{dec}}^3
\end{align*}
and thus

\[
\frac{g_{ss,\text{vis}}(t) T_{\text{vis}}^3(t)}{g_{ss,\text{vis}}(t_{\text{dec}})} = \frac{g_{ss,D}(t) T_{D}^3(t)}{g_{ss,D}(t_{\text{dec}})}.
\]  

(2.5)

By estimating the number of degrees of freedom in both sectors, today and at decoupling, we can therefore infer their temperature ratio \(T_D/T_{\text{vis}}\) at the time of BBN. Fan et al. assumed that \(g_{ss,D}\) would be unchanged between the two times, which implies

\[
\frac{g_{ss,\text{vis}}(t_{\text{BBN}}) T_{\text{vis}}^3(t_{\text{BBN}})}{g_{ss,\text{vis}}(t_{\text{dec}})} = T_{D}^3(t_{\text{BBN}}).
\]  

(2.6)

They also assumed that the decoupling temperature was somewhere between the \(b\) and \(W\) mass. This means that \(g_{ss,\text{vis}}(t_{\text{dec}}) = 86.25\). Using the value \(g_{ss,\text{vis}}(t_{\text{BBN}}) = 10.75\), we have, at BBN,

\[
\frac{T_D}{T_{\text{vis}}} = \left(\frac{10.75}{86.25}\right)^{1/3} \simeq 0.5.
\]  

(2.7)

Since the radiation density is given by

\[
\rho_R = \rho_\gamma + \rho_\nu + \rho_D
\]  

(2.8)
The dark sector would have 2 photons and $2C, \bar{C}$, giving $g_{s,s,D} = 2 + 7/8 \times 4 = 5.5$. Thus,

$$\rho_D = g_{s,s,D} \frac{\pi^2}{30} T_D^4 = 5.5 \frac{\pi^2}{30} \left( \frac{10.75}{86.25} \right)^{1/3} T_{\text{vis},\text{BBN}}^4$$

$$\rho_D = \frac{11}{4} \left( \frac{10.75}{86.25} \right)^{4/3} \rho_\gamma$$

$$\rho_D = \frac{7}{8} \left[ \frac{22}{7} \left( \frac{10.75}{86.25} \right)^{4/3} \right] \rho_\gamma$$

$$\simeq \frac{7}{8} 0.20 \rho_\gamma$$

implying that the dark sector contributes

$$\Delta N_{\nu,\text{eff}}^{\text{BBN}} \simeq 0.2$$

to the effective number of neutrino species at BBN. This, when combined the 3.046 species of the standard model (Mangano et al. 2005), agrees with the recent Planck results (Planck Collaboration et al. 2016) of

$$N_{\text{eff}} = 3.15 \pm 0.23.$$
2.2 Relic Density

Here we will compute the non-equilibrium dynamics during the decoupling of the dark sector from the visible sector which lead to thermal relic density of the dark $U(1)$ sector. The non-equilibrium dynamics of the phase space density $f(x^\tilde{\mu}, p^\tilde{\mu}, x)$ of the dark species are governed by the Boltzmann Equation:

$$\hat{L}[f] = C[f]. \quad (2.17)$$

The Liouville operator along a path parametrized by $\tau$ is given by

$$\hat{L}_\tau[f(x^\tilde{\mu}, \dot{x}^\tilde{\mu})] = \dot{x}^\mu \frac{\partial f}{\partial x^\mu} + \ddot{x}^\mu \frac{\partial f}{\partial \dot{x}^\mu} \quad (2.18)$$

where $\dot{}$ represents the derivative with respect to $\tau$. By the geodesic equation, we can replace

$$\ddot{x}^\mu = -\Gamma_{\alpha\beta}^\mu \dot{x}^\alpha \dot{x}^\beta \quad (2.19)$$

to obtain an energy-multiplied, relativistic Liouville operator:

$$E \hat{L}[f] = p^\mu \frac{\partial f}{\partial x^\mu} - \Gamma_{\alpha\beta}^\mu p^\alpha p^\beta \frac{\partial f}{\partial p^\mu}. \quad (2.20)$$
The relativistic Liouville operator for a homogeneous and isotropic phase space density $f(E, t)$ is given by (Kolb & Turner 1990)

\[
\hat{L}[f(E, t)] = \frac{\partial f}{\partial t} - \frac{1}{E} \Gamma^0_{ij} p^i p^j \frac{\partial f}{\partial E} \tag{2.21}
\]

\[
= \frac{\partial f}{\partial t} - \frac{1}{E} \dot{a} \eta_{ij} p^i p^j \frac{\partial f}{\partial E} \tag{2.22}
\]

\[
= \frac{\partial f}{\partial t} - H \frac{p^2}{E} \frac{\partial f}{\partial E} \tag{2.23}
\]

\[
= \frac{\partial f}{\partial t} - H p \frac{\partial f}{\partial p} \tag{2.24}
\]

Using the relation between the number density and phase space density

\[
n(t) = \int d^3 p \, f(p, t), \tag{2.25}
\]

we can integrate the Liouville term to obtain

\[
\int d^3 p \, \hat{L}[f] = \frac{dn}{dt} - H \int d^3 p \, p \frac{\partial f}{\partial p} \tag{2.26}
\]

\[
= \frac{dn}{dt} - H 4 \pi \int dp \, p^3 \frac{\partial f}{\partial p} \tag{2.27}
\]

\[
= \frac{dn}{dt} + 3H \int dp \, p^2 f(p, t) \tag{2.28}
\]

\[
= \frac{dn}{dt} + 3H \int d^3 p \, f(p, t) \tag{2.29}
\]

\[
= \frac{dn}{dt} + 3H n(t). \tag{2.30}
\]
Now, the collision term is given by

\[
\int d^3p \ C[f] = - \int d^3p_X d^3p_{\bar{X}} d^3p_{\text{vis}} d^3p_{\text{vis}} (2\pi)^4 \delta^{(4)}(p_X + p_{\bar{X}} - p_{\text{vis}} - p_{\text{vis}}) (p_X + p_{\bar{X}} - p_{\text{vis}} - p_{\text{vis}})
\]

\[
\times |M_{X\bar{X}\rightarrow \text{vis} \text{vis}}|^2 [f_X f_{\bar{X}} - f_{\text{vis}} f_{\text{vis}}].
\]

Now, we do not know \( f_X \) because it is not in equilibrium. However, we do know that the visible sector is in equilibrium, so that

\[
f_{\text{vis}} f_{\text{vis}} \simeq \exp[-(E_{\text{vis}} + E_{\text{vis}})/T]
\]

(2.33)

where we have approximated the distributions as classical Maxwell-Boltzmann. However, because energy conservation, we know that

\[
\exp[-(E_{\text{vis}} + E_{\text{vis}})/T] = \exp[-(E_X + E_{\bar{X}})/T] \simeq f_X^{\text{EQ}} f_{\bar{X}}^{\text{EQ}}.
\]

(2.34)

Using

\[
\int \frac{d^3p_X d^3p_{\bar{X}} d^3p_{\text{vis}} d^3p_{\text{vis}}}{(2\pi)^{12}} (2\pi)^4 \delta^{(4)}(p_X + p_{\bar{X}} - p_{\text{vis}} - p_{\text{vis}}) = \sigma v,
\]

(2.35)

we have

\[
\int d^3p \ C[f] = \langle \sigma v \rangle [n_{X}^2 - (n_{X}^{\text{EQ}})^2]
\]

(2.36)
so that the Boltzmann equation finally reads

$$\frac{dn}{dt} = -3Hn(t) - \langle \sigma v \rangle [n_X^2 - (n_X^{\text{EQ}})^2]. \tag{2.37}$$

This means that there is a change in the comoving number density only when there is a departure from equilibrium. If we change to dimensionless variables

$$t \rightarrow x \equiv \frac{m_X}{T_{\text{vis}}} \quad n_X \rightarrow Y = \frac{n_X}{s} \tag{2.38}$$

we can recast the Boltzmann equation as

$$\frac{dY}{dx} = -\lambda [Y^2 - Y_{\text{EQ}}^2] \tag{2.39}$$

where $\lambda$ is defined in Eq. 27 of Feng et al. (2008) and is proportional to $\langle \sigma v \rangle$. Feng et al. also gives analytic approximation formulas.

$$n_X(T) \sim T^{3/2}e^{-m_X/T} \tag{2.40}$$

becomes

$$Y_{\text{EQ}} = 0.145(g/g_{*})x^{3/2}\xi^{3/2}e^{-x/\xi}, \tag{2.41}$$
where $\xi = T_D/T_{\text{vis}}$, and assumed S-wave annihilation to give the temperature at freezeout

$$x_f \simeq \xi \log V - \frac{1}{2} \xi \log[\xi \log V] \quad (2.42)$$

where

$$V = 0.038 M_{\text{Pl}} m_X \sigma_0 (g/\sqrt{g_{\text{tot}}^*}) \xi^{3/2} \delta (\delta + 2) \quad (2.43)$$

where $\sigma_0$ is a typical cross-section defined in Feng et al. (2008) and $\delta$ is a tuneable parameter of order 0.3. This means, schematically,

$$T_f \sim \frac{m_X}{\xi}. \quad (2.44)$$

For S-wave annihilation, the final relic abundance is given by (Feng et al. 2008)

$$Y_0 \simeq \frac{3.79 x_f}{(g_{*s}/\sqrt{g_{\text{tot}}^*}) M_{\text{Pl}} m_X \sigma_0} \quad (2.45)$$

or, again schematically,

$$Y_0 \sim \xi \quad (2.46)$$
\[ n_0 \sim \xi T^3. \] (2.47)

We see that although the comoving equilibrium density falls off with time \( x \) as \( x^{3/2} e^{-x} \), the comoving freezeout density stays roughly constant.

Fan et al. (2013) showed that for reasonable values of dark \( U(1)_D \) coupling \( \alpha_D \), one can achieve a relic density which saturates the bound implied by gravitational interactions with local Milky Way stars (the Oort bound, explained more in Chapter 3). They also showed that the annihilation rate was lower than the Hubble rate for \( \alpha_D \lesssim 0.01 \) and \( m_X \gtrsim 1 \text{GeV} \). They also argued that a non-thermal asymmetric production of \( C \) particles was necessary in order for the \( C \) particles, which freeze out much later, to have a non-negligible relic density.

### 2.3 Cooling and Disk Formation

Fan et al. (2013) argued that the cooling of the dark dissipative sector proceeds similarly to the cooling of the baryonic sector. They would first cool adiabatically through Hubble expansion, and then, after virialization through shock heating, cool through bremsstrahlung and Compton scattering. The latter require the presence of the light particle \( C \).
They argued that shock heating would give a dark matter gas with virial temperature of the order

\[ T_{\text{vir}} \sim \frac{G_N M m_X}{5 R_{\text{vir}}} \sim 10\text{keV}. \] (2.48)

For a cluster of a given mass \( M \) virial radius, dissipative dark matter would therefore be hotter than baryonic matter by a factor \( m_X/m_p \). Assuming this temperature to be above the binding energy for the \( X \) atom

\[ B_{XC} = \frac{1}{2} \alpha_D^2 m_C \] (2.49)

the virial cluster will be completely ionized.

To check for consistent cooling, Fan et al. (2013) looked at the mean-free-path for radiation to escape, given by \( n_C = 1/(\ell \sigma) \). Using the Thomson cross-section for radiation, we have:

\[ \ell = \frac{1}{\sigma_T n_C} = \frac{3m_C^2}{8\pi \alpha_D^2 n_C} \sim 10^8\text{kpc} \] (2.50)

where a virial radius of 110 kpc was assumed, along with a dissipative dark matter density \( \Omega_{DD} \sim 10^{-2} \).

Fan et al. (2013) also calculated the time scale for bremsstrahlung and Compton cooling and showed that cooling was efficient in enough of the parameter space to allow cooling of
the $C$ particles. In order for the cooling of $C$ to allow cooling of $X$, $X$ and $C$ need to be thermally coupled. Thermal coupling occurs when the $X-C$ Rutherford scattering rate exceeds the cooling rate. They showed that this is indeed the case in reasonable parts of the parameter space.

By angular momentum conservation, the dissipative dark matter should cool to form a disk. This is confirmed by simulations even without stellar and supernova feedback (Vogelsberger et al. 2012; Torrey et al. 2012). Assuming an isothermal disk with exponential radial profile gives

$$
\rho(R, z) = \rho_0 e^{-R/R_d} \text{sech}^2(z/2h).
$$

(2.51)

For an isolated self-gravitating disk, we have (Spitzer 1942)

$$
h = \frac{\sigma}{\sqrt{8\pi G \rho_0}}
$$

(2.52)

where $\sigma = \sqrt{\bar{v}_z^2}$ is the rms velocity of the $X$ particles after the cooling. They estimated

$$
\sigma^2 \sim T_{\text{cooled}}/m_X.
$$

(2.53)
In order to estimate $T_{\text{cooled}}$, they used the Saha equation

$$\frac{n_X^2}{n_{XC} n} = \frac{1}{n} \left( \frac{T m_C}{2\pi} \right)^{3/2} \exp \left( -\frac{B_{XC}}{T} \right)$$  \hspace{1cm} (2.54)

where $n = n_X + n_{XC} = \rho/m_X$. They found

$$T_{\text{cooled}} = (0.02 - 0.2) B_{XC}. \hspace{1cm} (2.55)$$

For the relevant parts of the parameter space, this is on the order 10 pc.
3.1 Introduction

Since the original study by Oort (1932, 1960), the question of disk dark matter has been a subject of controversy. Over the years, several authors have suggested the idea of a dark disk to explain various phenomena. Kalberla et al. (2007) proposed a thick dark disk as a way to explain the flaring of the interstellar gas layer. Read et al. (2008) showed using cosmological simulations that a thick dark disk is formed naturally in a $\Lambda$CDM cosmology as a consequence of satellite mergers. A phenomenologically very different idea is that of a thin dark disk. Fan et al. (2013) put forward a model for dark matter, coined Double Disk Dark Matter (DDDM), where a small fraction of the total dark matter is self-interacting and dissipative, forming a thin disk. Following the 2013 DDDM paper, Randall & Reece (2014) showed that a dark matter disk of surface density $\sim 10\, M_\odot\,\text{pc}^{-2}$ and scale height $\sim 10\,\text{pc}$ could potentially explain periodicity of comet impacts on earth, giving a target surface density and scale height. If such a dark disk were found to exist, we would want to know what are its surface mass density $\Sigma_D$ and its scale...
Its density in the plane would then be approximately $\rho_D(0) \simeq \Sigma_D/4h_D$.\(^*\) A dark disk model allows us to test the viability of these benchmark values. We do not assume this model is favored, but since the literature clearly supports no dark disk as a possibility, we ask whether it also allows for a disk with this or greater density.

A possible concern in a thin dark-disk model is if the system can emerge dynamically when instabilities and fragmentation are accounted for. These concerns may be resolved with simulations and more careful theoretical considerations, or possibly additional model building. For the purposes of this chapter, however, we consider the dark disk from a purely phenomenological perspective: we ask only what the data tell us about the presence of a thin dark disk, and how to hope to better determine this in the future.

The existing literature suggests that such a model is highly constrained. In principle, there are many ways in which one might aim to constrain a dark disk kinematically. All of these rely on the Poisson-Jeans theory (cf. Section 3.2). Based on our survey of the literature, we categorize the attempted constraints into three main categories:

1. Using stellar kinematics to fit a model for known matter containing interstellar gas, Galactic disk, and dark halo. The parameter being fit here is the surface density of the Galactic disk. Once this is found, one has a measure of the total surface density of the galactic disk. The authors then compare this result to the inventories of known matter to try to constrain the dark disk surface density.

2. Using stellar kinematics to directly measure the vertical gravitational acceleration $K_z$ as a function of height $z$ above the Galactic plane. By the Poisson equation, this is proportional to the total surface density $\Sigma_z$ integrated to height $z$ above the plane. Again

\(^*\)Note that in our convention for scale height $h$, a self-gravitating isothermal disk (Spitzer 1942) is defined to have the spatial dependence $\rho(z) \sim \text{sech}^2(z/2h)$. At large $z$, this is proportional to $e^{-|z|/h}$. 

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an attempt is made to compare this result to the inventories of known matter.

3. Using an isothermal component model (Bahcall 1984a,b) for known matter and fitting the remaining dark matter mass self-consistently using stellar kinematics.

Table 3.1 contains a list of such constraints and which category they fall into. In the face of all these constraints, one may wonder what hope is left for a dark disk. We claim, however, that, at least in their present state, the majority if not all of the constraints in Table 3.1 do not apply.

Method 1 assumes that there is no dark disk and fits a model using known matter to stellar kinematics to show that a dark disk is not necessary. For example, Kuijken & Gilmore (1989a,b, 1991) showed that the density and velocity distributions of K-dwarfs in the Milky Way were consistent with little or no disk dark matter (besides the usual dark halo). Although it is correct that the stellar kinematics do not require a dark disk, the converse is not correct: they do not exclude a dark disk. One might then try to exclude a dark disk by comparing the total Galactic disk surface density found in this way to the inventories of known matter, which at face value seems to be correct. For example, using the distribution function techniques developed by Binney (Binney & Tremaine 2008), Bovy & Rix (2013) dynamically constrained the amount of matter in the Galactic disk, claiming that their results, which are consistent with only standard Galactic components, “leave little room for a dark disk component” in the Milky Way.

However, there is an important detail that has so far been overlooked: the dark disk can actually make room for itself. This is because the distributions of known components in the Galactic disk are extrapolated from observations near the midplane. However, depending on the gravitational pull in the disk, this extrapolation can produce thinner scale heights for these
components. For example, suppose we know the density $\rho_i$ of certain mass components near the plane. The surface density contributions of these components will be $\Sigma_i \sim \rho_i h_i$, where $h_i$ is the thickness of components $i$. The dark disk will affect this value by ‘pinching’ the matter distributions and reducing the thickness $h_i$ of the components, resulting in a lower surface density determination. The surface density estimation for known matter is therefore lower in the presence of a dark disk than without one. Qualitatively, we can write this ‘triangle inequality’ as

$$\Sigma (\text{visible matter + dark disk}) \leq \Sigma (\text{visible matter alone}) + \Sigma (\text{dark disk}).$$

(3.1)

We therefore need to keep this in mind when comparing kinematic determinations of the total surface density to known mass inventories. Figure 3.1 compares the various bounds in the literature to models with and without a dark disk, as well as to a dark disk model that does not take into account the pinching effect back-reaction of the dark disk.

An additional problem with this method is that the specific models that are fit to the kinematics do not allow for a dark disk. Most of the studies using Method 1 (Kuijken & Gilmore 1989a,b, 1991; Bienaymé et al. 2006; Zhang et al. 2013; Bienaymé et al. 2014) fit models containing only and halo, but no thin component. Other studies (Zhang et al. 2013; Bovy & Rix 2013) considered a thin gas component as well but kept its mass fixed and did not allow for any additional mass in a thin component. The study of Creze et al. (1998), in fact, did not consider a stellar disk component at all and assumed a constant density potential. Their result is therefore
very difficult to compare to the literature and in particular to a thin dark disk model.

A similar argument would apply to Method 2. Indeed, the results of Bovy & Tremaine (2012) are consistent with a thin dark disk scenario for precisely the same reason. However, it should be noted that the results of Bovy & Tremaine were based on kinematic data high above the Galactic midplane (1.5-4.5 kpc). Uncertainties in asymmetric drift at these heights could possibly allow total surface densities that are much lower. In fact, analyzing the same data, Moni Bidin et al. (2012a) find that the data do not even allow a dark halo, let alone a dark disk. In Moni Bidin et al. (2015), however, they point out that this method is effective only in constraining the mass above $z = 1.5$ kpc, i.e. $\Sigma_{\text{tot}}(z) - \Sigma_{\text{tot}}(1.5 \text{kpc})$. (This must be the case, as their 2012 value of $\Sigma_{\text{tot}}(1.5 \text{kpc})$ is significantly lower than all literature measurements of $\Sigma_{\text{tot}}(1.1 \text{kpc})$.) This argument applies to the results of Binney & Tremaine (2008) as well. These results therefore do not apply in constraining a thin dark disk.

Another study measuring $\Sigma_{\text{tot}}(z)$ directly is that of Korchagin et al. (2003). Their result $\Sigma_{\text{tot}}(10 \text{pc}) = 10 \pm 1 \ M_\odot \text{pc}^{-2}$ is highly dependent on their choice for the form of $\rho_{\text{tot}}(z)$. By assuming that $\rho_{\text{tot}}(z) \sim \text{sech}(z/2h)$ or $\text{sech}^2(z/2h)$, they are actually not allowing a thin massive component from the outset. Their only robust result is therefore $\Sigma_{\text{tot}}(350 \text{pc}) = 42 \pm 6 \ M_\odot \text{pc}^{-2}$ since it does not depend on the type of extrapolation to $z = 0$ used. Even at 350 pc, these results seem to be in disagreement with a dark disk of the size we are considering, as can be seen in Figure 3.1. However, it was pointed out to us (T. Girard, private communication) that the extinction corrections were applied with the wrong sign. Intuitively, without reddening corrections, we expect preferential reddening near the Galactic plane to increase the apparent
number of red giants near the plane, giving a cuspy profile such as we would expect as arising in
the presence of a thin dark disk. Although reddening corrections should remove this potentially
large bias, applying the reddening corrections with the wrong sign should make it twice as large.
It is therefore unclear why Korchagin et al. did not obtain a tighter bound on $\Sigma_{\text{tot}}(50 \text{ pc})$. At
any rate, we cannot take their present results at face value.

On the other hand, the older studies of Oort (1932, 1960) seem to particularly favor a dark
disk model, as can be seen in Figure 3.1. (Here, we assume the robust result to be $\Sigma_{\text{tot}}(100 \text{ pc})$
and ignore the endpoint result $\Sigma_{\text{tot}}(50 \text{ pc})$.) However, Read (2014) argues that these results
were based on ‘poorly calibrated photometric distances’, stars that were too young to be in
equilibrium, and other questionable assumptions.

An important lesson here is that, as with particle physics experimental studies, it helps to
have a definite model in order to find a self-consistent bound. Previous analyses may not apply
because of changes in measured parameters, but also because the constraint itself depends on
the model, which determines the distribution of other components. On top of this, most previous
studies didn’t use the dark disk thickness as an independent parameter. With a model, it becomes
clear that it makes sense to analyze the data with such a parameter included. Given the large
amount of data becoming available, it makes sense to view the data as a way of measuring
standard parameters, but also constraining—or hopefully discovering—new ones. It is also true
that, were one to analyze a different model aside from the dark disk model we have in mind,
one may have to do the analysis differently to account for any different parameters that we (and
previous authors) did not include.
Figure 3.1: $\Sigma_{tot}(z)$ curves for models with $\Sigma_D = 0$ (dashed line) and $\Sigma_D = 10 \, M_\odot/pc^2$ (solid line). Curve for inconsistent inclusion of dark disk without pinching back-reaction also shown (grey, dot-dashed line).
Method 3, on the other hand, can include a dark disk in a manner that is gravitationally self-consistent. For example, Bahcall (1984a) used this method to show that a thin dark disk similar in size to the interstellar gas disk had to be lighter than $\Sigma_D = 17 \, M_\odot \text{pc}^{-2}$ in the solar region. Bienayme et al. (1987) also used this technique to show that a thick dark disk had to be lighter than $\rho_D(0) \leq 0.03 \, M_\odot \text{pc}^{-3}$, as did Flynn & Fuchs (1994) to show that a model with no dark halo but a very light thick dark disk, with a total galactic surface density $\Sigma_{\text{tot}} = 52 \pm 13$, was a good match to the kinematics of K giants in the solar region. Most importantly, this method was used by Holmberg & Flynn (2000) to show that the kinematics of A and F stars in the solar region were consistent with visible matter distributions alone, without the need for a dark disk. By adding dark matter to known components, they were able to compute a midplane density for the total matter of $\rho(0) = 0.100 \pm 0.006 \, M_\odot \text{pc}^{-3}$. Since they considered dark matter as thin as the molecular hydrogen mass component, their result effectively sets a bound on a dark disk of scale height of 40 pc.

We will see that, using Method 3, the bounds on a dark disk become even tighter as the scale height is decreased. However, there are at least two questions to be asked on the interpretation of this result. i) Can the combined errors in visible components and kinematics allow for a dark disk on their own? ii) Do the assumptions of their kinematic methods (e.g. of thermal equilibrium) hold?

To answer the first question we note that a non-zero dark disk mass could certainly have been hiding in the previously assumed error bars on $\Sigma_{\text{vis}}$, including the errors on the interstellar gas, which had until now been around 50%. Bovy & Rix (2013) reported an uncertainty on $\Sigma_{\text{tot}}$ of
Table 3.1: Bounds on the surface density and local density of total matter and visible matter in the galaxy.

