Asymptotic Symmetries in Four-Dimensional Gauge and Gravity Theories

Asymptotic Symmetries in Four-Dimensional Gauge and Gravity Theories

A DISSERTATION PRESENTED
BY
PRAHAR MITRA
TO
THE DEPARTMENT OF PHYSICS

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY
IN THE SUBJECT OF
PHYSICS

HARVARD UNIVERSITY
CAMBRIDGE, MASSACHUSETTS
MAY 2017

Thesis advisor: Professor Andrew Strominger Prahar Mitra

Asymptotic Symmetries in Four-Dimensional Gauge and Gravity Theories

Abstract

Recent developments have uncovered a deep relationship between soft theorems in quantum field theories and asymptotic symmetries. We investigate five explicit examples wherein these connections are studied and verified.

First, we show that the Weinberg's soft-photon theorem may be recast as the Ward identity for CPT-invariant large $U(1)$ gauge transformations that asymptotically approach an arbitrary function ε of the conformal sphere at null infinity, but are independent of retarded time. The symmetries for which $\varepsilon \neq$ constant are spontaneously broken in the perturbative quantum field theory vacuum and the associated Goldstone modes are the zero-momentum photons. These comprise a $U(1)$ boson living on the conformal sphere.

Second, we generalize the construction to non-abelian gauge theories with gauge group $\mathcal{G}$ and show that the massless tree-level soft-gluon theorem is the Ward identity of a holomorphic two-dimensional $\mathcal{G}$-Kac-Moody symmetry acting on these correlation functions. Holomorphic Kac-Moody current insertions are positive helicity soft-gluon insertions. These symmetries are also spontaneously broken and the soft-gluons are the Goldstone modes.

Third, we generalize to supersymmetric $\mathcal{N} = 1$ abelian gauge theories with massless charged

matter and establish the existence of infinitely many fermionic asymptotic symmetries at null infinity, parametrized by a function on S^2, whose Ward identities give rise to the soft photino theorem. Unlike large gauge transformations, these symmetries are not manifest at the level of the Lagrangian. They are spontaneously broken, and the soft photinos are the associated Goldstone fermions. Unbroken global supersymmetry relates this fermionic charge to the $U(1)$ large gauge charge.

Fourth, we consider gravitational theories and show that Weinberg's soft-graviton theorem is the Ward identity corresponding to a certain infinite-dimensional "diagonal" subgroup of BMS supertranslations acting on past and future null infinity ($\mathscr{I}^-$ and $\mathscr{I}^+$). The soft-gravitons are the Goldstone bosons of spontaneously broken supertranslation invariance.

Finally, we use the sub-leading soft-graviton theorem to construct an operator T_{zz} whose insertion in the four-dimensional tree-level quantum gravity $\mathcal{S}$-matrix obeys the Virasoro-Ward identities of the energy momentum tensor of a two-dimensional conformal field theory (CFT$_2$). The celestial sphere at Minkowskian null infinity plays the role of the Euclidean sphere of the CFT$_2$, with the Lorentz group acting as the unbroken $SL(2,\mathbb{C})$ subgroup.

Contents

Acknowledgments

I would like to thank my advisor, Andrew Strominger, for his mentorship, guidance and for the many penetrating discussions on physics and otherwise. It has truly been a privilege to learn physics from him. More importantly, I have gained immensely from his approach towards physics and research. He has the remarkable ability to reduce complicated physical system to their simplest of elements and I am grateful to have been able to learn this from him.

I would like to thank all my collaborators, Thomas Dumitrescu, Temple He, Daniel Kapec, Vyacheslav Lysov, Achilleas Porfyriadis, Ana-Maria Raclariu and Andrew Strominger who co-authored the papers that comprise this thesis. I have gained immeasurably from the time we spent thinking together.

My Ph.D. would not have been such an amazing learning experience if not for the warmth and nurturing environment provided by the high energy theory group at Harvard. The string theory faculty, particularly Daniel Jafferis and Xi Yin have always been open to a variety of discussions and it has been a pleasure to engage with them. Many of the ideas presented here have been clarified by intriguing discussions with the string theory postdocs, particularly Clay Cordova, Thomas Dumitrescu, Burkhard Schwab and Alexander Zhiboedov. I am extremely grateful also to Lisa Cacciabado, Nicole D'Aleo and Nancy Partridge without whom it would be impossible to have a comfortableface stay at Harvard.

One of my fondest memories at Harvard arise largely from the amazing graduate student community with whom I have had the pleasure of learning together. I would particularly like to thank Anders Andreassen, Will Frost, Monica Kang, Eric Kramer, Alex Lupsasca, Andrew Marantan, Sabrina Pasterski, Monica Pate, Abhishek Pathak, Gim Seng Ng, Shu-Heng Shao and Siddharth Venkat.

I would additionally like to thank my wonderful friends – Anders, Ajay, Will, Vijeet, I-Chun, Amar, Abhishek, Albert, Alex, Andrew, Himaanee, Abhishek, Krishna, Ankit, Rahul, Siddharth and Letian – that have made my life immensely enjoyable!

I would like to extend immense gratitude and love to my partner – Stacy – who has been with me throughout my Ph.D. during the ups and the downs. She has supported me and has made my life better in ways that I am yet to appreciate.

Finally, and most importantly, I would like to thank my parents, Tapas and Madhumita, and my sister, Priyanka, for their constant support throughout my life. Without their constant encouragement, I would not be where I am today. They have always been there for me whenever I have needed them.

Citations to previously published work

This dissertation contains research originally reported in the following articles:

1. The discussion in Chapter 3 were obtained in collaboration with Temple He, Achilleas Porfyriadis and Andrew Strominger. The text has appeared previously in [1].

   [1] T. He, P. Mitra, A. P. Porfyriadis, and A. Strominger, "New Symmetries of Massless QED," *JHEP* 10 (2014) 112, `arXiv:1407.3789 [hep-th]`.

   I am grateful to F. Cachazo, T. Dumitrescu, I. Feige, D. Kapec, V. Lysov, J. Maldacena, S. Pasterski, A. Pathak, G. S. Ng, M. Schwartz and A. Zhiboedov for valuable discussions related to the material in this chapter.

2. The discussion in Chapter 4 were obtained in collaboration with Temple He and Andrew Strominger. The text has appeared previously in [2].

   [2] T. He, P. Mitra, and A. Strominger, "2D Kac-Moody Symmetry of 4D Yang-Mills Theory," *JHEP* 10 (2016) 137, `arXiv:1503.02663 [hep-th]`.

   I am grateful to A. Andreassen, I. Feige, S. Caron-Huot, D. Kapec, E. Kramer, V. Lysov, J. Maldacena, G. S. Ng, S. Pasterski, A. Pathak, A. Porfyriadis, M. Schwartz and A. Zhiboedov for valuable discussions related to the material in this chapter.

3. The discussions of Chapter 5 were obtained in collaboration with Thomas Dumitrescu, Temple He and Andrew Strominger. The text has appeared previously in [3].

   [3] T. T. Dumitrescu, T. He, P. Mitra, and A. Strominger, "Infinite-Dimensional Fermionic Symmetry in Supersymmetric Gauge Theories," arXiv:1511.07429 [hep-th].

   I am grateful to H. Elvang, D. Kapec, V. Lysov, S. Naculich, B. Schwab and S. Pasterski for valuable discussions related to the material in this chapter.

4. The discussions of Chapter 6 were obtained in collaboration with Temple He, Vyacheslav Lysov and Andrew Strominger. The text has appeared previously in [4].

   [4] T. He, V. Lysov, P. Mitra, and A. Strominger, "BMS supertranslations and Weinberg's soft graviton theorem," *JHEP* 05 (2015) 151, arXiv:1401.7026 [hep-th].

   I am grateful to G. Barnich, G. Compere, J. Maldacena and A. Zhiboedov for valuable discussions related to the material in this chapter.

5. The discussions of Chapter 7 were obtained in collaboration with Daniel Kapec, Ana-Maria Raclariu and Andrew Strominger. The text has appeared previously in [5].

   [5] D. Kapec, P. Mitra, A.-M. Raclariu, and A. Strominger, "A 2D Stress Tensor for 4D Gravity," arXiv:1609.00282 [hep-th].

   I am grateful to T. Dumitrescu, J. Maldacena, S. Pasterski, B. Schwab, and A. Zhiboedov for valuable discussions related to the material in this chapter.

1

Introduction

Quantum field theory is the central mathematical framework that is used in modern day particle physics research. The Standard Model, which is a particular quantum field theory, describes the dynamics and interactions of all known elementary particles. It is one of the most widely and accurately tested theories to date (the other being Einstein's General Theory of Relativity). Experimental verifications of the Standard Model typically come from scattering experiments. In a scattering experiment, two or more collimated beams of particles are accelerated to very high energies and collided. The immense energy released in the collision process is almost instantaneously converted into a plenitude of new particles which are then collected and their properties measured by particle detectors. By studying the type and amount of each particle that is released as well as their momenta, one can obtain information of the structure of the subatomic world as well as the laws that govern it. In general, these processes are exceedingly complicated and a lot of incredible theoretical, experimental and technological ideas are needed to extract useful information from these collisions.

The fundamental quantity that theoretical physicists like to use to describe scattering processes is called the *scattering amplitude* or *scattering matrix* or $\mathcal{S}$*-matrix* or often, simply *amplitude*. The $\mathcal{S}$-matrix is defined as the overlap between the quantum state before the collision $|\,\mathrm{in}\,\rangle$ (the incoming or *in* state) and a possible quantum state after the collision $|\,\mathrm{out}\,\rangle$ (the outgoing or *out* state), i.e.

$$\mathscr{A}_{\mathrm{out,in}} = \langle\,\mathrm{out}\,|\,\mathrm{in}\,\rangle\ .$$

The $\mathcal{S}$-matrix amplitude $\mathscr{A}$ is a function of all the quantum numbers that describe the *in* and *out* states, e.g. the momentum, spin, charge, flavor, color, etc of each particle. It describes almost all aspects of the collision process and consequently, it is extremely vital that we have a good understanding of the structure of the $\mathcal{S}$-matrix and more importantly, how it is determined in quantum field theory. A large part of particle physics research in the past 80 years has been devoted to this endeavor.

Despite being incredibly complicated in general (as expected from the complexity of the collision process it describes), there are certain limits in which the $\mathcal{S}$-matrix simplifies. These simplifications are often due special properties of the $\mathcal{S}$-matrix or of the underlying quantum field theory such as unitarity, locality, causality, Poincaré invariance, etc. In this thesis, we will study a particularly interesting kinematic limit of the $\mathcal{S}$-matrix known as the *soft* limit, in which the energy of one or more of the massless particles involved in the collision is taken to be small compared to the energy or masses of the other particles in the process. It has been well known since the work of Bloch and Nordseick [6], Low [7, 8], Yennie, Frautschi and Suura [9] and Weinberg [10] that the $\mathcal{S}$-matrix factorizes in this limit into a so-called *soft-factor* and another $\mathcal{S}$-matrix that involves fewer particles. Roughly

$$\mathscr{A}_n \quad \xrightarrow{\text{soft-limit}} \quad S_m \cdot \mathscr{A}_{n-m} \,.$$

Here, $\mathscr{A}_n$ denotes the $\mathcal{S}$-matrix of a collision process involving a total of n particles (incoming + outgoing), m is the number of soft-particles and S_m is the *soft-factor* which may be a number, a matrix or differential operator that depends, in general, on the precise structure of $\mathscr{A}_n$. There are, however, certain aspects of this soft factor that are independent of these details and are therefore universal. The leading, subleading or in some cases, the subsubleading terms in the soft expansion (expansion in the energies of all the soft particles) of S_m are universal! Soft limits which extract these universal structures are referred to as *soft theorems*.

Soft theorems characterize universal properties of $\mathcal{S}$-matrices. They often imply severe constraints on the amplitude, such as conservation of (color) charge, momentum and angular momentum [10, 11]. They additionally imply that long range interactions cannot be mediated by particles of helicity $|s| > 2$ [10]. They also ensure infrared finiteness of cross sections and decay rates [6, 12, 13], which are directly measured at colliders. Despite their immense usefulness, the origin of soft theorems has not been clear. They are often determined, as in the case of [6–10], by an explicit computation of soft-limit in each case. This case-by-case derivation makes it difficult to understand when soft theorems may exist in general. For instance, while the leading soft-photon and soft-graviton theorems were known since 1965 due to seminal work of Weinberg [10], the subleading soft-graviton theorem was derived only as recently as 2014 [11].

Recent developments in this area, which form the central topic of this thesis, have shown that soft theorems are consequences of infinite-dimensional symmetries of the $\mathcal{S}$-matrix. In some cases, these infinite-dimensional symmetries have been connected to previously known symmetries and in other cases have turned out to be completely new! Once established, this connection allows us to deduce new infinite-dimensional symmetries from soft theorems and vice versa. In fact, the subleading soft-graviton theorem was conjectured to exist only by first connecting it to a previously known symmetry known as superrotations.

The infinite-dimensional symmetries referred to above are known as *asymptotic symmetries*. These are *exact* symmetries of the theory which are highly non-trivial in the bulk of spacetime, but take on a rather simplified form at infinity. From the perspective of the scattering amplitudes, these are symmetries that act in a simple way on the $\mathcal{S}$-matrix but in a non-trivial way on the action. Asymptotic symmetries have been studied in the context of general relavity for a long time, starting from the work of Bondi, van der Burg, Metzner and Sachs in the early 60s [14, 15]. The authors were interested in understanding the structure of gravitational waves at infinity, i.e. far away from any sources. In particular, they were interested in determining the symmetry group that acts on such gravitational waves and expected to find the Poincaré group – since general relativity ought to reduce to special relativity when spacetime is weakly curved. However, what they surprisingly found instead was the so-called BMS group, an infinite-dimensional extension of the Poincaré group. At the time, most were puzzled by the result and strived to impose stronger constraints than the ones BMS used to reduce the asymptotic symmetry group down from BMS to Poincaré. On the other hand, there significant interest in the the structure of the BMS group and its implications on gravitational physics, and in particular the gravitational $\mathcal{S}$-matrix (see [16–26] and references therein.). However, it wasn't until quite recently [4, 27] that the consequences of the BMS group on the $\mathcal{S}$-matrix were understood. The study of asymptotic symmetries and their consequences on the $\mathcal{S}$-matrix in non-gravitational systems such as QED or non-abelian gauge theories is more recent [1, 28–31] and is a subject of ongoing research [32–39].

Soft theorems and asymptotic symmetries have been independently studied over the past 60 years with significant developments in both. The language and notation employed in these distinct fields have been wildly different and yet – as we will argue in this thesis – they are in fact completely equivalent. This equivalence will come in the form of conservation laws. In particular, due to the seminal work of Noether [40], it is known that the existence of symmetries implies conservation laws or Ward identities for the $\mathcal{S}$-matrix. We will show, in several examples, that the Ward identities corresponding to these asymptotic symmetries are precisely the soft-theorems in quantum field theory.

The implications of the new found connection between soft theorems and asymptotic symmetries are deep.

Firstly, connecting two disparate fields is often in and of itself a useful development. Results in one field can be translated into potentially new results in the other field. The connection opens up the possibility of new calculational techniques and new insights into physical phenomena.

Secondly, it is known that soft theorems constrain the IR dynamics of a quantum field theory which therefore implies that the IR sector is governed by infinitely many symmetries! This may provide new light into infrared problems in quantum field theory, which currently is treated technically by introducing a IR cutoff and then removing it at the end of the calculation. This procedure explicitly breaks these symmetries and obscures the interesting physics. In particular, it is not clear with this technique how to define an IR finite $\mathcal{S}$-matrix in gauge theories (the cutoffs are removed from decay rates and cross-sections, but cannot be removed from the $\mathcal{S}$-matrix.). In fact, as we will see, in theories with such infinite dimensional asymptotic symmetries, the vacuum is not unique so that a basic assumption of perturbative quantum field theory breaks down. It is believed that this infinite vacuum degeneracy might be the cause of IR divergences.

Thirdly, such a connection implies deep insights on the long-sought-after flat space holography, i.e. a holographic description of quantum gravity in asymptotically flat spacetimes. With the advent of AdS/CFT [41, 42], substantial progress was made in understanding quantum gravity in AdS spacetimes by holographically mapping it to a conformal field theory in one lower dimension. Various attempts to extrapolate this holographic principle to Minkowski spacetime have been made by taking the infinite radius limit of AdS [43, 44], but not much progress has been made on this front. For instance, while it has been possible to recover the three-dimensional BMS group by taking such a limit, attempts to recover BMS_4 have failed. Nonetheless a quantum theory of gravity in flat space is much desired and one may begin to answer this question holographically by studying the asymptotic symmetries of a spacetime. It was recently noted [45] that the original analysis of BMvS allows for another infinite-dimensional extension of the four-dimensional Lorentz group $SO(1,3) \cong SL(2,\mathbb{C})$ to the local two-dimensional Virasoro group. Existence of the local two-dimensional conformal group implies a possibility of the description of quantum gravity in flat spacetimes in terms of a CFT_2. We are still in the process of understanding how such a holographic correspondence would come about and we will touch upon some developments on this front in this thesis.

Another interesting motivation to study this subject lies in understanding the so-called *miracles of* $\mathcal{N} = 4$ *supersymmetric Yang-Mills theory*. Detailed calculations of the $\mathcal{S}$-matrix in this complicated theory show that while intermediate steps of the calculation are often intricate, there are miraculous cancellations so that the final answer is exceedingly simple. There is no apriori explanation for these simplifications. One possible suggestion, and the one we advocate, is that the cancellations occur due the asymptotic symmetries discussed here. Using a recently developed technique called inverse soft [46], one is able to construct a large class of amplitudes in $\mathcal{N} = 4$ SYM with the knowledge of only the soft theorems. It is then natural to suspect that the same symmetries that constrain the soft sector also have implications for hard amplitudes.

Finally, another interesting application of this connection that has emerged due to the work of Hawking, Perry and Strominger [47, 48] is to the black hole information paradox. It turns out that the same infinite-dimensional asymptotic symmetries that BMvS obtained by studying the structure of asymptotically flat spacetimes at infinity appear also on the horizon of black holes and therefore constrain the formation and evaporation of a black hole. In particular, complete specification of a black hole now requires not only the mass, charge and angular momentum (as dictated by the no-hair theorem), but also these infinitely many charges. In other words, black holes have infinitely many hair! This idea is intriguing and seems to significantly affect the original argument of the information paradox [49]. However, it is not clear if this resolves the paradox or simply reformulates it. The answer to this question is currently being investigated [47, 48, 50–66].

While the connection between soft theorems and asymptotic symmetries is interesting and rich enough, it has additionally been revealed that these two fields are related to a third field, namely the study of *memory*. A memory effect is a change in the average value of some quantity before and after a certain process occurs. The first instance of memory was understood in 1974 in the context of gravitational physics in [67–71] known as the Christodoulou memory. The Christodoulou memory effect describes the net change in the geodesic distance between two inertial detectors before and after a gravitational wave passes through. In other words, when a gravitational wave passes through two inertial detectors, there is a temporary oscillation while the wave passes, followed by a permanent relative displacement of the two detectors. This net change in the distance is a DC effect captures important non-radiative data about the passing wave. It is now understood that the Christodoulou memory effect is related to the BMS group [72] and therefore also to the corresponding soft theorem. In particular, the formula that relates the DC shift to the gravitational wave is simply a Fourier transform of the soft theorem. The DC shift can also be understood as a transition between the infinitely degenerate vacua of systems with asymptotic symmetries. These relationships are concisely portrayed in the so-called infrared triangle shown in Figure 1.1. Similar memory effects and their

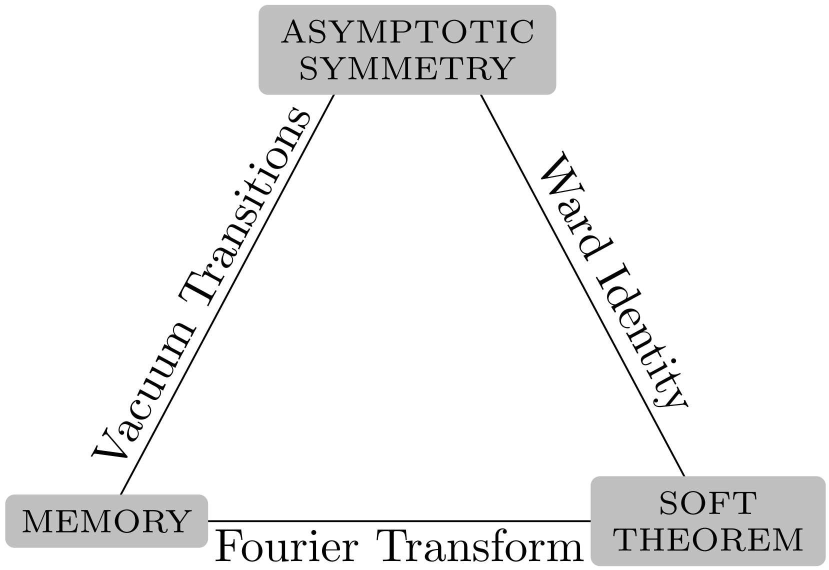

Figure 1.1: The Infrared Triangle: The triangular equivalence of three phenomena that characterize the infrared structure of all theories.

relationships to soft theorems and asymptotic symmetries have now been shown to exist for gauge theories [73, 74] as well as new ones in gravity [75–84].

Instances of the infrared triangle appear ubiquitously in all physical systems – gauge, gravitational or supersymmetric theories and in all dimensions. They are present in classical theories as well as in quantum theories. Further, there is a version of this triangle corresponding to all types of soft theorems – leading, subleading, subsubleading, double soft theorems, etc. Current research is slowly uncovering various instances of this triangle in different theories. In many cases, only one vertex of the triangle is known which can be used to deduce the remaining two vertices.

In this thesis, we will discuss the detailed relationship between asymptotic symmetries and soft theorems. It is organized as follows. In Chapter 2, we present a basic introduction to fields in flat space and their asymptotic structure. This chapter sets up the notations and conventions that we use in the rest of the thesis and introduces all preliminary material. In Chapter 3, we study asymptotic symmetries in massless QED and relate it to the leading Weinberg's soft-photon theorem. In Chapters 4 and 5, we generalize the discussion of Chapter 3 to non-abelian gauge theories and supersymmetric theories respectively. In Chapter 6, we move away from Minkowski space and consider gravitational fluctuations thereof. We show that the asymptotic symmetries derived by BMvS [14, 15] are related to Weinberg's soft-graviton theorem. Finally, in Chapter 7, we discuss a recently proposed infinite-dimensional extension of the BMS group – the extended BMS group. The extended BMS group includes a Virasoro subgroup which is shown to be related to a newly discovered subleading soft-graviton theorem [11] and we construct the corresponding two-dimensional stress tensor.

2

Asymptotics of Minkowski Spacetime

In this chapter, we study the geometric structure of Minkowski space. In particular, we will be interested in the asymptotic structure of Minkowski space. We also discuss free fields and their asymptotic structure. A lot of the discussion in this chapter is found in quantum field theory and general relativity textbook, albeit in a slightly different form. We reproduce it here to setup our notations and conventions.

2.1 Causal Structure

A useful representation of the asymptotic structure of a $(d+2)$-dimensional Lorentzian spacetime (M, g) is given by the Penrose diagram (or more precisely, the Penrose-Carter diagram), which captures the causal relation between different points in M, i.e. whether two points are spacelike, null or timelike separated. The idea is to perform a conformal transformation on the metric $g \to \widetilde{g} = \Omega^2 g$ which brings the entire spacetime M into a compact region which can then be conveniently represented on a two-dimensional diagram. Note that on the asymptotic boundary of M, we have $\Omega = 0$ on the boundary of M. Since conformal transformations preserve causal relationships, the causal structure of the *unphysical* spacetime $(M, \widetilde{g})$ is the same as that of (M, g). Distances are not accurately represented in the Penrose diagram.

In practice, one obtains the Penrose diagram by finding compact timelike and spacelike coordinates T and R and then choosing Ω so that $\widetilde{g} = -\mathrm{d}T^2 + \mathrm{d}R^2 + \widetilde{g}_{ab}\mathrm{d}x^a\mathrm{d}x^b$ where x^a are the remaining d spatial coordinates. The Penrose diagram is obtained by plotting the coordinate ranges of T and R on the usual Cartesian plane. The remaining d spatial directions x^a are suppressed in this representation of M.

We now carry out this program for Minkowski spacetime $M_{D=d+2}$. This is globally described in Cartesian coordinates $y^A = (y^0, y^1, y^2, \cdots, y^{d+1})$, $y^A \in \mathbb{R}$ by the metric

$$\mathrm{d}s^2 = \eta_{AB}\mathrm{d}y^A\mathrm{d}y^B = -\left(\mathrm{d}y^0\right)^2 + \left(\mathrm{d}y^1\right)^2 + \left(\mathrm{d}y^2\right)^2 + \cdots + \left(\mathrm{d}y^{d+1}\right)^2 . \tag{2.1.1}$$

To find the compact coordinates T and R, we move to spherical coordinates $(t, r, \theta^1, \cdots, \theta^d)$,

$$y^0 = t\,, \qquad y^i = r\,\hat{y}^i(\theta)\,, \qquad i = 1, \cdots, d+1\,. \tag{2.1.2}$$

Here, θ^a, $a = 1, \cdots, d$ are generalized coordinates on S^d and $\hat{y}^i(\theta)$ is the unit-vector in $\mathbb{R}^{d+1}$ pointing towards $\theta \in S^d$ (which is embedded in $\mathbb{R}^{d+1}$ in the standard way).

The metric of Minkowski spacetime in spherical coordinates is

$$ds^2 = -dt^2 + dr^2 + r^2\gamma_{ab}(\theta)d\theta^a d\theta^b\,, \qquad \gamma_{ab}(\theta) = \partial_a\hat{y}^i(\theta)\,\partial_b\hat{y}_i(\theta)\,. \tag{2.1.3}$$

where $\gamma_{ab}(\theta)$ is the round metric on the unit S^d.[1]

We may then find T and R by performing the following chain of coordinate transformations

$$\begin{aligned}
&u = t - r\,, && v = t + r\,, && -\infty < u < v < \infty\,,\\
&U = \tan^{-1} u\,, && V = \tan^{-1} v\,, && -\frac{\pi}{2} < U \leq V < \frac{\pi}{2}\,,\\
&T = U + V\,, && R = V - U\,, && R \geq 0\,, \quad R + |T| < \pi\,.
\end{aligned} \tag{2.1.4}$$

u and v are null coordinates in the sense that null geodesics in Minkowski spacetime are defined by varying u keeping (v, θ) fixed and varying v keeping (u, θ) fixed. Similarly, U and V are null coordinates. The metric in the (T, R, θ) coordinates is

$$ds^2 = \frac{1}{(\cos T + \cos R)^2}\left(-dT^2 + dR^2 + \sin^2 R\gamma_{ab}d\theta^a d\theta^b\right)\,. \tag{2.1.5}$$

We then obtain the *unphysical* metric $\widetilde{g}$ by choosing $\Omega = \cos T + \cos R$. We note that the resultant unphysical spacetime $(M, \widetilde{g})$ is compact and may be represented as shown in Figure 2.1.

[1]For the unit S^d, we have

$$R_{abcd}[\gamma] = \gamma_{ac}\gamma_{bd} - \gamma_{ad}\gamma_{bc}\,, \qquad R_{ab}[\gamma] = (d-1)\gamma_{ab}\,, \qquad R[\gamma] = d(d-1)\,.$$

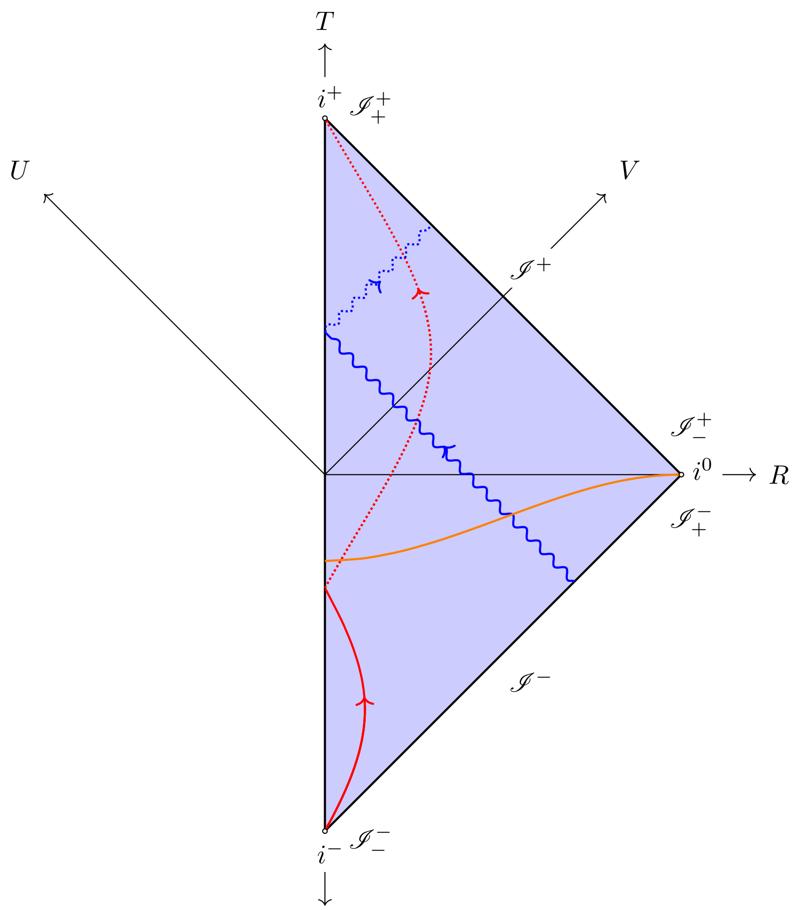

Figure 2.1: Penrose diagram of Minkowski spacetime: The coordinate ranges $|T| + R \leq \pi$ and $R \geq 0$ are plotted above. The angular coordinates θ is frozen $\theta = \theta_0$ so that each point in the diagram above is an S^d. Timelike, spacelike and null geodesics that pass through the origin are shown in red, orange and blue respectively. The dotted segment indicates that the geodesic that starts at θ_0 crosses the origin $r = 0$ over to the antipodal point $\widetilde{\theta}_0$.

The asymptotic boundaries of Minkowski spacetime are given by $\Omega\big|_{\partial M_D} = 0$. There are three different types of asymptotic regions (shown in Figure 2.1):

Time-like Infinity $i^\pm$ This is described by $T = \pm(\pi - \varepsilon)$, $R = \frac{1}{2}r\varepsilon^2$ with $\varepsilon \to 0$. In spherical coordinates, this corresponds to the limit $t \to \pm\infty$ with (r, θ) fixed. All time-like curves begin at past timelike infinity i^- and end at future timelike infinity i^+.

Spatial Infinity i^0 This is described by $T = \frac{1}{2}t\varepsilon^2$, $R = \pi - \varepsilon$ with $\varepsilon \to 0$. In spherical coordinates, this corresponds to the limit $r \to \infty$ keeping (t, θ) fixed. The end-points of all spatial curves lies in i^0.

Null Infinity $\mathscr{I}^\pm$ Future null infinity $\mathscr{I}^+$ is described by $V = \frac{\pi}{2}$ whereas past null infinity $\mathscr{I}^-$ is $U = -\frac{\pi}{2}$. In spherical coordinates, $\mathscr{I}^+$ corresponds to the limit $v \to \infty$ keeping (u, θ) fixed whereas $\mathscr{I}^-$ is the limit $u \to -\infty$ keeping (v, θ) fixed. These have the topology of $S^d \times \mathbb{R}$. All null geodesics begin at $\mathscr{I}^-$ and end at $\mathscr{I}^+$. In this thesis, we will focus our discussion primarily on null boundaries $\mathscr{I}^\pm$. $\mathscr{I}^+$ has further d-dimensional boundaries located at $u = \pm\infty$ ($U = \pm\frac{\pi}{2}$) which we denote by $\mathscr{I}^+_\pm$. Similarly, $\mathscr{I}^-$ has boundaries at $v = \pm\infty$ ($V = \pm\frac{\pi}{2}$) which we denote by $\mathscr{I}^-_\pm$. Note that $\mathscr{I}^+_+$ ($\mathscr{I}^-_-$) is distinct from i^+ (i^-) and $\mathscr{I}^\pm_\mp$ are distinct from i^0.

Note that null (timelike) geodesics that begin at $\mathscr{I}^-$ (i^-) at a point $\theta = \theta_0$ end at $\mathscr{I}^+$ (i^+) at the antipodal point $\theta = \widetilde{\theta}_0$. For this reason, it is often more convenient to draw the Penrose diagram of Minkowski space by including also the antipodal angle as shown in Figure 2.2.

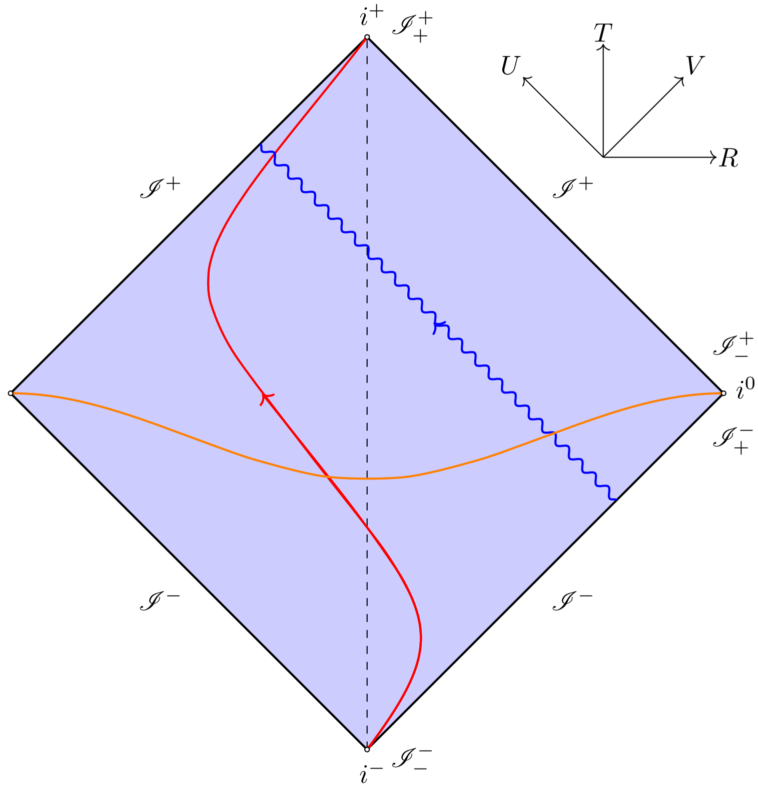

Figure 2.2: Alternative Penrose diagram of Minkowski spacetime: In the diagram above, we have frozen the angular coordinates, with the right side corresponding to a fixed point $\theta = \theta_0$ on S^d and the left side the antipodal point, $\theta = \widetilde{\theta}_0$. Thus, each point in the diagram above is a hemisphere. Timelike, null and spacelike geodesics passing through the origin are shown in red, blue and orange respectively.

2.2 Retarded and Advanced Coordinates

In this section, we introduce two coordinate systems that are more naturally adapted to $\mathscr{I}^+$ and $\mathscr{I}^-$. These are the so-called retarded coordinates (u, r, θ) and advanced coordinates $(v, r, \widetilde{\theta})$ where the metric of Minkowski spacetime takes the form

$$ds^2 = -\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + r^2\gamma_{ab}(\theta)\mathrm{d}\theta^a\mathrm{d}\theta^b = -\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r + r^2\gamma_{ab}(\widetilde{\theta})\mathrm{d}\widetilde{\theta}^a\mathrm{d}\widetilde{\theta}^b\,. \tag{2.2.1}$$

$\mathscr{I}^+$ is best described in retarded coordinates as the boundary located at $r = \infty$ keeping (u, θ) fixed whereas $\mathscr{I}^-$ is best described in advanced coordinates as the boundary located at $r = \infty$ keeping $(v, \widetilde{\theta})$ fixed. Here, θ and $\widetilde{\theta}$ are antipodal coordinate systems on the asymptotic S^d, i.e.

$$\hat{y}^i\big(\widetilde{\theta}(\theta)\big) = -\hat{y}^i(\theta)\,, \qquad \gamma_{ab}(\theta) = \partial_a\widetilde{\theta}^c\partial_b\widetilde{\theta}^d\gamma_{cd}\big(\widetilde{\theta}(\theta)\big)\,. \tag{2.2.2}$$

The Christoffel symbols in the retarded coordinates are

$$\Gamma^u_{ab}[g] = -\Gamma^r_{ab}[g] = r\gamma_{ab}\,, \qquad \Gamma^a_{rb}[g] = \frac{1}{r}\delta^a_b\,, \qquad \Gamma^a_{bc}[g] = \Gamma^a_{bc}[\gamma]\,. \tag{2.2.3}$$

These coordinates are shown in Figure 2.3

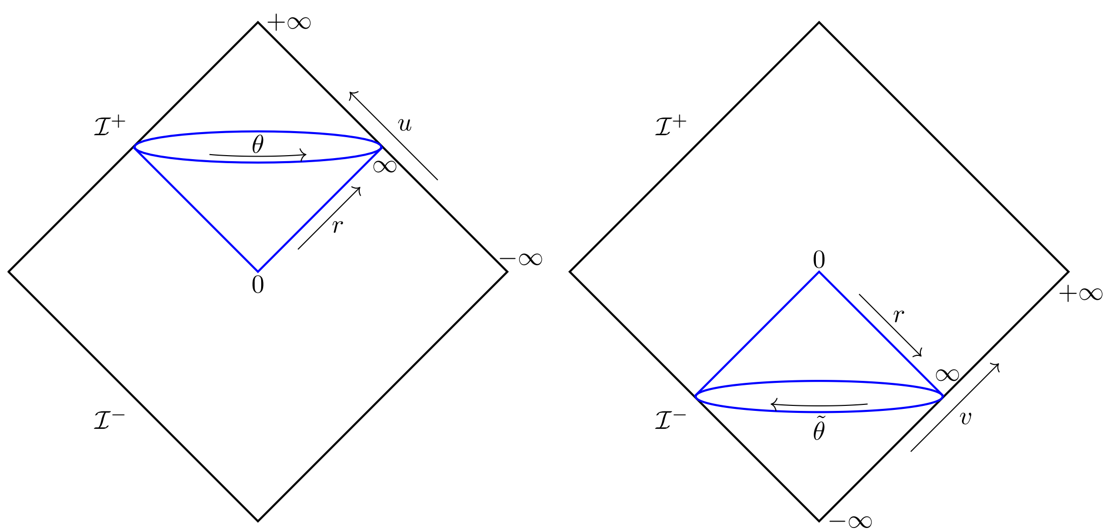

Figure 2.3: Retarded (left) and Advanced Coordinates (right) shown on the Penrose diagram of Minkowski space

The null normal vector and volume element on $\mathscr{I}^+$ is

$$n = \partial_u - \frac{1}{2}\partial_r\,, \qquad \int_{\mathscr{I}^+} d\Sigma^\mu = \lim_{r\to\infty} r^d \int_{-\infty}^{\infty} du \int_{S^d} d^d\theta\sqrt{\gamma}\left(\delta^\mu_u - \frac{1}{2}\delta^\mu_r\right)\,. \tag{2.2.4}$$

In §2.4.3, we will be discussing the structure of spinor fields near $\mathscr{I}^+$. For this purpose, we will need to introduce a vielbein. We will work with the flat vielbein,

$$e^A_\mu \mathrm{d}x^\mu = \mathrm{d}y^A = \frac{\partial y^A}{\partial x^\mu}\mathrm{d}x^\mu\,, \qquad \omega_\mu{}^A{}_B = 0\,. \tag{2.2.5}$$

We will discuss these vielbein more explicitly when we discuss spinors in §2.4.3.

Special Coordinates in $D = 4$ So far we have discussed retarded and advanced coordinates in general dimensions $D \geq 3$ and have not made any particular choice for coordinates on the asymptotic sphere, S^d. In $D = 4$, the asymptotic sphere is two-dimensional and it is extremely useful to work in stereographic coordinates $\theta^a = (z, \overline{z})$. These are related to the standard angular coordinates (θ, ϕ) by

$$z = e^{i\phi}\tan\frac{\theta}{2}\,, \qquad \overline{z} = e^{-i\phi}\tan\frac{\theta}{2}\,. \tag{2.2.6}$$

Alternatively, we may describe them by relating $(z, \overline{z})$ to the unit vector $\hat{y}^i(\theta)$ as

$$\hat{y}^i(z,\overline{z}) = \left(\frac{z+\overline{z}}{1+z\overline{z}}\,, \frac{-i(z-\overline{z})}{1+z\overline{z}}\,, \frac{1-z\overline{z}}{1+z\overline{z}}\right)\,. \tag{2.2.7}$$

The S^2-metric takes the form

$$d\Omega_2^2 = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2 = 2\gamma_{z\overline{z}}\mathrm{d}z\mathrm{d}\overline{z}\,, \qquad \gamma_{z\overline{z}} = \frac{2}{(1+z\overline{z})^2}\,. \tag{2.2.8}$$

The volume form on S^2 is

$$d^2\Omega = \sin\theta \mathrm{d}\theta \wedge \mathrm{d}\phi = i\gamma_{z\overline{z}}\mathrm{d}z \wedge \mathrm{d}\overline{z} \equiv d^2z\gamma_{z\overline{z}}\,. \tag{2.2.9}$$

The S^2 Christoffel symbols are

$$\Gamma^z_{zz}[\gamma] = \gamma^{z\overline{z}}\partial_z\gamma_{z\overline{z}}\,, \qquad \Gamma^{\overline{z}}_{\overline{z}\overline{z}}[\gamma] = \gamma^{z\overline{z}}\partial_{\overline{z}}\gamma_{z\overline{z}}\,, \tag{2.2.10}$$

and all others vanish.

A convenience of working in stereographic coordinates is that one may describe spinor and tensor representations together. These are classified by their weights $(h, \overline{h})$. The covariant derivative of a tensor $T_{(h,\overline{h})}$ is given by

$$\begin{aligned} D_z T_{(h,\overline{h})} &= \partial_z T_{(h,\overline{h})} - h\Gamma^z_{zz} T_{(h,\overline{h})} \,, \\ D_{\overline{z}} T_{(h,\overline{h})} &= \partial_{\overline{z}} T_{(h,\overline{h})} - \overline{h}\Gamma^{\overline{z}}_{\overline{z}\overline{z}} T_{(h,\overline{h})} \,. \end{aligned} \tag{2.2.11}$$

For integer and positive h and $\overline{h}$, $T_{(h,\overline{h})}$ can be thought of as a covariant tensor (or contravariant if h or $\overline{h}$ are negative) with h z-indices and $\overline{h}$ $\overline{z}$ indices. However, (2.2.11) holds for both integer and half-integer h and $\overline{h}$.

