Bargaining in Markets

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Eduard Talamas

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Abstract

In the first chapter I describe a method to construct fair—or symmetry-preserving—stable sets of compound simple games from fair stable sets of their quotient and components, and I discuss how it contributes to the characterization of the set of simple games that admit a fair stable set.

In the second chapter I study an infinite-horizon model of coalitional bargaining in stationary markets with strategic choice of bargaining partners: In each period, a player is selected to be the proposer and selects a coalition as well as how to share its surplus among its members. Players respond in sequence; if all players accept, they leave the market and—to maintain stationarity—are replaced by replicas. I describe an algorithm that characterizes the essentially-unique stationary subgame-perfect equilibrium. This algorithm reveals that the coalitions that form in equilibrium have a tier structure, with equilibrium payoffs determined from the top tier down.

In the third chapter I present an application of the model of the second chapter to networked markets. In the unique subgame-perfect equilibrium, each player has a preferred neighbor to whom she always extends offers in equilibrium. Each component (or submarket) of the preferred-neighbor network has exactly one pair of mutually preferred neighbors, whose terms of trade determine the price at which all trades occur in their submarket. I describe a simple method to compute the highest and the lowest equilibrium price in the limit as bargaining frictions vanish, and I use it to formalize the idea that the law of one price holds if and only if the market is thick enough.
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Introduction

In the first chapter I study fair stable sets of simple games. Simple games are abstract representations of voting systems and other group-decision procedures. A stable set—or von Neumann-Morgenstern solution—of a simple game represents a “standard of behavior” that satisfies certain internal and external stability properties. Compound simple games are built out of component games, which are, in turn, “players” of a quotient game. I describe a method to construct fair—or symmetry-preserving—stable sets of compound simple games from fair stable sets of their quotient and components. This method is closely related to the composition theorem of Shapley (1963c), and contributes to the answer of a question that he formulated: what is the set \( \mathcal{G} \) of simple games that admit a fair stable set? In particular, this method shows that the set \( \mathcal{G} \) includes all simple games whose factors—or quotients in their “unique factorization” of Shapley (1967)—are in \( \mathcal{G} \), which suggests a path to characterize \( \mathcal{G} \).

In the second chapter I study an infinite-horizon model of coalitional bargaining in stationary markets that—in contrast to standard models—features strategic choice of bargaining partners: In each period, a player is selected to be the proposer and selects a coalition as well as how to share its surplus among its members. Players respond in sequence; if all players accept, they leave the market and—to maintain stationarity—are replaced by replicas. I describe an algorithm that characterizes the essentially-unique stationary subgame-perfect equilibrium. This algorithm reveals that the coalitions that form in equilibrium have a tier structure, with equilibrium payoffs determined from the top tier down. In the special case of matching markets, the algorithm characterizes the essentially-unique subgame-perfect equilibrium, and reveals that submarkets form endogenously in equilibrium. Generically,
each submarket has exactly one top-tier match, whose terms of trade are as in the canonical bilateral model of Rubinstein (1982) and influence the prices throughout their submarket.

In the third chapter I study an infinite horizon model of bargaining in a stationary networked market for a homogeneous good that—in contrast to standard models—features strategic choice of bargaining partners: In each period, a player is selected to be the proposer and selects the recipient as well as the terms of her offer. Players who reach agreement leave the market and—to maintain stationarity—are replaced by replicas. Each player has a preferred neighbor to whom she always extends offers in equilibrium. Each component (or submarket) of the preferred-neighbor network has exactly one pair of mutually preferred neighbors, whose terms of trade are as in Rubinstein’s (1982) alternating-offers model and determine the price at which all trades occur in their submarket. I describe a simple method to compute the highest and the lowest equilibrium price in the limit as bargaining frictions vanish, and I use it to formalize the idea that the law of one price holds if and only if the market is thick enough.
Chapter 1

Fair Stable Sets of Simple Games

1.1 Introduction

Simple games—in which every coalition of players either wins or loses—represent political structures in which “power” is the fundamental driving force. For example, a legislature in which any two of three parties have enough parliamentary seats to form a new government can be represented by the three-player simple majority game. A fair stable set of a simple game gives a set of distributions of power among the players that (i) satisfies certain internal and external stability properties, and (ii) does not discriminate among players based on their names. For example, the unique fair stable set of the three-player simple majority game consists of the three possible outcomes in which power is divided equally among two of the three players; intuitively, this stable set reflects the idea that the coalition forming government is vulnerable when one of its parties receives less than the other.\footnote{See Taylor and Zwicker (1999) for an excellent exposition of the theory of simple games. Stable sets of simple games were first studied by von Neumann and Morgenstern (1944), who—among many other things—showed that all simple games have a stable set. In fact, they showed that many interesting games admit multiple stable sets, which lead to the advancement of several refinements of the theory. The fairness requirement is one such refinement; see also Shapley (1952), Luce and Raiffa (1957), Vickrey (1959), Harsanyi (1974), Roth (1976), Muto (1980), Greenberg (1990), Bogomolnaia and Jackson (2002), Béal et al. (2008), Mauleon et al. (2011), Jordan and Obadia (2015), Ray and Vohra (2015) and Dutta and Vohra (forthcoming).}

In 1978, Lloyd Shapley asked which simple games admit a fair stable set. Rabie (1985) showed that the answer is not “all,” but—to the best of my knowledge—the set $\mathcal{G}$ of simple
games that admit a fair stable set has not been characterized.\(^2\) In this article I describe a method to construct fair stable sets of compound simple games from fair stable sets of their quotient and components. I also discuss how this method contributes to the characterization of the set \(G\) of simple games that admit a fair stable set.

Compound simple games are built out of component games, which are, in turn, “players” of a quotient game. An example of a compound simple game is the multimillion-person game of “Presidential Election”—whose quotient is a weighted majority game (the Electoral College), and whose components are symmetric majority games of assorted sizes (the electorates of the 50 states and the District of Columbia). A game is said to be prime if it does not have any non-trivial compound representation.\(^3\)

Sums of games—whose winning coalitions are exactly those that win in at least one of these games—and products of games—whose winning coalitions are exactly those that win in all of these games—are particular classes of compound simple games. For example, the game “Congress” can be represented as the product of two simple majority games (the Senate and the House of Representatives).

The main result of this article is closely related to the composition theorem of Shapley (1963c). To emphasize this connection, I quote his verbal description of his result (Shapley, 1963c, pages 3-4) adding words in italics that transform it into a description of my result:

If we divide the proceeds of a regular\(^4\) [compound simple] game among the components, in accordance with a fair [stable set] of the quotient, and then subdivide them among the players of each component according to a scaled-down fair [stable set] of that component, using isomorphic fair stable sets for isomorphic components, then the resulting set of imputations is a fair [stable set] of

\(^2\)Shapley raised this question during the Fourth International Workshop in Game Theory; see Lucas (1978) and Rabie (1985). Rabie’s result is analogous to those of Stearns (1964) and Lucas (1968), who showed that not all coalitional games with non-transferable and transferable utility, respectively, have a stable set (see Lucas (1992) for an illuminating review of these and other results in the theory of stable sets).

\(^3\)Compound simple games and their stable sets were first studied by Shapley (1963a, 1963b, 1963c, 1967). See Shapley (1962) for an illuminating introduction to this theory.

\(^4\)A compound simple game is regular if either its quotient is prime, or it is a sum of products that are not themselves sums, or it is a product of games which are not themselves products.
the compound.\footnote{Stable sets were originally called “solutions.” In this quote, I have replaced “solution” for “[stable set]” to reflect the modern terminology.}

The “unique factorization” theorem of Shapley (1967, Theorem 8) shows how every simple game can be uniquely decomposed into a hierarchical arrangement of compound simple games that use only prime quotients and the operations of sums and products; the prime quotients in this decomposition are the factors of the simple game.

The combination of Shapley’s unique factorization theorem with my composition result reveals that the set $\mathcal{G}$ of simple games that admit a fair stable set includes all simple games whose factors are in $\mathcal{G}$. In other words, a game that does not admit a fair stable set has at least one factor that does not admit a fair stable set. It is an open question whether the converse holds; if it does, the composition result presented in this article implies that characterizing the set of all prime games in $\mathcal{G}$ is equivalent to characterizing the set $\mathcal{G}$.

The rest of this article is organized as follows. In section 1.2 I provide background material: The definitions of simple game, compound simple game, committee and stable set, and Shapley’s unique factorization and composition theorems. In section 1.3 I define the notion of fair stable set, and I state and prove the main result of this article: A composition theorem that shows how to construct fair stable sets of compound simple games from fair stable sets of its quotient and components. I conclude in section 1.4 by discussing the implications of this result for the characterization of the set of simple games that admit a fair stable set and for the problem of aggregation in the theory of fair stable sets.

### 1.2 Preliminaries

In this section I review the two fundamental results of Lloyd Shapley that this article builds on. In subsection 1.2.1 I review the definitions of simple game, compound simple game, dual of a game and committee of a game, and I present the unique factorization theorem of Shapley (1967). In subsection 1.2.2 I review the definition of stable set of a game and the composition theorem presented in Shapley (1963c).
1.2.1 Compound Simple Games

Simple Games

Let $P$ be the set of all players that might ever come under consideration. A (simple) game $\mathcal{W}$ is a collection of subsets of $P$ (the winning coalitions). I require that $\mathcal{W}$ includes the grand coalition $P$ and excludes the empty coalition, and that it be monotonic—in the sense that every superset of a winning coalition is also winning.

The monotonicity of a simple game $\mathcal{W}$ implies that it can be identified with the set $\mathcal{W}^m$ that contains only its minimal winning coalitions. For example, $\{ab, ac\}$ represents the game in which only the coalitions that contain both player $a$ and one of players $b$ and $c$ win, and $\{ab, ac, bc\}$ represents the three-person simple majority game.\(^6\) A player is said to be a dummy of a game if it is not in any of the minimal winning coalitions of the game.\(^7\)

Compound Simple Games

Throughout this article, let there be given $m$ non-overlapping component games $\mathcal{W}_i$, $i = 1, 2, \ldots, m$, together with an $m$-person quotient game $\mathcal{W}$, whose players are identified with the integers $1, 2, \ldots, m$.\(^8\)

**Definition 1.2.1.** For each set $S \subset P$, let $K(S)$ be the set $\{i \mid S \in \mathcal{W}_i\}$; intuitively, the set $K(S)$ consists of the set of all components that $S$ wins. The compound simple game $\mathcal{W}|\mathcal{W}_1, \ldots, \mathcal{W}_m|$ is defined by the following condition:

$$S \in \mathcal{W}|\mathcal{W}_1, \ldots, \mathcal{W}_m| \text{ if and only if } K(S) \in \mathcal{W}.$$ 

Thus, a coalition wins in the compound if and only if it wins enough of the components to make up a winning coalition of the quotient. The game $\mathcal{W}|\mathcal{W}_1, \ldots, \mathcal{W}_m|$ is a compound

\(^6\)For brevity I write $ab$ for $\{a, b\}$, etc., and I denote by $a$ the one-player game with non-dummy player $a$.

\(^7\)Abusing terminology slightly, I often refer to the non-dummy players of a game as its players.

\(^8\)Non-overlapping in the sense that their non-dummy player sets do not overlap; that is, for any $i \neq j$, the union of $\mathcal{W}^m_i$ is disjoint from the union of $\mathcal{W}^m_j$. 

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representation of game \( M \) if it satisfies

\[
S \in M \text{ if and only if } S \in W[W_1, \ldots, W_m].
\]

For example, the game \( \{ab, ac\} \) can be represented as the compound game whose quotient is a two-player unanimity game, and whose components are the games \( a \) and \( \{b, c\} \). In contrast, the three-player majority game is prime, in the sense that it does not have a non-trivial compound representation.

**Sums and Products**

Quotients having the maximum and minimum possible number of winning coalitions play a central part in the theory of compound simple games. It is useful to represent them as operations on games, as follows. The *sum of \( m \geq 2 \) non-overlapping games*

\[
W_1 \oplus W_2 \oplus \cdots \oplus W_m
\]

is the compound game \( S_m[W_1, W_2, \ldots, W_m] \) where the minimal winning coalitions of the game \( S_m \) are all singleton subsets of \( \{1, 2, \ldots, m\} \). That is, a coalition wins in the sum of games whenever it contains a winning contingent from at least one of these games.

Similarly, the *product of \( m \geq 2 \) non-overlapping games*

\[
W_1 \otimes W_2 \otimes \cdots \otimes W_m
\]

is the compound game \( P_m[W_1, W_2, \ldots, W_m] \) where the only minimal winning coalition of the game \( P_m \) is \( \{1, 2, \ldots, m\} \). That is, a coalition wins in the product of games whenever it contains winning contingents from all of them.

**Definition 1.2.2.** A compound representation \( W[W_1, \ldots, W_m] \) of a simple game is regular if either its quotient \( W \) is prime, or its quotient \( W \) is a sum and none of its components \( W_i \) is a sum, or its quotient \( W \) is a product and none of its components \( W_i \) is a product.

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\(^9\)The quotient of a sum of games has the maximum possible number of winning coalitions (all nonempty coalitions win) and the quotient of the product has the minimum number of winning coalitions (only the coalition containing all non-dummy players wins).
For example, the compound representation $S_3[a, b, c]$ of the game $\{a, b, c\}$ is regular, but the compound representation $S_2[a \oplus b, c]$ of the same game is not.

**Dual Games**

The following duality between sums and products of games is useful to prove the main result of this paper. The dual $M^*$ of a game $M$ is the set of all coalitions $B$ that block in $M$; that is, the set of all coalitions $B$ whose complement $P - B$ does not win in $M$. For example, the dual of the sum of games (1.1) is $W_1^* \otimes W_2^* \otimes \cdots \otimes W_m^*$, and the dual of the product of games (1.2) is $W_1^* \oplus W_2^* \oplus \cdots \oplus W_m^*$.

**Committees**

Shapley’s unique factorization theorem (Theorem 1.2.1 below) describes how a simple game can be decomposed into a hierarchical arrangement of committees, defined as follows.

**Definition 1.2.3.** A committee of a game $M$ is another game $M_C$ (with non-dummy player set $C$) which is related to the first as follows: for every coalition $S$ such that

$$S \cup C \in M$$

and

$$S - C \notin M,$$

we have

$$S \in M$$

if and only if

$$S \cap C \in M_C.$$

A committee of a game $M$ is proper if it is not the committee of the whole set of its non-dummy players or a committee that consists of only one individual.

For example, denoting the three player majority game by $M_3$, the game $M_3[b, c, d]$ is a proper committee of the game $M_3[a, M_3[b, c, d], e]$. Prime games are exactly those that do not possess proper committees (Shapley, 1967, Section 7).\(^{10}\)

\(^{10}\)The games that have only two non-dummy players are an exception: even though they do not have any proper committee, they are not regarded to be prime; see Shapley (1967, page 5).
Shapley’s Unique Factorization Theorem

The main result of Shapley (1967) is that—just like every natural number can be uniquely expressed as the product of prime numbers—every simple game has a unique compound representation that uses prime quotients and the operations of sums and products.

**Theorem 1.2.1** (Shapley, 1967, Theorem 8). *Every simple game has a compound representation that uses nothing but prime quotients and the associative operations \( \oplus \) and \( \otimes \) and that is unique except for the arbitrariness in the ordering of the players.*

Continuing the analogy with the natural numbers, the *factors* of a simple game are its quotients in the above compound representation.

**1.2.2 Stable Sets of Compound Simple Games**

**Stable Sets of Simple Games**

For any set of players \( Q \), the set of *imputations* \( A_Q \) is the simplex of real nonnegative vectors \( x \) with \( x_j = 0 \) for any \( j \notin Q \), and whose entries sum to one. Geometrically, the simplex of

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\(^{11}\)Shapley’s statement adds an additional exception regarding “the disposition of dummy players.” This is because he defines a simple game to be a finite set \( N \) (the players) and a set of subsets of \( N \) (the winning coalitions), so in order to define a compound simple game, he needs to specify to which component each dummy player belongs. In contrast, I identify a simple game with the set of its minimal winning coalitions, so I do not need to assign dummy players to components.
imputations $A_Q$ is the set of convex combinations of the vectors that divide a unit of surplus among the players in $Q$. Figure ?? illustrates some imputation simplices.\textsuperscript{12}

Fix a simple game $\mathcal{M}$. An imputation $x$ dominates another imputation $y$ via the coalition $S$ if $S \in \mathcal{M}$ and if each of the players in $S$ gets strictly more payoff in $x$ than in $y$. For example, in the game $\{ab, ac, bc\}$, the imputation that gives one half of the payoff to each of the players $a$ and $b$ dominates the imputation that gives all the payoff to player $c$ via the coalition $ab$. An imputation $x$ dominates another imputation $y$ if $x$ dominates $y$ via some coalition.

\textbf{Definition 1.2.4.} Given a simple game $\mathcal{M}$, a set $X$ of imputations is (i) \textit{internally stable} if no imputation in $X$ is dominated by any imputation in $X$, (ii) \textit{externally stable} if each imputation that is not in $X$ is dominated by some imputation in $X$, and (iii) \textit{stable} if it is internally and externally stable.

Figure ?? depicts a set of imputations that constitutes a stable set of both $\{ab, ac, bc\}$ and $\{ab, ac\}$. Stable sets are the classical solutions of von Neumann and Morgenstern (1944).

\textbf{Shapley’s Composition Theorem}

The main contribution of Shapley (1963c) is the description of a method to construct a stable set of a compound simple game from stable sets of its quotient and components. I now formally describe this method. For each $i = 1, 2, \ldots, m$, let $X_i$ be a stable set of the game $\mathcal{W}_i$, and let $\chi$ be a stable set of the quotient game $\mathcal{W}$.

\textbf{Definition 1.2.5.} The \textit{compound set} $\chi[X_1, \ldots, X_m]$ is the set of all imputations of the form

$$x = \sum_{i=1}^{m} a_i x_i, \alpha \in \chi, x_i \in X_i, i = 1, 2, \ldots, m.$$ 

\textbf{Theorem 1.2.2} (Shapley, 1963c, Part II, Section 2, Theorem 1). The compound set $\chi[X_1, \ldots, X_m]$ is a stable set of the compound game $\mathcal{W}[\mathcal{W}_1, \ldots, \mathcal{W}_m]$.