<table>
<thead>
<tr>
<th>Authors</th>
<th>Year</th>
<th>Bound $[M_\odot \text{pc}^{-n}]$</th>
<th>Category</th>
</tr>
</thead>
<tbody>
<tr>
<td>Oort</td>
<td>1932</td>
<td>$\Sigma_{\text{tot}}(100 \text{pc}) = 31$</td>
<td>2</td>
</tr>
<tr>
<td>Oort</td>
<td>1960</td>
<td>$\Sigma_{\text{tot}}(100 \text{pc}) = 29 \pm 10%$</td>
<td>2</td>
</tr>
<tr>
<td>Bahcall</td>
<td>1984a</td>
<td>$\Sigma_{D,\text{thin}} \leq 17$ $\rho_{\text{tot}}(0) \leq 0.24$</td>
<td>3</td>
</tr>
<tr>
<td>Bahcall</td>
<td>1984b</td>
<td>$\Sigma_{\text{tot}} = 55 - 83$ $\rho_{\text{tot}}(0) = 0.17 - 0.25$</td>
<td>3</td>
</tr>
<tr>
<td>Bienayme et al.</td>
<td>1987</td>
<td>$\rho_{\text{DM}}(0) \leq 0.03$ for thick dark disk</td>
<td>3</td>
</tr>
<tr>
<td>Kuijken &amp; Gilmore</td>
<td>1991</td>
<td>$\Sigma_{\text{tot}}(1.1 \text{kpc}) = 71 \pm 6$</td>
<td>1</td>
</tr>
<tr>
<td>Bahcall et al.</td>
<td>1992</td>
<td>$\Sigma_{\text{tot}} = 70^{+24}_{-16}$</td>
<td>3</td>
</tr>
<tr>
<td>Flynn &amp; Fuchs</td>
<td>1994</td>
<td>$\Sigma_{\text{tot}} = 52 \pm 13$</td>
<td>3*</td>
</tr>
<tr>
<td>Pham</td>
<td>1997</td>
<td>$\rho_{\text{tot}}(0) = 0.11 \pm 0.01$</td>
<td>NA</td>
</tr>
<tr>
<td>Creze et al.</td>
<td>1998</td>
<td>$\rho_{\text{tot}}(0) = 0.076 \pm 0.015$ (assumed constant density)</td>
<td>1*</td>
</tr>
<tr>
<td>Holmberg &amp; Flynn</td>
<td>2000</td>
<td>$\rho_{\text{tot}}(0) = 0.102 \pm 0.010$ $\rho_{\text{vis}} = 0.095$</td>
<td>3*</td>
</tr>
<tr>
<td>Korchagin et al.</td>
<td>2003</td>
<td>$\Sigma_{\text{tot}}(350 \text{pc}) = 42 \pm 6$</td>
<td>2</td>
</tr>
<tr>
<td>Siebert et al.</td>
<td>2003</td>
<td>$\Sigma_{\text{tot}}(800 \text{pc}) = 76^{+25}_{-12}$</td>
<td>1</td>
</tr>
<tr>
<td>Holmberg &amp; Flynn</td>
<td>2004</td>
<td>$\Sigma_{\text{tot}}(1.1 \text{kpc}) = 74 \pm 6$</td>
<td>3</td>
</tr>
<tr>
<td>Bienaymé et al.</td>
<td>2006</td>
<td>$\Sigma_{\text{tot}}(800 \text{pc}) = 57 - 66$</td>
<td>1</td>
</tr>
<tr>
<td>Garbåri et al.</td>
<td>2011</td>
<td>$\rho_{\text{halo}} = 0.003 - 0.033$</td>
<td>3</td>
</tr>
<tr>
<td>Moni Bidin et al.</td>
<td>2012b</td>
<td>$\Sigma_{\text{tot}}(1.5 \text{kpc}) = 55.6 \pm 4.7$</td>
<td>2</td>
</tr>
<tr>
<td>Bovy &amp; Tremaine</td>
<td>2012</td>
<td>$\rho_{\text{halo}} = 0.008 \pm 0.003$</td>
<td>2</td>
</tr>
<tr>
<td>Zhang et al.</td>
<td>2013</td>
<td>$\Sigma_{\text{tot}}(1 \text{kpc}) = 67 \pm 6$ $\rho_{\text{halo}}(0) = 0.0065 \pm 0.0023$</td>
<td>1</td>
</tr>
<tr>
<td>Bovy &amp; Rix</td>
<td>2013</td>
<td>$\Sigma_{1100} = 68 \pm 4$</td>
<td>1</td>
</tr>
<tr>
<td>Bienaymé et al.</td>
<td>2014</td>
<td>$\Sigma_{\text{tot}}(1.1 \text{kpc}) = 68.5 \pm 1$ $\Sigma_{\text{tot}}(350 \text{pc}) = 44.2^{+2.3}_{-2.9}$</td>
<td>1</td>
</tr>
</tbody>
</table>

* denotes bounds derived using HF technique
±4 \, M_\odot\,\text{pc}^{-2}, \text{and an uncertainty on } \Sigma_{\text{stars + remnants}} \text{ also of } ±4 \, M_\odot\,\text{pc}^{-2}. \text{ If we combine this with the traditionally assumed } ±7 \, M_\odot\,\text{pc}^{-2} \text{ error on the interstellar gas, we can already allow a dark disk with mass } \sqrt{4^2 + 4^2 + 7^2} \approx 9 \, M_\odot\,\text{pc}^{-2}.

The baryonic matter components of the Galaxy have, however, been more carefully measured in recent years. Revised values for the interstellar gas parameters are the subject of Chapter 4. We also include new values for the stellar components, revised by McKee et al. (2015). We repeated the study of HF2000 including radial velocities (which were not available at the time) and using the recent three-dimensional reddening map of Schlafly et al. (2014). With these updates, we show in this chapter that the traditional HF2000 method would not allow a dark disk heavier than \sim 4 \, M_\odot\,\text{pc}^{-2}.

Regarding the second question, however, we note that one of the major assumptions in the kinematic method of HF2000 is that the populations under study are in statistical-mechanical equilibrium. We claim (and show in Appendix 8.4) that the definition of equilibrium needed for these analyses is one that precludes the possibility of oscillating solutions, in which the tracer population is in a distribution whose center oscillates above and below the Galactic plane. Such behavior, if present, would smooth out the effects of any thin Galactic component, in which case incorrectly assuming a static distribution would result in an upper bound on a dark disk that is too low.

We show that the tracer populations studied by HF2000 indeed do possess a net vertical velocity as well as a center that is vertically displaced from the Galactic midplane, which are evidence for such deviations from equilibrium. The HF2000 analysis, which assumes that the
tracer populations are in equilibrium, produces an overly strong bound. We show here how a consistent analysis can be performed without any need for the assumption of equilibrium. Analyzed in this way, we show that the kinematics currently allow for a thin dark disk of up to $14 \, M_{\odot} \, \text{pc}^{-2}$, with a weaker bound for thicker disks. (These considerations apply equally to the studies of Flynn & Fuchs (1994) and of Creze et al. (1998). The latter sample, in fact, does show deviations from equilibrium including a negative net vertical displacement from the midplane, as reported by the authors, and a net vertical velocity, as can be inferred from their numbers. This may, along with the reasons described above, account for the unusually low values of $\rho_{\text{tot}}(0)$ found by these authors.)

3.2 Theory

The Poisson-Jeans theory will be important both for constructing a model of the Galactic potential and for gaining a qualitative understanding of the effect of a thin dark disk. We will now explain the Poisson-Jeans theory, following which we will use the P-J theory to solve a simple toy model. We will then explain the theory behind the HF2000 study, and derive the Holmberg & Flynn relation.
3.2.1 Poisson-Jeans Theory

Consider the phase-space distribution $f_i(x, v)$ for a stellar population $i$, satisfying

$$\int d^3v f_i(x, v) = \rho_i(x) \quad (3.2)$$

and, for a more general function $g$,

$$\int d^3v f_i(x, v)g(v) = \rho_i(x) \langle g \rangle \quad (3.3)$$

where $\rho_i(x)$ is the population density. The $f_i$ must also satisfy the collisionless Boltzmann equation (Liouville’s theorem)

$$\frac{Df_i}{Dt} \equiv \frac{\partial f_i}{\partial t} + \dot{x} \cdot \frac{\partial f_i}{\partial x} + \dot{v} \cdot \frac{\partial f_i}{\partial v} = 0. \quad (3.4)$$

If we assume our stellar population is in equilibrium then the first term on the right vanishes. Also, we can replace $\dot{x}$ and $\dot{v}$ by $v$ and $-\frac{\partial \Phi}{\partial x}$ respectively, where $\Phi$ is the gravitational potential. From the Boltzmann equation, we can derive the vertical Jeans equation in the standard way, found e.g. in Binney & Tremaine (2008). The vertical Jeans equation then reads:

$$\frac{1}{R} \frac{\partial}{\partial R} \left( R \rho_i \sigma_{i,Rz} \right) + \frac{\partial}{\partial z} \left( \rho_i \sigma_{i,z}^2 \right) + \rho_i \frac{\partial \Phi}{\partial z} = 0 \quad (3.5)$$
The first term in Equation 3.5 describes the rate of change of the correlation between $v_z$ and $v_R$, so it is commonly referred to as the ‘tilt’ term. The tilt term is expected to be very small near the plane, since, being asymmetric in $z$, it must vanish at $z = 0$ (Moni Bidin et al. 2012b). If we restrict our analysis to heights that are small compared to the tracer population’s scale height, we can safely neglect this term. We thus have (dropping the subscript on $\sigma_{i,z}$)

$$\frac{\partial}{\partial z} (\rho_i \sigma_i^2) + \rho_i \frac{\partial \Phi}{\partial z} = 0$$

(3.6)

which we can rewrite as

$$\frac{1}{\rho_i \sigma_i^2} \frac{\partial}{\partial z} (\rho_i \sigma_i^2) + \frac{1}{\sigma_i^2} \frac{\partial \Phi}{\partial z} = 0,$$

(3.7)

from which we can immediately see that the solution is

$$\rho_i \sigma_i^2 \propto \exp \left( - \int dz \frac{1}{\sigma_i^2} \frac{\partial \Phi}{\partial z} \right).$$

(3.8)

If we additionally take our stellar population to be isothermal, that is $\sigma_i(z) = \text{constant}$, then the solution reduces further to

$$\rho_i(R, z) = \rho_i(R, 0) e^{-\frac{\Phi(R, z) - \Phi(R, 0))}{\sigma_i^2}}.$$

(3.9)
We will now consider only $R = R_\odot$, where we will fix $\Phi(R_\odot, 0) = 0$. This gives

$$
\rho_i(z) = \rho_i(0) e^{-\Phi(z)/\sigma_i^2}.
$$

(3.10)

Indeed, once we have assumed equilibrium, vanishing tilt, and isothermality, what we are left with is simply a gas at temperature per unit mass $kT_i/M = \sigma_i^2$, and Equation 4.2 is merely a Boltzmann factor. A possible objection to the method is that it was found by Garbari et al. (2011) that neglecting the tilt term in Equation 3.5 in general leads to a biased determination of $\rho_{dm}$. However, this was only found to be a problem at heights of $z \gtrsim 0.5 \text{kpc}$ or greater. Garbari et al. describe the HF2000 sample, which is relatively close to the plane, as ‘unlikely to be biased’. With this solution to the Jeans equation, we can now relate the potential $\Phi$ to the mass distribution consisting of the different components $\rho_i$. By the Poisson equation,

$$
\nabla^2 \Phi = 4\pi G \sum_i \rho_i.
$$

(3.11)

We can split up the Laplacian in the standard way, following Kuijken & Gilmore (1989b):

$$
\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial z^2} + \frac{1}{R} \frac{\partial}{\partial R} \left( R \frac{\partial \Phi}{\partial R} \right)
$$

(3.12)

$$
= \frac{\partial^2 \Phi}{\partial z^2} + \frac{1}{R} \frac{\partial}{\partial R} V_c^2
$$

(3.13)

$$
= \frac{\partial^2 \Phi}{\partial z^2} + 2 \frac{V_c}{R} \frac{\partial V_c}{\partial R}
$$

(3.14)

$$
= \frac{\partial^2 \Phi}{\partial z^2} + 2(B^2 - A^2)
$$

(3.15)
where $V_c$ is the local circular velocity and $A$, $B$ are the ‘Oort constants’.

\begin{equation}
A \equiv \frac{1}{2} \left( -\frac{\partial V_c}{\partial R} + \frac{V_c}{R} \right) \quad (3.16)
\end{equation}

\begin{equation}
B \equiv -\frac{1}{2} \left( \frac{\partial V_c}{\partial R} + \frac{V_c}{R} \right). \quad (3.17)
\end{equation}

We have assumed azimuthal symmetry in Equation 3.12. Although in reality the azimuthal symmetry of the Galaxy is spoiled by spiral arms and other structures, we will assume that these are not relevant over the time scales needed for our tracer populations to relax to their current distributions. At the Sun’s position, the second term in Equation 3.12 is very small, so we shall rename it $4\pi G\delta \rho$. From Catena & Ullio (2010), we have $A - B = 29.45 \pm 0.15 \text{ km} \text{s}^{-1} \text{kpc}^{-1}$ and $A + B = 0.18 \pm 0.47 \text{ km} \text{s}^{-1} \text{kpc}^{-1}$, giving

\begin{equation}
\delta \rho = (-2 \pm 5) \times 10^{-4} M_\odot \text{pc}^{-3} \quad (3.18)
\end{equation}

near the $z = 0$ plane. This is comparable in magnitude to the density of red giant stars and to the stellar halo density (Holmberg & Flynn 2000), but is about 2-3 orders of magnitude smaller than the total density, and can safely be neglected:

\begin{equation}
\frac{\partial^2 \Phi}{\partial z^2} = 4\pi G \sum_i \rho_i. \quad (3.19)
\end{equation}
Combining Equation 3.19 with the Jeans equation (4.2), we have the Poisson-Jeans equation for the potential $\Phi$:

$$\frac{\partial^2 \Phi}{\partial z^2} = 4\pi G \sum_i \rho_i(0) e^{-\Phi/\sigma_i^2}.$$  \hspace{1cm} (3.20)

where we have dropped the $R$ coordinate label. This can also be cast in integral form (assuming $z$-reflection symmetry)

$$\frac{\rho_i(z)}{\rho_i(0)} = \exp \left( -\frac{4\pi G}{\sigma_i^2} \sum_k \int_0^z dz' \int_0^{z'} dz'' \rho_k(z'') \right)$$  \hspace{1cm} (3.21)

which is the form used in our Poisson-Jeans solver.

### 3.2.2 A Toy Model

We now study the Poisson-Jeans theory in some simple models following Spitzer (1942). This will be useful in understand the qualitative effects of a thin dark disk. We first study a self-gravitating, single-component galaxy. In this case, Equation 4.3 reads

$$\frac{1}{4\pi G} \frac{d^2 \Phi}{dz^2} = \rho_0 e^{-\Phi(z)/\sigma^2}$$  \hspace{1cm} (3.22)

which has solution

$$\frac{d\Phi(z)}{dz} = \frac{\sigma^2}{h} \tanh(z/2h)$$  \hspace{1cm} (3.23)
or

\[ \rho(z) = \rho_0 \operatorname{sech}^2\left(z/2h\right) \]  

(3.24)

where

\[ h = \frac{\sigma}{\sqrt{8\pi G \rho_0}}. \]  

(3.25)

Although this was a toy model, it illustrates two very important features of gravitating disks:

1) The scale height \( h \) of the disk grows with its vertical dispersion \( \sigma \), approximately as \( h \sim \sigma \), and

2) The scale height decreases with the total midplane density. These features will remain qualitatively true even when more components are included.

We can now ask what happens if we include an additional very thin, delta-function component (as a toy approximation to a dark disk). In this case, we have

\[ \nabla^2 \Phi = 4\pi G \rho_s + 4\pi G \Sigma_D \delta(z) \]  

(3.26)

and the Poisson-Jeans equation takes the form

\[ \frac{1}{4\pi G} \frac{d^2\Phi(z)}{dz^2} - \Sigma_D \delta(z) = \rho_{s0} \exp\left(-\frac{\Phi(z)}{\sigma^2}\right) \]  

(3.27)

where \( \rho_s \) stands for the density of stars and \( \Sigma_D \) is the dark disk surface density. Defining
\( Q \equiv \Sigma_D / 4\rho_s h_s \), (with \( h_s \) defined as in Equation 3.25), we can write down the exact solution, which is

\[
\rho_s(z) = \rho_s(0) (1 + Q^2) \sech^2 \left( \frac{\sqrt{1 + Q^2}}{2h_s} (|z| + z_0) \right) \tag{3.28}
\]

with

\[
z_0 \equiv \frac{2h_s}{\sqrt{1 + Q^2}} \arctanh \left( \frac{Q}{\sqrt{1 + Q^2}} \right) \tag{3.29}
\]

Effectively, the dark disk has the effect of ‘pinching’ the density distribution of the other components, as we can see in Figure 4.1. In particular, it reduces their scale heights, and for a fixed midplane density \( \rho_i(0) \), it implies that their surface density \( \Sigma_i \) is less than what it would be without the disk. Actually, this is what would happen were we to include any other additional mass component.

\[\textbf{Figure 3.2:} \quad \text{A plot of the exact solutions without and with a dark disk of } Q = 1. \text{ The density is 'pinched' by the disk, in accordance with Equation 4.5.}\]
An important consequence of this for the stellar disk, especially in the context of our analysis, is that for a fixed midplane density, the effect of the dark disk is to decrease the stellar surface density. Integrating Equation 4.5, the stellar surface density find is

\[ \Sigma_s(Q) = \Sigma_s(0) \left( \sqrt{1 + Q^2} - Q \right) \]  

\[ = \sqrt{\Sigma_s(0)^2 + \Sigma_D^2} - \Sigma_D \]  

where \( \Sigma_s(0) \equiv 4\rho_s h_s \) is what the surface density would have been without the dark disk. We can see that the stellar surface density, \( \Sigma_s(Q) \) is monotonically decreasing with \( Q \). We can then write the total surface density as

\[ \Sigma_{\text{tot}} = \Sigma_s(Q) + \Sigma_D \]  

\[ = \sqrt{\Sigma_s(0)^2 + \Sigma_D^2} \]  

verifying the triangle inequality (Equation 3.1) for this simple case.

### 3.2.3 The HF study

We now explain the analysis of Holmberg & Flynn (2000), and derive the HF relation. This analysis, developed under different forms by Kuijken & Gilmore (1989a,b), Fuchs & Wielen (1992), Flynn & Fuchs (1994), Holmberg & Flynn (2000), is related to the P-J theory. Near
where \( f(z, v_z) \) represents the one-dimensional phase space density in \( z, v_z \). We therefore work solely with \( f_z(z, v_z) \) and drop the subscript \( z \). This function satisfies

\[
\int dv_z f(z, v_z) g(v_z) = \rho(z) \langle g \rangle.
\]

The Boltzmann equation also tells us that

\[
v_z \frac{\partial f}{\partial z} - \frac{\partial \Phi}{\partial z} \frac{\partial f}{\partial v_z} = 0
\]

which has solution

\[
f(z, v_z) = F \left( \frac{1}{2} v_z^2 + \Phi(z) \right).
\]

That is, since this function is constant it must be a function solely of the ‘vertical energy’ \( \frac{1}{2} v_z^2 + \Phi(z) \). Using Eqs. 3.36 and 3.37 we can write

\[
\rho_i(z) = \int_{-\infty}^{\infty} dw \ f_i(z, w)
\]

\[
= \int_{-\infty}^{\infty} dw \ f_i(0, \sqrt{w^2 + 2\Phi(z)})
\]
and since we can write

\[ f_i(z, w) = \rho_i(z) f_{z;i}(w) \quad (3.40) \]

where \( f_z(w) \) is the vertical velocity probability distribution at height \( z \), normalized so that \( \int dw f_z(w) = 1 \), we have

\[ \frac{\rho_{f_i}(z)}{\rho_i(0)} = \int_{-\infty}^{\infty} dw f_{z=0;i} \left( \sqrt{w^2 + 2\Phi(z)} \right). \quad (3.41) \]

By extracting the in-the-plane velocity distribution \( f_{z=0}(w) \) from tracer populations of A and F-stars, and integrating to find \( \rho_{A,F}(z) \) for specific potentials \( \Phi(z) \), Holmberg & Flynn were able to test models for \( \Phi \) by comparing the \( \rho_{A,F}(z) \) obtained in this fashion to the observed tracer densities. The model they considered was a modified version of the ‘Bahcall models’ (Bahcall 1984a,b,c). The Bahcall model consists of splitting the visible matter into a series of isothermal components, each with a distinct vertical dispersion \( \sigma_i \). By assuming dispersions and densities in the plane \( \rho_i(0) \) for all components, the Poisson-Jeans equation (4.3) then gives a unique solution for \( \Phi(z) \). The Bahcall model used by HF2000 is shown in Table 1 of the latter. An updated version of this model, featuring slight modifications, was used in Flynn et al. (2006). Holmberg & Flynn found that a mass model with little or no dark matter in the disk was in good agreement with the data, as did previously Kuijken & Gilmore (1989a,b) using a similar method. By adding or subtracting invisible mass to the various components in the model, HF
then obtained a range on the acceptable mass models, which gave a range of acceptable densities as a function of height. Figure 3.3 (top) compares the tracer densities with those predicted using the HF technique with an updated mass model containing no dark disk. We also show in Figure 3.3 (bottom) the same result with a dark disk of surface density $\Sigma_D = 10 \, M_\odot/pc^2$ and $h_D = 10$ pc. Clearly it is of interest to know what is the maximum value of $\Sigma_D$ compatible with the data for a given $h_D$, within statistical errors.
3.3 Sample

We now describe the numerical procedures used in the selection of our sample, as well as our methods for performing extinction corrections and for extracting the tracers’ velocity distribution.

3.3.1 Sample Selection

We worked with the new reduction of the Hipparcos data (van Leeuwen 2008). As our sample, we used mostly the same stars as did HF2000, which contained A and F-stars. We performed the same cuts as HF2000. That is, for A-type stars, $-0.2 < B - V < 0.6$, and $0.0 < M_V < 1.0$, and the completeness limit $V \leq 7.9 + 1.1 \sin |b|$. By the completeness limit we have, using the definition of absolute magnitude,

$$M_V + 5 \log_{10} \left( \frac{d}{10 \text{pc}} \right) \leq 7.9 + 1.1 \sin |b|$$

(3.42)

and so

$$\frac{d}{10 \text{pc}} \leq 10^{(7.9 + 1.1 \sin |b| - M_V)/5}$$

(3.43)

and since $|b| \geq 0$ and $M_V < 1.0$, we have for our A-star sample, $d \lesssim 240 \text{ pc}$ as our completeness limit. We therefore limited our sample to a vertical cylinder centered at the Sun with radius and half-height of $r_c = h_c = 170 \text{ pc}$. This ensures that the diagonals of the cylinder will be
shorter than 240 pc. Note that HF2000 used a cylinder with radius 200 pc. The part of their A-star sample higher than 130 pc was therefore not complete. (By the same reasoning, their F-star sample was only complete to a height of 66 pc.) Using this method, our data set contained approximately 1500 A-stars. The precise numbers depend on the reddening corrections, as will be explained below.

For the F-star sample, the cuts in color were also $-0.2 < B - V < 0.6$, but here we used $1.0 < M_V < 2.5$. Here the maximum distance was found to be 120 pc. We therefore restricted our analysis to a cylinder of radius 50 pc and half-height of 109 pc. Additionally, to avoid any bias from the Coma Star Cluster (Trumpler 1938), we further cut off this sample from above at +40 pc. The final sample contained only about 500 stars, with the exact number depending on the extinction/reddening corrections and the value of the height of the Sun above the Galactic plane, $Z_\odot$.

### 3.3.2 Extinction Corrections

HF2000 used the extinction model of Hakkila et al. (1997). We used the more recent 3D reddening map of Schlafly et al. (2014), which computes the reddening for most of the stars at a distance of 63 pc from the Sun or farther. For stars closer than this, we interpolated from 63 pc using the interpolation map of Chen et al. (1999), but with a sech$^2$ profile for the reddening material instead of exponential. This interpolation model takes the dust scale height $h_{\text{dust}}$ as an input. We assumed a sech$^2$ scale height of 100 pc. For stars farther than 63 pc but not covered
by the map, we extrapolated from the 2D dust map of Schlegel et al. (1998).

### 3.3.3 The Velocity Distributions

For the vertical velocity of their stars, HF2000 used the approximation

\[
w = \frac{k \mu_b}{\pi \cos b} + u \cos l \tan b + v \sin l \tan b.
\]  

(3.44)

where \( \tilde{\pi} \) is the parallax and \( \kappa = 4.74047 \) km \( s^{-1} \cdot \text{mas}^{-1} \cdot \text{mas} \cdot \text{yr}^{-1} \). This approximation is valid for stars with \( \sin b \ll 1 \). However, we make use of the exact equation

\[
w - w_0 = \frac{k \mu_b}{\pi} \cos b + V_R \sin b
\]

(3.45)

where \( w_0 = 7.25 \) km s\(^{-1} \) is the vertical velocity of the sun (Schönrich et al. 2010) relative to the LSR. Since radial velocities are not known, we simply take an average:

\[
\langle w \rangle = w_0 + \frac{k \mu_b}{\pi} \cos b + \langle V_R \rangle \sin b,
\]

(3.46)

where we take

\[
\langle V_R \rangle = -V_{\text{sun}} \cdot \hat{r} = -u_0 \cos l \cos b - v_0 \sin l \cos b - w_0 \sin b
\]

(3.47)
with \( u_0 = 11.1 \text{ km s}^{-1} \) and \( v_0 = 12.24 \text{ km s}^{-1} \), also given by Schönrich et al. (2010). This follows from the fact that from the LSR frame, we expect \( \langle V_{R}^{\text{LSR}} \rangle = 0 \) for any axisymmetric stellar population. Since we are also working with stars close to the plane (\( \sin b \ll 1 \)), approximating \( V_R \) by its average still gives a good approximation, and this approximation will still be better than Equation 3.44. This gives

\[
\langle w \rangle = \frac{k \mu_b}{\pi} \cos b + w_0 \cos^2 b - u_0 \cos l \cos b \sin b - v_0 \sin l \cos b \sin b.
\] (3.48)

We can do even better since radial velocities have been measured for many stars in the solar region since 2000. Therefore, rather than simply use Equation 3.48 for all our stars, we supplemented the Hipparcos data with radial velocities from Barbier-Brossat & Figon (2000). We could thereby use measured radial velocities for 52% of our stars within \( |b| < 12^\circ \). It was reported in Binney et al. (1997) that stars with measured radial velocities constitute a kinematically biased sample. However, Korchagin et al. (2003) showed, for their red giant sample with measured \( V_R \), the result only appears biased when judging from their proper motions; once these are converted to cartesian \( x, y, \) and \( z \)-velocities, the kinematic bias is no longer observed. Moreover, we did not restrict our analysis to stars with measured radial velocities, since we completed the sample using the average \( \langle V_R \rangle \). In light of these factors, along with the fact that we restrict our analysis to low \( b \), we expect the radial velocities to introduce very little bias to our results.
3.3.4 The Height of the Sun

Because the Poisson-Jeans equation assumes thermal equilibrium, and because we have assumed Galactic azimuthal symmetry, we are considering only models with $z$-reflection symmetry across the Galactic plane; nonetheless, more general models are certainly possible. In these $z$-symmetric models, all disk-like components will be centered at the Galactic plane at $z = 0$.

In order to know the coordinates of the stars in our tracer populations, we need to know the height of the Sun relative to this Galactic midplane. Various measurements of this quantity have been made, but the precise result depends on the data set used. The value inferred from classical Cepheid variables is $Z_\odot = +26 \pm 3$ pc (Majaess et al. 2009). However, other measurements have found values as low as $Z_\odot = +6$ pc (Joshi 2007). In fact, for our tracer sample we find a best fit of $Z_\odot = +7 \pm 1$ pc. For values of $Z_\odot$ very different from this, the data in our sample were no longer consistent with any model for the Galactic potential. HF2000 also seem to have used a value of $Z_\odot$ very close to $Z_\odot \approx 0$. They may have been using the value $Z_\odot = +7$ pc, which they measured several years earlier in Holmberg et al. (1997). Interpreting this value in the context of DDDM implies that the Sun is currently within the dark disk, which could have important consequences for comet impacts (Matese et al. 2001; Randall & Reece 2014; Rampino 2015; Shaviv et al. 2014).

