Geometric Variational Problems for Mean Curvature

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Geometric Variational Problems for Mean Curvature

A dissertation presented

by

Jonathan J. Zhu

to

The Department of Mathematics

in partial fulfillment of the requirements
for the degree of
Doctor of Philosophy
in the subject of
Mathematics

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Abstract

This thesis investigates variational problems related to the concept of mean curvature on submanifolds. Our primary focus is on the area functional, whose critical points are the minimal submanifolds and whose gradient flow is the mean curvature flow. We study these geometric objects from the perspectives of existence, regularity and rigidity; their variational stability plays an essential role for each of these perspectives.

As direct minimisation tends to produce stable critical points, the natural method to construct unstable critical points becomes the min-max procedure, first developed for the area functional by Almgren-Pitts. We extend the min-max theory to produce constrained critical points of the area functional - hypersurfaces with prescribed constant mean curvature. A key concern is the regularity of the resulting hypersurface, which derives from local stability properties.

To study the mean curvature flow, for which singular behaviour is in fact inevitable, we study self-shrinking solitons, which model any singularities of the flow and are themselves minimal submanifolds with respect to a Gaussian metric. We study the dynamical stability of these self-shrinkers, particularly in the case that the soliton is itself singular, and in relation to the Colding-Minicozzi entropy functional for the flow in Euclidean space. We use similar techniques to analyse the principal eigenvalue of certain geometric operators on potentially
singular minimal hypersurfaces. Furthermore, we prove several strong rigidity theorems for self-shrinkers assuming certain almost-stability conditions.

Finally, we describe two monotonicity results: The first is a ‘moving-centre’ monotonicity formula for the area of minimal submanifolds; the second is a new entropy functional for mean curvature flow in the spherical space form. Both yield a priori estimates for the area of minimal submanifolds.
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Bibliography
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CHAPTER 1

Introduction

This thesis concerns variational problems for submanifolds, of two broad flavours corresponding to the major themes of existence and uniqueness (or rigidity) in geometric partial differential equations (PDE). These flavours are the construction of geometrically interesting submanifolds by variational methods, and analysis of the spectrum of geometric operators on such submanifolds. We focus on problems dealing with the concept of mean curvature, which is a natural notion of curvature for submanifolds; mean curvature also arises in several important physical phenomena such as surface tension and horizons in general relativity. Mathematically, mean curvature governs the first variation of the area functional on a submanifold, which is the primary focus of our variational theory.

Indeed, if a submanifold $\Sigma^k$ in an ambient Riemannian manifold $M^n$ is deformed by a vector field $X$, the first variation of area is

\[
\left. \frac{d}{dt} \right|_{t=0} \text{Area}(\Sigma_t) = - \int_{\Sigma} \langle \bar{H}, X \rangle ,
\]

where $\bar{H}$ is the mean curvature vector. Critical points of the area functional are termed minimal, and are thus characterised by the elliptic PDE given by the vanishing of mean curvature,

\[
\bar{H} \equiv 0.
\]

The study of minimal surfaces is a classical topic in geometry and in mathematics, stretching back to Euler in 1744 and Lagrange in 1762, who first studied the problem of finding a least area surface spanning a given curve. Minimal surface theory has been at the forefront of
the elliptic theory for geometric PDE, with direct and indirect applications to many other
geometric settings such as Einstein manifolds or positive scalar curvature [98].

If $\Sigma$ is a closed hypersurface - that is, a codimension 1 submanifold - then one may also
consider critical points of the area under the constraint of fixed enclosed volume. Calculating
the first variation of volume reveals that such critical points are defined by having constant
(scalar) mean curvature

\begin{equation}
H \equiv c.
\end{equation}

In addition to being of classical interest, both the unconstrained and constrained variational
problems for area have various applications ranging from modelling interface phenomena
[112, 84] to black holes and general relativity [99, 30, 68].

Perhaps the most natural method for constructing critical points is to minimise the
functional under study. For the area functional, however, direct minimisation may fail to
produce nontrivial minimal submanifolds. Even if a critical point is found, one would like
to investigate its stability or second variation, and in principle minimisation should only
detect stable critical points. To construct critical points of higher index, one naturally turns
to min-max methods, in which the maximum area of a family of submanifolds is minimised
over all suitable families. Min-max theory has enjoyed a remarkable amount of success in this
decade, led by Marques and Neves’ resolution of the Willmore conjecture [86] and their more
recent resolution of Yau’s conjecture for infinitely many minimal hypersurfaces in positive
Ricci curvature [88] and for generic metrics [72].