\textsuperscript{12}In the figures, I write $a, b, c, d$ for $A_a, A_b, A_c$ and $A_d$ respectively.
For an illustration of Theorem 1.2.2, consider the game \{ab, ac\}. This game can be represented as a compound simple game, whose quotient is the two-player unanimity game, and whose components are \(a\) and \(\{b, c\}\). A stable set of its quotient consists of all imputations that share the payoff among its two players, and \(A_a\) and \(A_b\) are stable sets of its components \(a\) and \(\{b, c\}\), respectively. Shapley’s composition theorem then says that the set \(A_{ab}\) (illustrated in Figure ??) is a stable set of the game \{ab, ac\}. Similarly, since any singleton set that consists of an imputation that shares the payoff arbitrarily between players \(b\) and \(c\) is a stable set of the game \{b, c\}, any straight line in \(A_{abc}\) from \(A_a\) to any point in \(A_{bc}\) is a stable set of the game \{ab, ac\}; the right diagram of Equation A.9 illustrates another stable set of this game that can be constructed in this way.\(^{13}\)

\[\text{1.3 Fair Stable Sets of Compound Simple Games}\]

*Fair stable sets* are those that do not discriminate among players based on their names. Not all stable sets are fair; for example, the stable set of the three-player majority game depicted in Figure ?? is not fair, since it discriminates among its three non-dummy players, even if they play exactly the same role in this game. In subsection 1.3.1 I formally describe what it means for a stable set to be *fair*, and in subsection 1.3.2 I provide the main result of this article: A composition theorem that shows how to construct a fair stable set of a compound simple game from fair stable sets of its quotient and components.

\(^{13}\)Shapley (1963c) also presents a generalization of his composition theorem that shows that the requirement that such a line be straight is not necessary.
1.3.1 Fair Stable Sets of Simple Games

A permutation $\pi$ of the set of players $P$ acts on an imputation by permuting its indices—for example, $^{14}(ab)A_b = A_a$—and it acts on a set of imputations by acting on each of the imputations of this set—for example, $^{14}(ab)A_{bc} = A_{ac}$.

**Definition 1.3.1.** A permutation $\pi$ of the player set $P$ is an isomorphism between an imputation set $X$ and an imputation set $Y$ if $\pi X = Y$.

An imputation set $X$ is said to be isomorphic to an imputation set $Y$ if there exists an isomorphism between $X$ and $Y$. For example, the permutation $(bc)$ is an isomorphism between $A_{ab}$ and $A_{ac}$. Isomorphisms between an imputation set $X$ and itself are symmetries of $X$. For example, the permutation $(ab)$ is a symmetry of $A_{abc}$.

A permutation $\pi$ of the set of players $P$ acts on a game $M$ by permuting the players in each of the coalitions in $M$. For example, $(ac)\{ab, ac\} = \{cb, ca\}$.

**Definition 1.3.2.** A permutation $\pi$ of the player set $P$ is an isomorphism between a game $M_1$ and a game $M_2$ if $\pi M_1 = M_2$, or, equivalently, $\pi M_1^m = M_2^m$.

A game $M_1$ is said to be isomorphic to a game $M_2$ if there is an isomorphism between $M_1$ and $M_2$. For example, the permutation $(ac)$ is an isomorphism between the game $\{ab, ac\}$ and the game $\{cb, ca\}$. Isomorphisms between a game $M$ and itself are symmetries of $M$. For example, the permutation $(bc)$ is a symmetry of the game $\{ab, ac\}$, and every permutation of the players is a symmetry of the three-player majority game.

**Definition 1.3.3.** A stable set $X$ of a game $M$ is fair if every symmetry of $M$ is also a symmetry of $X$.

The stable set of the games $\{ab, ac, bc\}$ and $\{ab, ac\}$ illustrated in Figure ?? is not fair, since it is not invariant under the permutation $(bc)$ (which is a symmetry of both these games). The left and right diagrams in Equation A.9 illustrate the unique fair stable set

\[ ^{14} \text{denote by } (a_1a_2a_3\ldots a_n) \text{ the permutation that maps } a_1 \text{ to } a_2, a_2 \text{ to } a_3, \ldots, a_{n-1} \text{ to } a_n, a_n \text{ to } a_1, \text{ and every other player to herself.} \]
Figure 1.3: The fair stable sets of \{ab, ac, bc\} (left) and \{ab, ac\} (right).

of the game \{ab, ac, bc\} and \{ab, ac\}, respectively. The fair stable set of \{ab, ac, bc\} is a set of three imputations; in each of these imputations, two players divide the payoff equally, leaving the remaining player with zero payoff. The fair stable set of \{ab, ac\} is the set of all imputations in the simplex \(A_{abc}\) that give the same payoff to both players \(b\) and \(c\).

1.3.2 A New Composition Theorem

Let \(\chi\) be a fair stable set of the (quotient) game \(\mathcal{W}\) and, for each \(i = 1, 2, \ldots, m\), let \(X_i\) be a fair stable set of the (component) game \(\mathcal{W}_i\). Note that, in contrast to section 1.2.2, these stable sets are required to be fair.

**Theorem 1.3.1.** If the compound representation \(\mathcal{W}[\mathcal{W}_1, \ldots, \mathcal{W}_m]\) is regular and if \(X_i\) is isomorphic to \(X_j\) when \(\mathcal{W}_i\) is isomorphic to \(\mathcal{W}_j\), then the compound set \(\chi[X_1, \ldots, X_m]\) is a fair stable set of the compound game \(\mathcal{W}[\mathcal{W}_1, \ldots, \mathcal{W}_m]\).

The condition that the compound representation be regular is to avoid situations like the one described in Example 1.3.2.1 below, in which the compound representation “hides” certain symmetries of the game by having a component \(\mathcal{W}_i\) that is isomorphic to a component of another component \(\mathcal{W}_j\).

The condition that \(X_i\) be isomorphic to \(X_j\) when \(\mathcal{W}_i\) is isomorphic to \(\mathcal{W}_j\) is natural.\(^{15}\) This condition would not be needed if every simple game had at most one fair stable set, but this is not the case. For example, both \(A_{bc}\) and \(A_{abc} \cup A_{bcd}\) are fair stable sets of the

\(^{15}\)In fact, it is enough to require the use of isomorphic fair stable sets for isomorphic components \(\mathcal{W}_i\) and \(\mathcal{W}_j\) for which there is a symmetry of the quotient game \(\mathcal{W}\) that maps \(i\) to \(j\); in this case, the non-dummy players of \(i\) and \(j\) play exactly the same role in the game, so they should be treated in the same way by a fair stable set.
Example 1.3.2.1. The following example shows that the conclusion of Theorem 1.3.1 is not generally true without the requirement that the compound representation be regular. Consider the compound simple game $S_2 [a \oplus b, c]$, where $S_2$ is the sum of components 1 and 2. This compound representation is not regular, since both its quotient and one of its components are sums of games.

Both $S_2$ and $a \oplus b$ have a unique fair stable set, which consists of the imputation that divides the unit payoff equally among their two players; denote them by $\eta$ and $Y_1$, respectively. Similarly, the game $c$ has a unique fair stable set, which consists of the imputation that gives the unit payoff to player $c$; denote it by $Y_2$.

The compound set $\eta[Y_1, Y_2]$ consists of the singleton set containing the imputation that gives $1/4$ to each of players $a$ and $b$, and $1/2$ to player $c$. But this is not a fair stable set of the game; for example, $(ac)$ is a symmetry of the game but not a symmetry of $\eta[Y_1, Y_2]$.

1.3.3 Proof of Theorem 1.3.1

Let $\pi$ be a symmetry of the compound game in regular form $W[W_1, \ldots, W_m]$, and assume that, for every two isomorphic component games $W_i$ and $W_j$, we have that $X_i$ is isomorphic to $X_j$. Given Theorem 1.2.2, we just need to show that $\pi$ is a symmetry of the compound set $X[X_1, \ldots, X_m]$, which follows from Proposition 1.3.4, Proposition 1.3.5 and Proposition 1.3.6 below.

Outline of the Proof

The first step (Proposition 1.3.4) is the most subtle one: for every component $i$, there exists a component $j$ such that the map $\pi$ is an isomorphism between $W_i$ and $W_j$. When a map has this property, I say that it is compatible with this compound representation. The requirement that the compound game be regular is used in this first step. Indeed, a symmetry of a game need not be compatible with one of its non-regular compound representations; for example,
is a symmetry of the game in Example 1.3.2.1, but this map is not an isomorphism between any of its components.

A corollary of the first step is that $\pi$ naturally defines a permutation $\pi^*$ of the players of the quotient. The second step (Proposition 1.3.5) is to show that $\pi^*$ is a symmetry of the quotient. The third and final step (Proposition 1.3.6) is to show that any permutation of the players with the two properties above is a symmetry of the compound set $\chi[X_1, \ldots, X_m]$. The requirement that $X_i$ is isomorphic to $X_j$ when $W_i$ be isomorphic to $W_j$ is used in this final step.

**Intuition for the Proof**

Before proving the three steps of the proof, I present Example 1.3.3.1 to help build intuition for why the statement proved in each step holds.

**Example 1.3.3.1.** Consider the nine-player regular compound simple game

$$
\mathcal{M}[\mathcal{M}_3, \mathcal{M}_3, \mathcal{M}_3],
$$

where $\mathcal{M}$ denotes the three-player game $\{\{1, 2\}, \{1, 3\}\}$ and $\mathcal{M}_3$ denotes the three-player majority game. Let the set of non-dummy players in the first, second, and third component be $\{a, b, c\}, \{d, e, f\}$ and $\{g, h, i\}$ respectively.

The two sets of players $\{d, e, f\}$ and $\{g, h, i\}$ play the same role, in the sense that any two players in one of these sets combined with any two players in $\{a, b, c\}$ win. In fact, the set of all minimal winning coalitions of this compound game are exactly the set of all four-player coalitions just described.

To gain intuition for first step of the proof of Theorem 1.3.1, consider the maps $\pi_L$, $\pi_M$ and $\pi_R$ depicted in Equation A.7. The map $\pi_L$ is not compatible with the compound representation (1.3), since it maps players $d$ and $e$ to one component and player $f$ to a different component. To see why $\pi_L$ is not a symmetry of the compound game, note that it maps the winning coalition $\{a, b, e, f\}$ to the losing coalition $\{a, b, e, g\}$. In contrast, the maps $\pi_M$ and $\pi_R$ are compatible with the compound representation (1.3).
(a) The map $\pi_L$  
(b) The map $\pi_M$  
(c) The map $\pi_R$

Figure 1.4: The maps $\pi_L$, $\pi_M$ and $\pi_R$ of Example 1.3.3.1.

To gain intuition for the second step of the proof of Theorem 1.3.1, note that since the maps $\pi_M$ and $\pi_R$ are compatible with the compound representation (1.3), they define the following two maps on the players of $\mathcal{M}$: The map $\pi^*_M$ interchanges players 1 and 2 and keeps 3 fixed (so it is not a symmetry of the quotient $\mathcal{M}$), and the map $\pi^*_R$ interchanges players 2 and 3 and keeps 1 fixed (so it is a symmetry of the quotient $\mathcal{W}$). To see why $\pi_M$ is not a symmetry of the compound game (1.3), note that it maps the winning coalition $\{a, b, g, h\}$ to the losing coalition $\{d, e, g, h\}$.

To gain intuition for the third step of the proof of Theorem 1.3.1, note that the map $\pi_R$ (which is compatible with the compound representation and defines a map on the quotient that is a symmetry of the quotient) is a symmetry of the compound set that uses the unique fair stable of the quotient and components in (1.3); this compound set consists of the set of all imputations that give $0 \leq x \leq 1/2$ of surplus to each of two players in component 1, and $1/2 - x$ to two players in one of the other two components.\textsuperscript{16}

\textsuperscript{16}Note that, since the three-player majority game has a unique fair stable set, the condition that the fair stable sets of isomorphic components be isomorphic has no bite in this example.
Terminology

Abusing terminology slightly, I refer to the intersection of a coalition \( A \) of players with the non-dummy player set of a given component \( i \) as the \textit{intersection of coalition \( A \) with component \( i \)}, and I say that \textit{coalition \( A \) intersects with component \( i \)} when the intersection of \( A \) with \( i \) is not empty. Also, I often denote component \( \mathcal{W}_i \) by its index \( i \).

Two Auxiliary Results

Lemma 1.3.2 is useful to reduce the number of cases to be considered in the proof of Theorem 1.3.1.\(^{17}\)

\textbf{Lemma 1.3.2.} \textit{A permutation is an isomorphism between two games if and only if it is an isomorphism between their duals.}

\textit{Proof.} Let \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) be two simple games and let \( \mu \) be a permutation such that \( \mu \mathcal{M}_1 = \mathcal{M}_2 \). Let \( B \) be a blocking coalition of \( \mathcal{M}_1 \); that is, suppose that \( P - B \notin \mathcal{M}_1 \). Then \( \mu(P - B) \notin \mathcal{M}_2 \), so \( P - \mu(B) \notin \mathcal{M}_2 \); that is, \( \mu(B) \) is also a blocking coalition of \( \mathcal{M}_2 \). Since \( \mu \) is one to one, this implies that \( \mu \mathcal{M}_1^* = \mathcal{M}_2^* \). The converse follows from the fact that every game is the dual of its dual. \( \blacksquare \)

Lemma B.2.3 gives information about the map \( \pi \) that I use extensively in the sequel.

\textbf{Lemma 1.3.3.} \textit{Let \( A \) and \( B \) be two arbitrary minimal winning coalitions of a given component. If \( \pi(A) \) intersects with a given component \( j \) but \( \pi(B) \) does not, then the intersection of \( \pi(A) \) with component \( j \) is a minimal winning coalition of \( j \).}

\textit{Proof.} Let \( A \) and \( B \) be two minimal winning coalitions of a given component, and suppose that \( \pi(A) \) intersects with a given component \( j \) but \( \pi(B) \) does not. Let \( C \) be a minimal winning coalition (of the compound) that includes \( A \) (such a coalition \( C \) can be found because \( i \) is not a dummy of the quotient). Since \( \pi \) is a symmetry of the compound and \( \pi(C) \) intersects with \( j \), \( \pi(C) \) is also a minimal winning coalition of the compound.

\(^{17}\)In particular, it is because of this result that \textit{Case 2} in section 1.3.3 follows from \textit{Case 1}.
Hence, the intersection of $\pi(C)$ with $j$ is in fact a minimal winning coalition in $j$. Since the intersection of $\pi(C)$ with $j$ is the union of the intersection of $\pi(C - A)$ with $j$ and the intersection of $\pi(A)$ with $j$, it is enough to show that the intersection of $\pi(C - A)$ with $j$ is empty.

Note that—since $\pi(B)$ does not intersect with $j$—the intersection of $\pi(C - A)$ with $j$ is equal to the intersection of $\pi((C - A) \cup B)$ with $j$. Suppose for contradiction that this intersection is not empty. Then, since $(C - A) \cup B$ is a minimal winning coalition of the compound, this intersection must in fact be a minimal winning coalition in $j$, which contradicts the fact that it is a strict subset of the intersection of $\pi(C)$ with $j$ (which is itself a minimal winning coalition in $j$).

\[\square\]

**Step 1 of the Proof**

**Proposition 1.3.4.** The map $\pi$ is compatible with the representation $W[W_1, \ldots, W_m]$.

**Proof.** Fix an arbitrary component $i$ with non-dummy player set $I$. It is enough to show that there exists a component $j$ such that $\pi$ is an isomorphism between $W_i$ and $W_j$; that is, $\pi W_i^m = W_j^m$. Since the compound representation $W[W_1, \ldots, W_m]$ is regular, the following three cases are exhaustive.

**Case 1: The quotient $W$ is a sum and the component $W_i$ is not a sum:** Let $A$ and $B$ be two coalitions in $W_i^m$. Since the set of minimal winning coalitions of the compound consists of the union of the set of minimal winning coalitions of each component—and $\pi$ is a symmetry of the compound—we can assume without loss of generality that $\pi$ maps $A$ to a minimal winning coalition of some component $j$. Since $W_i$ is not a sum, we cannot partition its set of non-dummy players into two nonempty sets in such a way that there is no minimal winning coalition that intersects both of them. In other words, there is a set $\{A_1 = A, A_2, \ldots, A_l = B\}$ of minimal winning coalitions of $i$ with the property that $A_i$ intersects with $A_{i+1}$ (for all $i = 1, 2, \ldots, l - 1$). Since $\pi$ maps $A_1$ to minimal winning coalition of $j$, and $A_1$ overlaps with $A_2$, $\pi$ maps $A_2$ to a minimal winning coalition of component $j$ as well. Indeed, since $\pi$ is a symmetry of the compound, and $A_2$ is a minimal winning coalition of a component (and
hence the compound), \(\pi(A_2)\) must also be a minimal winning coalition of the compound (and hence of some component). Since \(A_2\) intersects with \(A_1\), and \(\pi(A_1)\) is a minimal winning coalition of \(j\), \(\pi(A_2)\) intersects with component \(j\), and hence it must be a minimal winning coalition of \(j\) as well. Iterating on this observation, we conclude that \(\pi\) maps \(B\) to a minimal winning coalition of component \(j\).

**Case 2: The quotient \(W\) is a product and the component \(W_i\) is not a product:** Since the dual of the product of games is the sum of their duals (see section 1.2.1), the combination of Case 1 and Lemma 1.3.2 shows that \(\pi\) is an isomorphism between \(W_i^m\) and \(W_j^m\) for some component \(j\).

**Case 3: The quotient \(W\) is prime:** Let \(T\) be the union of all components \(j\) for which there is a minimal winning coalition \(A\) of \(i\) such that \(\pi(A)\) intersects with \(j\). First, I show that \(T\) is not the set of all components. Second, I show that there is a unique component \(u\) with the property that, for all minimal winning coalitions \(A\) of component \(i\), \(\pi(A)\) intersects with \(u\). Third, I use this fact to show that \(T\) is a committee of the quotient. Since the quotient is prime, it follows that \(T\) is a singleton. The same logic then shows that every minimal winning coalition of \(u\) intersects only with \(i\), so \(\pi\) is an isomorphism between \(i\) and \(u\).