On the other hand, in Section 3.5.3, we introduce the non-equilibrium version of the HF method. In this way of constraining the distribution, we can choose the initial position of the Sun, $Z_\odot$, in accordance with the measured value $Z_\odot = 26 \pm 3$ or any other value since the
population is assumed to be oscillating. The value affects the bound, because a larger initial value for $z$ for the stars will mean that they have a higher kinetic energy when they cross the midplane. This will cause them to spend a smaller fraction of their period interacting with the disk, predicting a weaker effect from a dark disk, and thus implying a looser bound. Conversely, a lower value for $Z_\odot$ will lead to a tighter constraint on the surface density $\Sigma_D$ of the dark matter disk. We will compute the bound assuming $Z_\odot = 7$ pc and $Z_\odot = 26$ pc separately. The currently favored value, $Z_\odot = 26 \pm 3$ pc, being the highest measured value, should yield the most persistent bound. Assuming too small a height would yield a bound that could disappear if the Sun height is found to be bigger.

### 3.4 Galactic Disk Components

As previously stated, one of the main differences between our analysis and that of HF2000 is that we update the mass model for the Galactic disk in light of more recent observations. One important feature of this update is the use of midplane densities for the interstellar gas as inputs to the Poisson-Jeans equation. These were not available at the time of the HF2000 study. The gas surface densities can also be used to place further constraints on the dark disk by demanding self-consistency of the model. This is explained in Chapter 4 where we show how the combination of midplane and surface densities, which depend on the density profile in the vertical direction, can also set a bound on a dark matter disk. For now, we will discuss Holmberg & Flynn’s model and how we modify it.
Table 3.2: The Bahcall model used by Flynn et al. (2006). The $\Sigma_i$ were calculated by Flynn et al. from the solution to the Poisson-Jeans equation, except in the case of the interstellar gas, where they were held fixed by HF and the midplane densities were chosen to give the correct $\Sigma_i$. Note that revised values of $\rho_i(0)_{\text{new}}$ give revised values of $\Sigma_i$ and that these are dependent on $\Sigma_D$.

<table>
<thead>
<tr>
<th>$i$</th>
<th>Description</th>
<th>$\rho_i(0)$ ($M_\odot \text{pc}^{-3}$)</th>
<th>$\sigma_i$ (km s$^{-2}$)</th>
<th>$\Sigma_i$ ($M_\odot \text{pc}^{-2}$)</th>
<th>$\rho_i(0)<em>{\text{new}}$ ($M</em>\odot \text{pc}^{-3}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>H$_2$</td>
<td>0.021</td>
<td>4.0</td>
<td>3.0</td>
<td>0.014*</td>
</tr>
<tr>
<td>2</td>
<td>H$_i$(1)</td>
<td>0.016</td>
<td>7.0</td>
<td>4.1</td>
<td>0.015*</td>
</tr>
<tr>
<td>3</td>
<td>H$_i$(2)</td>
<td>0.012</td>
<td>9.0</td>
<td>4.1</td>
<td>0.005*</td>
</tr>
<tr>
<td>4</td>
<td>warm gas</td>
<td>0.0009</td>
<td>40.0</td>
<td>2.0</td>
<td>0.0011*</td>
</tr>
<tr>
<td>5</td>
<td>giants</td>
<td>0.0006</td>
<td>20.0</td>
<td>0.4</td>
<td>0.0006*</td>
</tr>
<tr>
<td>6</td>
<td>$M_V &lt; 2.5$</td>
<td>0.0031</td>
<td>7.5</td>
<td>0.9</td>
<td>0.0018</td>
</tr>
<tr>
<td>7</td>
<td>2.5 &lt; $M_V$ &lt; 3.0</td>
<td>0.0015</td>
<td>10.5</td>
<td>0.6</td>
<td>0*</td>
</tr>
<tr>
<td>8</td>
<td>3.0 &lt; $M_V$ &lt; 4.0</td>
<td>0.0020</td>
<td>14.0</td>
<td>1.1</td>
<td>0.0018*</td>
</tr>
<tr>
<td>9</td>
<td>4.0 &lt; $M_V$ &lt; 5.0</td>
<td>0.0022</td>
<td>18.0</td>
<td>1.7</td>
<td>0.0029</td>
</tr>
<tr>
<td>10</td>
<td>5.0 &lt; $M_V$ &lt; 8.0</td>
<td>0.007</td>
<td>18.5</td>
<td>5.7</td>
<td>0.0072</td>
</tr>
<tr>
<td>11</td>
<td>$M_V &gt; 8.0$</td>
<td>0.0135</td>
<td>18.5</td>
<td>10.9</td>
<td>0.0216</td>
</tr>
<tr>
<td>12</td>
<td>white dwarfs</td>
<td>0.006</td>
<td>20.0</td>
<td>5.4</td>
<td>0.0056</td>
</tr>
<tr>
<td>13</td>
<td>brown dwarfs</td>
<td>0.002</td>
<td>20.0</td>
<td>1.8</td>
<td>0.0015</td>
</tr>
<tr>
<td>14</td>
<td>thick disk</td>
<td>0.0035</td>
<td>37.0</td>
<td>7.0</td>
<td>0.0035</td>
</tr>
<tr>
<td>15</td>
<td>stellar halo</td>
<td>0.0001</td>
<td>100.0</td>
<td>0.6</td>
<td>0.0001</td>
</tr>
</tbody>
</table>

* marks components whose dispersions $\sigma_i$ have been revised.

The Galactic disk model we update is Flynn et al.’s slightly updated model from their 2006 study, which includes fainter stars than HF2000 and allows for the presence of a thick stellar disk. It is shown in Table 3.2. In the far right column, we show the updated values, which we discuss in Sections 4.4 and 3.4.2.
3.4.1 **Interstellar Gas**

Until now, one of the largest uncertainties in the visible mass model has been that of the interstellar medium. This is also the uncertainty most relevant to a dark disk, since it is the component most similar in scale height. The density and dispersion parameters used by HF2000 (and Flynn et al. (2006), also quoted in Binney & Tremaine 2008) for the gas layer were taken from Scoville & Sanders (1987) for the $\text{H}_2$ layer and from Kulkarni & Heiles (1987) in the case of HI. Following Kulkarni & Heiles, HF separate HI into the Cold Neutral Medium (CNM) and Warm Neutral Medium (WNM), due to their different scale heights. Based on the results of these studies, HF obtained a total gas surface density of about $13 \, M_\odot \, \text{pc}^{-2}$, with an estimated uncertainty of about 50%. In Chapter 4, we show that these parameters have changed in recent years. This compilation, which is discussed in detail in Chapter 4, is cumbersome and tangential to our discussion here. We therefore simply list the old and new values for the various disk components in Table 3.2 and leave the discussion of measurements to Chapter 4, in which we also discuss how these measurements can be used independently to provide additional constraints on a dark disk model. We present the results with both the old and the new values. We find that the corrections to the various parameters tend to compensate each other, so that, somewhat surprisingly, the new values do not change the results significantly.

The table shows the gas parameters including a factor of 1.4 (Ferrière 2001) to account for the presence of helium and other elements. We also include the updated dispersions for the interstellar gas components, based on a number of more recent studies, to $3.7 \, \text{km} \, \text{s}^{-1}$, $6.7 \, \text{km} \, \text{s}^{-1}$, $10.7 \, \text{km} \, \text{s}^{-1}$, $22 \, \text{km} \, \text{s}^{-1}$, and $43 \, \text{km} \, \text{s}^{-1}$.
s$^{-1}$, and 13.1 km $s^{-1}$, and 22 km $s^{-1}$ for H$_2$, HI(1), HI(2), and HII respectively as explained in Chapter 4. We also include a contribution for thermal pressure, magnetic pressure, and cosmic ray pressure for HI(2) and HII as explained in Chapter 4.

### 3.4.2 Other Components

For the mass of the stellar components, we use the values reported by McKee et al. (2015). These are shown in Table 3.2 (right column). Some of these are very different from those of Flynn et al. (2006), such as the M-dwarf density (row 11). Differences in the other stellar components and in the gas components as well tend to compensate these changes so that the total mass of the galactic disk is roughly equal to the HF value. The value ‘0’ in row 7 reflects the fact that McKee et al. grouped all the stars with $M_V < 3$ into one category with a scale height given roughly by that of row 6. In addition to the midplane densities, we adjusted the dispersions of both the giants and the $3 < M_V < 4$ stars to 15.5 km $s^{-1}$ and 12.0 km $s^{-1}$ respectively to agree with the scale heights of McKee et al. (2015). For the dark halo, we follow Bovy & Tremaine (2012, Equation 28), who approximate the dark halo as a disk-like component with vertical dispersion $\sigma \simeq 130$ km $s^{-1}$. Its midplane density was chosen so that $\rho_{\text{halo}}(z = 2.5\, \text{kpc}) = 0.008\, M_\odot\, \text{pc}^{-3}$ and depends on the specific values of $\Sigma_D, h_D$. We used this particular measurement because it relied on data high above the Galactic midplane and would be least biased by the existence or non-existence of a thin dark disk.
3.4.3 Poisson-Jeans Solver

We implement our Poisson-Jeans solver with the densities and dispersions as inputs, as did HF2000. Our solver is implemented by using Equation 4.4 in the following way. An initial distribution is assumed for each component $\rho_i^{(0)}(z)$. The potential $\Phi(z)$ due to these components is then calculated via

$$\Phi^{(0)}(z) = 4\pi G \sum_i \int_0^z dz' \int_0^{z'} dz'' \rho_i^{(0)}(z'')$$

which then gives the next iteration of $\rho_i(z)$:

$$\rho_i^{(N+1)} = \rho_i(0) \exp \left( -\Phi^{(N)}(z)/\sigma_i^2 \right)$$

and this process is repeated until the solution converges. Remarkably, in only 5 iterations, the solution for a single component disk converges to the Spitzer (1942) solution (Equation 3.24) to better than one part in $10^7$! This is affected only very slightly by the initial distributions $\rho_i^{(0)}$ assumed.
3.5 Our Analysis

3.5.1 Traditional Method - Statistics

In order to compare the stellar kinematics to a given dark disk model, we define a $\chi^2$-type statistic $X$ that measures the distance between the predicted and observed densities:

$$X[\Phi] \equiv \int_{z_{\text{min}}}^{z_{\text{max}}} dz \frac{|\rho_{\text{obs}}(z) - \rho_{f,\text{obs}}[\Phi(z)]|^2}{\Delta^2 \rho(z)}$$

where

$$\rho_{f,\text{obs}}[\Phi(z)] \equiv \rho_f(z) = \rho(0) \int_{-\infty}^{\infty} dw f_{z=0} \left( \sqrt{w^2 + 2\Phi(z)} \right)$$

as defined in 3.41, $\Delta \rho(z)$ represents the uncertainty in $\rho(z)$ at the position $z$, and where $z_{\text{min}}$, $z_{\text{max}}$ are given by the completeness limits, $Z_{\odot} \pm 170$ pc for A stars and $Z_{\odot} - 92$ pc, $Z_{\odot} + 40$ pc for F stars.

As explained in Appendix 8.1, we can then bootstrap from the data to obtain the probability distribution $P(X)$ and obtain total probabilities on the A and F-star data $p(X_A, X_F | \Phi)$ given a dark matter disk model contained in the gravitational potential $\Phi$.

3.5.2 A Cross-Check: Measuring $\Phi_{\text{obs}}(z)$ Directly

It is worth pointing out that besides the traditional HF analysis which compares measured and model densities, it is also possible to use the HF equation to measure the Galactic potential
Φ_{\text{obs}}(z) directly from the data. This has the advantage that it treats the errors in density and velocity on more equal footing. This can be done by interpreting Equation 3.41 as an equation for \( \rho_{A,F}(\Phi) \) (as in Kuijken & Gilmore 1989b, Equation 20):

\[
\frac{\rho_f(\Phi)}{\rho(0)} = \int_{-\infty}^{\infty} dw_0 f_{z=0} \left( \sqrt{w_0^2 + 2\Phi} \right)
\]

(3.52)

and then inverting this to obtain \( \Phi_f(\rho) \). The ‘observed’ gravitational potential is then given by combining this with the observed density \( \rho_{A,F}(z) \):

\[
\Phi_{\text{obs}}(z) \equiv \Phi_f(\rho_{\text{obs}}(z))
\]

(3.53)

Note that Equation 3.52 always gives \( \rho(\Phi)/\rho(0) \leq 1 \) for real \( w \). For values of the observed density \( \rho_{\text{obs}}(z) > \rho(0) \), we analytically continued \( f_{z=0}(w) \) to imaginary \( w \) (negative \( \Phi \)) by fitting \( f_{z=0}(w) \) to a sum of three analytic Gaussians. Once we obtain \( \Phi_{\text{obs}}(z) \), we can then perform a cross-check on our analysis by comparing the measured \( \Phi_{\text{obs}}(z) \) with our model \( \Phi(z) \) directly. The statistics can be accounted for similarly as in Appendix 8.1, by sampling \( \Phi_{\text{obs}}(z) \) and computing their distribution.

### 3.5.3 Non-Equilibrium Method

As was mentioned in Section 3.1, our tracer populations show evidence for deviations from equilibrium behavior. This evidence is a non-zero mean velocity and a displacement from the
assumed position of the Galactic plane. Figure 3.4 shows the velocity and density distributions of the A stars. They feature a net velocity of $1.3 \pm 0.3$ km s$^{-1}$ and a net displacement from the Galactic plane of $19 \pm 5$ pc. A bulk velocity and vertical displacement from the plane can clearly be seen.

**Figure 3.4:** (Left) The A-star velocity distribution possesses a peak value of $1.3 \pm 0.3$ km s$^{-1}$. (Right) The A-star density distribution has a non-zero central value of $19 \pm 5$ pc relative to the Galactic plane, assuming a value for the solar position of $Z_0 = 26$ pc.

Because of these features, the assumption $\partial f / \partial t = 0$ in Equation 3.4 will not be satisfied. Consequently, Equations 3.36 and 3.37 no longer hold and the HF relation (3.41) that in principle constrains the potential will no longer be useful for constraining the Galactic model. On the other hand, we might expect that, if the tracer distribution is oscillating about the plane in the Galactic potential, that the long-time average of $\partial f / \partial t$ will vanish. We therefore expect that the HF relation will hold for long-time averages. However, the only way to compute the long-time average of the star dynamics is to assume some form for the potential $\Phi(z)$. We show...
in Appendix 8.4 that in this case, the HF relation (Equation 3.41) is trivially satisfied for the potential \( \Phi(z) \):

\[
\overline{\rho(0) - 1} \frac{\Phi(z)}{\Phi} = \int \text{d}w \overline{f_{z=0}(\sqrt{w^2 + 2\Phi(z)})}
\]  \hspace{1cm} (3.54)

where \( \overline{\cdot} \) represents the time average under evolution in the potential \( \Phi(z) \). Since Equation 3.54, the correct statement of the HF relation, is trivially satisfied for any potential, it therefore cannot be used to constrain a dark matter model. Although we do indeed expect a dark disk to ‘pinch’ the tracer distributions as explained in Section 4.2.1, this pinch cannot be measured with the HF method if the sample is oscillating.

The solution we propose is to observe over time the shape of the distribution relative to the position of its center \( z_0(t) \), i.e. \( \rho(z - z_0(t), t) \), and to predict its time average, \( \overline{\rho(z - z_0)} \). Figure 3.5 shows this time average for a model with \( \Sigma_D = 0 \) as well as with \( \Sigma_D = 20 \) \( M_\odot \) \( \text{pc}^{-2} \).

Comparing this to the Hipparcos ‘snapshot’, which in our case is used as initial data, gives a method of constraining \( \Sigma_D \) by assuming that the instantaneous distribution should not deviate strongly from its time average. We therefore define an appropriate \( \chi^2 \) parameter:

\[
\chi^2 = \int \text{d}z \left| \frac{\rho(z - z_0(0), 0) - \overline{\rho(z - z_0)}}{\Delta\rho(z)} \right|^2 
\]  \hspace{1cm} (3.55)

Where \( \Delta\rho(z) \) represents the expected variance in \( \rho(z - z_0) \). For this, we use the variance \( \Delta\rho^2(z) \) computed in Section 3.5.1. This assumes that the distribution \( \rho(z - z_0) \) is static in time, and that
at any time, including the present $t = 0$, we expect $\rho_{\text{obs}}(z - z_0)$ to be equal to its average up to sampling error $\Delta_\rho(z)$. Although a more careful analysis would include the aforementioned sampling variance $\Delta^2_\rho$ as well as the variance due to the time dependence of $\rho(z - z_0)$, and will therefore be larger than $\Delta^2_\rho$, for the purposes of this analysis we neglected this contribution and considered only the sampling error $\Delta^2_\rho$. More careful determinations may be important in the future.

An alternate proposal for how the present distribution $\rho(z - z_0)$ fits the model is to measure the stability of the distribution over time. Were we to observe that the initial (possibly oscillating) distribution $\rho(z - z_0)$, decayed to a different (possibly oscillating) distribution, we would conclude that there was a dynamical mismatch between the distribution and the potential. On the other hand, were we to observe that the initial distribution retained its behavior over time, we would conclude that the distribution and potential were appropriately matched. Such an analysis would also have to account for spiral arm crossing and is beyond the scope of this manuscript. In any case, if evidence persists of nonstatic distributions, more careful analyses using nonstatic distributions will be required.
Figure 3.5: (Left) Comparison of observed tracer distribution to distribution predicted with no dark disk. (Right) Comparison of observed tracer distribution to distribution predicted with dark disk of surface density $\Sigma_D = 20\, M_\odot\, \text{pc}^{-2}$, $h_D = 10\, \text{pc}$.

3.6 Results and Discussion

Static Method

We assigned probabilities to models with various $\Sigma_D$ and $h_D$ in the manner described in Section 3.5.1. The scale height $h_D$ was defined so that the value of the density of the dark disk at $z = h_D$ is $\rho(z = 0) \sech^2(1/2)$.

The probabilities and probability density derived without applying reddening corrections are shown in Figure 3.6. For low scale heights $h_D$, the probability has a 95% upper bound at $\Sigma_D \simeq 2\, M_\odot\, \text{pc}^{-2}$ and a 95% lower bound of $1\, M_\odot\, \text{pc}^{-2}$. For larger scale heights, the bound is slightly weaker, with the 95% upper bound growing beyond $\Sigma_D \simeq 5\, M_\odot\, \text{pc}^{-2}$ at $h_D = 80\, \text{pc}$.

However, without reddening corrections, the entire parameter space appears to be disfavored at
**Figure 3.6:** Standard HF analysis, without reddening corrections. *Top:* Relative probability density in DDDM parameter space as described in Appendix 8.3. *Bottom:* 95% bounds on DDDM parameter space using conventional HF method for both A and F stars. The blue region is ruled out at 94% to 95% confidence. White regions are ruled out at >95%.

94%. Moreover, without reddening corrections, the value $\Sigma_D = 0$ appears to be excluded at greater than 95% confidence.

Figure 3.7, on the other hand, shows, on the left, the results using the standard HF method used in Figure 3.6 but with reddening corrections applied. This plot shows the results determined using the old values for the mass parameters from Flynn et al. (2006). We see that the
upper bound is roughly the same with as without reddening corrections. Now, however, the entire parameter space is only disfavored at 77%. On the right, we have the bounds using the new values for the gas parameters (Chapter 4) for comparison. The 95% upper bound in this case (for low scale height) is around $4 \, M_\odot \, pc^{-2}$. Figure 3.8, on the left, shows the values determined by using both the updated gas values and the updated values for the stellar components (McKee et al. 2015). Here, the bound is $3 \, M_\odot \, pc^{-2}$, which is remarkably similar to the bound using the 2006 values. This is because changes the changes in the various mass parameters tend to compensate each other on average, as would be expected statistically.

In terms of midplane densities, the dark matter densities are less than $0.02 \, M_\odot \, pc^{-3}$ for thick scale heights ($h_D \gtrsim 75 \, pc$) but are unbounded for low scale heights, varying as $h_D^{-1}$. Note that according to this analysis, the value $\Sigma_D = 0$ is still ruled out at more than 70% confidence, so one might still question the robustness of this method as applied to this data set and mass parameters.

In order to know whether the uncertainty in the interstellar gas mass parameters have an important effect on the bound, we compute the bounds using gas densities one standard deviation below their average values. The results are shown in Figure 3.8 on the right. Here, the 95% upper bound at low scale height ($h_D \sim 10 \, pc$) is $3.5 \, M_\odot \, pc^{-2}$, which is slightly higher than the bound using the mean gas values. The value $\Sigma_D = 0$ is still ruled out at more than 70% confidence.

The plot in Figure 3.9 shows the results of the cross-check mentioned in Section 3.5.2 where the HF equation was interpreted as an equation for $\Phi(\rho)$. The results of this method are consis-
Figure 3.7: *Left:* Results of traditional HF analysis using mass model parameters of Flynn et al. (2006). *Right:* Results of traditional HF analysis using updated gas parameters of Chapter 4.

tent with the results from the standard analysis, although the bounds are weaker. This is because $\Phi(\rho)$ blows up as $\rho \to 0^+$, causing large fluctuations in $\Phi(z)$ at high $z$ and thus reducing the sensitivity of this method relative to the standard method.

**Non-Equilibrium Constraint**

Figure 3.10 shows the results obtained using the non-equilibrium HF method described in Section 3.5.3 and Appendix 8.4, applied to the A star sample of Section 3.3.1. Here, we assume the distribution moves in the gravitational potential determined by the Galactic model and ask that the shape of the measured distribution not be far from that of the average distribution. The top plot shows the results computed assuming the low value of $Z_\odot = 7$ pc (*cf.* Section 3.3.4), i.e. assuming our distribution is centered at the Galactic plane. The bottom plot shows the results
Figure 3.8: *Left:* Results of traditional HF analysis using both updated gas parameters (Chapter 4) and updated stellar parameters (McKee et al. 2015). *Right:* 95% bounds on DDDM parameter space using updated gas parameters one standard deviation below mean values.

computed assuming the more accepted value of $Z_\odot = 26$ pc. Note that the upper subplots in both figures show relative probability density, which, for $Z_\odot = 26$ pc, very slightly seems to favor $\Sigma_D \approx 2 \, M_\odot \text{pc}^{-2}$. In any case, the absolute probabilities clearly show that any values between $\Sigma_D = 0$ and quite high density are allowed.

The bound here is significantly weaker than that of the static method. One reason is that the oscillations reduce the amount of time that the tracers spend in the dark disk, thus reducing their sensitivity to it. Applying the static method to such an oscillating population would not yield a bound on the parameter space since in this case, as explained in Section 3.5.3, the static HF relation is trivially satisfied. Another reason the bound here is weaker simply depends on the amount of usable data in this method. This is reduced because the stars in the data are oscillating vertically, but in order to take a time average of the stars’ distribution, we can include only values of $z$ that are covered by the data at all times. In other words, the oscillation of the
Figure 3.9: 68% and 95% bounds on DDDM parameter space using $\Phi(z)$ method.

For low scale heights ($h_D = 10$ pc), in the non-equilibrium method, we find that the 95% upper bound is between $\Sigma_D \simeq 10 M_\odot pc^{-2}$ and $\Sigma_D \simeq 14 M_\odot pc^{-2}$, and that the bound grows with $h_D$ as for the static case.
Galactic Disk Parameters

Another important result of our analysis is the value for the surface density of the Galactic disk, which we compare against existing measurements. Figure 3.11 shows the values of $\Sigma_{\text{ISM}}$, the surface density of the interstellar medium; $\Sigma_*$, the surface density of the stellar disk (not including brown dwarfs and stellar remnants); and $\Sigma_{1.1}$ the total surface density to 1.1 kpc, including all visible and dark components (including e.g. brown dwarfs, dark halo, and dark disk). These were computed for a dark disk model with scale height 10 pc, and were obtained by integrating the self-consistent Poisson-Jeans solutions of each isothermal component. For the value of $\Sigma_D = 10 \, M_\odot \, \text{pc}^{-2}$ at this scale height (black dashed line in Figure 3.11), we find $\Sigma_{\text{ISM}} \simeq 10 \pm 2 \, M_\odot \, \text{pc}^{-2}$, the uncertainty being attributed to uncertainty in gas midplane densities, consistent with the value $\Sigma_{\text{ISM}} = 11.0 \pm 0.8 \, M_\odot \, \text{pc}^{-2}$ of Chapter 4 and with the value $\Sigma_{\text{ISM}} = 12.8 \pm 1.5 \, M_\odot \, \text{pc}^{-2}$ of McKee et al. (2015). At the $\Sigma_D$ upper bound obtained from the non-equilibrium method (gray dashed line), we have $\Sigma_{\text{ISM}} = 9.7 \pm 0.7 \, M_\odot \, \text{pc}^{-2}$. Values of $\Sigma_D$ higher than about $\Sigma_D = 20 \, M_\odot \, \text{pc}^{-2}$ give lower values of $\Sigma_{\text{ISM}}$ that are at odds with the literature. In Chapter 4, we perform a more detailed analysis of the self-consistency of the interstellar gas parameters and derive an independent bound on the DDDM parameter space.

We also find $\Sigma_* = 30 \pm 2 \, M_\odot \, \text{pc}^{-2}$ for the stellar surface density (not including brown dwarfs and stellar remnants) for $\Sigma_D = 10 \, M_\odot \, \text{pc}^{-2}$, $h_D = 10 \, \text{pc}$. This agrees with the value $\Sigma_* = 30 \pm 1 \, M_\odot \, \text{pc}^{-2}$ measured by Bovy et al. (2012). (On the other hand, the latter measurement was made assuming exponential profiles for the stellar disk components. Sublead-
ing corrections to this value obtained by assuming more realistic disk profiles near the plane
give anywhere between 26 and 28 \( M_\odot pc^{-2} \) depending on the specific model.) Note that our
value was computed using the recent values of McKee et al. (2015), and is somewhat higher
than the values inferred using Holmberg & Flynn’s 2006 model. The latter’s parameters yield
\( \Sigma_\ast = 30 \pm 2 \ M_\odot pc^{-2} \) without the dark disk, and \( \Sigma_\ast = 26 \pm 2 \ M_\odot pc^{-2} \) when it is included.
The uncertainties on our value of \( \Sigma_\ast \) were estimated from those on the individual components
given in McKee et al. At the upper bound, we find \( \Sigma_\ast = 28.5 \pm 0.2 \ M_\odot pc^{-2} \). Note that
\( \Sigma_D = 0 \) gives \( \Sigma_\ast = 34 \pm 2 \ M_\odot pc^{-2} \), which is too large. The \( \Sigma_\ast \) values therefore seem to favor
\( \Sigma_D \gtrsim 4 \ M_\odot pc^{-2} \).