Finally, we introduce a complex zweibein on S^2,

$$E^+_z = E^-_{\overline{z}} = \sqrt{2\gamma_{z\overline{z}}} \,, \qquad E^+_{\overline{z}} = E^-_z = 0 \,. \tag{2.2.12}$$

Note that $(E^+_z)^* = E^-_{\overline{z}}$. The flat metric is $\eta_{\pm\pm} = 0$, $\eta_{+-} = \eta_{-+} = \frac{1}{2}$. The corresponding spin connection has the following non-vanishing components

$$\Omega_z{}^{\pm}{}_{\pm} = \pm\Omega_z \,, \qquad \Omega_{\overline{z}}{}^{\pm}{}_{\pm} = \mp\Omega_{\overline{z}} \,. \tag{2.2.13}$$

where

$$\Omega_z = \frac{1}{2}\gamma^{z\overline{z}}\partial_z\gamma_{z\overline{z}} = \frac{1}{2}\Gamma^z_{zz}[\gamma] \,, \qquad \Omega_{\overline{z}} = \frac{1}{2}\gamma^{z\overline{z}}\partial_{\overline{z}}\gamma_{z\overline{z}} = \frac{1}{2}\Gamma^{\overline{z}}_{\overline{z}\overline{z}}[\gamma] \,. \tag{2.2.14}$$

Now, given any $(h, \overline{h})$ tensor, we may change basis and define a tensor w.r.t the *internal flat metric* $\hat{T}_{(h,\overline{h})}$ as

$$\hat{T}_{(h,\overline{h})} = \left(E^z_+\right)^h \left(E^{\overline{z}}_-\right)^{\overline{h}} T_{(h,\overline{h})} \,. \tag{2.2.15}$$

$\hat{T}_{(h,\overline{h})}$ does not transform under spacetime diffeomorphisms, but transforms as a $(h, \overline{h})$ tensor under the internal $SO(2)$ rotation. The action of the covariant derivative on this tensor is

$$\begin{aligned} D_z \hat{T}_{(h,\overline{h})} &= \partial_z \hat{T}_{(h,\overline{h})} - s\Omega_z \hat{T}_{(h,\overline{h})} \,, \\ D_{\overline{z}} \hat{T}_{(h,\overline{h})} &= \partial_{\overline{z}} \hat{T}_{(h,\overline{h})} + s\Omega_{\overline{z}} \hat{T}_{(h,\overline{h})} \,, \end{aligned} \tag{2.2.16}$$

where $s = h - \overline{h}$ is the spin of the tensor.

Finally, we note that the antipodal stereographic coordinates $(\widetilde{z}, \widetilde{\overline{z}})$ is related to $(z, \overline{z})$ as

$$\widetilde{z} = -\frac{1}{\overline{z}} \,, \qquad \widetilde{\overline{z}} = -\frac{1}{z} \,. \tag{2.2.17}$$

which corresponds to $\widetilde{\theta} = \pi - \theta$, $\widetilde{\phi} = \pi + \phi$. It may easily be verified that

$$\hat{y}^i(\widetilde{z}, \widetilde{\overline{z}}) = -\hat{y}^i(z, \overline{z}) \,. \tag{2.2.18}$$

2.3 Poincaré Generators

Particle dynamics are constrained by symmetries, via Noether's theorem. In particular, Killing vector fields of Minkowski spacetime give rise to translational and Lorentz invariance, which correspond to momentum conservation and angular momentum conservation. In Cartesian coordinates, these vector fields are

$$\zeta^T_A = \partial_A \,, \qquad \zeta^L_{AB} = x_A \partial_B - x_B \partial_A \,. \tag{2.3.1}$$

Together, these generators form the Poincaré algebra, $T^{d+2} \ltimes \mathfrak{so}(1, d+1)$. In our discussion of null infinity, we will find it more convenient to rewrite these generators in retarded (or advanced) coordinates,

$$\begin{aligned} \zeta_f &= f\partial_u + \frac{1}{d}D^2 f\partial_r - \frac{1}{r}D^a f\partial_a \, , \\ \zeta_Y &= \psi \left[u\partial_u - (u+r)\,\partial_r \right] + \left[Y^a - \frac{u}{r}D^a\psi \right] \partial_a \, . \end{aligned} \tag{2.3.2}$$

where

$$\psi = \frac{1}{d}D^c Y_c \, . \tag{2.3.3}$$

Here, $f(\theta)$ is a function and $Y^a(\theta)$ is a vector field on S^d that satisfy

$$D_a D_b f - \frac{1}{d}\gamma_{ab} D^2 f = 0 \, , \quad (d > 1) \tag{2.3.4}$$

$$D_a (D^2 + d) f = 0 \, , \quad (d \geq 1) \tag{2.3.5}$$

$$D_a Y_b + D_b Y_a - \frac{2}{d}\gamma_{ab} D^c Y_c = 0 \, , \quad (d \geq 2) \tag{2.3.6}$$

$$\big(D_a D_b + \gamma_{ab}\big) D^c Y_c = 0 \, . \quad (d \geq 2) \tag{2.3.7}$$

Here, D_a is the γ-covariant derivative and $D^2 = D^c D_c$. In $d > 1$, (2.3.4) implies (2.3.5) and in $d > 2$, (2.3.6) implies (2.3.7).

The Killing vectors (2.3.2) are related to the usual translation and Lorentz transformation generators as follows. We may decompose the function $f(\theta)$ into spherical harmonic modes as

$$f(\theta) = \sum_{\ell=0}^{\infty} \sum_J a_{\ell,J} \mathrm{Y}_{\ell,J}(\theta) \, , \qquad D^2 \mathrm{Y}_{\ell,J}(\theta) = -\ell\,(\ell + d - 1)\,\mathrm{Y}_{\ell,J}(\theta) \, . \tag{2.3.8}$$

where J are all the remaining quantum numbers[2]. (2.3.5) implies that only the $\ell = 0$ and $\ell = 1$ modes of $f(\theta)$ are non-zero. The single $\ell = 0$ mode corresponds to time translation whereas the

[2]For S^d, $J = \{m_1 \cdots m_{d-1}\}$ with $m_i \in \mathbb{Z}$ and $|m_1| \leq m_2 \leq \cdots \leq m_{d-1} \leq \ell$. The number of spherical harmonics with quantum number ℓ is $N(d,\ell) = \frac{d+2\ell-1}{d-1}\binom{d+\ell-2}{\ell}$. Note that $N(d,0) = 1$ and $N(d,1) = d+1$.

$(d+1)$ $\ell = 1$ modes correspond to the $(d+1)$ spatial translations in $\mathbb{R}^{1,d+1}$.

In a similar way, (2.3.6) implies that Y^a is a conformal Killing vector field of S^d. The Lie algebra of these vectors is $\mathfrak{so}(1, d+1)$ which is also the Lorentz algebra in $(d+2)$-dimensions. In $d = 2$, the conformal algebra is infinite-dimensional. In this case, (2.3.7) implies that Y^a is a *global conformal Killing vector* of S^2 which reduces to the symmetry algebra to $\mathfrak{so}(1,3) \cong \mathfrak{sl}(2,\mathbb{C})$. Spatial rotations are generated by Killing vector fields of S^d, namely those satisfying $\psi = 0$, whereas Lorentz boosts correspond to vector fields with $\psi \neq 0$.

The algebra of these vector fields is the Poincaré algebra,

$$\big[\zeta_f, \zeta_{f'}\big] = 0\,, \qquad \big[\zeta_Y, \zeta_f\big] = \zeta_{Y(f)-\psi f}\,, \qquad \big[\zeta_Y, \zeta_{Y'}\big] = \zeta_{[Y,Y']}\,. \tag{2.3.9}$$

In a similar way, one may also determine the Killing vectors in advanced coordinates as

$$\begin{aligned} \zeta^-_{\widetilde{f}^-} &= \widetilde{f}^-\partial_v - \frac{1}{d}\widetilde{D}^2\widetilde{f}^-\partial_r - \frac{1}{r}\widetilde{D}^a\widetilde{f}^-\widetilde{\partial}_a\,, \\ \zeta^-_{\widetilde{Y}^-} &= \widetilde{\psi}^-\big[v\partial_v + (v-r)\partial_r\big] + \Big[\widetilde{Y}^{-a} + \frac{v}{r}\widetilde{D}^a\widetilde{\psi}^-\Big]\,\widetilde{\partial}_a\,. \end{aligned} \tag{2.3.10}$$

where $\widetilde{\psi}^- = \frac{1}{d}\widetilde{D}^c\widetilde{Y}^-_c$ and $\widetilde{Y}^{-a}$ is a conformal Killing vector of S^d. Of course, these Killing vectors are not independent of those in retarded coordinates (2.3.2). Rather,

$$f(\theta) = \widetilde{f}^-(\theta)\,, \qquad Y^a(\theta) = \widetilde{Y}^{-a}(\theta)\,. \tag{2.3.11}$$

The identification above implies that f and Y^a are *antipodally* identified with $\widetilde{f}^-$ and $\widetilde{Y}^{-a}$ as functions. For instance, the value of the function f at the north pole (say $\theta = 0$) is equal to the value of $\widetilde{f}$ at the south pole (which is now $\widetilde{\theta} = 0$).

In $D = 4$ and in stereographic coordinates, the conditions (2.3.4)–(2.3.7) read

$$D_z^2 f = 0 = D_{\bar{z}}^2 f\,, \qquad D_{\bar{z}} Y^z = 0 = D_z Y^{\bar{z}}\,, \qquad D_z^3 Y^z = 0 = D_{\bar{z}}^2 Y^{\bar{z}}\,. \tag{2.3.12}$$

These are solved by

$$f_a = a^0 + a^1 \frac{z+\bar{z}}{1+z\bar{z}} + a^2 \frac{-i(z-\bar{z})}{1+z\bar{z}} + a^3 \frac{1-z\bar{z}}{1+z\bar{z}}\,, \qquad Y^z = \alpha + \beta z + \gamma z^2\,, \tag{2.3.13}$$

where $a^\mu \in \mathbb{R}$ and $\alpha, \beta, \gamma \in \mathbb{C}$.

Stereographic coordinates make the group isomorphism $SO(1,3) \cong SL(2,\mathbb{C})$ explicit. A general finite Lorentz transformation takes the form of a Mobiüs transformation,

$$z \to z' = \frac{az+b}{cz+d}\,, \qquad u \to u' = \frac{u(1+z\bar{z})}{|cz+d|^2 + |az+b|^2}\,, \qquad ad - bc = 1\,. \tag{2.3.14}$$

The antipodal map of the Poincaré generators (2.3.11) in these coordinates is

$$f(z,\bar{z}) = \widetilde{f}^-(z,\bar{z})\,, \qquad Y^z(z) = \widetilde{Y}^{-\tilde{z}}(z)\,. \tag{2.3.15}$$

2.4 Asymptotics of Massless Fields

Having understood the geometry of Minkowski spacetime, we may consider the asymptotic dynamics of fields on M_4. We only consider massless particles and will therefore be interested in the structure near $\mathscr{I}^+$ and $\mathscr{I}^-$. For most of this section, we discuss the structure near $\mathscr{I}^+$. The analogous structure near $\mathscr{I}^-$ is almost identical and only briefly discussed in §2.4.5.

In particular, the goal of this section is to solve the Cauchy problem for and canonically quantize massless scalar, vector and spinor fields on $\mathscr{I}^+$. The discussion of free fields trivially extends to the interacting case as long as all interactions die off sufficiently fast near $\mathscr{I}^+$, i.e. as long as all interactions are either marginal or irrelevant. We briefly discuss this in §2.4.6.

2.4.1 Scalar Field

A massless free scalar field is governed by the action

$$S[\Phi] = -\frac{1}{2}\int_{M_4} d^4x\sqrt{-g}\nabla_\mu\Phi\nabla^\mu\Phi\,. \tag{2.4.1}$$

We vary the action w.r.t. Φ to determine the equations of motion

$$\delta S[\Phi,\delta\Phi] = \int_{M_4} d^4x\sqrt{-g}\delta\Phi\nabla^2\Phi - \int_{M_4} d^4x\sqrt{-g}\nabla^\mu\left(\delta\Phi\nabla_\mu\Phi\right)\,. \tag{2.4.2}$$

The first term above gives us the equations of motion

$$\nabla^2\Phi = \left[\partial_r^2 - 2\partial_u\partial_r + \frac{2}{r}(\partial_r - \partial_u) + \frac{1}{r^2}D^2\right]\Phi = 0\,. \tag{2.4.3}$$

The first step towards studying the asymptotics of the scalar field is to determine its large r behavior. To do this, we consider (2.4.3) at large r,

$$\nabla^2\Phi = 0 \quad \overset{\text{large } r}{\rightarrow} \quad -2\left[\partial_r - \frac{1}{r}\right]\partial_u\Phi = 0\,, \tag{2.4.4}$$

which implies that near $\mathscr{I}^+$, $\partial_u\Phi = \mathcal{O}\left(r^{-1}\right)$. This motivates the boundary condition,

$$\Phi = \mathcal{O}\left(r^{-1}\right) \quad \text{at large } r. \tag{2.4.5}$$

This boundary condition is also consistent with finiteness of momentum and angular momentum flux through $\mathscr{I}^+$.

We now solve the Cauchy problem on $\mathscr{I}^+$, which is to determine the *data* on $\mathscr{I}^+$ that must be prescribed in order to have a unique solution to the wave equation (2.4.3). To do this, we assume that the scalar field Φ admits a large r Taylor expansion satisfying (2.4.5),

$$\Phi(u,r,\theta) = \sum_{n=1}^{\infty} \frac{\Phi^{(n)}(u,\theta)}{r^n} \,. \tag{2.4.6}$$

Plugging this expansion into (2.4.3) and expanding in large r, we find the following equations order-by-order in large r,

$$\partial_u \Phi^{(n+1)} = -\frac{1}{2n}\left[D^2 + n(n-1)\right]\Phi^{(n)}\,, \qquad n \geq 1\,. \tag{2.4.7}$$

Up to u-independent integration constants $\Phi^{(n)}(\theta)$ for $n > 1$[3], the full scalar field is determined by the leading order coefficient $\Phi^{(1)}(u,\theta)$. Thus, the boundary data for the massless scalar is

$$\phi(u,\theta) = \lim_{r\to\infty}\left[r\Phi(u,r,\theta)\right]\,. \tag{2.4.8}$$

where we have now relabelled $\Phi^{(0)} \to \phi$. In the language of S^2 tensors introduced in §2.2, this field has $h = \bar{h} = 0$ and is denoted $\phi_{(0,0)}$.

Now that we have solved the Cauchy problem on $\mathscr{I}^+$, we may proceed with the canonical quantization of the theory. To do this, we follow the procedure described in [85]. Let us briefly review this here.

[3]Solutions in which these integration constants are non-zero are of the form

$$\sum_{n=1}^{\infty} \frac{1}{r^n}\, {}_2F_1\left(n - \frac{1}{2} - \sqrt{\frac{1}{4} - D^2}, n - \frac{1}{2} + \sqrt{\frac{1}{4} - D^2}, n, -\frac{u}{2r}\right)\Phi^{(n)}(\theta)\,.$$

These solutions are singular at the origin $r = 0$ and are not considered here.

Aside – Review of Wald and Zoupas: Let φ be the set of all fields in a theory which is described by an action $S[\varphi]$,

$$S[\varphi] = \int d^4x\sqrt{-g}\mathcal{L}(\varphi)\,. \tag{2.4.9}$$

Varying this action, we find the form

$$\delta S[\varphi] = \int d^4x\sqrt{-g}E(\varphi)\delta\varphi + \int d^4x\sqrt{-g}\nabla_\mu\Theta^\mu(\varphi,\delta\varphi)\,. \tag{2.4.10}$$

The bulk term implies the Euler-Lagrange equations of motion $E(\varphi) = 0$. From the bulk term, we read off the symplectic current potential density, $\Theta^\mu(\varphi,\delta\varphi)$. This quantity is defined only up to a total derivative

$$\Theta'^\mu(\varphi,\delta\varphi) = \Theta^\mu(\varphi,\delta\varphi) + \nabla_\nu B^{\nu\mu}(\varphi,\delta\varphi)\,. \tag{2.4.11}$$

The symplectic form on a hypersurface Σ is given by

$$\Omega_\Sigma(\varphi,\delta\varphi,\delta'\varphi) = \int_\Sigma d\Sigma_\mu\big[\delta\Theta^\mu(\varphi,\delta'\varphi) - \delta'\Theta^\mu(\varphi,\delta\varphi)\big]\,. \tag{2.4.12}$$

Note that if we use Θ' instead of Θ, the symplectic form is modified to

$$\Omega'_\Sigma(\varphi,\delta\varphi,\delta'\varphi) = \Omega_\Sigma(\varphi,\delta\varphi,\delta'\varphi) + \int_{\partial\Sigma} d\Sigma_{\mu\nu}\big[\delta B^{\nu\mu}(\varphi,\delta'\varphi) - \delta' B^{\nu\mu}(\varphi,\delta\varphi)\big]\,. \tag{2.4.13}$$

Thus, the ambiguity in Θ affects the boundary symplectic form. In this thesis, we will not discuss this ambiguity.

Let us now work this out for the scalar case. The symplectic current potential density from the boundary term in the variation of the action (2.4.2)

$$\Theta_\mu\left(\Phi,\delta\Phi\right) = \partial_\mu\Phi\delta\Phi\,. \tag{2.4.14}$$

Finally, the symplectic form on a Cauchy surface Σ is given by

$$\Omega_\Sigma = \int d\Sigma^\mu \left[\delta_1 \Theta_\mu \left(\Phi, \delta_2 \Phi\right) - \delta_2 \Theta_\mu \left(\Phi, \delta_1 \Phi\right)\right] = \int d\Sigma^\mu \partial_\mu \delta\Phi \wedge \delta\Phi \,. \tag{2.4.15}$$

where we define $\delta a \wedge \delta b = \delta_1 a \delta_2 b - \delta_2 a \delta_1 b$.

As an example of this procedure, let us take Σ to be a $t =$ constant hypersurface, $\mathcal{H}_t$. The Cauchy data on this hypersurface is $\phi = \Phi|_\Sigma$ and its time-derivative $\pi = \partial_t \Phi|_\Sigma$. The symplectic form is

$$\Omega_{\mathcal{H}_t} = \int d^3x \, \delta\pi(t, \vec{x}) \wedge \delta\phi(t, \vec{x}) \,. \tag{2.4.16}$$

The quantum commutators on $\mathcal{H}_t$ are then determined by inverting Ω[4]

$$\big[\phi(t, \vec{x}), \pi(t, \vec{x}\,')\big] = i\delta^{d-1}\big(\vec{x} - \vec{x}\,'\big) \,, \quad \big[\phi(t, \vec{x}), \phi(t, \vec{x}\,')\big] = \big[\pi(t, \vec{x}), \pi(t, \vec{x}\,')\big] = 0 \,. \tag{2.4.17}$$

Thus, by quantizing the theory on $\mathcal{H}_t$, we retrieve the usual equal time commutators of quantum field theory.

We now move back to the case of interest, namely $\Sigma = \mathscr{I}^+$. The symplectic form is

$$\Omega_{\mathscr{I}^+} = \int du d^2\theta \sqrt{\gamma} \partial_u \delta\phi \wedge \delta\phi \,. \tag{2.4.18}$$

Recall that when we quantized the theory on $\mathcal{H}_t$, ϕ was paired with π both of which are independent data on $\mathcal{H}_t$. On $\mathscr{I}^+$ however, ϕ is paired with $\partial_u\phi$, which are not independent data. We must therefore be careful about the way we read off quantum commutators. In particular, we must be very careful about u-independent modes of $\phi(u, \theta)$. We start by considering the zero mode $C(\theta) = \phi(+\infty, \theta) + \phi(-\infty, \theta)$. This zero mode corresponds to the divergent $n = 0$ solution

[4] On a general symplectic manifold, if $\Omega = \frac{1}{2}\Omega_{\mu\nu} dq^\mu \wedge dq^\nu$ then $[q^\mu, q^\nu] = i\Omega^{\mu\nu}$ where $\Omega^{\mu\nu}$ is the inverse matrix of $\Omega_{\mu\nu}$.

mentioned in footnote 3. We therefore discard this zero mode and set $C = 0$. For all such solutions, we may write

$$\phi(u,\theta) = \frac{1}{2}\int_{-\infty}^{\infty} du' \Theta(u-u')\partial_{u'}\phi(u',\theta) \,. \tag{2.4.19}$$

where $\Theta(x)$ is the sign function. We now move to Fourier space

$$N(\omega,\theta) = \int_{-\infty}^{\infty} du e^{i\omega u}\partial_u \phi(u,\theta) \,. \tag{2.4.20}$$

The symplectic form is then

$$\Omega_{\mathscr{I}^+} = \frac{i}{\pi}\int_0^{\infty} d\omega d^2\theta \sqrt{\gamma}\frac{1}{\omega}\delta N(\omega,\theta)\wedge \delta N(-\omega,\theta) \,. \tag{2.4.21}$$

We can then easily read off the quantum commutators

$$\big[N(\omega,\theta), N(\omega',\theta')\big] = -\pi\omega\delta\big(\omega+\omega'\big)\delta^2\big(\theta,\theta'\big) \,. \tag{2.4.22}$$

where $\delta^2(\theta,\theta')$ is the Dirac Delta function on S^2 normalized as

$$\int d^2\theta\sqrt{\gamma}\delta^2(\theta,\theta') = 1 \,. \tag{2.4.23}$$

Moving back to position space, we find

$$\big[\phi(u,\theta), \phi(u',\theta')\big] = -\frac{i}{4}\Theta(u-u')\delta^2\big(\theta,\theta'\big) \,. \tag{2.4.24}$$

(2.4.24) is the canonical commutation relation on $\mathscr{I}^+$. The non-standard Θ-function that appears on the RHS is due to the fact that $\mathscr{I}^+$ is a null hypersurface. Note that (2.4.24) is the quantum commutator one would obtain if we naively invert the symplectic form (2.4.18) without discussing the zero mode issues. This is due to the fact that the potential zero mode C vanishes on the space of solutions that we are considering. When dealing with gauge fields, the analogous zero mode is generically non-vanishing and we must follow through with the analogous argument as above in order to determine the correct commutators.

2.4.2 Vector Field

We now consider the quantization of a free $U(1)$ gauge field $\mathcal{A}$ whose dynamics is governed by the Maxwell action

$$S[\mathcal{A}] = -\frac{1}{2e^2}\int_{M_4} \mathcal{F}\wedge *\mathcal{F}\,. \tag{2.4.25}$$

where $\mathcal{F} = \mathrm{d}\mathcal{A}$ is the field strength. Under $U(1)$ gauge transformations

$$\mathcal{A} \;\rightarrow\; \mathcal{A} + \mathrm{d}\lambda\,. \tag{2.4.26}$$

Varying the action, we find

$$\delta S[\mathcal{A},\delta\mathcal{A}] = \frac{1}{e^2}\int_{M_4}(\mathrm{d}*\mathcal{F})\wedge\delta\mathcal{A} - \frac{1}{e^2}\int_{M_4}\mathrm{d}\big(*\mathcal{F}\wedge\delta\mathcal{A}\big)\,. \tag{2.4.27}$$

The bulk term above gives us the equations of motion

$$\mathrm{d}(*\mathcal{F}) = \mathrm{d}(*\mathrm{d}\mathcal{A}) = 0\,. \tag{2.4.28}$$

As in the case with the scalar field, we start by determining the boundary conditions for the gauge field. This can be done by studying the equations of motion (2.4.28) and further imposing finiteness of momentum and angular momentum flux through $\mathscr{I}^+$, analogous to the scalar case. Near $\mathscr{I}^+$, we find

$$\mathcal{F}_{ur} = \mathcal{O}\big(r^{-2}\big)\,, \qquad \mathcal{F}_{ra} = \mathcal{O}\big(r^{-2}\big)\,, \qquad \mathcal{F}_{ua} = \mathcal{O}(1)\,, \qquad \mathcal{F}_{ab} = \mathcal{O}(1)\,. \tag{2.4.29}$$

These boundary conditions are also consistent with finiteness of momentum and angular momentum flux through $\mathscr{I}^+$.

For the gauge field, this motivates the following boundary conditions

$$\mathcal{A}_u = \mathcal{O}(r^{-1}) , \qquad \mathcal{A}_r = \mathcal{O}(r^{-2}) , \qquad \mathcal{A}_a = \mathcal{O}(1) . \tag{2.4.30}$$

Next, we solve the Cauchy problem on $\mathscr{I}^+$. Here, the gauge field $\mathcal{A}$ is defined only up to gauge transformations so we need to fix a gauge. A convenient gauge for our purposes is the retarded radial gauge

$$\mathcal{A}_r = 0 . \tag{2.4.31}$$

In this gauge, (2.4.28) takes the form

$$\begin{aligned}
\nabla^\mu \mathcal{F}_{\mu u} &= \frac{1}{r^2}\left[(\partial_r - \partial_u)\big(r^2 \partial_r \mathcal{A}_u\big) + D^2 \mathcal{A}_u - \partial_u D^a \mathcal{A}_a\right] = 0 , \\
\nabla^\mu \mathcal{F}_{\mu r} &= \frac{1}{r^2}\partial_r\big(r^2 \partial_r \mathcal{A}_u - D^a \mathcal{A}_a\big) = 0 , \\
\nabla^\mu \mathcal{F}_{\mu a} &= \partial_r\big(\partial_r - 2\partial_u\big)\mathcal{A}_a + D_a \partial_r \mathcal{A}_u + \frac{1}{r^2}\big(D^2 - 1\big)\mathcal{A}_a - \frac{1}{r^2} D_a D^b \mathcal{A}_b = 0 .
\end{aligned} \tag{2.4.32}$$

To solve these equations, we Taylor expand the gauge field near $\mathscr{I}^+$,

$$\mathcal{A}_u(u,r,\theta) = \sum_{n=1}^{\infty} \frac{\mathcal{A}_u^{(n)}(u,\theta)}{r^n} , \qquad \mathcal{A}_a(u,r,\theta) = \sum_{n=0}^{\infty} \frac{\mathcal{A}_a^{(n)}(u,\theta)}{r^n} . \tag{2.4.33}$$

The equations at each order in large r takes the form (for $n \geq 0$)

$$\begin{aligned}
\partial_u\big[\mathcal{A}_u^{(1)} - D^a \mathcal{A}_a^{(0)}\big] &= 0 , \\
2\partial_u \mathcal{A}_a^{(1)} - D_a \mathcal{A}_u^{(1)} - D^b\big(D_a \mathcal{A}_b^{(0)} - D_b \mathcal{A}_a^{(0)}\big) &= 0 , \\
(n+2)\mathcal{A}_u^{(n+2)} + D^a \mathcal{A}_a^{(n+1)} &= 0 , \\
2(n+2)\partial_u \mathcal{A}_a^{(n+2)} + \big(D^2 + (n+1)(n+2) - 1\big)\mathcal{A}_a^{(n+1)} &= 0 .
\end{aligned} \tag{2.4.34}$$

The equations imply that up to u-independent integration constants $\mathcal{A}_u^{(1)}$ and $\mathcal{A}_a^{(n)}$ for $n \geq 1$[5], the full gauge field is determined by $\mathcal{A}_a^{(0)}(u, \theta)$. Thus, the boundary data for the gauge field is

$$A_a(u, \theta) = \lim_{r \to \infty} \mathcal{A}_a(u, r, \theta) \,. \tag{2.4.35}$$

where have now relabelled $\mathcal{A}_a^{(0)} \to A_a$. In stereographic coordinates, the data is $A_z \equiv A_{(1,0)}$ and $A_{\overline{z}} \equiv A_{(0,1)}$. As we will see in §2.5, A_z corresponds to a positive helicity photon and $A_{\overline{z}}$ corresponds to a negative helicity photon.

Finally, we consider canonical quantization of the gauge field on $\mathscr{I}^+$. From the boundary term of (2.4.27), we find

$$*\Theta[\mathcal{A}, \delta\mathcal{A}] = -\frac{1}{e^2} * \mathcal{F} \wedge \delta\mathcal{A} \,, \tag{2.4.36}$$

which implies the symplectic form

$$\Omega_{\mathscr{I}^+} = -\frac{1}{e^2} \int_{\mathscr{I}^+} *\delta\mathcal{F} \wedge \delta\mathcal{A} = \frac{1}{e^2} \int_{\mathscr{I}^+} dud^2\theta \sqrt{\gamma}\, \partial_u \delta A_a \wedge \delta A^a \,. \tag{2.4.37}$$

Now, as in the scalar case, it is important to be careful about u-independent *zero* modes of the gauge field. For a scalar field, such a zero mode was forced to be zero by requirement of regularity at the origin. For the gauge field, we may allow for a pure gauge zero mode which does not affect the structure of the solution at the origin. Define,

$$C_a(\theta) = \frac{1}{2} \left[A_a(+\infty, \theta) + A_a(-\infty, \theta) \right] = e^2 \partial_a C(\theta) \,. \tag{2.4.38}$$

[5] Similar to the scalar field case (see footnote 3), gauge field solutions in which these integration constants are non-zero are singular at the origin and will not be considered.

We define the zero-mode-stripped gauge field as

$$\widehat{A}_a(u,\theta) = A_a(u,\theta) - C_a(\theta)\,. \tag{2.4.39}$$

Plugging these into the symplectic form, we find

$$\Omega_{\mathscr{I}^+} = \frac{2}{e^2}\int_{\mathscr{I}^+} dud^2\theta\sqrt{\gamma}\partial_u\delta\widehat{A}_a \wedge \delta\widehat{A}^a - \int_{S^2} d^2\theta\sqrt{\gamma}\partial_a\delta C \wedge \delta N^a\,. \tag{2.4.40}$$

where we have defined

$$N_a(\theta) = \int du\partial_u A_a(u,\theta)\,. \tag{2.4.41}$$

Note that C_a and N_a are symplectically paired. To understand further the structure of the symplectic form, we decompose N_a into two pieces

$$N_a = e^2\partial_a N + e^2\varepsilon_{ab}D^a N'\,. \tag{2.4.42}$$

Using this, we find

$$\Omega_{\mathscr{I}^+} = \frac{2}{e^2}\int_{\mathscr{I}^+} dud^2\theta\sqrt{\gamma}\partial_u\delta\widehat{A}_a \wedge \delta\widehat{A}^a - e^2\int_{S^2} d^2\theta\sqrt{\gamma}\delta C \wedge D^2\delta N\,. \tag{2.4.43}$$

Note that the mode N' does not enter the symplectic form and is therefore non-dynamical. We can therefore set $N' = 0$[6] which then implies that N_a is flat.

We can finally read-off the quantum commutators as

$$\begin{aligned}\big[\widehat{A}_a(u,\theta),\widehat{A}_b(u',\theta')\big] &= -\frac{ie^2}{4}\gamma_{ab}\Theta(u-u')\delta^2(\theta,\theta')\,,\\ \big[N(\theta),C(\theta')\big] &= -\frac{i}{e^2}G(\theta,\theta')\,,\end{aligned} \tag{2.4.44}$$

[6]This is no longer true when one includes magnetically charged matter. We will not consider this case in this thesis, but has been discussed in [86].

where $G(\theta,\theta')$ is the Green's function on S^2,

$$\Box_\theta G(\theta,\theta') = \delta^2(\theta,\theta') \,. \tag{2.4.45}$$

For later use, we simplify the result above in stereographic coordinates. Here, we find it convenient to normalize the Dirac delta function as

$$\delta^2(z,\overline{z};w,\overline{w}) = \gamma^{z\overline{z}}\delta^2(z-w)\,, \qquad \int_{S^2} d^2z\,\delta^2(z-w) = 1\,. \tag{2.4.46}$$

The non-zero commutators take the form

$$\begin{aligned} \big[\widehat{A}_z(u,z,\overline{z}), \widehat{A}_{\overline{z}'}(u',z',\overline{z}')\big] &= -\frac{ie^2}{4}\Theta(u-u')\delta^2(z-z')\,, \\ \big[N(z,\overline{z}), C(z',\overline{z}')\big] &= \frac{i}{4\pi e^2}\log|z-z'|^2\,. \end{aligned} \tag{2.4.47}$$

2.4.3 Spinor Field

The final field we consider is a two-component spinor Ψ_α in four dimensions. The dynamics of a massless two-component spinor is described by the action

$$S[\Psi] = -\frac{i}{2}\int_{M_4} d^4x\sqrt{-g}\left[\overline{\Psi}\overline{\sigma}^\mu\nabla_\mu\Psi - \nabla_\mu\overline{\Psi}\overline{\sigma}^\mu\Psi\right]\,. \tag{2.4.48}$$

Here, we are using the spinor conventions of [87], which we review in Appendix A.2. The σ-matrices in retarded coordinates are given by $\sigma^\mu = e^\mu_A\sigma^A$. Varying the action, we find

$$\begin{aligned} \delta S[\Psi] = -i\int d^4x\sqrt{-g}\left[\delta\overline{\Psi}\overline{\sigma}^\mu\nabla_\mu\Psi - \nabla_\mu\overline{\Psi}\overline{\sigma}^\mu\delta\Psi\right] \\ -\frac{i}{2}\int d^4x\sqrt{-g}\nabla_\mu\left[\overline{\Psi}\overline{\sigma}^\mu\delta\Psi - \delta\overline{\Psi}\overline{\sigma}^\mu\Psi\right]\,. \end{aligned} \tag{2.4.49}$$

The bulk term above gives the equations of motion

$$\overline{\sigma}^\mu\nabla_\mu\Psi = 0\,, \qquad \nabla_\mu\Psi\overline{\sigma}^\mu = 0\,. \tag{2.4.50}$$

To simplify these equations and solve the Cauchy problem, it is convenient to use a helicity basis for the spinors,

$$\sigma_z{}^z \xi_\alpha^{(\pm)} = \pm\frac{1}{2}\xi_\alpha^{(\pm)}\,, \qquad \xi^{(+)}\xi^{(-)} = 1\,. \tag{2.4.51}$$

Complex conjugation changes the helicity, so we denote $\big(\xi_\alpha^{(\pm)}\big)^* = \overline{\xi}_{\dot\alpha}^{(\mp)}$. Using the explicit form of the Lorentz matrices given in Appendix A.2, we find

$$\xi_\alpha^{(+)} = \sqrt{\frac{1}{1+z\overline{z}}}\begin{pmatrix}1\\ z\end{pmatrix}\,, \qquad \xi_\alpha^{(-)} = \sqrt{\frac{1}{1+z\overline{z}}}\begin{pmatrix}\overline{z}\\ -1\end{pmatrix} \tag{2.4.52}$$

We expand the spinor Ψ in this basis as

$$\Psi_\alpha = \hat\Psi_{(+)}\xi_\alpha^{(+)} + \hat\Psi_{(-)}\xi_\alpha^{(-)}\,, \qquad \hat\Psi_{(\pm)} = \mp\xi^{(\mp)}\Psi\,. \tag{2.4.53}$$

The fields $\hat\Psi_{(+)}$ has $h=\frac{1}{2}, \overline{h}=0$ and $\hat\Psi_{(-)}$ has as $h=0, \overline{h}=\frac{1}{2}$ w.r.t the internal flat metric. In particular, these fields do not transform under S^2 diffeomorphisms.

(2.4.50) then take the form

$$\begin{aligned} \frac{1}{r}\big[(\partial_r - 2\partial_u)(r\hat\Psi_{(-)}) + 2E_-^{\overline{z}}D_{\overline{z}}\hat\Psi_{(+)}\big] = 0\,, \\ \frac{1}{r}\big[-\partial_r\big(r\hat\Psi_{(+)}\big) + 2E_+^{z}D_z\hat\Psi_{(-)}\big] = 0\,. \end{aligned} \tag{2.4.54}$$

First, we determine the large r fall-off of the $\hat\Psi_{(\pm)}$. The first equation in (2.4.54) implies that $\hat\Psi_{(-)}$ falls-off one power of r faster than $\hat\Psi_{(+)}$. The second equation then implies

$$\hat\Psi_{(+)} = \mathcal{O}\big(r^{-1}\big)\,, \qquad \hat\Psi_{(-)} = \mathcal{O}\big(r^{-2}\big) \qquad \text{at large } r. \tag{2.4.55}$$

These fall-offs are also consistent with finite energy and angular momentum flux through $\mathscr{I}^+$. To determine the boundary data, we Taylor expand

$$\hat{\Psi}_{(+)}(u,r,z,\overline{z}) = \sum_{n=1}^{\infty} \frac{\hat{\Psi}^{(n)}_{(+)}(u,z,\overline{z})}{r^n}\,, \qquad \hat{\Psi}_{(-)}(u,r,z,\overline{z}) = \sum_{n=2}^{\infty} \frac{\hat{\Psi}^{(n)}_{(-)}(u,z,\overline{z})}{r^n}\,. \tag{2.4.56}$$

At each order, we find the equations

$$\begin{aligned} \partial_u \hat{\Psi}^{(2)}_{(-)} &= E^{\overline{z}}_{-} D_{\overline{z}} \hat{\Psi}^{(1)}_{(+)}\,, \\ 2\partial_u \hat{\Psi}^{(n+1)}_{(-)} &= 2E^{\overline{z}}_{-} D_{\overline{z}} \hat{\Psi}^{(n)}_{(+)} - (n-1)\hat{\Psi}^{(n)}_{(-)}\,, \qquad n \geq 2\,, \\ \hat{\Psi}^{(n)}_{(+)} &= -\frac{2}{n-1} E^{z}_{+} D_z \hat{\Psi}^{(n)}_{(-)}\,, \qquad n \geq 2 \end{aligned} \tag{2.4.57}$$

Up to u-independent integration constants, the full spinor field is determined in terms of $\hat{\Psi}^{(1)}_{(+)}$. Thus, the boundary data is

$$\psi_{(+)}(u,z,\overline{z}) = \lim_{r\to\infty} \left(r\hat{\Psi}_{(+)}(u,r,z,\overline{z}) \right) = -\lim_{r\to\infty} \left(r\xi^{(-)}\Psi(u,r,z,\overline{z}) \right)\,. \tag{2.4.58}$$

where we have no relabelled $\hat{\Psi}^{(1)}_{(+)} \to \psi_{(+)}$. As described previously, in terms of S^2 tensor notation, this field is denoted $\psi_{(\frac{1}{2},0)}$.

Finally, we consider canonical quantization of the spinor. The symplectic potential current density is

$$\Theta^\mu[\Psi,\delta\Psi] = -\frac{i}{2}\left[\overline{\Psi}\overline{\sigma}^\mu \delta\Psi - \delta\overline{\Psi}\overline{\sigma}^\mu \Psi \right]\,. \tag{2.4.59}$$

Then, the symplectic form on $\mathscr{I}^+$ is

$$\Omega_{\mathscr{I}^+} = i\int_{\mathscr{I}^+} du d^2 z \gamma_{z\overline{z}} \delta\overline{\psi}_{(+)} \wedge \delta\psi_{(+)}\,. \tag{2.4.60}$$

Note that now since ψ is a fermionic field, the wedge product is symmetric. The quantum anti-commutator can be read off as

$$\{\psi_{(+)}(u,z,\bar{z}),\overline{\psi}_{(-)}(u',z',\bar{z}')\}=\gamma^{z\bar{z}}\delta(u-u')\delta^{2}(z-z')\,. \tag{2.4.61}$$

2.4.4 Generalization to Fields of Arbitrary Spin

Without proof, we now present the large r fall-offs and the boundary data of general spin fields, though the procedure to determine these is the identical to that of the previous three sections.

We recall that the the Lorentz algebra in $D=4$, $\mathfrak{so}(1,3)\cong\mathfrak{su}(2)_L\times\mathfrak{su}(2)_R$ so that a general field representation of $SO(1,3)$ is defined by two half-integers $(j,\bar{j})$. The scalar representation is $(0,0)$. The left- and right-handed spinor representations are $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ respectively. The gauge field is described by the representation $(1,0)\oplus(0,1)$ corresponding to the self-dual and anti-self-dual field strength tensor. Finally, a vector representation is $(\frac{1}{2},\frac{1}{2})$.

Working in the spinor notation introduced in §2.4.3, a field that transforms as $(j,\bar{j})$ has index structure $\mathcal{V}_{(\alpha_1\cdots\alpha_{2j})(\dot\beta_1\cdots\dot\beta_{2\bar{j}})}$. As we did with the spinor, it is natural to expand it in terms of the helicity eigenspinors

$$\begin{aligned}\mathcal{V}_{(\alpha_1\cdots\alpha_{2j})(\dot\beta_1\cdots\dot\beta_{2\bar{j}})}=\sum_{m=-j}^{m}\sum_{\overline{m}=-\bar{j}}^{\bar{j}}\xi^{(+)}_{(\alpha_1}\cdots\xi^{(+)}_{\alpha_{j+m}}\xi^{(-)}_{\alpha_{j+m+1}}\cdots\xi^{(-)}_{\alpha_{2j})}\\ \times\,\bar{\xi}^{(-)}_{(\dot\beta_1}\cdots\bar{\xi}^{(-)}_{\dot\beta_{\bar{j}+\overline{m}}}\bar{\xi}^{(+)}_{\dot\beta_{\bar{j}+\overline{m}+1}}\cdots\bar{\xi}^{(+)}_{\dot\beta_{2\bar{j})}}\mathcal{V}_{(m,\overline{m})}\,.\end{aligned} \tag{2.4.62}$$

The fields $\mathcal{V}_{(m,\overline{m})}$ have left- and right- J_3 values m and $\overline{m}$ respectively. To see this, we note that under Lorentz transformations

$$\begin{aligned}\delta_Y\mathcal{V}_{(m,\overline{m})}=&\left(\zeta^\mu_Y\partial_\mu+mD_zY^z+\overline{m}D_{\overline{z}}Y^{\overline{z}}\right)\mathcal{V}_{(m,\overline{m})}\\&+\frac{1}{2}\left(j-m+1\right)D_z^2Y^z\mathcal{V}_{(m-1,\overline{m})}+\frac{1}{2}\left(\overline{j}-\overline{m}+1\right)D_{\overline{z}}^2Y^{\overline{z}}\mathcal{V}_{(m,\overline{m}-1)}\\&-\frac{u}{2r}\left(j+m+1\right)D_{\overline{z}}^2Y^{\overline{z}}\mathcal{V}_{(m+1,\overline{m})}-\frac{u}{2r}\left(\overline{j}+\overline{m}+1\right)D_z^2Y^z\mathcal{V}_{(m,\overline{m}+1)}\,.\end{aligned}\tag{2.4.63}$$

To find the left-handed J_3 value, we set $Y^z=z$ and $Y^{\overline{z}}=0$. For this choice

$$\delta_Y\mathcal{V}_{(m,\overline{m})}=\left(zD_z+m\right)\mathcal{V}_{(m,\overline{m})}\,.\tag{2.4.64}$$

which implies that the field above has a left-handed J_3 value m. Similarly, the right-handed J_3 value is $\overline{m}$.

The coefficient fields $\mathcal{V}_{(m,\overline{m})}$ obey simple falloff conditions near null infinity. In order to state these conditions, we need to introduce a conformal scaling dimension Δ for $\mathcal{V}_{(\alpha_1\cdots\alpha_{2j})(\dot{\beta}_1\cdots\dot{\beta}_{2\overline{j}})}$, even though the theory under consideration need not be conformally invariant. Nevertheless, we expect its long-distance behavior near null infinity to be governed by a conformally invariant IR fixed point, and we take Δ to be the scaling dimension of $\mathcal{V}_{(\alpha_1\cdots\alpha_{2j})(\dot{\beta}_1\cdots\dot{\beta}_{2\overline{j}})}$ at that fixed point. In cases where the IR theory is free, Δ coincides with the mass dimension of $\mathcal{V}_{(\alpha_1\cdots\alpha_{2j})(\dot{\beta}_1\cdots\dot{\beta}_{2\overline{j}})}$. The behavior of the coefficient field $\mathcal{V}_{(m,\overline{m})}$ near $\mathscr{I}^+$ is governed by Δ and its Lorentz quantum numbers $m,\overline{m}$,

$$\mathcal{V}_{(m,\overline{m})}(u,r,\theta)=\mathcal{O}\left(r^{-\tau}\right)\,,\qquad\tau=\Delta-m-\overline{m}\,.\tag{2.4.65}$$

The quantity τ is known as the collinear twist: it is the eigenvalue of the conformal generator $D+M$, which stabilizes the null vector field p^μ.[7]

[7] Here $D=u\partial_u+r\partial_r$ is a dilatation, which satisfies $[D,p^\mu\partial_\mu]=-p^\mu\partial_\mu$, and $M=u\partial_u-r\partial_r+z\partial_z+$

As a simple example, consider an IR free, massless scalar field Φ of scaling dimension $\Delta_\Phi = 1$. It has just one component with $m = \overline{m} = 0$ so that $\tau = 1$. Thus, it falls off as r^{-1} at large r.