**\(T\) is not the set of all components:** For contradiction, suppose otherwise. Then, the image under \(\pi\) of each minimal winning coalition of \(i\) intersects with exactly one component.\(^{18}\)

Let \(C\) be a minimal winning coalition (of the compound) that contains a minimal winning coalition of \(i\) and of some other component \(j\) (such a coalition can be found, because \(i\) is not a dummy—so it is in at least one minimal winning coalition of the quotient—and the quotient is not a sum of \(i\) and another game—so the coalition containing only \(i\) does not win in the quotient). Let \(S\) denote the set of components \(k\) such that the intersection of \(\pi(C - I)\) with \(k\) is a minimal winning coalition. \(S\) is a committee of the quotient, but it can neither be empty nor the set of all components, a contradiction of the assumption that the quotient is prime.

---

\(^{18}\)To see this, let \(A\) be minimal winning coalition (of the compound) that does not win \(i\) (such a coalition can be found because the quotient is prime, and hence it is not the product of \(i\) and some other game), let \(B\) be a minimal winning coalition (of the compound) such that \(\pi^{-1}(B) = A\), let \(j\) be a component that \(B\) wins, let \(B_j\) denote the intersection of \(B\) with \(j\), and let \(A_j\) be a minimal winning coalition of \(j\) whose image under \(\pi^{-1}\) intersects with \(i\). Then the image under \(\pi^{-1}\) of \((B - B_j) \cup A_j\) is a minimal winning coalition of the compound, and it intersects with component \(i\); hence, this intersection must be a minimal winning coalition of \(i\).
There is a unique component $u$ with the property that, for all minimal winning coalitions $A$ of component $i$, $\pi(A)$ intersects with $u$: On the one hand, suppose for contradiction that there is no such component $u$ with the property that, for all minimal winning coalitions $A$ of component $i$, $\pi(A)$ intersects with $u$. Then, there are two minimal winning coalitions $A$ and $B$ in component $i$ and two components $j$ and $k$ such that $\pi(A)$ intersects with $j$ and not with $k$, and $\pi(B)$ intersects with $k$ and not with $j$.\textsuperscript{20} I show that the sum $j \oplus k$ is a committee of the quotient, which contradicts the assumption that it is prime.\textsuperscript{21} Let $S$ be a set (of components) such that $S \cup \{j,k\} \in \mathcal{W}$ and $S - \{j,k\} \notin \mathcal{W}$. By Lemma B.2.3, the intersection of $\pi(A)$ with $j$ is a minimal winning coalition in $j$, the intersection of $\pi(A)$ with $k$ is empty, the intersection of $\pi(B)$ with $j$ is empty and the intersection of $\pi(A)$ with $k$ is a minimal winning coalition in $k$. This means that no minimal winning coalition $C$ (of the compound) that intersects with $j$ can intersect with $k$ (and vice versa).\textsuperscript{22} Hence, $S$ wins in the quotient if and only if $S$ contains either $l$ or $k$ (or both).

On the other hand, suppose for contradiction that there are (at least) two different components $u_1$ and $u_2$ with the property that, for all minimal winning coalitions $A$ in $i$, $\pi(A)$ intersects with both $u_1$ and $u_2$. I show that the product $u_1 \otimes u_2$ is a committee of

\textsuperscript{19}For each component $k$, let $A_k$ be a minimal winning coalition whose image under $\pi$ is a minimal winning coalition of $k$. To see that $S$ is a committee of the quotient, note that since the image under $\pi$ of $C$ wins in the compound, for any component $k$, the image under $\pi$ of $(C - I) \cup A_k$ wins in the compound. Hence, $S \cup \{k\}$ wins in the compound for any component $k$. It also follows from this observation that $S$ cannot be empty. Finally, $S$ cannot be the set of all components, since otherwise $\pi(C - I)$ would win in the compound, a contradiction of the assumption that $C$ is a minimal winning coalition.

\textsuperscript{20}Indeed, if this was not the case, then, for every two minimal winning coalitions $C$ and $D$ of $i$, either $\pi(C)$ would intersect only with a subset of those components that $\pi(D)$ intersects with, or vice versa. But this would imply that there is a minimal winning coalition $C$ of $i$ such that, for every minimal winning coalition $D$ of $i$, $\pi(D)$ intersects with all the components that $\pi(C)$ intersects with (so any component that $\pi(C)$ intersects with would serve as $u$).

\textsuperscript{21}Indeed, since the quotient is prime, it has three or more players (see footnote 11), so $j \oplus k$ is a proper committee of the quotient, a contradiction.

\textsuperscript{22}To see this, note that if a minimal winning coalition of the compound intersects with $k$, then it actually contains a minimal winning coalition in $k$. So, if $C$ is a minimal winning coalition (of the compound) that contains a minimal winning coalition of $j$ and that intersects with $k$, then—letting $J$ and $K$ denote the non-dummy player sets of $j$ and $k$—the coalition $H = (C - I - K) \cup \pi(B) \cup \pi(A)$ is minimal winning of the compound, so $\pi^{-1}(H)$ is also a minimal winning coalition of the compound. But this cannot be, since $\pi^{-1}(H)$ contains $A \cup B$, which is a strict superset of any minimal winning coalition.
the quotient, which is again a contradiction of the assumption that the quotient is prime. Let $S$ be a set (of components) such that $S \cup \{u_1, u_2\} \in \mathcal{W}$ and $S - \{u_1, u_2\} \notin \mathcal{W}$. Every minimal winning coalition $C$ (of the compound) that contains a minimal winning coalition $C_1$ in $u_1$ also contains a minimal winning coalition in $u_2$ (and vice versa). So $S$ wins in the compound if and only if $S$ contains both $u_1$ and $u_2$.

$T$ is a committee: For contradiction, suppose otherwise. That is, suppose that there exist sets (of components) $S_1, S_2$ and $Q$ such that both $S_1 \cup T$ and $S_2 \cup T$ win, both $S_1 - T$ and $S_2 - T$ lose, and $(S_1 - T) \cup Q$ wins but $(S_2 - T) \cup Q$ loses in the quotient. Note that $u$ must be an element of $Q$, because the image under $\pi^{-1}$ of a coalition (of players) that does not intersect with $u$ but intersects with $\pi(A)$, for some minimal winning coalition $A$ of $i$, cannot be a winning coalition of the compound (since the image of such coalition would have to intersect with $u$). Let $B_1$ and $B_2$ be two coalitions (of players) that contain one minimal winning coalition from each of the components in $S_1 \cup T$ and $S_2 \cup T$, respectively, and let $C$ be a coalition (of players) that contains one minimal winning coalition from each of the components in $Q$. Since the union of $S_1 - T$ with $Q$ wins in the quotient, $B_1 \cup C$ wins in the compound, so the image of $C$ under $\pi^{-1}$ wins in $i$. Since the union of $S_2 - T$ with $Q$ does not win in the quotient, $B_2 \cup C$ does not win in the compound. But since the union of $S_2$ with $T$ wins in the quotient, the union of $B_2 \cup C$ with another set of players $D$ that intersect only with components in $T - Q$ wins in the compound, but this contradicts the fact that $\pi^{-1}(B_2 \cup C \cup D)$ wins in exactly the same components as does $\pi^{-1}(B_2 \cup C)$.

\[\square\]

### Step 2 of the Proof

Given Proposition 1.3.4, we can define, for every symmetry $\mu$ of the compound game, the map $\mu^*$ from the set of components to itself such that $\mu^*(i) = j$ if $\mu W_i = W_j$.

**Proposition 1.3.5.** The map $\pi^*$ is a symmetry of the quotient $\mathcal{W}$.

---

23This is because, since $\pi^{-1}$ is a symmetry of the compound and $\pi^{-1}(C)$ intersects with $i$, the intersection of $\pi^{-1}(C)$ with $i$ is in fact a minimal winning coalition in $i$, which implies that $C$ intersects with $u_2$ (since the image of every minimal winning coalition of $i$ under $\pi$ intersects with the non-dummy player set of $u_2$).
Proof. Let $S$ be a minimal winning coalition of the quotient. Since $\pi^*$ is one-to-one, it is enough to show that the image of $A$ under $\pi^*$ is a minimal winning coalition of the quotient. For contradiction, suppose otherwise. Let the coalition $A$ contain exactly one minimal winning coalition of each of the component games with index in $S$. Then $A$ is a minimal winning coalition of the compound, but $\pi$ maps it to a non-minimal winning coalition of the compound, a contradiction. \hfill \Box

Step 3 of the Proof

Proposition 1.3.6. Assume that for every two isomorphic components $W_i$ and $W_j$, $X_i$ is isomorphic to $X_j$. If a permutation $\mu$ is compatible with the compound representation $W[W_1, \ldots, W_m]$ and $\mu^*$ is a symmetry of $W$, then $\mu$ is a symmetry of the compound set $\chi[X_1, \ldots, X_m]$.

Proof. Let $\mu$ be a permutation that is compatible with the compound $W[W_1, \ldots, W_m]$, and that is such that $\mu^*$ is a symmetry of the quotient $W$. Let $x$ be in the compound set $\chi[X_1, \ldots, X_m]$. Since $\mu$ is one-to-one, it is enough to show that $\mu(x)$ is also in this compound set. By definition,

$$x = \sum_{i=1}^{m} a_i x_i, \text{ and } \mu(x) = \sum_{i=1}^{m} a_i \mu(x_i).$$

for some $a \in \chi$ and $x_i \in X_i$ for each $i = 1, \ldots, m$. Since $\mu$ is an isomorphism between the components $W_i$ and $W_{\pi^*(i)}$, $X_i$ is isomorphic to $X_{\mu^*(i)}$, and hence $\mu(x_i) \in X_{\mu^*(i)}$ for each $i = 1, \ldots, m$.

Also, since $\chi$ is a fair stable set of $W$ and $\mu^*$ is a symmetry of $W$, there exists $\beta \in \chi$ such that $\beta_{\mu^*(i)} = \alpha_i$ for all $i = 1, 2, \ldots, m$. Hence, we can write $\mu(x)$ as

$$\mu(x) = \sum_{i=1}^{m} \beta_{\mu^*(i)} \mu(x_i),$$

for some $\beta \in \chi$ and $\mu(x_i) \in X_{\mu^*(i)}$, for $i = 1, \ldots, m$; that is, $\mu(x)$ is in the compound $\chi[X_1, \ldots, X_m]$. \hfill \Box

\footnote{To see this, note that ...}
1.4 Conclusion

Lloyd Shapley made fundamental contributions to the theory of simple games. In particular, he was the first to define and study compound simple games. One of the reasons he thought compound simple games are interesting is that they allow us to tackle the problem of aggregation of players in game theory. In his own words (Shapley, 1963c, pages 4-5):

An important question in the application of \( n \)-person game theory is the extent to which it is permissible to treat firms, committees, political parties, labor unions, nations, etc., as though they were individual players. Behind every game model played by such aggregates, there lies another, more detailed model: a compound game of which the original is the quotient. Given any solution concept, it is legitimate to ask how well it stands up under the aggregation—or disaggregation—of its players. How sensitive are its theoretical predictions to the detail adopted in constructing the model?

Shapley’s (1963c) composition theorem shows that the stable sets proposed by von Neumann and Morgenstern (1944) stand up well under disaggregation: a stable set of the gross model (the quotient), with details added at the component level, becomes a stable set of the refined model (the compound game).

In this article, I have shown that fair stable sets—that is, stable sets that do not discriminate among players based on their names—stand up well under disaggregation in a similar manner.

The composition theorem presented in this article can also be used to shed light on a question that Lloyd Shapley asked in 1978 and that remains open to this day: what is the set \( \mathcal{G} \) of simple games that admit a fair stable set? Rabie (1985) showed that \( \mathcal{G} \) is not the set of all simple games. The composition result presented in this article implies that a game that does not admit a fair stable set must have a factor—or prime quotient in its unique factorization (Shapley, 1967, Theorem 8)—that does not admit a fair stable set.

This raises several natural questions that I leave for future research. For example: Is there any game that admits a fair stable set some of whose factors do not admit a fair stable set? Or: What is the set of prime games that admit a fair stable set? Answers to these
questions might provide the key to the characterization of the set of simple games that admit a fair stable set. In particular, the composition result presented in this article implies that if the answer to the first question is negative, answering the second question would be equivalent to characterizing the set of simple games that admit a fair stable set.
Chapter 2

Coalitional Bargaining with Strategic Choice of Partners

2.1 Introduction

Understanding the determinants of prices and allocations in decentralized markets is one of the central questions in economics. A key behavioral assumption that has proved useful to understand large “competitive” markets is that traders take prices as given, so that the price of each good adjusts to clear the market. But while the properties of markets in which traders take prices as given are well understood, the circumstances under which traders actually take—or act as if they took—prices as given are not.

Therefore, an important and natural endeavor is to understand the strategic incentives of traders in decentralized markets. One stream of literature considers trading in non-stationary environments, in which traders leave the market upon trading and are never replaced, so the market shrinks over time. Recent work along these lines includes Abreu and Manea (2012b, 2012a), Okada (2011) and Elliott and Nava (2016), who study the conditions under which traders have incentives to trade efficiently in networked markets.¹

One issue with the study of strategic behavior in non-stationary environments is that the endogenous evolution of the composition of traders in the market typically leads to multiplicity of equilibria. This is the case in the models of Abreu and Manea (2012b), Okada (2011) and Elliott and Nava (2016), for example, so the focus in these studies is understanding the conditions under which there exists an efficient equilibrium. Therefore, another important stream of research abstracts away from the evolution of the composition of the market, and focuses instead on understanding the incentives in large markets in which traders’ bargaining positions are roughly stationary. An advantage of this approach is that it allows sharp characterizations of the equilibrium behavior. Manea (2011), for example, presents a model of bargaining in stationary networks that has an essentially-unique subgame-perfect equilibrium, and he describes a simple algorithm that characterizes this equilibrium.2

In Manea’s model, bilateral trading opportunities arise stochastically over time and—when these opportunities arise—players bargain over how to share their gains from trade. Nguyen (2015) shows that a generalization of Manea’s model in which trading opportunities involve coalitions of players (of two or more players) has an essentially-unique stationary subgame-perfect equilibrium, and he develops a method to characterize this equilibrium.

In this paper I propose an alternative model of strategic bargaining in stationary markets in which players can strategically choose whom to bargain with. As in Nguyen (2015), I consider the general case in which coalitions of arbitrary size can form, and I allow arbitrary heterogeneity in coalitional productivities and players’ impatience. In contrast to Manea (2011) and Nguyen (2015), however, I allow players to strategically choose their bargaining partners: each player’s opportunities to trade arise stochastically over time and—when a player has such an opportunity—she can choose which coalition of players to propose trading with.

The model I present has an essentially-unique stationary subgame-perfect equilibrium.

---

The structure of this equilibrium is simple: Each non-dummy player—that is, each player that ever trades in equilibrium—has a threshold wage and a set of preferred coalitions such that she always accepts trades that give her more than her threshold wage, and she always proposes that (i) one of her preferred coalitions form, and (ii) each of the members of this coalition receives his threshold wage.

The main contribution of this paper is the description of an algorithm that finds the threshold wage and the preferred coalitions of each player. This algorithm reveals that the coalitions that form in equilibrium have a tier structure, with equilibrium payoffs determined from the top tier down, in the following sense: The payoffs of players that are members of a tier-1 coalition are locally pinned down by the productivity of this coalition and the preferences of its members. Similarly, for every $k \geq 0$, the payoffs of players that are members of a tier-$k$ coalition are locally pinned down by the productivity of this coalition, the preferences of its members, and the threshold wage of at least one player in tier-$s$ coalition, for every $1 \leq s < k$.

In the special case of matching markets—that is, when only coalitions of at most two players are allowed to form—the algorithm characterizes the essentially-unique subgame-perfect equilibrium of the game. The equilibrium structure in this case is described by the preferred-neighbor network, which has a link from node $i$ to node $j$ if the coalition $\{i, j\}$ is the preferred-coalition of player $i$. The algorithm reveals that, generically, each component of the preferred-neighbor network (or submarket) has exactly two mutually-preferred neighbors, whose terms of trade are as in (the random-proposer version of) the canonical bilateral model of Rubinstein (1982). The top-tier coalitions in this case are the two mutually-preferred neighbors in each submarket, so their terms of trade influence the prices of all other trades in their submarket.

The remainder of this paper is organized as follows. In section 3.3 I present the model. In section 2.3 I define algorithm $A$ and I show how it characterizes the essentially-unique stationary equilibrium of the model. In section 2.4 I focus on the special of matching markets. Finally, in section 3.6 I conclude.
2.2 Framework

In subsection 3.3.1 I describe the market, which consists of a set of players together with their preferences, the specification of the productivity of each coalition of players, and an exogenous measure of players’ bargaining power. In subsection 3.3.2 I describe the bargaining protocol that turns the market into a well-defined non-cooperative game. In subsection 3.3.3 I discuss the bargaining frictions in this game. In subsection 3.3.4, subsection 3.3.5 and subsection 3.3.6 I describe the histories, strategies and the notion of equilibrium that I use throughout this article, respectively. Finally, in subsection 2.2.7 I describe the example that I use to illustrate the ideas throughout.

2.2.1 The Market $M$

The (finite) set of players is denoted by $N$, and the power set—that is, the set of all subsets—of $N$ is denoted by $\mathcal{P}(N)$. The worth function $v : \mathcal{P}(N) \rightarrow \mathbb{R}_{\geq 0}$ specifies, for each coalition $T \subseteq N$, the amount of surplus $v(T) \geq 0$ that this coalition generates when it forms. I normalize the worth function so that $v(S) = 0$ for any set $S$ with strictly less than two players.

Players’ discount rate profile is denoted by $r \in \mathbb{R}^N_{>0}$. Fix a probability distribution $q$ on $N$, which can be interpreted as the distribution of some measure of ex-ante bargaining power among players. I refer to the set $M = (N, v, r, q)$ as the market.

2.2.2 The Game $\Gamma$

Given the market $M$ and a period length $\Delta > 0$, I study the following infinite-horizon bargaining game $\Gamma(M, \Delta)$. In each period $t = 0, 1, \ldots$, one player is selected at random—according to the distribution $q$—to be the proposer.

The proposer chooses one coalition $S$ that she is a member of, and proposes a split of the surplus $v(S)$. The members in $S$ respond sequentially in (a pre-specified) order until one of them rejects or all of them accept it. In the former case, no trade occurs this period.
and all players stay in the market for the next period. In the latter case, coalition $S$ forms
and its members exit the market with the agreed shares.