Another natural variational problem for the area functional is the mean curvature flow,
in which submanifolds $\Sigma_t$ are deformed by their mean curvature vector, that is,

\begin{equation}
\frac{dx}{dt} = \vec{H}_{\Sigma_t}.
\end{equation}

From the first variation of area, one can see that the mean curvature flow is the (negative) $L^2$
gradient flow for the area functional. Thus, one might hope to analyse and produce critical
points by running this flow. Mean curvature flow is also of significant interest in its own right and as the most natural geometric heat flow for submanifolds. Geometric flows have also enjoyed striking success, the most prominent example being Perelman’s resolution of the Poincaré conjecture via the analogous intrinsic flow, Hamilton’s Ricci flow.

As alluded to above, once a critical point or a flow is produced, we would like to study its geometric properties, including regularity, stability and rigidity. The first major issue is the possibility of singular behaviour of solutions. The singular set $\text{sing}(\Sigma)$ of a submanifold $\Sigma$ is the subset of points where $\Sigma$ fails to be embedded. Even for minimisers of the area functional, singularities can be unavoidable - for instance, the so-called Simons’ cone $\{ (x, y) \in \mathbb{R}^4 \times \mathbb{R}^4 : |x|^2 = |y|^2 \}$ is area-minimising, yet has a singularity at the cone point. Singularities are also inevitable for the mean curvature flow, as the maximum principle implies that any closed hypersurface must collapse in finite time. A fundamental technique to study singularities is blowup analysis, where we rescale or ‘zoom in’ at a singular point. With suitable a priori control one obtains a limit which has additional symmetries, and understanding this blowup limit is a model for the singular behaviour.

Such control is often provided by a monotonicity formula. On minimal submanifolds, for instance, we have the classical monotonicity formula:

**Proposition 0.1.** Let $\Sigma^k$ be a minimal submanifold in $\mathbb{R}^n$. Then so long as $\partial \Sigma \cap \overline{B^m_r} = \emptyset$, we have

\begin{equation}
\frac{d}{dr} (r^{-k} |\Sigma \cap B^m_r|) = r^{-k-1} \int_{\Sigma \cap \partial B^m_r} \frac{|x|^4}{|x|^2} \geq 0.
\end{equation}

A consequence is that singularities of minimal submanifolds are modelled by dilation-invariant minimal cones. Similarly, for a mean curvature flow $\Sigma_t$, one has Huisken’s monotonicity formula [65], which consider for a given spacetime centre $(x_0, t_0)$ the quantity $F_{x_0, t_0-t}(\Sigma_t) = \int_{\Sigma_t} \rho_{x_0, t_0-t}$. Here

\begin{equation}
\rho_{x_0, t_0}(x) = (4\pi t_0)^{-\frac{k}{2}} \exp \left( -\frac{|x-x_0|^2}{4t_0} \right).
\end{equation}
Huisken’s monotonicity formula is then
\[
(0.7) \quad \frac{d}{dt} \int_{t_0} Z_{t_0-t} = -\int_{\Sigma_t} \left( \bar{H} + \frac{(x - x_0)^\perp}{2(t_0-t)} \right)^2 \rho_{x_0,t_0-t},
\]
and it implies that the singularities of the flow are modelled by self-shrinking solitons (self-shrinkers for short) - submanifolds \( \Sigma \) that contract homothetically under the flow. That is, up to rescaling in spacetime, \( \Sigma_t = \sqrt{-t} \Sigma \) should be a mean curvature flow for \( t < 0 \). With this normalisation, self-shrinkers are critical points the \( F = F_{0,1} \) functional, are minimal with respect to a Gaussian metric on \( \mathbb{R}^n \) and are defined by the elliptic PDE
\[
(0.8) \quad \bar{H} = -\frac{1}{2} x^\perp.
\]