For the total surface density to 1.1 kpc (still with \( h_D = 10 \) pc), we find \( \Sigma_{1.1} \) grows between
73.5 \pm 6.0 \ M_\odot pc^{-2} \) for \( \Sigma_D = 0 \) and \( \Sigma_D = 76.5 \pm 6.0 \ M_\odot pc^{-2} \) for \( \Sigma_D = 10 \ M_\odot pc^{-2} \). These
are slightly higher than the values in Bovy & Rix (2013) who measured \( 68 \pm 4 \ M_\odot pc^{-2} \) but
agree within their combined uncertainty. The major source of uncertainty in our measurement
of this quantity is that of the dark halo density, which we set as described in Section 3.4.2. Here,
too, we find that \( \Sigma_D \gtrsim 20 \ M_\odot pc^{-2} \) is at odds with the literature. For thicker scale heights
(\( h_D \gtrsim 50 \) pc), we find that even for \( \Sigma_D \gtrsim 13 \ M_\odot pc^{-2} \), \( \Sigma_{1.1} \) is too large.

### 3.7 Conclusions

It is of interest to use existing and future kinematical data to ascertain the possible existence
of a dark disk component in the Milky Way disk. Previous analyses argued that a dark disk
is not necessary to match the data but as has been often demonstrated, that is a far cry from ruling it out. Inspired by the Holmberg & Flynn (2000) study, we have rederived the kinematic constraints on a dark disk by considering a dark disk in a self-consistent manner. Our analysis features updated kinematics and extinction corrections, an updated model for the interstellar gas, and careful statistics.

We find that for a dark disk of sech² scale height of 10 pc, the static method rules out surface densities greater than $3 \, M_\odot \, \text{pc}^{-2}$ at 95% confidence. In terms of midplane density, the favored value is $\rho_D = 0.0^{+0.1} \, M_\odot \, \text{pc}^{-3}$, giving a total matter density in the plane of $\rho = 0.1^{+0.1} \, M_\odot \, \text{pc}^{-3}$. These bounds increase with scale height. We note, however, that in the static method, even a model with $\Sigma_D = 0$ is disfavored at around 70%, pointing to the inadequacy of the method for the current data set. Using values of gas midplane densities a standard deviation lower than their average moves this bound closer to $4 \, M_\odot \, \text{pc}^{-2}$. We also find that the updated values of the mass model increase the bound relative to the old Flynn et al. (2006) values by about 50% from $2 \pm 5 \, M_\odot \, \text{pc}^{-2}$ to $3 \pm 1 \, M_\odot \, \text{pc}^{-2}$.

These results were derived by arbitrarily removing the non-equilibrium features from our sample, which consisted of a bulk vertical velocity and a net vertical displacement from the Galactic midplane. However, if these features are instead taken into account using a non-equilibrium version of the HF analysis, the bound increases to $\Sigma_D \leq 14 \, M_\odot \, \text{pc}^{-2}$, assuming $Z_\odot = 26 \, \text{pc}$. This is partly because, in this method, it becomes more difficult to constrain a dark disk for the same amount of data.

We also showed that the static HF method can be modified to directly measure the Galactic
potential.

For thin dark disk models, the total surface density of the Galactic disk to 1.1 kpc, \( \Sigma_{1.1} \), as well as the surface densities of visible matter, \( \Sigma_* \) and \( \Sigma_{\text{ISM}} \), were found to be consistent with literature values both at the target values of \( \Sigma_D \), \( h_D \) and at their 95\% upper bound. According to our Poisson-Jeans solver, literature \( \Sigma_* \) values appear to favor a dark disk model.

Since the non-equilibrium analysis allows surface densities of low as zero and up to 14\( M_\odot \)pc\(^{-2} \), it follows that a dark disk may account for the comet periodicity that was extrapolated from the crater measurements. The results using the Gaia data promise to be more constraining, as more data close to the Galactic midplane will be available over a much larger area. However, it will be important to verify whether the sample used is in equilibrium. The correct method to use may indeed have to take account of the motion of the star populations, in which case non-equilibrium methods will be more appropriate.
Figure 3.10: 68% and 95% bounds on DDDM parameter space using the non-equilibrium version of the HF method for A stars only.
Figure 3.11: Values of surface densities of the Galactic disk for a dark disk with scale height $10 \text{ pc}$. Black vertical dashed line corresponds to benchmark values $\Sigma_D = 10 \, M_\odot \text{pc}^{-2}$, $h_D = 10 \, \text{pc}$. The NEQ dashed line corresponds to the current limits using the non-equilibrium method.
4.1 Introduction

Fan, Katz, Randall, and Reece in 2013 proposed the existence of thin disks of dark matter in spiral galaxies including the Milky Way, in a model termed Double Disk Dark Matter (DDDM). In this model, a small fraction of the dark matter is interacting and dissipative, so that this sector of dark matter would cool and form a thin disk. More recently Randall & Reece (2014) showed that a dark matter disk of surface density \( \sim 10 \, M_\odot \, pc^{-2} \) and scale height \( \sim 10 \, pc \) could possibly explain the periodicity of comet impacts on earth. It is of interest to know what values of dark disk surface density and scale height are allowed by the current data, and whether these particular values are allowed.

Since the original studies by Oort (1932, 1960), the question of disk dark matter has been a subject of controversy. Over the years, several authors have suggested a dark disk to explain various phenomena. Kalberla et al. (2007) proposed a thick dark disk as a way to explain the flaring of the interstellar gas layer. It has also been argued that a thick dark disk is formed naturally
in a ΛCDM cosmology as a consequence of satellite mergers (Read et al. 2008). Besides these, there are also models arguing for a thin dark disk. Fan et al. (2013) put forward a model for dark matter where a small fraction of the total dark matter could be self-interacting and dissipative, necessarily forming a thin dark disk. In Chapter 3 we investigated the constraints on such a disk from stellar kinematics. In this chapter we investigate the constraint imposed by demanding consistency between measurements of midplane densities and surface densities interstellar gas.

We assume a Bahcall-type model for the vertical distributions of stars and gas (Bahcall 1984a,b,c) as in Chapter 3, with various visible mass components, as well as a dark disk. We investigate the visible components in detail given more recent measurements of both the surface and midplane densities. A dark disk affects the relationship between the two as argued in Chapter 3 and as we review below. Although current measurements are insufficiently reliable to place strong constraints on or identify a disk, we expect this method will be useful in the future when better measurements are achieved.

4.2 Poisson-Jeans Theory

As explained in detail in Chapter 3, for an axisymmetric self-gravitating system, the vertical Jeans equation near the $z = 0$ plane reads

$$\frac{\partial}{\partial z}(\rho_i \sigma_z^2) + \rho_i \frac{\partial \Phi}{\partial z} = 0.$$ 

(4.1)
For an isothermal population ($\sigma_i(z) = \text{constant}$), the solution reduces to

$$\rho_i(z) = \rho_i(0) e^{-\Phi(R,z)/\sigma_i^2}. \quad (4.2)$$

Combining this with the Poisson equation gives the Poisson-Jeans equation for the potential $\Phi$

$$\frac{\partial^2 \Phi}{\partial z^2} = 4\pi G \sum_i \rho_i(0) e^{-\Phi/\sigma_i^2}, \quad (4.3)$$

which can also be cast in integral form (assuming $z$-reflection symmetry)

$$\frac{\rho_i(z)}{\rho_i(0)} = \exp \left( -\frac{4\pi G}{\sigma_i^2} \sum_k \int_0^z dz' \int_0^{z'} dz'' \rho(z'') \right). \quad (4.4)$$

This is the form used in our Poisson-Jeans solver.

### 4.2.1 A toy model

In Chapter 3, we showed that the exact solution to the Poisson-Jeans equation for a thick component (in this case, the interstellar gas which is thick compared to the dark disk) with midplane density $\rho_0$ and vertical dispersion $\sigma$ interacting with an infinitely thin (delta-function profile) dark disk was

$$\rho(z) = \rho_0(1 + Q^2) \text{sech}^2 \left( \frac{\sqrt{1 + Q^2}}{2h} (|z| + z_0) \right) \quad (4.5)$$
where
\[ Q \equiv \Sigma_D / 4\rho_0 h, \]  
(4.6)

\[ h \equiv \frac{\sigma}{\sqrt{8\pi G \rho_0}}, \]  
(4.7)

and
\[ z_0 \equiv \frac{2h}{\sqrt{1 + Q^2}} \text{arctanh} \left( \frac{Q}{\sqrt{1 + Q^2}} \right). \]  
(4.8)

Thus, the effect of the dark disk is to ‘pinch’ the density distribution of the other components, as we can see in Figure 4.1. Thus, although the scale height of the gas disk is proportional to its velocity dispersion according to Equation 4.7, a dark disk will reduce the gas disk’s thickness relative to this value, and for a fixed midplane density \( \rho_i(0) \), it implies that their surface densities \( \Sigma_i \) are less than what it would be without the dark disk or any other mass component. In this approximation, the gas distribution will have a cusp at the origin, but in general, the dark disk will have a finite thickness and the solution will be smooth near \( z = 0 \).

Integrating (4.5) gives the surface density of the visible component as

\[ \Sigma_{\text{vis}}(\Sigma_D) = \sqrt{\Sigma_{\text{vis}(0)}^2 + \Sigma_D^2} - \Sigma_D \]  
(4.9)

where \( \Sigma_{\text{vis}}(0) \equiv 4\rho_0 h \) is what the surface density would have been without the dark disk. This expression is monotonically decreasing with \( \Sigma_D \).
Figure 4.1: A plot of the exact solutions without and with a dark disk of $Q = 1$. The density is 'pinched' by the disk, in accordance with Equation 4.5.

Another way of explaining this is that the dark disk ‘pinches’ the visible matter disk, reducing its thickness $H_{\text{vis}}$. Since the surface density of the visible disk scales roughly as $\Sigma_{\text{vis}} \sim \rho_{\text{vis}} H_{\text{vis}}$, the effect of the dark disk is to reduce the total surface density for a given midplane density $\rho_{\text{vis}}$.

4.3 Analysis

Here we explain how we compare the surface densities of the various gas components estimated in the next section to those predicted by their midplane densities and velocity dispersions in order to place self-consistency constraints on the mass model. Section 4.2.1 explains how the presence of the dark disk decreases the surface density of each component if the midplane density is held fixed (as it is in a Poisson-Jeans solver). Thus, given fixed midplane densities, we can assign a probability to a model with any dark disk surface density $\Sigma_D$ and scale height.
Based on how well the predicted surface densities $\Sigma_i$ determined from the Poisson-Jeans solver match the observed values.

Starting with the midplane densities and dispersions of Section 4.4, we solved the Poisson-Jeans equation for $\Sigma_D$ values between 0 and $24 M_\odot pc^{-2}$. Each time the Poisson-Jeans equation was solved, the density distributions were integrated to give the total surface densities of H$_2$ and HI. Each model was then assigned a probability, according to the chi-squared distribution with 2 degrees of freedom, based on the deviation of this surface density from the measured values.

We did this using different central values and uncertainties for the midplane densities. Thus, for example, using $n_{H_2} = 0.19 \text{ cm}^{-3}$, we would take $\rho_{H_2+He}(0) = 1.42 \times m_{H_2} \times 0.19 \text{ cm}^{-3} = 0.013 M_\odot pc^{-3}$. If for a model with this value of $\rho_{H_2}(0)$ and with a certain dark disk surface density value of $\Sigma_D$ we find an H$_2$ surface density $\Sigma_{H_2} = 1.0 M_\odot pc^{-2}$, then according to Section 4.4.1, we should assign this model a chi-squared value $\chi^2_{H_2} = (1.0 - 1.3)^2/\Delta^2_{\Sigma_{H_2}}$. We would then assign a probability to this model according to the Gaussian cumulative distribution,

$$p_{H_2} = \int d\chi \exp(-\chi^2/2)/\sqrt{2\pi},$$

where the limits of integration are from $-\infty$ to $-\chi_{H_2}$ and $\chi_{H_2}$ to $\infty$. We would similarly compute probabilities $p_{HI}$, $p_{HI}$, and a combined probability $p = p_{H2} \times p_{HI} \times p_{HI}$. We note here that this is not the absolute probability of the model given the data; rather, it is the probability of the data given the model. We define a model for which the data is less probable than 5% to be excluded.

An important question is what to use for $\Delta^2_{\Sigma}$. There are two uncertainties here. Namely, 1) the uncertainty in the surface density measurements, $\Delta_{\hat{\Sigma}_i}$, and 2) the uncertainty in our output values of $\Sigma(\hat{\rho}_i, \Sigma_D)$, resulting from the uncertainty in the input midplane density measurements.
\[ \rho_i. \] Formally, this is \( |\partial \Sigma_i / \partial \rho_i| \Delta \hat{\rho}_i \). Assuming Gaussian distributions for the measurements \( \hat{\Sigma}_i \) and \( \hat{\rho}_i \), and a uniform prior for \( \Sigma_D \), one can show that

\[
p(\hat{\Sigma}_i, \hat{\rho}_i | \Sigma_D) \sim \int d\rho_i \exp \left( -\frac{(\hat{\Sigma}_i - \Sigma(\rho_i, \Sigma_D))^2}{2\Delta_{\Sigma_i}^2} \right) \exp \left( -\frac{(\rho_i - \hat{\rho}_i)^2}{2\Delta_{\rho_i}^2} \right) \quad (4.10)
\]

where \( \rho_i \) are the true midplane densities, and that, expanding \( \Sigma(\rho_i, \Sigma_D) \) to first order in \( \rho_i \), this integrates to give an approximately Gaussian distribution for \( \hat{\Sigma}_i - \Sigma(\hat{\rho}_i, \Sigma_D) \), with width

\[
\Delta_{\Sigma_i} \simeq \sqrt{\Delta_{\Sigma_i}^2 + \left( \frac{\partial \Sigma_i}{\partial \rho_i} \right)^2 \Delta_{\hat{\rho}_i}^2}.
\quad (4.11)
\]

We computed \( |\partial \Sigma_i / \partial \rho_i| \) by sampling values of \( \rho_i \) and computing the output values \( \Sigma_i \). The effect of the uncertainties in the vertical dispersions of the different components was also included in \( \Delta_{\Sigma_i} \) in the same way as those in \( \hat{\rho}_i \). The formula for \( \Delta_{\Sigma_i} \) that we adopt (Equation 4.11) therefore contains an extra term under the square root to include this uncertainty:

\[
\Delta_{\Sigma_i} \simeq \sqrt{\Delta_{\Sigma_i}^2 + \left( \frac{\partial \Sigma_i}{\partial \rho_i} \right)^2 \Delta_{\hat{\rho}_i}^2 + \left( \frac{\partial \Sigma_i}{\partial \sigma_{i,\text{eff}}} \right)^2 \Delta_{\sigma_{i,\text{eff}}}^2}.
\quad (4.12)
\]

### 4.4 Gas parameters

The purpose of this section is to determine accurate values for \( \rho_i(0), \sigma_i, \) and \( \Sigma_i \) (midplane density, velocity dispersion, and surface density) for the different components of interstellar gas
based on existing measurements. Using Equation 4.2, these can then be compared for a given
dark disk model in order to check for self-consistency.

We now discuss in detail the various measurements of the gas parameters and the uncertain-
ties in each. Our starting point is the Bahcall model used by Flynn et al. (2006, Table 2). These
values are updated from the ones used in Holmberg & Flynn (2000). Values for the stellar com-
ponents were updated using the values of McKee et al. (2015), and are shown in rows 5-15 in
Table 2 of Chapter 3.

In these models, the gas and stars are both separated into approximately isothermal compo-
nents as in Bahcall (1984b), so that each component \( i \) is characterized by a midplane density \( \rho_{i0} \)
and a vertical dispersion \( \sigma_{i} \). Using only these values for all of the components, we can solve the
Poisson-Jeans equation (4.4) for the system. A major difference between our model and that of
Flynn et al. (2006) is that their gas midplane densities were fixed by the values needed to give
the correct surface densities in accordance with the Poisson-Jeans equation. We, on the other
hand, use measured values of the midplane densities as we explain in this section.

We explain the various literature values that were included in the determination of the gas
parameters. We also compare these to the recent values of McKee et al. (2015). In Section 5.3,
the analysis is conducted separately for the values we determine by combining the results in the
literature and the values obtained solely from the recent paper by McKee et al. (2015).
4.4.1 Molecular hydrogen

We now explain the various measurements of the molecular hydrogen volume density and surface densities and how they are corrected. As molecular hydrogen cannot be observed directly, it must be inferred from the amount of CO present, derived from the intensity of the $J = 1 - 0$ transition photons. These are related by the so-called $X$-factor, defined by

$$N_{\text{H}_2} \equiv X W_{\text{CO}}$$

(4.13)

where $N_{\text{H}_2} = N_{\text{l.o.s.}}$ is the line-of-sight column density of $\text{H}_2$ molecules and $W_{\text{CO}}$ is the total, velocity-integrated CO intensity along the line of sight (Draine 2011). Column densities perpendicular to the galactic plane can then be obtained by simple trigonometry:

$$N_\perp = N_{\text{l.o.s.}} \sin b$$

(4.14)

and volume densities can be obtained by dividing the intensity density in velocity space $dW_{\text{CO}}/dv$ by the rotation curve gradient $dv/dR$, or by estimating the distance along the line of sight using other means. The volume and surface densities can also both be found by fitting an assumed distribution to measurements of the gas’ vertical scale height $\Delta z$. Surface densities can then be given, for example, by

$$\Sigma_{\text{H}_2} = m_{\text{H}_2} N_{\perp,\text{H}_2} = m_{\text{H}_2} X W_{\text{CO}} \sin b.$$ 

(4.15)
On the other hand, a certain reference may not be measuring surface density directly. Instead, they may be measuring the emissivity,

\[ J(r) \equiv \frac{dW_{CO}}{dr} \]  \hspace{1cm} (4.16)

from which, according to Equation 8.1, we can obtain the volume density as

\[ n(r) = X J(r). \]  \hspace{1cm} (4.17)

If the authors also measured the vertical (z–direction) distribution of the molecular hydrogen, then the surface mass density can be obtained according to

\[ \Sigma_{H_2} = m_{H_2} \int n_{H_2}(z) \, dz. \]  \hspace{1cm} (4.18)

For example, the full width at half maximum (FWHM) of the molecular hydrogen distribution gives the surface density as

\[ \Sigma_{H_2} = m_{H_2} C_{\text{shape}} n_{H_2}(0) \times \text{FWHM} \]  \hspace{1cm} (4.19)
where $C_{\text{shape}}$ is given by 1.06, 1.13, or 1.44 for a Gaussian, sech$^2$, or exponential profile respectively. For our calculations, we used

$$C_{\text{shape}} = 1.10$$  \hspace{1cm} (4.20)

as a reasonable estimate for the shape of the distribution.

In the literature, mass values are often quoted including the associated helium, metals, and other gaseous components such as CO, etc. The amount of helium accompanying the hydrogen is typically assumed in the range 36-40% of the hydrogen alone by mass (Kulkarni & Heiles 1987; Bronfman et al. 1988). Including other gas components increases this number to about 42% (Ferrière 2001). Thus, the total mass of any component of the ISM should be about 1.42 times the mass of its hydrogen. These will be distinguished by using, e.g. $\Sigma_{\text{H}_2}$, $\Sigma_{\text{H}_2+\text{He}}$ to refer to the bare values and and the values including their associated helium respectively. Thus,

$$\Sigma_{\text{H}_2+\text{He}} = 1.42 \Sigma_{\text{H}_2}.$$  \hspace{1cm} (4.21)

Note that Binney & Merrifield (1998, p.662) did not include helium in the total ISM mass. Also Read (2014) did not distinguish between HI results including and not including helium.

We now explain how we obtain midplane volume densities $n_{\text{H}_2}(z = 0)$ and surface densities $\Sigma_{\text{H}_2+\text{He}}$ from the various references in the literature. Bronfman et al. (1988) measured the molecular hydrogen over different radii within the solar circle. Their data are shown as one of
the data sets in Figure 4.2. Averaging the values from the Northern and Souther Galactic plane in Table 4 of the latter, we find, for the measurements closest to the Sun, $\Sigma_{\text{H}_2} = 2.2 \, M_\odot \text{pc}^{-2}$ and $n_{\text{H}_2} = 0.2 \, \text{cm}^{-3}$. Since surface densities depend only on the total integrated intensity along the line of sight, they are independent of the value of $R_\odot$, the Sun’s radial position from the center of the Galaxy. On the other hand, it follows from this that old values for volume densities (which scale as $R_\odot^{-1}$) must be rescaled by $R_\odot^{-1}$ (Scoville & Sanders 1987, p.31). Since Bronfman et al. used the old value $R_\odot = 10 \, \text{kpc}$, this value needs to be rescaled by $(0.833)^{-1}$ to take into account the new value of $R_\odot = 8.33 \pm 0.35 \, \text{kpc}$ (Gillessen et al. 2009). They also used an $X$-factor of $X = 2.8 \times 10^{20} \, \text{cm}^{-2} \, (\text{K}^{-1} \text{km s}^{-1})^{-1}$. We correct this using a more recent value of $X = 1.8 \pm 0.3 \times 10^{20} \, \text{cm}^{-2} \, (\text{K}^{-1} \text{km s}^{-1})^{-1}$, obtained by Dame et al. (2001). The most recent value of $X$, obtained by Okumura & Kamae (2009), is $X = 1.76 \pm 0.04 \times 10^{20} \, \text{cm}^{-2} \, (\text{K}^{-1} \text{km s}^{-1})^{-1}$, although the value of Dame et al. that we use is still cited by Draine (2011) as the most reliable. These corrections give $n_{\text{H}_2} = 0.15 \, M_\odot \text{pc}^{-3}$ and $\Sigma_{\text{H}_2} = 1.4 \, M_\odot \text{pc}^{-2}$. Including helium gives $\Sigma_{\text{H}_2 + \text{He}} = 2.0 \, M_\odot \text{pc}^{-2}$.

On the other hand, Clemens et al. (1988), found the local CO emissivity $J = \frac{dW_{\text{CO}}}{dr}$ in the first galactic quadrant for radii through $R_\odot$. For $R < R_\odot$ and $R > R_\odot$ respectively, they found these to be $J = 3.1$ and $2.3 \, \text{K km s}^{-1} \text{ kpc}^{-1}$, which, using $X = 1.8 \times 10^{20} \, \text{cm}^{-2} \, (\text{K}^{-1} \text{km s}^{-1})^{-1}$, and rescaling for $R_\odot$ by $(0.833)^{-1}$, gives interpolated density $n_{\text{H}_2}(R_\odot) = 0.19 \, \text{cm}^{-3}$. Using their FWHM measurements for $\text{H}_2$, we can convert their measurements to surface density values according to Equation 4.19. As before, the surface density values are independent of $R_\odot$. We have, interpolating to $R_\odot$, $\Sigma_{\text{H}_2 + \text{He}} = 1.1 \, M_\odot \text{pc}^{-2}$. The
rescaled data are shown in Figure 4.2.

Another measurement is provided by Burton & Gordon (1978), who had already measured Galactic CO emissivity $J = dW_{\text{CO}}/dr$ between $R \sim 2 - 16$ kpc, assuming $R_\odot = 10$ kpc, from which we obtain $n_{\text{H}_2}(R)$ after correcting for $R_\odot$, shown in Figure 4.2. Interpolating linearly, this gives $n(R_\odot) = 0.31 \text{ cm}^{-3}$. Sanders et al. (1984) also measured CO in the first and second Galactic quadrants within and outside the solar circle. They used the values $R_\odot = 10$ kpc and $X = 3.6 \times 10^{20} \text{ cm}^{-2} (\text{K}^{-1} \text{ km s}^{-1})^{-1}$. Their results for both volume and surface density, corrected to $R_\odot = 8.33$ kpc and $X = 1.8 \times 10^{20} \text{ cm}^{-2} (\text{K}^{-1} \text{ km s}^{-1})^{-1}$, are also shown in Figure 4.2. In particular, after rescaling and interpolating their volume densities, we have $n(0.95R_\odot) = 0.39 \text{ cm}^{-3}$. For surface density, we obtain $\Sigma_{\text{H}_2+\text{He}} = 2.7 M_\odot \text{ pc}^{-2}$. This is the highest value in the literature. Grabelsky et al. (1987) also measured CO in the outer Galaxy, which, with a 1.8/2.8 correction factor for $X$, as well as correcting $R_\odot$ from 10 to 8.33 kpc, their results near the Sun read $n(1.05R_\odot) = 0.14 \text{ cm}^{-3}$ and $\Sigma_{\text{H}_2}(1.05R_\odot) = 1.4 M_\odot \text{ pc}^{-2}$.

Digel (1991) also measured H$_2$ in the outer Galaxy. Using his results, we find $n(1.06R_\odot) = 0.13 \text{ cm}^{-3}$ and $\Sigma_{\text{H}_2}(1.06R_\odot) = 2.1 M_\odot \text{ pc}^{-2}$.

Dame et al. (1987), by directly observing clouds within 1 kpc of the Sun only, found local volume density $n_{\text{H}_2} = 0.10 \text{ cm}^{-3}$ and surface density $\Sigma_{\text{H}_2+\text{He}} = 1.3 M_\odot \text{ pc}^{-2}$, which, correcting for $X = 2.7$ to $1.8 \times 10^{20} \text{ cm}^{-2} (\text{K}^{-1} \text{ km s}^{-1})^{-1}$, gives $0.08 \text{ cm}^{-3}$ and $0.87 M_\odot \text{ pc}^{-2}$. This volume density is lower than many other measurements, and may represent a local fluctuation in the Solar region on a larger scale than the Local Bubble. On the other hand, their surface density value is not the lowest. Luna et al. (2006), using $X = 1.56 \times 10^{20} \text{ cm}^{-2} (\text{K}^{-1} \text{ km s}^{-1})^{-1}$,
found $\Sigma_{\text{H}_2 \text{He}}(0.975R_\odot) = 0.24 M_\odot \text{pc}^{-2}$. Correcting for $X$ gives 0.29 $M_\odot \text{pc}^{-2}$, which is the lowest value in the literature. However, they admit that their values beyond 0.875 $R_\odot$ are uncertain. Another determination from 2006 (Nakanishi & Sofue 2006) gives, after interpolation, $n_{\text{H}_2}(R_\odot) = 0.17 \text{ cm}^{-3}$ and $\Sigma_{\text{H}_2}(R_\odot) = 1.4 M_\odot \text{pc}^{-2}$, or $\Sigma_{\text{H}_2}(R_\odot) = 2.0 M_\odot \text{pc}^{-2}$.