Every player who leaves the market in period $t$ is replaced by a replica in period $t + 1$. Formally, there exists a sequence $i_0, i_1, \ldots, i_T, \ldots$ of players of type $i \in N$. If player $i_T$ exits
the game (following an agreement), player $i_{T+1}$ replaces her in the next period.

All players have common knowledge of the game and perfect information about all the
events preceding any of their decision nodes in the game.

2.2.3 Discount Factors and Bargaining Frictions

Players in $\Gamma(M, \Delta)$ typically have to wait several periods before they can trade. Since they
are impatient, they find this costly. Exactly how costly it is for players to wait a given
amount of periods depends on the length $\Delta$ of time periods. Hence, I often refer to $\Delta$ as the
measure of bargaining frictions.

Given $\Delta$, player $i$’s discount factor $\delta_i$ is given by $e^{-r_i\Delta}$. Player $i$’s period-$T$ utility of
getting surplus $w$ in period $T + \tau$ is $e^{-r_i\Delta\tau}w$.

2.2.4 Histories

There are three types of histories. I denote by $h_t$ a history of the game up to (but not including) time $t$, which is a sequence of $t$ pairs of proposers and responders with corresponding
proposals and responses. I denote by $(h_t; i)$ the history that consists of $h_t$ followed by player
$i$ being selected to be the proposer. I denote by $(h_t; i \to S; z; j)$ the history that consists of
$h_t$ followed by player $i$ proposing that coalition $S$ splits its surplus according to the profile
$z \in \mathbb{R}^S_{\geq 0}$, and all players responding before player $j$ having accepted.

2.2.5 Strategies

A strategy $\sigma_i$ for player $i$ specifies, for all possible histories $h_t$, the offer $\sigma_i(h_t; i)$ that she makes
following history $(h_t; i)$ and her response $\sigma_i(h_t; j \to S; z; i)$ following history $(h_t; j \to S; z; i)$.
I allow for mixed strategies, so \( \sigma_i(h_t; i) \) and \( \sigma_i(h_t; j \rightarrow S; z; i) \) are probability distributions over \( \mathcal{P}(N) \times \mathbb{R}^N_{\geq 0} \) and \{Yes, No\}, respectively.

### 2.2.6 Equilibrium

The strategy profile \((\sigma_i)_{i \in N}\) is a stationary (Markovian-perfect) equilibrium of game \( \Gamma(M, \Delta) \) if it induces a Nash equilibrium in the subgame following every history, and if no player’s strategy conditions behavior on the history of the game except—in the case of a response—on the going proposal. I often refer to a stationary subgame-perfect equilibrium simply as an equilibrium.

### 2.2.7 Example G

I use the following example to illustrate the ideas throughout. The market is \( M_G = (N_G, v_G, r_G, q_G) \) where

- \( N_G = \{1, 2, 3\} \),
- \( v_G(S) \) is negligible except for the three coalitions whose worth is described in Figure 2.1, and
- \( r_G, i = .00i \) and \( q_G, i = 1/3 \) for all \( i \in N_G \).

That is, the market consists of players 1, 2 and 3 with discount rates of .1\%, .2\% and .3\%, respectively. The grand coalition can create 100 units of surplus; the most impatient player
can create 60 units of surplus with the most patient player; and the two most impatient players can create 80 units of surplus. In contrast, the two most patient players generate negligible surplus. All players are equally likely to be proposers each period. I refer to the game $\Gamma(M_G, 1)$ as Game $G$.

### 2.3 Equilibrium Characterization

In section 3.4 I show that the game $\Gamma(M, \Delta)$ admits an essentially-unique stationary subgame-perfect equilibrium, and I introduce the preferred-coalition hypergraph, which is useful to understand the structure of this equilibrium. In subsection 2.3.2 I describe algorithm $A$, which defines each player’s $A$-wage and set of $A$-coalitions. In subsection 2.3.3 I show that algorithm $A$ ends after finitely many steps, and in subsection 2.3.4 I show that this algorithm characterizes the equilibrium of the game $\Gamma(M, \Delta)$. Finally, in subsection 2.3.5, I describe the tier structure of the coalitions that form in equilibrium.

#### 2.3.1 Essentially-Unique Equilibrium

Theorem 2.3.1 shows that the game $\Gamma(M, \Delta)$ admits an essentially-unique subgame-perfect equilibrium.

**Theorem 2.3.1.** Each player $i \in N$ has a threshold wage $w_i \geq 0$ such that, in every stationary subgame-perfect equilibrium of the game $\Gamma(M, \Delta)$,

1. she always accepts offers that give her strictly more than $w_i$, and
2. she always rejects offers that give her strictly less than $w_i$.

**Proof.** Fix a stationary equilibrium of $\Gamma(M, \Delta)$. Let $z_i$ denote player $i$’s expected equilibrium payoff at the beginning of a period before the proposer is selected. Similarly, let $x_i$ denote player $i$’s expected equilibrium payoff when she is selected to be the proposer. We have that

$$z_i := \chi_i x_i$$
where \( \chi_i := \frac{\delta q_i}{1 - \Delta_i + \delta q_i} \). Player \( i \) is indifferent between accepting and rejecting an offer that gives her \( z_i \). Hence, we have that

\[
x_i = \max_{T \subseteq N : i \in T} \left( v(T) - \sum_{j \in T - i} z_j \right),
\]

or, equivalently,

\[
z_i = \chi_i \max_{T \subseteq N : i \in T} \left( v(T) - \sum_{j \in T - i} z_j \right)
\]

The proof now follows from the fact that there exists a unique profile \( z \in \mathbb{R}^N \) that satisfies Equation 2.1. The existence of \( z \) relies on a standard argument based on Brouwer’s fixed point theorem, so I omit it. Proposition A.1.1 in the appendix proves uniqueness.

For each player \( i \in N \), I refer to the set of coalitions that she ever makes offers to in equilibrium—that is, the set of coalitions \( S \) that include \( i \) for which \( v(S) - \sum_{j \in S - i} w_j \) is greatest—as the set of her preferred coalitions. I say that player \( i \) is a dummy if \( v(S) - \sum_{j \in S - i} w_j \leq 0 \) for all coalitions \( S \) that include \( i \).

The game \( \Gamma(M, \Delta) \) admits an essentially-unique equilibrium in the following sense: In every period in which a non-dummy player is the proposer, she proposes that one of her preferred coalitions forms, and offers each of its members their threshold wage; and this proposal is accepted.

Hence, the equilibrium structure can be visualized by depicting the preferred-coalition hypergraph; that is, the directed graph that has a link from player \( i \) to coalition \( S \subseteq N \) if and only if \( S \) is a preferred coalition of player \( i \). Figure ?? depicts the preferred-coalition hypergraph in Example \( G \), together with players’ threshold wages.

I now turn to describing algorithm \( A \), which finds the threshold wages and preferred-coalition hypergraph of \( \Gamma(M, \Delta) \).

### 2.3.2 Description of Algorithm \( A \)

In section 2.3.2 I provide a formal description of algorithm \( A \). In section 2.3.2, I illustrate this description using example \( G \).
Formal Description of Algorithm $A$

Definition 2.3.4 below defines algorithm $A$ inductively in steps. Each step of this algorithm assigns to a subset of players their $A$-wage and their $A$-coalitions, which coincide with their threshold prices and preferred coalitions in $\Gamma(M, \Delta)$, respectively (Proposition 2.3.3). In order to define algorithm $A$, I need three pieces of notation, which I provide in Definition 2.3.1, Definition 2.3.2 and Definition 2.3.3.

**Definition 2.3.1.** I say that player $i$ becomes tied at step $k$ of algorithm $A$ if she is assigned her $A$-wage $y_i$ at this step. For each $k \geq 1$, I say that player $i$ is $k$-tied if she has been tied at some step $k' < k$; otherwise I say that she is $k$-free. For each coalition $P \subseteq N$ and each step $k \geq 1$, I denote by $P_k$ the set of $k$-free players in $P$, and by $v(P_k)$ the amount of surplus that the $k$-free players in $P$ can share after giving to each of the $k$-tied players in $P$ their $A$-wage. That is,

$$v(P_k) := v(P) - \sum_{i \in P - P_k} y_i.$$  

**Definition 2.3.2.** For each coalition $P \subseteq N$ and each $k$-free player $i \in P$, $i$'s $(P, k)$-Rubinstein wage is

$$x_i(P, k) := \frac{\chi_i \prod_{j \in P - i} (1 - \chi_j)}{\prod_{j \in P_k} (1 - \chi_j) + \sum_{j \in P_k} \chi_j \prod_{i \in P_k - j} (1 - \chi_i)} v(P_k), \tag{2.2}$$

where $\chi_i := \frac{\delta_i q_i}{\delta_i + \kappa_i q_i}$ for each $i \in N$.  

---

**Figure 2.2:** The preferred-coalition hypergraph and players’ (approximate) threshold prices in Example G.
Note 2.3.1. For each $k \geq 0$, each coalition $P \subseteq N$, and each $k$-free player $i$, $i$’s $(P,k)$-Rubinstein wage is $i$’s threshold wage in the (essentially-unique) stationary equilibrium of the modification $\Gamma_{P_k}$ of the game $\Gamma(M,\Delta)$ in which the worth of the coalition $P_k$ is $\nu(P_k)$ and the worth of all other coalitions is 0.

Definition 2.3.3. Fix $k \geq 0$. The $k$-best wage of a player is her maximum $(P,k)$-Rubinstein wage over all coalitions $P \subseteq N$. The set of player $i$’s $k$-best coalitions is the set of all coalitions $P \subseteq N$ for which $i$’s $(P,k)$-Rubinstein wage is her $k$-best wage. Coalition $P \subseteq N$ is a $k$-perfect coalition if it is a $k$-best coalition of all its $k$-free members.

Note 2.3.2. Intuitively, for each $k \geq 0$ and each $k$-free player $i$, $i$’s $k$-best coalition is the coalition that $i$ would choose if she were allowed to choose any coalition $S$ and bargain with its $k$-free members in the game $\Gamma_{S_k}$ just described, and $i$’s $k$-best wage is her threshold wage in this hypothetical situation.

Definition 2.3.4. Algorithm $A$ is defined inductively as follows: At each step $k \geq 1$, set the $A$-wage of every player in a $k$-perfect coalition to be her $k$-best wage. End as soon as all players have been assigned their $A$-wage.

For each player $i$, I refer to the set of her $k$-best coalitions in the step $k$ of algorithm $A$ in which she has been tied as her $A$-wage.

Algorithm $A$ in Example $G$

The left panel of Figure ?? depicts all the relevant $(S,1)$-Rubinstein wages, 1-best coalitions, and 1-best wages of each player in example $G$.

Step 1 of algorithm $A$ assigns to the two players in the 1-perfect coalition $\{2,3\}$, their $A$-wage and $A$-coalition to be their 1-best wage and 1-best coalition, respectively. Hence, the $A$-wages of players 2 and 3 are 48 and 32, respectively, and both these players’ $A$-coalition is $\{2,3\}$.

Player 1 is the only 2-free player in step 2. The right panel of Figure ?? depicts her relevant $(S,2)$-Rubinstein wages. The only 2-perfect coalition is $\{1,3\}$, so algorithm $A$ ties.
the 2-free player in this coalition. Hence, player 1’s \( A \)-wage is her 2-best wage 28, and her \( A \)-coalition is her 2-best coalition \( \{1, 3\} \).

Note that—in this example—each player’s \( A \)-wage and \( A \)-coalition coincides with her threshold wage and preferred coalition in game \( G \). Proposition 2.3.3 shows that this is true in general. Before showing this, however, I verify that the \( A \)-wage and \( A \)-coalition of each player are well defined, by showing that algorithm \( A \) always ends after finitely many steps.

### 2.3.3 Algorithm \( A \) Ends in Finitely Many Steps

Proposition 2.3.2 implies that each step of algorithm \( A \) ties at least one player, so algorithm \( A \) ends after finitely many steps.

**Proposition 2.3.2.** For any \( k \geq 1 \), if the set \( N_k \) of \( k \)-free players is not empty, then there is at least one \( k \)-perfect coalition.

**Proof.** Suppose for contradiction that \( k \geq 1 \) is such that \( N_k \neq \emptyset \) and that no coalition is a \( k \)-best coalition of all its \( k \)-free members. Then, since there are finitely many players, there exists a family of coalitions \( \{ P_i \}_{1 \leq i \leq m} \) such that, for each \( i \in \{1, 2, \ldots, m\} \), there exists a \( k \)-free player \( p^i \in P^i \) for whom \( P^i \) is not one of her \( k \)-best coalitions and \( P^{i+1} \) is one of her
k-best coalitions (where superscripts are modulo \( m \)). In other words, for all \( i \in \{1, 2, \ldots, m\} \),

\[
\frac{\chi_i}{\prod_{j \in P_i} (1 - \chi_j)} \left( 1 - \frac{\prod_{j \in P_i} (1 - \chi_j)}{\prod_{j \in P_i} (1 - \chi_j) + \sum_{j \in P_i} \chi_j \prod_{i \in P_i - j} (1 - \chi_i)} \right) v(P_i^j) < \frac{\chi_i}{\prod_{j \in P_i^{j+1}} (1 - \chi_j)} \left( 1 - \frac{\prod_{j \in P_i^{j+1}} (1 - \chi_j)}{\prod_{j \in P_i^{j+1}} (1 - \chi_j) + \sum_{j \in P_i^{j+1}} \chi_j \prod_{i \in P_i^{j+1} - j} (1 - \chi_i)} \right) v(P_i^{j+1})
\]

Multiplying these \( m \) inequalities gives an inequality with identical left- and right-hand sides, a contradiction.

2.3.4 Algorithm \( \mathcal{A} \) Characterizes the Equilibrium of the Game \( \Gamma \)

**Proposition 2.3.3.** Player \( i \)'s \( \mathcal{A} \)-wage is her threshold wage \( w_i \) in the game \( \Gamma(M, \Delta) \). Similarly, the set of \( i \)'s \( \mathcal{A} \)-coalitions is the set of her preferred coalitions in the game \( \Gamma(M, \Delta) \).

**Proof.** Let \( i \in N \). Given that the profile of threshold wages is the unique profile that solves the system in Equation 2.1, we just need to show that for every coalition \( W \subseteq N \) containing \( i \), we have that

\[
y_i \geq \chi_i \left[ v(W) - \sum_{j \in W-i} y_j \right].
\]  

with equality holding if and only if only \( W \) is one of player \( i \)'s \( \mathcal{A} \)-coalitions.

It is easy to check that Equation 2.3 holds with equality when \( W \) is one of \( i \)'s \( \mathcal{A} \)-coalitions. Hence, letting \( S \subseteq N \) be an arbitrary coalition that contains \( i \) and that is not one of \( i \)'s \( \mathcal{A} \)-coalitions, we only need to prove that Equation 2.3 holds with strict inequality for \( S \).

For each \( k \geq 1 \), define

\[
f(i, k) = \chi_i \left( v(S) - \sum_{j \in S-i-S_k} y_j(S) - \sum_{j \in S_k-i} \chi_j(S, k) \right).
\]  

Let \( a \) be the step of algorithm \( \mathcal{A} \) at which player \( i \) becomes tied. Since \( S \) is not one of \( i \)'s \( \mathcal{A} \)-coalition, we have that

\[
y_i > f(i, a).
\]  

Let step \( \beta \geq a \) be such that all players in \( S \) are tied. Since the function \( f(i, k) \) is decreasing, we have that

\[
y_i > f(i, a).
\]  

\( ^3 \)To see why \( f(i, k) \) is decreasing in \( k \), note that every player \( j \) that becomes tied at step \( k \) has an \( \mathcal{A} \)-wage at least as large as \( j \)'s \((W, k)-Rubinstein wage x_j(W, k)\).
in $k$, we get
\[ f(i, \alpha) \geq f(i, \beta) = \chi_i \left( v(S) - \sum_{j \in S-i} y_j \right). \] (2.6)
Combining Equation 2.5 and Equation 2.6 shows that Equation 2.3 holds with inequality for $S$. \hfill \square

2.3.5 The Tier Structure of the Set of Coalitions that Form in Equilibrium

Proposition 2.3.3 shows how algorithm $A$ provides a simple way to compute the equilibrium and to understand its structure. Algorithm $A$ reveals, for example, that the coalitions that form in equilibrium have a tier structure, with players’ threshold wages being determined from the top tier down, in the following sense.

Say that the coalitions that are the $A$-coalition of all its members are in tier 1. The threshold wages of all members of a tier-1 coalition $C$ are their $(C,1)$-Rubinstein wages; in particular, they are locally pinned down by the productivity of the coalition $C$ and the impatience of each of its members.

Arguing inductively in the index $t$ of the tiers, suppose that we have determined both the tier-$t$ coalitions and players\(^4\) for all $t \leq s - 1$. Let the tier-$s$ coalitions be those that are the $A$-coalition of all its members who are not in a tier $h$ coalition for any $h < s$. It follows from the mechanics of algorithm $A$ that the threshold wages of all tier-$s$ players are locally pinned down by the productivity of at least one tier-$k$ coalition for each $k \leq s$, and the discount factors and proposer probabilities of the members of these coalitions.

To gain intuition for this tier structure, consider example $G$. There is one coalition in tier 1 (coalition $\{2, 3\}$) and one coalition in tier 2 (coalition $\{1, 3\}$). Hence, the threshold wages of players 2 and 3 are locally pinned down by the productivity of coalition $\{2, 3\}$ and the time preferences of its members. In contrast, the threshold wage of player 1 is locally pinned down by both the productivity of coalition $\{2, 3\}$ and the time preferences of its members, the productivity of coalition $\{1, 3\}$, and her own impatience.

\[^4\text{Abusing terminology slightly, I refer to players that are members of a tier-$k$ coalition—and are not members of a tier-$h$ coalition for any } h < k \text{ as tier-$k$ players.}\]
2.4 Matching Markets

Let $M^*$ denote market $M$ after imposing that, for any coalition $S$ of more than two players, the worth $v(S)$ be 0. Elliott and Nava (2016) study the non-stationary version of the game $\Gamma(M^*, \Delta)$; that is, the version of this game in which players are not replaced by replicas after they trade. Their model has multiple Markov-perfect equilibria, so they focus on understanding the conditions under which an efficient equilibrium exists. In contrast, Theorem 2.3.1 shows that the game $\Gamma(M^*, \Delta)$ admits an essentially-unique subgame-perfect equilibrium.