The photon is described by an anti-symmetric field strength $\mathcal{F}_{\mu\nu}$, whose IR scaling dimension is $\Delta_\mathcal{F} = 2$. It decomposes into self-dual and anti-self-dual parts $\mathcal{F}^{\text{SD}}_{\mu\nu}$ and $\mathcal{F}^{\text{ASD}}_{\mu\nu}$, which transform as $(1,0)$ and $(0,1)$ representations of the Lorentz group. According to (2.4.65), the different components of $\mathcal{F}^{\text{SD}}_{\mu\nu}$ behave as follows near $\mathscr{I}^\pm$,

$$\begin{aligned} \mathcal{F}^{\text{SD}}_{(1,0)} &\sim \frac{1}{r}\mathcal{F}_{uz} = \mathcal{O}\left(r^{-1}\right) \ , \\ \mathcal{F}^{\text{SD}}_{(0,0)} &\sim \mathcal{F}_{ur} - \frac{1}{r^2}\gamma^{z\bar{z}}\mathcal{F}_{z\bar{z}} = \mathcal{O}\left(r^{-2}\right) \ , \\ \mathcal{F}^{\text{SD}}_{(-1,0)} &\sim \frac{1}{r}\mathcal{F}_{r\bar{z}} = \mathcal{O}\left(r^{-3}\right) \ . \end{aligned} \tag{2.4.66}$$

This is consistent with the following asymptotic expansion (2.4.30) near $\mathscr{I}^+$.

Similar, the spinor field also satisfies (2.4.65). A left-handed spinor field Ψ has $\Delta_\Psi = \frac{3}{2}$ and transforms as $(\frac{1}{2}, 0)$. Then,

$$\Psi_{(\frac{1}{2},0)} \sim \Psi_{(+)} = \mathcal{O}(r^{-1})\,, \qquad \Psi_{(-\frac{1}{2},0)} \sim \Psi_{(-)} = \mathcal{O}(r^{-2})\,, \tag{2.4.67}$$

which is precisely (2.4.55).

As another example, we may apply (2.4.65) to determine the fall-off of currents that the fields couple to. For instance, a scalar current $\mathcal{J}$ has $\Delta = 3$, $m = \overline{m} = 0$ and falls off as r^{-3} at large r.

$\bar{z}\partial_{\bar{z}}$ is a boost along p^μ, which satisfies $[M, p^\mu\partial_\mu] = p^\mu\partial_\mu$.

The gauge field current $\mathcal{J}_\mu$ also has $\Delta = 3$. This transforms as $(\frac{1}{2}, \frac{1}{2})$. Using (2.4.65), we find

$$\begin{aligned}
\mathcal{J}_{(\frac{1}{2},\frac{1}{2})} &\sim \mathcal{J}_u = \mathcal{O}(r^{-2})\,, \\
\mathcal{J}_{(\frac{1}{2},-\frac{1}{2})} &\sim \frac{1}{r}\mathcal{J}_z = \mathcal{O}(r^{-2})\,, \\
\mathcal{J}_{(-\frac{1}{2},\frac{1}{2})} &\sim \frac{1}{r}\mathcal{J}_{\overline{z}} = \mathcal{O}(r^{-2})\,, \\
\mathcal{J}_{(-\frac{1}{2},-\frac{1}{2})} &\sim \mathcal{J}_r = \mathcal{O}(r^{-4})\,.
\end{aligned} \tag{2.4.68}$$

The spinor field Ψ couples to a spinor current $\overline{\mathcal{K}}_{\dot\alpha}$ which has $\Delta_{\mathcal{K}} = \frac{5}{2}$ and transforms as $(0, \frac{1}{2})$. Then, expanding this current as

$$\overline{\mathcal{K}}_{\dot\alpha} = \overline{\mathcal{K}}_{(+)}\overline{\xi}^{(+)}_{\dot\alpha} + \overline{\mathcal{K}}_{(-)}\overline{\xi}^{(-)}_{\dot\alpha}\,. \tag{2.4.69}$$

Then,

$$\overline{\mathcal{K}}_{(0,\frac{1}{2})} \sim \overline{\mathcal{K}}_{(-)} = \mathcal{O}(r^{-2})\,, \qquad \overline{\mathcal{K}}_{(0,-\frac{1}{2})} \sim \overline{\mathcal{K}}_{(+)} = \mathcal{O}(r^{-3})\,. \tag{2.4.70}$$

Massless fields have $\overline{j} = 0$ (left-handed fields) or $j = 0$ (right-handed fields). Free left-handed massless fields of arbitrary spin satisfy the equations of motion

$$\left(\overline{\sigma}^\mu\right)^{\dot\alpha\alpha_1}\partial_\mu \mathcal{V}_{(\alpha_1\cdots\alpha_{2j})} = 0\,. \tag{2.4.71}$$

For free fields, we always have $\Delta = j + 1$.

The boundary data for these fields may be determined just as we have done previously. We first expand the fields in the spinor basis (2.4.62). The coefficient fields $\mathcal{V}_{(m,0)}$ are then Taylor expanded near $\mathscr{I}^+$ using the boundary fall-offs (2.4.65). Equations are then solved order-by-order in large r. Following this procedure, we find that the boundary data is the leading coefficient of $\mathcal{V}_{(j,0)}$, which we denote $V_{(+)}$,

$$
\begin{aligned}
V_{(+)}(u,\theta) &= \lim_{r\to\infty} r\mathcal{V}_{(j,0)}(u,r,\theta) \\
&= (-1)^{2j} \lim_{r\to\infty} r\xi^{(-)\alpha_1}\cdots\xi^{(-)\alpha_{2j}}\mathcal{V}_{\alpha_1\cdots\alpha_{2j}}(u,r,\theta)\,.
\end{aligned}
\tag{2.4.72}
$$

Under Lorentz transformation,

$$
\delta_Y \mathcal{V}_{(+)}(u,z,\bar{z}) = \left[Y^a\partial_a + \frac{1}{2}D_aY^a\left(u\partial_u + \Delta_{\mathcal{V}} - j\right) + mD_zY^z\right]\mathcal{V}_{(+)}(u,z,\bar{z})\,, \tag{2.4.73}
$$

where recall that $\Delta_{\mathcal{V}} - j = 1$.

2.4.5 Asymptotic Structure at $\mathscr{I}^-$

In the previous sections, we have completely determined the boundary data on $\mathscr{I}^+$. We may analogously determine the boundary data on $\mathscr{I}^-$ as well. For scalar, vector and spinor fields the boundary data on $\mathscr{I}^-$ is

$$
\begin{aligned}
\widetilde{\phi}(v,\widetilde{\theta}) &= \lim_{r\to\infty}\left(r\Phi(v,r,\widetilde{\theta})\right)\,, \\
\widetilde{A}_{\widetilde{z}}(v,\widetilde{\theta}) &= \lim_{r\to\infty}\left(r\mathcal{A}_{\widetilde{z}}(v,r,\widetilde{\theta})\right)\,, \\
\widetilde{\psi}_{(+)}(v,\widetilde{\theta}) &= -\lim_{r\to\infty}\left(r\xi^{(-)}\hat{\Psi}(v,r,\widetilde{\theta})\right)\,.
\end{aligned}
\tag{2.4.74}
$$

Note that the boundary data on $\mathscr{I}^-$ is not independent of the data on $\mathscr{I}^+$. Both independently determine the full bulk field uniquely. The precise map that describes ϕ, A, ψ in terms of $\widetilde{\phi}$, $\widetilde{A}$, $\widetilde{\psi}$ is known as the classical $\mathcal{S}$-matrix. We will not discuss the precise structure of the classical or the quantum $\mathcal{S}$-matrix, but we will later discuss certain features which will allow us to relate asymptotic symmetries in to soft theorems.

2.4.6 Boundary Data for Interacting Fields

We now briefly comment on the boundary data for interacting fields. We will require that the interactions do not change the fall-offs of the fields described above. To be concrete, we discuss scalar fields. The extension to fields with spin is quite similar and will be omitted.

We start by coupling the scalar field to a background current $\mathcal{J}(x)$,

$$\Box\Phi(x) = \mathcal{J}(x)\,. \tag{2.4.75}$$

The scalar current falls-off as r^{-3} near $\mathscr{I}^+$ as discussed in §2.4.4. We Taylor expand the current

$$\mathcal{J}(u,r,\theta) = \sum_{n=3} \frac{\mathcal{J}^{(n)}(u,\theta)}{r^n}\,. \tag{2.4.76}$$

The wave equation (2.4.7) is then modified to

$$2n\partial_u\Phi^{(n+1)} = -[D^2 + n(n-1)]\Phi^{(n)} + \mathcal{J}^{(n+2)}\,, \qquad n \geq 1\,. \tag{2.4.77}$$

Again, up to u-independent integration constants, we may determine the full scalar field in terms of $\phi = \Phi^{(0)}$. The only difference is that the solution at each order in large r is more involved due to the presence of the background current $\mathcal{J}$.

We now generalize to the case when the current $\mathcal{J}$ is dynamical. The requirement that the current falls of at least as fast as $\frac{1}{r^3}$ at large r implies that the interaction term that generates $\mathcal{J}$ must be marginal or irrelevant. This is consistent with the requirement that the theory remains free in the infrared. For instance, consider the case in which the scalar field Φ couples to itself. In this case, the RHS of (2.4.77) involves only the fields with $m < n$, which implies that one can determine the full scalar field Φ order-by-order in large r. Again, the boundary data remains unchanged, though the equations at each subsequent order in large r become increasingly complex. This statement trivially extends to the gauge and spinor fields as well. We note in particular that in the presence of currents, the leading constraint equations for the gauge field (2.4.34) and spinor field (2.4.57) is modified to

$$\partial_u \mathcal{A}_u^{(1)} = \partial_u D^a A_a + e^2 j_u \,, \tag{2.4.78}$$

$$\partial_u \hat{\Psi}_{(-)}^{(2)} = E_-^{\bar{z}} D_{\bar{z}} \psi_{(+)} - \frac{e^2}{2} \overline{k}_{(-)} \,. \tag{2.4.79}$$

where

$$\begin{aligned} j_u(u,z,\bar{z}) &= \lim_{r\to\infty} r^2 \mathcal{J}_u(u,r,z,\bar{z}) \,, \\ \overline{k}_{(-)}(u,z,\bar{z}) &= \lim_{r\to\infty} r^2 \overline{\mathcal{K}}_{(-)}(u,r,z,\bar{z}) \,, \end{aligned} \tag{2.4.80}$$

are the leading terms in current expansion (they satisfy the fall-off (2.4.68) and (2.4.70)). Equations (2.4.78) and (2.4.79) will play a very important role in the discussions of Chapter 3 and 5.

Similarly, the boundary data for fields on $\mathscr{I}^-$ remains unchanged. In other words, the bulk field Φ is determined uniquely either in terms of $\mathscr{I}^+$ data or in terms of $\mathscr{I}^-$ data and the relationship between the two is now the full interacting $\mathcal{S}$-matrix.

2.5 Free Field Mode Expansions on $\mathscr{I}^+$

In this section, we relate the boundary data derived in the previous three sections to the creation and annihilation operators that are more standard in quantum field theory. This will important to connect our discussion of asymptotic symmetries – which will be on $\mathscr{I}^+$ and in terms of the boundary data described previously – to soft theorems, which are derived in perturbative quantum field theory using Feynman diagrams.

Scalar Field We start with a massless complex scalar field $\Phi(y)$ satisfying $\Box\Phi(y) = 0$. Assuming sufficiently fast fall-offs for $\Phi(y)$ at infinity (we will make this precise soon), we may mode expand the scalar field as

$$\Phi(y) = \int \frac{d^3q}{(2\pi)^3} \frac{1}{2\omega_q} \left[a_\Phi(\vec{q}) e^{iq\cdot y} + a^\dagger_{\overline{\Phi}}(\vec{q}) e^{-iq\cdot y} \right] . \tag{2.5.1}$$

where $\omega_q = |\vec{q}\,|$.

Here, we use the notation that the annihilation operator that appears in a field $f(y)$ is denoted $a_{f,s}(\vec{q}\,)$ where s is the helicity of the particle that it annihilates. For the scalar field, $s = 0$ and we drop this label. These operators may carry additional labels (such as Lie algebra indices) which we have dropped here. The creation and annihilation operators satisfy[8]

$$\left[a_{f,s}(\vec{q}\,), a^\dagger_{f',s'}(\vec{q}\,') \right\} = (2\pi)^3 (2\omega_q) \delta_{f,f'} \delta_{s,s'} \delta^3(\vec{q} - \vec{q}\,') . \tag{2.5.2}$$

One-particle states are defined as

$$|\, \vec{q}, f, s \,\rangle = a^\dagger_{\overline{f},s}(\vec{q}\,) |\, 0 \,\rangle , \tag{2.5.3}$$

which satisfy

$$\langle\, \vec{q}, f, s \,|\, \vec{q}\,', f', s' \,\rangle = (2\pi)^3 (2\omega_q) \delta_{f,f'} \delta_{s,s'} \delta^3(\vec{q} - \vec{q}\,') . \tag{2.5.4}$$

We now determine the structure of the scalar field (2.5.1) near $\mathscr{I}^+$. We write out the field explicitly in retarded coordinates

$$\Phi(u,r,z,\overline{z}) = \frac{1}{2(2\pi)^3} \int_0^\infty d\omega_q \omega_q \int_{S^2} d\Omega_{\hat{q}} \left[a_\Phi(\vec{q}\,) e^{-i\omega_q u - i\omega_q r(1-\hat{q}\cdot\hat{y})} + a^\dagger_{\overline{\Phi}}(\vec{q}\,) e^{i\omega_q u + i\omega_q r(1-\hat{q}\cdot\hat{y})} \right] . \tag{2.5.5}$$

[8] $[\ ,\ \}$ is a commutator if the operators are bosonic and an anti-commutator if they are fermionic.

Now, consider the mode expansion in the limit $r \to \infty$. In the integrand, in the stationary phase approximation, we have an oscillating exponent which is localized to $\hat{q}\cdot\hat{y} = 1$ in the large r limit. In particular

$$\int_{S^2} d\Omega_{\hat{q}} f(\vec{q}) e^{\pm i\omega_q r(1-\hat{q}\cdot\hat{y})} \quad \overset{r\to\infty}{\to} \quad \pm\frac{2\pi i}{\omega_q r} f(\omega_q \hat{y}) \,. \tag{2.5.6}$$

Using this, we find

$$\Phi(u,r,z,\bar{z}) \quad \overset{r\to\infty}{\to} \quad -\frac{i}{8\pi^2 r}\int_0^\infty d\omega_q \left[a_\Phi(\omega_q\hat{y})e^{-i\omega_q u} - a^\dagger_{\overline{\Phi}}(\omega_q\hat{y})e^{i\omega_q u}\right] \,. \tag{2.5.7}$$

Here, $\hat{y}$ is to be understood to be related to $z, \bar{z}$ according to (2.2.7). Then, the boundary field $\phi(u,z,\bar{z})$ (defined in (2.4.8)) is given by

$$\begin{aligned}\phi(u,z,\bar{z}) &= \lim_{r\to\infty}\big(r\Phi(u,r,z,\bar{z})\big) \\ &= -\frac{i}{8\pi^2}\int_0^\infty d\omega_q \left[a_\Phi(\omega_q\hat{y})e^{-i\omega_q u} - a^\dagger_{\overline{\Phi}}(\omega_q\hat{y})e^{i\omega_q u}\right] \,.\end{aligned} \tag{2.5.8}$$

The complex conjugate field is

$$\bar{\phi}(u,z,\bar{z}) = -\frac{i}{8\pi^2}\int_0^\infty d\omega_q \left[a_{\overline{\Phi}}(\omega_q\hat{y})e^{-i\omega_q u} - a^\dagger_{\Phi}(\omega_q\hat{y})e^{i\omega_q u}\right] \,. \tag{2.5.9}$$

Gauge Field The gauge field $\mathcal{A}_A(x)$ is not a gauge invariant operator and in particular does not admit a unique mode expansion. We instead talk about the field strength tensor $\mathcal{F}_{AB}$, which is mode expanded as

$$\mathcal{F}_{AB}(x) = e\sum_{s=\pm}\int\frac{d^3q}{(2\pi)^3}\frac{1}{2\omega_q}\left[f^{(s)}_{AB}(\vec{q})^* a_{\mathcal{F},s}(\vec{q})e^{iq\cdot x} + f^{(s)}_{AB}(\vec{q})a^\dagger_{\mathcal{F},s}(\vec{q})e^{-iq\cdot x}\right] \,. \tag{2.5.10}$$

The explicit factor of e is present due to our non-standard normalization of the gauge field in (2.4.25). This ensures that are creation and annihilation operators satisfy (2.5.2). Since $\mathcal{F} = \mathrm{d}\mathcal{A}$, the wave-functions $f^{(s)}_{AB}(\vec{q})$ may be written in terms of polarizations as

$$f^{(s)}_{AB}(\vec{q}) = -i\big[q_A\varepsilon^{(s)}_B(\vec{q}) - q_B\varepsilon^{(s)}_A(\vec{q})\big] \,. \tag{2.5.11}$$

The polarization tensors $\varepsilon^{(s)}_A(\vec{q})$ satisfy

$$q^A\varepsilon^{(s)}_A(\vec{q}) = 0 \,, \qquad \varepsilon^{(\pm)}_A(\vec{q})\varepsilon^{A(\pm)}(\vec{q}) = 0 \,, \qquad \varepsilon^{(+)}_A(\vec{q})\varepsilon^{A(-)}(\vec{q}) = 1 \,. \tag{2.5.12}$$

We may pick any gauge to describe the polarizations in $f_{AB}(\vec{q})$ since $\mathcal{F}$ is gauge invariant. A convenient choice for the polarization tensors is made as follows. We start by parameterizing the null momentum q^A in terms of $(\omega_q, w, \overline{w})$ as

$$q^A = \omega_q\left(1\,,\ \frac{w+\overline{w}}{1+w\overline{w}}\,,\ \frac{-i(w-\overline{w})}{1+w\overline{w}}\,,\ \frac{1-w\overline{w}}{1+w\overline{w}}\right)\,. \tag{2.5.13}$$

In this parameterization, we choose the polarization to be

$$\begin{aligned}\varepsilon^{(+)}_A(\vec{q}) &= \frac{1}{\sqrt{2}}\left(-\overline{w}\,,\ 1\,,\ -i\,,\ -\overline{w}\right)\,,\\ \varepsilon^{(-)}_A(\vec{q}) &= \frac{1}{\sqrt{2}}\left(-w\,,\ 1\,,\ i\,,\ -w\right)\,.\end{aligned} \tag{2.5.14}$$

The field strength component $F^{(0)}_{uz}$ on $\mathscr{I}^+$ can be determined as

$$F^{(0)}_{uz}(u,z,\overline{z}) = \lim_{r\to\infty}\partial_u y^A\partial_z y^B\mathcal{F}_{AB}(u,r,z,\overline{z})\,. \tag{2.5.15}$$

Using the stationary phase approximation as before, we find

$$F^{(0)}_{uz}(u,z,\overline{z}) = -\frac{e}{8\sqrt{2}\pi^2}E^+_z\int_0^\infty d\omega_q\omega_q\big[a_{\mathcal{F},+}(\omega_q\hat{y})e^{-i\omega_q u} + a^\dagger_{\mathcal{F},-}(\omega_q\hat{y})e^{i\omega_q u}\big] \tag{2.5.16}$$

Recall that $F^{(0)}_{uz} = \partial_u A_z$. Then, analogous to (2.4.19), we may now determine $\widehat{A}_z$ as

$$\begin{aligned}\widehat{A}_z(u,\theta) &= \frac{1}{2}\int_{-\infty}^{\infty} du'\Theta(u-u')F^{(0)}_{u'z}(u',z,\overline{z}) \\ &= -\frac{ie}{8\sqrt{2}\pi^2}E^+_z\int_0^\infty d\omega_q\big[a_{\mathcal{F},+}(\omega_q\hat{y})e^{-i\omega_q u} - a^\dagger_{\mathcal{F},-}(\omega_q\hat{y})e^{i\omega_q u}\big]\end{aligned} \tag{2.5.17}$$

Note that we can determine only $\widehat{A}_z$, not the full A_z. In particular, the zero mode $C_z = e^2 D_z C$ is not determined in terms of the creation and annihilation modes. This mode is typically not considered in standard quantum field theory. As we will see, it is precisely the inclusion of this mode that will lead us to an enhancement of $\mathcal{S}$-matrix symmetries which are related the soft theorems.

In particular, (2.5.17) implies $\widehat{A}_z$ creates outgoing positive helicity states. On the other hand, the mode expansion for $\widehat{A}_{\overline{z}}$ is

$$\widehat{A}_{\overline{z}}(u,\theta) = -\frac{ie}{8\sqrt{2}\pi^2}E^-_{\overline{z}}\int_0^\infty d\omega_q\big[a_{\mathcal{F},-}(\omega_q\hat{y})e^{-i\omega_q u} - a^\dagger_{\mathcal{F},+}(\omega_q\hat{y})e^{i\omega_q u}\big] \tag{2.5.18}$$

which creates outgoing negative helicity states.

Spinor Field The final mode expansion we consider is that of the spinor field $\Psi(y)$. This is

$$\Psi_\alpha(y) = \int \frac{d^3q}{(2\pi)^3}\frac{1}{2\omega_q}\eta_\alpha(\vec{q})\left[a_{\Psi,+}(\vec{q})e^{iq\cdot y} + a^\dagger_{\Psi,-}(\vec{q})e^{-iq\cdot y}\right] \tag{2.5.19}$$

Here, $\eta_\alpha(\vec{q})$ is a momentum space spinor that satisfies[9]

$$q^A\big(\overline{\sigma}_A\big)^{\dot{\alpha}\beta}\eta_\beta(\vec{q}) = 0\,, \qquad q_A\big(\sigma^A\big)_{\alpha\dot{\beta}} = \eta_\alpha(\vec{q})\overline{\eta}_{\dot{\beta}}(\vec{q})\,. \tag{2.5.20}$$

When, we parameterize the momentum q^A as (2.5.13), these wave-functions have the explicit form

$$\eta_\alpha(\vec{q}) = \sqrt{\frac{2\omega_q}{1+w\overline{w}}}\begin{pmatrix}1\\ w\end{pmatrix} = \sqrt{2\omega_q}\,\xi^{(+)}_\alpha\big|_{z=w}\,. \tag{2.5.21}$$

[9]These spinors are precisely equivalent to the square and angle brackets that are used in the study of scattering amplitudes. Precisely, $|\,q\,\rangle_\alpha = \eta_\alpha(\vec{q})$, $\langle\,q\,|^\alpha = \eta^\alpha(\vec{q})$, $[\,q\,|_{\dot{\alpha}} = \overline{\eta}_{\dot{\alpha}}(\vec{q})$ and $|\,q\,]^{\dot{\alpha}} = \overline{\eta}^{\dot{\alpha}}(\vec{q})$.

The large r limit of the field may then be taken just as before. We may then extract the boundary data $\psi(u,z,\bar{z})$ as

$$\psi(u,z,\overline{z}) = -\frac{i}{8\pi^2}\int_0^\infty d\omega_q\sqrt{2\omega_q}\left[a_{\Psi,+}(\omega_q\hat{y})e^{-i\omega_q u} - a^\dagger_{\overline{\Psi},-}(\omega_q\hat{y})e^{i\omega_q u}\right] . \tag{2.5.22}$$

Again, we note that ψ creates positive helicity outgoing spinors and its complex conjugate

$$\overline{\psi}(u,z,\overline{z}) = -\frac{i}{8\pi^2}\int_0^\infty d\omega_q\sqrt{2\omega_q}\left[a_{\overline{\Psi},-}(\omega_q\hat{y})e^{-i\omega_q u} - a^\dagger_{\Psi,+}(\omega_q\hat{y})e^{i\omega_q u}\right] . \tag{2.5.23}$$

creates negative helicity outgoing spinors.

2.6 The Perturbative Quantum $\mathcal{S}$-matrix

In perturbative quantum field theory, the classical $\mathcal{S}$-matrix is elevated to the $\mathcal{S}$-matrix operator when working in the interaction picture (where all the one- and multi-particle states are free and non-interacting and the $\mathcal{S}$-matrix operator captures all the interactions of the theory). The $\mathcal{S}$-matrix amplitude is given by

$$\mathscr{A}_n = \langle\, 0\,|a_{f_1,s_1}(\vec{p}_1)\cdots a_{f_m,s_m}(\vec{p}_m)\mathcal{S}a^\dagger_{f_{m+1},s_{m+1}}(\vec{p}_{m+1})\cdots a^\dagger_{f_n,s_n}(\vec{p}_n)|\,0\,\rangle\,. \tag{2.6.1}$$

The extra indices on $\mathscr{A}_n$ are described by the RHS of the equation above, but are dropped on the LHS to have simplified expressions. We will reinstate them if and when required. The n-point amplitude here includes a momentum conserving Dirac delta function, which we can extract as

$$\mathscr{A}_n = i(2\pi)^4\delta^4\left(\sum_{i=1}^m p_i^\mu - \sum_{i=m+1}^n p_i^\mu\right)\mathscr{M}_n\,. \tag{2.6.2}$$

In this thesis, we will focus our discussion on the amplitude $\mathscr{A}_n$.

A further simplification occurs by using a now common convention of describing all particles as outgoing. This is done using CPT invariance of the $\mathcal{S}$-matrix, which implies

$$\mathcal{S}a^{\dagger}_{f,s}(p^0,\vec{p}) = a_{\overline{f},-s}(-p^0,\vec{p})\mathcal{S} \tag{2.6.3}$$

In particular, an incoming particle with helicity s and 3-momentum $\vec{p}$ and $p^0 > 0$ can be equivalently described as an outgoing particle of the opposite helicity and with the same 3-momentum $\vec{p}$ but now with $p^0 < 0$. Other quantum numbers of the particle are also conjugated. For instance, if the ingoing one-particle state transforms under a representation R of some internal Lie algebra, then the outgoing particle transforms w.r.t. the conjugate representation $\overline{R}$. For a $U(1)$ gauge group this implies that a positive charged incoming particle is mapped to a negatively charged outgoing particle and vice versa.

In this convention, the amplitude $\mathscr{A}_n$ may be written as

$$\mathscr{A}_n = \langle\, 0\, | a_{f_1,s_1}(p_1) \cdots a_{f_n,s_n}(p_n)\mathcal{S} |\, 0\, \rangle\,. \tag{2.6.4}$$

where now the sign of the energy p^0_i determines whether the particle is ingoing or outgoing.

3

New Symmetries in Massless QED

In this chapter, we study the simplest – though historically, not the first – non-trivial example of the relationship between soft theorems and asymptotic symmetries. This chapter is a modified extract of [1].

3.1 Introduction

The purpose of this chapter is to argue that the soft photon theorem in massless QED [9, 10, 88] can be understood as a new asymptotic symmetry. The symmetry is generated by "large" $U(1)$ gauge transformations which approach an arbitrary function $\varepsilon(z, \bar{z})$ on the conformal sphere at $\mathscr{I}$ but are constant along the null generators, even as they antipodally cross from $\mathscr{I}^-$ to $\mathscr{I}^+$ through spatial infinity. Except for the constant transformation, these symmetries are spontaneously broken in the conventional vacuum. The soft photons appear as Goldstone modes living on the sphere at the boundary of $\mathscr{I}$.

The relation between soft theorems and asymptotic symmetries of $\mathscr{I}^+$ (but not of the $\mathcal{S}$-matrix), was described already in [28], which in turn was inspired by [89]. Two "simplifying" restrictions were made in the analysis of [28]: the incoming state was required to be invariant under the large gauge symmetries, and the parameter $\varepsilon(z, \bar{z})$ was required to be locally holomorphic. However, far from simplifying the analysis, these restrictions obscured the underlying structure. The present analysis both simplifies and generalizes that of [28].

This chapter considers theories in which there are no stable massive charged particles, and the quantum state begins and ends in the vacuum at past and future timelike infinity. Of course, in real-world QED the electron is a stable massive charged particle, so it is highly desirable to generalize our analysis to this case.[1] However, stable massive charges create technical complications because the charge current has no flux through future null infinity. Rather, there is charge flux across timelike

[1] The present analysis is relevant to hard scattering in QED when the electron mass becomes negligible.

infinity which becomes a singular point in the conformal compactification of Minkowski space. In principle a systematic treatment of this singularity should be possible – the fields disperse and are weakly interacting; nonetheless, this is well beyond the scope of the present chapter.[2]

This chapter is organized as follows. In §3.2 we review the classical final data formulation at $\mathscr{I}^+$, study the asymptotic symmetries and constructs the associated charges. §3.3 gives the corresponding formulae for $\mathscr{I}^-$. In §3.4 we give conditions which tie the data of $\mathscr{I}^-$ to that of $\mathscr{I}^+$ and thereby defines the scattering problem. The conditions are shown to break the separate asymptotic symmetries to a diagonal subgroup preserving the $\mathcal{S}$-matrix. In §3.5 the quantum Ward identity of this symmetry is shown to relate scattering amplitudes with and without a soft photon insertion. Finally in §3.6 we show that this Ward identity is the soft photon theorem.

3.2 Large gauge transformations on $\mathscr{I}^+$

In this subsection we briefly review the canonical final data formulation of $U(1)$ electrodynamics coupled to massless charged matter at future null infinity ($\mathscr{I}^+$), introduce the large gauge transformations and construct the corresponding charges.

We recall that the boundary data for the gauge field is $A_z = \lim_{r\to\infty} \mathcal{A}_z$. When coupled to a current $\mathcal{J}_\mu$, the leading constraint equation is given by (2.4.78)

$$\partial_u \mathcal{A}_u^{(1)} = \partial_u D^a A_a + e^2 j_u \,, \tag{3.2.1}$$

[2]The massive case has been studied in [1, 39].

where

$$j_u(u,z,\bar{z}) = \lim_{r\to\infty}\left[r^2\mathcal{J}_u(u,r,z,\bar{z})\right] \ . \tag{3.2.2}$$

Further, since we are interested in systems with massless particles only, we consider configurations which revert to the vacuum in the far future, i.e.

$$F_{ur}|_{\mathscr{I}^+_+} = F_{ua}|_{\mathscr{I}^+_+} = 0 \, . \tag{3.2.3}$$

The analogous structure at $\mathscr{I}^-$ is described in §3.3 below.

Gauge theories have a local gauge symmetry which acts on the gauge field as

$$\delta_{\hat{\varepsilon}}\mathcal{A}_\mu = \partial_\mu\hat{\varepsilon} \, , \qquad \hat{\varepsilon} \sim \hat{\varepsilon} + \frac{2\pi}{e^2} \, . \tag{3.2.4}$$

Radial gauge (2.4.31) leave unfixed residual gauge transformations generated by an arbitrary function approaching $\hat{\varepsilon} = \varepsilon(\theta)$ on the conformal sphere at $r = \infty$. We will refer to these as *large gauge transformations*. The action on Γ^+ is

$$\delta_\varepsilon A_a(u,\theta) = \partial_a\varepsilon(\theta) \, . \tag{3.2.5}$$

These comprise the asymptotic symmetries considered in this chapter. We can construct the charge that generates this symmetry on $\mathscr{I}^+$ following the procedure discussed in [85]. In particular, the charge $Q_{\Sigma,\Delta}$ on a hypersurface Σ that generates a particular symmetry transformation which acts on fields as $\Delta\varphi$ satisfies

$$\delta Q_{\Sigma,\Delta} = -\Omega_\Sigma\big(\varphi,\delta\varphi,\Delta\varphi\big) \, . \tag{3.2.6}$$

The charge Q_Δ is then constructed by integrating the equation above. If it is possible to integrate the RHS, the charge is called integrable, else it is non-integrable.

Let us now use this procedure to deduce the charge for large gauge transformations (3.2.5). We will do so in a particular example when the gauge field couples to a scalar field of charge Q. In this case, the symplectic form on $\mathscr{I}^+$ is

$$\Omega_{\mathscr{I}^+} = \int dud^2\theta\sqrt{\gamma}\big[\frac{1}{e^2}\partial_u\delta A_a \wedge \delta A^a + \partial_u\delta\overline{\phi}\wedge\delta\phi + \partial_u\delta\phi\wedge\delta\overline{\phi}\big] \tag{3.2.7}$$

Now, under large gauge transformations $\delta_\varepsilon\phi = iQ\varepsilon\phi$ and $\delta_\varepsilon\overline{\phi} = -iQ\varepsilon\overline{\phi}$. In this case, (3.2.6) takes the form

$$\delta Q^+_\varepsilon = \delta\int dud^2\theta\sqrt{\gamma}\big[-\frac{1}{e^2}\partial_u A_a D^a\varepsilon - iQ\varepsilon(\partial_u\overline{\phi}\phi - \overline{\phi}\partial_u\phi)\big] \tag{3.2.8}$$

Note that for a scalar field the leading component of the current is

$$j_u = -iQ(\partial_u\overline{\phi}\phi - \overline{\phi}\partial_u\phi) \tag{3.2.9}$$

Additionally, the charge is integrable so that

$$Q^+_\varepsilon = \int dud^2\theta\sqrt{\gamma}\big[-\frac{1}{e^2}\partial_u A_a D^a\varepsilon + \varepsilon j_u\big]\,. \tag{3.2.10}$$

The same formula for the charge holds for arbitrary matter fields coupled to the gauge field. Integrating this by parts and using (3.2.3), we may write

$$Q^+_\varepsilon = \frac{1}{e^2}\int dud^2\theta\sqrt{\gamma}\varepsilon\mathcal{F}^{(2)}_{ru}\big|_{\mathscr{I}^+_-}\,. \tag{3.2.11}$$

Thus, the charge that generates (3.2.5) is precisely the weighted integral of radial electric field at $\mathscr{I}^+_-$. Using (2.4.41), we may also write the charge as

$$Q^+_\varepsilon = \int d^2\theta\sqrt{\gamma}\varepsilon D^2 N + \int dud^2\theta\sqrt{\gamma}\varepsilon j_u\,. \tag{3.2.12}$$

For the special case $\varepsilon = 1$, Q_1^+ is the total final electric charge which obeys

$$Q_1^+ = \int_{\mathscr{I}^+} dud^2\theta\sqrt{\gamma}j_u \,. \tag{3.2.13}$$

For the choice $\varepsilon(z,\overline{z}) = \delta^2(z-w)$ one has the fixed-angle charge

$$Q_{w\overline{w}}^+ = 2\partial_w\partial_{\overline{w}}N + \int_{-\infty}^{\infty} du\gamma_{w\overline{w}}j_u \,. \tag{3.2.14}$$

This is the total outgoing electric charge radiated into the fixed angle $(w,\overline{w})$ on the asymptotic S^2. The first term is a linear "soft" photon (by which we mean momentum is strictly zero, as opposed to just small) contribution to the fixed-angle charge. It does not contribute to the total charge Q_1^+ as it is a total derivative. The second term is the accumulated matter charge flux at the angle $(w,\overline{w})$. Q_ε^+ generates the large gauge transformation on matter fields

$$\left[Q_\varepsilon^+, \phi(u,\theta)\right] = -Q\varepsilon(\theta)\phi(u,\theta) \,, \tag{3.2.15}$$

where ϕ is any massless charged matter field operator on $\mathscr{I}^+$ with charge q.

Using the commutators (2.4.44), we find

$$[Q_\varepsilon^+, \widehat{A}_a(u,\theta)] = i\partial_a\varepsilon(\theta) \,, \qquad [Q_\varepsilon^+, C(\theta)] = \frac{i}{e^2}\varepsilon(\theta) \,. \tag{3.2.16}$$

Moreover, the charges satisfy the Abelian algebra

$$\left[Q_\varepsilon^+, Q_{\varepsilon'}^+\right] = 0. \tag{3.2.17}$$

Periodicity of ε implies that C lives on a circle of radius $\frac{1}{e^2}$:

$$C \sim C + \frac{2\pi}{e^2}. \tag{3.2.18}$$

Exponentials of C obey

$$\left[Q^+_\varepsilon, e^{ine^2C(\theta)}\right] = -n\varepsilon(\theta)e^{ine^2C(\theta)}\,, \qquad n\in\mathbb{Z}\,. \tag{3.2.19}$$

Such operators do not in themselves create physical states. Rather states with charge n are created by products of these operators with neutral matter-sector operators. This is virtually the same operator product decomposition familiar in 2D CFT when factoring a $U(1)$ current algebra boson, or in 4D soft collinear effective field theory (SCET) involving the so-called jet field [90, 91].

A vacuum wave function for the Goldstone mode which we take to be C can be defined by the condition

$$C(\theta)|\,0\,\rangle = 0\,. \tag{3.2.20}$$

(3.2.16) implies that the large gauge symmetries are broken in this vacuum. The symmetries transform (3.2.20) into more general C eigenstates obeying

$$C(\theta)|\,\alpha\,\rangle = \alpha(\theta)|\,\alpha\,\rangle\,. \tag{3.2.21}$$

Up to an undetermined normalization, the inner products are

$$\langle\,\alpha\,|\,\alpha'\,\rangle = \prod_\theta \delta\left(\alpha(\theta)-\alpha'(\theta)\right)\,. \tag{3.2.22}$$

Other zero-energy states are

$$|\,\beta\,\rangle = \int[d\alpha]e^{2i\int d^2\theta\sqrt{\gamma}D^a\alpha D_a\beta}|\,\alpha\,\rangle\,. \tag{3.2.23}$$

These are zero-mode eigenstates

$$N_a|\,\beta\,\rangle = \partial_a\beta|\,\beta\,\rangle\,. \tag{3.2.24}$$

obeying

$$Q_\varepsilon^+|\beta\rangle = \frac{2}{e^2}\int d^2\theta\sqrt{\gamma}\varepsilon D^2\beta|\,\beta\,\rangle\,. \tag{3.2.25}$$

In particular, any state with β = constant has unbroken large gauge symmetry. These vacua are annihilated by the zero mode and are not the ones usually employed in QED analyses: it might be of interest to consider scattering in such states.[3] Finally there are normalizable, symmetry-breaking vacua annihilated by complex linear combinations such as $C + iN$.

3.3 Asymptotic structure at $\mathscr{I}^-$

A similar structure exists near $\mathscr{I}^-$ and is needed to discuss scattering. Recall that the boundary data is $\widetilde{A}_a(v,\widetilde{\theta}) = \lim_{r\to\infty}\mathcal{A}_a(v,r,\widetilde{\theta})$. This forms the coordinate on the asymptotic phase space Γ^-. The leading order constraint equation is (obtained from the $\mu = v$ component of the equations of motion)

$$\partial_v\mathcal{A}_v^{(1)} = -\partial_v D^a\widetilde{A}_a - e^2 j_v\,, \qquad j_v(v,\widetilde{\theta}) = \lim_{r\to\infty}\left[r^2\mathcal{J}_v(v,r,\widetilde{\theta})\right]\,. \tag{3.3.1}$$

Unfixed large gauge transformations are parameterized by $\varepsilon^-(\widetilde{\theta})$ under which

$$\delta_{\varepsilon^-}\widetilde{A}_a = D_a\varepsilon^-\,. \tag{3.3.2}$$

The associated charge is

$$\begin{aligned} Q^-_{\varepsilon^-} &= -\frac{1}{e^2}\int_{\mathscr{I}^-_+} d^2\theta\sqrt{\gamma}\varepsilon^-\mathcal{F}^{(2)}_{vr} \\ &= \int d^2\theta\sqrt{\gamma}\varepsilon^- D^2\widetilde{N} + \int_{-\infty}^{\infty} dv d^2\theta\sqrt{\gamma}\varepsilon^- j_v\,. \end{aligned} \tag{3.3.3}$$

[3]For example, such states might be related to the vacua considered in [92, 93].

3.4 Matching $\mathscr{I}^+_-$ to $\mathscr{I}^-_+$

The classical scattering problem is to find the map from Γ^- to Γ^+, i.e. to determine the final data A_a on $\mathscr{I}^+$ which arises from a given set of initial data $\widetilde{A}_a$ on $\mathscr{I}^-$. Given a field strength everywhere on Minkowski space, this data is so far determined only up to the large gauge transformations which are generated by both ε and ε^- and act separately on Γ^+ and Γ^-. Clearly there can be no sensible scattering problem without imposing a relation between ε and ε^-. Any relation between them should preserve Lorentz invariance. Under Lorentz transformation

$$\begin{aligned} \delta_Y A_a\big|_{\mathscr{I}^+_-} &= \left(Y^b\partial_b A_a + A_b\partial_a Y^b\right)\big|_{\mathscr{I}^+_-} , \\ \delta_Y \widetilde{A}_a\big|_{\mathscr{I}^-_+} &= \left(\widetilde{Y}^b\widetilde{\partial}_b \widetilde{A}_a + \widetilde{A}_b\widetilde{\partial}_a \widetilde{Y}^b\right)\big|_{\mathscr{I}^-_+} . \end{aligned} \tag{3.4.1}$$

Now, recall that Y and $\widetilde{Y}$ are related to each other as (2.3.15). Then, the symmetry is preserved by the natural requirement

$$A_z(-\infty, z, \overline{z}) = \widetilde{A}_{\overline{z}}(+\infty, z, \overline{z}) . \tag{3.4.2}$$

(3.4.2) in turn requires

$$\varepsilon(z, \overline{z}) = \varepsilon^-(z, \overline{z}), \tag{3.4.3}$$

as well as the generalization to finite gauge transformations.

In addition to the gauge field, the radial electric fields are also antipodal matched[4]

$$F^{(2)}_{ur}(-\infty, \theta) = \widetilde{F}^{(2)}_{vr}(+\infty, \theta) . \tag{3.4.4}$$

[4] It is easily checked that the Liénard-Wiechert electric field of moving charged particle satisfies this antipodal matching condition as shown in [94].

Note that, because of the antipodal identification of the null generators of $\mathscr{I}^+$ and $\mathscr{I}^-$, this means the gauge parameter is not the limit of a function which depends only on the angle in Minkowskian (r, t) coordinates. Rather it goes to the same value at the beginning and end of light rays crossing through the origin of Minkowski space.[5] ε is then a function on the space of null generators of $\mathscr{I}$.

This antipodal matching condition can also be understood by looking at the Penrose diagram of compactified Minkowski space as shown in Figure 3.1.[6]

Both the gauge field strength and the charge current are invariant under these symmetries. The phases they generate on matter fields are classically unobservable. Hence (unlike the case of gravitational supertranslations considered in [4]), they have little import for the usual discussion of classical scattering. It simply (antipodally) equates the final data for $\widetilde{A}_a\big|_{\mathscr{I}^-_+}$ with that of the initial data for $A_a\big|_{\mathscr{I}^+_-}$. However in the quantum theory, where phases matter, they have significant consequences to which we now turn.

3.5 Quantum Ward identity

In this section, we consider the consequences of the large gauge symmetry on the semi-classical $\mathcal{S}$-matrix. Following the conventions introduced in §2.4.6, we consider an amplitude (2.6.4) with n outgoing particles with charges Q_k outgoing to points $z_k, \bar{z}_k$ with energies ω_k so that $\langle\, \text{out}\,| \equiv \langle\, z_1, \cdots z_n\,|$ and $|\,\text{in}\,\rangle \equiv |\,0\,\rangle$. The $\mathcal{S}$-matrix elements are then denoted as $\langle\, \text{out}\,|\mathcal{S}|\,\text{in}\,\rangle$. The quan-

[5]Such gauge transformations were considered in [95].