The second major issue is that of stability, including the variational stability of a critical point, dynamical stability under a geometric flow or the stability of singular behaviour under perturbation. On minimal or constant mean curvature hypersurfaces \( \Sigma \), the second variation of area is governed by the Jacobi or stability operator
\[
(0.9) \quad L = \Delta_\Sigma + |A|^2 + \text{Ric}(\nu, \nu),
\]
where \( A \) is the second fundamental form and \( \text{Ric}(\nu, \nu) \) is the ambient Ricci curvature in the normal direction. Self-shrinking hypersurfaces \( \Sigma \) also have second variation controlled by a stability operator, given by
\[
(0.10) \quad L = \mathcal{L} + |A|^2 + \frac{1}{2},
\]
where
\[
(0.11) \quad \mathcal{L} = \Delta - \frac{1}{2} \langle x, \nabla \rangle
\]
is the drift Laplacian on \( \Sigma \). The spectra of these stability operators thus directly control the variational stability of the corresponding critical points.
In order to address the dynamical stability of singularities under mean curvature flow, Colding and Minicozzi \cite{37} introduced an entropy functional measuring the Gaussian areas at all scales in spacetime, which is a Lyapunov functional for mean curvature flow:

\[
\Lambda(\Sigma) = \sup_{x_0, t_0 > 0} F_{x_0, t_0}(\Sigma),
\]

where \( F_{x_0, t_0}(\Sigma) = \int_{\Sigma} \rho_{x_0, t_0} \) as above. They showed that any smooth non-round self-shrinker is unstable for the entropy functional and hence dynamically unstable under the flow; a crucial part of their theory is a careful analysis of the stability operator \( L \). Since it is decreasing under the flow, the Colding-Minicozzi entropy is a measure for the complexity of a submanifold and has become an important functional in its own right. In particular, there is significant interest in understanding submanifolds of low entropy.

Aside from applications to stability analysis, the spectrum of a Schrödinger operator \( \Delta + V \) on a manifold is itself an interesting variational problem, corresponding to critical points of an energy \( \int (|\nabla u|^2 - Vu^2) \) for constrained \( L^2 \)-norm \( \int u^2 \). Indeed, the eigenvalues have a min-max characterisation which is the focus of a deep ongoing programme pioneered by Gromov \cite{57}, Guth \cite{61} and more recently Liokumovich, Marques and Neves \cite{82}. The least eigenvalue \( \lambda_1 \) is particularly fundamental, and has a variational characterisation in terms of the Rayleigh quotient,

\[
\lambda_1 = \inf \frac{\int (|\nabla u|^2 - Vu^2)}{\int u^2}.
\]

On a submanifold \( \Sigma \) itself arising from a variational problem, one expects the fundamental tone \( \lambda_1 \) for natural operators such as the Laplacian \( \Delta_\Sigma \) (or the drift Laplacian \( \mathcal{L}_\Sigma \)) to be rigid in some geometric sense. For instance, a famous conjecture of Yau asks whether the first Laplace eigenvalue of any minimal surface in the round 3-sphere must be equal to 2. (See also \cite{100} for some discussion.)
In the remainder of this section we describe the specific problems studied in this thesis as well as our main theorems. To begin, we return to the variational theory for the area functional.

1. Min-max theory

For finding critical points of the area functional - that is, minimal hypersurfaces - the min-max method has been greatly successful. In [3], Almgren initiated a celebrated program to develop a variational theory for minimal submanifolds in Riemannian manifolds of any dimension and co-dimension using geometric measure theory, namely the min-max theory for minimal submanifolds. He was able to prove the existence of a nontrivial weak solution as stationary integral varifolds [4]. Higher regularity was established in the co-dimension-one case by the seminal work of Pitts [94] (for \(2 \leq n \leq 5\)) and later extended by Schoen-Simon [37] (for \(n \geq 6\)). Colding-De Lellis [34] established the corresponding theory using smooth sweepouts based on ideas of Simon-Smith [107].

Very recently, in a series of works, Marques-Neves [86, 1, 88] found surprising applications of the Almgren-Pitts min-max theory to solve a number of longstanding open problems in geometry, including their celebrated proof of the Willmore conjecture. Due to these tremendous successes, there have been a vast number of developments of this programme in various contexts, including [42, 90, 60, 76, 87, 82, 23, 41, 80, 75, 109].