**Theorem 2.4.1.** Each player $i \in N$ has a threshold price $w_i \geq 0$ such that, in every subgame-perfect equilibrium of the game $\Gamma(M^*, \Delta)$,

1. she always accepts offers that give her strictly more than $w_i$, and
2. she always rejects offers that give her strictly less than $w_i$.

Manea (2017a) proves an analog of Theorem 3.4.1 in the context of a similar model with a random-match bargaining protocol—that is, a model in which, in each period, one match is randomly selected, and the players in this match bargain over their gains from trade. The proof of Theorem 3.4.1 is analogous to the one in Manea (2017a), so I relegate it to section A.2.

The statement of Theorem 3.4.1 can be strengthened, in the sense that iterated conditional dominance—which is a weaker concept than subgame-perfect equilibrium—determines behavior in the same way.\(^5\) Moreover, iterated conditional dominance defines a procedure that can be used to compute the threshold prices up to any degree of accuracy.

I say that player $j$ is a preferred neighbor of player $i$ in $\Gamma(M^*, \Delta)$ if $\{i, j\}$ is one $i$’s preferred coalitions (or, equivalently, one of $i$’s $A$-coalitions). Note that, generically, each player has exactly one $A$-coalition, which implies that each node of the preferred-neighbor network—that

---

\(^5\)In games of perfect information—like the bargaining game $\Gamma(M^*, \Delta)$—the notion of iterated conditional dominance is weaker than the concept of subgame-perfect equilibrium in the following sense: Every subgame-perfect equilibrium survives the process of iterated conditional dominance (Theorem 4.3 in Fudenberg and Tirole, 1991).
is, the network that has a link from $i$ to $j$ if and only if $j$ is a preferred neighbor of $i$—has one out-link. Combining this observation with the fact that (i) each component of the preferred-neighbor network has at least two mutually-preferred neighbors (which follows directly from the structure of algorithm $\mathcal{A}$) and that (ii) player $a$’s $(\{a, b\}, 1)$-Rubinstein wage is her threshold price in the (random-proposer version of the) canonical bilateral model of Rubinstein (1982) with players $a$ and $b$, we obtain the following corollary of Proposition 2.3.3.

**Corollary 2.4.2.** Generically, each component of the preferred-neighbor network of $\Gamma(M^*, \Delta)$ has exactly two mutually-preferred neighbors, whose terms of trade are as in the (random-proposer version of the) canonical bilateral model of Rubinstein (1982) with players $a$ and $b$.

Corollary 2.4.2 implies that only two players in each submarket are in the top tier. These two players’ terms of trade are as if they were alone in the market and they influence the terms of trade of all other players in the submarket, in the sense that small perturbations in the terms of trade of the top-tier players affect all terms of trade in their submarket.

In Talamàs (2017a), I use this characterization of the equilibrium of the game $\Gamma(M^*, \Delta)$ to explore the connections between market thickness and price dispersion in decentralized networked markets.

### 2.5 Conclusion

I study an infinite-horizon model of coalitional bargaining in a stationary market that—in contrast to standard models—features strategic choice of bargaining partners: In each period, a player is selected to be the proposer and selects a coalition as well as how to share its surplus among its members. Players respond in sequence; if all players accept, they leave the market and—to maintain stationarity—are replaced by replicas.

I show that this model has a unique stationary subgame-perfect equilibrium with a simple structure, and I describe an algorithm that characterizes it. This algorithm reveals that the coalitions that form in equilibrium have a tier structure, with equilibrium payoffs determined from the top tier down.
In the special case of matching markets, the algorithm characterizes the essentially-unique subgame-perfect equilibrium, and it reveals that submarkets endogenously form in equilibrium, with exactly one match in each submarket influencing the terms of trade of all transactions in its submarket.

The algorithm presented in this article is useful to understand the structure of equilibrium behavior in decentralized markets. For example, this algorithm proves very useful to characterize the equilibrium in Talamàs (2017a), where I explore the relationship between market thickness and price dispersion in decentralized networked markets.
Chapter 3

Bargaining in Networked Markets and the Law of One Price

3.1 Introduction

Many markets have a network structure that effectively restricts who can trade with whom. For example, social connections play an important role in labor markets (Granovetter, 1995), in over-the-counter securities’ markets (Li and Schürhoff, 2014), in the apparel industry (Uzzi, 1996), in developing countries’ insurance markets (Fafchamps and Minten, 1999), and even in fish markets (Kirman and Vriend, 2000).

But the classical Walrasian paradigm abstracts away from a market’s network structure and assumes that (i) traders take prices as given and (ii) the price of each good adjusts so that the market clears. The traditional justification for this paradigm is that the fine details of a market’s network structure are not likely to be relevant as long as the market is thick enough. But, of course, this begs the question of how thick is “thick enough.” To answer this question, we need a theory of how prices and allocations are determined in decentralized networked markets.

In this article I develop one such theory, and I use it to explore the conditions under which traders take—or act as if they took—prices as given. In particular, I present an
analytically-tractable non-cooperative model of bargaining in stationary networked markets with strategic choice of partners. This model is similar to the one in Manea (2011); the main difference is that here players strategically choose whom to make offers to, whereas in Manea (2011) players bargain over trading opportunities that arise stochastically over time. As in that paper, an important assumption that makes the model analytically tractable is that the market is stationary: To achieve this as simply as possible, players that leave the market are immediately replaced by replicas. In contrast to Manea (2011), I focus on a buyer-seller network with a homogeneous good, and I allow arbitrary heterogeneity in players’ valuations and discount rates.

As shown in Talamàs (2017b), this framework admits an essentially-unique subgame-perfect equilibrium. The structure of this equilibrium is simple: Generically, each player has a preferred neighbor to whom she always extends offers in equilibrium. Each component (or submarket) of the preferred-neighbor network has exactly one pair of mutually preferred neighbors, whose terms of trade are as in Rubinstein’s (1982) alternating-offers model and determine the price at which all trades occur in their submarket.

This model has bargaining frictions, in the sense that players have to wait for their turn to make offers and—since they are impatient—they find this costly. In order to focus on the friction created by the network structure of the market, I focus on the limit as bargaining frictions vanish.

The main result of this article is a simple method to compute the highest and the lowest equilibrium price in the limit as bargaining frictions vanish. Using this method, I explore the relationship between market thickness and price dispersion in networked markets.

On the one hand, I derive an upper bound on price dispersion that does not depend on the fine details of the market’s network structure. This upper bound suggests a natural measure of market thickness with which the idea that the law of one price holds if the market is thick enough can be formalized.

On the other hand, in the context of a random market with an Erdös and Rényi’s (1959) network structure, I provide a threshold on network thickness above which the law of one
price holds—and below which price dispersion is arbitrary large—with probability one in the limit as the market grows large. This threshold is the same as the one that governs whether the network is connected—that is, whether for every two nodes there is a sequence of links that connects them—in the limit as it grows large. Hence, an unexpected connection arises between network connectedness and the law of one price: In big-enough markets, we expect the law of one price to hold when and only when we expect its network structure to be connected. This is surprising, since no such relationship exists in finite markets.

The remainder of this paper is organized as follows. In section 3.2 I discuss the related literature. In section 3.3 I present the model and the notions of Rubinstein prices, best prices and best mates, which play a key role in the analysis. In section 3.4 I show that the model admits an essentially-unique equilibrium, I characterize this equilibrium, and I present the main result of this paper: A simple method to compute tight bounds on limit market prices. In section 3.5 I provide an upper bound on price dispersion, I explore the connection between market thickness and price dispersion in random markets, and I derive the threshold on market thickness that governs whether the law of one price holds in the limit as the market grows large. Finally, in section 3.6 I conclude.

3.2 Related Literature

The literature on two-sided matching—where the central solution concept is stability—goes back to Gale and Shapley (1962). The most closely related part of this literature features wages and endogenous job characteristics, and includes Crawford and Knoer (1981), Kelso Jr and Crawford (1982), Gul and Stacchetti (1999), Hatfield and Milgrom (2005), Sun and Yang (2006), Ostrovsky (2008), Hatfield et al. (2013, 2015) and Fleiner et al. (2016). A recent strand of this literature including Kakade et al. (2004), Hassidim and Romm (2014), Kanoria et al. (2015) and Donna et al. (2017) focuses on understanding the conditions under which the law of one price holds. In particular, Kakade et al. (2004) and Donna et al. (2017) share the motivation of the present paper to explore the price dispersion that arises in markets whose network structure is modeled as a random graph. In contrast to this literature, the present
paper takes a non-cooperative approach to bargaining in markets.

The literature on non-cooperative bargaining in networks goes back to Calvó-Armengol (2001a, 2001b, 2003), Corominas-Bosch (2004) and Polanski (2007). The main objective of these early studies—as well as that of more recent work including Abreu and Manea (2012b, 2012a), Polanski (2016), Elliott and Nava (2016)—is to determine the relation between network structure, market power and efficiency when prices and allocations are determined by bargaining in non-stationary networked markets.

The stream of literature that is most closely related to this paper features non-cooperative bargaining in stationary markets. Rubinstein and Wolinsky (1985) studied the first such bargaining model, in the case of a market with symmetric buyers and sellers.¹ In the equilibrium of this model, the law of one price holds trivially in the limit as bargaining frictions vanish, and the equilibrium price depends on the relative number of buyers and sellers in the market. Gale (1987) and De Fraja and Sakovics (2001) showed that the law of one price also holds in a version of this model with heterogeneous buyers and sellers.² Manea (2011) challenges this conclusion by showing that—when an exogenous network restricts which buyer-seller pairs can trade—the law of one price holds only under very special conditions. In particular, he shows that submarkets endogenously form in equilibrium, and the price in each submarket depends—as in Rubinstein and Wolinsky (1985)—on the relative number of buyers and sellers that trade in it. I borrow much of the analytical framework from this literature. The key difference is that I allow players to strategically choose whom to make offers to, which leads to substantially different economic forces. For example, in my model—as in Manea (2011)—submarkets endogenously form in equilibrium, but—in contrast to Manea (2011)—the price in each submarket is determined by the preferences of exactly two players that trade in it.

¹The main innovation of Rubinstein and Wolinsky (1985) with respect to the previous literature—including Diamond and Maskin (1979), Diamond (1981) and Diamond (1982)—is the study of a sequential bargaining framework in which market prices arise in equilibrium, instead of trades being concluded with an instantaneous agreement which divides the surplus in an arbitrary predetermined way.

²See Osborne and Rubinstein (1990) and Gale (2000) for textbook treatments of these ideas.
In the original model of Manea (2011), each pair of players can generate either one or zero units of surplus. Nguyen (2015) generalizes this model to a setting in which coalitions of various sizes can create different amounts of surplus, and he characterizes the unique stationary equilibrium payoffs as the solution to a convex optimization problem. Polanski and Vega-Redondo (2016) use Nguyen’s methodology to characterize the conditions under which the law of one price holds in a generalization of the model of Manea (2011) in which players have heterogeneous preferences. In this paper I provide the analogous conditions in a setting in which players strategically choose whom to bargain with. In addition, I derive an upper bound on price dispersion that does not depend on the fine details of the market’s network structure, and I use it to provide a formal version of the idea that the law of one price holds if the market is thick enough. Also, in the context of a random market with an Erdös-Rényi network structure, I explore the relationship between market thickness and price dispersion, and I derive conditions under which the law of one price holds in the limit as it grows large.

3.3 Framework

In subsection 3.3.1 I describe the market, which consists of a set of buyers and sellers in a network together with their preferences and an exogenous measure of players’ bargaining power. In subsection 3.3.2 I describe the bargaining protocol that turns the market into a well-defined non-cooperative game. In subsection 3.3.3 I discuss the bargaining frictions in this game. In subsection 3.3.4, subsection 3.3.5 and subsection 3.3.6 I describe the histories, strategies and the notion of equilibrium that I use throughout this article, respectively. In subsection 3.3.7 I describe an example. Finally, in subsection 3.3.8, I describe the notions of Rubinstein prices, best prices and best mates, which play a key role in the analysis.

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3See also Polanski and Vega-Redondo (2013).

4In Talamàs (2017b), I show that this model can be generalized to a coalitional setting.
3.3.1 The Market $M$

The set of players is $N = B \cup S$, where $B = \{b_1, \ldots, b_n\}$ and $S = \{s_1, \ldots, s_m\}$ are disjoint sets of buyers and sellers, respectively. All sellers have one unit of a homogeneous good, and each player $i$ has a reservation value $v_i \in \mathbb{R}_{\geq 0}$ for that good. Player $i$’s discount rate is $r_i \in \mathbb{R}_{>0}$.

Fix a buyer–seller network $g$ with vertex set $N$; that is, a graph $(N, E)$ with link set

$$E \subset \{(i,j) \in N \times N \mid i \in B \text{ if and only if } j \in S\} \text{ s.t. } (j,i) \in E \text{ whenever } (i,j) \in E.$$ 

In order to simplify the formal statements that follow, I adopt the convention that each player is her own neighbor, so the set of player $i$’s neighbors is

$$N_i := \{i\} \cup \{j \in N \mid (i,j) \in E\}.$$

Fix a probability distribution $q$ on $N$, which can be interpreted as the distribution of some measure of ex-ante bargaining power among players. I refer to the set $M = (N, v, r, g, q)$ as the market.

3.3.2 The Game $\Gamma$

Given the market $M$ and a period length $\Delta > 0$, I study the following infinite-horizon bargaining game $\Gamma(M, \Delta)$. In each period $t = 0, 1, \ldots$, one player is selected at random—according to the distribution $q$—to be the proposer. The proposer then chooses one of her neighbors to be the receiver, and offers her a trade at a certain price $p \in \mathbb{R}_{\geq 0}$. If the receiver accepts the offer, the proposer and the receiver trade at the specified price and exit the market. Otherwise, they stay in the market for the next period.\(^5\)

Every player who leaves the market in period $t$ is replaced by a replica in period $t + 1$. Formally, there exists a sequence $i_0, i_1, \ldots, i_T, \ldots$ of players of type $i \in N$ (player $i$’s type

\(^5\)Note that I only allow players to leave the market after trading. For example, if a seller wants to consume her own good, she has to wait to become the proposer and make an offer to herself. This is for simplicity; nothing of substance would change if I allowed players to leave the market at any point.
is determined by her position in the network, the probability \( q_i \) that she is selected to be the proposer, her reservation value \( v_i \), and her discount rate \( r_i \). If player \( i \) exits the game (following an agreement), player \( i_{\tau+1} \) replaces her in the next period.

All players have common knowledge of the game and perfect information about all the events preceding any of their decision nodes in the game.

### 3.3.3 Discount Factors and Bargaining Frictions

Players in \( \Gamma(M, \Delta) \) typically have to wait several periods before they can trade. Since they are impatient, they find this costly. Exactly how costly it is for players to wait a given amount of periods depends on the length of time periods \( \Delta \). Hence, I often refer to \( \Delta \) as the measure of bargaining frictions.

Given \( \Delta \), player \( i \)'s discount factor \( \delta_i \) is given by \( e^{-r_i \Delta} \). Buyer \( b \)'s period \( T \) utility of trading at price \( p \) in period \( T + \tau \) is \( e^{-r_b \Delta \tau} (v_b - p) \). Similarly, seller \( s \)'s period \( T \) utility of trading at price \( p \) in period \( T + \tau \) is \( e^{-r_s \Delta \tau} (p - v_s) \).

### 3.3.4 Histories

There are three types of histories. I denote by \( h_t \) a history of the game up to (but not including) time \( t \), which is a sequence of \( t \) pairs of proposers and responders with corresponding proposals and responses. I denote by \( (h_t; i) \) the history that consists of \( h_t \) followed by player \( i \) being selected to be the proposer. I denote by \( (h_t; i \rightarrow j; p) \) the history that consists of \( h_t \) followed by player \( i \) offering player \( j \) to trade at price \( p \in \mathbb{R}_{\geq 0} \).

### 3.3.5 Strategies

A strategy \( \sigma_i \) for player \( i \) specifies, for all possible histories \( h_t \), the offer \( \sigma_i(h_t; i) \) that she makes following history \( (h_t; i) \) and the response \( \sigma_i(h_t; j \rightarrow i; p) \) that she gives to \( j \) following history \( (h_t; j \rightarrow i; p) \). I allow for mixed strategies, so \( \sigma_i(h_t; i) \) and \( \sigma_i(h_t; j \rightarrow i; p) \) are probability distributions over \( N \times \mathbb{R}_{\geq 0} \) and \( \{ \text{Yes, No} \} \), respectively.
3.3.6 Equilibrium

The strategy profile \((\sigma_i)_{i \in N}\) is a subgame-perfect equilibrium in \(\Gamma(M, \Delta)\) if it induces a Nash equilibrium in each of its subgames. I often refer to a “subgame-perfect equilibrium” simply as an “equilibrium.”

3.3.7 Example W

Throughout this article, I use the following example to illustrate the key ideas.\(^6\) The market in this example is \(M_W = (N_W, v_W, r_W, g_W, q_W)\) where

- \(N_W = \{b_1, b_2, b_3, s_1, s_2\}\),
- \(v_W, g_W\) are described in Figure ??, and
- \(r_{W,i} = .02\) and \(q_{W,i} = .2\) for all \(i \in N_W\).

That is, the market consists of two sellers and three buyers with a common discount rate of 2%. All players are equally likely to become proposers each period. Both sellers can trade with buyer \(b_3\), who places relatively low value on the good. In addition, seller \(s_2\) can trade with buyer \(b_2\), who values the good more than buyer \(b_3\) does, and seller \(s_1\) can trade with buyer \(b_1\), who values the good the most. I refer to the game \(\Gamma(M_W, 1)\) as Game W.

\(^6\)I thank Michael Creel for the observation that the natural name for this example—given Figure ??—is \(W\).
3.3.8 Rubinstein Prices, Best Prices and Best Mates

In this section I define the notions of Rubinstein prices, best prices and best mates—in section 3.3.8 for arbitrary bargaining frictions and, in section 3.3.8, in the limit as bargaining frictions vanish. These concepts play a central role in the characterization of the equilibrium of $\Gamma(M,\Delta)$.