[6]We are assuming that the theory under consideration be conformally invariant in the deep IR.

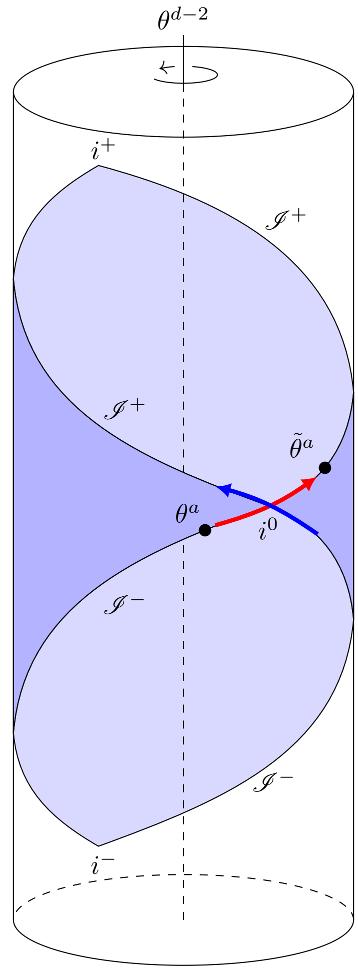

Figure 3.1: The Einstein Cylinder: Minkowski space $\mathbb{R}^{1,3}$ is conformally equivalent to the Einstein cylinder $\mathbb{R} \times S^3$. In the diagram above, we have compactified Minkowski space onto the cylinder and shown the range of coordinates. Note that the null generators that move from $\mathscr{I}^-$ to $\mathscr{I}^+$ across i^0 are antipodally related. From this perspective, the antipodal matching condition on the gauge field and large gauge parameter is quite natural.

tum version of the classical invariance of scattering under large gauge transformations is

$$\langle \, \mathrm{out} \, | \left(Q_\varepsilon^+ \mathcal{S} - \mathcal{S} Q_\varepsilon^- \right) | \, \mathrm{in} \, \rangle = 0. \tag{3.5.1}$$

The semi-classical charge obeys the quantum relations (from (3.2.12) and (3.3.3))

$$\begin{aligned}\langle\, \mathrm{out}\,|Q_\varepsilon^+ &= \langle\, \mathrm{out}\,|F^+[\varepsilon] + \sum_{k=1}^{n} Q_k \varepsilon(z_k, \overline{z}_k)\langle\, \mathrm{out}\,|\,,\\ Q_\varepsilon^-|\,\mathrm{in}\,\rangle &= F^-[\varepsilon]|\,\mathrm{in}\,\rangle\,,\end{aligned} \tag{3.5.2}$$

where

$$F^+[\varepsilon] \equiv -2\int d^2w \partial_{\overline{w}}\varepsilon\partial_w N\,, \qquad F^-[\varepsilon] \equiv -2\int d^2w \partial_{\overline{w}}\varepsilon\partial_w \widetilde{N}\,. \tag{3.5.3}$$

Defining

$$F[\varepsilon] \equiv F^+[\varepsilon] - F^-[\varepsilon]\,. \tag{3.5.4}$$

and the time ordered product

$$:F[\varepsilon]\mathcal{S}: \,= F^+[\varepsilon]\mathcal{S} - \mathcal{S}F^-[\varepsilon]\,, \tag{3.5.5}$$

equation (5.1.1) becomes

$$\langle\, \mathrm{out}\,|:F[\varepsilon]\mathcal{S}:|\,\mathrm{in}\,\rangle = -\sum_{k=1}^{n} Q_k \varepsilon(z_k, \overline{z}_k)\langle\, \mathrm{out}\,|\mathcal{S}|\,\mathrm{in}\,\rangle. \tag{3.5.6}$$

This Ward identity relates the insertion of a soft photon with polarization and normalization given in (3.5.4) into any $\mathcal{S}$-matrix element to the same $\mathcal{S}$-matrix element without a soft photon insertion.

For an incoming state which happens to be the vacuum, (3.5.2) reduces to

$$Q_\varepsilon^-|\,0\,\rangle = F^-[\varepsilon]|\,0\,\rangle\,. \tag{3.5.7}$$

Hence, Q_ε^- does not annihilate the vacuum unless $\varepsilon =$ constant, implying that all but the constant mode of the large gauge symmetries are spontaneously broken. Moreover (3.2.16) identifies C as the corresponding Goldstone boson.

This result may seem surprising for the following reason. Soft photons are labelled by a spatial direction and a polarization. This suggests two modes for every point on the sphere, which is twice the number predicted by Goldstone's theorem. In fact the positive and negative helicity modes are not independent. As spelled out in Appendix 3.7, there are non-local (on the asymptotic S^2) linear combinations of positive and negative helicity photons whose associated soft factor cancels exactly at leading order.[7] To leading order, these linear combinations of soft modes decouple from all $\mathcal{S}$-matrix elements and hence are truly pure gauge. This relation reduces the two modes for every point on the sphere to the single one predicted by Goldstone theorem.

3.6 Soft photon theorem

In this subsection, we show that the Ward identity (3.5.6) is the soft photon theorem in disguise. In order to do so we must rewrite everything in momentum space. We start with the mode expansion on $\mathscr{I}^+$ (2.5.17),

$$\widehat{A}_z(u,z,\bar{z}) = -\frac{ie^2}{8\sqrt{2}\pi^2}E_z^+\int_0^\infty d\omega_q\left[a_{\mathcal{F},+}(\omega_q\hat{y})e^{-i\omega_q u} - a^\dagger_{\mathcal{F},-}(\omega_q\hat{y})e^{i\omega_q u}\right]. \tag{3.6.1}$$

Defining the energy eigenmodes

$$N_a^\omega(\theta) \equiv \int_{-\infty}^\infty du e^{i\omega u}F_{ua}(u,\theta)\,, \tag{3.6.2}$$

we find

$$N_z^\omega(z,\bar{z}) = -\frac{e}{4\sqrt{2}\pi}E_z^+\left[a_{\mathcal{F},+}(\omega_q\hat{y})H(\omega) + a^\dagger_{\mathcal{F},-}(\omega_q\hat{y})H(-\omega)\right]. \tag{3.6.3}$$

[7] Sub-leading orders are considered in [96].

where $H(x)$ is the Heaviside theta function. When $\omega > 0$ ($\omega < 0$) only the first (second) term contributes. We define the zero mode by the hermitian expression

$$N_a = \lim_{\omega \to 0} \frac{1}{2} \left(N_a^{\omega} + N_a^{-\omega} \right) . \tag{3.6.4}$$

It follows that

$$N_z = -\frac{e}{8\sqrt{2}\pi} E_z^{+} \lim_{\omega \to 0^+} \left[\omega a_{\mathcal{F},+}(\omega \hat{y}) + \omega a_{\mathcal{F},-}^{\dagger}(\omega \hat{y}) \right] . \tag{3.6.5}$$

Similarly on $\mathscr{I}^-$, we define

$$\widetilde{N}_z(\widetilde{z}, \widetilde{\overline{z}}) \equiv \int_{-\infty}^{\infty} dv \widetilde{F}_{vz} = -\frac{e}{8\sqrt{2}\pi} E_z^{+} \lim_{\omega \to 0^+} \left[\omega a_{\mathcal{F},+}(\omega \hat{y}) + \omega a_{\mathcal{F},-}^{\dagger}(\omega \hat{y}) \right] , \tag{3.6.6}$$

It follows from (3.6.2) and (3.6.6) that

$$N_z - \widetilde{N}_z = \int_{-\infty}^{\infty} du F_{uz} - \int_{-\infty}^{\infty} dv \widetilde{F}_{vz} = \frac{e^2}{4\pi} F\left[\frac{1}{z-w} \right] , \tag{3.6.7}$$

where $F[\varepsilon]$ is defined in (3.5.4). Setting $\varepsilon(w, \overline{w}) = \frac{1}{z-w}$, the Ward identity (3.5.6) becomes

$$\langle \, \text{out} \, | \colon \; (N_z - M_z) \, \mathcal{S} \colon | \, \text{in} \, \rangle = -\frac{e^2}{4\pi} \sum_{k=1}^{n} \frac{Q_k}{z - z_k} \langle \, \text{out} \, | \mathcal{S} | \, \text{in} \, \rangle . \tag{3.6.8}$$

Using (3.6.5) and (3.6.6), the above equations become

$$\lim_{\omega \to 0^+} \left[\omega \langle \, \text{out} \, | a_{\mathcal{F},+}(\omega \hat{y}) \mathcal{S} | \, \text{in} \, \rangle \right] = \sqrt{2} e E_{+}^{z} \sum_{k=1}^{n} \frac{Q_k}{z - z_k} \langle \, \text{out} \, | \mathcal{S} | \, \text{in} \, \rangle , \tag{3.6.9}$$

where we have used the fact

$$\lim_{\omega \to 0^+} \left[\omega \langle \, \text{out} \, | \mathcal{S} a_{\mathcal{F},-}^{\dagger}(\omega \hat{y}) | \, \text{in} \, \rangle \right] = - \lim_{\omega \to 0^+} \left[\omega \langle \, \text{out} \, | a_{\mathcal{F},+}(\omega \hat{y}) \mathcal{S} | \, \text{in} \, \rangle \right] . \tag{3.6.10}$$

(3.6.9) takes on precisely the form of a soft theorem where the particle that is taken to be soft is a photon. It is then natural to expect that this is precisely the soft theorem derived by Weinberg in

1965 [97]. We now show that this is indeed the case.

First, we review the derivation of the soft-photon theorem. The standard derivation utilizes Feynman diagrams. However, we present a derivation here that will readily generalize to arbitrary theories with arbitrary soft particles easily.

We consider a scattering amplitude $\mathscr{A}_{n+1}^{\text{out},\pm}$ involving an outgoing photon of momentum q^μ and polarization $\varepsilon_\mu^{(\alpha)}(q)$, as well as n other hard asymptotic states (some of which may also be photons),

$$\mathscr{A}_{n+1}^{\text{out},\alpha} = \langle\, \text{out}\, | a_{\mathcal{F},\alpha}(q) \mathcal{S}\, |\, \text{in}\, \rangle\ . \tag{3.6.11}$$

The leading behavior of this amplitude when the momentum of the photon is taken to zero, $q \to 0$, is governed by a universal soft theorem,

$$\mathscr{A}_{n+1}^{\text{out},\alpha} \quad \to \quad e \sum_{k=1}^{n} Q_k \frac{p_k \cdot \varepsilon^{(\alpha)}(q)}{p_k \cdot q} \mathscr{A}_n\ . \tag{3.6.12}$$

Here Q_k is the electric charge of kth particle, and $\mathscr{A}_n = \langle\, \text{out}\, |\mathcal{S}|\, \text{in}\, \rangle$ is the hard amplitude without the soft photon. The only assumptions that are needed to derive the soft photon theorem are Lorentz symmetry and gauge invariance.

At tree level, the pole at $q = 0$ on the right-hand side of (3.6.12) can only arise when an internal propagator goes on shell, which happens precisely when the soft photon attaches to one of the external lines. This is described by a single insertion of the interaction term $-\mathcal{A}^\mu \mathcal{J}_\mu \subset \mathscr{L}_{\text{int}}$. Factorizing on the propagators that go on shell when $q \to 0$ then leads to

$$\begin{aligned} \mathscr{A}_{n+1}^{\text{out},\alpha} \quad \longrightarrow \quad & -i \langle\, 0\, | a_{\mathcal{F},\alpha}(q) \mathcal{A}^\mu(0) |\, 0\, \rangle \times \\ & \sum_{f,s} \sum_{k=1}^{n} \frac{-i}{2 p_k \cdot q} \langle\, i\, | \mathcal{J}_\mu(0) |\, f, p_k, s\, \rangle \left\langle 1\, ;\, \ldots\, ;\, f, p_k, s\, ;\, \ldots\, ; n \right| \mathcal{S} \left| \text{in} \right\rangle\ . \end{aligned} \tag{3.6.13}$$

It follows from the mode expansions in §2.5 that

$$\langle\, 0\,| a_{\mathcal{F},\alpha}(q)\mathcal{A}^{\mu}(0)|\, 0\,\rangle = e\varepsilon^{(\alpha)}_{\mu}(q)\;. \tag{3.6.14}$$

Lorentz invariance and current conservation completely determine the matrix elements of the electric current $\mathcal{J}_\mu(x)$ between states of equal momentum (i.e. in the forward limit) in terms of their electric charge (see for instance [97], chapter 10),

$$\langle\, f, p, s\,|\mathcal{J}_\mu(0)|\, f', p, s'\,\rangle = -2Q_f\, p_\mu \delta_{ff'}\delta_{ss'}\;. \tag{3.6.15}$$

Substituting (3.6.14) and (3.6.15) into (3.6.13) establishes the soft photon theorem (3.6.12).

We can rewrite (3.6.12) in the following form

$$\lim_{\omega\to 0^+}\left[\omega\mathscr{A}^{\text{out},\alpha}_{n+1}(q)\right] = e\lim_{\omega\to 0^+}\omega\sum_{k=1}^{n} Q_k \frac{p_k\cdot\varepsilon^{(\alpha)}(q)}{p_k\cdot q}\mathscr{A}_n. \tag{3.6.16}$$

Using the parametrization of the momenta discussed earlier

$$\begin{aligned} p_k^\mu &= \omega_k\left(1, \frac{z_k+\bar{z}_k}{1+z_k\bar{z}_k}, \frac{-i\left(z_k-\bar{z}_k\right)}{1+z_k\bar{z}_k}, \frac{1-z_k\bar{z}_k}{1+z_k\bar{z}_k}\right), \\ q^\mu &= \omega\left(1, \frac{z+\bar{z}}{1+z\bar{z}}, \frac{-i\left(z-\bar{z}\right)}{1+z\bar{z}}, \frac{1-z\bar{z}}{1+z\bar{z}}\right), \\ \varepsilon^{(+)}_\mu(q) &= \frac{1}{\sqrt{2}}\left(-\bar{z}, 1, -i, -\bar{z}\right), \end{aligned} \tag{3.6.17}$$

we find

$$\lim_{\omega\to 0^+}\omega\mathscr{A}^{\text{out},+}_{n+1}(q) = \sqrt{2}eE^z_+\sum_{k=1}^{n}\frac{Q_k}{z-z_k}\mathscr{A}_n\;, \tag{3.6.18}$$

which is precisely (3.6.9). Thus, we have shown that the Ward identity, (3.5.6) is equivalent to the soft-photon theorem. This argument can be run backwards to show that (3.6.16) implies (3.5.6) with

$\varepsilon = \frac{1}{z-w}$. However, since any function $\varepsilon(z,\overline{z})$ can be written as

$$\varepsilon(w,\overline{w}) = \frac{1}{2\pi}\int d^2z\varepsilon(z,\overline{z})\partial_{\overline{z}}\frac{1}{z-w} \tag{3.6.19}$$

and $F[\varepsilon]$ is linear in ε, the soft-photon theorem implies (3.5.6) for any $\varepsilon(z,\overline{z})$.

3.7 Appendix: Decoupled soft photons

It is possible to see directly from the soft photon theorem that a particular combination of positive and negative helicity photons decouples from the theory. This is seen easiest in the $(z,\overline{z})$ coordinates. We start with the soft photon theorem in this parameterization (3.6.9) for positive helicity insertions

$$\lim_{\omega\to 0^+}\omega\langle\,\text{out}\,|a_{\mathcal{F},+}(\omega\hat{y})\mathcal{S}|\,\text{in}\,\rangle = \sqrt{2}eE^z_+\sum_{k=1}^{n}\frac{Q_k}{z-z_k}\langle\,\text{out}\,|\mathcal{S}|\,\text{in}\,\rangle\,. \tag{3.7.1}$$

Consider now the amplitude involving the following linear combination of the positive helicity soft photons

$$\mathcal{O}_-(z,\overline{z}) = \frac{1}{2\pi}E^{\overline{z}}_-\int d^2w\frac{1}{\overline{z}-\overline{w}}\partial_{\overline{w}}\left[E^+_w\lim_{\omega\to 0^+}\left\{\omega a_{\mathcal{F},+}\big(\omega\hat{y}(w,\overline{w})\big)\right\}\right]\,, \tag{3.7.2}$$

where $\hat{y}$ points towards $(w,\overline{w})$. Insertions of this operator is given by (3.7.1) as

$$\langle\,\text{out}\,|\mathcal{O}_-(z,\overline{z})\mathcal{S}|\,\text{in}\,\rangle = \sqrt{2}eE^{\overline{z}}_-\sum_{k=1}^{n}\frac{Q_k}{\overline{z}-\overline{z}_k}\langle\,\text{out}\,|\mathcal{S}|\,\text{in}\,\rangle\,. \tag{3.7.3}$$

This is precisely the soft photon theorem for a negative-helicity soft photon with momentum pointing towards $(z,\overline{z})$. We therefore conclude that the linear combination

$$\lim_{\omega\to 0}\left[\omega a_{\mathcal{F},-}(\omega\hat{x})\right] - \frac{1}{2\pi}E^{\overline{z}}_-\int d^2w\frac{1}{\overline{z}-\overline{w}}\partial_{\overline{w}}\left[E^+_w\lim_{\omega\to 0}\left\{\omega a_{\mathcal{F},+}\big(\omega\hat{y}(w,\overline{w})\big)\right\}\right] \tag{3.7.4}$$

has no poles and decouples from the $\mathcal{S}$-matrix. In the more familiar momentum space variables, this is

$$a_{-}^{\text{out}}(\omega\hat{p}_{\gamma})+\frac{1}{2\pi\left(1+\cos\theta_{p_{\gamma}}\right)}\int d\Omega_{q}\frac{1+\cos\theta_{q}}{\left(\varepsilon^{(+)}(\hat{p}_{\gamma})\cdot\hat{q}\right)^{2}}a_{+}^{\text{out}}(\omega\hat{q})\,, \tag{3.7.5}$$

where the integral is over the angular distribution of $\hat{q}$.

4

2D Kac-Moody Symmetries of 4D Yang-Mills Theory

4.1 Introduction

In this chapter, we study the nonabelian generalization of the large gauge symmetry. This chapter is a modified extract of [2].

The n-particle scattering amplitudes $\mathscr{A}_n$ of any four-dimensional quantum field theory (QFT_4) can be described as a collection of n-point correlation functions on the two-sphere (S^2) with coordinates $(z, \bar{z})$

$$\mathscr{A}_n = \langle\, \mathcal{O}_1(\omega_1, z_1, \bar{z}_1) \cdots \mathcal{O}_n(\omega_n, z_n, \bar{z}_n) \,\rangle\,, \tag{4.1.1}$$

where $\mathcal{O}_k$ creates (if $\omega_k < 0$) or annihilates (if $\omega_k > 0$) an asymptotic particle with energy $|\omega_k|$ at the point $(z_k, \bar{z}_k)$ where the particle crosses the asymptotic S^2 at null infinity ($\mathscr{I}$). The alternate description (4.1.1) is obtained from the usual momentum space description by simply trading the three independent components of the on-shell four momentum p_k^μ (subject to $p_k^2 = -m_k^2$) with the three quantities $(\omega_k, z_k, \bar{z}_k)$. This is shown in Figure 4.1.

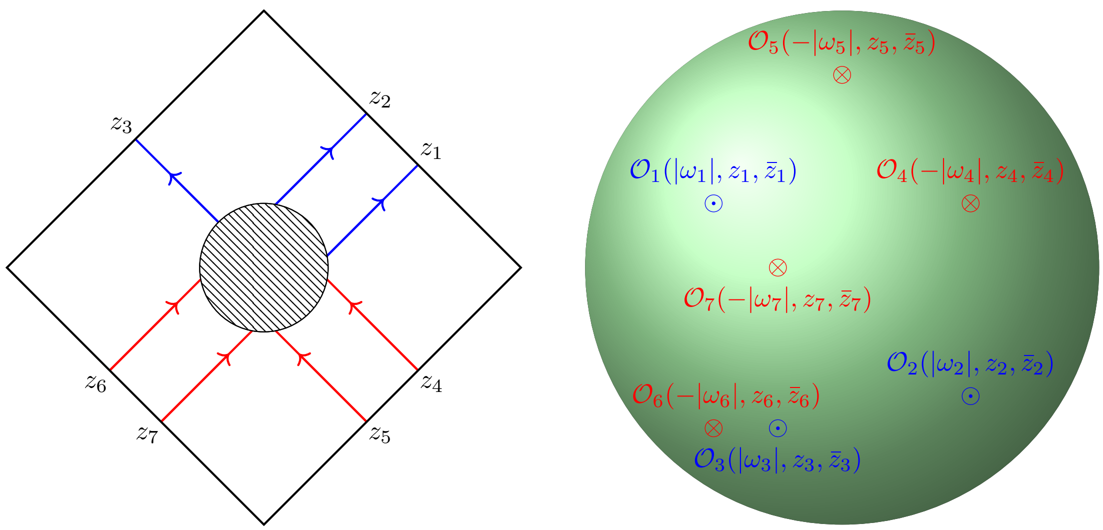

Figure 4.1: An $\mathcal{S}$-matrix amplitude represented as a correlation function on the sphere. Operators in red have negative energy and represent incoming states and operators in blue have positive energy and represent outgoing states.

As discussed previously (2.3.14), the Lorentz group $SL(2, \mathbb{C})$ acts as the global conformal group on the asymptotic S^2 according to

$$z \to \frac{az+b}{cz+d}, \qquad ad - bc = 1 . \tag{4.1.2}$$

Hence, in this respect, *Minkowskian* QFT_4 amplitudes resemble *Euclidean* two-dimensional conformal field theory (CFT_2) correlators. It is natural to ask what other properties QFT_4 scattering amplitudes, expressed in the form (4.1.1), have in common with conventional CFT_2 correlators, and more generally whether a holographic relation of the form *Minkowskian* QFT_4 $=$ *Euclidean* CFT_2 might plausibly exist when gravity is included.[1] In this paper, we consider tree-level scattering of massless particles in 4D nonabelian gauge theories with gauge group $\mathcal{G}$. A salient feature of all such amplitudes is that soft gluon scattering is controlled by the soft gluon theorem [104]. A prescription is given for completing the hard $\mathcal{S}$-matrix (in which all external states have $E_k \neq 0$) to an $\mathcal{S}$-correlator which includes positive helicity soft gluons at strictly zero energy. It is shown that the content of the soft gluon theorem at tree-level is that the positive helicity soft gluon insertions are holomorphic 2D currents which generate a 2D $\mathcal{G}$-Kac-Moody algebra in the $\mathcal{S}$-correlator! Turning the argument around, the soft gluon theorem can be derived as a tree-level Ward identity of the Kac-Moody symmetry.[2]

Moreover, we show that the Kac-Moody symmetries are equivalent to the asymptotic symmetries of 4D gauge theories described in [28]. They are CPT-invariant gauge transformations, which are independent of advanced or retarded time and take angle-dependent values on $\mathscr{I}$. CPT invariance

[1] The results of [11, 45, 98–103] suggest that for quantum gravity scattering amplitudes the $SL(2,\mathbb{C})$ Lorentz symmetry (4.1.2) is enhanced to the infinite-dimensional local 2D conformal symmetry.

[2] A similar Kac-Moody algebra was studied in [105] in the context of MHV amplitudes.

requires that the gauge transformation at any point on $\mathscr{I}^+$ equals that at the PT antipode on $\mathscr{I}^-$. Such transformations act nontrivially on the asymptotic physical states and comprise the asymptotic symmetry group. These are the gauge theory analogs of BMS transformations in asymptotically flat gravity [4, 14, 15, 25, 27, 45, 100–103, 106, 107]. The abelian $U(1)$ case was discussed in [1, 30, 96] and related recent discussions of symmetries, infrared divergences and soft theorems are in [11, 35, 108–125].

The asymptotic symmetries of QED are spontaneously broken in the perturbative vacuum and the soft photons were shown to be the resulting Goldstone bosons [1]. Analogously, the standard rules of Yang-Mills perturbation theory presume a trivial flat color frame on $\mathscr{I}$. In this paper we see that this trivial frame is not invariant under the non-constant Kac-Moody transformations and the large gauge symmetry is spontaneously broken, with the soft gluons being the corresponding Goldstone bosons.

The nonabelian interactions of Yang-Mills theory lead to some surprising new features that are not present in parallel analyses of QED [1]. As pointed out to us by S. Caron-Huot [126, 127], the double-soft limit of the $\mathcal{S}$-matrix involving one positive and one negative helicity gluon is ambiguous. The result depends on the order in which the gluons are taken to be soft. Hence a prescription must be given for defining the double-soft boundary of the $\mathcal{S}$-matrix. We adopt the prescription that positive helicities are always taken soft first. With this prescription there is one holomorphic $\mathcal{G}$-Kac-Moody from positive helicity soft gluons, but not a second one from negative helicity soft gluons.

The soft gluon theorem has well-understood universal corrections due to IR divergences which appear only at one loop, see e.g. [91]. These will certainly affect any extension of the present discussion beyond tree-level. Since an infinite number of relations among $\mathcal{S}$-matrix elements remain, an asymptotic symmetry may survive these corrections. However it is not clear if it can still be understood as a Kac-Moody symmetry. Corrections do *not* appear at the level of the integrands studied in the amplitudes program [111, 112, 128, 129] or in contexts requiring the soft limit to be taken prior to the removal of the IR regulator [91, 111, 128]. Hence the Kac-Moody symmetry is relevant in some contexts to all loops. We leave this issue, as well as the generalization to massive particle scattering, to future investigations.

This chapter is organized as follows. §4.2 establishes our notation and conventions. In §4.3, we introduce the various asymptotic fields used in the paper and discuss the asymptotic symmetries of nonabelian gauge theories. In §4.4, we show that the soft gluon theorem is the Ward identity of a holomorphic Kac-Moody symmetry which can also be understood as an asymptotic gauge symmetry. In §4.5, we show that the double-soft ambiguity of the $\mathcal{S}$-matrix obstructs the appearance of a second antiholomorphic Kac-Moody. Finally, §4.6 contains a preliminary discussion of Wilson line insertions, SCET fields and an operator realization of the flat gauge connection on $\mathscr{I}$.

4.2 Notations and Conventions

We consider a nonabelian gauge theory with group $\mathcal{G}$ and associated Lie algebra $\mathfrak{g}$. Elements of $\mathcal{G}$ in representation R_k are denoted by g_k, where k labels the representation. The corresponding hermitian generators of $\mathfrak{g}$ obey

$$\left[T_k^a, T_k^b\right] = i f^{abc} T_k^c, \tag{4.2.1}$$

where $a = 1, \cdots$, dim $\mathfrak{g}$ and the sum over repeated Lie algebra indices is implied. The adjoint elements of $\mathcal{G}$ and generators of $\mathfrak{g}$ are denoted by g and T^a respectively with $(T^a)_{bc} = -i f^{abc}$. The real antisymmetric structure constants f^{abc} are normalized so that

$$f^{acd} f^{bcd} = \delta^{ab} = \mathrm{tr}\left[T^a T^b\right]. \tag{4.2.2}$$

The four-dimensional matrix valued gauge field is $\mathcal{A}_\mu = \mathcal{A}_\mu^a T^a$. The field strength corresponding to $\mathcal{A}_\mu$ is

$$\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu - i[\mathcal{A}_\mu, \mathcal{A}_\nu] = \mathcal{F}_{\mu\nu}^a T^a. \tag{4.2.3}$$

The theory is invariant under gauge transformations

$$\mathcal{A}_\mu \to g \mathcal{A}_\mu g^{-1} + i g \partial_\mu g^{-1}, \qquad \phi_k \to g_k \phi_k, \qquad \mathcal{J}_\mu \to g \mathcal{J}_\mu g^{-1}, \tag{4.2.4}$$

where ϕ_k are matter fields in representation R_k and $\mathcal{J}_\mu$ is the matter current that couples to the gauge field. The infinitesimal gauge transformations with respect to $\hat{\varepsilon} = \hat{\varepsilon}^a T^a$ (where $g = e^{i\hat{\varepsilon}}$) are

$$\delta_{\hat{\varepsilon}} \mathcal{A}_\mu = \partial_\mu \hat{\varepsilon} - i[\mathcal{A}_\mu, \hat{\varepsilon}], \qquad \delta_{\hat{\varepsilon}} \phi_k = i \hat{\varepsilon}^a T_k^a \phi_k, \qquad \delta_{\hat{\varepsilon}} \mathcal{J}_\mu = -i[\mathcal{J}_\mu, \hat{\varepsilon}]. \tag{4.2.5}$$

The bulk equations that govern the dynamics of the gauge field are

$$\nabla^\nu \mathcal{F}_{\nu\mu} - i\left[\mathcal{A}^\nu, \mathcal{F}_{\nu\mu}\right] = g_{\text{YM}}^2 \mathcal{J}_\mu, \tag{4.2.6}$$

where ∇^μ is the covariant derivative with respect to the spacetime metric.

4.3 Asymptotic fields and symmetries

In this section, we give our conventions for the asymptotic expansion around $\mathscr{I}$ (see [28] for more details), specify the gauge conditions and boundary conditions, and describe the residual large gauge symmetry.

For the purposes of this chapter, we find it convenient to work in temporal gauge

$$\mathcal{A}_u = 0\,. \tag{4.3.1}$$

In this gauge, we can expand the gauge fields near $\mathscr{I}^+$ as

$$\mathcal{A}_a(r,u,\theta) = A_a(u,\theta) + \mathcal{O}\left(1/r\right)\,, \qquad \mathcal{A}_r(r,u,\theta) = \frac{1}{r^2}A_r^{(2)}(u,\theta) + \mathcal{O}\left(1/r^3\right)\,, \tag{4.3.2}$$

where the leading behavior of the gauge field is chosen so that the charge and energy flux through $\mathscr{I}^+$ is finite. The full four-dimensional gauge field is determined by the equations of motion in terms of $A_a(u,\theta)$, which forms the boundary data of the theory.

The leading behavior of the field strength is $\mathcal{F}_{ur} = \mathcal{O}(1/r^2)$ and $\mathcal{F}_{ua}, \mathcal{F}_{ab} = \mathcal{O}(1)$ with leading coefficients

$$F_{ur} = \partial_u A_r^{(2)}\,, \qquad F_{ua} = \partial_u A_a\,, \qquad F_{ab} = \partial_a A_b - \partial_b A_a - i[A_a, A_b]\,. \tag{4.3.3}$$

We will be interested in configurations that revert to the vacuum in the far future, i.e.

$$F_{ur}|_{\mathscr{I}^+_+} = F_{ua}|_{\mathscr{I}^+_+} = F_{ab}|_{\mathscr{I}^+_+} = 0. \tag{4.3.4}$$

(4.3.4) implies

$$\mathcal{U}_a \equiv A_a|_{\mathscr{I}^+_+} = i\mathcal{U}\partial_a\mathcal{U}^{-1}, \tag{4.3.5}$$

where $\mathcal{U}(\theta) \in \mathcal{G}$. A residual gauge freedom near $\mathscr{I}^+$ is generated by an arbitrary function $\varepsilon(\theta)$ on the asymptotic S^2. These create zero-momentum gluons and will be referred to as large gauge transformations. Under finite large gauge transformations $\mathcal{U} \to g\mathcal{U}$. We also define the soft gluon operator

$$N_a \equiv \int_{-\infty}^{\infty} du F_{ua} = \mathcal{U}_a - A_a|_{\mathscr{I}^+_-}. \tag{4.3.6}$$

Near $\mathscr{I}^-$, the temporal gauge condition implies

$$\mathcal{A}_v = 0. \tag{4.3.7}$$

We expand the gauge fields as $\mathcal{A}_a = \widetilde{A}_a + \mathcal{O}(r^{-1})$, $\mathcal{A}_r = \frac{1}{r^2}\widetilde{A}_r^{(2)} + \mathcal{O}(r^{-3})$. The field strength has leading behavior $\mathcal{F}_{vr} \sim \mathcal{O}(1/r^2)$ and $\mathcal{F}_{va}, \mathcal{F}_{ab} = \mathcal{O}(1)$ with leading coefficients

$$\widetilde{F}_{vr} = \partial_v \widetilde{A}_r\,, \qquad \widetilde{F}_{va} = \partial_v \widetilde{A}_a\,, \qquad \widetilde{F}_{ab} = \partial_a \widetilde{A}_b - \partial_b \widetilde{A}_a - i[\widetilde{A}_a, \widetilde{A}_b]\,. \tag{4.3.8}$$

Configurations that begin from the vacuum in the far past satisfy

$$\widetilde{F}_{ur}|_{\mathscr{I}^-_-} = \widetilde{F}_{va}|_{\mathscr{I}^-_-} = \widetilde{F}_{ab}|_{\mathscr{I}^-_-} = 0. \tag{4.3.9}$$

The four-dimensional gauge field is uniquely determined by the boundary data $\widetilde{A}_a(v, \widetilde{\theta})$.

Residual gauge freedom near $\mathscr{I}^-$ is generated by an arbitrary function $\varepsilon^-(\widetilde{\theta})$ on the asymptotic S^2. Furthermore, (4.3.9) implies

$$\mathcal{V}_a \equiv B_a|_{\mathscr{I}^-_-} = i\mathcal{V}\partial_a \mathcal{V}^{-1}, \tag{4.3.10}$$

On $\mathscr{I}^-$, we define the soft gluon operator

$$\widetilde{N}_a \equiv \int_{-\infty}^{\infty} dv \widetilde{F}_{va} = B_a|_{\mathscr{I}^-_+} - \mathcal{V}_a\,. \tag{4.3.11}$$

The classical scattering problem, i.e. to determine the final data A_a given a set of initial data $\widetilde{A}_a$ is defined only up to the large gauge transformations generated by both ε and ε^- that act separately on the initial and final data. Clearly, there can be no sensible scattering problem without imposing some relation between ε and ε^-. To do this, we match the gauge field at i^0. Lorentz invariant matching conditions are

$$A_z|_{\mathscr{I}^+_-} = \widetilde{A}_z|_{\mathscr{I}^-_+}. \tag{4.3.12}$$

This is preserved by

$$\varepsilon(z, \bar{z}) = \varepsilon^-(z, \bar{z}). \tag{4.3.13}$$

Note that because of the antipodal identification of the null generators of $\mathscr{I}^\pm$ across i^0, the gauge parameter $\varepsilon(z, \bar{z})$ is *not* the limit of a function that depends on the angle in Minkowskian (t, r) coordinates. Rather, it goes to the same value at the beginning and end of light rays crossing through the origin of Minkowski space. ε is then a Lie algebra valued function (or section) on the space of null generators of $\mathscr{I}$.

4.4 Holomorphic soft gluon current

In this section, we show that the soft theorem for outgoing positive helicity gluons (or equivalently incoming negative helicity gluons) is the Ward identity of the holomorphic large gauge transformations and takes the form of a holomorphic $\mathcal{G}$-Kac-Moody symmetry acting on the S^2 on $\mathscr{I}$.

Let $\mathcal{O}_k(\omega_k, z_k, \bar{z}_k)$ denote an operator which creates or annihilates a colored hard particle with energy $\omega_k \neq 0$ crossing the S^2 on $\mathscr{I}$ at the point z_k.[3] We denote the standard n-particle hard amplitudes by

$$\mathscr{A}_n(z_1, \dots, z_n) = \langle\, \mathcal{O}_1 \cdots \mathcal{O}_n \,\rangle_{U=1}. \tag{4.4.1}$$

There are no traces here, so $\mathscr{A}_n$ has n suppressed color indices. Since the gauge field vanishes at infinity, the asymptotic S^2 has a flat connection $U_z = iU\partial_z U^{-1}$, where $U \in \mathcal{G}$.[4] In order to compare the color of particles emerging at different points on the S^2, this connection must be specified. The $U = 1$ subscript here indicates the fact that the standard perturbation theory presumes the trivial connection $U_z = 0$.[5]

The hard $\mathcal{S}$-matrix has soft boundaries where gluon momenta vanish. We wish to give a prescription to extend, or 'compactify' the $\mathcal{S}$-matrix to a larger object that includes these boundaries. Since zero-energy gluons are not obviously either incoming or outgoing, the $\mathcal{S}$-matrix so compactified is not obviously a matrix mapping in states to out states. Hence we will refer to the compactified $\mathcal{S}$-matrix as the $\mathcal{S}$-correlator.[6]

---

[3]For instance, for scalar particles

$$\mathcal{O}_k(\omega_k, z_k, \bar{z}_k) = -\frac{4\pi}{\omega_k}\int_{-\infty}^{\infty} du e^{i\omega_k u}\partial_u \lim_{r\to\infty}\left[r\phi_k(u, r, z_k, \bar{z}_k)\right] .$$

[4]U_z should not be confused with $\mathcal{U}_z$ (defined in (4.3.5)).

[5]For $U_z = 0$ or $U = 1$ an outgoing configuration with a red quark at the north pole and a $\overline{\text{red}}$ quark at the south pole is a color singlet state which can be created by a colorless incoming state. For more general choices of U_z this will not be the case.

[6]In the abelian examples of gravity and QED [1, 4, 27, 103], it is possible to view the $\mathcal{S}$-correlator as a conventional $\mathcal{S}$-matrix. However, the non-commutativity (see (4.5.2)) of the multi-gluon soft limits persists even if one gluon is outgoing ($q^0 > 0$) and the other incoming ($q'^0 < 0$). This means that the soft limit on an out state does not commute with the soft limit on an in state, creating difficulties for the reinterpretation of the $\mathcal{S}$-correlator as an $\mathcal{S}$-matrix.

4.4.1 Soft gluon theorem

In this section, we will show that insertions of the soft gluon current J_z, defined by

$$J_z \equiv -\frac{4\pi}{g_{\text{YM}}^2}\left(N_z - \widetilde{N}_z\right) = \frac{4\pi}{g_{\text{YM}}^2}\left(\int dv \widetilde{F}_{vz} - \int du F_{uz}\right), \tag{4.4.2}$$

into the hard tree-level $\mathcal{S}$-matrix are determined by the soft gluon theorem. In its conventional momentum space form, this theorem states

$$\begin{aligned}&\langle\, \mathcal{O}_1(p_1)\cdots\mathcal{O}_n(p_n); \mathcal{O}^a(q,\varepsilon)\,\rangle_{U=1}\\ &\qquad = g_{\text{YM}}\sum_{k=1}^{n}\frac{p_k\cdot\varepsilon}{p_k\cdot q}\langle\, \mathcal{O}_1(p_1)\cdots T_k^a\mathcal{O}_k(p_k)\cdots\mathcal{O}_n(p_n)\,\rangle_{U=1} + \mathcal{O}(q^0),\end{aligned} \tag{4.4.3}$$

where $\mathcal{O}^a(q,\varepsilon) = \text{tr}\,[T^a\mathcal{O}(q,\varepsilon)]$ creates or annihilates, depending on the sign of q^0, a soft gluon with momentum $\vec{q}$ and polarization ε^μ, and T_k^a is a generator in the representation carried by $\mathcal{O}_k$. Gauge invariance of the theory requires that the right hand side vanishes when $\varepsilon = q$. This implies

$$\sum_{k=1}^{n}\langle\, \mathcal{O}_1(p_1)\cdots T_k^a\mathcal{O}_k(p_k)\cdots\mathcal{O}_n(p_n)\,\rangle_{U=1} = 0, \tag{4.4.4}$$

which is global color conservation. Using the notation of our present paper and assuming $\varepsilon \neq q$, for a positive helicity gluon with massless particles ($p_k^2 = 0$), (4.4.3) becomes

$$\langle\, J_z^a\mathcal{O}_1\cdots\mathcal{O}_n\,\rangle_{U=1} = \sum_{k=1}^{n}\frac{1}{z-z_k}\langle\, \mathcal{O}_1\cdots T_k^a\mathcal{O}_k\cdots\mathcal{O}_n\,\rangle_{U=1}, \tag{4.4.5}$$

where $J_z^a \equiv \text{tr}\,[T^a J_z]$. This was shown in [28] and is reviewed in the appendix. The collinear $q\cdot p_k \to 0$ singularities of (4.4.3) become the poles at $z = z_k$ in (4.4.5). The soft pole in (4.4.3) is absent in (4.4.5) simply because the definition of J_z^a involves the zero mode of the field strength rather than the gauge field and hence an extra factor of the soft energy.

4.4.2 Kac-Moody symmetry

Since $\partial_{\bar{z}} J_z = 0$ away from operator insertions, J_z is a holomorphic current. Consider a contour $\mathcal{C}$ and an infinitesimal gauge transformation $\varepsilon^a(z)$ which is holomorphic ($\partial_{\bar{z}}\varepsilon^a = 0$) inside $\mathcal{C}$. It follows from (4.4.5) that

$$\langle\, J_{\mathcal{C}}(\varepsilon)\mathcal{O}_1 \cdots \mathcal{O}_n \,\rangle_{U=1} = \sum_{k\in\mathcal{C}} \langle\, \mathcal{O}_1 \cdots \varepsilon_k(z_k)\mathcal{O}_k \cdots \mathcal{O}_n \,\rangle_{U=1}, \tag{4.4.6}$$

where $\varepsilon_k(z_k) = \varepsilon^a(z_k)T_k^a$ and

$$J_{\mathcal{C}}(\varepsilon) \equiv \oint_{\mathcal{C}} \frac{dz}{2\pi i} \mathrm{tr}\left[\varepsilon J_z\right], \tag{4.4.7}$$

and the sum $k \in \mathcal{C}$ includes all insertions inside the contour $\mathcal{C}$. Moreover from the soft theorem with multiple J_z insertions one finds

$$\begin{aligned}\langle\, J_{\mathcal{C}}(\varepsilon)J_w\mathcal{O}_1 \cdots \mathcal{O}_n \,\rangle_{U=1} = &\sum_{k\in\mathcal{C}} \langle\, J_w\mathcal{O}_1 \cdots \varepsilon_k(z_k)\mathcal{O}_k \cdots \mathcal{O}_n \,\rangle_{U=1} \\ &+ \langle\, \varepsilon(w)J_w\mathcal{O}_1 \cdots \mathcal{O}_n \,\rangle_{U=1},\end{aligned} \tag{4.4.8}$$

where the last term is added only when w is also inside $\mathcal{C}$.

(4.4.8) is a very familiar formula in two-dimensional conformal field theory. It is the Ward identity of a holomorphic Kac-Moody symmetry for the group $\mathcal{G}$. The absence of a term with no J_w on the right hand side of (4.4.8) indicates that the Kac-Moody level is zero (at tree-level). Hence the $\mathcal{S}$-correlators for any massless theory with nonabelian gauge group $\mathcal{G}$ transform under a holomorphic level-zero $\mathcal{G}$-Kac-Moody action!