Figure 4.2 shows the various measurements described here, as well as the overall average and standard error. Although not all measurements are equally certain, in computing average values for $n_{\text{H}_2}$ and $\Sigma_{\text{H}_2}$ we treated all measurements with equal weight. We estimated the resulting uncertainty as the standard deviation divided by the square root of the number of measurements available at each $R$. We found the mean values and standard errors of volume and surface densities near the Sun to be

$$n_{\text{H}_2}(R_\odot) = 0.19 \pm 0.03 \text{ cm}^{-3} \quad (4.22)$$

$$\Sigma_{\text{H}_2 \text{He}}(R_\odot) = 1.55 \pm 0.32 M_\odot \text{pc}^{-2} \quad (4.23)$$

This analysis has not yet taken into account the more recent observations of a significant component of molecular gas that is not associated with CO (Heyer & Dame 2015; Hessman 2015). Planck Collaboration et al. (2011) estimates this “dark gas” density to be 118% that of the CO-associated $\text{H}_2$. Pineda et al. (2013), on the other hand, found roughly 40% at Solar radius. We therefore include the dark molecular gas with a mean value of 79% and with an uncertainty of
39%. This gives total molecular gas estimates of

\[
    n_{H_2+DG}(R_\odot) = 0.34 \pm 0.09 \, \text{cm}^{-3} \quad (4.24)
\]

\[
    \Sigma_{H_2+He+DG}(R_\odot) = 2.8 \pm 0.8 \, M_\odot \text{pc}^{-2} \quad (4.25)
\]

which are the values we assume for our analysis. It should be noted, however, that in propagating
the errors for dark gas, \(n_{H_2}\) and \(\Sigma_{H_2+He}\) always vary together. We take this into account in
the statistical analysis by considering only the error on the ratio \(\Sigma/\rho\). The same would apply to
the error in \(X_{CO}\) although this error is much smaller.

Besides the molecular hydrogen’s volume density \(n_{H_2}\) and surface density \(\Sigma_{H_2}\), another
important quantity is its cloud-cloud velocity dispersion \(\sigma_{H_2}\), since this is one of the inputs
in the Poisson-Jeans equation. The velocity dispersions of the molecular clouds containing
\(H_2\) can be inferred from that of their CO, which was found by Liszt & Burton (1983) to be
\(\sigma_{H_2} = 4.2 \pm 0.5 \, \text{km s}^{-1}\). Belfort & Crovisier (1984) found \(\sigma_{CO} = \sigma_{H_2} = 3.6 \pm 0.2 \, \text{km s}^{-1}\).
Scoville & Sanders (1987) found \(\sigma_{H_2} = 3.8 \pm 2 \, \text{km s}^{-1}\). The weighted average of these is
approximately given by

\[
    \sigma_{H_2} = 3.7 \pm 0.2 \, \text{km s}^{-1}. \quad (4.26)
\]
4.4.2 The Atomic Hydrogen

We now discuss the various measurements of atomic hydrogen HI volume density $n_{\text{HI}}(z)$ and surface density $\Sigma_{\text{HI}}$. These typically are made by observing emissions of hydrogen’s 21 cm hyperfine transition. Kulkarni & Heiles (1987) estimate an HI surface density of $8.2 \ M_\odot \text{pc}^{-2}$ near the Sun. They separate HI into the Cold Neutral Medium (CNM) and Warm Neutral Medium (WNM).

Dickey & Lockman (1990), summarizing several earlier studies, describe the Galactic HI as having approximately constant properties over the range $4 \text{kpc} < R < 8 \text{kpc}$. Their best estimate for the HI parameters over this range is a combination of subcomponents, one thin Gaussian component with central density $n(0) = 0.40 \text{cm}^{-3}$ and FWHM = 212 pc (and surface density $2.2 \ M_\odot \text{pc}^{-2}$), which we identify with the CNM, and a thicker component with central density $n(0) = 0.17 \text{cm}^{-3}$ and surface density $2.8 \ M_\odot \text{pc}^{-2}$, which we identify as the WNM. This gives a total of $\Sigma_{\text{HI}} = 5.0 \ M_\odot \text{pc}^{-2}$, or $\Sigma_{\text{HI}+\text{He}} = 7.1 \ M_\odot \text{pc}^{-2}$. Another measurement is provided by Burton & Gordon (1978), who measured volume densities for $R \sim 2 - 16 \text{kpc}$. We interpolate their data (and correct for $R_\odot = 10 \text{kpc} \rightarrow 8.33 \text{kpc}$) to obtain $n_{\text{HI}} = 0.49 \text{cm}^{-3}$. Although they did not determine surface densities, we can estimate them by assuming a single Gaussian component with FWHM given Dickey & Lockman (220 - 230 pc). A better estimate is perhaps obtained by assuming, rather than a Gaussian distribution, a distribution with the same shape as Dickey & Lockman. This amounts to assuming an effective Gaussian FWHM of $\sim 330 \text{pc}$. This gives a surface density near the Sun of $\Sigma_{\text{HI}+\text{He}} = 5.9 \ M_\odot \text{pc}^{-2}$. Liszt (1992),
however, argues that the midplane density of Dickey & Lockman was artificially enhanced to give the correct surface density. He measures midplane density $n_{\text{HI}} = 0.41 \text{cm}^{-3}$, which, assuming as for Burton & Gordon a Gaussian distribution with effective FWHM 330 pc, gives a surface density of only $\Sigma_{\text{HI+He}} = 5.1 \, M_{\odot}\text{pc}^{-2}$. Nakanishi & Sofue (2003) also measured the Galactic HI, from the Galactic center out to $\sim 25 \text{kpc}$. Their results are shown in Figure 4.3. Interpolating to $R_\odot$, we have $n_{\text{HI}}(R_\odot) = 0.28 \text{cm}^{-3}$ and $\Sigma_{\text{HI+He}} = 5.9 \, M_{\odot}\text{pc}^{-2}$, in agreement with the value of Burton & Gordon.

On the other hand, there are several authors who report much larger mass parameters for Galactic HI. They are Wouterloot et al. (1990) and Kalberla & Dedes (2008). Wouterloot et al. used 21 cm observations from outside the Solar circle. Their data are shown in Figure 4.3. Closest to the Sun, their data show $\Sigma_{\text{HI+He}}(1.06R_\odot) = 8.6 \, M_{\odot}\text{pc}^{-2}$ with a FWHM of 300 pc. This corresponds to a midplane density of roughly $n_{\text{HI}} = 0.73 \text{cm}^{-3}$. The Kalberla & Dedes data (also shown in Figure 4.3) show $\Sigma_{\text{HI+He}} \simeq 10 \, M_{\odot}\text{pc}^{-2}$. A more refined estimate gives $\Sigma_{\text{HI+He}} \simeq 9 \, M_{\odot}\text{pc}^{-2}$ (McKee et al. 2015). This is consistent with a midplane density of roughly $0.8 \text{cm}^{-3}$. This is much higher than the value of Kalberla & Kerp (1998), who obtained $n_{\text{CNM}} = 0.3 \text{cm}^{-3}$ and $n_{\text{WNM}} = 0.1 \text{cm}^{-3}$. However, there is reason to expect a relatively high HI midplane density. Based on extinction studies, Bohlin et al. (1978) find a total hydrogen nucleus density $2n_{\text{H}_2} + n_{\text{HI}} = 1.15 \text{cm}^{-3}$.

According to the average midplane density determined for molecular hydrogen in Section 4.4.1, $n_{\text{H}_2} = 0.19 \pm 0.03 \text{cm}^{-3}$, and including an additional $0.15 \pm 0.07 \text{cm}^{-3}$ for the dark molecular
hydrogen, we therefore expect an atomic hydrogen density $n_{\text{HI}} = 0.70 \pm 0.18 \, \text{cm}^{-3}$. Optical thickness corrections, which we explain below, increase this number to $0.84 \, \text{cm}^{-3}$. The results are shown in Figure 4.3. As in the case of molecular hydrogen, all measurements were treated with equal weight and the uncertainty was estimated as the standard error at each $R$.

Combining all these results, we have, in the absence of optical thickness corrections,

\begin{align}
      n_{\text{HI}}(R_\odot) & = 0.53 \pm 0.10 \, \text{cm}^{-3} \hspace{1cm} (4.27) \\
      \Sigma_{\text{HI+He}}(R_\odot) & = 7.2 \pm 0.7 \, M_\odot \text{pc}^{-2}. \hspace{1cm} (4.28)
\end{align}

In the Dickey & Lockman (1990) model, 70% of this HI midplane density is in CNM and the remaining 30% is WNM. In Kalberla & Kerp (1998), the numbers are 75% and 25%. We will take the average of these two results, 72.5% and 27.5%, which give $n_{\text{CNM}} = 0.38 \, \text{cm}^{-3}$ and $n_{\text{CNM}} = 0.15 \, \text{cm}^{-3}$

McKee et al. (2015) pointed out that these numbers must be corrected for the optical depth of the CNM. Assuming the CNM to be optically thin leads to an underestimation of the CNM column density by a factor $R_{\text{CNM}}$. McKee et al. (2015) estimate this factor to be $R_{\text{CNM}} = 1.46$, which they translate, for the total HI column density, to $R_{\text{HI}} = 1.20$. Correcting for this gives

\begin{align}
      n_{\text{CNM}} & \simeq 0.56 \, \text{cm}^{-3} \hspace{1cm} (4.29) \\
      n_{\text{WNM}} & \simeq 0.15 \, \text{cm}^{-3}. \hspace{1cm} (4.30)
\end{align}
with totals

\[ n_{\text{HI}}(R_\odot) = 0.71 \pm 0.13 \text{ cm}^{-3} \]  \hspace{1cm} (4.31)
\[ \Sigma_{\text{HI+He}}(R_\odot) = 8.6 \pm 0.8 \text{ M}_\odot \text{pc}^{-2}. \]  \hspace{1cm} (4.32)

which we use for this analysis.

On the other hand, McKee et al. (2015) argues that the model of Heiles et al. (1981) is more accurate, and recommends increasing the amount of HI in the ISM by a factor of 7.45/6.2. McKee et al. (2015)’s values are therefore \( n_{\text{CNM}} = 0.69 \text{ cm}^{-3} \), \( n_{\text{WNM}} = 0.21 \text{ cm}^{-3} \), \( n_{\text{HI}} = 0.90 \text{ cm}^{-3} \), and \( \Sigma_{\text{HI}} = 10.0 \pm 1.5 \text{ M}_\odot \text{pc}^{-2} \). Although these numbers are different from our average of conventional measurements (Equations 4.29-4.32), it agrees with the extinction result of Bohlin et al. (1978) mentioned above once the latter is corrected for the optical depth of the CNM. To account for any discrepancy, we perform our analysis separately using the values of Equations 4.29 to 4.32 and the results of McKee et al. (2015). We present both results in Section 5.3.

For the atomic hydrogen’s velocity dispersion, Heiles & Troland (2003), found \( \sigma_{\text{CNM}} = 7.1 \text{ km s}^{-1} \) and \( \sigma_{\text{WNM}} = 11.4 \text{ km s}^{-1} \), while Kalberla & Dedes (2008) found \( \sigma_{\text{CNM}} = 6.1 \text{ km s}^{-1} \) and \( \sigma_{\text{WNM}} = 14.8 \text{ km s}^{-1} \). Earlier, Belfort & Crovisier (1984) measured \( \sigma_{\text{HI}} = 6.9 \pm 0.4 \text{ km s}^{-1} \), and Dickey & Lockman (1990) found \( \sigma_{\text{HI}} = 7.0 \text{ km s}^{-1} \) but did not specify if the gas was CNM or WNM. Since these are comparable to more recent measurements of the CNM component of HI, we assume both of these to correspond to \( \sigma_{\text{CNM}} \). The averages of these
values are

\[ \sigma_{\text{CNM}} = 6.8 \pm 0.5 \, \text{km s}^{-1} \]  \hspace{1cm} (4.33)

\[ \sigma_{\text{WNM}} = 13.1 \pm 2.4 \, \text{km s}^{-1} \]  \hspace{1cm} (4.34)

### 4.4.3 Ionized Hydrogen

Besides the \( H_2 \) and the two types of HI (CNM and WNM), there is a fourth, warm, ionized component of interstellar hydrogen, denoted HII. Holmberg & Flynn (2000) and Flynn et al. (2006) included this component. Binney & Merrifield (1998) did not include the ionized component in the value for \( \Sigma_{\text{ISM}} \), possibly because of its very large scale height. Its density is typically obtained by measuring the dispersion of pulsar signals that have passed through the HII clouds.

The time delay for a pulse of a given frequency is proportional to the dispersion measure

\[ DM = \int n_e ds \]  \hspace{1cm} (4.35)

where the integral is performed along the line of sight to the pulsar, and where \( n_e \) is the electron number density, equal to the number density of ionized gas. The dispersion measure perpendicular to the plane of the Galaxy, \( DM_\perp = DM/\sin b \), therefore corresponds to the half surface density \( 1/2 \Sigma_{\text{HII}} \). Fitting a spatial distribution (e.g. exponential profile), provides midplane den-
density information. For its midplane density, Kulkarni & Heiles (1987) found \( n_{\text{HII}} = 0.030 \, \text{cm}^{-3} \); Cordes et al. (1991) found \( n_{\text{HII}} = 0.024 \, \text{cm}^{-3} \); Reynolds (1991) found \( n_{\text{HII}} = 0.040 \, \text{cm}^{-3} \). The average of these values is

\[
 n_{\text{HII}} = 0.031 \pm 0.008 \, \text{cm}^{-3}. \tag{4.36}
\]

This agrees with the traditional model of Taylor & Cordes (1993), refined by Cordes & Lazio (2003), who found a midplane density of

\[
 n_{\text{HII}} = 0.034 \, \text{cm}^{-3}. \tag{4.37}
\]

For the HII surface density, Reynolds (1992) reports \( \Sigma_{\text{HII+He}} = 1.57 \, M_\odot \text{pc}^{-2} \). This is slightly higher that what was found by Taylor & Cordes (1993), who found a one-sided column density \( 1/2 \, N_{\perp,\text{HII}} = 16.5 \, \text{cm}^{-3} \text{pc} \), or \( \Sigma_{\text{HII+He}} = 1.1 \, M_\odot \text{pc}^{-2} \), but it is slightly lower than the more recent value of Cordes & Lazio (2003), who found \( 1/2 \, N_{\perp,\text{HII}} = 33 \, \text{cm}^{-3} \text{pc} \), or \( \Sigma_{\text{HII+He}} = 2.3 \, M_\odot \text{pc}^{-2} \). Assuming an exponential profile, with the scale height of 0.9 kpc of Taylor & Cordes (1993), the Reynolds (1992) result agrees with the midplane densities of Equations 4.36 and 4.37. However, Gaensler et al. (2008) argued for a scale height of 1.8 kpc that a distribution with midplane density of

\[
 n_{\text{HII}} = 0.014 \, \text{cm}^{-3}. \tag{4.38}
\]
Similarly, Schnitzeler (2012) also argues for large scale heights of $\sim 1.4$ kpc. For DM values between 20 and 30 cm$^{-3}$ pc, this gives a midplane density of $\sim 0.015$ cm$^{-3}$ pc, as preferred by McKee et al. (2015). As we explain in Section 5.3, we do not find our model to be consistent with these large scale heights, even without a dark disk. We therefore do not use HII parameters in this chapter as a constraint.

For its velocity dispersion, Holmberg & Flynn (2000) used the value $\sigma_{\text{HII}} = 40$ km s$^{-1}$. This value seems to have been inferred from scale height measurements of the electrons associated with this ionized gas from Kulkarni & Heiles (1987). From the data in Reynolds (1985), however, we find a turbulent component to the dispersion of only $\sigma_{\text{HII}} = 21 \pm 5$ km s$^{-1}$. On the other hand, temperatures between 8,000 K and 20,000 K give a thermal contribution of $\sigma_{\text{HII,thermal}} = \sqrt{2.1 k_B T/m_p} \approx 12 - 19$ km s$^{-1}$ (Ferrière 2001, p.14). Summing these in quadrature gives $\sigma_{\text{HII}} = 25 - 29$ km s$^{-1}$. As we will explain in Section 4.4.4, including magnetic and cosmic ray pressure contributions pushes this up to 42 km s$^{-1}$. Similarly, Kalberla (2003) also finds $\sigma_{\text{HII}} = 27$ km s$^{-1}$ while assuming $p_{\text{mag}} = p_{\text{cr}} = 1/3 p_{\text{turb}}$, for a total effective dispersion of 35 km s$^{-1}$ but did not include a thermal contribution. This gives an average total effective dispersion of $39 \pm 4$ km s$^{-1}$, which, removing magnetic, cosmic ray, and thermal contributions, gives a turbulent dispersion of $\sigma_{\text{HII}} = 22 \pm 3$ km s$^{-1}$.

The new gas parameter estimates, obtained in this work by incorporating a broad range of literature values, are summarized in Table 4.1 alongside the old (Flynn et al. 2006) values. The values of McKee et al. (2015) are also included for comparison.
Table 4.1: Old values (Flynn et al. 2006) and new values (including all the references mentioned in Section 4.4) estimated in this thesis. We also include the values of McKee et al. (2015).

<table>
<thead>
<tr>
<th>Component</th>
<th>Flynn et al. (2006) $n(0)$ [cm$^{-3}$]</th>
<th>This reference $n(0)$ [cm$^{-3}$]</th>
<th>McKee et al. (2015) $n(0)$ [cm$^{-3}$]</th>
</tr>
</thead>
<tbody>
<tr>
<td>H$_2^*$</td>
<td>0.30</td>
<td>0.19</td>
<td>0.15</td>
</tr>
<tr>
<td>HI(CNM)</td>
<td>0.46</td>
<td>0.56</td>
<td>0.69</td>
</tr>
<tr>
<td>HI(WNM)</td>
<td>0.34</td>
<td>0.15</td>
<td>0.21</td>
</tr>
<tr>
<td>HII</td>
<td>0.03</td>
<td>0.03</td>
<td>0.0154</td>
</tr>
</tbody>
</table>

* does not include dark molecular gas

4.4.4 Other Forces

Boulares & Cox (1990) considered the effect of magnetic forces and cosmic ray pressure on the interstellar gas. The effect of the magnetic field is a contribution to the force per unit volume on the $i^{th}$ component of the gas:

\[ f_i = J_i \times B \]  

(4.39)

where $J_i$ is the current density associated with gas component $i$, $B_i$ is the magnetic induction field due to component $i$, and where

\[ B = \sum_i B_i \]  

(4.40)

is the total magnetic field from all the gas components. Using Ampere’s law, we can rewrite
the z-component of the force as

\[ f_{zi} = \frac{1}{\mu_0} \left( (\nabla \times B_i) \times B \right)_z \]

(4.41)

\[ = \frac{1}{\mu_0} \left( B \cdot \text{div} \right) B_{iz} - \frac{1}{\mu_0} B \cdot \frac{\partial B_i}{\partial z} \]

(4.42)

Since according to Parker (1966), the magnetic field is, on average, parallel to the plane of the Galaxy, \( B_z = 0 \) we will make the approximation that the first term vanishes in equilibrium. The second term couples each gas component to the remaining components, since \( B \) represents the total magnetic field. However, summing all components, we have

\[ f_z \equiv \sum_i f_{zi} = -\frac{1}{\mu_0} B \cdot \frac{\partial B}{\partial z} \]

(4.43)

\[ = -\frac{\partial}{\partial z} \left( \frac{B^2}{2\mu_0} \right). \]

(4.44)

We recognize the form of this expression as the gradient of the magnetic pressure \( p_B = B^2/2\mu_0 \).

To include this effect in the Poisson-Jeans Equation, we note that the first term on the left-hand-side of Equation 4.1 has the interpretation (up to an overall mass factor) as the gradient of a ‘vertical pressure’. This pressure term is a correct description of a population of stars or of gas clouds. In a warm gas, this term has the interpretation as the turbulent pressure of the gas. However, in this case, one also needs to take into account the thermal pressure of the gas

\[ p_{\text{thermal}} = c_i n_i k_B T_i \]

(4.45)
where $c_i$ is a factor that takes into account the degree of ionization of the gas, and $n_i = \rho_i/m_p$ is the number density of the gas atoms. The correct Poisson-Jeans equation in this case therefore reads

$$\frac{\partial}{\partial z} (\rho_i \sigma_i^2 + \rho_i c_i k_B T_i) + \rho_i \frac{\partial \Phi}{\partial z} = 0. \quad (4.46)$$

If we define a ‘thermal dispersion’ as

$$\sigma_{i,T}^2 \equiv c_i k_B T_i \quad (4.47)$$

then we can rewrite this as

$$\frac{\partial}{\partial z} (\rho_i (\sigma_i^2 + \sigma_{i,T}^2)) + \rho_i \frac{\partial \Phi}{\partial z} = 0. \quad (4.48)$$

Clearly, to account for the magnetic pressure, we would include the average of the magnetic pressure term in precisely the same manner:

$$\sum_i \frac{\partial}{\partial z} (\rho_i (\sigma_i^2 + \sigma_{i,T}^2)) + \frac{\partial}{\partial z} \left\langle B^2 \right\rangle + \rho \frac{\partial \Phi}{\partial z} = 0. \quad (4.49)$$

where $\rho$ is the total mass density of the gas. In the following subsections, we describe how we model this magnetic pressure term.
Magnetic Pressure: Thermal Scaling Model

An important phenomenon noted by Parker (1966) is that the magnetic field $B$ is confined by the weight of the gas through which it penetrates. We therefore would like to solve this equation by following Parker in assuming that the magnetic pressure is proportional to the thermal pressure term, $p_i = \rho_i c_s k_B T_i$. Since each gas component contributes to the total thermal pressure with a different temperature $T_i$, we write:

$$\left\langle \frac{B^2(z)}{2\mu_0} \right\rangle = \alpha \sum_i \rho_i(z) \sigma_{i,T}^2$$  \hspace{1cm} (4.50)$$

$$= \sum_i \rho_i(z) \sigma_{i,B}^2$$  \hspace{1cm} (4.51)

where $\alpha$ is a proportionality constant fixed by $\left\langle B^2(0) \right\rangle$ and $\sum_i \sigma_{i,T}^2$, and where we have defined the ‘magnetic dispersion’ $\sigma_{i,B}^2 = \alpha \sigma_{i,T}^2$ i.e. the effective dispersion arising from the magnetic pressure. The Poisson-Jeans equation then reads

$$\sum_i \frac{\partial}{\partial z} \left( \rho_i \left( \sigma_i^2 + \sigma_{i,T}^2 + \sigma_{i,B}^2 \right) \right) + \rho \frac{\partial \Phi}{\partial z} = 0.$$  \hspace{1cm} (4.52)

The above equation admits many solutions. However, we will assume that the unsummed equation

$$\frac{\partial}{\partial z} \left( \rho_i \left( \sigma_i^2 + \sigma_{i,T}^2 + \sigma_{i,B}^2 \right) \right) + \rho_i \frac{\partial \Phi}{\partial z} = 0.$$  \hspace{1cm} (4.53)
holds for each component individually. This amounts to assuming that all gas components confine the magnetic field equally. Other solutions can be found by substituting $\sigma^2_{i, B} \rightarrow \sigma^2_{i, B} + S_i(z)$, such that $\sum_i \rho_i(z)S_i(z) = 0$. However, if we restrict our analysis to ‘isothermal’ solutions (constant $\sigma^2_{i, B}$) the solution $S_i = 0$ will be unique. We can also include the effects of cosmic ray pressure in a similar way, by assuming that the partial cosmic ray pressure is also proportional to the density

$$p_{i, cr}(z) = \rho_i(z) \sigma^2_{i, cr}$$

(4.54)

and where $\sigma^2_{i, cr} = \beta \sigma^2_{i, T}$ for some other constant $\beta$. The Poisson-Jeans Equation then reads

$$\frac{\partial}{\partial z}(\rho_i \sigma^2_{i,\text{eff}}) = \sum_i \frac{\partial}{\partial z} \left( \rho_i \left( \sigma_i^2 + \sigma^2_{i, T} + \sigma^2_{i, B} + \sigma^2_{i, cr} \right) \right) + \rho_i \frac{\partial \Phi}{\partial z} = 0,$$

(4.55)

where we have defined

$$\sigma^2_{i,\text{eff}} = \sigma_i^2 + \sigma^2_{i, T} + \sigma^2_{i, B} + \sigma^2_{i, cr}.$$  

(4.56)

The solution to the Poisson-Jeans Equation for each component will then be

$$\rho_i(z) = \rho_i(0) \exp \left( -\frac{\Phi(z)}{\sigma^2_{i,\text{eff}}} \right).$$

(4.57)

Note that since the pressure is additive, the dispersions add in quadrature. Boulares & Cox (1990) estimate for the magnetic pressure $p_B \simeq (0.4 - 1.4) \times 10^{-12}$ dyn cm$^{-2}$. For the cosmic
Table 4.2: Intrinsic and effective dispersions for ISM components

<table>
<thead>
<tr>
<th>Component</th>
<th>$\sigma_T$ [km s$^{-1}$]</th>
<th>$\sigma_B$ [km s$^{-1}$]</th>
<th>$\sigma_{cr}$ [km s$^{-1}$]</th>
<th>$\sigma_{\text{eff}}$ (thermal scaling) [km s$^{-1}$]</th>
<th>$\sigma_{\text{eff}}$ (warm equipartition) [km s$^{-1}$]</th>
</tr>
</thead>
<tbody>
<tr>
<td>H$_2$</td>
<td>3.7</td>
<td>0.2</td>
<td>0.3</td>
<td>3.7</td>
<td>6.4</td>
</tr>
<tr>
<td>HI(CNM)</td>
<td>6.8</td>
<td>0.8</td>
<td>1.2</td>
<td>7.1</td>
<td>11.8</td>
</tr>
<tr>
<td>HI(WNM)</td>
<td>13.1</td>
<td>6.7</td>
<td>10.3</td>
<td>21</td>
<td>23.7</td>
</tr>
<tr>
<td>HII</td>
<td>22</td>
<td>11.8</td>
<td>18.1</td>
<td>36.2</td>
<td>39.9</td>
</tr>
</tbody>
</table>

Ray pressure, they estimate $p_{cr} \simeq (0.8 - 1.6) \times 10^{-12}$ dyn cm$^{-2}$. The dispersions for each component are shown in Table 4.2 for comparison, as well as the effective dispersions in this model.

There is, however, no clear evidence to support this model. Although $p_{\text{mag}} \propto p_{cr} \propto nk_BT$ has been assumed in the past (Parker 1966), this was when the entire gas was treated as a single component. Equation 4.50, however, is much more specific. We therefore supplement this model with a second model in the next subsection for comparison.