**Arbitrary Bargaining Frictions**

**Definition 3.3.1.** For any two players $i$ and $j$, let the $(i,j)$-Rubinstein price be

$$p_{i,j} := \lambda_{i,j}v_j + (1 - \lambda_{i,j})v_i$$

where $\lambda_{i,j} := \frac{a_i - a_j}{1 - a_i a_j}$ and $\alpha_i := \frac{\delta_i a_i}{1 - \delta_i a_i + \delta_j a_j}$.

When $i$ is a buyer, $j$ is a seller, and $v_i > v_j$, the $(i,j)$-Rubinstein price $p_{i,j}$ is exactly player $i$’s threshold price (that is, the maximum price that she accepts) in the unique subgame-perfect equilibrium of the version of the game $\Gamma(M,\Delta)$ in which only players $i$ and $j$ are allowed to make and accept offers. The analogous statement holds for sellers. This modified game is a variation of the canonical alternating-offers bilateral model of Rubinstein (1982). I refer to the equilibrium outcome of this modified game as the outcome when players $i$ and $j$ bargain à la Rubinstein.

Figure ?? depicts the relevant Rubinstein prices in Example W, together with each player’s best price and best mate, defined as follows.

**Definition 3.3.2.** The best price of buyer (seller) $i$ is the minimum (maximum) $(i,j)$-Rubinstein price over all her neighbors $j$. Player $j$ is a best mate of player $i$ if the $(i,j)$-Rubinstein price is equal to $i$’s best price.

Intuitively, a player’s best mate is the player that she would choose if she could pick one player and engage in bilateral trading à la Rubinstein with her; and her best price is her threshold price in this hypothetical situation.
Figure 3.2: Rubinstein Prices, Best Prices and Best Mates in Game W.

Note 3.3.1. Since each player is her own neighbor, each buyer’s best price is bounded above by her reservation value. In particular, if all sellers that buyer $i$ has access to have a reservation value higher than $v_i$, then buyer $i$ is her own best mate. The analogous statement holds for sellers.

**Limit as Bargaining Frictions Vanish**

I refer to the $(i,j)$-Rubinstein price in the limit as bargaining frictions vanish as the *limit $(i,j)$-Rubinstein price* $p_{ij}^\ast$. From Equation 3.1, we see (using l’Hôpital’s rule) that

$$p_{ij}^\ast := \frac{r_i/q_i}{r_i/q_i + r_j/q_j} v_i + \frac{r_j/q_j}{r_i/q_i + r_j/q_j} v_j.$$  \hspace{1cm} (3.2)

This limit makes transparent the fact that the $(i,j)$-Rubinstein price is a measure of the relative bargaining power of players $i$ and $j$ (when they have no other option but to bargain bilaterally). The more *effectively impatient* buyer $i$ is relative to seller $j$, the higher the price $i$ and $j$ trade at in equilibrium when they bargain à la Rubinstein.

The relevant measure of $i$’s effective impatience when bargaining frictions are negligible is $r_i/q_i$. This reflects the fact that the effective impatience of a player depends on both her genuine impatience and her ability to make take-it-or-leave-it offers.

Note 3.3.2. For each pair of players $(i,j)$, the limit $(i,j)$-Rubinstein price coincides with the
limit \((j,i)\)-Rubinstein price. Intuitively, the bargaining advantage of the proposer vanishes as bargaining frictions vanish; in the limit, equilibrium trade occurs at the same price independently of who is the proposer.

Figure ?? depicts the relevant limit Rubinstein prices in Example \(W\), together with each player’s limit best price and limit best mate; that is, her best price and her best mate in the limit as bargaining frictions vanish.

### 3.4 Equilibrium

In this section I characterize the essentially-unique subgame-perfect equilibrium of the game \(\Gamma(M,\Delta)\). In subsection 3.4.1 I present the result that shows that the game \(\Gamma(M,\Delta)\) indeed has an essentially-unique equilibrium. In subsection 3.4.2 I characterize the structure of this equilibrium. Finally, in subsection 3.4.3 I present the main result of this article: A simple method to compute the highest and the lowest market price in the limit as bargaining frictions vanish.
3.4.1 Essentially-Unique Equilibrium

In this section I show that the game \( \Gamma(M, \Delta) \) admits an essentially-unique subgame-perfect equilibrium, and I introduce the \textit{preferred-neighbor network}, which is useful to understand the structure of this equilibrium.

\textbf{Theorem 3.4.1.} Each buyer \( i \) has a threshold price \( r_i \leq v_i \) such that in every subgame-perfect equilibrium of the game \( \Gamma(M, \Delta) \),

1. she always accepts trades at price \( p < r_i \), and
2. she always rejects trades at price \( p > r_i \).

Similarly, each seller \( i \) has a threshold price \( r_i \geq v_i \) such that in every subgame-perfect equilibrium of the game \( \Gamma(M, \Delta) \),

1. she always accepts trades at price \( p > r_i \), and
2. she always rejects trades at price \( p < r_i \).

Theorem 3.4.1 is a special case of Theorem 4.1 in Talamàs (2017b), which shows—using techniques developed in Manea (2017a)—the analog result in a more general model that the one described in section 3.3 (in which each pair of players \( \{i, j\} \) can generate a pair-specific amount \( s_{i,j} \in \mathbb{R} \) of surplus).

The statement of Theorem 3.4.1 can be strengthened, in the sense that \textit{iterated conditional dominance}—which is a weaker solution concept than subgame-perfect equilibrium—determines behavior in the same way.\footnote{In games of perfect information—like the bargaining game \( \Gamma(M, \Delta) \)—the notion of iterated conditional dominance is weaker than the concept of subgame-perfect equilibrium in the following sense: Every subgame-perfect equilibrium survives the process of iterated conditional dominance (Theorem 4.3 in Fudenberg and Tirole, 1991).} Moreover, iterated conditional dominance defines a procedure that can be used to compute the threshold prices up to any degree of accuracy.

A corollary of Theorem 3.4.1 is that a buyer (seller) all of whose neighbors (other than herself) have a threshold price strictly higher (lower) than her reservation value never trades.
in equilibrium.\(^8\) I refer to these players as *dummies*.

Another corollary of Theorem 3.4.1 is that, in equilibrium, each non-dummy buyer (seller) \(i\) always extends offers to one her *preferred neighbors*—that is, one of her neighbors with lowest (highest) threshold price—and all these offers are always accepted.

Hence, the equilibrium outcome can be visualized by depicting the *preferred-neighbor network* \(G^\Delta\), which is a directed network that has a link from player \(i\) to player \(j\) if and only if player \(j\) is a preferred neighbor of player \(i\). Figure 3.4 depicts the preferred-neighbor network and the threshold prices in Game \(W\).

### 3.4.2 Equilibrium Structure

Talamàs (2017b) describes an algorithm that can be used to find both the equilibrium threshold prices and the preferred neighbors of each player in the bargaining game \(\Gamma(M, \Delta)\) in a more general setup than the one presented in section 3.3. In section B.1, I prove Proposition 3.4.2 below using this algorithm.

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\(^8\) Clearly, such a player do not make acceptable offers in equilibrium. To see that she never receives acceptable offers either, note that—since players are impatient—their threshold prices are less beneficial for them than the price at which they can trade when they are the proposers. Hence, in equilibrium all the neighbors of the dummy buyer (seller) \(i\) make offers that involve a price above (below) \(i\)'s reservation value.
Proposition 3.4.2. Generically:9

1. Each player has a unique preferred neighbor.

2. Each component of the preferred-neighbor network $G^\Delta$ has exactly two mutually-preferred neighbors.

3. If $i$ and $j$ are mutually-preferred neighbors, $i$’s threshold price is the $(i, j)$-Rubinstein price.

4. There exists $\Delta > 0$ and $G$ such that $G^\Delta = G$ for all $\Delta < \Delta$.

5. As bargaining frictions vanish, the threshold price of each player in a component of $G$ with mutually-preferred neighbors $i$ and $j$ converges to the limit $(i, j)$-Rubinstein price.

Proposition 3.4.2 implies that mutually-preferred neighbors in $G$ determine the equilibrium prices in the limit as bargaining frictions vanish; for this reason, I refer to them as key players.

3.4.3 Identifying the Extreme Key Players

Let $H$ and $L$ denote the components of $G$ with the highest and the lowest limit prices, respectively. In this section I present Proposition 3.4.4, which shows how to identify the key players in $H$ and $L$. By Proposition 3.4.2, the highest and the lowest limit prices are determined by the limit Rubinstein prices of these key players. Hence, Proposition 3.4.4 provides tight bounds on market prices in the limit as bargaining frictions vanish.

I focus on the case in which the generic Condition 3.4.3 holds.

Condition 3.4.3. No two limit Rubinstein prices coincide; that is, if $\{i, j\} \neq \{k, l\}$, then $p_{i,j}^* \neq p_{k,l}^*$.

Under Condition 3.4.3, there is a unique buyer $\beta$ with highest limit best price, and a unique seller $\sigma$ with lowest limit best price.

Proposition 3.4.4. Assume Condition 3.4.3 holds. Buyer $\beta$ and her limit best mate are the key players in $H$. Similarly, seller $\sigma$ and her limit best mate are the key players in $L$.

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9With probability one if independent error terms are drawn from a continuous, atomless distribution and added to players’ valuations (or discount rates or probability distributions).
I present the proof of Proposition 3.4.4 in section B.2. Its intuition is simple. On the one hand, the buyer with the highest threshold price receives offers from all her neighbors, so her preferred-neighbor is her best mate, and her threshold price is her best price. Hence, the highest limit best price among buyers is an upper bound on the highest limit threshold price among buyers. On the other hand, a similar argument implies that the buyer with highest limit best price is a key player, which implies that the highest limit best price among buyers is a lower bound on the highest limit threshold price among buyers. The first part of Proposition 3.4.4 follows from combining these two observations with Condition 3.4.3. The second part of Proposition 3.4.4 is proved analogously.

In Game $W$, for example, the buyer with highest limit best price is $b_1$ (see Figure ??). So $b_1$ and her limit best mate $s_1$ are the key players in the component of the limit preferred neighbor network of Game $W$ with the highest limit price, which is equal to $p^*_{b_1,s_1} = 50$. Similarly, the seller with lowest limit best price is $s_2$. So $s_2$ and her limit best mate $b_2$ are the key players in the component of the limit preferred neighbor network with lowest limit price, which is equal to $p^*_{b_2,s_2} = 40$. Buyer $b_3$ is in the component with key players $b_2$ and $s_2$, since she prefers to trade at price 40 than 50.

3.5 Market Thickness and Price Dispersion

Proposition 3.4.4 states that—in the limit as bargaining frictions vanish—the highest and the lowest equilibrium prices in $\Gamma(M,\Delta)$ are the highest limit best price among buyers and the lowest limit best price among sellers, respectively. In this section, I use this result to provide formal versions of the idea that the law of one price holds if the market is thick enough.

In subsection 3.5.1, I provide an upper bound on price dispersion that does not depend on the fine details of the market’s network structure. In subsection 3.5.2 I explore the connection between market thickness and price dispersion in the context of a random market. Finally, in subsection 3.5.3 I characterize the conditions under which these bounds converge to the same value as the network grows large.
3.5.1 An Upper Bound on Price Dispersion

In this section I present a simple corollary of Proposition 3.4.4 that gives an upper bound on the difference between the highest and the lowest limit market price. This upper bound does not depend on the fine details of the network structure, and it provides a useful measure of market thickness.

Consider for simplicity a market with homogeneous discount rates and proposer probabilities, and normalize the highest and the lowest values in the market to 0 and 1, respectively. Let \( \sigma \geq 0 \) be the smallest number such that:

- each buyer has a link to at least one player \( i \) with \( v_i \leq \sigma \), and
- each seller has a link to at least one player \( i \) with \( v_i \geq 1 - \sigma \).

Proposition 3.5.1 is a simple corollary of Proposition 3.4.4.

**Proposition 3.5.1.** The difference between the highest and the lowest limit market price is bounded above by \( \sigma \).

**Proof.** By Proposition 3.4.4, it is enough to show that the difference between the highest limit best price among buyers and the lowest limit best price among sellers is bounded above by \( \sigma \). Note that no buyer can have a higher limit best price than a buyer who values the good at 1 and who does not have access to any seller with value smaller than \( \sigma \). Hence, \( \frac{1 + \sigma}{2} \) is an upper bound on the highest limit best price among buyers. Similarly, \( \frac{1 - \sigma}{2} \) is a lower bound on the lowest limit best price among sellers. Hence, \( \sigma \) is an upper bound on the difference between the highest and the lowest limit best price among buyers and sellers, respectively. \( \square \)

Proposition 3.5.1 can be regarded as a formal version of the idea that the law of one price holds if the market is thick enough. A condition that guarantees that the law of one price holds is that all buyers have access to a seller with lowest valuation, and that all sellers have access to a buyer with highest valuation—which is a substantially weaker requirement than the complete connectivity of the network structure of the market.
Beyond this, we can use $\sigma$ to define a measure $\tau$ of market thickness, such that the upper bound on price dispersion is smaller the thicker is the market. For example, $\tau := 1/(1 + \sigma)$ serves as such a measure, where $1/2 \leq \tau \leq 1$. First, when no buyer-seller pair can trade, market thickness $\tau$ is equal to its lowest value $1/2$. Second, adding a link in the market’s network structure always (weakly) increases $\tau$. Third, in the case in which the network structure of the market is complete (that is, every buyer-seller pair can trade) market thickness $\tau$ is equal to its maximum value 1.

### 3.5.2 Market Thickness and Price Dispersion in Random Market

In this section I explore the relationship between market thickness and price dispersion by simulating a random market generated as follows: The market consists of $n \in \{10, 30, 50, 100, 200\}$ buyers and $n$ sellers. The valuations of sellers are independently and identically distributed; sellers’ valuations are drawn from the uniform distribution with support $[0, 5]$, and buyers’ valuations are drawn from the uniform distribution with support $[5, 10]$.

Figure ?? plots the average price dispersion (difference between the highest and lowest price) over 500 realizations of this market when the expected number of partners is $k \in \{1, 2, \ldots, 18\}$. Figure ?? shows that, in large markets, the expected number of partners needed for the law of one price to approximately hold is a small fraction of the total number of potential partners. For example, a market with 200 buyers and 200 sellers in which each player expects to be able to trade with only 5% of the agents in the other side of the market has an expected price dispersion of less than 20% of the range of valuations in the market.

### 3.5.3 Law of One Price in Large Random Market

Figure ?? suggests the existence of a threshold on the rate of growth of the expected number of partners that guarantees that the law of one price holds as the market grows large. I now turn to formalizing this idea in a slightly more general framework than the one considered in subsection 3.5.2.
Random Market $\tilde{M}$

Consider the following random market $\tilde{M} = (N, \tilde{v}, r, \tilde{g}, q)$, where

- the player set $N$ consists of $bn$ buyers and $sn$ sellers, where $n$, $b$, and $s$ are natural numbers,
- each player $i$’s value $\tilde{v}_i$ is drawn from an atomless distribution $f$ with full support $[\underline{v}, \overline{v}] \subset \mathbb{R}_{\geq 0}$,
- $r$ is a constant vector; that is, all players are equally impatient,
- for any buyer-seller pair $(i, j)$, $\tilde{g}$ has a link from $i$ to $j$ with probability $0 < p(n) < 1$ (independent across links).
- $q$ is the uniform distribution on $N$.

Market $\tilde{M}$ satisfies the assumptions of Proposition 3.4.2 and Proposition 3.4.4 with probability one, so these two propositions hold with probability one in $\tilde{M}$ as well.
The Law of One Price

**Definition 3.5.1.** The law of one price holds with accuracy \( \kappa > 0 \) in market \( M \) if—for all sufficiently small values of \( \Delta \)—the difference between the highest and the lowest threshold price in the game \( \Gamma(M, \Delta) \) is smaller than \( \kappa \).

Market Thickness and the Law of One Price

**Definition 3.5.2.** A function \( t(n) \) is a threshold function for the law of one price if, as \( n \to \infty \), the probability that the law of one price holds for any degree of accuracy \( 0 < \kappa < \frac{\bar{\nu} + \underline{\nu}}{2} \) in \( \tilde{M} \) converges to:

- 1 when \( p(n)/t(n) \) converges to \( \infty \), and
- 0 when \( p(n)/t(n) \) converges to 0.

Proposition 3.5.2 below states that \( \log(n)/n \) is a threshold for the law of one price. The same threshold governs whether the network is connected—that is, whether for every two nodes there is a sequence of links that connects them.\(^{10}\) Hence, an unexpected connection arises between network connectedness and the law of one price: In big-enough markets we expect the law of one price to hold when and only when we expect its network structure to be connected. This is surprising, since no such relationship exists in finite markets. Recall, for example, that the law of one price does not hold in the limit as bargaining frictions in the game \( W \), even though its network structure is connected.

**Proposition 3.5.2.** The function \( \frac{\log(n)}{n} \) is a threshold function for the law of one price.

It is well known that \( \log(n)/n \) is a threshold for the non-existence of isolated players. Lemma 3.5.3—which is the key behind Proposition 3.5.2 and is proved in section B.3 using standard techniques—is a variation of this result.

**Definition 3.5.3.** Given an interval \( P \subset [\underline{\nu}, \bar{\nu}] \), I say that a player is \( P \)-isolated if she has no link to any player with preferences in \( P \).

\(^{10}\)See Jackson (2008); the classical reference is Erdős and Rényi (1959).
Lemma 3.5.3. Given any two nonempty intervals $I_1, I_2 \subset [\underline{v}, \overline{v}]$, the function

$$t(n) = \frac{\log(n)}{n}$$

is the threshold for the non-existence of any $I_1$-isolated player in $I_2$. That is, as $n \to \infty$, the probability that there exists an $I_1$-isolated player in $I_2$ converges to:

- 1 when $p(n)/t(n)$ converges to 0, and
- 0 when $p(n)/t(n)$ converges to $\infty$.