4.4.3 Asymptotic symmetries

In this subsection, the Kac-Moody symmetry is identified with holomorphic large gauge symmetry of the gauge theory. According to (4.2.4) under the action of the asymptotic symmetry transformation U

$$\mathcal{O}_k(z_k, \overline{z}_k) \to U_k(z_k, \overline{z}_k)\mathcal{O}_k(z_k, \overline{z}_k), \tag{4.4.9}$$

where U_k acts in the representation of $\mathcal{O}_k$. $\mathcal{S}$-correlators for general U are simply related to those for $U = 1$

$$\langle\, J_z^a \mathcal{O}_1^{i_1} \cdots \rangle_U = U(z, \overline{z})^{ab} U_1(z_1, \overline{z}_1)^{i_1 j_1} \cdots \langle\, J_z^b \mathcal{O}_1^{j_1} \cdots \rangle_{U=1}. \tag{4.4.10}$$

To compare the asymptotic symmetry action (4.4.10) with the Kac-Moody action (4.4.6), consider infinitesimal complexified transformations of the form

$$U(z, \overline{z}) = 1 + i\varepsilon(z) + \cdots, \tag{4.4.11}$$

which are holomorphic inside the contour $\mathcal{C}$ and vanish outside. In that case (4.4.10) linearizes to

$$\delta_\varepsilon \langle\, \mathcal{O}_1 \cdots \mathcal{O}_n \,\rangle_{U=1} = i \sum_{k \in \mathcal{C}} \langle\, \mathcal{O}_1 \cdots \varepsilon_k(z_k)\mathcal{O}_k \cdots \mathcal{O}_n \,\rangle_{U=1}, \tag{4.4.12}$$

where the operator insertions could also include a positive-helicity soft gluon. Comparing with (4.4.6) we see that

$$-i\delta_\varepsilon \langle\, \mathcal{O}_1 \cdots \mathcal{O}_n \,\rangle_{U=1} = \langle\, J_{\mathcal{C}}(\varepsilon)\mathcal{O}_1 \cdots \mathcal{O}_n \,\rangle_{U=1}. \tag{4.4.13}$$

Hence, $J_{\mathcal{C}}(\varepsilon)$ generates holomorphic asymptotic symmetry transformations

$$J_{\mathcal{C}}(\varepsilon) = \int_{D_{\mathcal{C}}} d^2z \gamma_{z\bar{z}} \varepsilon^a \frac{\delta}{\delta U^a}, \tag{4.4.14}$$

where $D_{\mathcal{C}}$ is the region inside $\mathcal{C}$ and U^a is the Lie algebra element corresponding to U, that is, $U = e^{iU^aT^a}$.

Let $\mathcal{C}_0$ be any contour that divides the incoming and outgoing particles as shown in Figure 4.2.

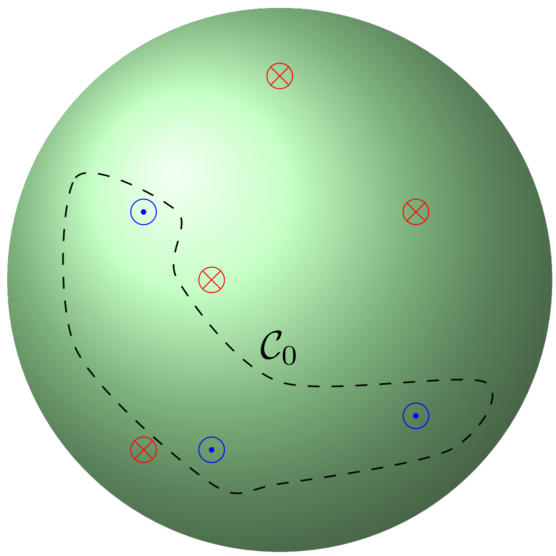

Figure 4.2: The contour $\mathcal{C}_0$ which separates incoming particles and outgoing particles.

For ε holomorphic on the incoming side of $\mathcal{C}_0$, the corresponding $J_{\mathcal{C}_0}(\varepsilon)$ is then the charge that generates the asymptotic symmetries on the incoming state. If ε is holomorphic and non-constant on the incoming side of $\mathcal{C}_0$, it extends to a meromorphic section which must have poles on the outgoing side whose locations we denote $w_1, \ldots, w_p$. We may also evaluate the contour integral by pulling it over the outgoing state. Equating this with (4.4.6) one finds, for any meromorphic section

ε

$$-i\delta_\varepsilon\langle\, \mathcal{O}_1\cdots\mathcal{O}_n\,\rangle_{U=1} = -\sum_{i=1}^{p}\langle\, \operatorname{tr}\left[\varepsilon J_w\right]_{w_i} \mathcal{O}_1\cdots\mathcal{O}_n\,\rangle_{U=1}, \tag{4.4.15}$$

where

$$\operatorname{tr}\left[\varepsilon J_w\right]_{w_i} = \operatorname*{Res}_{w\to w_i} \operatorname{tr}\left[\varepsilon J_w\right]. \tag{4.4.16}$$

This is another form of the soft gluon theorem. It states that $\mathcal{S}$-correlators are invariant under the asymptotic symmetries up to insertions of the soft gluon current. The appearance of the inhomogeneous term on the right hand side implies that the $U = 1$ vacuum spontaneously breaks the symmetry. The soft gluons are the associated Goldstone bosons. Indeed, when $p = 0$, i.e. when ε is a globally holomorphic function on the sphere (and therefore a constant), we have

$$\delta_\varepsilon\langle\, \mathcal{O}_1\cdots\mathcal{O}_n\,\rangle_{U=1} = 0, \tag{4.4.17}$$

which is precisely (4.4.4). This indicates that the subgroup of constant global asymptotic color rotations is not spontaneously broken, as expected.

One might think that the Kac-Moody symmetry does not capture *all* of the asymptotic symmetry group, since the transformations are restricted to be holomorphic within some contour $\mathcal{C}$. However, this is an irrelevant restriction. The $\mathcal{S}$-correlator identities depend only on the n values $\varepsilon_k = \varepsilon(z_k)$ of ε at the n operator insertions. For any choice of ε_k there exists a holomorphic $\varepsilon(z)$ inside some $\mathcal{C}$ such that $\varepsilon(z_k) = \varepsilon_k$ at the positions of operator insertions. Hence the holomorphicity does not preclude consideration of any gauge transformation on Fock space states, and all nontrivial relations among $\mathcal{S}$-correlation functions can be derived from the Kac-Moody symmetry. In particular the soft gluon theorem (4.4.5) is itself a Ward identity of the the Kac-Moody symmetry.

4.5 Antiholomorphic current

We have seen that positive helicity soft gluon currents J_z generate a holomorphic Kac-Moody symmetry. Naively one might expect that negative helicity soft gluon currents $J^a_{\bar{z}}$ generate a second Kac-Moody symmetry which is antiholomorphic. This turns out *not* to be the case for a very interesting reason.

The crucial observation is due to [126, 127]. Consider a boundary of the $\mathcal{S}$-matrix near which two gluons become soft. One finds

$$\begin{aligned}
&\mathscr{A}_{n+2}(p_1, \ldots, p_n; q, \varepsilon, a; q', \varepsilon', b) \\
&\qquad = g^2_{\text{YM}} \sum_{k=1}^{n} \frac{\varepsilon \cdot p_k}{q \cdot p_k} \sum_{j=1}^{n} \frac{\varepsilon' \cdot p_j}{q' \cdot p_j} \langle\, \mathcal{O}_1 \cdots T^a_k \mathcal{O}_k \cdots T^b_j \mathcal{O}_j \cdots \mathcal{O}_n \,\rangle_{U=1} \\
&\qquad\qquad - i g^2_{\text{YM}} f^{abc} \sum_{j=1}^{n} \frac{\varepsilon' \cdot p_j}{q' \cdot p_j} \frac{\varepsilon \cdot q'}{q \cdot q'} \langle\, \mathcal{O}_1 \cdots T^c_j \mathcal{O}_j \cdots \mathcal{O}_n \,\rangle_{U=1} + \mathcal{O}(q^0, q'^0)\,,
\end{aligned} \tag{4.5.1}$$

where the above limit has been computed by taking $q \to 0$ first. Surprisingly, the right hand side actually depends on the order of limits and

$$\begin{aligned}
&\left[\lim_{q \to 0}, \lim_{q' \to 0} \right] \mathscr{A}_{n+2}(p_1, \ldots, p_n; q, \varepsilon, a; q', \varepsilon', b) \\
&\qquad = i g^2_{\text{YM}} f^{abc} \sum_{k=1}^{n} \left(\frac{\varepsilon \cdot p_k}{p_k \cdot q} - \frac{\varepsilon \cdot q'}{q \cdot q'} \right) \left(\frac{\varepsilon' \cdot p_k}{q' \cdot p_k} - \frac{\varepsilon' \cdot q}{q \cdot q'} \right) \langle\, \mathcal{O}_1 \cdots T^a_k \mathcal{O}_k \cdots \mathcal{O}_n \,\rangle_{U=1} \\
&\qquad\qquad + \mathcal{O}\left(q^0, q'^0\right)\,.
\end{aligned} \tag{4.5.2}$$

In the special case that the helicities are the same, then the right hand side of the above expression vanishes and the limits commute. In this case, the $\mathcal{S}$-matrix can be extended to its soft boundaries unambiguously. When the helicities are not the same, the value of the $\mathcal{S}$-matrix at the soft boundary is ambiguous. In terms of currents, taking the positive helicity gluon to zero first gives

$$J_z^a J_{\overline{w}}^b \sim -\frac{if^{abc}}{z-w} J_{\overline{w}}^c, \tag{4.5.3}$$

while in the other order we have

$$J_z^a J_{\overline{w}}^b \sim -\frac{if^{abc}}{\overline{z}-\overline{w}} J_z^c. \tag{4.5.4}$$

Thus, the extension (or 'compactification') of the $\mathcal{S}$-matrix to all soft boundaries requires a prescription. In this paper we adopt the prescription that positive helicity gluon momenta are always taken to zero *before* negative helicity gluon momenta. With this prescription, it follows from (4.5.3) that the current J_z^a generates a Kac-Moody symmetry, under which $J_{\overline{z}}^a$ transforms in the adjoint. $J_{\overline{z}}^a$ itself does not generate a symmetry. A prescription which treats J_z^a and $J_{\overline{z}}^a$ symmetrically yields no symmetry, while taking negative helicity momenta to zero first gives one antiholomorphic Kac-Moody symmetry generated by $J_{\overline{z}}^a$.

The situation is reminiscent of three-dimensional Chern-Simon gauge theory on a manifold with a boundary parameterized by $(z, \overline{z})$. A priori, one might have expected A_z and $A_{\overline{z}}$ to generate both holomorphic and antiholomorphic $\mathcal{G}$-Kac-Moody symmetries. However a more careful analysis reveals that boundary conditions must be chosen to eliminate one or the other. Indeed, this may be more than an analogy. The current J_z^a has no time dependence and lives on the S^2 at the boundary of the 3-manifold $\mathscr{I}$, and the addition of a $\theta F \wedge F$ term to the 4D gauge theory action induces a Chern-Simons term on $\mathscr{I}$. It would be interesting to understand how such a term affects the present analysis.

4.6 Wilson lines and the flat connection on $\mathscr{I}$

Other types of $\mathcal{S}$-correlator insertions besides soft gluon currents are of physical interest and have been considered in the literature. This section contains preliminary observations on a few such insertions.

Consider the Wilson line operator

$$W_{\mathcal{C}}(u, z_1, z_2) = P \exp \left(i \int_{\mathcal{C}} dx^\mu \mathcal{A}_\mu \right), \tag{4.6.1}$$

where P denotes path-ordering and the contour $\mathcal{C}$ is chosen such that it initially enters $\mathscr{I}^+$ at $(u, z_1, \bar{z}_1)$ and leaves at $(u, z_2, \bar{z}_2)$ along null lines of varying r and fixed $(u, z, \bar{z})$. Under holomorphic large gauge transformations

$$W_{\mathcal{C}}(u, z_1, z_2) \to g(z_1) W_{\mathcal{C}}(u, z_1, z_2) g(z_2)^{-1}, \tag{4.6.2}$$

where $g(z) \in \mathcal{G}$. Insertions of J_z in the presence of the Wilson lines are given by the soft theorem[7]

$$\begin{aligned} \langle J_z^a W_{\mathcal{C}}(u, z_1, z_2) \cdots \rangle_{U=1} = \frac{1}{z - z_1} \langle T^a W_{\mathcal{C}}(u, z_1, z_2) \cdots \rangle_{U=1} \\ - \frac{1}{z - z_2} \langle W_{\mathcal{C}}(u, z_1, z_2) T^a \cdots \rangle_{U=1} + \cdots . \end{aligned} \tag{4.6.3}$$

From this, we can construct

$$A_z(u, z, \bar{z}) = -i \lim_{z' \to z} \partial_z W_{\mathcal{C}}(u, z, z'), \tag{4.6.4}$$

where we take $\mathcal{C}$ to be a short contour from $z' \to z$. It follows from (4.6.3)

$$\langle J_z^a A_w^b \mathcal{O}_1 \cdots \rangle_{U=1} = -\frac{i\delta^{ab}}{(z-w)^2} \langle \mathcal{O}_1 \cdots \rangle - \frac{i f^{abc}}{z - w} \langle A_w^c \mathcal{O}_1 \cdots \rangle + \cdots . \tag{4.6.5}$$

[7] See §36.3.2 of [130] for details.

Hence the action of J_z indeed transforms A_z as a connection on $\mathscr{I}$ as expected. A similar discussion applies to fields on $\mathscr{I}^-$.

Recall that J_z was constructed from zero modes of the past and future field strengths (see (4.4.2)). However, $A_z(u)$ has an inhomogeneous term in its gauge transformation and has a soft u-independent piece that cannot be constructed from J_z. To see this, we expand on $\mathscr{I}^+$

$$A_z(u,z,\overline{z}) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega u} A_z^{\omega}(z,\overline{z}) + C_z, \tag{4.6.6}$$

where C_z is defined in (2.4.38)

Here we have used the fact that functions whose boundary values at $\pm\infty$ do not sum to zero do not have a Fourier transform given in terms of ordinary functions. Radiative insertions in an $\mathcal{S}$-matrix involve A_z^{ω} and

$$N_z = -\frac{i}{2} \lim_{\omega\to 0^+} \left(\omega A_z^{\omega} - \omega A_z^{-\omega}\right). \tag{4.6.7}$$

Under a large gauge transformation,

$$\delta_\varepsilon A_z^{\omega} = -i[A_z^{\omega}, \varepsilon], \qquad \delta_\varepsilon \mathcal{U}_z = \partial_z \varepsilon - i[\mathcal{U}_z, \varepsilon]. \tag{4.6.8}$$

Hence the Fourier modes of A_z transform in the adjoint of the asymptotic symmetry group, while the constant piece $\mathcal{U}_z$ is a connection on S^2. Further, (4.3.5) and (4.6.5) imply that we have

$$\langle\, J_z^a \mathcal{U}(w,\overline{w}) \mathcal{O}_1 \cdots \rangle_{U=1} = \frac{T^a}{z-w} \langle \mathcal{U}(w,\overline{w}) \mathcal{O}_1 \cdots \rangle_{U=1}. \tag{4.6.9}$$

A parallel structure on $\mathscr{I}^-$ also exists.

The flat connection U_z is related to the SCET or Wilson line fields used to study jet physics [91]. In CFT_2 with a Kac-Moody symmetry, correlations functions factorize into a hard part and a soft part computed by the current algebra. 4D gauge theory amplitudes also factorize into a hard and a soft part, with the latter computed by Wilson line correlators. It would interesting to relate this soft part to $\mathcal{U}$-correlators and compare it to the structure in CFT_2.

5

Infinite-dimensional Fermionic Symmetries of $\mathcal{N} = 1$ supersymmetric QED

In this chapter, we study our final example of asymptotic symmetries in flat spacetime – in a supersymmetric extension of QED. Unlike the previous two chapters, where the asymptotic symmetry was a subgroup of previously known symmetry, namely gauge symmetry, this will not be the case here. Instead, our approach will be to start with the soft-photino theorem and deduce an asymptotic symmetry. This chapter is a modified extract of [3].

5.1 Introduction

The universal soft behavior of gauge boson amplitudes was recently traced back to the existence of infinitely many symmetries that act on asymptotic scattering states at Minkowskian null infinity, i.e. asymptotic symmetries, whose Ward identities are equivalent to the soft theorems [1, 2, 4, 27–31, 35, 86, 96, 103, 125, 131–135]. Typically, these asymptotic symmetries can be viewed as large gauge transformations, which do not vanish at infinity and therefore act non-trivially on physical states.[1] For instance, the symmetries that give rise to the leading soft photon theorem [1, 2, 28, 31, 86, 132, 135] are parametrized by a function $\varepsilon(z, \overline{z})$ on an asymptotic S^2 (with complex coordinates $z, \overline{z}$) inside the null boundary of Minkowski space. The corresponding charges Q_ε are higher-harmonic generalizations of the electric charge, to which they reduce when $\varepsilon(z, \overline{z}) = 1$. Transformations with non-constant $\varepsilon(z, \overline{z})$ inhomogeneously shift the gauge field $\mathcal{A}_\mu$ by $\partial_\mu \varepsilon$, and hence they are spontaneously broken. The corresponding Goldstone bosons are soft, zero-momentum photons.

It is natural to ask whether soft theorems for massless particles that are not gauge bosons have similar interpretations in terms of asymptotic symmetries. Here we will explore this question in the context of rigid supersymmetric gauge theories, where the gauge fields are accompanied by massless spin-$\frac{1}{2}$ superpartners. For simplicity, we confine our attention to $U(1)$ gauge theories with $\mathcal{N} =$

[1] Here we follow the terminology of [28]: large gauge transformations are assumed to act non-trivially on physical states, because they do not vanish sufficiently rapidly at the boundary of spacetime. However, they may be topologically trivial, i.e. deformable to the identity gauge transformation.

1 supersymmetry and massless charged matter in four dimensions. The $U(1)$ photon $\mathcal{A}_\mu$ has an electrically neutral, fermionic superpartner – the photino Λ_α – whose couplings to charged matter are related to those of the photon by supersymmetry.

In this paper, we will establish the existence of infinitely many fermionic asymptotic symmetries, parametrized by a chiral spinor-valued function $\chi_\alpha(z, \bar{z})$ on S^2, whose Ward identities give rise to the soft photino theorem. The corresponding anti-commuting charges $\mathscr{F}_\chi$ act on the asymptotic fields at null infinity. However, unlike the infinity of bosonic charges Q_ε, they are not a subgroup of any obvious symmetry of the Lagrangian.[2] The usual Lagrangian only displays a finite number of manifest fermionic symmetries – the global supersymmetries generated by $\mathbb{Q}_\alpha$ and $\overline{\mathbb{Q}}_{\dot{\alpha}}$. It is perhaps surprising that even rigid supersymmetric gauge theories can support an infinite number of fermionic asymptotic symmetries.[3] By contrast, this is expected in supergravity, where local supersymmetry is a gauge symmetry [137, 138].

Under the action of $\mathscr{F}_\chi$ we find that the photino Λ_α shifts inhomogeneously. Hence these symmetries are spontaneously broken, and the soft photini are interpreted as the corresponding Goldstone fermions. Interestingly, supersymmetry relates the fermionic charges $\mathscr{F}_\chi$ to the bosonic

[2] The asymptotic symmetries related to the magnetic generalization of the leading soft photon theorem [86] or the subleading soft photon theorem [96] are also not manifest at the level of the Lagrangian.

[3] A similar phenomenon occurs in three-dimensional, supersymmetric Chern-Simons theory in the presence of a suitably supersymmetric boundary, which supports a supersymmetric Kac-Moody current algebra. (As we will see below, the asymptotic symmetries Q_ε and $\mathscr{F}_\chi$ also give rise to just such a current algebra.) The bosonic Kac-Moody symmetries are conventional gauge transformations that do not vanish at the boundary. The Kac-Moody fermions can be understood as a remnant of the full super gauge symmetry that is present before fixing Wess-Zumino (WZ) gauge (see for instance [136]). It is plausible that our asymptotic symmetries $\mathscr{F}_\chi$ have a similar interpretation, but we will not show it here. Instead, we will exhibit the charges $\mathscr{F}_\chi$ directly in WZ gauge and explore their properties.

charges Q_ε. We find (see (5.3.14) below),

$$\{\zeta^\alpha \mathbb{Q}_\alpha, \mathscr{F}_\chi\} = iQ_{\zeta^\alpha \chi_\alpha}\,, \qquad \{\overline{\mathbb{Q}}_{\dot\alpha}, \mathscr{F}_\chi\} = 0\,. \tag{5.1.1}$$

Here the supersymmetry transformation is parametrized by a commuting, constant spinor ζ_α, and $\chi_\alpha(z,\bar z)$ is also taken to be commuting. The charges Q_ε commute with $\mathbb{Q}_\alpha$ and $\overline{\mathbb{Q}}_{\dot\alpha}$.

The soft photon theorem implies that the insertion of a zero-momentum, positive-helicity photon into a scattering amplitude can be interpreted as the Ward identity for a $U(1)$ Kac-Moody current, which transforms in a $(1,0)$ representation of the $SL(2,\mathbb{C})$ conformal symmetry acting on the S^2 at null infinity [1, 28]. Similarly, we will see that the insertion of a positive-helicity photino behaves like a $(\frac{1}{2},0)$ current on S^2. The two currents are related by supersymmetry, as was the case for the charges in (5.1.1).

In §5.2, we begin by reviewing basic aspects of abelian gauge theories with $\mathcal{N}=1$ supersymmetry, focusing on the structure of the supermultiplet that contains the electric current $\mathcal{J}_\mu$, which couples to the photon, and its fermionic superpartner $\mathcal{K}^F_\alpha$, which couples to the photino. We present a current-algebra derivation of the tree-level soft photon and photino theorems that utilizes the properties of $\mathcal{J}_\mu$ and $\mathcal{K}^F_\alpha$ matrix elements between asymptotic states. This derivation emphasizes the universality of the two soft theorems, as well as their relation via supersymmetry. In §5.3 we analyze the classical dynamics of the supersymmetric gauge theory near null infinity. This is facilitated by a convenient choice of coordinates and spinor basis, in which the asymptotic behavior of massless fields near null infinity is simply related to their quantum numbers with respect to the conformal group that governs the deep IR behavior of the theory. After reviewing the results of [1] on the asymptotic dynamics of the photon and the associated bosonic charges Q_ε, we repeat the analysis for the photino. We construct the fermionic asymptotic charges $\mathscr{F}_\chi$ and establish some of their basic properties. In §5.4 we show that the Ward identity for the fermionic symmetries $\mathscr{F}_\chi$ reproduces the soft photino theorem derived in §5.2.

5.2 Soft Theorems

5.2.1 Aspects of $\mathcal{N} = 1$ Gauge Theories

Unless stated otherwise, we will use the conventions of [87]. As was stated in the introduction, we will consider $U(1)$ gauge theories with $\mathcal{N} = 1$ supersymmetry. After fixing Wess-Zumino (WZ) gauge, the vector multiplet $\mathcal{V}$ is given by

$$\mathcal{V} = \left(\mathcal{A}_\mu\,, \Lambda_\alpha\,, \overline{\Lambda}_{\dot{\alpha}}\,, \mathcal{D}\right)\ . \tag{5.2.1}$$

Here $\mathcal{A}_\mu$ is the $U(1)$ gauge field (i.e. the photon) with field strength $\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu$. It is subject to conventional $U(1)$ gauge transformations, which remain unfixed in WZ gauge. The spin-$\frac{1}{2}$ superpartner of the photon is the photino, which is described by a left-handed Weyl fermion Λ_α and its right-handed Hermitian conjugate $\overline{\Lambda}_{\dot{\alpha}}$. The vector multiplet also contains a real scalar $\mathcal{D}$, which is a non-propagating auxiliary field.

In WZ gauge, the non-vanishing (anti-) commutators of the component fields in the vector multiplet $\mathcal{V}$ with the supercharges $\mathbb{Q}_\alpha, \overline{\mathbb{Q}}_{\dot\alpha}$ are given by

$$[\mathbb{Q}_\alpha, \mathcal{A}_\mu] = -\sigma_{\mu\alpha\dot\alpha}\overline{\Lambda}^{\dot\alpha}\,, \qquad [\overline{\mathbb{Q}}_{\dot\alpha}, \mathcal{A}_\mu] = \Lambda^\alpha\sigma_{\mu\alpha\dot\alpha}\,, \tag{5.2.2a}$$

$$\{\mathbb{Q}_\alpha, \Lambda_\beta\} = \varepsilon_{\alpha\beta}\mathcal{D} - i\,(\sigma^{\mu\nu})_{\alpha\beta}\,\mathcal{F}_{\mu\nu}\,, \qquad \{\overline{\mathbb{Q}}_{\dot\alpha}, \overline{\Lambda}_{\dot\beta}\} = \varepsilon_{\dot\alpha\dot\beta}\mathcal{D} - i\,(\overline{\sigma}^{\mu\nu})_{\dot\alpha\dot\beta}\,\mathcal{F}_{\mu\nu}\,, \tag{5.2.2b}$$

$$[\mathbb{Q}_\alpha, \mathcal{D}] = -i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\overline{\Lambda}^{\dot\alpha}\,, \qquad [\overline{\mathbb{Q}}_{\dot\alpha}, \mathcal{D}] = -i\partial_\mu\Lambda^\alpha\sigma^\mu_{\alpha\dot\alpha}\,. \tag{5.2.2c}$$

The dynamics of the gauge multiplet is described by a Lagrangian $\mathscr{L}_{\text{gauge}}$, which is invariant (up to a total derivative) under the supersymmetry transformations in (5.2.2),

$$\mathscr{L}_{\text{gauge}} = -\frac{1}{4e^2}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - \frac{i}{e^2}\overline{\Lambda}\,\overline{\sigma}^\mu\partial_\mu\Lambda + \frac{1}{2e^2}\mathcal{D}^2 + (\text{higher-derivative terms})\ . \tag{5.2.3}$$

In addition to the standard two-derivative kinetic terms for the gauge multiplet, we are allowing for the possibility of higher-derivative terms, e.g. terms such as $\mathcal{F}^4 + (\text{fermions})$, which arise in supersymmetric Born-Infeld actions. The soft theorems discussed below remain valid in the presence of such terms.

The interaction of the gauge field $\mathcal{A}_\mu$ with matter proceeds through a conserved current $\mathcal{J}_\mu$, which resides in a real linear multiplet $\mathcal{J}$,[4]

$$\mathcal{J} = \left(\mathcal{K}^B\,, \mathcal{K}^F_\alpha\,, \overline{\mathcal{K}}^F_{\dot\alpha}\,, \mathcal{J}_\mu\right)\,, \qquad \partial^\mu\mathcal{J}_\mu = 0\,. \tag{5.2.4}$$

Here $\mathcal{K}^B$ is a real scalar, while $\mathcal{K}^F_\alpha$ and its Hermitian conjugate $\overline{\mathcal{K}}^F_{\dot\alpha}$ are left- and right-handed Weyl spinors. Unlike $\mathcal{J}_\mu$, neither $\mathcal{K}^B$ nor $\mathcal{K}^F_\alpha, \overline{\mathcal{K}}^F_{\dot\alpha}$ obey a differential constraint, i.e. they are not conserved currents. All fields in the current supermultiplet $\mathcal{J}$ are gauge invariant. Their non-vanishing

[4] In superspace, a real linear multiplet is described by a real superfield $\mathcal{J}$ that satisfies the constraints $D^2\mathcal{J} = \overline{D}^2\mathcal{J} = 0$, where $D_\alpha, \overline{D}_{\dot\alpha}$ are the usual super-covariant derivative operators defined in [87].

supersymmetry transformations take the following model-independent form,

$$\left[\mathbb{Q}_\alpha, \mathcal{K}^B\right] = i\mathcal{K}^F_\alpha\,, \qquad \left[\overline{\mathbb{Q}}_{\dot\alpha}, \mathcal{K}^B\right] = i\overline{\mathcal{K}}^F_{\dot\alpha}\,, \tag{5.2.5a}$$

$$\left\{\overline{\mathbb{Q}}_{\dot\alpha}, \mathcal{K}^F_\alpha\right\} = -i\sigma^\mu_{\alpha\dot\alpha}\left(\mathcal{J}_\mu + i\partial_\mu\mathcal{K}^B\right)\,, \qquad \left\{\mathbb{Q}_\alpha, \overline{\mathcal{K}}^F_{\dot\alpha}\right\} = i\sigma^\mu_{\alpha\dot\alpha}\left(\mathcal{J}_\mu - i\partial_\mu\mathcal{K}^B\right)\,, \tag{5.2.5b}$$

$$\left[\mathbb{Q}_\alpha, \mathcal{J}_\mu\right] = -2(\sigma_{\mu\nu})_\alpha{}^\beta\partial^\nu\mathcal{K}^F_\beta\,, \qquad \left[\overline{\mathbb{Q}}_{\dot\alpha}, \mathcal{J}_\mu\right] = 2\left(\overline{\sigma}_{\mu\nu}\right)_{\dot\alpha\dot\beta}\partial^\nu\overline{\mathcal{K}}^{F\dot\beta}\,. \tag{5.2.5c}$$

At first order, the interaction of the fields in the vector multiplet $\mathcal{V}$ with matter proceeds via the following universal couplings to the operators in the current multiplet $\mathcal{J}$,[5]

$$\mathscr{L}_{\text{int}} = -\mathcal{A}^\mu\mathcal{J}_\mu - i\Lambda\mathcal{K}^F + i\overline{\Lambda}\,\overline{\mathcal{K}}^F - \mathcal{D}\mathcal{K}^B + (\text{higher order})\,. \tag{5.2.6}$$

The higher-order terms are required by gauge invariance and supersymmetry.

In general, the current multiplet $\mathcal{J}$ encodes all couplings of the gauge theory to charged matter, as well as possible self-interactions due to higher-derivative terms, such as those indicated in (5.2.3). For simplicity, we will take all matter fields to reside in massless chiral multiplets. Most of the results below only rely on general properties of the current multiplet $\mathcal{J}$, e.g. its supersymmetry transformations (5.2.5), but do not depend on the detailed form of the interaction terms. Nevertheless, it is helpful to keep in mind the simplest theory in this class, which consists of a single massless, minimally coupled chiral multiplet of charge q, with canonical kinetic terms and no superpotential or

[5] For instance, this means that $\mathcal{J}_\mu(x) = -\dfrac{\delta S_{\text{int}}}{\delta\mathcal{A}^\mu(x)}$, where $S_{\text{int}} = \int d^4x\,\mathscr{L}_{\text{int}}$.

higher-derivative interactions.[6] In this theory, the operators in the current multiplet $\mathcal{J}$ are given by

$$\mathcal{K}^B = Q\overline{\Phi}\Phi \,, \tag{5.2.7a}$$

$$\mathcal{K}^F_\alpha = Q\sqrt{2}\,\overline{\Phi}\Psi_\alpha \,, \qquad \overline{\mathcal{K}}^F_{\dot\alpha} = Q\sqrt{2}\,\Phi\overline{\Psi}_{\dot\alpha} \,, \tag{5.2.7b}$$

$$\mathcal{J}_\mu = Q\left(i\overline{\Phi}\overleftrightarrow{D}_\mu\Phi + \overline{\Psi}\,\overline{\sigma}_\mu\Psi\right) \,. \tag{5.2.7c}$$

Here Φ, Ψ_α are the propagating component fields in the chiral multiplet (their Hermitian conjugates $\overline{\Phi}$, $\overline{\Psi}_{\dot\alpha}$ reside in an anti-chiral multiplet) and $D_\mu = \partial_\mu - iQ\mathcal{A}_\mu$ is the gauge-covariant derivative. In our conventions, the electric charge Q_{el} is given by

$$Q_{\text{el}} = \int d^3x\,\mathcal{J}_0 \,, \tag{5.2.8}$$

and the statement that Φ, Ψ_α both have charge Q means that

$$[Q_{\text{el}}, \Phi(x)] = -Q\Phi(x) \,, \qquad [Q_{\text{el}}, \Psi_\alpha(x)] = -Q\Psi_\alpha(x) \,. \tag{5.2.9}$$

This implies that a state $\Phi(x)|0\rangle$ has charge $-Q$.

5.2.2 Soft Photino Theorem

In this section, we derive the soft-photino theorem following very closely the derivation of the soft-photon theorem in §3.6. In the supersymmetric case, we can study scattering amplitudes $\mathscr{A}^{\text{out},+}_{n+1}$ involving an outgoing photino Λ of momentum q and positive helicity, as well as n other hard particles,

$$\mathscr{A}^{\text{out},+}_{n+1} = \left\langle \text{out}\middle| a_{\Lambda,+}(q)\,\mathcal{S}\,\middle|\text{in}\right\rangle \,. \tag{5.2.10}$$

[6] This theory is quantum mechanically anomalous. The anomaly can be cancelled by including additional chiral multiplets with suitable $U(1)$ charge assignments.

In order for the amplitude to be non-zero, the total number of fermions involved in the scattering process (including the photino) must be even. We are interested in the leading behavior of this amplitude when the photino momentum is taken to zero, $q \to 0$. As in §3.6, this arises from single insertions of the interaction terms $-i\Lambda\mathcal{K}^F + i\overline{\Lambda}\,\overline{\mathcal{K}}^F \subset \mathscr{L}_{\text{int}}$ in (5.2.6) that attach only to external lines. For a positive-helicity photino, insertions of $-i\Lambda\mathcal{K}^F$ do not contribute, since $\langle\, 0\,| a_{\Lambda,+}(q)\Lambda_\alpha(0)|\, 0\,\rangle = 0$. Therefore, the amplitude obeys the following soft theorem,

$$\begin{aligned}\mathscr{A}^{\text{out},+}_{n+1} \quad &\longrightarrow \quad -\langle\, 0\,|a_{\Lambda,+}(q)\overline{\Lambda}_{\dot\alpha}(0)|\, 0\,\rangle \times \\ &\sum_{f,s}\sum_{i=1}^{n}\frac{-i}{2p_i\cdot q}(-1)^{\sigma_i}\langle\, i\,|\overline{\mathcal{K}}^{F\dot\alpha}(0)|\, f, p_i, s\,\rangle\,\big\langle 1\,;\,\ldots\,;\, f, p_i, s\,;\,\ldots\,; n\big|\,\mathcal{S}\,\big|\text{in}\big\rangle\ .\end{aligned} \tag{5.2.11}$$

Here $(-1)^{\sigma_i}$ is a fermion sign factor that comes from anti-commuting $\overline{\mathcal{K}}^{F\dot\alpha}$ across multi-particle states.[7]

The photino wavefunction is given by (see §2.5)

$$\langle\, 0\,|a_{\Lambda,+}(q)\overline{\Lambda}_{\dot\alpha}(0)|\, 0\,\rangle = e\overline{\eta}_{\dot\alpha}(q)\ . \tag{5.2.12}$$

We must now evaluate the matrix elements of the fermionic operator $\overline{\mathcal{K}}^{F\dot\alpha}$ between single-particle states, in the forward limit. In general, the matrix elements of such an operator may be model-dependent. However, $\overline{\mathcal{K}}^{F\dot\alpha}$ resides in the same supermultiplet (5.2.4) as the conserved electric current $\mathcal{J}_\mu$, whose forward matrix elements are universal, as discussed around (3.6.15). Explicitly, we can evaluate the following commutation relation from (5.2.5),

$$\left\{\mathbb{Q}_\alpha, \overline{\mathcal{K}}^F_{\dot\alpha}\right\} = i\sigma^\mu_{\alpha\dot\alpha}\left(\mathcal{J}_\mu - i\partial_\mu\mathcal{K}^B\right)\ , \tag{5.2.13}$$

[7] For a state $\langle 1\,;\,\ldots\,;\, i\,;\,\ldots\,;\, n|$ we define σ_i to be the number of fermionic states in positions $i+1$ through n.

between single-particle states in the forward limit, where we can drop the total derivative $\partial_\mu \mathcal{K}^B$. Using (3.6.15) then leads to

$$\langle\, f, p, s\,|\big\{\mathbb{Q}_\alpha, \overline{\mathcal{K}}^F_{\dot\alpha}\big\}|\, f', p, s'\,\rangle = -2iQ_f\, p_\mu \sigma^\mu_{\alpha\dot\alpha}\, \delta_{ff'}\delta_{ss'}\,. \tag{5.2.14}$$

The appearance of $\delta_{ff'}$ on the right-hand side shows that only single-particle states that reside in the same supermultiplet can can lead to non-vanishing matrix elements for $\overline{\mathcal{K}}^F_{\dot\alpha}$. When the supercharges act on the left or the right, they lead to other states in this supermultiplet, in a way that is completely determined by representation theory. This can be used to derive all matrix elements of $\overline{\mathcal{K}}^F_{\dot\alpha}$ between massless or massive single-particle states of arbitrary spin.

Here we explicitly work this out for a massless chiral multiplet Φ, Ψ_α of charge Q, and its conjugate anti-chiral multiplet $\overline{\Phi}$, $\overline{\Psi}_{\dot\alpha}$ of charge $-Q$. The relevant single-particle states are

$$|\overline{\Phi}, p\rangle\,, \quad |\overline{\Psi}, p, -\rangle \qquad \text{and} \qquad |\Phi, p\rangle\,, \quad |\Psi, p, +\rangle\,. \tag{5.2.15}$$

On these states, the supersymmetry algebra is represented as follows,[8]

$$\begin{aligned} &\overline{\mathbb{Q}}_{\dot\alpha}|\overline{\Phi}, p\rangle = 0\,, && \mathbb{Q}_\alpha|\overline{\Phi}, p\rangle = \sqrt{2}i\,\eta_\alpha(p)|\overline{\Psi}, p, -\rangle\,, \\ &\overline{\mathbb{Q}}_{\dot\alpha}|\overline{\Psi}, p, -\rangle = -\sqrt{2}i\,\overline{\eta}_{\dot\alpha}(p)|\overline{\Phi}, p\rangle\,, && \mathbb{Q}_\alpha|\overline{\Psi}, p, -\rangle = 0\,. \end{aligned} \tag{5.2.16}$$

The action of the supercharges on the conjugate anti-chiral states is obtained by exchanging $\mathbb{Q}_\alpha \leftrightarrow \overline{\mathbb{Q}}_{\dot\alpha}$, $|\Phi, p\rangle \leftrightarrow |\overline{\Phi}, p\rangle$, $\eta_\alpha(p) \leftrightarrow \overline{\eta}_{\dot\alpha}(p)$, and $|\Psi, p, -\rangle \leftrightarrow |\overline{\Psi}, p, +\rangle$.

We can now implement the procedure described after (5.2.14) to obtain all non-vanishing matrix

[8] This follows from the non-vanishing commutation relations for a free chiral multiplet, $[\mathbb{Q}_\alpha, \Phi] = i\sqrt{2}\Psi_\alpha$ and $\{\overline{\mathbb{Q}}_{\dot\alpha}, \Psi_\alpha\} = \sqrt{2}\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\Phi$.

elements of $\overline{\mathcal{K}}^F_{\dot\alpha}$,

$$\langle \overline{\Phi}, p | \overline{\mathcal{K}}^F_{\dot\alpha}(0) | \overline{\Psi}, p, - \rangle = \langle \Psi, p, + | \overline{\mathcal{K}}^F_{\dot\alpha}(0) | \Phi, p \rangle = \sqrt{2} Q \overline{\eta}_{\dot\alpha}(p) \,. \tag{5.2.17}$$

Substituting into (5.2.11), we obtain the final form of the soft photino theorem,

$$\mathscr{A}^{\text{out},+}_{n+1} \quad \longrightarrow \quad \sqrt{2} i e \sum_{i=1}^{n} \frac{Q_i}{\eta(q)\eta(p_i)} \left(\mathscr{F}_i \mathscr{A}_n \right) \,. \tag{5.2.18}$$

Here the Q_i are the electric charges of the asymptotic states. The n-particle amplitude $\mathscr{A}_n$ is obtained from $\mathscr{A}^{\text{out},+}_{n+1}$ by deleting the photino, but since it has an odd number of fermion external states, it vanishes. The non-vanishing n-point amplitude $\mathscr{F}_i \mathscr{A}_n$ is obtained from $\mathscr{A}_n$ by acting on the ith single-particle state with a fermionic operator $\mathscr{F}$, which satisfies

$$\begin{aligned} \langle \overline{\Phi}, p | \mathscr{F} &= -\langle \overline{\Psi}, p, - | \,, \qquad \langle \Psi, p, + | \mathscr{F} = \langle \Phi, p | \,, \\ \mathscr{F} | \Phi, p \rangle &= | \Psi, p, + \rangle \,, \qquad \mathscr{F} | \overline{\Psi}, p, - \rangle = -| \overline{\Phi}, p \rangle \,. \end{aligned} \tag{5.2.19}$$

The action of $\mathscr{F}$ on all other single-particle states vanishes, and we take $\mathscr{F}$ to act from the right on out states and from the left on in states. Since $\mathscr{F}$ is a fermionic operator, it picks up a sign whenever it moves past another fermionic operator or state. This accounts for the factors $(-1)^{\sigma_i}$ in (5.2.11).

So far we have only discussed an outgoing soft photino of positive helicity. The negative helicity case can similarly be shown to satisfy

$$\mathscr{A}^{\text{out},-}_{n+1} \longrightarrow -\sqrt{2} i e \sum_{i=1}^{n} \frac{Q_i}{\overline{\eta}(q)\overline{\eta}(p_i)} \left(\mathscr{F}^\dagger_i \mathscr{A}_n \right) \,, \tag{5.2.20}$$

where the fermionic operator $\mathscr{F}^\dagger$ is the Hermitian conjugate of the operator $\mathscr{F}$ defined in (5.2.19). Finally, the soft theorems for ingoing photini can be obtained from (5.2.18) and (5.2.20) by crossing symmetry.

5.3 Asymptotic Symmetries

5.3.1 Photino Asymptotics

The photino Λ_α is a left-handed spinor that transforms as $(\frac{1}{2}, 0)$. We have performed the boundary analysis for such a spinor in §2.4.3. The boundary data is $\lambda_{(+)}$. The leading constraint equation when it is coupled to a current is given by (2.4.79)

$$\partial_u \lambda_{(-)} = E_-^{\overline{z}} D_{\overline{z}} \lambda_{(+)} - \frac{e^2}{2} \overline{k}_{(-)} \,. \tag{5.3.1}$$

where we have denoted $\lambda_{(-)} = \hat{\Lambda}^{(2)}_{(-)}$. As for the bosonic current, we assume that the fermionic current vanishes at large u.

We would now like to know how supersymmetry relates the fermionic boundary fields $\lambda_{(+)}$ and $\overline{k}_{(-)}$ to the bosonic boundary fields A_z and j_u. Even though all four supercharges remain unbroken at null infinity, we will focus on supersymmetry transformations with constant spinor parameter $\xi^{(-)}_\alpha$ and their complex conjugates, which are generated by the following supercharges

$$\mathcal{Q}^{(-)} = \xi^{(-)\alpha} \mathbb{Q}_\alpha \,, \qquad \overline{\mathcal{Q}}^{(+)} = \overline{\xi}^{(+)\dot{\alpha}} \overline{\mathbb{Q}}_{\dot{\alpha}} \,, \qquad \left\{ \mathcal{Q}^{(-)}, \overline{\mathcal{Q}}^{(+)} \right\} = -4i\partial_u \,. \tag{5.3.2}$$

They are the position space analogues of the supercharges that act non-trivially on massless particle representations, as in (5.2.16). The only non-vanishing commutators of $\mathcal{Q}^{(-)}$ with the boundary photon field A_z and $A_r^{(2)}$ are given by

$$\left[\mathcal{Q}^{(-)}, E_-^{\overline{z}} A_{\overline{z}} \right] = \overline{\lambda}_{(-)} \,, \qquad \left[\mathcal{Q}^{(-)}, A_r^{(2)} \right] = -\overline{\lambda}_{(+)} \,, \tag{5.3.3a}$$

while the only non-vanishing anti-commutators of $\mathcal{Q}$ with the boundary photino $\lambda_{(\pm)}$ are

$$\big\{\mathcal{Q}^{(-)}, \lambda_{(+)}\big\} = 4iE_+^z \partial_u A_z \;, \qquad \big\{\mathcal{Q}^{(-)}, \lambda_{(-)}\big\} = -D - i\big(F_{ur} - \gamma^{z\bar{z}} F_{z\bar{z}}\big) \,. \tag{5.3.4a}$$

The action of $\overline{\mathcal{Q}}^{(+)}$ on these fields can be obtained by taking the Hermitian conjugates of these formulas. The auxiliary field $\mathcal{D}$ in the vector multiplet (5.2.1) is a Lorentz scalar with IR scaling dimension $\Delta_{\mathcal{D}} = 2$, which according to (2.4.65) falls off like $\mathcal{D} = \frac{D}{r^2} + \mathcal{O}(r^{-3})$. The fact that there are no residual powers of r in these formulas shows that the assumed large-r falloffs are consistent with supersymmetry.