**Magnetic Pressure: Warm Equipartition Model**

Here we describe a second possible model to describe the magnetic and cosmic ray pressures in the interstellar medium. Namely, it has been observed that within the CNM, energy densities in magnetic fields and in turbulence are often roughly equal (Heiles & Troland 2003; Heiles & Crutcher 2005). Although the ratio between these energies is observed to vary greatly over different molecular clouds, this so-called “energy equipartition” seems to be obeyed on average. Physically, this happens because the turbulence amplifies the magnetic field until it becomes strong enough to dissipate through van Alfvén waves. Similarly, we expect magnetic fields to
trap cosmic rays within the gas until they become too dense and begin to escape. We might therefore expect the cosmic ray and magnetic field energy densities to be similar. For these reasons, an alternative to the first model (Equation 4.50) would be to assume equipartition of pressure between turbulence, magnetic fields, and cosmic rays:

\[
\sigma_i^2 = \sigma_{i,B}^2 = \sigma_{i,cr}^2
\]  

(4.58)

for each component \(i\). The effective dispersion, which is the sum of turbulent, magnetic, cosmic ray, and thermal contributions, would then be

\[
\sigma_{i,\text{eff}}^2 = \sigma_i^2 + \sigma_{i,B}^2 + \sigma_{i,cr}^2 + \sigma_{i,T}^2
\]

\[
\simeq 3 \sigma_i^2 + \sigma_{i,T}^2
\]  

(4.59) \hspace{1cm} (4.60)

for each component. One important factor that we should not overlooked here, however, is that the molecular hydrogen and CNM condense to form clouds. Thus, although turbulence, magnetic fields, and cosmic rays may affect the size of the individual clouds, we expect the overall scale height of the cold components to be determined only by the cloud-cloud dispersion and not by these forces. We therefore assume Equations 4.58-4.60 only for the warm components WNM and HII. For the cold components (H\(_2\) and CNM), we assume \(\sigma_{i,\text{eff}} = \sigma_i\). We perform calculations separately for the two different magnetic field and cosmic ray models. The effective dispersions in both models are shown in Table 4.2. As we will see, the results from both
models are in good agreement with one another.

### 4.5 Results and Discussion

We now present the results of the analysis described above. Using the midplane densities of Section 4.4, we calculate according to the Poisson-Jeans equation the corresponding $H_2$ and HI surface densities, and from these we compute the chi-squared value, $(\Sigma - \hat{\Sigma})^2 / \Delta \Sigma^2$, from the disagreement between these values and the measured values. This is done over a range of values for $\Sigma_D$ and $h_D$. We thereby determine the regions of parameter space where the disagreement exceeds the 68% and 95% bounds, as will be displayed in the plots below. The scale height $h_D$ is defined such that

$$\rho(Z = h_D) = \rho(Z = 0) \text{sech}^2(1/2).$$

(4.61)

We begin by determining the bounds without including the contribution from magnetic fields and cosmic rays. The result is shown in Figure 4.4. The uncertainty $H_2$ is dominated by that of the dark molecular gas, while the uncertainty of HI is dominated by that of the WNM velocity dispersion (18%). We can see that although the $H_2$ parameters are consistent with dark disk surface densities of (for low scale height) up to $10 - 12 M_\odot \text{pc}^{-2}$, the HI parameters point toward lower surface densities, and that the combined probabilities are lower than 9% for all models. These results make it apparent that the model without magnetic fields is inconsistent.
On the other hand, when we include the pressure contribution from the magnetic fields and from cosmic rays, we find that both the HI and $\text{H}_2$ parameters allow non-zero surface densities $\Sigma_D$, with an upper bound of $\Sigma_D \simeq 10 \, M_\odot \text{pc}^{-2}$ in both models, for low scale heights. Higher scale heights are consistent with even higher dark disk surface densities. The results are shown in Figure 4.5.

For comparison, we also include the corresponding results using the values of McKee et al. (2015). Using these values and including magnetic fields and cosmic ray contributions, the data favors a non-zero surface density for the dark disk of between 5 and 15 $M_\odot \text{pc}^{-2}$. Note that when neglecting magnetic field and cosmic rays pressures, only low dark disk surface densities seem consistent with the data.

**Ionized Hydrogen Results and Issues**

As was mentioned in Section 4.4.3, various authors have measured DM values for the HII component of the Milky Way in the range 20-30 cm$^{-3}$ pc. Older models favored low scale height with midplane densities as high as $0.034 \, \text{cm}^{-3}$, while newer models favor large scale height models with midplane densities as low as $0.014 \, \text{cm}^{-3}$. However, using our model, the results of our Poisson-Jeans solver are consistent with only low scale heights. Following the models described in Section 4.4.4, we find scale heights for the HII of 0.9 kpc for the thermal scaling model and 1.0 kpc for the warm equipartition model, assuming $\Sigma_D = 0$. Incorporating a more massive dark disk makes these scale heights smaller. Possible reasons for this might be:

1) The magnetic field model must be modified to include a different value of $\alpha$ for HII. This
could be in correspondence with the result of Beuermann et al. (1985), who found that Galactic magnetic fields contained two components, one with short scale height and one with larger scale height. These two components would likely be described by different $\alpha$. We could then attribute the low scale height component to the molecular and atomic gas and the large scale height component to the ionized gas. We know of no such alternative in the warm equipartition model.

2) The isothermal assumption may not be valid for HII. In fact, as explained in Gaensler et al. (2008), the volume filling fraction of HII may also vary a lot with scale height. If this is the case, then it would be incorrect to treat the HII as an isothermal component as the degrees of freedom that the temperature describes (the gas clouds) vary with distance from the Galactic midplane.

**Stability Issues and Kinematic Constraints**

In Figure 7 we show the bound we obtained from the kinematics of A stars in the Solar region, accounting for nonequilibrium features of the population, namely a net displacement and vertical velocity relative to the Galactic midplane. We also note that there will exist disk stability bounds. The true analysis is subtle, but a step toward the analysis is done by Shaviv (2016b) who develops the stability criterion for a heterogeneous Milky Way disk including a thin dark matter disk. We convert his bound to a bound in the $h_D - \Sigma_D$ plane and superimpose this bound on the gas parameter bound of the present work. We see that a disk with significant mass ($\Sigma_D$) and $h_D > 30$ pc is consistent with all current bounds.

In addition to stability issues, Hessman (2015) has argued that there exist other issues with
using the vertical Jeans equation to constrain the dynamical mass in the MW disk. In particular, spiral structure must be taken into account when performing these analyses. Indeed, Shaviv (2016a) has pointed out that the effect of spiral arm crossing is to induce a ‘ringing’ in the dynamics of tracer stars. However, the present analysis assumes that the time scales for this ringing are much shorter in gas components so that the analysis is valid. Spiral arm crossing could also induce non-equilibrium features in the tracer population, such as discussed in Chapter 3, but as in there, including this effect would allow for more dark matter.

4.6 Conclusion

In this chapter we have shown how to use measured midplane and surface densities of various galactic plane components to constrain or discover a dark disk. Although literature values of atomic hydrogen midplane densities are discordant, their mean value is consistent with the remaining gas parameters when magnetic and cosmic ray pressures are included. Using the global averages of literature values of gas parameters that we compiled, we find the data are consistent with dark disk surface densities as high as $10\ M_\odot\ pc^{-2}$ for low scale height, and as low as zero. The gas parameters of McKee et al. (2015) seem to favor an even higher non-zero dark disk surface density. Current data are clearly inadequate to decide this definitively. Further measurements of visible and dark $H_2$ density and WNM density and dispersion, as well as further refinements of magnetic field and cosmic ray models for cold gas could allow placing more robust bounds on a dark disk.
Figure 4.2: Molecular hydrogen midplane densities and surface densities determined by various authors between 1984 and 2006.
Figure 4.3: Atomic hydrogen midplane densities and surface densities determined by various authors between 1978 and 2008.
Figure 4.4: Confidence bounds on DDDM parameter space as a function of $h_D$, the dark disk \( \text{sech}^2(z/2h_D) \) scale height, using averages and uncertainties from Sections 4.4.1 to 4.4.3. Solid lines represent 95% bounds and dashed lines represent 68% bounds.
Figure 4.5: Bounds on DDDM parameter space as in Figure 4.4, but including contributions from magnetic fields and from cosmic rays. *Black*: computed assuming ‘thermal scaling model’. *Red*: computed assuming ‘warm equipartition model’.
Figure 4.6: Confidence bounds as in Figure 4.4 but using the values of McKee et al. (2015). Left: Not including magnetic field and cosmic ray contributions. Right: including magnetic field and cosmic ray contributions as in Figure 4.5.
Figure 4.7: The red shaded region, delimited by the solid red line, denotes the parameters allowed by the stability bound of Shaviv (2016b). The blue shaded region, delimited by the solid blue line, denotes the parameters allowed by the kinematic bound of Chapter 3. The grey shaded region, delimited by the solid black line, denotes the parameters allowed by the gas parameters as determined in the current chapter. As in Figure 4.6, the dashed and solid black lines denote the 68% and 95% bounds obtained from the combined gas bound, including magnetic field and cosmic ray contributions, and using the parameters of McKee et al. (2015).
5 Evidence from the Crater Record

5.1 Introduction

Many authors have underlined a potential periodicity in the rate of comet impacts on Earth. A possible cause hypothesized to explain this periodicity is the oscillation of the Sun about the galactic midplane. Indeed, this is not a new problem and the possibility has been pursued from
a variety of angles (Hut et al. 1987; Stothers 1988; Matese et al. 2001; Rampino 2015) and the many references therein. Accordingly, many authors (Stothers 1998; Shaviv et al. 2014; Randall & Reece 2014) and others suggest the existence of a significant distribution of dark matter in the galactic disk. However, Shaviv (2016b) suggests that this disk is too thick to cause comet showers. Matese et al. (2001) also suggests, based on Holmberg & Flynn (2000), that the amount of dark matter required to cause periodic comet impacts is ruled out by local stellar kinematics. The goal of this chapter is to examine the question of the dark matter-dinosaur connection more carefully, with a global analysis incorporating all of its currently known observable constraints. These include:

1. A dark disk model motivated by particle physics, and that is also
2. Consistent with the remaining Galactic disk parameters under the Poisson-Jeans equation
3. Recently updated Galactic disk parameters of Chapter 4 and McKee et al. (2015).
4. Inclusion of prior probabilities from kinematic data, taking into account vertical epicyclic oscillations of the star populations from Chapter 3
5. Inclusion of prior probabilities from the distribution of interstellar gas from Chapter 4.
6. Inclusion of radial oscillations
7. Inclusion of spiral arm crossings

Performing a global analysis including these together will give a more conclusive answer on the question of the dark matter-dinosaur connection. A Poisson-Jeans model was used in Bahcall & Bahcall (1985), but this model was not up-to-date. Radial oscillations were included in Shaviv (2016b), but not spiral arm crossings. Lastly, the prior probabilities are important because they allow the evidence of the crater record to be interpreted in the context of the
existing literature and constraints. Based on the above analysis, we find that the likelihood ratio between the dark matter cause for the dinosaur extinction and the random cause to be close to 12. Including prior probabilities from stellar kinematics and the distribution of local interstellar gas increases this likelihood ratio to 10. This indicates that this connection should be taken seriously.

5.2 Analysis

5.2.1 Galactic Mass Model

We compute the oscillations of the Sun through the galactic plane by integrating the motion of the Sun in the potential computed assuming the mass model described in Chapters 3 and 4. As explained in Chapter 3, this Bahcall-type model Bahcall (1984b) was obtained by solving the Poisson-Jeans equation for a superposition of isothermal components. The gas parameters used are explained in Chapter 4. These were an amalgam of a number of sources. For comparison, the more widely accepted parameters of McKee et al. (2015) were also used.

5.2.2 Crater Record

We followed the Bayesian analysis of Randall & Reece (2014) in comparing the predicted comet rates from oscillations to a constant comet rate. We used the craters dated to the past 250 My from the Earth Impact Database, developed and maintained by the Planetary and Space Science
Center of the University of New Brunswick (Earth Impact Database 2016). We also follow Renne et al. (2013) in correcting the age of the Chicxulub crater from 64.98 ± 0.05 My to 66.04 ± 0.05 My*. We included only craters larger than a certain diameter; we performed the analysis for diameter cutoffs of \( D = 20 \) km, 40 km, and 50 km, giving sample sizes of \( N_D = 26, 13, \) and 7 craters respectively. Although the 50 km sample is small, according to Shoemaker et al. (1988), it contains craters with a high probability of being caused by comets and not asteroids. The maximum likelihood ratio, however, was found for \( D \geq 20 \) km.

For an impulse-like disturbance in the tidal force, a comet shower profile was computed by Hut et al. (1987). Despite using many updates in the crater record and the Milky Way’s mass distribution, we assume the comet shower profile of Hut et al. (1987) to still be accurate. We therefore convolved this comet shower profile with the predicted tidal force to give the predicted comet rate as a function of age. In the constant rate model, we assumed a constant rate of \( N_D = 26, 13, \) or 7 comets per 250 My.

### 5.2.3 Statistics

We derived the likelihood for a given model based on Poisson statistics. In a time period \( dt \), an expected number of comets \( \lambda \) will generate an observed number of comets \( k \) with probability

\[
P(k|\lambda) = \frac{\lambda^k e^{-\lambda}}{k!}.
\]

\[\text{(5.1)}\]

*This may also explain why no \(^3\)He flux increase, associated with comet showers, was observed between 63.9 and 65.4 Mya (Mukhopadhyay et al. 2001).
According to Bayes’ theorem, the likelihood of a model $\lambda$ given the observed comet number $k$ will therefore be:

$$P(\lambda|k) = \frac{P(k|\lambda)P(\lambda)}{P(k)} = \frac{\lambda^k e^{-\lambda} P(\lambda)}{k!} \frac{P(\lambda)}{P(k)}$$  \hspace{1cm} (5.2)$$

where $P(\lambda)$ is the prior probability distributions for $\lambda$ and for $k$. For two different models $\lambda$ and $\lambda'$ for this expected number, the likelihood ratio between these two models will therefore be given by

$$L(\lambda, \lambda') = \left(\frac{\lambda}{\lambda'}\right)^k e^{-(\lambda-\lambda')} \frac{P(\lambda)}{P(\lambda')}.$$ \hspace{1cm} (5.3)$$

In our case, we will be interested in the likelihood ratio between the dark matter explanation for the crater record vs. the constant rate explanation. The expected number of comet impacts in these two models will be

$$\lambda = dt \, r(t) \hspace{1cm} (5.4)$$
$$\lambda' = dt \, \frac{N}{T}.$$ \hspace{1cm} (5.5)$$

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Here, Equation 5.4 reflects our choice of the model where the comet rate is proportional to the instantaneous density $\rho(t)$, convolved with the Oort cloud response function $h(t)$:

$$r(t) = \kappa [h \ast \rho](t). \quad (5.6)$$

The constant $\kappa$ is fixed by demanding that the total integrated rate be equal to the total number of comets $N$. Equation 5.5 simply gives the expected number of comets in time interval $dt$ for a constant rate giving total number of comets impacts $N$ in total time $T = 250$ My. Likewise, the observed number of comets in time interval $dt$ is given by the crater record:

$$k = dt \cdot C(t) \quad (5.7)$$

where $C(t)$ is the crater rate history in events per My. It is a sum of Gaussian distributions for each crater with area 1, centered at its age and with width given by the uncertainty in its age.

Equation 5.3 defines the likelihood ratio at a given time $t$. To obtain the likelihood ratio over all times, we can assume the impacts probabilities at all times are independent and multiply the likelihood $L$ over all times. The log likelihood ratio over all times will therefore be given by the integral

$$\log L = \sum_t k \log \left( \frac{\lambda}{\lambda'} \right) + \lambda' - \lambda + \log \left( \frac{P(\lambda)}{P(\lambda')} \right) \quad (5.8)$$

$$= \int dt \cdot C(t) \log \left( \frac{r(t)}{N/T} \right) + \log \left( \frac{P(\Sigma D)}{P(0)} \right). \quad (5.9)$$
The $\lambda' - \lambda$ term vanishes by the restriction that the total integrated rates are equal for the two models. $\Sigma_D$ is the surface mass density of the dark disk. $P(0)$ represents a model in which the dark disk surface density is zero. $P(\Sigma_D)/P(0)$ therefore represents the prior likelihood ratio between a model with and without a dark matter disk. We are thus comparing only two models: 1) a model where the comet rate is caused by solar oscillations with a dark matter disk of surface density $\Sigma_D$, and 2) a model in which $\Sigma_D = 0$ and in which the comets arrive at a constant average rate $N/T$. The expression for the likelihood ratio is therefore

$$
\mathcal{L} = \exp \left\{ \int dt C(t) \log \left( \frac{r(t)}{N/T} \right) \right\} \frac{P(\Sigma_D)}{P(0)}.
$$

(5.10)

For any given dark disk model $\Sigma_D$, we chose the thickness of the disk to be the minimum value permitted under the stability bound derived by Shaviv (2016b). Under these assumptions, we find a posterior likelihood ratio of 10.

### 5.2.4 Prior Probabilities

For the ratio of prior probabilities $P(\Sigma_D)/P(0)$ used in Equation 5.10, we rely on the results of Chapters 3 and 4. These contain a contribution from the stellar kinematics of A stars, taking into account the vertical epicyclic oscillations of these stars. The other contribution is from the gravothermal equilibrium of the Milky Way’s interstellar gas. The values of $P(\Sigma_D)/P(0)$ are shown in Figure 5.1 The priors inferred using the gas parameters of McKee et al. (2015) are also shown. As can be seen, the latter favor a higher value of $\Sigma_D$.  

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Figure 5.1: The prior probabilities for a model with dark disk surface density $\Sigma_D$ and thickness given by the stability bound of Shaviv (2016b).
5.2.5 **Spiral Arm Crossing**

Another effect we account for here is the Sun’s periodic crossing of the Milky Way’s rotating spiral arm pattern. We assume spiral arms with a sinusoidal amplitude \( A = 0.25 \) (Hessman 2015). This corresponds to an arm-interarm density ratio of 1.7 (Binney & Tremaine 2008, Ch.6). As we are currently entering the Orion spur, we assume the current locally measured density to represent the azimuthally averaged density of the disk at the Solar radius. The Sun was assumed to move through the spiral arm pattern according to the ages in Shaviv (2002).

5.2.6 **Radial Oscillations**

For radial oscillations, epicycle theory (Binney & Tremaine 2008, Ch.3) gives an estimate of the mean radial position \( R_g \) of the Sun:

\[
R_\odot - R_g = \frac{V_\odot - V_c}{2B}
\]

(5.11)

where \( B = -12.37 \pm 0.64 \) km s\(^{-1}\) kpc\(^{-1}\) is one the Oort constants (Feast & Whitelock 1997).

From Schönrich et al. (2010), we have

\[
(U_\odot, V_\odot - V_c, W_\odot) = (11.1 \pm 0.74, 12.24 \pm 0.47, 7.25 \pm 0.37) \text{ km s}^{-1}, \quad (5.12)
\]
This gives, according to Equation 5.11:

$$R_\odot - R_g = -0.49 \pm 0.03 \text{ kpc.} \quad (5.13)$$

The radial oscillations will are assumed to vary with a frequency of $1.35 \Omega_0$ (Binney & Tremaine 2008, Ch. 3), where $\Omega_0 = V_c/R_\odot$ is the angular velocity of the circular orbit at the position of the Sun. With the measured Oort constant $A - B = 29.45 \pm 0.15 \text{ km s}^{-1} \text{ kpc}^{-1}$ (Reid & Brunthaler 2004) and Solar radius $R_\odot = 8.33 \pm 0.35 \text{ kpc}$ (Gillessen et al. 2009), we find the circular velocity at the Solar radius to be

$$V_c = 245 \pm 10 \text{ km s}^{-1}. \quad (5.14)$$

We therefore have

$$1.35 \Omega_0 = 1.35 V_c/R_\odot = 39.7 \pm 2.3 \text{ km s}^{-1} \text{ kpc}^{-1} \quad (5.15)$$

Combining this radial velocity $U_\odot$ of 5.12 therefore gives a radial oscillation amplitude $0.57 \pm 0.03 \text{ kpc}$ and a present phase of $1.06 \pm 0.15 \text{ rad.}$ Assuming an exponential scale radius $R_{MW} \simeq 3.5 \pm 0.5 \text{ kpc}$ for the Galactic mass (Binney & Tremaine 2008), the radial oscillations will generate oscillations in density an amplitude of approximately $\simeq 16.3 \pm 2.0\%$. The radial oscillations are therefore assumed to vary the density with an amplitude of 16.3% and a period of 155 My.
5.2.7 Solar Parameters

Since there is a lot of uncertainty in the position of the Sun relative to the galactic plane, this will certainly affect whether the model can account for the dinosaur extinction. We therefore compute the likelihood for the value \( Z_\odot = 26 \text{ pc} \) (Majaess et al. 2009) as well as for other values of \( Z_\odot \). We also varied the vertical velocity of the Sun \( W_0 \) (Schönrich et al. 2010) and the dark halo density \( \rho_{\text{halo}} \) (Bovy & Tremaine 2012).

5.2.8 Stability Parameters

The stability bound used here was that estimated by Shaviv (2016b). For any dark disk surface density \( \Sigma_D \), we estimated the disk’s thickness \( h_D(\Sigma_D) \) to be the minimum value consistent with the stability bound, i.e.

\[
h_D = h_{D,\text{min}}(\Sigma_D)
\]

To estimate the effect of the uncertainty in this bound, we also used the values equal to half and double the value implied by the bound, i.e. \( h_D = 0.5 \) or \( 1.5 h_{D,\text{min}} \).

5.3 Results and Discussion

Figure 5.2 shows the likelihood ratio for the two models as a function of dark disk surface density \( \Sigma_D \), including the prior probabilities from Figure 5.1. The figure also shows the relative rate of comets at the date of 66 Mya as a function of \( \Sigma_D \). For these calculations, the Galactic
disk parameters for visible matter were taken from Chapter 3. The height of the Sun above the
Galactic plane to be $Z_\odot = 26$ pc, and a crater diameter cutoff of 20 km was imposed. We see
that the likelihood ratio is as large as 10 in favor of an oscillatory model, at the best fit value
$\Sigma_D \simeq 9 M_\odot pc^{-2}$. We also see that the relative rate of predicted comet impacts at age 66 My
also is quite high near this value, indicating that the model correctly predicts the date of the 66
Mya Chicxulub crater.

Figure 5.3 shows the predicted comet rate in the oscillatory model superposed on the crater
record. The time spent within spiral arms is shown by the double arrows. We can see that the
the comet rate is consistent with the oscillations of the Sun when spiral arm crossing is taken
into account. In particular, the Chicxulub crater matches the predicted rate.

Figure 5.4 shows the effect of varying the visible mass parameters, the dark disk scale height
$h_D$, the present height of the Sun above the Galactic plane $Z_\odot$, the Sun’s vertical velocity $W_0$,
and the Sun’s vertical velocity according to the ranges in Section 5.2. We see that varying the
visible mass parameters has a very minor effect on the results. Varying $h_D$ affects the best-fit
value of $\Sigma_D$, with larger scale heights allowing more mass, as should be expected. lower values
of $Z_\odot$ favor a lower value of $\Sigma_D$. Varying $W_0$ does not significantly affect the results.

Figure 5.5 shows the result of the computation1) including neither spiral arm crossing nor
radial oscillations, 2) including radial oscillations but not spiral arms, and 3) including both. We
can see that the radial oscillations decrease the likelihood of a dark disk model, while the spiral
arms greatly enhance it. In both cases we see that although the data favors a higher dark disk
surface density $\Sigma_D$. 118
Figure 5.2: The solid line shows the likelihood ratio of oscillating Sun model relative to the constant rate model. It shows a peak near $\Sigma_D = 9 \, M_\odot \, pc^{-2}$. Dashed and notted lines show the posterior likelihood ratio after including the prior probabilities of Figure 5.1. The dot-dshed line shows the predicted comet rate at age 66 My as function of $\Sigma_D$. 
**Figure 5.3:** The figure shows the history of craters larger than 20 km in diameter over the past 250 My as a probability density in age, as well as the predicted comet rate (in arbitrary units) assuming Solar oscillations with a best fit dark disk surface density of $\Sigma_D = 9 M_\odot \text{pc}^{-2}$. Each blue spike represents a cratering shower. The recent craters of Popigai (36 Mya) and Chicxulub (66 Mya) are separated by only 29 My. The figure shows that the 50 My spent crossing the Carina / Sagittarius arm reduces the period for long enough to account for this smaller interval.
Figure 5.4: Plots of likelihood ratio vs. $\Sigma_D$. The first plot on the left shows the effect of varying the parameters of the interstellar gas disk between those of Chapter 3 and the right figure uses those of McKee et al. (2015). The second plot shows the effect of varying the disk scale height relative to its minimum size $h_{\text{min}}(\Sigma_D)$ under the stability bound of Shaviv (2016b). The third plot and fourth plots show the effect of varying the present height of the Sun, $Z_{\odot}$ and its vertical velocity $W_{\odot}$.

Figure 5.6 shows the effect of varying the crater diameter cutoff. We find that the best fit values do not depend on the diameter cutoff, although a much higher significance is obtained for the 20 km cutoff, containing 26 craters.

5.4 Conclusions

We have shown that a dark disk consistent with stability bounds can indeed account for the observed comet rate on Earth. A dark disk model was found to be 10 times more likely than the constant rate model after prior probabilities were included. We have also shown that spiral arm crossings are crucial in accounting for variations in the comet rate. In particular, the crating history on earth, including the date of the Chicxulub crater 66 Mya, is consistent with the
Figure 5.5: We show here the inferred likelihood ratio as a function of dark disk surface density 1) including neither spiral arm crossing nor radial oscillations, 2) including radial oscillations but not spiral arms, and 3) including both. We can see that the radial oscillations decrease the likelihood of a dark disk model, while the spiral arms greatly enhance it.
Figure 5.6: Plots as in Figure 5.2. Here we vary the diameter cutoff between 20 km, 40 km, and 50 km respectively. Although the 20 km diameter cut may include objects that are not comets, a much higher significance is obtained.
predicted comet shower rate when the spiral arms are taken into account.
6.1 Introduction

Dark matter is 85% of the matter in the universe and we still do not know what it is made of. A popular candidate is the WIMP (Weakly Interacting Massive Particle). The reason is that in order to obtain the correct abundance of dark matter today, a self-annihilation cross-section...
of $\langle \sigma v \rangle \simeq 3 \times 10^{-26} \text{cm}^3\text{s}^{-1}$ would be needed, which coincides with the cross-section for a weakly interacting particle with mass roughly 100 GeV. This is known as the “WIMP miracle” (Jungman et al. 1996). A promising way to search for these WIMPs is through so-called “direct detection” experiments. In these, a target of nuclei, sitting in the presumed dark matter halo, would experience recoils from colliding WIMPs, with a rate dependent on the halo density and interaction cross-section. Examples of these include the LUX, XENON, PandaX, and CDMS experiments (Akerib et al. 2017; Aprile et al. 2012; Fu et al. 2017; Agnese et al. 2017). However, thus far, searches for WIMPs have turned up nothing. Although experiments continue to search for WIMPs with lower interaction cross-section, WIMPs with lighter masses are emerging as the most viable possibility. Much attention has therefore recently been given to direct detection experiments sensitive in the keV - 100 MeV range, such as those of Essig et al. (2017a); Essig et al. (2012); Schutz & Zurek (2016); Hochberg et al. (2016b,a).