In [126], together with Prof. Xin Zhou, we extended the min-max method to the CMC setting. Specifically, we were able to prove existence of closed hypersurfaces with any prescribed constant mean curvature:

**Theorem 1.1.** Let \(M^{n+1}\) be a smooth, closed Riemannian manifold of dimension \(3 \leq n + 1 \leq 7\). Given any \(c \in \mathbb{R}\), there exists a nontrivial, smooth, closed, almost embedded hypersurface \(\Sigma^n\) of constant mean curvature \(c\).
We now give a heuristic overview of the min-max method. In the actual proofs, for technical reasons we will work with discrete families as in Almgren-Pitts, but here we will describe the key ideas using continuous families to elucidate those ideas.

Let \( M, c \) be as in Theorem 1.1. We only need to consider \( c > 0 \). We consider the \( \mathcal{A}^c \) functional, defined on open sets \( \Omega \) with smooth boundary by \( \mathcal{A}^c(\Omega) = \text{area}(\partial \Omega) - c \text{Vol}(\Omega) \). Denote \( I = [0, 1] \). Consider a continuous 1-parameter family of sets with smooth boundary

\[
\{ \Omega_x : x \in I \}, \quad \text{with} \quad \Omega_0 = \emptyset \quad \text{and} \quad \Omega_1 = M.
\]

Fix such a family \( \{ \Omega_x^0 \} \), and consider its homotopy class \( \{ [\Omega_x^0] \} = \{ \{ \Omega_x \} \sim \{ \Omega_x^0 \} \} \). The \( c \)-min-max value is defined as

\[
L^c = \inf_{\{ \Omega_x \} \sim \{ \Omega_x^0 \}} \max_{x \in I} \mathcal{A}^c(\Omega_x).
\]

A sequence \( \{ \{ \Omega_x^i \} : i \in \mathbb{N} \} \) with \( \max_{x \in I} \mathcal{A}^c(\Omega_x^i) \to L^c \) is typically called a minimising sequence, and any sequence \( \{ \Omega_{x_i}^i : x_i \in (0, 1), i \in \mathbb{N} \} \) with \( \mathcal{A}^c(\Omega_{x_i}^i) \to L^c \) is called a min-max sequence.

Our main result then says that there is a nice minimising sequence \( \{ \{ \Omega_x^i \} : i \in \mathbb{N} \} \), and some min-max sequence \( \{ \Omega_{x_i}^i : x_i \in (0, 1), i \in \mathbb{N} \} \), such that:

**Theorem 1.2.** The sequence \( \partial \Omega_{x_i}^i \) converges as varifolds with multiplicity one to a non-trivial, smooth, closed, almost embedded hypersurface \( \Sigma \) of constant mean curvature \( c \).

Our proof broadly follows the Almgren-Pitts scheme, but with several important difficulties. This scheme proceeds generally as follows:

- Construct a sweepout with positive width, and extract a minimising sequence;
- Apply a ‘tightening’ map to construct a new sequence whose varifold limit satisfies a variational property and an ‘almost-minimising’ property;
- Use these properties to construct ‘replacements’ on annuli which must be regular;
Apply successive concentric annular replacements to the min-max limit and show that they coincide with each other, and hence extend to the centre;

- Show that the min-max limit coincides with the replacement near the centre.

2. Mean curvature flow and entropy

As mentioned above, the starting point for much of the singularity analysis for the mean curvature flow is Huisken’s monotonicity formula [65], which implies (see for instance [65, 69, 116]) that the tangent flow at any singular point of a mean curvature flow is modelled by a critical point of the $F$-functional; these critical points are referred to as self-shrinkers, and they are also critical points of the entropy functional. Because they model the singularities in this blow-up sense, the study of self-shrinkers is essential to understanding the mean curvature flow. A further consequence of the monotonicity formula is that entropy is non-increasing under mean curvature flow; as such, the entropy may be interpreted as a useful measure of the complexity of a hypersurface.

Indeed, the Colding-Minicozzi entropy forms a fundamental component of their theory of generic mean curvature flow: In [37] they showed that the only complete, smoothly embedded, entropy-stable self-shrinkers are the generalised cylinders $S^k(\sqrt{2k}) \times \mathbb{R}^{n-k}$, so that under suitable conditions other such singularities may be perturbed away. Here $S^k(r)$ denotes the round $k$-sphere of radius $r$, and we say that a self-shrinker $\Sigma$ is entropy-stable if it is a local minimum for the entropy functional amongst $C^2$ graphs over $\Sigma$. The entropy functional has recently been studied by various other authors; see for instance [6], [15], [14], [13], [35], [76] and [83]. It has also been adapted to other geometric flows; see for example [124] and [74].