Hence, below the threshold $\log(n)/n$, for any $\epsilon > 0$, the probability that there is a buyer $i$ with $v_i \in [\overline{v} - \epsilon, \overline{v}]$ all of whose neighbors have preferences in $[\overline{v} - \epsilon, \overline{v}]$ converges to 1 as $n \to \infty$. This implies that the probability that the highest limit best price among buyers is arbitrarily close to $\overline{v}$ converges to 1 as $n \to \infty$. The analogous argument shows that the probability that the lowest limit best price among sellers is arbitrarily close to $\underline{v}$ converges to 1 as $n \to \infty$. Combining this observation with Proposition 3.4.4, we conclude that the probability that the law of one price holds with accuracy $\kappa < \overline{v} - \underline{v}$ converges to 0 as $n \to \infty$.

In contrast, above the threshold $\log(n)/n$, for any $\epsilon > 0$, the probability that there is a buyer $i$ with no neighbors whose value is in $[\underline{v}, \underline{v} + \epsilon]$ converges to 0 as $n \to \infty$. This implies that the probability that the highest limit best price among buyers is arbitrarily close to $\frac{\overline{v} + \underline{v}}{2}$ converges to 1 as $n \to \infty$. The analogous argument shows that the probability that the lowest limit best price among sellers is arbitrarily close to $\frac{\overline{v} + \underline{v}}{2}$ converges to 1 as $n \to \infty$. Combining this observation with Proposition 3.4.4, we conclude that the probability that the law of one price holds with accuracy $0 < \kappa < \overline{v} - \underline{v}$ converges to 1.

3.6 Conclusion

I study an infinite horizon model of bargaining in a stationary networked market for a homogeneous good that—in contrast to standard models—features strategic choice of bargaining partners.
In this context, I describe a simple method to compute the highest and the lowest equilibrium price in the limit as bargaining frictions vanish, and I use this method to explore the conditions under which the law of one price holds (at least approximately) in decentralized networked markets.

On the one hand, I provide an upper bound on price dispersion that does not depend on the fine details of the market’s network structure, and I show how this upper bound can be used to formalize the idea that the law of one price holds if the market is thick enough.

On the other hand, in order to understand the conditions under which price dispersion is low, I study a random market with an Erdős-Rényi network structure. In this context, I obtain a threshold on network thickness above which the law of one price holds—and below which price dispersion is arbitrary large—with probability one in the limit as the market grows large. This threshold is the same as the one that governs whether the network is connected—that is, whether for every two nodes there is a sequence of links that connects them—in the limit as it grows large. Hence, an unexpected connection arises between network connectedness and the law of one price: In big-enough markets, we expect the law of one price to hold when and only when we expect its network structure to be connected. This is surprising, since no such relationship exists in finite markets.

While in this article I have focused on bounding market prices in the limit as bargaining frictions vanish, the equilibrium structure of the model can be fully characterized for all bargaining frictions using the algorithm described in Talamàs (2017b) based on the notions of Rubinstein prices, best prices and best mates defined here. Indeed, this algorithm characterizes the stationary subgame-perfect equilibrium for any degree of bargaining frictions in a more general setup in which coalitions of arbitrary size can generate surplus and write contracts on how to share it.

Two important assumptions of this article are that players are replaced immediately after trading and that the model is common knowledge and features perfect information. I leave the exploration of the consequences of relaxing these assumptions for future research. On the one hand, the replica assumption is made to simplify the analysis, and I expect the main
ideas of this article to go through in more realistic models of endogenous entry; for example, see Manea (2017b) for the version of Manea (2011) featuring endogenous entry. On the other hand, the consequences of relaxing the perfect information assumption—and especially the common knowledge assumption—are less clear; but see Condorelli et al. (2017) for a recent study of non-cooperative bargaining in networks featuring incomplete information.
References


Appendix A

Appendix to Chapter 2

A.1 Supplement to the Proof of Theorem 2.3.1

Proposition A.1.1. The system in Equation 2.1 has at most one solution.

Proof. This proof follows closely the proof of Proposition 7.1 in Ray (2007). Suppose, contrary to the claim, that there are two solutions \( z \) and \( z' \) to Equation 2.1. Define \( K \) to be the set of all indices in which the solutions differ; that is,

\[
K := \{ i \in S \mid z_i \neq z'_i \},
\]

pick the index \( k \in K \) for which \( \frac{1 - \chi_k}{\lambda_k} z_k \) is greatest, and suppose without loss of generality that \( \frac{1 - \chi_k}{\lambda_k} z_k \) is an upper bound on \( \{ \frac{1 - \chi_i}{\lambda_i} z'_i \}_{i \in K} \).

Choose \( T \subseteq N \) with \( k \in T \) such that

\[
z_k = \chi_k \left( v(T) - \sum_{j \in T - k} z_j \right) \tag{A.1}
\]

Of course,

\[
z'_k \geq \chi_k \left( v(T) - \sum_{j \in T - k} z'_j \right) \tag{A.2}
\]

By Lemma A.1.2 below, for all \( j \in T \) we have

\[
\frac{1 - \chi_j}{\lambda_j} z_j \geq \frac{1 - \chi_k}{\lambda_k} z_k
\]
So, given our choice of $k \in K$, it must be that $z'_j \leq z_j$ for all $j \in T$. But then, Equation A.1 and Equation A.2 together imply that $z'_k \geq z_k$, a contradiction. \hfill \Box

**Lemma A.1.2.** Let $z$ be a solution to Equation 2.1, and suppose that

$$z_i = \chi_i \left( v(T) - \sum_{k \in T-i} z_k \right).$$

Then, for all $j \in T$ we have

$$\frac{1 - \chi_i}{\chi_j} z_j \geq \frac{1 - \chi_i}{\chi_i} z_i.$$ (A.3)

**Proof.** Let the coalition $T \subseteq N$ attain the maximum in Equation 2.1. We have that

$$z_i = \chi_i \left[ v(T) - \sum_{k \in T-i} z_k \right]$$ (A.4)

and that, for all $j \in T - i$,

$$z_j \geq \chi_j \left[ v(T) - \sum_{k \in T-j} z_k \right].$$ (A.5)

Adding $-\chi_j z_j$ to both sides of Equation A.5 and using Equation A.4, we see that

$$(1 - \chi_j) z_j \geq \chi_j \left[ v(T) - \sum_{k \in T} z_k \right] = \frac{\chi_j}{\chi_i} [z_i - \chi_i z_i]$$

which is equivalent to Equation A.3. \hfill \Box

**A.2 Proof of Theorem 3.4.1**

Fudenberg and Tirole (1991, page 128) define iterated conditional dominance on the class of multi-stage games with observed actions as follows.

**Definition A.2.1.** Action $a^t_i$ available to some player $i$ at information set $H_i$ is conditionally dominated if every strategy of player $i$ that assigns positive probability to action $a^t_i$ in the information set $H_i$ is strictly dominated. Iterated conditional dominance is the process that, at each round, deletes every conditionally-dominated action given the strategies that have survived all the previous rounds.
Fudenberg and Tirole (1991) show how iterated conditional dominance solves the alternating-offers bilateral model of Rubinstein (1982), and Manea (2017a) shows how iterated conditional dominance also solves a wide class of models similar to the one considered in this article. I prove Theorem A.2.1 using the techniques developed in Manea (2017a).

**Theorem A.2.1.** Every player $i$ has a wage $w_i$ such that—after the process of iterated conditional dominance—she always accepts (rejects) an offer that gives her strictly more (less) than $w_i$.

**Proof.** The proof consists of two steps. First, I define recursively two sequences $(m^k_i)_{i \in N}$ and $(M^k_i)_{i \in N}$, and show by induction on $k$ that after every step $s$ of iterated conditional dominance, each player $i$ always rejects any offer that gives her strictly less than $\delta_i m^s_i$ and always accepts any offer that gives her strictly more than $\delta_i M^s_i$ (regardless of the identity of the proposer). Second, I show that both sequences $(m^k_i)_{i \in N}$ and $(M^k_i)_{i \in N}$ converge to the same point $(w_i)_{i \in N}$.

(i) **Iterated Conditional Dominance Procedure**

Let me start by showing how the process of iterated conditional dominance works in the context of the game $G(r,s,q,D)$. To simplify notation, I break up the procedure into steps $0, 1, \ldots$, each step containing three rounds.

**Step 0.**

**Round 0a.** Note that a strategy that ever accepts with positive probability a negative share is strictly dominated by the strategy

“reject all offers and make only offers that give me a positive share.”

These are all the actions that are eliminated in Round 0a. Hence, after this round *every* player $i$ always rejects an offer that gives her strictly less than $\delta_i m^0_i$ (regardless of the identity of the proposer), where

---

3 See below for the definition of a step of iterated conditional dominance.
\[m_i^0 := 0,\]  \hspace{1cm} (A.6)

**Round 0b.** Given the actions that survive round 0a, player \(i\) has an expected payoff (at the beginning of the period, i.e. before the proposer has been chosen) of at most \(M_i^0\), where

\[M_i^0 := \max_j \{s_{i,j}\}.\]  \hspace{1cm} (A.7)

because (i) by assumption no player \(j\) can ever offer player \(i\) a payoff higher than \(s_{i,j}\), and (ii) by round 0a no player ever accepts a negative payoff. Hence, every strategy \(S\) of player \(i\) that ever rejects with positive probability an offer \(a\) that gives her strictly more than \(\delta_i M_i^0\) is strictly dominated by a similar strategy \(S'\) that specifies

“accept \(a\) with probability \(\pi\).”

in every instance in which \(S\) specifies

“reject \(a\) with probability \(\pi\).”

These are all the actions that are eliminated in Round 0b; so after this round every player \(i\) always accepts an offer that gives her strictly more than \(\delta_i M_i^0\) (regardless of the identity of the proposer).

**Round 0c.** Given the actions that survive rounds 0a and 0b, every strategy \(S\) of player \(i\) that ever makes an offer with positive probability that gives \(y > \delta_j M_j^0\) to player \(j\) is strictly dominated by a similar strategy \(S'\) that specifies

“offer \(y - \epsilon > \delta_j M_j^0\) to player \(j\) with probability \(\pi\).”

in every instance in which \(S\) specifies

“offer \(y\) to player \(j\) with probability \(\pi\).”
since player \( j \) must accept both \( y \) and \( y - \epsilon \). These are all the actions that are eliminated in round 0c; after this round no player ever makes an offer giving \( y > \delta_j M_j^0 \) to any player \( j \).

To continue on, imagine that after step \( s = k \in \mathbb{Z}_{\geq 0} \), we have concluded (as we have just done for the case \( s = 0 \)) that every player \( i \):

1. rejects any offer that gives her strictly less than \( \delta_i m_i^s \) (regardless of the identity of the proposer);
2. has an expected payoff (at the beginning of each period) of at most \( M_i^s \);
3. accepts any offer that gives her strictly more than \( \delta_i M_i^s \) (regardless of the identity of the proposer);
4. does not make offers that give strictly more than \( \delta_j M_j^s \) to any player \( j \).

I now show that points (1)-(4) also hold at step \( s = k + 1 \).

**Step \( k + 1 \).**

I refer to strategies that assign positive probability only to actions that have survived all previous rounds of iterated conditional dominance as “surviving strategies.”

**Round (k+1)a.** Given the surviving strategies it is conditionally dominated for player \( i \) to ever accept an offer that gives her a surplus strictly lower than \( \delta_i m_i^{k+1} \), where \( m_i^{k+1} \) is defined as follows:

\[
m_i^{k+1} := q_i \max \left( \max_{j \in N} \left( s_{ij} - \delta_j M_j^k \right), \delta_i m_i^k \right) + (1 - q_i) \delta_i m_i^k
\]

(A.8)

To see this, consider a period-\( t \) subgame where some player \( i \) has to respond to an offer \( x < \delta_i m_i^{k+1} \). We argue that, for sufficiently small \( \epsilon > 0 \), accepting this offer is conditionally dominated by the following plan of action\(^2\):

\(^2\)This plan of action is designed so that it gives a time-\( t \) expected payoff that approaches \( m_i^{k+1} \) as \( \epsilon \) goes to 0.
"Reject all offers received at dates \( t' \geq t \). When selected to be the proposer at time \( t_0 \), offer 
\[
\delta_j M_i^{k+t+1-t'} + \epsilon
\]
if \( t' \in [t+1, t+k+1] \) and \( \max_{j \in N} (s_{ij} - \delta_j M_j^{k+t+1-t'}) > \delta_i m_i^{k+t+1-t'} \), and make an unacceptable offer otherwise.\(^3\)"

Note that since \( t' \geq t+1 \), we have that \( k+t+1-t' \leq k \), so by the induction hypothesis all players \( j \) accept the offer \( \delta_j M_i^{k+t+1-t'} + \epsilon \) at period \( t' \in [t+1, t+k+1] \). Moreover, note that equation (A.8) can be written as

\[
m_{i+1}^k = \begin{cases} 
\delta_i m_i^k & \text{if } \max_{j \in N} (s_{ij} - \delta_j M_j^k) \leq \delta_i m_i^{k+t+1-t'} \\
q_i \max_{j \in N} (s_{ij} - \delta_j M_j^k) + (1-q_i) \delta_i m_i^k & \text{otherwise}
\end{cases}
\]

and an analogous equation can be used to expand the term \( m_i^k \) in the expression above, and then \( m_{i-1}^k \) in the resulting equation, and so on until reaching \( m_0 = 0 \). The resulting formula for \( m_{i+1}^k \) proves that the strategy constructed above generates an expected period-\( t \) payoff for \( i \) of \( \delta_i m_i^{k+1} \) as \( \epsilon \to 0 \) under the surviving strategies. Hence, this strategy conditionally dominates accepting \( x \) in period \( t \) if \( \epsilon > 0 \) is sufficiently small. These are the actions eliminated in round \((k+1)a\); after this round no player \( i \) ever accepts an offer that gives her a surplus lower than \( \delta_i m_i^{k+1} \).

**Round \((k+1)b\).** Given the surviving strategies, it is conditionally dominated for player \( i \) to reject an offer that gives her strictly more than \( \delta_i M_i^{k+1} \), where \( M_i^{k+1} \) is defined by

\[
M_i^{k+1} := q_i \max_{j \in N} \left( \max_{j \in N} (s_{ij} - \delta_j m_j^k), \delta_i M_i^k \right) + (1-q_i) \delta_i M_i^k \quad \text{(A.9)}
\]

To prove this, I show that for each player \( i \), all surviving strategies deliver expected payoffs of at most \( M_i^{k+1} \) at the beginning of period \( t \). First, consider a period-\( t \) subgame where \( i \) is the proposer. Note that \( i \) cannot make an offer that generates an expected payoff greater

\(^3\)E.g. offer a negative amount to some player.
than
\[
\max \left( \max_{j \in N} (s_{i,j} - \delta_j m_i^k), \delta_i M_i^k \right).
\]

To see this note that, under the surviving strategies, all players \( j \) reject all offers lower than \( \delta_j m_i^k \), and when \( j \) rejects an offer, \( i \) can expect a period-\((t+1)\) payoff of at most \( M_i^k \). Second, consider a period-\( t \) subgame where \( i \) is not the proposer; under the surviving strategies, \( i \) can expect a period-\( t \) payoff of at most \( M_i^k \). Therefore, \( \text{player } i \text{ has an expected payoff (at the beginning of each period) of at most } M_i^{k+1} \); These are all the actions that are eliminated in round \((k+1)b\); after this round, \( \text{no player ever offers strictly more than } \delta_j M_j^{k+1} \) to player \( j \).

**Round \((k+1)c\).** Given the surviving strategies, every strategy \( S \) of player \( i \) that ever makes an offer that gives \( y > \delta_j M_j^{k+1} \) to player \( j \) is strictly dominated by a similar strategy \( S' \) that specifies

\[
\text{“offer } y - \epsilon > \delta_j M_j^{k+1} \text{ to player } j \text{ with probability } \pi.”
\]

in every instance in which \( S \) specifies

\[
\text{“offer } y \text{ to player } j \text{ with probability } \pi.”
\]

since player \( j \) must accept both \( y \) and \( y - \epsilon \). These are all the actions that are eliminated in round \((k+1)c\); after this round \( \text{no player ever makes an offer giving } y > \delta_j M_j^{k+1} \) to any player \( j \).

**(ii) The sequences \((m_i^k)_{i \in N} \text{ and } (M_i^k)_{i \in N} \text{ converge to the same limit.}**

First, we prove by induction in \( k \) that for all \( i \in N \),

- the sequence \((m_i^k)_{k \geq 0}\) is increasing in \( k \);

- the sequence \((M_i^k)_{k \geq 0}\) is decreasing in \( k \);

- \( \max_{j \in N} (s_{i,j}) \geq M_i^k \geq m_i^k \geq 0 \) for all \( k \geq 0 \).

which implies that both sequences \((m_i^k)_{i \in N} \text{ and } (M_i^k)_{i \in N} \text{ converge. Note that } m_i^0 = 0 \text{ and } M_i^0 := \max_j \{s_{i,j}\}, \text{ and that equations (A.8) and (A.9) imply that } m_i^1 \geq 0 \text{ and } M_i^1 \leq \max_j \{s_{i,j}\},\)
so $m_i^1 \geq m_i^0$ and $M_i^1 \leq M_i^1$. Now suppose that for some $l \in \mathbb{N}$:

$$m_i^l \geq m_i^{l-1} \text{ and } M_i^l \leq M_i^{l-1}.$$ 

We show that

$$m_i^{l+1} \geq m_i^l \text{ and } M_i^{l+1} \leq M_i^l.$$ 

Note that, by the induction hypothesis, every summand in equation (A.8) for $k = l + 1$ is smaller than for $k = l$, which implies that $m_i^{l+1} \leq m_i^l$. Similarly, every summand in equation (A.9) for $k = l + 1$ is bigger than for $k = l$, which implies that $M_i^{l+1} \geq M_i^l$. We conclude that the sequence $(m_i^k)_{k \geq 0}$ is increasing in $k$ and the sequence $(M_i^k)_{k \geq 0}$ is decreasing in $k$, which, since

$$\max_{j \in \mathbb{N}} (s_{ij}) = M_i^0 > m_i^0 = 0$$

implies that

$$\max_{j \in \mathbb{N}} (s_{ij}) \geq M_i^k \geq m_i^k \geq 0$$

for all $k \geq 0$.