Another important question is whether the dark matter sector could be as complex as the visible sector, with interactions and dissipation. One possible signature of dissipative interactions is the formation of a dark disk, which would be co-rotating with our Galactic disk (Fan et al. 2013). In models such as these, so-called “slow dark matter” models, the relative velocity between dark matter and our experiments is an order of magnitude lower than in the traditional WIMP case. Another case where slow dark matter could be important is in Earth-bound dark matter (Adler 2009, 2008; Catena & Kouvaris 2016). Since direct detection experiments rely on the dark matter particles depositing their kinetic energy $\frac{1}{2}mv^2$ into the target material, experiments that have been devised to test light dark matter can in fact also be used to discover or constrain slow dark
matter models with portals to the standard model. The question of direct detection of disk dark matter has been previously studied (Bruch et al. 2009; McCullough & Randall 2013; Fan et al. 2014; Reece & Roxlo 2016). In this chapter we explain how recoil experiments can constrain slow dark matter. We give brief survey of light dark matter experiments, and explain how the results are modified for slow dark matter.

6.2 Recoil Experiments: The Importance of Target Mass

As a function of center-of-mass scattering angle $\theta$ the momentum transfer in a WIMP-target scattering process can be shown to be given by

$$q = \mu v_X (1 - \cos \theta)$$  \hspace{1cm} (6.1)

where $\mu = m_X m_T / (m_X + m_T)$ is the WIMP-nucleus reduced mass, and $v_X$ is the WIMP’s velocity in the halo before the scattering. The maximum recoil energy of the target nucleus will therefore be

$$E_R = \frac{q^2}{2m_T} \leq \frac{2\mu^2 v_X^2}{2m_T}. \hspace{1cm} (6.2)$$
Figure 6.1: Target recoil energy compared to incoming WIMP kinetic energy. The recoil energy is always less than the incident energy and for $m_X \gg m_T$ is bounded by $2m_T v_X^2$.

This is always less than or equal to the incoming WIMP energy $E_X = m_X v_X^2 / 2$ and is shown in Figure 6.1. This has the limits:

$$E_R \leq \begin{cases} 
2v_X^2 \frac{m_X^2}{m_T} = E_X \left( \frac{m_X}{m_T} \right) & m_X \ll m_T \\
2m_T v_X^2 & m_X \gg m_T.
\end{cases} \quad (6.3)$$

Maximum energy deposition can occur when the target mass and WIMP mass are comparable. This will generally be near the WIMP mass where the recoil experiment is most sensitive.

The recoil rate will be given by

$$\text{Rate} = N_T \langle n_X \sigma_{XT} v_{\text{rel}} \rangle \quad (6.4)$$
where $N_T$ is the number of target particles, $n_X$ is the number density of WIMPs, $\sigma_{XT}$ is the WIMP-target cross-section, and $v_{\text{rel}}$ is the WIMP-target relative velocity. The expectation value is taken in a Maxwell-Boltzmann distribution with the appropriate temperature associated with the halo WIMPs. For a given WIMP mass $m_X$, $n_X$ will be given in terms of halo mass density $\rho_X$. The cross section will be an integral over the possible momentum transfers detectable by the apparatus:

$$\text{Rate} = N_T \left\langle \frac{\rho_X}{m_X} \int_{2m_Y E_\text{th}}^{4\mu^2 v^2} \frac{d\sigma_{XT}}{dq^2} v_{\text{rel}} \right\rangle$$

(6.5)

where $E_{\text{th}}$ is the threshold energy of the apparatus.

Clearly, we lose the ability probe the halo particles when $E_R < E_{\text{th}}$. For example, the LUX experiment uses liquid Xenon, with $m_T \simeq 122$ GeV, and has a detection threshold of $E_{\text{th}} \sim 2$ keV. This means that LUX cannot be used to detect anything lighter than

$$m_X \simeq \sqrt{\frac{m_T E_{\text{th}}}{2v_X^2}} \sim 10 \text{ GeV}$$

(6.6)

where a velocity of $v_X \sim 10^{-3}c$ was used. Since no WIMPs have been observed yet, it makes sense to look for experiments with both lower threshold energies and lower target masses in order to probe lower WIMP masses. Below, we will go through various recently proposed experiments that do just this. We will then discuss their detection prospects in the context of both halo dark matter as well as dissipative, disk-like dark matter. The implications would also
Figure 6.2: The latest results from the LUX experiment. For $m_X \sim 10$ GeV, sensitivity decreases very sharply due to the 2 keV threshold of the experiment. For high $m_X$, sensitivity decreases as the first power of $m_X$ due to the factor $n_X = \rho_X/m_X$.

be useful in the context of any slow sectors of dark matter. Figure 6.2 shows the most recent bounds from the LUX experiment (Akerib et al. 2017). Near $m_X \sim 10$ GeV, we can see where the threshold energy causes the experiment to lose sensitivity. For higher $m_X$, sensitivity only decreases with the first power of $m_X$ due to the lower number density $n_X$ implied by fixed $\rho_X$. 

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6.3 Slow Dark Matter

Here we describe the slow dark matter scenario in detail, including a careful analysis of the relative velocity distribution.

6.3.1 The Density of Slow Dark Matter

Dissipative dark matter models predict a fraction of dark matter in a disk-like distribution (Fan et al. 2013). Although the density of the dark halo is very constrained at

$$\rho_{\text{halo}} = (0.39 \pm 0.03) \cdot (1.2 \pm 0.2) \text{ GeV cm}^{-3}$$  \hspace{1cm} (6.7)

(Drees & Gerbier 2016, Ch. 26), the disk-like component can have a density an order of magnitude higher than this (Chapters 3, 4):

$$\rho_{\text{dark disk}} \simeq 0.1 M_\odot \text{ pc}^{-3} \simeq 3.8 \text{ GeV cm}^{-3}.$$  \hspace{1cm} (6.8)

6.3.2 Velocity distribution

One tell-tale signal of dark matter direct-detection is the annual modulation signal expected in the recoil rate. The plane of rotation of the Earth has a radius pointing roughly towards the Galactic center and is at a 60 degree angle from the Galactic plane. We will label the unit
vectors of these axes as

\[ \hat{b} = -\hat{r} \]  \hspace{1cm} (6.9)

\[ \hat{a} = \sin \theta \hat{z} + \cos \theta \hat{\phi} \]  \hspace{1cm} (6.10)

Where the \( r, \phi, z \) directions refer to the radial, azimuthal, and vertical directions in Galactic coordinates (see Figure 6.3).

With the measured Oort constant \( A - B = 29.45 \pm 0.15 \text{ km s}^{-1} \text{ kpc}^{-1} \) Reid & Brunthaler (2004) and Solar radius \( R_\odot = 8.33 \pm 0.35 \text{ kpc} \) Gillessen et al. (2009), we find the circular velocity at the Solar radius to be

\[ V_c = 245 \pm 10 \text{ km s}^{-1}. \]  \hspace{1cm} (6.11)

This value, together with Solar peculiar velocity Schönrich et al. (2010)

\[ (U, V, W) = (11.1 \pm 0.74, 12.24 \pm 0.47, 7.25 \pm 0.37) \text{ km s}^{-1}, \]  \hspace{1cm} (6.12)
agrees with the total measured Solar azimuthal velocity \( V_\odot = 255 \pm 5 \, \text{km s}^{-1} \) Reid et al. (2014). On the other hand, this is higher than the IAU value of \( V_c = 220 \, \text{km s}^{-1} \) (I.A.U. 1985; Catena & Ullio 2010). The measurement of Bovy et al. (2012), relying on global rather than local measurements however, was slightly higher, at \( V_\odot = 26 \pm 3 \, \text{km s}^{-1} \), which, together with the total velocity measurement \( V_\odot = 255 \pm 5 \, \text{km s}^{-1} \) of Reid et al. (2014), gives \( V_c = 229 \pm 6 \, \text{km s}^{-1} \), in better agreement with the IAU value. In any case, for the purpose of this chapter, we will use the first value of Schönrich et al. (2010). The velocity of the Earth is therefore

\[
\vec{v}_E = (V_c + V_\odot) \hat{\phi} + v_E (\hat{r} \cos t + \sin \theta \hat{z} + \cos \theta \hat{\phi}) \sin t
\]  

(6.13)

where \( t \) is the time in radians relative to when the Sun is in Sagittarius, near Dec 10 (horoscopes.com). For corotating dark matter, we will take the dark matter to have velocity

\[
\vec{u}_{DM} = \eta V_c \hat{\phi} + \vec{u}(\sigma)
\]  

(6.14)

Where we have defined the corotation parameter \( \eta \), equal to zero for halo dark matter or 1 for corotating dark matter. We have also defined a random vector \( \vec{u}(\sigma) \) with magnitude a Gaussian random variable with width \( \sigma \) and direction a uniform random variable over the 2-sphere. By
equating the total energy, we must have

$$\eta^2 V_c^2 + \sigma(\eta)^2 = V_c^2 + \sigma_0^2$$

(6.15)

where $\sigma_0$ is the velocity dispersion of the stars in the Galactic disk, on the order 5 km s$^{-1}$. The relative velocity between the Earth and the dark matter will therefore be

$$\vec{v}_{rel} = (1 - \eta) V_c + V_\odot \hat{\phi} - U_\odot \hat{r} + W_\odot \hat{z} + v_E (-\hat{r} \sin t + (\sin \theta \hat{z} + \cos \theta \hat{\phi}) \cos t) + \vec{u}(\sigma)$$

(6.16)

$$= \begin{pmatrix} -U_\odot & -v_E \sin t & +u_x(\sigma) \\ (1 - \eta) V_c + V_\odot & +v_E \cos \theta \cos t & +u_y(\sigma) \\ W_\odot & +v_E \sin \theta \cos t & +u_z(\sigma) \end{pmatrix}.$$  

(6.17)

Using the Maxwell-Boltzmann distribution for the vector $\vec{u}(\sigma)$:

$$f(\vec{u}(\sigma)) = \frac{1}{(\pi \sigma^2)^{3/2}} \exp \left( -\frac{\vec{u}^2}{\sigma^2} \right),$$

(6.18)
we can derive the probability distribution for the relative velocity $\vec{v} = (u, v, w)$:

$$f(\vec{v})d^3v \simeq \frac{du \, dv \, dw}{(\pi((1-\eta^2)V_c^2 + \sigma_0^2))^{3/2}} \exp\left(-\frac{(u + U_\odot + v \sin t)^2}{(1-\eta^2)V_c^2 + \sigma_0^2}\right) \times \exp\left(-\frac{(v + (1-\eta)V_c + \eta_0 + v \cos \theta \cos t)^2}{(1-\eta^2)V_c^2 + \sigma_0^2}\right) \times \exp\left(-\frac{(w + W_\odot + v \sin \theta \cos t)^2}{(1-\eta^2)V_c^2 + \sigma_0^2}\right).$$

(6.19)

For complete corotation ($\eta = 1$), we have

$$f(\vec{v}) \simeq \frac{1}{(\pi\sigma_0^2)^{3/2}} \exp\left(-\frac{(u + U_\odot + v \sin t)^2}{\sigma_0^2}\right) \times \exp\left(-\frac{(v + V_\odot + v \cos \theta \cos t)^2}{\sigma_0^2}\right) \times \exp\left(-\frac{(w + W_\odot + v \sin \theta \cos t)^2}{\sigma_0^2}\right).$$

(6.20)

An approximate expression for the relative speed is:

$$\langle v \rangle \simeq \left( (U_\odot + v \sin t)^2 + (V_\odot + v \cos \theta \cos t)^2 + (W_\odot + v \sin \theta \cos t)^2 + \frac{4}{\pi} \sigma_0^2 \right)^{1/2}$$

(6.21)

$$\simeq \left( (12.2 + 13.5 \cos t)^2 + (11.1 + 29.8 \sin t)^2 + (7.25 + 26.6 \cos t)^2 + 1.27 \cdot 5^2 \right)^{1/2} \text{km s}^{-1}.$$

(6.22)

The $4/\pi$ comes from the fact that $\pi^{-3/2} \int d^3v \, v \exp(-v^2) = 4/\pi$. An exact plot of $\langle v(t) \rangle$ is shown in Figure 6.4. We see that the relative speed can be as low as 17 km s$^{-1}$ near the
Figure 6.4: Dark matter-earth relative speed vs. time from Dec. 10.

beginning of July. On the other hand, near the end of January it has the value $47\, \text{km s}^{-1}$. So the relative velocity will always be between $0.5 - 2 \times 10^{-4}c$. Interestingly, it has an average value of about $30\, \text{km s}^{-1}$.

6.4 Nuclear Recoil for Slow Dark Matter

In this section, we discuss the potential results of nuclear recoil experiments in the context of slow dark matter. Since, as explained in Section 6.3, slow dark matter has velocities an order of magnitude lower than halo dark matter, it is of interest to know how this may affect direct detection results.

For halo dark matter with speeds of $10^{-3}c$, nuclear recoil experiments will give recoils with
energies of

\[ E_R \leq 2m_T v_X^2 \sim 100 \text{ keV} \quad (6.23) \]

for heavy dark matter particles, for xenon targets. However, when the dark matter mass \( m_X \) becomes lower than the target mass, the recoil energy is instead given by

\[ E_R = 2v_X^2 \left( \frac{m_X}{m_T} \right)^2 \sim 1 \text{ keV} \left( \frac{m_X}{10 \text{ GeV}} \right) \quad (6.24) \]

which means that the recoil energy will dip below the keV threshold for dark masses below 10 GeV. Although for helium nuclei the recoil energies are only a few keV, they have the advantage that this is maintained down to dark matter masses of a few GeV.

For slow dark matter, however, the velocities are an order of magnitude lower, as explained in Section 6.3. That means that for heavy dark matter particles, we have

\[ E_R \leq 2m_T v_X^2 \sim 1 \text{ keV}, \quad (6.25) \]

which can be detected for dark matter particles in the high tail of the Maxwell-Boltzmann distribution. On the other hand, for dark matter particles much lighter than the nuclear target mass of \( m_T \simeq 123 \text{ GeV} \), the recoils will not be detectible. If we instead use helium nuclei as the targets, the recoil energies will be of order 10 eV and will be indetectible. The detectible mass ranges for xenon targets are shown in the solid and dotted red curves in Figure 6.5.
Figure 6.5: Recoil energies are shown for both halo dark matter (solid lines) and disk dark matter (dotted lines). Only the mass ranges that can give rise to detectable recoil energies are shown.

Another interesting point here is that a recoil experiment with threshold $E_{\text{th}}$ will have a threshold velocity for incoming dark matter particles below which no recoil can be detected. For heavy dark matter particles, this is simply given by Equation 6.25:

$$v_{\text{threshold}} = \sqrt{\frac{E_{\text{th}}}{2m_X}} \simeq 27 \text{ km s}^{-1}. \quad (6.26)$$

For a threshold energy of 2 keV. Interestingly, this is very near the average relative velocity of Equation 6.21 and Figure 6.4. Figure 6.6 shows how the rate varies over the course of the year. We can see that there is no signal in the summer months because the recoil energy is below the threshold. On the other hand, if the threshold is lowered by an order of magnitude, the annual modulation signal can be measured the whole year. It is also interesting that the rates in Figure 6.6 are similar to what would be observed for nuclear recoil. Although the recoil rate
Figure 6.6: The recoil rate, in recoils per day, per ton of target material per zeptobarn of cross section, are shown. The annual modulation in the recoil rate can clearly be seen.

6.4 scales like $v_{\text{rel}}$, which is reduced by an order of magnitude in the disk dark matter scenario, it also scales like $n_X$, which, as explained in Section 6.3.1, is in turn enhanced by an order of magnitude. The resulting exclusion plot is shown in Figure 6.7. Here, we can see that while the sensitivities are similar to those of halo dark matter because of the density enhancement, the mass threshold is also increased by an order of magnitude because of the lower relative velocity according to Equation 6.6.

6.5 Experiments for Light Dark Matter

In this section we briefly review recent light dark matter experiments that have been proposed.
6.5.1 Light Dark Matter: Electron Recoil

It was shown in Essig et al. (2012) that the XENON10 experiment could also be used to constrain WIMPs in the MeV-GeV range from the fact that no significant number of electron recoils were observed.

Here, for $m_X \gg m_e$, we have

$$E_R = 2m_e v_X^2$$

(6.27)

For halo velocities of order $10^{-3}c$, this implies energy deposits of order 1 eV. Many semiconductors, such as Ge, have band gaps near this value (Hochberg et al. 2016a). Recoil energies of this size will be produced as long as $m_X \gg m_e$. For dark matter particles lighter than $m_e$,
however, electron recoil experiments will not be useful.

For ionization in xenon atoms, however, one might expect that these small energy deposits are not enough to overcome their ionization energy of 12 eV. However, reverse-Sommerfeld enhancement actually allows the process to happen and bounds can even be placed in this way (Essig et al. 2012; Essig et al. 2012; Essig et al. 2017b)

For disk dark matter, however, the recoil energy will again be two orders of magnitude lower, roughly 10 meV. This is far below any semiconductor gap or atomic ionization energy and is therefore completely indetectible. Electron recoil is therefore not a viable detection method for disk dark matter. The solid green curve for halo dark matter is shown in Figure 6.5.

6.5.2 Ultra-Light Dark Matter: Phonon Excitation

More recently, some experiments have been designed to probe even lighter forms of dark matter. These typically rely on low-energy phonon excitations.

Superconductors

One way to bypass the limit (6.27) is to use the velocity of the target particles (Hochberg et al. 2016b,a).

As explained in Section 6.5.1, electron recoil in semiconductors is only sensitive to DM masses down to an MeV. To access the keV range, experiments with lower sensitivity must be used. It would seem that should be unachievable with electrons. Fortunately, in superconduc-
tors, it is possible to make use of the intrinsic velocity of the electrons to achieve large energy deposits. Consider the equation for the deposited energy on an electron (Hochberg et al. 2016a):

\[ E_D \simeq \frac{q^2}{2m_e} + \vec{q} \cdot \vec{v}_T + \delta \]  

(6.28)

where \( \delta \) is the energy gap for the system and \( v_T \) is the velocity of the target electron. Although for keV dark matter the first term is of order \( \mu \text{eV} \), the second term, in a metal like aluminum where the Fermi velocity can be of order \( 10^{-2}c \), will be of order \( 10^{-2} \text{eV} \), and will scale roughly linearly with \( m_X \). That is, for dark matter at the Ly-\( \alpha \) limit, \( \sim 1 \text{ keV} \), the deposited energy will be of the order 10 meV. Superconductivity happens when electrons near the Fermi energy form Cooper pairs. In aluminum, the binding energy of these cooper pairs is roughly 0.6 meV. As long as the energy deposited in the recoil is much larger than this, we can treat the dark matter particles as scattering off individual electrons. The excited electron generally produces an athermal phonon (Hochberg et al. 2016a). Since Transition Edge Sensors are sensitive down to energies of \( \mathcal{O}(10^2) \) meV or lower (Hochberg et al. 2016a), recoil near the Ly-\( \alpha \) limit can potentially be detected.

For halo dark matter, the maximum deposited energy will be

\[ E_D = 2m_e v_X v_F \sim 10 \text{eV}. \]  

(6.29)

This will be true down to dark matter masses near \( m_e \). Below this, the recoil energy scales more
like $m_X$ and the recoil energies can be as low as a few 100 meV, and this is within the sensitivity of current detectors. In fact, while energy deposits below a few hundred meV typically produce a phonon, above this, the electron behaves as free and can break other Cooper pairs (Kurakado 1982). The author is therefore uncertain about the detection prospect for these recoils. The detection range for halo dark matter using superconductors might therefore be for mass values lower than about 100 keV.

For disk dark matter, however, the recoil energies are an order of magnitude lower, on the order eV for heavy $m_X$ and order 1 meV for $m_X$ near 1 keV. Hochberg et al. (2016a) argued that although current Transition Edge Sensors are only sensitive down to energies of $O(10^2)$ meV, it should be possible to lower this to $O(1)$ 1 meV or lower. These experiments could therefore be used to probe disk dark matter with masses lower than an MeV. In the context of DDDM (Fan et al. 2013) this may correspond to the mass of the $C$ particle.

**Liquid Helium - Double Phonon**

In traditional recoil experiments, the recoil energy is kinematically limited by Equation 6.3. For example, keV-mass halo DM can deposit at most $10^{-9}$ eV on a target nucleus like helium (Schutz & Zurek 2016). The dark matter particle cannot deposit all of its energy into the detector because the final energy is fixed by four-momentum conservation: it cannot both deposit momentum $q = m_X v_X$ and energy $q^2/2m_X$ because the recoil energy is fixed to be $q^2/2m_Y$. That is, we have two quantities to deposit on the target particle but only one degree of freedom to put them in. On the other hand, if a double-recoil is induced, we now have two degrees of
Figure 6.8: 1-body vs 2-body recoils

\[
\begin{align*}
(E_X, \vec{p}_X) & \quad \rightarrow \quad (0, \vec{0}) \\
(0, \vec{0}) & \quad \rightarrow \quad (E_X, \vec{p}_X)
\end{align*}
\]

freedom for the two quantities \( p_X \) and \( E_X \) (see Figure 6.8). Although this process is phase-space suppressed, a reasonable recoil rate can still be observed (Schutz & Zurek 2016). In this case, the phonon energies are

\[
E_\phi = \frac{1}{4} m_X v_X^2. \quad (6.30)
\]

The sensitivities for these experiments are over a small range, generally between \( \sim 0.5 - 1.5 \text{ meV} \) (Schutz & Zurek 2016). Above this, there is no phonon scattering, only ordinary He nuclear recoil. Double-phonon scattering thus probes DM masses between roughly 1-10 keV for halo dark matter.

For disk dark matter, the energies will be two orders of magnitude lower, and the mass ranges probed will therefore be roughly between 0.1-1 MeV. This is again in the range proposed for the \( C \) particle in DDDM.
6.6 Conclusions

In conclusion, we find that there is an interesting parameter space in dissipative dark matter models and other slow dark matter models. Nuclear recoil has the potential to detect dark matter particles down to roughly 100 GeV. Experiments for light dark matter can also be used. Although electron recoil experiments are not sensitive enough to detect these particles, experiments that rely on phonon scattering such as superconductor targets or liquid helium can detect dark matter particles down to the keV-MeV range. In the context of DDDM, nuclear recoil experiments can be used to detect the heavy $X$ particle while phonon scattering experiments could potentially be used to detect the lighter $C$ particle. In the latter case, however, it would be important to take into account the fact that the $C$-particle may be part of an $XC$ bound state. Reverse-Sommerfeld enhancement and the $XC$ binding energy are expected to play a role.
Conclusions and Future Directions

We have shown that dissipative dark matter is a viable model for a subsector of dark matter. Despite the fact that these models generally predict an enhanced local density, these are not constrained by most measurements of the local dark matter density because these generally apply only to the dark matter halo. Some studies of the Milky Way’s local kinematics seems to place tight constraints on the possible amount of local dark matter. However, these constraints
are found to be drastically weakened when non-equilibrium features in the tracer populations are taken into account. A dark disk with parameters typical of dissipative dark matter models ($\Sigma_D \simeq 10 \, M_\odot \, \text{pc}^{-2}$, $\sigma_D \simeq 4 \, \text{km s}^{-1}$) is actually consistent with the upper bound of these kinematic studies. When studying the distribution of the Milky Way’s local interstellar gas, it was found that a dark disk may be needed in order to balance the magnetic pressure of the gas’s magnetic fields. Also, if the apparent periodicity in the Earth’s crater record is to be explained by the Sun’s oscillations about the Galactic midplane inducing a comet flux from the Sun’s Oort cloud, a dark disk with parameters similar to those of DDDM would be required. Direct detections prospects are promising using nuclear recoil experiments. In particular, these would target the heavy $X$ particle. Experiments aimed at lighter dark matter could also target the lighter $C$ particle.

With the upcoming release of data from the Gaia satellite, we hope to have a much better idea of the distribution of dark matter in our Galaxy. Non-equilibrium features in the stellar kinematics will be important. Although equilibrium techniques have been used in the past, it would be of interest to see if machine learning techniques could shed light on deciphering these kinematics.
8.1 Appendix - Statistics

In order to compare the stellar kinematics to a given dark disk model, we define a $\chi^2$-type statistic $X$ that measures the distance between the predicted and observed densities:

$$X[\Phi] \equiv \int_{z_{\min}}^{z_{\max}} dz \frac{\left| \rho_{\text{obs}}(z) - \rho_{f,\text{obs}}[\Phi(z)] \right|^2}{\Delta^2_\rho(z)}$$  (8.1)

where

$$\rho_{f,\text{obs}}[\Phi(z)] \equiv \rho_f(z) = \rho(0) \int_{-\infty}^{\infty} dw f_{z=0} \left( \sqrt{w^2 + 2\Phi(z)} \right)$$  (8.2)

as defined in 3.41, $\Delta_\rho(z)$ represents the uncertainty in $\rho(z)$ at the position $z$, and where where $z_{\min}$, $z_{\max}$ are given by the completeness limits, $Z_\odot \pm 170$ pc for A stars and $Z_\odot - 92$ pc, $Z_\odot + 40$ pc for F stars. As explained in Section 3.3.1, for the F stars, we do not include any data higher than 40 pc above the Sun.

The observed densities $\rho_{\text{obs}}(z) \equiv \rho_{A,F}(z)$ were constructed using the kernel histogram technique. That is, we represented each star as a Gaussian with unit area in position and velocity space, and then obtained total distributions by summing these Gaussians. The uncertainties in
position and velocity of the individual stars, $\Delta z$ and $\Delta w$, do not however fully account for the error. The chief source of error on this sparse distribution is the likelihood that stars will actually fill in the purported distribution (i.e. Poisson error).