It is straightforward to repeat the preceding discussion near $\mathscr{I}^-$. The photino fields on $\mathscr{I}^+$ and $\mathscr{I}^-$ must then be matched at spatial infinity. The appropriate matching conditions can be determined from the matching conditions (3.4.2) for the photon using supersymmetry. Combining the supersymmetry variation in (5.3.3a) with the matching condition for A_z in (3.4.2) leads to

$$\lambda_{(+)}\big|_{\mathscr{I}^+_-} = \widetilde{\lambda}_{(+)}\big|_{\mathscr{I}^-_+} \,. \tag{5.3.5}$$

Similarly, the supersymmetry variation in (5.3.4a) and the matching condition for F_{ur} in (3.4.4) imply that the u-independent part of $\lambda_{(-)}$ should be matched across spatial infinity. However, the constraint equation (5.3.1) implies that $\lambda_{(-)}\big|_{\mathscr{I}^+_-}$ does not exist, since

$$\lambda_{(-)} \to u \left(E_-^{\bar{z}} D_{\bar{z}} \lambda_{(+)}\big|_{\mathscr{I}^+_-} \right) + \ell(z,\bar{z}) \qquad \text{as} \qquad u \to -\infty \,. \tag{5.3.6}$$

Instead, we should match the u-independent term across spatial infinity, $\ell(z,\bar{z}) = \ell^-(z,\bar{z})$, which can be expressed in terms of $\lambda_{(-)}$ as follows,

$$\left(1 - u\partial_u\right) \lambda_{(-)}\big|_{\mathscr{I}^+_-} = \left(1 - v\partial_v\right) \widetilde{\lambda}_{(-)}\big|_{\mathscr{I}^-_+} \,. \tag{5.3.7}$$

5.3.2 Fermionic Asymptotic Symmetries

Consider the following fermionic charges on $\mathscr{I}^+$ and $\mathscr{I}^-$, for any complex-valued $\chi(z,\bar{z})$,

$$\begin{aligned}\mathscr{F}_\chi &= \frac{1}{e^2}\int d^2z\gamma_{z\bar{z}}\,\chi(z,\bar{z})\,(1-u\partial_u)\,\lambda_{(-)}\big|_{\mathscr{I}^+_-}\,,\\ \mathscr{F}^-_\chi &= \frac{1}{e^2}\int d^2z\gamma_{z\bar{z}}\,\chi(z,\bar{z})\,(1-v\partial_v)\,\widetilde{\lambda}_{(-)}\big|_{\mathscr{I}^-_+}\,.\end{aligned} \tag{5.3.8}$$

We can express them in a more covariant form by introducing a commuting, chiral spinor-valued function on S^2,

$$\chi_\alpha(z,\bar{z}) = \chi(z,\bar{z})\xi^{(+)}_\alpha(z)\,. \tag{5.3.9}$$

We can then write the charge as

$$\mathscr{F}_\chi = -\frac{1}{e^2}\int d^2z\gamma_{z\bar{z}}\,\chi^\alpha(z,\bar{z})\left[(1-u\partial_u)\left(\lim_{r\to\infty} r^2\Lambda_\alpha(u,r,z,\bar{z})\right)\right]\Big|_{\mathscr{I}^+_-}\,, \tag{5.3.10}$$

and similarly for $\mathscr{F}^-_\chi$. Comparing the matching condition (5.3.7) to (5.3.8) implies the conservation law

$$\mathscr{F}_\chi = \mathscr{F}^-_\chi\,, \tag{5.3.11}$$

and hence a Ward identity for the tree-level $\mathcal{S}$-matrix,

$$\mathscr{F}_\chi\mathcal{S} - \mathcal{S}\mathscr{F}^-_\chi = 0\,. \tag{5.3.12}$$

In section 5.4 we will show that this identity gives rise to the positive-helicity soft photino theorem (5.2.18); the Hermitian conjugate charges $\mathscr{F}^\dagger_\chi$ lead to the negative-helicity case (5.2.20). In the remainder of this section we establish several basic properties of $\mathscr{F}_\chi$.

Supersymmetry relates the fermionic symmetries $\mathscr{F}_\chi$ defined in (5.3.8) to the bosonic asymptotic symmetries Q_ε in Chapter 3. For instance, we can use (5.3.4a) to determine the anti-commutators of the supercharges $\mathcal{Q}$, $\overline{\mathcal{Q}}$ that were singled out in (5.3.2) with $\mathscr{F}_\chi$,[9]

$$\left\{\mathcal{Q}^{(-)}, \mathscr{F}_\chi\right\} = -\frac{i}{e^2}\int d^2z\gamma_{z\bar{z}}\chi(z,\bar{z})\, F_{ur}\big|_{\mathscr{I}^+_-} = iQ_\chi\,, \qquad \left\{\overline{\mathcal{Q}}^{(+)}, \mathscr{F}_\chi\right\} = 0\,. \tag{5.3.13}$$

Note that the fermionic symmetry $\mathscr{F}_\chi$, with complex parameter $\chi(z,\bar{z})$, transforms into the bosonic symmetry Q_ε with the same parameter, $\varepsilon(z,\bar{z}) = \chi(z,\bar{z})$. This shows that it is natural to allow complex $\varepsilon(z,\bar{z})$, as was discussed in [2, 28, 86]. More generally, we can use (5.2.2) and (5.3.10) to express the commutator of an arbitrary supercharge with $\mathscr{F}_\chi$ in the covariant form quoted in (5.1.1),

$$\left\{\zeta^\alpha \mathbb{Q}_\alpha, \mathscr{F}_\chi\right\} = iQ\left[\zeta^\alpha\chi_\alpha\right]\,, \qquad \left\{\overline{\mathbb{Q}}_{\dot{\alpha}}, \mathscr{F}_\chi\right\} = 0\,. \tag{5.3.14}$$

Here ζ_α is a commuting, constant spinor and $\chi_\alpha(z,\bar{z})$ was defined in (5.3.9). It can similarly be shown that the bosonic charges Q_ε in Chapter 3 are annihilated by all supercharges. This is expected from their interpretation as conventional gauge transformations that do not vanish at $\mathscr{I}^+$, since the latter commute with supersymmetry.

Following the discussion of the bosonic case around (3.2.12), we can express $\mathscr{F}_\chi$ as an integral over $\mathscr{I}^+$ and use the constraint equation (5.3.1) to write it as a sum of hard and soft contributions,

$$\mathscr{F}_\chi = \mathscr{F}^{\mathrm{h}}_\chi + \mathscr{F}^s_\chi\,. \tag{5.3.15}$$

[9] Here we use the fact that $D|_{\mathscr{I}^+_\pm} = F_{z\bar{z}}|_{\mathscr{I}^+_\pm} = 0$. The first equation is obtained by solving for the auxiliary field $\mathcal{D}$ in (5.2.1) in terms of the bosonic source $\mathcal{K}^B$ in (5.2.4), which is assumed to vanish at $\mathscr{I}^+_\pm$. The second equation follows from the fact that there are no magnetic charges.

The hard charge is given by

$$\mathscr{F}_\chi^{\mathrm{h}} = \frac{1}{2}\int dud^2z\gamma_{z\bar{z}}\chi(z,\bar{z})u\partial_u\bar{k}_{(-)} + \frac{1}{e^2}\int d^2z\gamma_{z\bar{z}}\chi(z,\bar{z})\left(1-u\partial_u\right)\lambda_{(-)}\big|_{\mathscr{I}_+^+}. \tag{5.3.16}$$

Since we are considering theories without massive particles, we set $\left(1-u\partial_u\right)\lambda_{(-)}\big|_{\mathscr{I}_+^+} = 0$.[10] The supersymmetry transformation (5.3.4a) turns this condition into $F_{ur}\big|_{\mathscr{I}_+^+} = 0$, which was imposed also in the bosonic case in (3.2.3). We can compute the following anti-commutators with the supercharges singled out in (5.3.2),

$$\left\{\mathcal{Q}^{(-)},\mathscr{F}_\chi^{\mathrm{h}}\right\} = i\int dud^2z\gamma_{z\bar{z}}\chi(z,\bar{z})j_u = iQ^{\mathrm{h}}[\chi]\,, \qquad \left\{\overline{\mathcal{Q}}^{(+)},\mathscr{F}_\chi^{\mathrm{h}}\right\} = 0\,, \tag{5.3.17}$$

up to boundary terms at $\mathscr{I}_\pm^+$ that involve the sources and hence vanish by assumption. In section 5.4 we will use these relations to determine the action of the hard charges $\mathscr{F}_\chi^{\mathrm{h}}$ on asymptotic scattering states.

The soft charges in (5.3.15) are given by

$$\mathscr{F}_\chi^{\mathrm{s}} = \frac{1}{2\pi}\int d^2z\gamma_{z\bar{z}}E_-^{\bar{z}}\partial_{\bar{z}}\chi(z,\bar{z})\,\omega_{(+)}^{\mathrm{s}}\,, \qquad \omega_{(+)}^{\mathrm{s}} = \frac{\pi}{e^2}\int du\,u\partial_u\lambda_{(+)}\,. \tag{5.3.18}$$

Here we have defined a soft photino current ω^{s}.[11] Under a Lorentz transformation, it changes as follows,

$$\delta_Y\omega_{(+)}^{\mathrm{s}} = \left(\frac{1}{2}D_zY^z + Y^z\partial_z + Y^{\bar{z}}\partial_{\bar{z}}\right)\omega_{(+)}^{\mathrm{s}}\,, \tag{5.3.19}$$

up to boundary terms that vanish as long as $\lambda_{(+)}$ asymptotes to a u-independent function of $z,\bar{z}$

[10] Following [31, 132], it should be possible to incorporate massive particles by appropriately taking into account their semiclassical photino field as they pass through timelike infinity.

[11] Note that the operator $\int du\,\partial_u\lambda_{(+)}$, which is similar to soft photon current j_z^{s} defined above (5.3.20), can be shown to vanish inside $\mathcal{S}$-matrix elements by expressing it in terms of creation and annihilation operators and comparing to the soft photino theorem (5.2.18).

sufficiently rapidly at $\mathscr{I}^+_\pm$.[12] The Lorentz transformation (5.3.19) shows that the soft photino current ω^{s} is a two-dimensional field with $SL(2,\mathbb{C})$ conformal weights $h = \frac{1}{2}$ and $\overline{h} = 0$, i.e. it is a left-moving spin-$\frac{1}{2}$ current. Under the supercharges in (5.3.2), the soft photino current ω^{s} transforms into the soft photon current $j^{\mathrm{s}}_z = -\frac{4\pi}{e^2}N_z$ as follows,

$$\left\{\mathcal{Q}^{(-)}, \omega^{\mathrm{s}}_{(+)}\right\} = iE^z_+ j^{\mathrm{s}}_z\,, \qquad \left\{\overline{\mathcal{Q}}^{(+)}, \omega^{\mathrm{s}}_{(+)}\right\} = 0\,. \tag{5.3.20}$$

In order to understand the action of the soft charges (5.3.18) on the photino, it is convenient to rewrite them as follows,[13]

$$\mathscr{F}^{\mathrm{s}}_\chi = -\frac{1}{2e^2}\lim_{\omega\to 0}\int du d^2z\,\partial_{\overline{z}}\chi(z,\overline{z})\,\cos(\omega u)\lambda_{(+)}\,. \tag{5.3.22}$$

In terms of creation and annihilation operators (see §2.5.22),

$$\mathscr{F}^{\mathrm{s}}_\chi = \frac{i}{4\sqrt{2}e\pi}\lim_{\omega\to 0}\sqrt{\omega}\int d^2z\gamma_{z\overline{z}}E^{\overline{z}}_-\partial_{\overline{z}}\chi(z,\overline{z})\,\left(a_{\Lambda,+}(\omega,z,\overline{z}) - a^\dagger_{\Lambda,-}(\omega,z,\overline{z})\right)\,. \tag{5.3.23}$$

This shows that $\mathscr{F}^{\mathrm{s}}_\chi$ acts on zero-momentum photini. Using this expression, as well as the mode expansion for $\lambda_{(+)}$ and the anti-commutation relations for creation and annihilation operators, it can be checked that

$$\left\{\mathscr{F}^{\mathrm{s}}_\chi, \lambda_{(+)}(u,z,\overline{z})\right\} = 0\,, \qquad \left\{\mathscr{F}^{\mathrm{s}}_\chi, \overline{\lambda}_{(-)}(u,z,\overline{z})\right\} = -E^{\overline{z}}_-\partial_{\overline{z}}\chi(z,\overline{z})\,. \tag{5.3.24}$$

Thus, $\overline{\lambda}_{(+)}$ shifts inhomogeneously whenever $\partial_{\overline{z}}\chi(z,\overline{z}) \neq 0$. Just as in the bosonic case, we in-

---

[12] It is sufficient to assume that $\lambda_{(+)} = \lambda_{(+)}\big|_{\mathscr{I}^+_\pm} + \mathcal{O}\big(|u|^{-(1+\delta)}\big)$, with $\delta > 0$, as $u \to \pm\infty$.

[13] Given a function $f(u)$ such that $\lim\limits_{u\to\pm\infty} f(u)$ exists, but is nonzero, we have the following identity,

$$\int_{-\infty}^{\infty} du\, u f'(u) = -\lim_{\omega\to 0}\int_{-\infty}^{\infty} du\,\cos(\omega u) f(u)\,, \tag{5.3.21}$$

which amounts to integrating by parts but dropping the divergent boundary terms.

terpret this as spontaneous breaking of the corresponding charges $\mathscr{F}_\chi^s$. The u-independent part of $\overline{\lambda}_{(-)}$ furnishes the corresponding Goldstone fermions. Similar comments apply to $\mathscr{F}_\chi^\dagger$, which shifts $\lambda_{(+)}$ by $-E_+^z \partial_z \overline{\chi}(z, \overline{z})$.

5.4 Soft Photino Theorem from Asymptotic Fermionic Symmetries

5.4.1 Fermionic Ward Identity for Scattering Amplitudes

In the previous section we argued for the existence of a fermionic asymptotic symmetry $\mathscr{F}_\chi$, which is classically conserved (see (5.3.11)) and hence leads to a Ward identity (5.3.12) for the tree-level $\mathcal{S}$-matrix,

$$\mathscr{F}_\chi \mathcal{S} - \mathcal{S} \mathscr{F}_\chi^- = 0\,. \tag{5.4.1}$$

We will now show that this Ward identity is nothing but the soft photino theorem for the case of an outgoing positive-helicity photino (equivalently, by crossing symmetry, an ingoing negative helicity photino), which we repeat for convenience,

$$\mathscr{A}_{n+1}^{\text{out},+} \quad \longrightarrow \quad \sqrt{2} ie \sum_{i=1}^{n} \frac{Q_i}{\eta(q)\eta(p_i)} \left(\mathscr{F}_i \mathscr{A}_n\right)\,. \tag{5.4.2}$$

Here $q \to 0$ is the momentum of the soft photino. Analogously, the Ward identity for $\mathscr{F}_\chi^\dagger$ leads to the soft photino theorem (5.2.20) for an outgoing negative helicity photino.

We begin by translating (5.4.2) from momentum to position space. As in (3.6.17), we can express the null momenta p_i in terms of variables $\omega_i, z_i, \overline{z}_i$ and q in terms of $\omega, z, \overline{z}$. In particular, the spinor-helicity variables corresponding to the p_i are given by (2.4.52), so that

$$\eta^\alpha(q)\eta_\alpha(p_i) = \sqrt{\frac{(2\omega)(2\omega_i)}{(1+z\overline{z})(1+z_i\overline{z}_i)}}\,(z-z_i)\ , \qquad (i=1,\ldots,n)\,. \tag{5.4.3}$$

In this parametrization, the soft photino theorem can be written as follows,

$$\sqrt{2\omega}\mathscr{A}^{\text{out},+}_{n+1} \quad \longrightarrow \quad ie\sqrt{1+z\overline{z}}\sum_{i=1}^{n}\frac{\sqrt{1+z_i\overline{z}_i}}{\sqrt{\omega_i}}\frac{Q_i}{z-z_i}\,(\mathscr{F}_i\mathscr{A}_n)\ . \tag{5.4.4}$$

In order to reproduce this result, we take the matrix element of the Ward identity (5.4.1) between an n-particle out-state $\langle\, 1\,;\,\ldots\,;\,n\,|$ and the in state $|\,0\,\rangle$. All in- and outgoing particles (some of which could be photini) are hard, i.e. they have non-vanishing momenta. Writing $\mathscr{F}_\chi = \mathscr{F}^{\text{h}}_\chi + \mathscr{F}^{\text{s}}_\chi$ as a sum of hard and soft contributions, as in (5.3.15), and similarly for $\mathscr{F}^-_\chi$, we obtain

$$\langle\,\text{out}\,|\,\mathscr{F}^{\text{s}}_\chi\mathcal{S} - \mathcal{S}\mathscr{F}^{\text{s}-}[\chi]\,|\,\text{in}\,\rangle = -\langle\,\text{out}\,|\,\mathscr{F}^{\text{h}}_\chi\mathcal{S} - \mathcal{S}\mathscr{F}^{\text{h}-}[\chi]\,|\,\text{in}\,\rangle\ . \tag{5.4.5}$$

To proceed, we need to know the action of the soft and hard charges on asymptotic scattering states. The soft charge was expressed in terms of photino creation and annihilation operators in (5.3.23). It creates an outgoing positive-helicity photino and an ingoing negative-helicity photino of zero momentum. Crossing symmetry implies that these two contributions lead to identical $\mathcal{S}$-matrix elements, so that we can write the left-hand side of (5.4.5) as the $\omega\to 0$ limit of

$$\frac{i\sqrt{\omega}}{\sqrt{2}e\pi}\int d^2w\gamma_{w\overline{w}}E^{\overline{w}}_-\,\partial_{\overline{w}}\chi(w,\overline{w})\,\langle\,\text{out};\Lambda,p(\omega,w,\overline{w})\,|\mathcal{S}\big|\text{in}\big\rangle\ . \tag{5.4.6}$$

The action of the hard charges on asymptotic states will be derived section 5.4.2 below, where it is shown that

$$\begin{aligned}\mathscr{F}^{\text{h}}_\chi\big|f,p(\omega,z,\overline{z}),s\big\rangle &= -\frac{Q_f}{2\sqrt{\omega}}\chi(z,\overline{z})\mathscr{F}\big|f,p(\omega,z,\overline{z}),s\big\rangle\ ,\\ \mathscr{F}^{\text{h}\dagger}_\chi\big|f,p(\omega,z,\overline{z}),s\big\rangle &= -\frac{Q_f}{2\sqrt{\omega}}\overline{\chi}(z,\overline{z})\mathscr{F}^\dagger\big|f,p(\omega,z,\overline{z}),s\big\rangle\ .\end{aligned} \tag{5.4.7}$$

Here Q_f is the electric charge of the state labeled by $f \in \{\Phi, \overline{\Phi}, \Psi, \overline{\Psi}\}$. The operator $\mathscr{F}$ and its Hermitian conjugate $\mathscr{F}^\dagger$ appear in the soft theorem (5.4.2). Its action on chiral and anti-chiral matter states was defined in (5.2.19).

If we choose $\chi(w, \overline{w}) = \frac{1}{z-w}$, the Ward identity collapses to the soft theorem (5.4.2). As in the bosonic case [1], the argument can be reversed to deduce the Ward identity – and hence the underlying symmetries – from the soft theorem, which establishes their equivalence.

5.4.2 Action of the Fermionic Charges on Matter Fields

Here we show that the action of the hard fermionic charges $\mathscr{F}^{\text{h}}_\chi$ on asymptotic states is given by (5.4.7), thereby completing the argument of section 5.4.1. We will do this by using the supersymmetry relations (5.3.17),

$$\left\{\mathcal{Q}, \mathscr{F}^{\text{h}}_\chi\right\} = iQ^{\text{h}}_\chi\,, \qquad \left\{\overline{\mathcal{Q}}, \mathscr{F}^{\text{h}}_\chi\right\} = 0\,. \tag{5.4.8}$$

Here Q^{h}_χ are the hard bosonic charges, whose action on boundary fields $f_q(u, z, \overline{z})$ of electric charge q is given by

$$\left[Q_\varepsilon, f_Q(u, z, \overline{z})\right] = -Q\varepsilon(z, \overline{z}) f_Q(u, z, \overline{z})\,. \tag{5.4.9}$$

Given the action of the supercharges $\mathcal{Q}^{(-)}, \overline{\mathcal{Q}}^{(+)}$ on charged boundary fields, we can extract the action of $\mathscr{F}^{\text{h}}_\chi$ on such fields from (5.4.8) and (5.4.9). The same logic was used in section 5.2.2 to relate the matrix elements of the fermionic source $\overline{\mathcal{K}}^F_{\dot{\alpha}}$ to those of the electric current $\mathcal{J}_\mu$.

For our present purposes, all charged fields reside in massless chiral or anti-chiral multiplets. A chiral multiplet consists of a complex scalar Φ and left-handed spinor Ψ_α whose boundary data are

ϕ and $\psi_{(+)}$ respectively. If the chiral multiplet has charge Q, then so do the boundary fields ϕ and ψ, i.e.

$$\begin{aligned}\left[Q^{\mathrm{h}}_{\varepsilon}, \phi(u,z,\bar{z})\right] &= -Q\varepsilon(z,\bar{z})\phi(u,z,\bar{z})\,,\\ \left[Q^{\mathrm{h}}_{\varepsilon}, \psi_{(+)}(u,z,\bar{z})\right] &= -Q\varepsilon(z,\bar{z})\psi_{(+)}(u,z,\bar{z})\,.\end{aligned} \tag{5.4.10}$$

Given the asymptotic expansions in Chapter 2, we obtain the following transformation rules for the boundary fields ϕ, $\psi_{(+)}$ under the supercharges $\mathcal{Q}^{(-)}$, $\overline{\mathcal{Q}}^{(+)}$ singled out in (5.3.2),

$$\left[\mathcal{Q}^{(-)}, \phi\right] = \sqrt{2}i\psi_{(+)}\,, \qquad \left[\overline{\mathcal{Q}}^{(+)}, \phi(u,z,\bar{z})\right] = 0\,, \tag{5.4.11a}$$

$$\left\{\mathcal{Q}^{(-)}, \psi_{(+)}(u,z,\bar{z})\right\} = 0\,, \qquad \left\{\overline{\mathcal{Q}}^{(+)}, \psi_{(+)}(u,z,\bar{z})\right\} = -2\sqrt{2}\partial_u\phi(u,z,\bar{z})\,. \tag{5.4.11b}$$

Given the transformation properties (5.4.10) and (5.4.11) of the chiral multiplet fields under the bosonic symmetry $Q^{\mathrm{h}}_{\varepsilon}$ and the supersymmetries $\mathcal{Q}^{(-)}$, $\overline{\mathcal{Q}}^{(+)}$, the commutators in (5.4.8) are only consistent if the fermionic symmetry $\mathscr{F}^{\mathrm{h}}_{\chi}$ acts as follows,

$$\left[\mathscr{F}^{\mathrm{h}}_{\chi}, \phi(u,z,\bar{z})\right] = 0\,, \qquad \left\{\mathscr{F}^{\mathrm{h}}_{\chi}, \psi_{(+)}(u,z,\bar{z})\right\} = -\frac{Q}{\sqrt{2}}\chi(z,\bar{z})\phi(u,z,\bar{z})\,. \tag{5.4.12}$$

The first commutator can be understood as a consequence of the $U(1)_R$ symmetry that is expected to emerge at the superconformal IR fixed point that governs the dynamics near null infinity. Since $\mathscr{F}_{\chi}$ is linear in the photino (see (5.3.8)), it has R-charge $+1$. (We take the R-charge of $\mathbb{Q}_{\alpha}$ to be -1.) The electric and $U(1)_R$ charges of the first commutator in (5.4.12) are not consistent with any fermionic field in the chiral multiplet, and hence it must vanish.

For the anti-chiral multiplet of charge $-Q$, we similarly find

$$\left\{\mathscr{F}^{\text{h}}_{\chi}, \overline{\psi}_{(-)}(u,z,\overline{z})\right\} = 0\,, \tag{5.4.13a}$$

$$\left[\mathscr{F}^{\text{h}}_{\chi}, \partial_u\overline{\phi}(u,z,\overline{z})\right] = -\frac{iQ}{2\sqrt{2}}\chi(z,\overline{z})\overline{\psi}_{(-)}(u,z,\overline{z})\,, \tag{5.4.13b}$$

$$\left\{\mathcal{Q}^{(-)}, \left[\mathscr{F}^{\text{h}}_{\chi}, \overline{\phi}(u,z,\overline{z})\right]\right\} = iQ\chi(z,\overline{z})\overline{\phi}(u,z,\overline{z})\,. \tag{5.4.13c}$$

As above, the first equation (5.4.13a) is due to the electric and $U(1)_R$ charges of the fields. While (5.4.13b) shows that $\partial_u\overline{\phi}$ has a local transformation rule, it follows from (5.4.13c) that this does not lead to a local transformation rule for $\overline{\phi}$ itself. If it did, then $\overline{\phi}$ would be $\mathcal{Q}$-exact, which is not the case because $\overline{\phi}$ is the bottom component of the supermultiplet in (5.4.11).

The (anti-) commutators in (5.4.12), (5.4.13a), and (5.4.13b) are sufficient to establish the action of $\mathscr{F}^{\text{h}}_{\chi}$ and $\mathscr{F}^{\text{h}\dagger}_{\overline{\chi}}$ on asymptotic states. Using the mode expansions in §2.5 and the fact that $\mathscr{F}^{\text{h}}_{\chi}$ annihilates the vacuum,[14] we find that

$$\begin{aligned}
\mathscr{F}^{\text{h}}_{\chi}\left|\overline{\Phi}, p(\omega,z,\overline{z})\right\rangle &= \mathscr{F}^{\text{h}}_{\chi}\left|\Psi, p(\omega,z,\overline{z}), +\right\rangle = 0\,,\\
\mathscr{F}^{\text{h}}_{\chi}\left|\overline{\Psi}, p(\omega,z,\overline{z}), -\right\rangle &= -\frac{Q}{2\sqrt{\omega}}\chi(z,\overline{z})\left|\overline{\Phi}, p(\omega,z,\overline{z})\right\rangle\,,\\
\mathscr{F}^{\text{h}}_{\chi}\left|\Phi, p(\omega,z,\overline{z})\right\rangle &= -\frac{Q}{2\sqrt{\omega}}\chi(z,\overline{z})\left|\Psi, p(\omega,z,\overline{z}), +\right\rangle\,.
\end{aligned} \tag{5.4.14}$$

We can express (5.4.14) in terms of the operator $\mathscr{F}$, whose action on asymptotic states was defined in (5.2.19),

$$\mathscr{F}|\overline{\Phi},p\rangle = \mathscr{F}|\Psi,p,+\rangle = 0\,, \qquad \mathscr{F}|\Phi,p\rangle = |\Psi,p,+\rangle\,, \qquad \mathscr{F}|\overline{\Psi},p,-\rangle = -|\overline{\Phi},p\rangle\,. \tag{5.4.15}$$

[14] Recall that the soft charges $\mathscr{F}^{\text{s}}_{\chi}$ are spontaneously broken, since they shift the photino as in (5.3.24) and hence do not annihilate the vacuum. However, this is not the case for the hard charges $\mathscr{F}^{\text{h}}_{\chi}$.

Since $|\overline{\Phi}, p\rangle$ has charge q and $|\Psi, p, -\rangle$ has charge $-q$, we can express (5.4.14) as follows,

$$\mathscr{F}_{\chi}^{\mathrm{h}} |\, f, p, s\,\rangle = -\frac{Q_f}{2\sqrt{\omega}} \chi(z, \overline{z}) \mathscr{F} |\, f, p, s\,\rangle \,, \tag{5.4.16}$$

where Q_f is the electric charge of the state. It is straightforward to repeat the preceding discussion for the Hermitian conjugate charges. They obey

$$\mathscr{F}_{\overline{\chi}}^{\mathrm{h}\dagger} |\, f, p, s\,\rangle = -\frac{Q_f}{2\sqrt{\omega}} \overline{\chi}(z, \overline{z}) \mathscr{F}^{\dagger} |\, f, p, s\,\rangle \,, \tag{5.4.17}$$

where the action of $\mathscr{F}^{\dagger}$ on one-particle asymptotic states was defined in (5.2.19). Together with (5.4.16), this establishes the relations stated in (5.4.7).

6

BMS Supertranslations and the Soft Graviton Theorem

In Chapters 3, 4 and 5, we have studied three examples of the relationship between asymptotic symmetries and soft theorems in gauge theories. We now turn to a study of the relationship in gravitational theories. This chapter is a modified extract of [4].

6.1 Introduction

Weinberg's soft graviton theorem [10] is a universal formula relating any $\mathcal{S}$-matrix element in any quantum theory including gravity to a second $\mathcal{S}$-matrix element which differs only by the addition of a graviton whose four-momentum is taken to zero. Remarkably, the formula is blind to the spin or any other quantum numbers of the asymptotic particles involved in the $\mathcal{S}$-matrix element.

It is often the case that universal formulae are explained by symmetries. Recently [28], it was conjectured that the quantum gravity $\mathcal{S}$-matrix has an exact symmetry given by a certain infinite-dimensional "diagonal" subgroup of the asymptotic supertranslation symmetries of Bondi, van der Burg, Metzner and Sachs (BMS) [14, 15]. In this paper, we show that the universal soft graviton theorem of [10] is simply the Ward identity following from the diagonal BMS supertranslation symmetry of [28].

Put another way, it turns out that the deep discoveries made a half century ago about the structure of Minkowski scattering in theories with gravity by Weinberg and by BMS are equivalent, albeit phrased in very different languages.

The Ward identities following from the diagonal BMS supertranslations were expressed in [28] in terms of data at null infinity, namely the Bondi news representing gravitational radiation together with certain infrared modes. These are described in terms of their retarded times and positions on the asymptotic conformal sphere. The soft graviton theorem on the other hand is described [10] in terms of the scattering of momentum-space plane waves. The demonstration of this paper consists largely in transforming between these two different descriptions.

In the course of our demonstration it is necessary to carefully define the physical phase spaces $\Gamma^{\pm}$ of gravitational modes at past and future null infinity ($\mathscr{I}^-$ and $\mathscr{I}^+$). $\Gamma^{\pm}$ must include, in addition to the Bondi news, all soft graviton degrees of freedom which do not decouple from the $\mathcal{S}$-matrix. The latter we argue are constrained by boundary conditions at the boundaries of $\mathscr{I}^{\pm}$. The soft modes can be viewed as living on these boundaries, and the boundary conditions reduce their number by a crucial factor of 2. The reduced space of modes may then be identified (from their transformation law) as nothing but the Goldstone modes of spontaneously broken supertranslation invariance. The relevant physical phase spaces $\Gamma^{\pm}$ become simply the usual radiative modes plus the Goldstone modes.[1] The boundary constraint entails a modification of the naive Dirac bracket. After this modification canonical expressions for $T^{\pm}$ are given which generate supertranslations on all of $\Gamma^{\pm}$. While there has been much discussion of $T^{\pm}$ over the decades, the construction of generators which act properly on the infrared as well as radiative modes is new.

This paper is organized as follows. In §6.2 we present the full $\mathscr{I}^{\pm}$ phase spaces $\Gamma^{\pm}$ (including the boundary condition), present the Dirac brackets and supertranslation generators $T^{\pm}$ and identify the soft gravitons as Goldstone modes. §6.3 reviews the proposed relation [28] between $\mathscr{I}^+$ and $\mathscr{I}^-$ near where they meet at spatial infinity, together with the diagonal supertranslations which preserve this relation and provide a symmetry of the $\mathcal{S}$-matrix. §6.4 reviews the soft graviton theorem [10]. §6.5 describes the transformation between the asymptotic description of §6.3 and the momentum space description of §6.4. In §6.6 we show that Weinberg's soft graviton theorem is the Ward

[1] This is the minimal phase space required for a good action of supertranslations. We have not ruled out the possibility of further soft modes and a larger phase space associated to local conformal symmetries[45, 98, 100–102] which could lie in components of the metric not considered here.

identity following from diagonal supertranslation invariance.

We mainly consider only the case of pure gravity but expect the inclusion of massless matter or gauge fields to be straightforward. New elements may arise in theories which do not revert to the vacuum in the far past and future. We expect that parallel results apply to the gauge theory case [27]. Related results are in [89].

6.2 Supertranslation generators

In this section we construct the physical phase space, the symplectic form (or equivalently the Dirac bracket) and the canonical generators of supertranslations at $\mathscr{I}^{\pm}$.

6.2.1 Asymptotic vector fields

We consider asymptotically flat geometries in the finite neighborhood of Minkowski space defined in [139] and referred to in [28] as CK spaces. These have a large-r weak-field expansion near future null infinity ($\mathscr{I}^+$) in retarded Bondi coordinates (see [45, 100–102] for details)

$$\begin{aligned} ds^2 &= -du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} \\ &\qquad + \frac{2m_B}{r}du^2 + rC_{zz}dz^2 + rC_{\bar{z}\bar{z}}d\bar{z}^2 - 2U_z dudz - 2U_{\bar{z}}dud\bar{z} + \cdots , \end{aligned} \tag{6.2.1}$$

where[2]

$$U_z = -\frac{1}{2}D^z C_{zz}. \tag{6.2.2}$$

[2]The U_z defined here should not be confused with that defined after (4.4.1).

The retarded time u parameterizes the null generators of $\mathscr{I}^+$ and $(z,\bar{z})$ parameterize the conformal S^2. The Bondi mass aspect m_B and C_{zz} depend on $(u,z,\bar{z})$, $\gamma_{z\bar{z}} = \frac{2}{(1+z\bar{z})^2}$ is the round metric on unit S^2 and D_z is the γ-covariant derivative. Near past null infinity $\mathscr{I}^-$, CK spaces have a similar expansion in advanced Bondi coordinates

$$\begin{aligned} ds^2 = -dv^2 + 2dvdr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} \\ + \frac{2m_B^-}{r}dv^2 + rD_{zz}dz^2 + rD_{\bar{z}\bar{z}}d\bar{z}^2 - 2V_zdvdz - 2V_{\bar{z}}dvd\bar{z} + \cdots , \end{aligned} \tag{6.2.3}$$

where

$$V_z = \frac{1}{2}D^zD_{zz}. \tag{6.2.4}$$

We denote the future (past) of $\mathscr{I}^+$ by $\mathscr{I}^+_+$ ($\mathscr{I}^+_-$), and the future (past) of $\mathscr{I}^-$ by $\mathscr{I}^-_+$ ($\mathscr{I}^-_-$). These comprise the boundary of $\mathscr{I}$ ($\mathscr{I}^+ \cup \mathscr{I}^-$). We also define the outgoing and incoming Bondi news by

$$N_{zz} \equiv \partial_u C_{zz} , \qquad M_{zz} \equiv \partial_v D_{zz}. \tag{6.2.5}$$

BMS$^+$ transformations [14, 15] are defined as the subgroup of diffeomorphisms which act non-trivially on the radiative data at $\mathscr{I}^+$. These include the familiar Lorentz transformations and supertranslations. The latter are generated by the infinite family of vector fields[3]

$$f\partial_u - \frac{1}{r}(D^{\bar{z}}f\partial_{\bar{z}} + D^zf\partial_z) + D^zD_zf\partial_r, \tag{6.2.6}$$

for any function $f(z,\bar{z})$ on the S^2. BMS$^+$ acts on C_{zz} according to

$$\mathcal{L}_fC_{zz} = f\partial_uC_{zz} - 2D_z^2f. \tag{6.2.7}$$

[3]The subleading in $\frac{1}{r}$ terms depend on the coordinate condition: see [45, 100–102].

Similarly BMS$^-$ transformations act on $\mathscr{I}^-$ and contain the supertranslations parameterized by $f^-(z,\bar{z})$

$$f^-\partial_v + \frac{1}{r}(D^{\bar{z}}f^-\partial_{\bar{z}} + D^z f^-\partial_z) - D^z D_z f^-\partial_r, \tag{6.2.8}$$

under which

$$\mathcal{L}_{f^-}D_{zz} = f^-\partial_v D_{zz} + 2D_z^2 f^-. \tag{6.2.9}$$

6.2.2 Dirac brackets on $\mathscr{I}$

The Dirac bracket on the radiative modes (the non-zero modes of the Bondi news) at $\mathscr{I}^+$ was found in [20, 25, 106, 107]

$$\{N_{\bar{z}\bar{z}}(u,z,\bar{z}), N_{ww}(u',w,\bar{w})\} = -16\pi G\partial_u\delta(u-u')\delta^2(z-w)\gamma_{z\bar{z}}, \tag{6.2.10}$$

where G is Newton's constant. The generator of BMS$^+$ supertranslations on these modes is [20, 25, 45, 100–102, 106, 107]

$$\begin{aligned} T^+(f) &= \frac{1}{4\pi G}\int_{\mathscr{I}^+_-} d^2z\gamma_{z\bar{z}} f m_B \\ &= \frac{1}{16\pi G}\int dud^2z f\left[\gamma_{z\bar{z}}N_{zz}N^{zz} + 2\partial_u(\partial_z U_{\bar{z}} + \partial_{\bar{z}}U_z)\right], \\ \{T^+(f), N_{zz}\} &= f\partial_u N_{zz}, \end{aligned} \tag{6.2.11}$$

where in the second line we have used the constraints and assumed no matter fields.

Of course BMS$^+$ transformations acting on the radiative modes alone do not comprise an asymptotic symmetry. One must act on a larger phase space Γ^+ including some non-radiative modes. The obvious guess is to identify this larger space with that parametrized by C_{zz} itself, and define a bracket for all (u, u') by integrating (6.2.10) to

$$\{C_{\overline{z}\overline{z}}(u, z, \overline{z}), C_{ww}(u', w, \overline{w})\} = 8\pi G\Theta(u - u')\delta^2(z - w)\gamma_{z\overline{z}}, \tag{6.2.12}$$

where $\Theta(x) = \text{sign}(x)$. However if we use this we find, perhaps surprisingly,

$$\{T^+(f), C_{zz}\} = f\partial_u C_{zz} - D_z^2 f \neq \mathcal{L}_f C_{zz}. \tag{6.2.13}$$

The inhomogeneous term is off by a factor of 2. So clearly either the bracket (6.2.12) or the generator (6.2.11) is incorrect. This problem does not seem to have been addressed in the literature.

Here we solve this problem by motivating and imposing boundary conditions on C_{zz} at the boundaries of $\mathscr{I}^+$, and incorporating this boundary constraint into a modified Dirac bracket. Since the constraints apply only to the boundary degree of freedom, (6.2.12) will be unaltered unless either u or u' is on the boundary. However this will turn out to give us exactly the missing factor of 2 in (6.2.13)! The supertranslation invariant boundary conditions are

$$\left[\partial_z U_{\overline{z}} - \partial_{\overline{z}} U_z\right]_{\mathscr{I}^+_\pm} = 0, \tag{6.2.14}$$

$$N_{zz}|_{\mathscr{I}^+_\pm} = 0. \tag{6.2.15}$$

Equivalently the first condition may be written

$$\left[D_z^2 C_{\overline{z}\overline{z}} - D_{\overline{z}}^2 C_{zz}\right]_{\mathscr{I}^+_\pm} = 0. \tag{6.2.16}$$

This reduces the boundary degrees of freedom by a factor of two. It has a coordinate invariant expression in terms of the component of the Weyl tensor sometimes referred to as the magnetic mass aspect:

$$\text{Im}\ \Psi_2^{(0)}|_{\mathscr{I}^+_\pm} = 0. \tag{6.2.17}$$

There are two related motivations for this constraint besides the fact that it (as we will see momentarily) leads to a proper action of T^+. First, the boundary condition (6.2.14) is obeyed by CK spaces [139]. Second, operator insertions of $[\partial_z U_{\bar{z}}]^{\mathscr{I}^+_+}_{\mathscr{I}^+_-}$ and $[\partial_{\bar{z}} U_z]^{\mathscr{I}^+_+}_{\mathscr{I}^+_-}$ correspond to soft gravitons and have non-vanishing $\mathcal{S}$-matrix elements (due to Weinberg poles) even though they are pure gauge. Therefore they must be retained as part of the physical phase space. However these poles cancel in the difference $[\partial_z U_{\bar{z}}]^{\mathscr{I}^+_+}_{\mathscr{I}^+_-} - [\partial_{\bar{z}} U_z]^{\mathscr{I}^+_+}_{\mathscr{I}^+_-}$. Hence this combination decouples from all $\mathcal{S}$-matrix elements and should not be part of the physical phase space. Our constraint (6.2.14) projects out these fully decoupled modes.

The general solution of the constraints (6.2.16) can be expressed

$$C_{zz}|_{\mathscr{I}^+_-} = D_z^2 C, \tag{6.2.18}$$

$$\int_{-\infty}^{\infty} du N_{zz} = D_z^2 N, \tag{6.2.19}$$

where the boundary fields C, N are real.[4] We may then take as our coordinates on phase space the boundary and bulk fields[5]

$$\Gamma^+ \equiv \{C(z,\bar{z}),\ N(z,\bar{z}),\ C_{zz}(u,z,\bar{z}),\ C_{\bar{z}\bar{z}}(u,z,\bar{z})\}. \tag{6.2.20}$$

[4] These fields are not to be confused with the analogous fields that we defined for the gauge field in §2.4.2.

[5] C and N each have four zero modes of $\ell = 0$ and $\ell = 1$ which are projected out by D_z^2 and hence do not appear in the metric. They might be omitted from the definition of Γ^+ and do not play an important role in the present discussion. However we retain them for future reference: as will become apparent below the C zero modes have an interesting interpretation as the spatial and temporal position of the geometry.

The arguments u of the bulk fields terms are restricted to non-boundary (i.e. finite) values only. The bulk-bulk Dirac brackets remain (6.2.12). A priori it is not obvious how one extends the bulk-bulk bracket (or equivalently the symplectic form) over all of Γ^+. We do so by first imposing (6.2.19) as a relation between bulk-bulk and bulk-boundary brackets in the form

$$D_z^2\{N(z,\overline{z}), C_{\overline{w}\overline{w}}(u,w,\overline{w})\} = \int_{-\infty}^{\infty} du'\{N_{zz}(u',z,\overline{z}), C_{\overline{w}\overline{w}}(u,w,\overline{w})\}, \tag{6.2.21}$$

and then constraining the boundary-boundary bracket by continuity in the form

$$D_{\overline{w}}^2\{N(z,\overline{z}), C(w,\overline{w})\} = \lim_{u\to-\infty}\{N(z,\overline{z}), C_{\overline{w}\overline{w}}(u,w,\overline{w})\}. \tag{6.2.22}$$

The non-zero Dirac brackets following from the boundary constraints (6.2.15), (6.2.16) are then uniquely determined as[6]

$$\begin{aligned}
\{C_{\overline{z}\overline{z}}(u,z,\overline{z}), C_{ww}(u',w,\overline{w})\} &= 8\pi G\Theta(u-u')\delta^2(z-w)\gamma_{z\overline{z}},\\
\{C(z,\overline{z}), C_{ww}(u',w,\overline{w})\} &= -8GD_w^2(S\ln|z-w|^2),\\
\{N(z,\overline{z}), C_{ww}(u',w,\overline{w})\} &= 16GD_w^2(S\ln|z-w|^2),\\
\{N(z,\overline{z}), C(w,\overline{w})\} &= 16GS\ln|z-w|^2,
\end{aligned} \tag{6.2.23}$$

where u, u' are *not* on the boundary and

$$S \equiv \frac{(z-w)(\overline{z}-\overline{w})}{(1+z\overline{z})(1+w\overline{w})}. \tag{6.2.24}$$

[6]We note but do not pursue herein the interesting appearance of logarithms related to the four C and N zero modes. These are projected out by acting with D_z^2 and hence irrelevant to the supertranslation generators below.