Second, I show that the sequences $(m_i^k)_{i \in \mathbb{N}}$ and $(M_i^k)_{i \in \mathbb{N}}$ converge to the same limit. For this, I let $D^k = \max_{i \in \mathbb{N}}(M_i^k - m_i^k)$ and show that

$$D^k \leq \left( \max_{j \in \mathbb{N}} \delta_j \right)^k D^0 = \left( \max_{j \in \mathbb{N}} \delta_j \right)^k \max_{j, j' \in \mathbb{N}} (s_{ij, j'})$$

for all $k \geq 0$, i.e. $D^k$ converges to 0 as $k$ grows large. We have,
\[ D^{k+1} = \max_{i \in N} [M_i^{k+1} - m_i^{k+1}] \]
\[ = \max_{i \in N} \left[ q_i \max_{j \in N} \left( \max_{i \in N} (s_{ij} - \delta_j m_i^k), \delta_i M_i^k \right) + (1 - q_i) \delta_i M_i^k \right. \]
\[ \left. - q_i \max_{j \in N} \left( \max_{i \in N} (s_{ij} - \delta_j M_i^k), \delta_i m_i^k \right) + (1 - q_i) \delta_i m_i^k \right] \]
\[ = \max_{i \in N} \left[ q_i \left[ \max_{j \in N} \left( \max_{i \in N} (s_{ij} - \delta_j m_i^k), \delta_i M_i^k \right) - \max_{j \in N} \left( \max_{i \in N} (s_{ij} - \delta_j M_i^k), \delta_i m_i^k \right) \right] \right. \]
\[ \left. + (1 - q_i) \left[ \delta_i M_i^k - \delta_i m_i^k \right] \right] \]
\[ \leq \max_{i \in N} \left[ q_i \left[ \max_{j \in N} \left( \delta_j (M_j^k - m_j^k), \delta_j M_j^k \right) - \max_{j \in N} \left( \delta_j (M_j^k - m_j^k), \delta_j m_j^k \right) \right] \right. \]
\[ \left. + (1 - q_i) \left[ \delta_i M_i^k - \delta_i m_i^k \right] \right] \]
\[ \leq \max_{i \in N} \left[ q_i \max_{j \in N} \left( \delta_j (M_j^k - m_j^k), \delta_i (M_i^k - m_i^k) \right) + (1 - q_i) \delta_i D^k \right] \]
\[ \leq \max_{i \in N} \delta_i D^k \]

where \( j^* \) in the first inequality is any element of \( \arg\max_{j \in N} (s_{ij} - \delta_j M_i^k) \), and the second inequality is a consequence of Lemma A.2.2 below. \( \square \)

**Lemma A.2.2** (Manea (2017a)). For all \( w_1, w_2, w_3, w_4 \in \mathcal{R}, \)

\[ |\max(w_1, w_2) - \max(w_3, w_4)| \leq \max(|w_1 - w_3|, |w_2 - w_4|). \]
Appendix B

Appendix to Chapter 3

B.1 Proof of Proposition 3.4.2

This proof refers to algorithm \( \mathcal{A} \) described in Talamàs (2017b). Algorithm \( \mathcal{A} \) is defined inductively in steps; each step assigns to at least one player her \( \mathcal{A} \)-wage and set of \( \mathcal{A} \)-coalitions. In the context of the present article, each buyer’s (seller’s) threshold price in \( \Gamma(M, \Delta) \) is her value minus (plus) her \( \mathcal{A} \)-wage.

Points 1, 2 and 3 in Proposition 3.4.2 follow from the definition of algorithm \( \mathcal{A} \). Point 5 follows from points 3 and 4. I now prove point 4 using the fact that the threshold prices in \( \Gamma(M, \Delta) \) are continuous in \( \Delta \), which follows from the observation that \( \mathcal{A} \)-wages are continuous in \( \Delta \).

I show that the cases in which point 4 does not hold are not generic. Indeed, suppose that point 4 does not hold; that is, there exists no \( \Delta > 0 \) such that the preferred neighbor network is fixed for all \( \Delta < \Delta \). This implies that there exist two different walks

\[
P := (1, 2, \ldots, k - 1, k, k - 1)
\]

and

\[
P' := (1, 2', \ldots, k' - 1, k', k' - 1)
\]

in \( g \) whose intersection is \( \{1\} \) (Case 1) or whose intersection is \( \{1, k - l, \ldots, k - 2, k - 1, k\} \)
for some \( l \geq 2 \) (Case 2), and a strictly decreasing sequence \( D = \{ \Delta_z \}_{z \in \mathbb{N}} \), such that, for each \( z \in \mathbb{N} \):

1. \( P \) is—and \( P' \) is not—a walk of the preferred-neighbor network for all \( \Delta < \Delta_z \) sufficiently close to \( \Delta_z \), and

2. \( P' \) is—and \( P \) is not—a walk of the preferred-neighbor network for all \( \Delta > \Delta_z \) sufficiently close to \( \Delta_z \).

Let \( f(P, \Delta) \) and \( f(P', \Delta) \) be the threshold prices of player 1 implied by the paths \( P \) and \( P' \) of the preferred-neighbor network when bargaining frictions are \( \Delta \). For each player \( i \), let \( f_i(x) \) denote the threshold price of player \( i \) when her preferred neighbors’ threshold price is \( x \), that is, \( f_i(x) = a_i x + (1 - a_i) v_i \). Equation 3.1 and Equation B.2 imply that

\[
f(P, \Delta) = f_1 \circ f_2 \circ \cdots \circ f_{k-2} (p_{k-1,k}(\Delta))
\]

and

\[
f(P', \Delta) = f_{1'} \circ f_{2'} \circ \cdots \circ f_{k'-2} (p_{k'-1,k'}(\Delta))
\]

where \( p_{i,j}(\Delta) \) denotes the \((i,j)\)-Rubinstein price when bargaining frictions are \( \Delta \).

By construction, we have \( f(P, \Delta) \neq f(P', \Delta) \) for infinitely many values of \( \Delta > 0 \), and—since threshold prices are continuous in \( \Delta \)—we also have \( f(P, \Delta) = f(P', \Delta) \) for \( \Delta \in D \). This situation is not generic, since:

1. Case 1: \( f(P, \Delta) \) converges to the limit \((k - 1, k)\)-Rubinstein price \( p_{k-1,k}^* \), and \( f(P', \Delta) \) converges to the limit \((k' - 1, k')\)-Rubinstein price \( p_{k'-1,k'}^* \), which are generically different, and

2. Case 2: letting \( x(\Delta) := f_{k-1} \circ \cdots \circ f_{k-2} (p_{k-1,k}(\Delta)) \), we have that

\[
f_2 \circ \cdots \circ f_{k-1-1}(x(\Delta)) = f_{2'} \circ \cdots \circ f_{k'-1-1}(x(\Delta)) \quad \text{(B.1)}
\]

In the special case of homogenous discount rates (a similar logic holds in general), Equation B.1 is a polynomial function of \( \delta := e^{-r\Delta} \), which implies that it either holds
for all $\Delta$ or for at most finitely many values of $\Delta$.

\[ \square \]

### B.2 Proof of Proposition 3.4.4

I prove the first statement of Proposition 3.4.4; that is, that assuming that Condition 3.4.3 holds, the buyer $\beta$ with highest limit best price and her limit best mate are the key players in the component $H$ of $G$ with the highest limit price. The second statement is proved analogously.

In subsection B.2.1 I derive a bound on each player $i$’s threshold price as a function of $i$’s location in the preferred-neighbor network $G^{\Delta}$. In subsection B.2.2 I use this bound to identify the key players in $H$.

#### B.2.1 Bounding Threshold Prices

Lemma B.2.1 describes how the threshold price of a player is determined by the threshold price of her preferred-neighbor. This result is useful to prove Proposition B.2.2, which shows that—when buyer $i$ is the preferred neighbor of player $j$—buyer $i$’s threshold price is bounded above by the $(i, j)$-Rubinstein price.

**Lemma B.2.1.** If player $i$’s preferred neighbor is player $j$, then

\[
\rho_i = a_i \rho_j + (1 - a_i)v_i
\]

(B.2)

where $a_i := \frac{q_i d_i}{1 - q_i + a_i q_i}$.

**Proof.** Let $i$ be a buyer (an analogous argument works for a seller). Assume without loss of generality that $i$ not a dummy (otherwise, we have that $i = j$ and $\rho_i = v_i$, so Equation B.2 holds trivially).

Let $a_i$ denote buyer $i$’s expected equilibrium payoff at the beginning of each period (before the proposer is selected). Since buyer $i$ is indifferent between accepting and rejecting
to trade at her threshold price $\rho_j$, we have that

$$v_i - \rho_i = \delta_i a_i.$$  \hfill (B.3)

Consider how the equilibrium play unfolds from the perspective of buyer $i$: At the beginning of each period, with probability $q_i$ she is selected to be the proposer, in which case she offers $\rho_j$ to seller $j$, and her payoff is $v_i - \rho_j$, since $j$ accepts this offer. Otherwise (with probability $1 - q_i$), her expected payoff is $\delta_i a_i$ regardless of whether she receives an offer in this period. Hence, her the expected payoff $a_i$ at the beginning of each period satisfies

$$a_i = q_i(v_i - \rho_j) + (1 - q_i)\delta_i a_i.$$  \hfill (B.4)

Combining Equation B.3 and Equation B.4 gives Equation B.2. \hfill $\square$

**Proposition B.2.2.** If buyer $j$ is the preferred neighbor of player $i$, then $j$’s threshold price $\rho_j$ is smaller than the $(j, i)$-Rubinstein price $p_{j,i}$. Similarly, if seller $j$ is the preferred neighbor of player $i$, then $j$’s threshold price $\rho_j$ is greater than the $(j, i)$-Rubinstein price $p_{j,i}$.

**Proof.** I prove the first part of the statement; the second part is proved analogously. Suppose buyer $j$ is the preferred neighbor of player $i$. By Equation B.2 we have that

$$\rho_i = \alpha_i \rho_j + (1 - \alpha_i)v_i$$  \hfill (B.5)

Let $\kappa_j := \alpha_j \rho_i + (1 - \alpha_j)v_j$. The same argument that proves Lemma B.2.1 shows that $\kappa_j$ is the threshold price that $j$ would have if she were to trade at $i$’s threshold price when she is the proposer. Since player $j$ could trade at this threshold price when she is the proposer, we have that

$$\rho_j \leq \kappa_j$$  \hfill (B.6)

Hence, it is enough to show that $\kappa_j \leq p_{j,i}$.

Combining Equation B.5 and Equation B.6 gives

$$\rho_i \leq \alpha_i \kappa_j + (1 - \alpha_i)v_i,$$  \hfill (B.7)
Combining the definition of $k_j$ and Equation B.7 then gives

$$k_j \leq \alpha_j \left[ \alpha_i k_j + (1 - \alpha_i) v_i \right] + (1 - \alpha_i) v_j$$  \hspace{1cm} (B.8)

which rearranging gives

$$k_j \leq \alpha_j \alpha_i \left[ k_j - p_{ji,i} \right] + \alpha_j \left[ \alpha_i p_{ji,i} + (1 - \alpha_i) v_i \right] + (1 - \alpha_i) v_j.$$  \hspace{1cm} (B.9)

Noticing that $p_{ji,i} = a_j \left[ \alpha_i p_{ji,i} + (1 - \alpha_i) v_i \right] + (1 - \alpha_i) v_j$ gives

$$k_j \leq \alpha_j \alpha_i k_j + (1 - \alpha_i \alpha_j) p_{ji,i}$$  \hspace{1cm} (B.10)

that is, $k_j \leq p_{ji,i}$. \hspace{1cm} \Box

### B.2.2 Identifying the Key Players in $H$

Lemma B.2.3 and Lemma B.2.4 are useful to prove Proposition B.2.5, which is the first statement in Proposition 3.4.4. Lemma B.2.3 holds for all bargaining frictions $\Delta > 0$.

**Lemma B.2.3.** The threshold price of the buyer with the highest threshold price is her best price.

**Proof.** The buyer $h$ with the highest threshold price is clearly the preferred neighbor of all sellers that can trade with her. Hence—by Proposition B.2.2—buyer $h$’s threshold price is exactly her best price and buyer $h$’s preferred neighbor is her best mate. \hspace{1cm} \Box

**Lemma B.2.4.** The buyer $\beta$ with the highest limit best price is a key player.

**Proof.** Suppose for contradiction that the preferred neighbor $l$ of buyer $\beta$ in the limit preferred neighbor network $G$ extends offers in equilibrium to buyer $b \neq \beta$. By Proposition B.2.2 and Condition 3.4.3, $l$’s limit threshold price $\rho_j^*$ is strictly higher than $p_{l,\beta}^*$. Hence, buyer $\beta$’s limit threshold price is strictly higher than $p_{l,\beta}^*$ and hence strictly higher than her own limit best price, which contradicts the fact that—by Lemma B.2.3—buyer $\beta$’s limit best price is an upper bound on buyers’ limit threshold prices. \hspace{1cm} \Box

**Proposition B.2.5.** Buyer $\beta$ and her limit best mate are the key players in $H$. 81
Proof. Let $h$ denote the highest limit threshold price among buyers. On the one hand, Lemma B.2.3 implies that buyer $b$’s limit best price is an upper bound on $h$. On the other hand, Lemma B.2.4 implies that buyer $b$’s limit best price is a lower bound on $h$. Hence, buyer $b$’s limit best price is equal to the highest threshold price among buyers. The proof then follows from the assumption that there is a unique buyer $b$ with the highest limit best price, so the key players in $H$ must be $b$ and her best mate. 

B.3 Proof of Lemma 3.5.3

The argument is similar to the proof of Theorem 4.1 in Jackson (2008). Let $I_1$ and $I_2$ be two nonempty subsets of $[v, u]$. Let $p_1 = \int_{I_1} f(x) dx$ and $p_2 = \int_{I_2} f(x) dx$; that is, $p_1$ and $p_2$ denote the probabilities that values are drawn from $I_1$ and $I_2$, respectively.

First, we show that if $p(n) / t(n) \to 0$, then the probability that there are $I_1$-isolated nodes in $I_2$ tends to 1. The probability that any given node is $I_1$-isolated is $(1 - p(n))^{p_1(n-1)}$; or roughly $(1 - p(n))^{p_1 n}$ when $n$ is large (since $p(n)$ converging to 0), which goes to 

$$e^{-p_1 n p(n)}.$$ 

We can write $p(n) = \frac{\log(n) - f(n)}{n}$, where $f(n) \to \infty$ and $f(n) < \log(n)$, and then $e^{-p_1 n p(n)}$ becomes 

$$\left( \frac{e^{f(n)}}{n} \right)^{p_1}.$$ 

The expected number of $I_1$-isolated nodes in $I_2$ is then $p_2 n \left( \frac{e^{f(n)}}{n} \right)^{p_1}$, which tends to infinity.

We now use Chebyshev’s inequality to prove that the probability of there being $I_1$-isolated nodes in $I_2$ converges to 1.\(^1\) Let $X_n$ denote the number of $I_1$-isolated nodes in $I_2$. We have shown that $\mu := E[X_n] \to \infty$. We now show that the variance of $X_n$, $E[(X_n)^2] - E[X_n]^2$, is no higher than twice $E(X_n)$ for large enough $n$. This implies that

$$\text{Pr} \left( X_n < \mu - r \sqrt{2 \mu} \right) < \frac{1}{r^2} \text{ for all } r > 0$$

\(^1\)Chebyshev’s inequality states that for a random variable $X$ with mean $\mu$ and standard deviation $\sigma$, $\text{Pr} (|X - \mu| > r \sigma) < 1/r^2$ for every $r > 0$. 

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which—since \( \mu \) converges to \( \infty \)—implies that the probability that \( X^n \) is arbitrarily large converges to 1.

To obtain an upper bound on \( E \left[ (X^n)^2 \right] - E(X^n)^2 \), note that

\[
E(X^n) = p_2 n (1 - p)^{p_1(n-1)}.
\]

Since \( E[X^n(X^n - 1)] \) is the expected number of ordered pairs of \( I_1 \)-isolated nodes in \( I_2 \), and assuming for simplicity that \( I_2 \) is a subset of \( P_1 \) (similar arguments show the other cases), note also that

\[
E[X^n(X^n - 1)] = p_2 n (p_2 n - 1)(1 - p)^{2(n-2)p_1 + 1}
\]

since a pair of nodes in \( I_2 \) (there are \( p_2 n (p_2 n - 1) \) such different pairs in expectation) are \( I_1 \)-isolated if they are not linked and none of the links from either of them to \( I_1 \) (there are \( 2(n - 2)p_1 \) such potential links in expectation) is present. Thus,

\[
E \left[ (X^n)^2 \right] - E[X^n]^2 = p_2 n (p_2 n - 1)(1 - p)^{2(n-2)p_1 + 1} + E[X^n] - E[X^n]^2
\]

\[
= p_2 n (p_2 n - 1)(1 - p)^{2(n-2)p_1 + 1} - E[X^n] - p_2 n^2 (1 - p)^{2p_1(n-1)}
\]

\[
\leq E(X^n) + (1 - p)^{2(n-2)p_1 + 1} p_2 n^2 (1 - (1 - p)^{2p_1 - 1})
\]

\[
\leq E(X^n) + (1 - p)^{2(n-2)p_1 + 1} p_2 n^2 p
\]

\[
\leq E(X^n) \left( 1 + p_2 p n (1 - p)^{p_1(n-1) + 2p_1 + 1} \right)
\]

\[
\leq E(X^n) \left( 1 + p_2 (\log(n) - f(n)) e^{-p_1(\log(n) - f(n))} (1 - p)^{1-3p_1} \right)
\]

\[
\leq 2E(X^n)
\]

where the last equality holds for large enough \( n \).
Second, I show that if \( p(n)/t(n) \to \infty \), then the probability that there are \( I_1 \)-isolated nodes in \( I_2 \) converges to 0. It is enough to show this for \( p(n) = \log(n) + f(n) \), where \( f(n) \to \infty \) but \( f(n)/n \to 0 \).\(^2\) By a similar argument to the one above, the expected number of \( I_1 \)-isolated nodes in \( I_2 \) is tending to \( p_2 n^{1-p_1} e^{-p_1 f(n)} \), which tends to 0. Hence, the probability of having \( X^n \) be at least one then has to tend to 0 as well.

\(^2\)Having no \( P_1 \)-isolated nodes in \( P_2 \) is clearly an increasing property, so it holds for larger \( p(n) \). The reason for working with \( f(n)/n \to 0 \) is to ensure that the approximation of \( (1 - p(n))^n \) by \( e^{-np(n)} \) is valid asymptotically.