We estimate the latter in the following way. If, in the case of $\rho(z)$, we assume the data are arranged in ‘bins’ of half-width $\Delta z$, then the density $\rho(z)$ should be

$$\rho(z) = \frac{N(z)}{2A\Delta z} \quad (8.3)$$

where $N(z)$ is the number of stars in each ‘bin’ and $A$ is the cross-sectional area of the bin at height $z$. Both $\Delta z$ and $N(z)$ are unknown. However, we know that the fluctuations in $\rho(z)$ should be given, according to Poisson statistics, by

$$\Delta \rho(z) = \sqrt{\frac{N(z)}{2A\Delta z}} \quad (8.4)$$

Eliminating $N(z)$, we have

$$\Delta z = \frac{\rho}{2A\Delta \rho} \quad (8.5)$$

Since the determination of $\rho(z)$ and $\Delta \rho(z)$ themselves depend on the value of $\Delta z$, Equation 8.5 should be used recursively. Also, since the derivation of Equation 8.5 was heuristic, we estimate this computation of $\Delta z$ to be correct only to within a factor of two or so. In this way, we obtain values of $\Delta z$ in the range 6-12 pc. To this we must add the $\Delta z$ arising from measurement error. This is derived by propagating the error in parallax available in the Hipparcos catalogue and
grows with $z^2$. Summing these two contributions in quadrature gives values of $\Delta z$ between $\Delta z \simeq 7 \text{ pc}$ near $z = 0$ and $\Delta z \simeq 20 \text{ pc}$ near the extremeties in the case of A stars, and $\Delta z \simeq 7 - 13 \text{ pc}$ for the F stars. On the other hand, standard kernel density estimation techniques (which assume a constant kernel width and unimodal distribution) suggest widths closer to (Silverman 1986):

$$\Delta z \simeq \frac{\text{stddev}\{z_i\}}{N^{1/5}} \simeq 18 \text{ pc}$$

(8.6)

and

$$\Delta w \simeq \frac{\text{stddev}\{w_i\}}{N^{1/5}} \simeq 2.1 \text{ pc},$$

(8.7)

for A stars. We find similar values for the F stars. We find that the final result is roughly independent of this width for the range $\Delta z \simeq 6 - 30 \text{ pc}$. The weakest bound is obtained for $\Delta z \simeq 18 \text{ pc}$, before the width of the kernels becomes so large that it biases the distribution and begins to remove the signature of any potential structure near $z = 0$.

In order to construct the relevant uncertainty $\Delta \rho(z)$, we considered the density distributions $\rho_{\text{obs}}(z)$ obtained by repeatedly sampling a fraction $q < 1$ of the stars in the respective samples. We did this by either including or not including each star in the original data set with probability $q$. The variance between these distributions thus gives the spread $\Delta \rho^{(q)}(z)$ at every point $z$. In terms of $\Delta \rho^{(q=1/2)}(z)$ obtained from repeatedly sampling half the stars in the sample, we can obtain the uncertainty $\Delta \rho(z)$ on the original sample as (Equation 8.41)

$$\Delta \rho_{\text{obs}}(z) = 2 \Delta \rho^{(q=1/2)}(z)$$

(8.8)
which can be derived using the binomial distribution with probability $q$ (see Appendix 8.2). In numerical simulations, we find this factor of 2 is closer to 1.97. Although we do not know the source of this discrepancy, we note that its effect is small compared to the remaining errors in the analysis. The errors $\Delta_{\rho_f,\text{obs}}$ were constructed in a similar way by sampling velocities from the data set, giving (Equation 8.47)

$$
\Delta_{\rho_f,\text{obs}}(z) = \Delta_{\rho_f}^{(q=1/2)}(z).
$$

(8.9)

The errors were then added in quadrature:

$$
\Delta^2\rho(z) = \Delta^2\rho_{\text{obs}}(z) + \Delta^2\rho_{f,\text{obs}}(z).
$$

(8.10)

Uncertainties in the Galactic potential, obtained from Sections 4.4 in the case of interstellar gas and from McKee et al. in the case of stellar components were found to be negligible compared to the errors above.

In order to associate a value of $X[\Phi]$ with a probability, we constructed a probability distribution for the values of $X[\Phi]$ obtained from fluctuations of the density and velocity distributions. To do this, we note that, given the true potential $\Phi_{\text{true}}$ of the Galaxy, the value

$$
X[\Phi_{\text{true}}] = \int dz \frac{\rho_{\text{obs}} - \rho_{f,\text{obs}}[\Phi_{\text{true}}]^2}{\Delta^2\rho}
$$

(8.11)
is itself a fluctuation with respect to its equilibrium value

\[
X_{eq}[\Phi_{true}] = \int dz \left| \rho_{eq} - \rho_{f,eq}[\Phi_{true}] \right|^2 \Delta_\rho^2 = 0,
\]  

(8.12)
as \rho_{obs} and \( f_{obs} \) are assumed to be fluctuations of their equilibrium values \( \rho_{eq}, f_{eq} \). We can similarly define \( \Phi_{obs} \) so that

\[
\rho_{f,obs}[\Phi_{obs}] = \rho_{obs}
\]  

(8.13)
and \( X[\Phi_{obs}] = 0 \). We then compute fluctuations in \( X[\Phi_{obs}] \) by sampling fluctuations in \( \rho_{obs} \) and \( f_{obs} \). That is, each time \( k \) that we sample a set \{ \( z', w' \) \} from the parent populations \{ \( z, w \) \}, this gives an observed density \( \rho_{obs,k}(z) \) and \( f_{obs,k}(w) \). We therefore have

\[
X_k[\Phi_{obs}] = \int dz \left| \rho_{obs,k} - \rho_{f,obs,k}[\Phi_{obs}] \right|^2 \Delta_\rho^2.
\]  

(8.14)
Thus, by repeatedly sampling values of \( X_k \), we obtain the distribution \( P(X) \). We can then use the distribution \( P(X) \) to associate probabilities with every point in parameter space by computing the cumulative \( X \) distribution:

\[
C(X) = \int_0^X dX' P(X').
\]

(8.15)
\( C(X[\Phi]) \) represents the probability that the fluctuations in \( \rho_{obs}, \rho_{f,obs}[\Phi] \) with respect to the
equilibrium distributions $\rho_{\text{eq}}, f_{\text{eq}}$ assuming that $\Phi_{\text{true}} = \Phi$ could result in a value $X \leq X[\Phi]$. The probability $1 - C(X[\Phi])$ therefore represents the probability that $X \geq X[\Phi]$. In Bayesian terms, this is the probability $p(X|\Phi)$. Combining the independent results from the A and F stars, we have the probability

$$p(X_A, X_F|\Phi) = (1 - C(X_A[\Phi])) \times (1 - C(X_F[\Phi]))$$  \hspace{1cm} (8.16)

We reject any model for which $p(X_A, X_F|\Phi) < 0.05$. This will be our criterion for “exclusion at 95% confidence”. Note that this does not represent the probability of the model itself, but rather the probability of the results given the model.

As an aside, probability densities in model space $\Sigma_D, h_D$ can also be computed by fitting the $P(X)$ distribution to a $\chi^2$ distribution, as explained in Appendix 8.3. However, since this fit involves two slightly degenerate parameters, as explained therein, these probability densities are only approximate and are included in our results for illustration purposes only. We will see that, even so, they match qualitatively what is suggested by the absolute probabilities.

### 8.2 Appendix - Constructing Errors

In Appendix 8.1, we claimed that we could construct the errors on the total distributions $\Delta_{\rho}(z)$ or $\Delta_f(w)$ by repeatedly sampling a fraction $q$ of the stars. We now proceed to derive the relationship between these values and the values $\Delta_{\rho}^{(q)}(z), \Delta_f^{(q)}(w)$ obtained by sampling.
We model the positions and velocities \( \{ z_i, w_i \} \) of the stars as being drawn from some 'parent' probability distribution \( f(z, w) \). If we sample a very small fraction \( q \ll 1 \) of the stars, we expect the randomness in the resulting set \( \{ z'_i, w'_i \} \) relative to the set \( z_i, w_i \) to mimic the randomness of the set \( \{ z_i, w_i \} \) relative to the parent probability distribution \( f(z, w) \), with the the errors reduced by a Poisson factor \( \sqrt{q} \). We can therefore estimate the randomness in the parent set \( \{ z_i, w_i \} \) by repeatedly sampling a small fraction \( q \) from this set and dividing the standard deviation \( \Delta(q) \) in the resulting set by \( \sqrt{q} \):

\[
\Delta = \lim_{q \to 0} \frac{\Delta(q)}{\sqrt{q}}.
\] (8.17)

However, since the data set \( \{ z_i, w_i \} \) is of finite size, for \( q \ll 1 \) we cannot generate a large enough child set \( \{ z'_i, w'_i \} \) to be able to obtain reliable statistics. On the other hand, if we use a larger sampling fraction \( q \), Equation 8.17 will no longer hold because different samplings \( i' \) will repeatedly contain the same values. The trick will therefore be to relate \( \lim_{q \to 0} \Delta(q)/\sqrt{q} \) to a reference value \( \Delta(q_0) \) for some reference \( q_0 = \mathcal{O}(1) \).

In order to derive the dependence of \( \Delta(q) \) on \( q \), consider binning the data \( \{ z_i, w_i \} \) into bins in \( z \) and \( w \) space. For concreteness, let us consider only \( z \) space for now. Let \( N_k \) be the number of stars in bin \( k \). Each star in bin \( k \) has a probability \( q \) of being sampled, and a probability \( 1 - q \)
of not being sampled. The distribution of outcomes is therefore binomial:

\[
1 = (q + 1 - q)^{N_k}
\]

\[
= \sum_{r=0}^{N_k} \binom{N_k}{r} q^r (1 - q)^{N_k - r},
\]

\[
\text{(8.19)}
\]

in the sense that the probability of sampling \( r \) stars in the bin \( k \) is given by

\[
P_{k,r} = \binom{N_k}{r} q^r (1 - q)^{N_k - r}
\]

\[
= (1 - q)^{N_k} \left( \binom{N_k}{r} \left( \frac{q}{1 - q} \right)^r \right)
\]

\[
\equiv (1 - q)^{N_k} \left( \binom{N_k}{r} \tilde{q}^r \right)
\]

\[
\text{(8.22)}
\]

where we have defined \( \tilde{q} \equiv q/(1 - q) \). The average number of stars sampled in bin \( k \) will
therefore be

\[
\langle r_k \rangle = \sum_{r=1}^{N_k} p_{k,r} r \\
= (1 - q)^{N_k} \sum_{r=1}^{N_k} \left( \frac{N_k}{r} \right) \tilde{q}^r r \\
= (1 - q)^{N_k} \frac{d}{dq} \sum_{r=1}^{N_k} \left( \frac{N_k}{r} \right) \tilde{q}^r \\
= (1 - q)^{N_k} \frac{d}{dq} (1 + \tilde{q})^{N_k} \\
= (1 + \tilde{q})^{-N_k} \tilde{q} \frac{d}{dq} (1 + \tilde{q})^{N_k} \\
= (1 + \tilde{q})^{-N_k} \tilde{q} N_k (1 + \tilde{q})^{N_k - 1} \\
= \frac{\tilde{q}}{1 + \tilde{q}} N_k \\
= q N_k
\]

which just says that if we sample a fraction \( q \) of the total number of stars we expect to sample that same fraction \( q \) of the stars in each bin. We can similarly calculate

\[
\langle r_k^2 \rangle = (1 + \tilde{q})^{-N_k} \left( \frac{\tilde{q}}{d} \frac{d}{dq} \right)^2 (1 + \tilde{q})^{N_k - 1} \\
= (1 + \tilde{q})^{-N_k} \frac{\tilde{q}}{d} \frac{d}{dq} \tilde{q} N_k (1 + \tilde{q})^{N_k - 1} \\
= q N_k + q^2 N_k (N_k - 1)
\]
which gives

$$\Delta r_k^2 \equiv \langle r_k^2 \rangle - \langle r_k \rangle^2 = q(1-q)N_k. \quad (8.34)$$

Since we expect $\Delta^{(q)} \propto \Delta r$, we thus find that

$$\Delta^{(g)}(q) = C(z) \sqrt{q(1-q)} \quad (8.35)$$

for some function $C(z)$. We therefore find, in the limit $q \to 0$,

$$\Delta_\rho(z) = \lim_{q \to 0} \frac{\Delta^{(g)}}{\sqrt{q}} = \lim_{q \to 0} C(z) \sqrt{1-q} = C(z). \quad (8.36)$$

We therefore have

$$\Delta^{(g)}(q) = \Delta_\rho(z) \sqrt{q(1-q)}. \quad (8.39)$$

Which we can use to solve for $\Delta_\rho(z)$ by numerically calculating $\Delta^{(q_0)}(z)$ at some reference value $q_0$:

$$\Delta_\rho(z) = \frac{\Delta^{(q_0)}(z)}{\sqrt{q_0(1-q_0)}}. \quad (8.40)$$
For the reference value \( q_0 = 1/2 \) (sampling half the stars), we therefore have

\[
\Delta_\rho(z) = 2 \Delta_\rho^{(1/2)}(z).
\]  
(8.41)

When simulating this sampling technique with a Hipparcos sample containing more stars than our data set, and extrapolating to \( q = 0 \), we found that \( \Delta_\rho(z) \) was more correctly given by \( 1.97 \Delta_\rho^{(1/2)}(z) \). We are unsure of the reason for this discrepancy. At any rate, the effect of using the factor of 1.97 instead of 2 is small compared to the remaining errors in the analysis.

For \( f(w) \), the story is slightly different because of the normalization \( \int dw f(w) = 1 \). Although \( \Delta r_k \) (where here \( k \) represents the \( k^{th} \) bin in \( w \)-space) is still given by \( \sqrt{q(1-q)N_k} \), the quantity corresponding to \( f(w) \) here is \( N_k / \sum_{k'} N_{k'} \) in the parent set or \( r_k / \sum_{k'} r_{k'} \) in the sample set. Since for a large number of bins,

\[
\sum_k r_k \approx \sum_k \langle r_k \rangle = q \sum_{k'} N_{k'},
\]  
(8.42)

we have

\[
\Delta^{(q)} f_k = \frac{\sqrt{q(1-q)N_k}}{q \sum_{k'} N_{k'}}.
\]  
(8.43)

Similarly to the case of \( \Delta(z) \), we expect, for small \( q \), that

\[
\lim_{q \to 0} \Delta^{(q)} f_k = \lim_{q \to 0} \frac{\sqrt{qN_k}}{q \sum_{k'} N_{k'}}.
\]  
(8.44)
We therefore find that

\[ \Delta f(w) = \sqrt{\frac{q}{1 - q}} \Delta_f^{(q)}(w). \]  

(8.45)

Since by Equation 3.41, \( \rho_f(z) \sim \int f \) scales linearly with \( f(w) \), we therefore have the same relationship for \( \rho_f(z) \):

\[ \Delta \rho_f(z) = \sqrt{\frac{q}{1 - q}} \Delta_{\rho_f}^{(q)}(z) \]  

(8.46)

and, for \( q = 1/2 \),

\[ \Delta \rho_f(z) = \Delta_{\rho_f}^{(1/2)}(z). \]  

(8.47)

### 8.3 Appendix - Probability Densities

To compute approximate probability densities, we can define a distance between expected and observed density

\[ X_{\text{model}}^2 = \int_0^{z_{\text{max}}} \frac{dz}{\Delta z} \left| \frac{\rho_{\text{model}}(z) - \rho_{\text{obs}}(z)}{\Delta \rho(z)} \right|^2 \equiv \frac{X_{\text{model}}}{\Delta z}, \]  

(8.48)
where $\chi^2$ is defined in the usual way as

$$\chi^2 = \sum_{n=1}^{k} \frac{(x_n - \langle x_n \rangle)^2}{\Delta^2_n},$$  \hspace{1cm} (8.49)$$

with $k$ the number of relevant degrees of freedom, and with $x_n$ representing the degrees of freedom. The probability density in model space would then be proportional to

$$p(\chi^2) \sim \exp \left( -\frac{\chi^2}{2} \right).$$  \hspace{1cm} (8.50)$$

However, we do not know the proportionality factor $\Delta z$, since we do not know the number of relevant degrees of freedom for $\Phi(z)$. Although we know the number of points we are using, the correlation between nearby points reduces the number of relevant degrees of freedom. This factor is important because when in the exponential in Equation 8.50, it will affect the sharpness of the probability curve. A simple way to determine the degree of freedom length $\Delta z$ as well as the number $k$ of relevant degrees of freedom is to fit the probability distribution of $P(X/\Delta z)$ to a $\chi^2$ distribution:

$$P(\chi^2) = \frac{1}{2^k \Gamma \left( \frac{k}{2} \right)} \chi^{\frac{k}{2} - 1} e^{-\chi^2/2}. \hspace{1cm} (8.51)$$

The reason this method is only approximate is that there is degeneracy between the two parameters we are fitting, $k$ and $1/\Delta z$. To see this, note that the position of the peak of the $\chi^2$ distribution grows linearly with $k$. Clearly, the position of the peak of $P(X/\Delta z)$ also grows.
linearly with $1/\Delta z$. Although other aspects of $P(\chi^2)$ also depend on $k$, this degeneracy still persists and makes it difficult to fit the distributions. While it does not solve this degeneracy problem, it will prove slightly simpler to fit the cumulative distribution $C(X) = \int_0^X dX' P(X')$, given by the incomplete Gamma function:

$$C(\chi^2) = \frac{\Gamma\left(\frac{k}{2}, \chi^2\right)}{\Gamma\left(\frac{k}{2}\right)}, \quad (8.52)$$

Another factor that complicates this is that $\Phi(z)$ is a functional depending on all the values of $f(w)$. What this implies is that although the errors on $f(w)$ and $\rho(z)$ are approximately Gaussian, the errors on $\Phi(z)$ will not be. Figure 8.1 below shows the $X$ distributions for A and F stars with best fit $\Delta z$ and $k = 12$ and 8 (respectively) degrees of freedom. However, because of the degeneracy between $k$ and $1/\Delta z$, the best fit values are very approximate and the derived probability density $p(\Sigma_D)$ should therefore only be regarded as qualitative.

### 8.4 Appendix - Non-Equilibrium Method

As explained in Section 3.5.3, we expect that the HF relation will hold for long-time averages. We will demonstrate that

$$\overline{\rho(z)\rho(0)^{-1}} = \int dw \overline{f_{z=0}\Phi}(\sqrt{w^2 + 2\Phi(z)})$$
Figure 8.1: Probability density for $X$ computed from A and F stars by sampling using statistical procedure defined above. Superimposed $\chi^2$ distributions with 12 and 8 relevant degrees of freedom, respectively.

where the $\langle \cdot \rangle$ represents the time average under evolution in the potential $\Phi(z)$. However, if this potential $\Phi$ is the same potential as under the square root $\sqrt{w^2 + \Phi(z)}$, then 8.53 will be satisfied trivially for any initial conditions. This can be demonstrated by taking the time-dependent density and in-plane velocity distributions to be:

$$
\rho(z, t) = \frac{1}{A_0} \sum_i \delta(z - z_i(t)) \quad (8.53)
$$

$$
\rho_{z=0}(w, t) = \frac{\sum_i \delta(w - w_i(t)) \theta(|z_i(t)| < \epsilon/2)}{\sum_j \theta(|z_j(t)| < \epsilon/2)} \quad (8.54)
$$

where the sums are over all stars in the tracer population, $A_0$ is the cross-sectional area of our sample, and $\theta(|z_i(t)| < \epsilon/2)$ assures that $z_i(t)$ is within some appropriate small distance $\epsilon/2$ of
the plane. We can now compute the long-time average:

\[
\overline{\rho(z)} = \lim_{t \to \infty} \frac{1}{t} \int_0^t dt' \rho(z, t') \quad (8.55)
\]

\[
= \frac{1}{A_0} \sum_i \lim_{t \to \infty} \frac{1}{t} \int_0^t dt' \delta(z - z_i(t')) \quad (8.56)
\]

\[
= \frac{1}{A_0} \sum_i \lim_{N_i \to \infty} \frac{1}{N_i T_i/2} N_i \int_0^{T_i/2} dt \delta(z - z_i(t)) \quad (8.57)
\]

\[
= \frac{1}{A_0} \sum_i \frac{2}{T_i} \int_{-z_{\max}}^{z_{\max}} \frac{dz_i}{w_i(z_i)} \delta(z - z_i) \quad (8.58)
\]

\[
= \frac{1}{A_0} \sum_i \frac{2}{T_i |w_i(z)|} \quad (8.59)
\]

where we have used the fact that the trajectories of the stars are periodic with individual periods \(T_i\), and where \(|w_i(z_i)|\) is star \(i\)'s speed (fixed by energy conservation) at height \(z_i\). We can proceed similarly for the velocity distribution. For a large number of stars, we can write the denominator as

\[
\sum_i \theta(|z_i(t)| < \epsilon/2) = \epsilon \rho(0, t) A_0. \quad (8.60)
\]

We now proceed to write

\[
\overline{f_{z=0}(w)} = \lim_{t \to \infty} \frac{1}{t} \int_0^t dt' \overline{f_{z=0}(w, t')} \quad (8.61)
\]

\[
= \sum_i \lim_{t \to \infty} \frac{1}{t} \int_0^t dt' \frac{\delta(w - w_i(t')) \theta(|z_i(t')| < \epsilon/2)}{\epsilon \rho(0, t) A_0} \quad (8.62)
\]
Now, when star \( i \) is at \( z = 0 \), its velocity will be maximum and will be equal to \( w_i = \sqrt{2E_i} \).

The amount of time that star \( i \) spends between \( z = \pm \epsilon/2 \) will therefore be given by \( \epsilon/\sqrt{2E_i} \). In each period, the integral will therefore receive a contribution \( \epsilon/\sqrt{2E_i} \delta(w - \sqrt{2E_i}) \) on the way up and \( \epsilon/\sqrt{2E_i} \delta(w + \sqrt{2E_i}) \) on the way down. We can achieve this by replacing

\[
\theta(|z_i(t')| < \epsilon) = \frac{\epsilon}{\sqrt{2E_i}} \sum_{n_i=0}^{N_i} \delta(t - t_{0i} - n_iT_i/2) \tag{8.63}
\]

\[
= \frac{\epsilon}{\sqrt{2E_i}} \sum_{n_i=0}^{N_i/2} (\delta(t - t_{0i} - n_iT_i) + \delta(t - t_{0i} - (n_i + 1/2)T_i)) \tag{8.64}
\]

where the first crossing of the plane for star \( i \) happens at time \( t_{0i} \). We thus have

\[
f_{z=0}(w) = \frac{1}{\epsilon A_0} \sum_i \frac{\epsilon}{\sqrt{2E_i}} \lim_{N_i \to \infty} \frac{1}{N_iT_i/2} \int_0^{N_iT_i/2} dt \frac{1}{\rho(0,t)} \ldots \tag{8.65}
\]

\[
\times \left[ \delta(w - \text{sign}(w_i(t_{0i})))\sqrt{2E_i} \sum_{n_i=0}^{N_i/2} \delta(t - t_{0i} - n_iT_i) \right. \ldots \tag{8.66}
\]

\[
+ \delta(w + \text{sign}(w_i(t_{0i})))\sqrt{2E_i} \sum_{n_i=0}^{N_i/2} \delta(t - t_{0i} - (n_i + 1/2)T_i) \right] \tag{8.67}
\]

\[
= \frac{1}{A_0} \sum_i \frac{1}{\sqrt{2E_i}} \lim_{N_i \to \infty} \frac{1}{N_iT_i/2} \ldots \tag{8.68}
\]

\[
\times \left[ \delta(w - \text{sign}(w_i(t_{0i})))\sqrt{2E_i} \sum_{n_i=0}^{N_i/2} \rho(0,t_{0i} + n_iT_i) \right. \ldots \tag{8.69}
\]

\[
+ \delta(w + \text{sign}(w_i(t_{0i})))\sqrt{2E_i} \sum_{n_i=0}^{N_i/2} \rho(0,t_{0i} + (n_i + 1/2)T_i) \right] \tag{8.70}
\]
Now, unless the periods of \( \rho(0, t) \) and of the trajectories of the individual stars divide each other, \( \sum_{n_i=0}^{N_i/2} 1/\rho(0, t_{0i} + n_i T_i) \) will just be, in the large \( N_i \) limit, \( N_i/2 \) times a time average of \( 1/\rho(0, t) \). Since the stars with periods dividing that of \( \rho(0, t) \) is a set of measure zero, we have

\[
\frac{1}{A_0} \sum_i \frac{1}{\sqrt{2E_i}} \lim_{N_i \to \infty} \frac{1}{N_i T_i/2} \ldots
\]

\[
\times \left[ \delta(w - \text{sign}(w_i(t_{0i}))) \sqrt{2E_i} \frac{N_i}{2} \frac{\rho(0)^{-1}}{\rho(0)^{-1}} \ldots \right.
\]

\[
+ \delta(w + \text{sign}(w_i(t_{0i}))) \sqrt{2E_i} \frac{N_i}{2} \frac{\rho(0)^{-1}}{\rho(0)^{-1}} \right]
\]

\[
= \frac{1}{A_0} \sum_i \frac{1}{\sqrt{2E_i}} \frac{1}{T_i} \frac{\rho(0)^{-1}}{\rho(0)^{-1}} \left[ \delta(w - \sqrt{2E_i}) + \delta(w + \sqrt{2E_i}) \right]
\]

\[
= \frac{1}{A_0} \sum_i \frac{1}{\sqrt{2E_i}} \frac{1}{T_i} \frac{\rho(0)^{-1}}{\rho(0)^{-1}} 2\sqrt{2E_i} \delta(w^2 - 2E_i)
\]

\[
= \frac{\rho(0)^{-1}}{A_0} \sum_i \frac{2}{T_i} \delta(w^2 - 2E_i).
\]

We can now evaluate \( \overline{f_{z=0}(w')}_{w' = \sqrt{w^2 + 2\Phi(z)}} \), using the fact that \( w_i(z)^2/2 + \Phi(z) = E_i \):

\[
\overline{f_{z=0}(w')}_{w' = \sqrt{w^2 + 2\Phi(z)}} = \frac{\rho(0)^{-1}}{A_0} \sum_i \frac{2}{T_i} \delta(w^2 - 2(E_i - \Phi(z)))
\]

\[
= \frac{\rho(0)^{-1}}{A_0} \sum_i \frac{2}{T_i} \delta(w^2 - w_i(z)^2)
\]
where \(|w_i(z)|\) is the speed of star \(i\) at height \(z\). This gives

\[
\int dw \left[ \frac{\rho(0)}{A_0} \sum_i \frac{2}{T_i} \delta(w^2 - w_i(z)^2) \right]_{w' = \sqrt{w^2 + 2\Phi(z)}} = \int dw \left[ \frac{\rho(0)}{A_0} \sum_i \frac{2}{T_i} \delta(w^2 - w_i(z)^2) \right]_{w' = \sqrt{w^2 + 2\Phi(z)}} = \frac{\rho(0)}{A_0} \sum_i \frac{2}{T_i |w_i(z)|} \tag{8.79}
\]

which, by Equation 8.59, is equal to \(\overline{\rho(z)}\).

Q.E.D.
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