In [35], Colding, Ilmanen, Minicozzi and White conjectured that the entropy of any closed hypersurface should be at least that of the round sphere (see also [40]). In [128] we confirmed this conjecture for every dimension $n$; specifically we prove the following:
Theorem 2.1. Let $\Sigma$ be a smooth, closed, embedded hypersurface in $\mathbb{R}^{n+1}$. Then we have $\Lambda(\Sigma) \geq \Lambda(\mathbb{S}^n)$, with equality if and only if $\Sigma$ is a round sphere.

Note that for $n = 1$ the result follows immediately from the theorems of Gage-Hamilton [54] and Grayson [56], which imply that any smooth closed embedded curve shrinks to a round point. Previously Theorem 2.1 had been established in the cases $2 \leq n \leq 6$ by Bernstein and Wang [13], using a cleverly constructed weak flow that ensured the extinction time singularity was of a special type. Ketover and Zhou [76] also gave an independent proof for the $n = 2$, non-toric case using min-max theory for the $F$-functional. Our proof of Theorem 2.1 results from combining the insightful work of Bernstein-Wang together with our classification of entropy-stable singular self-shrinkers.

The main theorem of [128] was the following extension of Colding-Minicozzi’s classification of entropy-stable self-shrinkers [37] to the singular setting:

**Theorem 2.2.** Let $V$ be an $F$-stationary integral $n$-varifold in $\mathbb{R}^{n+1}$, which has orientable regular part and finite entropy, and satisfies the $\alpha$-structural hypothesis for some $\alpha \in (0, \frac{1}{2})$. Suppose that $V$ is not a generalised cylinder $\mathbb{S}^k(\sqrt{2k}) \times \mathbb{R}^{n-k}$. Then $V$ is entropy-unstable.

Furthermore, if $V$ also does not split off a line and is not a cone, the unstable variation may be taken to have compact support away from $\text{sing} V$. If $V$ is a stationary cone, the unstable variation may be taken to be a homogenous variation induced by variation of the link away from its singular set.

**2.1. Stability of minimal hypersurfaces in the round sphere.** To handle the case of stationary cones in the above theorem, we needed an upper bound for the first stability eigenvalue $\kappa_1$ of a non-flat minimal hypersurface in $\mathbb{S}^n$. Such a result was proven classically by J. Simons [106], but in the above we do not know a priori that the link is smooth. We proved the corresponding estimate for singular minimal hypersurfaces in [127]:

**Theorem 2.3.** Let $W$ be a stationary integral $(n-1)$-varifold in $\mathbb{S}^n$ which has orientable regular part and satisfies the $\alpha$-structural hypothesis for some $\alpha \in (0, \frac{1}{2})$. Further suppose
that $W$ is not totally geodesic in $S^n$. Then $\kappa_1(W) \leq -2(n-1)$, with equality if and only if $\text{spt } W$ is a Clifford hypersurface $S^k \left(\sqrt{\frac{k}{n-1}}\right) \times S^l \left(\sqrt{\frac{l}{n-1}}\right)$, where $k + l = n - 1$.

2.2. Techniques on singular hypersurfaces. In the theorems for singular hypersurfaces described above, the main issue is how to deal with the singular set. Moreover, the singular set is known to be small in the Hausdorff sense, which is technically difficult to work with: If one knew the singular set to be small in the (stronger) Minkowski sense, as proven for minimising hypersurfaces in work of Naber et. al. (see for instance [24]), one could easily construct good cutoff functions based on the distance to the singular set.

Instead, for Theorems 2.2 and 2.3 we had to develop careful cutoff techniques around the singular set, assuming Hausdorff codimension greater than 4. For completeness we also include full details of a cutoff construction of Morgan and Ritoré [92], which applies for Hausdorff codimension greater than 2. The most involved construction presented in this thesis is that used for Theorem 2.2.

3. Monotonicity formulae

For many geometric partial differential equations, monotonicity formulae play an essential role and their discovery often leads to deep and fundamental results for those systems. Monotonicity is a particularly useful tool in the study of variational problems, and for regularity theory (see for example [5, 9, 38, 53, 50, 105, 117] and references therein). These formulae often control the evolution of energy-type quantities with respect to changes in scale, or time.