S is the sine-squared of the angle between z and w on the sphere and obeys

$$\begin{aligned} D_w^2(S\ln|z-w|^2) &= \frac{S}{(z-w)^2}, \\ D_{\bar{z}}^2 D_w^2(S\ln|z-w|^2) &= \pi\gamma_{z\bar{z}}\delta^2(z-w). \end{aligned} \tag{6.2.25}$$

Similarly, on $\mathscr{I}^-$, the constraints $[\partial_z V_{\bar{z}} - \partial_{\bar{z}} V_z]_{\mathscr{I}^-_\pm} = 0$ can be solved by

$$D_{zz}|_{\mathscr{I}^-_+} = D_z^2 D, \qquad \int_{-\infty}^{\infty} dv M_{zz} = D_z^2 M. \tag{6.2.26}$$

The coordinates on the phase space at $\mathscr{I}^-$ can then be taken as

$$\Gamma^- \equiv \{D(z,\bar{z}),\ M(z,\bar{z}),\ D_{zz}(v,z,\bar{z}),\ D_{\bar{z}\bar{z}}(v,z,\bar{z})\}, \tag{6.2.27}$$

where v is *not* on the boundary. The non-zero Dirac brackets are

$$\begin{aligned} \{D_{\bar{z}\bar{z}}(v,z,\bar{z}), D_{ww}(v',w,\bar{w})\} &= 8\pi G\Theta(v-v')\delta^2(z-w)\gamma_{z\bar{z}}, \\ \{D(z,\bar{z}), D_{ww}(v',w,\bar{w})\} &= 8GD_w^2(S\ln|z-w|^2), \\ \{M(z,\bar{z}), D_{ww}(v',w,\bar{w})\} &= 16GD_w^2(S\ln|z-w|^2), \\ \{M(z,\bar{z}), D(w,\bar{w})\} &= 16GS\ln|z-w|^2, \end{aligned} \tag{6.2.28}$$

where v, v' are *not* on the boundary.

The demand of continuity (6.2.22) is not as innocuous as it looks because we see from (6.2.23), (6.2.28) that other brackets (in particular $\{N_{zz}, C_{ww}\}$) are not continuous as u is taken to the boundary. We have not ruled out the possibility that there are inequivalent extensions of the symplectic form on the radiative phase space to all of $\Gamma^\pm$ corresponding to inequivalent quantizations of the boundary sector. In an action formalism, this could arise from different choices of boundary terms. However an *a posteriori* justification of our choice is, as we now show, that it leads to a realization of supertranslations as a canonical transformation on $\Gamma^{\pm}$.

6.2.3 Canonical generators

The supertranslation generator may now be written in terms of bulk and boundary fields as

$$\begin{aligned} T^{+}(f) &= \frac{1}{4\pi G}\int_{\mathscr{I}^{+}_{-}} d^2z\gamma_{z\bar{z}} f m_B \\ &= \frac{1}{16\pi G}\int du d^2z f\gamma_{z\bar{z}} N_{zz}N^{zz} - \frac{1}{8\pi G}\int d^2z\gamma^{z\bar{z}} f D_z^2 D_{\bar{z}}^2 N, \end{aligned} \tag{6.2.29}$$

where the integral over infinite u in the first term is the Cauchy principal value. Using the brackets (6.2.23) one finds

$$\begin{aligned} \{T^{+}(f), N_{zz}\} &= f\partial_u N_{zz}, \\ \{T^{+}(f), C_{zz}\} &= f\partial_u C_{zz} - 2D_z^2 f, \\ \{T^{+}(f), N\} &= 0, \\ \{T^{+}(f), C\} &= -2f, \end{aligned} \tag{6.2.30}$$

as desired.

Similarly on $\mathscr{I}^{-}$,

$$T^{-}(f^{-}) = \frac{1}{16\pi G}\int dv d^2z f^{-}\gamma_{z\bar{z}} M_{zz}M^{zz} + \frac{1}{8\pi G}\int d^2z\gamma^{z\bar{z}} f^{-} D_z^2 D_{\bar{z}}^2 M, \tag{6.2.31}$$

and

$$\begin{aligned} \{T^{-}(f^{-}), M_{zz}\} &= f^{-}\partial_v M_{zz}, \\ \{T^{-}(f^{-}), D_{zz}\} &= f^{-}\partial_v D_{zz} + 2D_z^2 f^{-}, \\ \{T^{-}(f^{-}), M\} &= 0, \\ \{T^{-}(f^{-}), D\} &= 2f^{-}, \end{aligned} \tag{6.2.32}$$

as desired.

At the quantum level supertranslations do not leave the usual in or out vacua invariant. Acting with T^+, the last term in (6.2.29) is linear in the graviton field operator and creates a new state with a soft graviton. The new state has energy degenerate with the out vacuum but different angular momentum. Hence supertranslation symmetry is spontaneously broken in the usual vacuum. The last line of (6.2.30) clearly identifies $-\frac{1}{2}C$ as the Goldstone mode associated with this symmetry breaking. It is conjugate to the soft graviton zero mode N.

In conclusion the construction of a generator of supertranslations on $\mathscr{I}^\pm$ is possible but subtle and requires a careful analysis of the zero mode structure and boundary conditions on the boundaries of $\mathscr{I}^\pm$.

6.3 Supertranslation invariance of the $\mathcal{S}$-matrix

In this section we summarize the supertranslation invariance of the $\mathcal{S}$-matrix conjectured in [28] as well as the associated Ward identity.

The first step is to understand how $\mathscr{I}^+$ and $\mathscr{I}^-$ may be linked near spatial infinity. In the conformal compactification of asymptotically flat spaces, the sphere at spatial infinity is the boundary of a point i^0. Null generators of $\mathscr{I}$ in the conformal compactification of asymptotically flat spaces run from $\mathscr{I}^-$ to $\mathscr{I}^+$ through i^0. We label all points lying on the same such generator with the same value of $(z, \bar{z})$. This gives an 'antipodal' identification of points on the conformal spheres at $\mathscr{I}^-$ with those on $\mathscr{I}^+$. For CK spaces one may identify geometric data on $\mathscr{I}^+_-$ with that at $\mathscr{I}^-_+$ via the continuity condition [28]

$$C_{zz}|_{\mathscr{I}^+_-} = -D_{zz}|_{\mathscr{I}^-_+}, \tag{6.3.1}$$

or equivalently

$$C(z,\overline{z}) = -D(z,\overline{z}). \tag{6.3.2}$$

In [28] it was conjectured that the "diagonal" subgroup of BMS$^+\times$BMS$^-$ which preserves the continuity condition (6.3.1) is an exact symmetry of both classical gravitational scattering and the quantum gravity $\mathcal{S}$-matrix. The diagonal supertranslation generators are those which are constant on the null generators of $\mathscr{I}$, $i.e.$

$$f^-(z,\overline{z}) = f(z,\overline{z}). \tag{6.3.3}$$

The conjecture states that $\mathcal{S}$-matrix obeys

$$T^+(f)\mathcal{S} - \mathcal{S}T^-(f) = 0. \tag{6.3.4}$$

A Ward identity is then derived by taking the matrix elements of (6.3.4) between states with n outgoing particles at z_k on the conformal sphere at $\mathscr{I}$. These carry energies ω_k, where

$$\sum_{k=1}^{n} \omega_k = 0\,. \tag{6.3.5}$$

by total energy conservation. We denote the out and in states by $\langle\,\mathrm{out}\,|$ and $|\,\mathrm{in}\,\rangle$. Choosing $f(w,\overline{w}) = \frac{1}{z-w}$, it was shown that the matrix element of (6.3.4) between such states implies

$$\langle\,\mathrm{out}\,|\!:\, P_z\mathcal{S}\!:\,|\,\mathrm{in}\,\rangle = \langle\,\mathrm{out}\,|\mathcal{S}|\,\mathrm{in}\,\rangle \sum_{k=1}^{n} \frac{\omega_k}{z-z_k}\,. \tag{6.3.6}$$

where the $:\ :$ denotes time-ordering and the "soft graviton current" is defined by

$$P_z \equiv \frac{1}{2G}\left(\int_{-\infty}^{\infty} dv\partial_v V_z - \int_{-\infty}^{\infty} du\partial_u U_z\right). \tag{6.3.7}$$

Since P_z involves zero-frequency integrals over $\mathscr{I}^{\pm}$ it creates and annihilates soft gravitons with a certain z-dependent wave function. The supertranslation Ward identity (6.3.6) relates $\mathcal{S}$-matrix elements with and without insertions of the soft graviton current. It can also easily be seen [28] that (6.3.6) implies the general Ward identities following from (6.3.4) for an arbitrary function $f(z,\bar{z})$.

6.4 The soft graviton theorem

In this section, we specify our conventions and briefly review Weinberg's derivation of the soft graviton theorem for the simplest case of a free massless scalar. For more details and general spin see [10].

Einstein gravity coupled to a free massless scalar is described by the action

$$S = -\int d^4x\sqrt{-g}\left[\frac{2}{\kappa^2}R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right], \tag{6.4.1}$$

where $\kappa^2 = 32\pi G$. In the weak field perturbation expansion $g_{AB} = \eta_{AB} + \kappa h_{AB}$ and the relevant leading terms are

$$\begin{aligned}\mathcal{L}_{\text{grav}} &= -\frac{2}{\kappa^2}R = -\frac{1}{2}\partial_C h_{AB}\partial^C h^{AB} + \frac{1}{2}\partial_A h\partial^A h + \partial^A h_{AB}\partial_\rho h^{B\rho} - \partial_A h^{AB}\partial_B h + \cdots,\\ \mathcal{L}_s &= -\frac{1}{2}\sqrt{-g}g^{AB}\partial_A\phi\partial_B\phi = -\frac{1}{2}\partial^A\phi\partial_A\phi + \frac{1}{2}\kappa h^{AB}\left[\partial_A\phi\partial_B\phi - \frac{1}{2}\eta_{AB}\partial^C\phi\partial_C\phi\right] + \cdots.\end{aligned} \tag{6.4.2}$$

In harmonic gauge $\partial^A h_{AB} = \frac{1}{2}\partial_B h$ the Feynman rules take the form (see [140])

Additional diagrams with the external graviton attached to internal lines cannot develop soft poles[10]. The contribution of these diagrams to the near-soft amplitude is

$$\begin{aligned}\mathscr{A}_{n+1}^{AB}(q,p_1,\cdots,p_n) = \sum_{k=1}^{m} \mathscr{A}_n(p_1,\cdots,p_k+q,\cdots,p_n)\frac{-i}{\left(p_k'+q\right)^2 - i\varepsilon} \\ \times\left[\frac{i\kappa}{2}\left(p_k^A(p_k+q)^B + p_k^B(p_k+q)^A - \eta^{AB}p_k\cdot(p_k+q)\right)\right]\end{aligned} \tag{6.4.3}$$

The soft graviton theorem is the leading term in q-expansion:

$$\mathscr{A}_{n+1}^{AB}(q,p_1,\cdots,p_n) = \frac{\kappa}{2}\sum_{k=1}^{n}\frac{p_k^A p_k^B}{p_k\cdot q}\mathscr{A}_n(p_1,\cdots,p_n)\,, \tag{6.4.4}$$

where $q \to 0$. While we reviewed the derivation here for a massless scalar, note that the pre-factor in square brackets is a universal soft factor and does not depend on the spin of the matter particles. Moreover the expression is actually gauge invariant. Under a gauge transformation $\delta\varepsilon_{AB} = q_A\Lambda_B + q_B\Lambda_A$ one finds

$$\delta\varepsilon_{AB}\mathscr{A}_{n+1}^{AB} = \kappa\Lambda^A\sum_{k=1}^{m}p_{kA}\mathscr{A}_n = 0 \tag{6.4.5}$$

by momentum conservation. Hence (6.4.4) is valid in any gauge.

6.5 From momentum to asymptotic position space

The supertranslation Ward identity (6.3.6) is expressed in terms of field operator P_z integrated along fixed-angle null generators of $\mathscr{I}$. Weinberg's soft graviton theorem (6.4.3) is expressed in terms of momentum eigenmodes of the field operators. In this section, in order to compare the two, we transform the field operator between these two bases.

We start with the mode expansion for the graviton field, h_{AB},

$$h_{AB}(x) = \sum_{\alpha=\pm}\int\frac{d^3q}{(2\pi)^3}\frac{1}{2\omega_q}\left[\varepsilon_{AB}^{(\alpha)}(\vec{q})^* a_{h,\alpha}(\vec{q})e^{iq\cdot x} + \varepsilon_{AB}^{(\alpha)}(\vec{q})a_{h,\alpha}^\dagger(\vec{q})e^{-iq\cdot x}\right], \tag{6.5.1}$$

where $q^0 = \omega_q = |\vec{q}|$, $\alpha = \pm$ are the two helicities and

$$\left[a_{h,\alpha}(\vec{q}), a^\dagger_{h,\beta}(\vec{q}\,')\right] = \delta_{\alpha\beta}(2\omega_q)(2\pi)^3\delta^3\big(\vec{q} - \vec{q}\,'\big). \tag{6.5.2}$$

The outgoing gravitons with momentum q and polarization α as in the amplitude (6.4.3) correspond to final-state insertions of $a_{h,\alpha}(\vec{q})$. In retarded Bondi coordinates, it follows from (6.2.1) that on $\mathscr{I}^+$

$$C_{zz}(u, z, \bar{z}) = \kappa \lim_{r\to\infty} \frac{1}{r} h_{zz}(r, u, z, \bar{z}). \tag{6.5.3}$$

Taking the large r expansion as in §2.5, we find

$$C_{zz} = -\frac{i\kappa}{16\pi^2}(E_z^+)^2 \int_0^\infty d\omega_q \left[a_{h,+}(\omega_q\hat{y})e^{-i\omega_q u} - a_{h,-}(\omega_q\hat{y})^\dagger e^{i\omega_q u}\right]. \tag{6.5.4}$$

where we have $\varepsilon^{(\pm)}_{AB}(q) = \varepsilon^{(\pm)}_A(q)\varepsilon^{(\pm)}_B(q)$ where $\varepsilon^{(\pm)}_A(q)$ is given by (2.5.14).

Defining

$$N^\omega_{zz}(z, \bar{z}) \equiv \int_{-\infty}^\infty du e^{i\omega u}\partial_u C_{zz}, \tag{6.5.5}$$

and using (6.5.4), we find

$$N^\omega_{zz}(z, \bar{z}) = -\frac{\kappa}{8\pi}(E_z^+)^2 \int_0^\infty d\omega_q\omega_q \left[a_{h,+}(\omega_q\hat{y})\delta(\omega_q - \omega) + a_{h,-}(\omega_q\hat{y})^\dagger\delta(\omega_q + \omega)\right]. \tag{6.5.6}$$

When $\omega > 0$ ($\omega < 0$), only the first (second) term contributes and we find

$$\begin{aligned} N^\omega_{zz}(z, \bar{z}) &= -\frac{\kappa\omega}{8\pi}(E_z^+)^2 a_{h,+}(\omega\hat{y}), \\ N^{-\omega}_{zz}(z, \bar{z}) &= -\frac{\kappa\omega}{8\pi}(E_z^+)^2 a^\dagger_{h,-}(\omega\hat{y}), \end{aligned} \tag{6.5.7}$$

where we have taken $\omega > 0$. In the case of the zero mode, we will define it in a hermitian way

$$N^0_{zz} \equiv \lim_{\omega \to 0^+} \frac{1}{2}(N^\omega_{zz} + N^{-\omega}_{zz}). \tag{6.5.8}$$

It follows that

$$N^0_{zz}(z,\bar{z}) = -\frac{\kappa}{16\pi}(E^+_z)^2 \lim_{\omega \to 0^+} \left[\omega a_{h,+}(\omega \hat{y}) + \omega a^\dagger_-(\omega \hat{y})^\dagger\right]. \tag{6.5.9}$$

A parallel construction is possible on $\mathscr{I}^-$. Defining

$$M^\omega_{zz}(z,\bar{z}) \equiv \int_{-\infty}^{\infty} dv e^{i\omega v}\partial_v D_{zz}, \tag{6.5.10}$$

we find for $\omega > 0$

$$\begin{aligned} M^\omega_{zz}(z,\bar{z}) &= -\frac{\kappa\omega}{8\pi}(E^+_z)^2 a_{h,+}(\omega \hat{y})\,, \\ M^{-\omega}_{zz}(z,\bar{z}) &= -\frac{\kappa\omega}{8\pi}(E^+_z)^2 a^\dagger_{h,-}(\omega \hat{y})\,, \end{aligned} \tag{6.5.11}$$

At $\omega = 0$,

$$M^0_{zz}(z,\bar{z}) \equiv -\frac{\kappa}{16\pi}(E^+_z)^2 \lim_{\omega \to 0^+} \left[\omega a_{h,+}(\omega \hat{y}) + \omega a^\dagger_{h,-}(\omega \hat{y})\right]. \tag{6.5.12}$$

From (6.5.5) and (6.5.10) we have also

$$\begin{aligned} N^0_{zz}(z,\bar{z}) &= D^2_z N, \\ M^0_{zz}(z,\bar{z}) &= D^2_z M. \end{aligned} \tag{6.5.13}$$

Defining

$$\mathcal{O}_{zz} \equiv N^0_{zz}(z,\bar{z}) + M^0_{zz}(z,\bar{z}) = D^2_z N + D^2_z M, \tag{6.5.14}$$

the soft graviton current (6.3.7) can be written

$$P_z = \frac{1}{2G}\left(V_z|^{\mathscr{I}^-_+}_{\mathscr{I}^-_-} - U_z|^{\mathscr{I}^+_+}_{\mathscr{I}^+_-}\right) = \frac{1}{4G}D^z\mathcal{O}_{zz}. \tag{6.5.15}$$

6.6 Soft graviton theorem as a Ward identity

Equations (6.5.12)-(6.5.15) express the soft graviton current P_z in terms of standard momentum space creation and annihilation operators. Amplitudes involving the latter are given by Weinberg's soft graviton theorem. In this section we simply plug this in and reproduce the supertranslation Ward identities.

We consider an $\mathcal{S}$-matrix element of n outgoing particles denoted by $\mathscr{A}_n = \langle \text{out} | \mathcal{S} | \text{in} \rangle$. We now consider the $\mathcal{S}$-matrix element $\langle \text{out} | : \mathcal{O}_{zz}\mathcal{S} : | \text{in} \rangle$ with a time ordered insertion. Using (6.5.12) and (6.5.14), this can be written as

$$\langle \text{out} | : \mathcal{O}_{zz}\mathcal{S} : | \text{in} \rangle = -\frac{\kappa (E_z^+)^2}{16\pi} \lim_{\omega\to 0} \omega \left[\langle \text{out} | a_{h,+}(\omega\hat{y})\mathcal{S} | \text{in} \rangle + \langle \text{out} | \mathcal{S} a^\dagger_{h,-}(\omega\hat{y}) | \text{in} \rangle \right] . \tag{6.6.1}$$

Here, we have used the fact that $a^\dagger_{h,-}(\omega\hat{y})$ $(a_{h,+}(\omega\hat{y}))$ annihilates the out (in) state for $\omega \to 0$.[7] The first term is the $\mathcal{S}$-matrix element with a single outgoing positive helicity soft graviton with spatial momentum $\omega\hat{y}$, while the second term is the $\mathcal{S}$-matrix element with a single incoming negative helicity soft graviton also with spatial momentum $\omega\hat{y}$. The two amplitudes are equal, and we get

$$\langle \text{out} | : \mathcal{O}_{zz}\mathcal{S} : | \text{in} \rangle = -\frac{\kappa}{8\pi}(E_z^+)^2 \lim_{\omega\to 0} \omega \langle \text{out} | a_{h,+}(\omega\hat{y})\mathcal{S} | \text{in} \rangle . \tag{6.6.2}$$

The soft graviton theorem (6.4.4) with a positive helicity outgoing graviton reads

$$\lim_{\omega\to 0} \omega \langle \text{out} | a_{h,+}(\vec{q})\mathcal{S} | \text{in} \rangle = \frac{\kappa}{2} \lim_{\omega\to 0} \sum_{k=1}^{n} \frac{\omega \left[p_k \cdot \varepsilon^+(q) \right]^2}{p_k \cdot q} \langle \text{out} | \mathcal{S} | \text{in} \rangle . \tag{6.6.3}$$

[7] This holds even if for example the initial state contains soft gravitons because of the factor of ω in (6.5.2).

Parameterizing the momenta p_i in terms of $(\omega_i, z_i, \bar{z}_i)$ and q in terms of $(\omega, z, \bar{z})$, we find

$$\langle\, \text{out}\, |\!:\, \mathcal{O}_{zz}\mathcal{S}\!:\, |\, \text{in}\,\rangle = \frac{8G}{(1+z\bar{z})}\langle\, \text{out}\, |\mathcal{S}|\, \text{in}\,\rangle \sum_{k=1}^{n} \frac{\omega_k\,(\bar{z}-\bar{z}_k)}{(z-z_k)\,(1+z_k\bar{z}_k)}\,. \tag{6.6.4}$$

Now, using (6.5.15), we can relate the insertion of P_z to that of $\mathcal{O}_{zz}$.

$$\begin{aligned}\langle\, \text{out}\, |\!:\, P_z\mathcal{S}\!:\, |\, \text{in}\,\rangle &= \frac{1}{4G}\gamma^{z\bar{z}}\partial_{\bar{z}}\langle\, \text{out}\, |\!:\, \mathcal{O}_{zz}\mathcal{S}\!:\, |\, \text{in}\,\rangle \\ &= \langle\, \text{out}\, |\mathcal{S}|\, \text{in}\,\rangle \sum_{k=1}^{n}\frac{\omega_k}{z-z_k} + \langle\, \text{out}\, |\mathcal{S}|\, \text{in}\,\rangle \sum_{k=1}^{n}\frac{\omega_k\bar{z}_k}{1+z_k\bar{z}_k}\end{aligned} \tag{6.6.5}$$

The very last square bracket vanishes due to total momentum conservation. We then have

$$\langle\, \text{out}\, |\!:\, P_z\mathcal{S}\!:\, |\, \text{in}\,\rangle = \sum_{k=1}^{n}\frac{\omega_k}{z-z_k}, \tag{6.6.6}$$

which reproduces exactly the supertranslation Ward identity (6.3.6) derived in [28]. We can also run the above argument backwards to show that this supertranslation Ward identity implies Weinberg's soft graviton theorem.

7

A 2D Stress Tensor for 4D Gravity

In this chapter, we study what is presumably the most interesting aspect of the relationship between soft theorems and asymptotic symmetries, namely the equivalence of the recently discovered subleading soft-graviton theorem [11] and BMS superrotations, which act as Virasoro transformations on the asymptotic S^2. This chapter is a modified extract of [5].

7.1 Introduction

Any quantum scattering amplitude of massless particles in four-dimensional (4D) asymptotically Minkowskian spacetime can be rewritten as a correlation function on the celestial sphere at null infinity. Asymptotic one-particle states are represented as operator insertions on the sphere at the points where they exit or enter the spacetime. The energy and other flavor or quantum numbers then label distinct operators. The $SL(2,\mathbb{C})$ Lorentz invariance acts as the global conformal group on the celestial sphere and implies that these correlators lie in $SL(2,\mathbb{C})$ representations.

In this paper we consider the $\mathcal{S}$-matrix for 4D quantum gravity in asymptotically Minkowskian spacetime. We construct an explicit soft-graviton mode, denoted T_{zz}, and prove that its insertions in the tree-level $\mathcal{S}$-matrix (with no other external soft insertions) obey all the Virasoro-Ward identities of a stress tensor insertion in a CFT$_2$ correlator on the sphere. Our main tool is the subleading soft-graviton theorem [11, 141–143]. Our construction refines and extends results and conjectures of [45, 98–101, 103]. It demonstrates that such quantum gravity scattering amplitudes are in Virasoro representations, as are CFT$_2$ correlators. This extends from gauge theory to gravity earlier work [1, 2] in which soft-photon and gluon insertions were shown to obey the Ward identities of a Kac-Moody algebra on the celestial sphere.

The current work has several limitations. We do not consider massive particles, but do expect the extension to the massive case to be possible along the lines of [31, 132, 134]. Qualitatively important issues arise - including a possible central term - when there are multiple soft insertions that are not addressed here. At the one-loop level, corrections to the Ward identity are expected as a consequence of corrections to the soft theorem [111, 112, 119]. We have not analyzed their implications. Finally, although our results imply that certain quantum gravity scattering amplitudes are in Virasoro representations, there is no reason to expect that they are the same kinds of unitary representations appearing in conventional 2D CFTs. We leave the nature of these representations to future work.

7.2 Soft-Graviton Limits

In this paper, we consider tree-level scattering amplitudes of massless particles in four dimensions. Let $\mathscr{A}_{n+1}^{(\pm)}(q)$ be an amplitude involving a graviton of momentum q^A and polarization $\varepsilon_{AB}^{(\pm)}(q)$ as well as n other massless asymptotic states

$$\mathscr{A}_{n+1}^{(\pm)}(q) = \langle\, \text{out}\,;\; q, \pm\, |\mathcal{S}|\; \text{in}\, \rangle\;. \tag{7.2.1}$$

The soft $q^0 \to 0$ limit of this amplitude is governed by the leading [10] and sub-leading [11, 141–143] soft-graviton theorems[1]

$$\mathscr{A}_{n+1}^{(\pm)}(q) \to \left[S_0^{(\pm)} + S_1^{(\pm)} + \mathcal{O}(q) \right] \mathscr{A}_n\;, \tag{7.2.2}$$

where $\mathscr{A}_n$ is the original amplitude without the soft-graviton and

$$S_0^{(\pm)} = \frac{\kappa}{2} \sum_{k=1}^{n} \frac{p_k^A p_k^B \varepsilon_{AB}^{(\pm)}(q)}{p_k \cdot q}\;, \qquad S_1^{(\pm)} = -\frac{i\kappa}{2} \sum_{k=1}^{n} \frac{\varepsilon_{AB}^{(\pm)}(q) p_k^A q_C}{p_k \cdot q} \mathcal{J}_k^{CB}\;, \tag{7.2.3}$$

where $\kappa = \sqrt{32\pi G}$ and $\mathcal{J}_{kAB}$ is the angular momentum operator acting on the kth outgoing state. It is the sum of the orbital angular momentum operator $\mathcal{L}_{kAB}$ and spin angular momentum $\mathcal{S}_{kAB}$.

[1] As shown in [11, 121, 122], tree-level graviton amplitudes are also constrained by a sub-subleading soft-graviton theorem.

Explicitly (see [97]),

$$\mathcal{L}_{kAB} = -i\left[p_{kA}\frac{\partial}{\partial p_k^B} - p_{kB}\frac{\partial}{\partial p_k^A}\right] ,$$

$$\mathcal{S}_{kAB} = -is_k\left[\varepsilon_A^{(+)}(p_k)\varepsilon_B^{(-)}(p_k) - \varepsilon_B^{(+)}(p_k)\varepsilon_A^{(-)}(p_k)\right] + s_k\varepsilon_C^{(+)}(p_k)\mathcal{L}_{kAB}\varepsilon^{(-)C}(p_k) . \tag{7.2.4}$$

$\varepsilon_A^{(\pm)}(p)$ are polarization vectors that satisfy[2]

$$\varepsilon^{(\pm)}(p)\cdot p = 0 , \qquad \varepsilon^{(\pm)}(p)\cdot\varepsilon^{(\pm)}(p) = 0 , \qquad \varepsilon^{(\pm)}(p)\cdot\overline{\varepsilon}^{(\pm)}(p) = 1 . \tag{7.2.5}$$

Equation (7.2.4) continues to hold for particles of half-integer helicity provided that the little group phase of the wavefunction is chosen consistently. Gauge invariance of the leading and subleading soft limits implies momentum and angular momentum conservation respectively,

$$\sum_{k=1}^{n} p_k^A \mathscr{A}_n = \sum_{k=1}^{n} \mathcal{J}_{kAB}\mathscr{A}_n = 0 . \tag{7.2.6}$$

To write out the soft factors explicitly, we parameterize the massless momenta and polarization vectors in terms of $(\omega_i, z_i, \overline{z}_i)$. In this parameterization, the soft factors (7.2.3) are given by

$$\begin{aligned}
S_0^{(+)} &= -\frac{\kappa}{2\omega}(1+z\overline{z})\sum_{k=1}^{n}\frac{\omega_k(\overline{z}-\overline{z}_k)}{(z-z_k)(1+z_k\overline{z}_k)} ,\\
S_0^{(-)} &= -\frac{\kappa}{2\omega}(1+z\overline{z})\sum_{k=1}^{n}\frac{\omega_k(z-z_k)}{(\overline{z}-\overline{z}_k)(1+z_k\overline{z}_k)} ,\\
S_1^{(+)} &= \frac{\kappa}{2}\sum_{k=1}^{n}\frac{(\overline{z}-\overline{z}_k)^2}{z-z_k}\left[\frac{2\overline{h}_k}{\overline{z}-\overline{z}_k} - \Gamma^{\overline{z}_k}_{\overline{z}_k\overline{z}_k}\overline{h}_k - \partial_{\overline{z}_k} + |s_k|\Omega_{\overline{z}_k}\right] ,\\
S_1^{(-)} &= \frac{\kappa}{2}\sum_{k=1}^{n}\frac{(z-z_k)^2}{\overline{z}-\overline{z}_k}\left[\frac{2h_k}{z-z_k} - \Gamma^{z_k}_{z_kz_k}h_k - \partial_{z_k} + |s_k|\Omega_{z_k}\right] .
\end{aligned} \tag{7.2.7}$$

[2]Note that (7.2.5) is invariant under $\varepsilon_A^{(\pm)}(q) \rightarrow e^{i\theta_\pm(q)}\varepsilon_A^{(\pm)}(q)$, i.e. (7.2.5) only determines the polarizations up to an overall momentum dependent phase. These correspond to the little group transformations.

Here Γ^z_{zz} is the connection with respect to the unit round metric $\gamma_{z\bar{z}} = 2(1+z\bar{z})^{-2}$ on the sphere, $\Omega_z = \frac{1}{2}\Gamma^z_{zz}$ is the spin connection, and we have defined the operators[3]

$$h_k \equiv \frac{1}{2}\left(s_k - \omega_k \partial_{\omega_k}\right) , \qquad \bar{h}_k \equiv \frac{1}{2}\left(-s_k - \omega_k \partial_{\omega_k}\right) . \tag{7.2.8}$$

In this parameterization, equation (7.2.6) takes the form

$$\begin{aligned}
\left(\sum_{k=1}^n \omega_k\right) \mathscr{A}_n &= \left(\sum_{k=1}^n \omega_k \frac{z_k+\bar{z}_k}{1+z_k\bar{z}_k}\right) \mathscr{A}_n = 0 \,, \\
-i\left(\sum_{k=1}^n \omega_k \frac{z_k-\bar{z}_k}{1+z_k\bar{z}_k}\right) \mathscr{A}_n &= \left(\sum_{k=1}^n \omega_k \frac{1-z_k\bar{z}_k}{1+z_k\bar{z}_k}\right) \mathscr{A}_n = 0 \,, \\
-i\sum_{k=1}^n \left[Y^{z_k}\left(\partial_{z_k} - |s_k|\Omega_{z_k}\right) + Y^{\bar{z}_k}\left(\partial_{\bar{z}_k} - |s_k|\Omega_{\bar{z}_k}\right) + D_{z_k}Y^{z_k}h_k + D_{\bar{z}_k}Y^{\bar{z}_k}\bar{h}_k\right] \mathscr{A}_n &= 0 \,,
\end{aligned} \tag{7.2.9}$$

where $Y^z(z) = a + bz + cz^2$ is a global conformal Killing vector and D_z is the covariant derivative on the unit sphere.

7.3 Mode Expansions and Zero Modes on $\mathscr{I}^+$

We now define certain zero-modes on $\mathscr{I}^+$ and rewrite the leading and subleading soft-graviton theorem in terms of the zero mode insertions. We recall from (6.5.4) that near $\mathscr{I}^+$, we have the following mode expansion

$$C_{\bar{z}\bar{z}}(u,z,\bar{z}) = -\frac{i\kappa}{16\pi^2}(E^-_{\bar{z}})^2 \int_0^\infty d\omega_q \left[a_{h,-}(\omega_q\hat{y})e^{-i\omega_q u} - a^\dagger_{h,+}(\omega_q\hat{y})e^{i\omega_q u}\right] . \tag{7.3.1}$$

[3]Single particle momentum eigenstates do not diagonalize the dilation operator $h_k + \bar{h}_k$. At tree-level, amplitudes are rational functions of the external momenta and we can formally define Mellin-transformed primary operators $\widetilde{\mathcal{O}}(m,z,\bar{z}) = \int_0^\infty d\omega \omega^{m-1}\mathcal{O}(\omega,z,\bar{z})$ with conformal weights $h = \frac{1}{2}(s+m)$, $\bar{h} = \frac{1}{2}(-s+m)$.

Let us define (as in (6.5.5),

$$N_{zz}^{\omega} \equiv \int du e^{i\omega u} N_{zz} \,, \qquad N_{\bar{z}\bar{z}}^{\omega} \equiv \int du e^{i\omega u} N_{\bar{z}\bar{z}} \,. \tag{7.3.2}$$

We now define the zero modes

$$\begin{aligned} N_{zz}^{(0)} &\equiv \int du N_{zz} = \frac{1}{2} \lim_{\omega \to 0} \left(N_{zz}^{\omega} + N_{zz}^{-\omega} \right) \\ &= -\frac{\kappa}{16\pi} (E_z^+)^2 \lim_{\omega \to 0} \left[\omega a_{h,+}(\omega \hat{y}) + \omega a_{h,-}(\omega \hat{y})^\dagger \right] \end{aligned} \tag{7.3.3}$$

and

$$\begin{aligned} N_{\bar{z}\bar{z}}^{(1)} &\equiv \int du u N_{\bar{z}\bar{z}} = -\frac{i}{2} \lim_{\omega \to 0} \partial_\omega \left[N_{\bar{z}\bar{z}}^{\omega} - N_{\bar{z}\bar{z}}^{-\omega} \right] \\ &= \frac{i\kappa}{16\pi} (E_{\bar{z}}^-)^2 \lim_{\omega \to 0} (1 + \omega \partial_\omega) \left[a_{h,-}(\omega \hat{y}) - a_{h,+}(\omega \hat{y})^\dagger \right] , \end{aligned} \tag{7.3.4}$$

along with similar definitions for $N_{\bar{z}\bar{z}}^{(0)}$ and $N_{zz}^{(1)}$. We note that $N_{\bar{z}\bar{z}}^{(1)}$ involves one less factor of ω than $N_{zz}^{(0)}$, but has the Weinberg pole projected out by the factor of $1 + \omega \partial_\omega$. Hence it has nonzero finite scattering amplitudes.

The insertion of the zero mode (7.3.4) is then given by (7.2.2) and (7.2.7) with

$$\begin{aligned} &\langle \, \text{out} \, | N_{\bar{z}\bar{z}}^{(1)} \mathcal{S} | \, \text{in} \, \rangle \\ &= \frac{4Gi}{(1 + z\bar{z})^2} \sum_{k=1}^{n} \frac{(z - z_k)^2}{\bar{z} - \bar{z}_k} \left[\frac{2h_k}{z - z_k} - \Gamma_{z_k z_k}^{z_k} h_k - \partial_{z_k} + |s_k| \Omega_{z_k} \right] \langle \, \text{out} \, | \mathcal{S} | \, \text{in} \, \rangle \,. \end{aligned} \tag{7.3.5}$$

7.4 A 2D Stress Tensor

Recall from Chapter 4 that massless scattering amplitudes $\mathscr{A}_n$ of any four-dimensional theory may always be recast as two-dimensional correlation functions of local operators on the asymptotic S^2 at null infinity as

$$\mathscr{A}_n = \langle\, \mathcal{O}_1(\omega_1, z_1, \bar{z}_1) \cdots \mathcal{O}_n(\omega_n, z_n, \bar{z}_n) \,\rangle \,. \tag{7.4.1}$$

The particle created by $\mathcal{O}_k$ intersects the asymptotic S^2 at the point $(z_k, \bar{z}_k)$[4]. The four-dimensional Lorentz group $SL(2,\mathbb{C})$ acts as the global conformal group on the asymptotic S^2 according to[5]

$$z \to z' = \frac{az+b}{cz+d}\,, \qquad ad - bc = 1\,. \tag{7.4.2}$$

This implies that all Minkowskian $\underset{\smile}{\text{QFT}}_4$ amplitudes are in representations of the same global conformal group as Euclidean CFT_2 correlators. In this section we will see that (hard) quantum gravity amplitudes are in representations of the full CFT_2 Virasoro group. Indeed it has already been shown that the leading soft-photon and graviton theorems are the Ward identities of abelian Kac-Moody current algebras acting on the asymptotic S^2 [1, 4, 27, 28]. A similar Kac-Moody structure for non-abelian gauge theory scattering amplitudes was studied in [105]. The leading soft-gluon theorem in a non-abelian gauge theory with gauge group $\mathcal{G}$ was shown in [2] to be equivalent to the Ward identity of a $\mathcal{G}$ Kac-Moody current algebra. In all of these cases, holomorphic Kac-Moody current insertions were related to positive helicity soft insertions. For instance, the soft-photon Kac-

[4] The same is not true for scattering amplitudes involving massive particles since a massive four-momentum does not localize to a point on $\mathscr{I}$. However following [31, 132, 134] we expect the analysis of this paper to have a suitable generalization to the massive case, as the subleading soft theorem [11, 141–143] remains valid for massive particles.

[5] This also acts on the energy as

$$\widetilde{\omega} \to \widetilde{\omega}|cz+d|^2\,, \qquad \widetilde{\omega} = \frac{\omega}{1+z\bar{z}}\,.$$

Moody current is

$$J_z = -\frac{8\pi}{e^2}F^{(0)}_{uz} = \frac{1}{e}E^+_z \lim_{\omega\to 0}\left[\omega a_{\mathcal{F},+}(\omega\hat{y}) + \omega a^\dagger_{\mathcal{F},-}(\omega\hat{y})\right], \tag{7.4.3}$$

where $F^{(0)}_{uz}$ is the zero mode of the photon field strength, and $a_{\mathcal{F},+}(\omega\hat{y})$ creates outgoing positive helicity photons. Insertions of this current take the form

$$\langle\, J_z\mathcal{O}_1\cdots\mathcal{O}_n\,\rangle = \sum_k \frac{Q_k}{z-z_k}\langle\, \mathcal{O}_1\cdots\mathcal{O}_n\,\rangle\,, \tag{7.4.4}$$

where eQ_k is the electric charge of the operator $\mathcal{O}_k$ and we have dropped the dependence of the operators on $(\omega_k, z_k, \bar{z}_k)$ for compactness.

In a similar vein, it has been shown [29, 103] that the subleading soft-graviton theorem is the Ward identity for the superrotations [45] which generate an infinite-dimensional Virasoro subgroup of the extended BMS group[6]. In the language of 2D correlators, the current corresponding to these local conformal transformations is the stress tensor. We now turn to an explicit construction of this operator.

Our starting point is (7.3.5) which has a form reminiscent of a stress tensor Ward identity. To bring this into the usual form, we define

$$T_{zz} \equiv \frac{i}{8\pi G}\int d^2w \frac{1}{z-w} D^2_w D^{\overline{w}} N^{(1)}_{\overline{ww}}\,. \tag{7.4.5}$$

Then (7.3.5) implies

$$\langle\, T_{zz}\mathcal{O}_1\cdots\mathcal{O}_n\,\rangle = \sum_{k=1}^n \left[\frac{h_k}{(z-z_k)^2} + \frac{\Gamma^{z_k}_{z_k z_k}}{z-z_k}h_k + \frac{1}{z-z_k}\left(\partial_{z_k} - |s_k|\Omega_{z_k}\right)\right]\langle\, \mathcal{O}_1\cdots\mathcal{O}_n\,\rangle\,, \tag{7.4.6}$$

[6]The sub-subleading soft-graviton theorem has also been recently recast as a symmetry of the S-matrix (see [144, 145]).

which is the precise form of the stress tensor correlator in a conformal field theory on a curved background. This can be brought to the more familiar form by dressing the operators with appropriate factors of the zweibein (see [146] for a more detailed discussion).

Define the charge

$$T_{\mathcal{C}}[Y] = \oint_{\mathcal{C}} \frac{dz}{2\pi i} Y^z T_{zz} \,, \tag{7.4.7}$$

where Y^z is a local CKV obeying $\partial_{\bar{z}} Y^z = 0$ with no singularities inside the contour. Insertions of (7.4.7) take the form

$$\langle\, T_{\mathcal{C}}[Y]\mathcal{O}_1 \cdots \mathcal{O}_n \,\rangle = \sum_{k \in \mathcal{C}} \left[D_{z_k} Y^{z_k} h_k + Y^{z_k} \left(\partial_{z_k} - |s_k| \Omega_{z_k} \right) \right] \langle\, \mathcal{O}_1 \cdots \mathcal{O}_n \,\rangle \,. \tag{7.4.8}$$

Thus, $T_{\mathcal{C}}[Y]$ generates a local conformal transformation on all operators inside $\mathcal{C}$[7].

Now, consider a contour $\mathcal{C}$ that encircles all z_k and a Y^z that is globally defined on the sphere, i.e. $Y^z = a + bz + cz^2$. Since we are on a compact S^2, insertions of $T_{\mathcal{C}}[Y]$ can be computed by either closing the contour towards $z = z_k$ or away from it. No poles are crossed when the contour is closed away from $z = z_k$ and these insertions must vanish. In other words,

$$\sum_{k=1}^{n} \left[D_{z_k} Y^{z_k} h_k + Y^{z_k} \left(\partial_{z_k} - |s_k| \Omega_{z_k} \right) \right] \langle\, \mathcal{O}_1 \cdots \mathcal{O}_n \,\rangle = 0 \,, \qquad Y^z = a + bz + cz^2 \,, \tag{7.4.9}$$

which is the statement of boost/angular momentum conservation (7.2.9).

The stress tensor (7.4.5) is non-local on S^2 in the news tensor zero mode $N^{(1)}_{\bar{z}\bar{z}}$. Nevertheless, we

[7] This operator is closely related to the soft part of the superrotation charge defined in [103]. More precisely if $\mathcal{C}$ is a contour that surrounds all z_k, then

$$Q_S^+ = -\frac{i}{2} T_{\mathcal{C}}[Y] \,.$$

have proven that insertions of T_{zz} are local on the S^2. In contrast, the construction of the boundary stress tensor in AdS/CFT [147, 148] is local in the bulk fields when written in terms of subleading terms in the metric expansion. Leading and subleading terms in the metric expansion have a gauge-dependent and generally nonlocal relation on the S^2 enforced by the Einstein equation. We have tried but failed to find, by rewriting $N^{(1)}_{\bar{z}\bar{z}}$ in terms of subleading metric components, such a local expression in Bondi gauge[8]. However it is possible that such a manifestly local expression exists in some other gauge. On the other hand, the non-locality may indicate that the Virasoro action in 4D quantum gravity has a different character than that in conventional 2D CFT. We leave this question unanswered for now.

Obviously an anti-holomorphic stress tensor $T_{\bar{z}\bar{z}}$ could be similarly constructed. However, a number of yet-unresolved issues arise for multiple soft-current insertions, even in the Maxwell case, as discussed in [1, 2]. The result of this paper is that insertions of a single T_{zz} generate local conformal transformations when all other insertions are hard.

[8] The $\mathcal{O}(r^0)$ term in g_{zz} is an obvious suspect.