As mentioned above, one important example is the classical monotonicity formula for minimal submanifolds - critical points of the area functional - which states:

**Proposition 3.1.** Let $\Sigma^k$ be a minimal submanifold in $\mathbb{R}^n$. Then so long as $\partial \Sigma \cap B^n_r = \emptyset$, we have

\[
\frac{d}{dr} \left( r^{-k} |\Sigma \cap B^n_r| \right) = r^{-k-1} \int_{\Sigma \cap \partial B^n_r} \frac{|x^+|^2}{|x|^2} \geq 0.
\]
In [130], we proved a new sharp ‘moving-centre’ monotonicity formula, in which the centres of the extrinsic balls are allowed to move, and the scale is adjusted in a particular manner:

**Definition 3.2.** Fix \( y \in B^n_R \). For \( s \geq 0 \) we let

\[
E_s = B^n((1-s)y, r(s))
\]

denote the ball with centre \((1-s)y\) and radius \( r(s) := \sqrt{s(R^2 - |y|^2) + s^2|y|^2} \).

**Theorem 3.3.** Let \( \Sigma^k \) be a minimal submanifold in \( \mathbb{R}^n \) and \( y \in B^n_R, E_s, r(s) \) be as above. Then so long as \( \partial \Sigma \cap \overline{E_s} = \emptyset \), we have

\[
\frac{d}{ds} \left( \frac{|\Sigma \cap E_s|}{(r(s)^2 - d(s)^2)^{\frac{n-1}{2}}} \right) = \frac{s^{-\frac{k+2}{2}}}{2(R^2 - |y|^2)^{\frac{n-1}{2}}} \int_{\Sigma \cap \partial E_s} \frac{|(x-y)^T|^2 + s^2|y|^2}{|(x-y + sy)^T|} \geq 0,
\]

where \( d(s) = s|y| \) is the distance from \( y \) to the centre of the ball \( E_s \).

In this thesis, we also find a new monotone quantity for mean curvature flows \( \Sigma^k_t \subset \mathbb{S}^{n+k} \), defined as the Colding-Minicozzi entropy when \( \Sigma_t \) is considered as a submanifold of \( \mathbb{R}^{n+k+1} \). As a consequence of the monotonicity of this new entropy functional, we are able to prove a gap theorem for the area of minimal hypersurfaces in \( \mathbb{S}^{n+1} \) which are not isotopic to the round sphere. This gap is larger than that proven by Cheng, Li and Yau [25], although theirs holds for any non-flat minimal hypersurface, and smaller than that proven by Ilmanen and White [71], although they require a stronger topological condition.

### 4. Rigidity of shrinking solitons

Since singularities of mean curvature flow are modelled by self-shrinkers, one of the most important questions in the study of mean curvature flow is to classify the possible singularities. The first major result is due to Huisken ([65], [66]), who showed that the only smooth complete embedded self-shrinkers in \( \mathbb{R}^{n+1} \) with \( H \geq 0 \), polynomial volume growth and \( |A| \) bounded are generalized cylinders \( S^k(\sqrt{2k}) \times \mathbb{R}^{n-k} \). Later, Colding-Minicozzi [37]
were able to remove the assumption of bounded curvature $|A|$. Consequently, they showed that the only generic shrinkers are the generalised cylinders $S^k(\sqrt{2k}) \times \mathbb{R}^{n-k}$, in the sense that all others can be perturbed away.

Another central problem in the singularity analysis for mean curvature flow is the uniqueness of blowups, namely, whether different sequences of dilations might give different blowups. For compact singularity models, this uniqueness problem is better understood; see for instance [101] and [102]. The first uniqueness theorem for blowups at noncompact singularities was obtained by Colding-Ilmanen-Minicozzi [31], who proved that if one blowup at a singularity of MCF is a multiplicity-one cylinder, then every subsequential limit is also a cylinder, and Colding-Minicozzi [39], who showed that the axis of the cylinder is also independent of the sequence of rescalings. Using this uniqueness in a fundamental way, Colding-Minicozzi [33] were able to give a quite complete description of the singular set for MCF having only generic singularities.

The key to proving the uniqueness at cylindrical singularities is the strong rigidity theorem of [31] Theorem 0.1, which says that any self-shrinker that is mean convex with bounded curvature $|A|$ on a large compact set must in fact be a cylinder.