A

Notations and Conventions

Most of the notations that are used in this thesis are described in the text when they are introduced. For quick reference, we also summarize our notations and conventions here along with some useful explicit formulae that are extensively used in the calculations.

A.1 Coordinate Conventions

Cartesian coordinates are denoted $y^A \equiv (y^0, y^1, y^2, y^3)$. Retarded coordinates $(u, r, z, \overline{z})$ and advanced coordinates are $(v, r, \widetilde{z}, \widetilde{\overline{z}})$ are related to the Cartesian coordinates as

$$\begin{aligned} t &= u + r = v - r \,, \\ y^1 &= \frac{r(z+\overline{z})}{1+z\overline{z}} = -\frac{r(\widetilde{z}+\widetilde{\overline{z}})}{1+\widetilde{z}\widetilde{\overline{z}}} \,, \\ y^2 &= \frac{-ir(z-\overline{z})}{1-z\overline{z}} = \frac{ir(\widetilde{z}-\widetilde{\overline{z}})}{1-\widetilde{z}\widetilde{\overline{z}}} \,, \\ y^3 &= \frac{r(1-z\overline{z})}{1+z\overline{z}} = -\frac{r(1-\widetilde{z}\widetilde{\overline{z}})}{1+\widetilde{z}\widetilde{\overline{z}}} \,. \end{aligned} \tag{A.1.1}$$

The unit vector is

$$\hat{y}(z, \overline{z}) = \frac{1}{1+z\overline{z}} \left(z+\overline{z}, -i(z-\overline{z}), 1 - z\overline{z} \right) \,. \tag{A.1.2}$$

We will often use the notation $\hat{y} \equiv \hat{y}(z, \overline{z})$ and $\hat{y}_k \equiv \hat{y}(z_k, \overline{z}_k)$.

$(z, \overline{z})$ coordinates are related to $(\widetilde{z}, \widetilde{\overline{z}})$ by the anti-podal map

$$\widetilde{z} = -\frac{1}{\overline{z}} \,, \qquad \widetilde{\overline{z}} = -\frac{1}{z} \qquad \Longrightarrow \qquad \hat{y}(z, \overline{z}) = -\hat{y}(\widetilde{z}, \widetilde{\overline{z}}) \,. \tag{A.1.3}$$

The metric of Minkowski space M_4 in each of these coordinates is

$$
\begin{aligned}
\mathrm{d}s^2 &= -\left(\mathrm{d}y^0\right)^2 + \left(\mathrm{d}y^1\right)^2 + \left(\mathrm{d}y^2\right)^2 + \left(\mathrm{d}y^3\right)^2 , \\
&= -\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + 2r\gamma_{z\bar{z}}\mathrm{d}z\mathrm{d}\bar{z} , \\
&= -\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r + 2r\gamma_{\tilde{z}\tilde{\bar{z}}}\mathrm{d}\tilde{z}\mathrm{d}\tilde{\bar{z}} ,
\end{aligned} \tag{A.1.4}
$$

where

$$
\gamma_{z\bar{z}} = \frac{2}{(1+z\bar{z})^2} . \tag{A.1.5}
$$

The non-vanishing Christoffel symbols are

$$
\begin{aligned}
&\Gamma^u_{z\bar{z}} = -\Gamma^r_{z\bar{z}} = r\gamma_{z\bar{z}} = \frac{2r}{(1+z\bar{z})^2} , \\
&\Gamma^z_{rz} = \frac{1}{r} , \\
&\Gamma^z_{zz} = \gamma^{z\bar{z}}\partial_z\gamma_{z\bar{z}} = -\frac{2\bar{z}}{1+z\bar{z}} .
\end{aligned} \tag{A.1.6}
$$

We also introduce four dimension vierbein

$$
e^A = \mathrm{d}y^A = \frac{\partial y^A}{\partial x^\mu}\mathrm{d}x^\mu , \qquad \omega^A{}_B(e) = 0 . \tag{A.1.7}
$$

Explicitly

$$
\begin{aligned}
e^0 &= \mathrm{d}u + \mathrm{d}r \,, \\
e^1 &= \frac{z+\bar{z}}{1+z\bar{z}}\mathrm{d}r + \frac{r(1-\bar{z}^2)}{(1+z\bar{z})^2}\mathrm{d}z + \frac{r(1-z^2)}{(1+z\bar{z})^2}\mathrm{d}\bar{z} \,, \\
e^2 &= \frac{-i(z-\bar{z})}{1+z\bar{z}}\mathrm{d}r - \frac{ir(1+\bar{z}^2)}{(1+z\bar{z})^2}\mathrm{d}z + \frac{ir(1+z^2)}{(1+z\bar{z})^2}\mathrm{d}\bar{z} \,, \\
e^3 &= \frac{1-z\bar{z}}{1+z\bar{z}}\mathrm{d}r - \frac{2r\bar{z}}{(1+z\bar{z})^2}\mathrm{d}z - \frac{2rz}{(1+z\bar{z})^2}\mathrm{d}\bar{z} \,.
\end{aligned}
\tag{A.1.8}
$$

We also define the zweibein on S^2,

$$
E^+ = \frac{2}{1+z\bar{z}}\mathrm{d}z \,, \qquad E^- = \frac{2}{1+z\bar{z}}\mathrm{d}\bar{z} \,. \tag{A.1.9}
$$

for which

$$
\Omega^{\pm}{}_{\pm}(E) = \pm\frac{1}{2}\left(\Gamma^z_{zz}\mathrm{d}z - \Gamma^{\bar{z}}_{\bar{z}\bar{z}}\mathrm{d}\bar{z}\right) = \mp\frac{\bar{z}}{1+z\bar{z}}\mathrm{d}z \pm \frac{z}{1+z\bar{z}}\mathrm{d}\bar{z} \,. \tag{A.1.10}
$$

A.2 Spinor Conventions

The four-dimensional sigma matrices are taken to be

$$
(\sigma^A)_{\alpha\dot{\beta}} = (-\mathbb{1}_{2\times 2}, \vec{\sigma}) \;, \qquad (\bar{\sigma}^A)^{\dot{\alpha}\beta} = (-\mathbb{1}_{2\times 2}, -\vec{\sigma}) \;. \tag{A.2.1}
$$

where $\vec{\sigma}$ are the Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \,, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \,, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \,. \tag{A.2.2}$$

Indices are raised and lowered using

$$\varepsilon^{\alpha\beta} = \varepsilon^{\dot{\alpha}\dot{\beta}} \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\varepsilon_{\alpha\beta} = -\varepsilon_{\dot{\alpha}\dot{\beta}} \,. \tag{A.2.3}$$

We note the following properties

$$\begin{aligned} \left(\sigma^A\right)_{\alpha\dot{\alpha}} \left(\overline{\sigma}_A\right)^{\dot{\beta}\beta} &= -2\delta_\alpha^\beta \delta_{\dot{\alpha}}^{\dot{\beta}} \,, \\ \left(\sigma^A\right)_{\alpha\dot{\alpha}} \left(\sigma_A\right)_{\beta\dot{\beta}} &= -2\varepsilon_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}} \,, \\ \left(\overline{\sigma}^A\right)^{\dot{\alpha}\alpha} \left(\overline{\sigma}_A\right)^{\dot{\beta}\beta} &= -2\varepsilon^{\alpha\beta}\varepsilon^{\dot{\alpha}\dot{\beta}} \,, \\ \left(\sigma^A\right)_{\alpha\dot{\alpha}} \left(\overline{\sigma}^B\right)^{\dot{\alpha}\alpha} &= -2\eta^{AB} \,, \\ \left(\sigma^A\overline{\sigma}^B + \sigma^B\overline{\sigma}^A\right)_\alpha{}^\beta &= -2\eta^{AB}\delta_\alpha^\beta \,, \\ \varepsilon_{\alpha\beta}\varepsilon^{\gamma\delta} &= \delta_\alpha^\delta \delta_\beta^\gamma - \delta_\alpha^\gamma \delta_\beta^\delta \,. \end{aligned} \tag{A.2.4}$$

We also define

$$\sigma_{AB} \equiv \frac{1}{4}\left(\sigma_A\overline{\sigma}_B - \sigma_B\overline{\sigma}_A\right) \,, \qquad \overline{\sigma}_{AB} \equiv \frac{1}{4}\left(\overline{\sigma}_A\sigma_B - \overline{\sigma}_B\sigma_A\right) . \tag{A.2.5}$$

We note following property that we use often

$$
\begin{aligned}
\overline{\sigma}_{AB}\overline{\sigma}_{C} &= \overline{\sigma}_{[AB}\overline{\sigma}_{C]} + \frac{1}{2}\left(\eta_{CA}\overline{\sigma}_{B} - \eta_{CB}\overline{\sigma}_{A}\right), \\
\sigma_{C}\overline{\sigma}_{AB} &= \sigma_{[C}\overline{\sigma}_{AB]} - \frac{1}{2}\left(\eta_{CA}\sigma_{B} - \eta_{CB}\sigma_{A}\right), \\
\sigma_{AB}\sigma_{C} &= \sigma_{[AB}\sigma_{C]} + \frac{1}{2}\left(\eta_{CA}\sigma_{B} - \eta_{CB}\sigma_{A}\right), \\
\overline{\sigma}_{C}\sigma_{AB} &= \overline{\sigma}_{[C}\sigma_{AB]} - \frac{1}{2}\left(\eta_{CA}\overline{\sigma}_{B} - \eta_{CB}\overline{\sigma}_{A}\right).
\end{aligned}
\tag{A.2.6}
$$

Any two-component spinor can be expanded in a basis of two spinors. It is convenient to choose the basis spinors

$$
\begin{aligned}
\xi_{\alpha}^{(+)} &= \sqrt{\frac{1}{1+z\overline{z}}}\begin{pmatrix} 1 \\ z \end{pmatrix}, \qquad & \xi_{\alpha}^{(-)} &= \sqrt{\frac{1}{1+z\overline{z}}}\begin{pmatrix} 1 \\ \overline{z} \end{pmatrix}, \\
\overline{\xi}_{\dot{\alpha}}^{(-)} &= \sqrt{\frac{1}{1+z\overline{z}}}\begin{pmatrix} 1 \\ \overline{z} \end{pmatrix}, \qquad & \overline{\xi}_{\dot{\alpha}}^{(+)} &= \sqrt{\frac{1}{1+z\overline{z}}}\begin{pmatrix} 1 \\ z \end{pmatrix}.
\end{aligned}
\tag{A.2.7}
$$

These basis spinors are also useful in describing the σ-matrices in retarded coordinates

$$
\begin{aligned}
\left(\sigma^{u}\right)_{\alpha\dot{\beta}} &= -2\xi_{\alpha}^{(+)}\xi_{\beta}^{(-)}, \\
\left(\sigma^{r}\right)_{\alpha\dot{\beta}} &= \xi_{\alpha}^{(+)}\xi_{\beta}^{(-)} - \xi_{\alpha}^{(-)}\xi_{\beta}^{(+)}, \\
\left(\sigma^{z}\right)_{\alpha\dot{\beta}} &= \frac{2}{r}E_{+}^{z}\xi_{\alpha}^{(+)}\xi_{\beta}^{(+)}, \\
\left(\sigma^{\overline{z}}\right)_{\alpha\dot{\beta}} &= \frac{2}{r}E_{-}^{\overline{z}}\xi_{\alpha}^{(-)}\xi_{\beta}^{(-)}.
\end{aligned}
\tag{A.2.8}
$$

Similar formulas may be obtained for $\overline{\sigma}^{\mu}$.

The Lorentz generators can be expressed as outer products as

$$
\begin{aligned}
(\sigma^{ur})_{\alpha\beta} &= -\frac{1}{2}\left[\xi^{(+)}_\alpha \xi^{(-)}_\beta + \xi^{(-)}_\alpha \xi^{(+)}_\beta\right] , \\
(\sigma^{uz})_{\alpha\beta} &= -\frac{2}{r} E^z_+ \xi^{(+)}_\alpha \xi^{(+)}_\beta , \\
(\sigma^{u\overline{z}})_{\alpha\beta} &= 0 , \\
(\sigma^{rz})_{\alpha\beta} &= \frac{1}{r} E^z_+ \xi^{(+)}_\alpha \xi^{(+)}_\beta , \\
(\sigma^{r\overline{z}})_{\alpha\beta} &= \frac{1}{r} E^{\overline{z}}_- \xi^{(-)}_\alpha \xi^{(-)}_\beta , \\
(\sigma^{z\overline{z}})_{\alpha\beta} &= -\frac{1}{r^2} E^z_+ E^{\overline{z}}_- \left[\xi^{(+)}_\alpha \xi^{(-)}_\beta + \xi^{(-)}_\alpha \xi^{(+)}_\beta\right] .
\end{aligned}
\tag{A.2.9}
$$

Similar formulas may be obtained for $\overline{\sigma}^{\mu\nu}$, since $\overline{\sigma}^{\mu\nu} = -(\sigma^{\mu\nu})^\dagger$.

The Lie derivative of a spinor w.r.t. a vector ζ^μ is given by

$$
\begin{aligned}
\mathcal{L}_\zeta \Psi_\alpha &= \zeta^\mu \nabla_\mu \Psi_\alpha - \frac{1}{2}\big(\nabla_\mu K_\nu\big)(\sigma^\mu)_\alpha{}^\beta \Psi_\beta , \\
\mathcal{L}_\zeta \overline{\Psi}^{\dot\alpha} &= \zeta^\mu \nabla_\mu \overline{\Psi}^{\dot\alpha} - \frac{1}{2}\big(\nabla_\mu K_\nu\big)(\overline{\sigma}^\mu)^{\dot\alpha}{}_{\dot\beta} \overline{\Psi}^{\dot\beta} .
\end{aligned}
\tag{A.2.10}
$$

A.3 Null Momenta

In this note, null momenta are parameterized as

$$
p^A = \frac{\omega}{1 + z\overline{z}} \left(1 + z\overline{z}, z + \overline{z}, -i(z - \overline{z}), 1 - z\overline{z}\right) . \tag{A.3.1}
$$

The Lorentz invariant δ-function in momentum space is then written as

$$(2p^0)\delta^3(\vec{p}-\vec{p}\,') = \frac{2}{\omega}\gamma^{z\bar{z}}\delta(\omega-\omega')\delta^2(z-z')\,. \tag{A.3.2}$$

Null momenta satisfy

$$p_A(\sigma^A)_{\alpha\dot{\beta}} = \eta_\alpha(p)\overline{\eta}_{\dot{\alpha}}(p)\,, \tag{A.3.3}$$

where

$$\eta_\alpha(p) = \sqrt{\frac{2\omega}{1+z\bar{z}}}\begin{pmatrix} 1 \\ z \end{pmatrix} = \sqrt{2\omega}\,\xi^{(+)}_\alpha\,. \tag{A.3.4}$$

A.4 One-Particle State Normalization

Here, we use the notation that the annihilation operator that appears in a field $f(y)$ is denoted $a_{f,s}(\vec{q})$ where s is the helicity of the particle that it annihilates. For the scalar field, $s = 0$ and we drop this label. These operators may carry additional labels (such as Lie algebra indices) which we have dropped here. The creation and annihilation operators satisfy

$$\big[a_{f,s}(\vec{q}), a^\dagger_{f',s'}(\vec{q}\,')\big\} = (2\pi)^3(2\omega_q)\delta_{f,f'}\delta_{s,s'}\delta^3(\vec{q}-\vec{q}\,')\,. \tag{A.4.1}$$

where $[\,,\,\}$ is a commutator if the operators are bosonic and an anti-commutator if they are fermionic.

One-particle states are defined as

$$| \, \vec{q}, f, s \, \rangle = a^{\dagger}_{f,s}(\vec{q}) | \, 0 \, \rangle \, , \tag{A.4.2}$$

which satisfy

$$\langle \, \vec{q}, f, s \, | \, \vec{q}\,', f', s' \, \rangle = (2\pi)^3 (2\omega_q) \delta_{f,f'} \delta_{s,s'} \delta^3(\vec{q} - \vec{q}\,') \, . \tag{A.4.3}$$

In this convention, the $\mathcal{S}$-matrix amplitude $\mathscr{A}_n$ is taken to be

$$\mathscr{A}_n = \langle \, 0 \, | a_{f_1,s_1}(p_1) \cdots a_{f_n,s_n}(p_n) \mathcal{S} | \, 0 \, \rangle \, . \tag{A.4.4}$$

where we use the convention that all particles are outgoing and that the sign of the energy p_i^0 determines whether the particle is actually ingoing or outgoing. We will often denote $a_{f_1,s_1}(p_1) \to \mathcal{O}_i(\omega_i, z_i, \overline{z}_i)$ and hence we write

$$\mathscr{A}_n = \langle \, \mathcal{O}_1(\omega_1, z_1, \overline{z}_1) \cdots \mathcal{O}_n(\omega_n, z_n, \overline{z}_n) \, \rangle \, . \tag{A.4.5}$$

References

[1] T. He, P. Mitra, A. P. Porfyriadis, and A. Strominger, "New Symmetries of Massless QED," *JHEP* 10 (2014) 112, arXiv:1407.3789 [hep-th].

[2] T. He, P. Mitra, and A. Strominger, "2D Kac-Moody Symmetry of 4D Yang-Mills Theory," *JHEP* 10 (2016) 137, arXiv:1503.02663 [hep-th].

[3] T. T. Dumitrescu, T. He, P. Mitra, and A. Strominger, "Infinite-Dimensional Fermionic Symmetry in Supersymmetric Gauge Theories," arXiv:1511.07429 [hep-th].

[4] T. He, V. Lysov, P. Mitra, and A. Strominger, "BMS supertranslations and Weinberg's soft graviton theorem," *JHEP* 05 (2015) 151, arXiv:1401.7026 [hep-th].

[5] D. Kapec, P. Mitra, A.-M. Raclariu, and A. Strominger, "A 2D Stress Tensor for 4D Gravity," arXiv:1609.00282 [hep-th].

[6] F. Bloch and A. Nordsieck, "Note on the Radiation Field of the electron," *Phys. Rev.* 52 (1937) 54–59.

[7] F. E. Low, "Scattering of light of very low frequency by systems of spin 1/2," *Phys. Rev.* 96 (1954) 1428–1432.

[8] F. E. Low, "Bremsstrahlung of very low-energy quanta in elementary particle collisions," *Phys. Rev.* 110 (1958) 974–977.

[9] D. R. Yennie, S. C. Frautschi, and H. Suura, "The infrared divergence phenomena and high-energy processes," *Annals Phys.* 13 (1961) 379–452.

[10] S. Weinberg, "Infrared photons and gravitons," *Phys.Rev.* 140 (1965) B516–B524.

[11] F. Cachazo and A. Strominger, "Evidence for a New Soft Graviton Theorem," arXiv:1404.4091 [hep-th].

[12] T. Kinoshita, "Mass singularities of Feynman amplitudes," *J. Math. Phys.* 3 (1962) 650–677.

[13] T. D. Lee and M. Nauenberg, "Degenerate Systems and Mass Singularities," *Phys. Rev.* 133 (1964) B1549–B1562. [,25(1964)].

[14] H. Bondi, M. G. J. van der Burg, and A. W. K. Metzner, "Gravitational waves in general relativity. 7. Waves from axisymmetric isolated systems," *Proc. Roy. Soc. Lond.* A269 (1962) 21–52.

[15] R. K. Sachs, "Gravitational waves in general relativity. 8. Waves in asymptotically flat space-times," *Proc. Roy. Soc. Lond.* A270 (1962) 103–126.

[16] P. J. McCarthy, "Representations of the bondi-metzner-sachs group. i. determination of the representations," *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences* 330 no. 1583, (1972) 517–535.

[17] P. J. McCarthy and M. Crampin, "Representations of the bondi-metzner-sachs group. iii. poincare spin multiplicities and irreducibility," *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences* 335 no. 1602, (1973) 301–311.

[18] P. J. McCarthy, "Representations of the bondi-metzner-sachs group. ii. properties and classification of the representations," *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences* 333 no. 1594, (1973) 317–336.

[19] M. Crampin and P. J. McCarthy, "Representations of the bondi-metzner-sachs group. iv. cantoni representations are induced," *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences* 351 no. 1664, (1976) 55–70.

[20] A. Ashtekar and R. O. Hansen, "A unified treatment of null and spatial infinity in general relativity. I - Universal structure, asymptotic symmetries, and conserved quantities at spatial infinity," *J. Math. Phys.* 19 (1978) 1542–1566.

[21] V. P. Frolov, "Null Surface Quantization and Quantum Field Theory in Asymptotically Flat Space-Time," *Fortsch.Phys.* 26 (1978) 455.

[22] P. J. McCarthy, "Lifting of projective representations of the bondi-metzner-sachs group," *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences* 358 no. 1693, (1978) 141–171.

[23] P. J. McCarthy, "Hyperfunctions and asymptotic symmetries," *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences* 358 no. 1695, (1978) 495–498.

[24] A. Ashtekar and T. Dray, "On the Existence of Solutions to Einstein's Equation With Nonzero Bondi News," *Commun. Math. Phys.* 79 (1981) 581–589.

[25] A. Ashtekar and M. Streubel, "Symplectic Geometry of Radiative Modes and Conserved Quantities at Null Infinity," *Proc. Roy. Soc. Lond.* A376 (1981) 585–607.

[26] A. Ashtekar, "Radiative Degrees of Freedom of the Gravitational Field in Exact General Relativity," *J. Math. Phys.* 22 (1981) 2885–2895.

[27] A. Strominger, "On BMS Invariance of Gravitational Scattering," *JHEP* 07 (2014) 152, arXiv:1312.2229 [hep-th].

[28] A. Strominger, "Asymptotic Symmetries of Yang-Mills Theory," arXiv:1308.0589 [hep-th].

[29] M. Campiglia and A. Laddha, "Asymptotic symmetries and subleading soft graviton theorem," *Phys. Rev.* D90 no. 12, (2014) 124028, arXiv:1408.2228 [hep-th].

[30] D. Kapec, V. Lysov, and A. Strominger, "Asymptotic Symmetries of Massless QED in Even Dimensions," arXiv:1412.2763 [hep-th].

[31] D. Kapec, M. Pate, and A. Strominger, "New Symmetries of QED," arXiv:1506.02906 [hep-th].

[32] G. Barnich and C. Troessaert, "Comments on holographic current algebras and asymptotically flat four dimensional spacetimes at null infinity," *JHEP* 11 (2013) 003, arXiv:1309.0794 [hep-th].

[33] G. Barnich and P.-H. Lambert, "Einstein-Yang-Mills theory: Asymptotic symmetries," *Phys. Rev.* D88 (2013) 103006, arXiv:1310.2698 [hep-th].

[34] D. Grumiller, M. Leston, and D. Vassilevich, "Anti-de Sitter holography for gravity and higher spin theories in two dimensions," *Phys. Rev.* D89 no. 4, (2014) 044001, arXiv:1311.7413 [hep-th].

[35] A. Mohd, "A note on asymptotic symmetries and soft-photon theorem," *JHEP* 02 (2015) 060, arXiv:1412.5365 [hep-th].

[36] C. Cardona, "Asymptotic Symmetries of Yang-Mills with Theta Term and Monopoles," arXiv:1504.05542 [hep-th].

[37] T. Adamo and E. Casali, "Perturbative gauge theory at null infinity," *Phys. Rev.* D91 no. 12, (2015) 125022, arXiv:1504.02304 [hep-th].

[38] A. Seraj, "Multipole charge conservation and implications on electromagnetic radiation," arXiv:1610.02870 [hep-th].

[39] B. Gabai and A. Sever, "Large gauge symmetries and asymptotic states in QED," *JHEP* 12 (2016) 095, arXiv:1607.08599 [hep-th].

[40] E. Noether, "Invariante variationsprobleme," *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse* 1918 (1918) 235–257. http://eudml.org/doc/59024.

[41] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz, "Large N field theories, string theory and gravity," *Phys. Rept.* 323 (2000) 183–386, arXiv:hep-th/9905111 [hep-th].

[42] J. M. Maldacena, "The Large N limit of superconformal field theories and supergravity," *Int. J. Theor. Phys.* 38 (1999) 1113–1133, arXiv:hep-th/9711200 [hep-th]. [Adv. Theor. Math. Phys.2,231(1998)].

[43] A. Bagchi and R. Fareghbal, "BMS/GCA Redux: Towards Flatspace Holography from Non-Relativistic Symmetries," *JHEP* 10 (2012) 092, arXiv:1203.5795 [hep-th].

[44] T. Andrade and D. Marolf, "Asymptotic Symmetries from finite boxes," *Class. Quant. Grav.* 33 no. 1, (2016) 015013, arXiv:1508.02515 [gr-qc].

[45] G. Barnich and C. Troessaert, "Supertranslations call for superrotations," *PoS* (2010) 010, arXiv:1102.4632 [gr-qc]. [Ann. U. Craiova Phys.21,S11(2011)].

[46] C. Boucher-Veronneau and A. J. Larkoski, "Constructing Amplitudes from Their Soft Limits," *JHEP* 09 (2011) 130, arXiv:1108.5385 [hep-th].

[47] S. W. Hawking, M. J. Perry, and A. Strominger, "Soft Hair on Black Holes," *Phys. Rev. Lett.* 116 no. 23, (2016) 231301, arXiv:1601.00921 [hep-th].

[48] S. W. Hawking, M. J. Perry, and A. Strominger, "Superrotation Charge and Supertranslation Hair on Black Holes," arXiv:1611.09175 [hep-th].

[49] S. W. Hawking, "Breakdown of Predictability in Gravitational Collapse," *Phys. Rev.* D14 (1976) 2460–2473.

[50] A. Averin, G. Dvali, C. Gomez, and D. Lust, "Gravitational Black Hole Hair from Event Horizon Supertranslations," *JHEP* 06 (2016) 088, `arXiv:1601.03725 [hep-th]`.

[51] M. Blau and M. O'Loughlin, "Horizon Shells and BMS-like Soldering Transformations," *JHEP* 03 (2016) 029, `arXiv:1512.02858 [hep-th]`.

[52] R.-G. Cai, S.-M. Ruan, and Y.-L. Zhang, "Horizon supertranslation and degenerate black hole solutions," *JHEP* 09 (2016) 163, `arXiv:1609.01056 [gr-qc]`.

[53] S. Carlip, "Black Hole Entropy from BMS Symmetry at the Horizon," `arXiv:1702.04439 [gr-qc]`.

[54] L. Donnay, G. Giribet, H. A. Gonzalez, and M. Pino, "Supertranslations and Superrotations at the Black Hole Horizon," *Phys. Rev. Lett.* 116 no. 9, (2016) 091101, `arXiv:1511.08687 [hep-th]`.

[55] L. Donnay, G. Giribet, H. A. González, and M. Pino, "Extended Symmetries at the Black Hole Horizon," *JHEP* 09 (2016) 100, `arXiv:1607.05703 [hep-th]`.

[56] C. Eling and Y. Oz, "On the Membrane Paradigm and Spontaneous Breaking of Horizon BMS Symmetries," *JHEP* 07 (2016) 065, `arXiv:1605.00183 [hep-th]`.

[57] M. Hotta, J. Trevison, and K. Yamaguchi, "Gravitational Memory Charges of Supertranslation and Superrotation on Rindler Horizons," *Phys. Rev.* D94 no. 8, (2016) 083001, `arXiv:1606.02443 [gr-qc]`.

[58] S. Hou, "Asymptotic Symmetries of the Null Infinity and the Isolated Horizon," `arXiv:1704.05701 [gr-qc]`.

[59] P. Mao, X. Wu, and H. Zhang, "Soft hairs on isolated horizon implanted by electromagnetic fields," *Class. Quant. Grav.* 34 no. 5, (2017) 055003, `arXiv:1606.03226 [hep-th]`.

[60] R. F. Penna, "Horizon symmetries as fluid symmetries," `arXiv:1703.07382 [hep-th]`.

[61] C. Shi and J. Mei, "Extended Symmetries at Black Hole Horizons in Generic Dimensions," `arXiv:1611.09491 [gr-qc]`.

[62] A. Averin, G. Dvali, C. Gomez, and D. Lust, "Goldstone origin of black hole hair from supertranslations and criticality," *Mod. Phys. Lett.* A31 no. 39, (2016) 1630045, arXiv:1606.06260 [hep-th].

[63] S. G. Avery and B. U. W. Schwab, "Soft Black Hole Absorption Rates as Conservation Laws," *JHEP* 04 (2017) 053, arXiv:1609.04397 [hep-th].

[64] G. Compère, "Bulk supertranslation memories: a concept reshaping the vacua and black holes of general relativity," *Int. J. Mod. Phys.* D25 no. 12, (2016) 1644006, arXiv:1606.00377 [hep-th].

[65] S. W. Hawking, "The Information Paradox for Black Holes," 2015. arXiv:1509.01147 [hep-th]. https://inspirehep.net/record/1391640/files/arXiv:1509.01147.pdf.

[66] A. Strominger and A. Zhiboedov, "Superrotations and Black Hole Pair Creation," *Class. Quant. Grav.* 34 no. 6, (2017) 064002, arXiv:1610.00639 [hep-th].

[67] Y. B. Zel'dovich and A. G. Polnarev, "Radiation of gravitational waves by a cluster of superdense stars," *Soviet Astronomy* 18 (1974) 17.

[68] V. B. Braginsky and L. P. Grishchuk, "Kinematic Resonance and Memory Effect in Free Mass Gravitational Antennas," *Sov. Phys. JETP* 62 (1985) 427–430. [Zh. Eksp. Teor. Fiz.89,744(1985)].

[69] D. Christodoulou, "Nonlinear nature of gravitation and gravitational wave experiments," *Phys. Rev. Lett.* 67 (1991) 1486–1489.

[70] A. G. Wiseman and C. M. Will, "Christodoulou's nonlinear gravitational wave memory: Evaluation in the quadrupole approximation," *Phys. Rev.* D44 no. 10, (1991) R2945–R2949.

[71] K. S. Thorne, "Gravitational-wave bursts with memory: The Christodoulou effect," *Phys. Rev.* D45 no. 2, (1992) 520–524.

[72] A. Strominger and A. Zhiboedov, "Gravitational Memory, BMS Supertranslations and Soft Theorems," *JHEP* 01 (2016) 086, arXiv:1411.5745 [hep-th].

[73] S. Pasterski, "Asymptotic Symmetries and Electromagnetic Memory," arXiv:1505.00716 [hep-th].

[74] L. Susskind, "Electromagnetic Memory," arXiv:1507.02584 [hep-th].

[75] L. Bieri, D. Garfinkle, and S.-T. Yau, "Gravitational wave memory in de Sitter spacetime," *Phys. Rev.* D94 no. 6, (2016) 064040, arXiv:1509.01296 [gr-qc].

[76] G. Compère and J. Long, "Classical static final state of collapse with supertranslation memory," *Class. Quant. Grav.* 33 no. 19, (2016) 195001, arXiv:1602.05197 [gr-qc].

[77] L. De Vittori, A. Gopakumar, A. Gupta, and P. Jetzer, "Memory effect from spinning unbound binaries," *Astrophys. Space Sci. Proc.* 40 (2015) 259–266, arXiv:1410.6605 [gr-qc].

[78] S. M. Du and A. Nishizawa, "Gravitational Wave Memory: A New Approach to Study Modified Gravity," *Phys. Rev.* D94 no. 10, (2016) 104063, arXiv:1609.09825 [gr-qc].

[79] C. Gomez and R. Letschka, "Memory and the Infrared," arXiv:1704.03395 [hep-th].

[80] S. Hollands, A. Ishibashi, and R. M. Wald, "BMS Supertranslations and Memory in Four and Higher Dimensions," arXiv:1612.03290 [gr-qc].

[81] D. A. Nichols, "Spin memory effect for compact binaries in the post-Newtonian approximation," *Phys. Rev.* D95 (2017) 084048, arXiv:1702.03300 [gr-qc].

[82] S. Pasterski, A. Strominger, and A. Zhiboedov, "New Gravitational Memories," *JHEP* 12 (2016) 053, arXiv:1502.06120 [hep-th].

[83] P. Mao, H. Ouyang, J.-B. Wu, and X. Wu, "New electromagnetic memories and soft photon theorems," arXiv:1703.06588 [hep-th].

[84] P. M. Zhang, C. Duval, G. W. Gibbons, and P. A. Horvathy, "The Memory Effect for Plane Gravitational Waves," arXiv:1704.05997 [gr-qc].

[85] R. M. Wald and A. Zoupas, "A General definition of 'conserved quantities' in general relativity and other theories of gravity," *Phys. Rev.* D61 (2000) 084027, arXiv:gr-qc/9911095 [gr-qc].

[86] A. Strominger, "Magnetic Corrections to the Soft Photon Theorem," *Phys. Rev. Lett.* 116 no. 3, (2016) 031602, arXiv:1509.00543 [hep-th].

[87] J. Wess and J. Bagger, *Supersymmetry and supergravity*. 1992.

[88] J. Grammer, G. and D. Yennie, "Improved treatment for the infrared divergence problem in quantum electrodynamics," *Phys.Rev.* D8 (1973) 4332–4344.

[89] J. Maldacena and A. Zhiboedov, "Notes on soft factors," *Unpublished and Private Communication* (2012) .

[90] C. W. Bauer, S. Fleming, D. Pirjol, and I. W. Stewart, "An Effective field theory for collinear and soft gluons: Heavy to light decays," *Phys.Rev.* D63 (2001) 114020, arXiv:hep-ph/0011336 [hep-ph].

[91] I. Feige and M. D. Schwartz, "Hard-Soft-Collinear Factorization to All Orders," arXiv:1403.6472 [hep-ph].

[92] P. Kulish and L. Faddeev, "Asymptotic conditions and infrared divergences in quantum electrodynamics," *Theor.Math.Phys.* 4 (1970) 745.

[93] J. Ware, R. Saotome, and R. Akhoury, "Construction of an asymptotic S matrix for perturbative quantum gravity," *JHEP* 1310 (2013) 159, arXiv:1308.6285 [hep-th].

[94] A. Strominger, "Lectures on the Infrared Structure of Gravity and Gauge Theory," arXiv:1703.05448 [hep-th].

[95] S. Caron-Huot, "When does the gluon reggeize?," arXiv:1309.6521 [hep-th].

[96] V. Lysov, S. Pasterski, and A. Strominger, "Low's Subleading Soft Theorem as a Symmetry of QED," *Phys. Rev. Lett.* 113 no. 11, (2014) 111601, arXiv:1407.3814 [hep-th].

[97] S. Weinberg, *The Quantum theory of fields. Vol. 1: Foundations*. Cambridge University Press, 2005.

[98] T. Banks, "A Critique of pure string theory: Heterodox opinions of diverse dimensions," arXiv:hep-th/0306074 [hep-th].

[99] J. de Boer and S. N. Solodukhin, "A Holographic reduction of Minkowski space-time," *Nucl. Phys.* B665 (2003) 545–593, arXiv:hep-th/0303006 [hep-th].

[100] G. Barnich and C. Troessaert, "Symmetries of asymptotically flat 4 dimensional spacetimes at null infinity revisited," *Phys. Rev. Lett.* 105 (2010) 111103, arXiv:0909.2617 [gr-qc].

[101] G. Barnich and C. Troessaert, "Aspects of the BMS/CFT correspondence," *JHEP* 05 (2010) 062, arXiv:1001.1541 [hep-th].

[102] G. Barnich and C. Troessaert, "BMS charge algebra," *JHEP* 12 (2011) 105, arXiv:1106.0213 [hep-th].

[103] D. Kapec, V. Lysov, S. Pasterski, and A. Strominger, "Semiclassical Virasoro symmetry of the quantum gravity $\mathcal{S}$-matrix," *JHEP* 08 (2014) 058, arXiv:1406.3312 [hep-th].

[104] F. A. Berends and W. T. Giele, "Multiple Soft Gluon Radiation in Parton Processes," *Nucl. Phys.* B313 (1989) 595–633.

[105] V. P. Nair, "A Current Algebra for Some Gauge Theory Amplitudes," *Phys. Lett.* B214 (1988) 215–218.

[106] A. Ashtekar, "Asymptotic Quantization of the Gravitational Field," *Phys. Rev. Lett.* 46 (1981) 573–576.

[107] A. Ashtekar, "Asymptotic Quantization: Based on 1984 Naples Lectures,".

[108] F. Cachazo, S. He, and E. Y. Yuan, "Scattering of Massless Particles in Arbitrary Dimensions," *Phys. Rev. Lett.* 113 no. 17, (2014) 171601, arXiv:1307.2199 [hep-th].

[109] E. Casali, "Soft sub-leading divergences in Yang-Mills amplitudes," arXiv:1404.5551 [hep-th].

[110] B. U. W. Schwab and A. Volovich, "Subleading Soft Theorem in Arbitrary Dimensions from Scattering Equations," *Phys. Rev. Lett.* 113 no. 10, (2014) 101601, arXiv:1404.7749 [hep-th].

[111] Z. Bern, S. Davies, and J. Nohle, "On Loop Corrections to Subleading Soft Behavior of Gluons and Gravitons," arXiv:1405.1015 [hep-th].

[112] S. He, Y.-t. Huang, and C. Wen, "Loop Corrections to Soft Theorems in Gauge Theories and Gravity," arXiv:1405.1410 [hep-th].

[113] A. J. Larkoski, "Conformal Invariance of the Subleading Soft Theorem in Gauge Theory," arXiv:1405.2346 [hep-th].

[114] A. J. Larkoski, D. Neill, and I. W. Stewart, "Soft Theorems from Effective Field Theory," *JHEP* 06 (2015) 077, arXiv:1412.3108 [hep-th].

[115] N. Afkhami-Jeddi, "Soft Graviton Theorem in Arbitrary Dimensions," arXiv:1405.3533 [hep-th].

[116] T. Adamo, E. Casali, and D. Skinner, "Perturbative gravity at null infinity," *Class. Quant. Grav.* 31 no. 22, (2014) 225008, arXiv:1405.5122 [hep-th].

[117] Y. Geyer, A. E. Lipstein, and L. Mason, "Ambitwistor strings at null infinity and (subleading) soft limits," *Class. Quant. Grav.* 32 no. 5, (2015) 055003, arXiv:1406.1462 [hep-th].

[118] B. U. W. Schwab, "Subleading Soft Factor for String Disk Amplitudes," arXiv:1406.4172 [hep-th].

[119] M. Bianchi, S. He, Y.-t. Huang, and C. Wen, "More on Soft Theorems: Trees, Loops and Strings," *Phys. Rev.* D92 no. 6, (2015) 065022, arXiv:1406.5155 [hep-th].

[120] A. Balachandran, S. Kurkcuoglu, A. de Queiroz, and S. Vaidya, "Spontaneous Lorentz Violation: The Case of Infrared QED," arXiv:1406.5845 [hep-th].

[121] J. Broedel, M. de Leeuw, J. Plefka, and M. Rosso, "Constraining subleading soft gluon and graviton theorems," arXiv:1406.6574 [hep-th].

[122] Z. Bern, S. Davies, P. Di Vecchia, and J. Nohle, "Low-Energy Behavior of Gluons and Gravitons from Gauge Invariance," arXiv:1406.6987 [hep-th].

[123] A. Campoleoni and M. Henneaux, "Asymptotic symmetries of three-dimensional higher-spin gravity: the metric approach," *JHEP* 03 (2015) 143, arXiv:1412.6774 [hep-th].

[124] P.-H. Lambert, *Conformal symmetries of gravity from asymptotic methods: further developments*. PhD thesis, Brussels U., 2014. arXiv:1409.4693 [gr-qc]. https://inspirehep.net/record/1317227/files/arXiv:1409.4693.pdf.

[125] M. Campiglia and A. Laddha, "New symmetries for the Gravitational S-matrix," *JHEP* 04 (2015) 076, arXiv:1502.02318 [hep-th].

[126] S. Caron-Huot *Private Communication* (2014) .

[127] J. Maldacena *Private Communication* (2014) .

[128] F. Cachazo and E. Y. Yuan, "Are Soft Theorems Renormalized?," arXiv:1405.3413 [hep-th].

[129] H. Elvang and Y.-t. Huang, "Scattering Amplitudes," arXiv:1308.1697 [hep-th].

[130] M. D. Schwartz, *Quantum Field Theory and the Standard Model*. Cambridge University Press, 2014. http://www.cambridge.org/us/academic/subjects/physics/theoretical-physics-and-mathematical-physics/quantum-field-theory-and-standard-model.

[131] D. Kapec, V. Lysov, S. Pasterski, and A. Strominger, "Higher-Dimensional Supertranslations and Weinberg's Soft Graviton Theorem," arXiv:1502.07644 [gr-qc].

[132] M. Campiglia and A. Laddha, "Asymptotic symmetries of QED and Weinberg's soft photon theorem," *JHEP* 07 (2015) 115, arXiv:1505.05346 [hep-th].

[133] S. G. Avery and B. U. W. Schwab, "Burg-Metzner-Sachs symmetry, string theory, and soft theorems," *Phys. Rev.* D93 (2016) 026003, arXiv:1506.05789 [hep-th].

[134] M. Campiglia and A. Laddha, "Asymptotic symmetries of gravity and soft theorems for massive particles," *JHEP* 12 (2015) 094, arXiv:1509.01406 [hep-th].

[135] S. G. Avery and B. U. W. Schwab, "Noether's second theorem and Ward identities for gauge symmetries," *JHEP* 02 (2016) 031, arXiv:1510.07038 [hep-th].

[136] N. Sakai and Y. Tanii, "Superwess-zumino-witten Models From SuperChern-Simons Theories," *Prog. Theor. Phys.* 83 (1990) 968–990.

[137] V. Lysov, "Asymptotic Fermionic Symmetry From Soft Gravitino Theorem," arXiv:1512.03015 [hep-th].

[138] S. G. Avery and B. U. W. Schwab, "Residual Local Supersymmetry and the Soft Gravitino," *Phys. Rev. Lett.* 116 no. 17, (2016) 171601, arXiv:1512.02657 [hep-th].

[139] D. Christodoulou and S. Klainerman, "The Global nonlinear stability of the Minkowski space,".

[140] S. Y. Choi, J. S. Shim, and H. S. Song, "Factorization and polarization in linearized gravity," *Phys. Rev.* D51 (1995) 2751–2769, arXiv:hep-th/9411092 [hep-th].

[141] D. J. Gross and R. Jackiw, "Low-Energy Theorem for Graviton Scattering," *Phys. Rev.* 166 (1968) 1287–1292.

[142] R. Jackiw, "Low-Energy Theorems for Massless Bosons: Photons and Gravitons," *Phys. Rev.* 168 (1968) 1623–1633.

[143] C. D. White, "Factorization Properties of Soft Graviton Amplitudes," *JHEP* 05 (2011) 060, arXiv:1103.2981 [hep-th].

[144] M. Campiglia and A. Laddha, "Sub-subleading soft gravitons and large diffeomorphisms," *JHEP* 01 (2017) 036, arXiv:1608.00685 [gr-qc].

[145] M. Campiglia and A. Laddha, "Sub-subleading soft gravitons: New symmetries of quantum gravity?," *Phys. Lett.* B764 (2017) 218–221, arXiv:1605.09094 [gr-qc].

[146] T. Eguchi and H. Ooguri, "Conformal and Current Algebras on General Riemann Surface," *Nucl. Phys.* B282 (1987) 308–328.

[147] J. D. Brown and J. W. York, Jr., "Quasilocal energy and conserved charges derived from the gravitational action," *Phys. Rev.* D47 (1993) 1407–1419, arXiv:gr-qc/9209012 [gr-qc].

[148] V. Balasubramanian and P. Kraus, "A Stress tensor for Anti-de Sitter gravity," *Commun. Math. Phys.* 208 (1999) 413–428, arXiv:hep-th/9902121 [hep-th].