In [59] and [58], together with Dr Qiang Guang, we proved several strong rigidity results, removing the bounded curvature assumption of [31] Theorem 0.1] and also extending it to the setting of graphical self-shrinkers:

**Theorem 4.1.** Given $n \leq 6$ and $\lambda_0$, there exists $R = R(n, \lambda_0)$ so that if $\Sigma^n \subset \mathbb{R}^{n+1}$ is a smoothly embedded self-shrinker with entropy $\lambda(\Sigma) \leq \lambda_0$ which satisfies

(†) $H \geq 0$ on $B_R \cap \Sigma$,

then $\Sigma$ is a generalised cylinder $S^k(\sqrt{2k}) \times \mathbb{R}^{n-k}$ for some $0 \leq k \leq n$.

**Theorem 4.2.** Given $n$, $\lambda_0$ and $\delta > 0$, there exists $R = R(n, \lambda_0, \delta)$ so that if $\Sigma^n \subset \mathbb{R}^{n+1}$ is a smoothly embedded self-shrinker with entropy $\lambda(\Sigma) \leq \lambda_0$ which satisfies

(‡) $H \geq \delta$ on $B_R \cap \Sigma$,

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then $\Sigma$ is a generalised cylinder $S^k(\sqrt{2k}) \times \mathbb{R}^{n-k}$ for some $1 \leq k \leq n$.

**Theorem 4.3.** Given $n \leq 6$ and $\lambda_0$, there exists $R = R(n, \lambda_0)$ so that if $\Sigma^n \subset \mathbb{R}^{n+1}$ is a smooth, complete self-shrinker with entropy $\lambda(\Sigma) \leq \lambda_0$ satisfying

$$(\dagger) \text{ } \Sigma \text{ is graphical in } B_R,$$

then $\Sigma$ is a hyperplane.

The above theorem for graphical self-shrinkers may be considered as the strong rigidity extension of the Bernstein theorem for self-shrinkers, in the same way that [31, Theorem 0.1] extended Huisken’s classification of mean convex self-shrinkers. The first such result was due to Ecker and Huisken [48], who showed that an entire self-shrinking graph with polynomial volume growth must be a hyperplane. Later, L. Wang [114] was able to remove the volume growth assumption.

5. **Yau’s conjecture for the first Laplace eigenvalue**

A famous open problem posed by S.-T. Yau states:

**Conjecture 5.1 (Yau).** Let $S^{n+1}$ be the unit $(n+1)$-sphere with its standard round metric. Then the first eigenvalue of the Laplacian on a closed embedded minimal hypersurface $\Sigma^n \subset S^{n+1}$ is precisely $n$.

This important conjecture appears as problem 100 in Yau’s 1982 problem list [121], in his lectures [122] and again in his more recent review [123]. It is a natural conjecture since when the ambient space $M$ is the round unit sphere $S^{n+1}$, the minimal surface equation shows that the embedding $X : \Sigma^n \hookrightarrow S^{n+1} \hookrightarrow \mathbb{R}^{n+2}$ satisfies $\tilde{\Delta}X + nX = 0$, where $\tilde{\Delta}$ is the Laplacian on $\Sigma$. Consequently, the first eigenvalue must satisfy $\lambda_1(\Sigma) \leq n$, and it is not difficult to see that equality holds for $\Sigma$ a great sphere, for instance. In fact, to the author’s knowledge, Yau’s conjecture has also been verified for all known examples of minimal hypersurfaces in spheres - see for instance [26] or more recently [111].
The most significant progress towards a proof of the conjecture, due to Choi and Wang [28], is indeed in this direction:

**Theorem 5.2 (Choi-Wang).** Let $\Sigma^n$ be a closed orientable embedded minimal hypersurface in a compact orientable manifold $(M^{n+1}, g)$. Suppose that the ambient Ricci curvature satisfies $\text{Ric}_g \geq kg$, $k > 0$. Then $\lambda_1(\Sigma) \geq k/2$.

**Remark 5.3.** The conclusion $\lambda_1 \geq k/2$ in fact holds for compact manifolds $M$ with boundary $\partial M = \Sigma$, so long as the second fundamental form $A$ and first eigenfunction $\phi_1$ on the boundary satisfy $\int_\Sigma A(\nabla\phi_1, \nabla\phi_1) \geq 0$. Indeed, the proof of Theorem 5.2 relies on choosing the side of $\Sigma$ for which the latter condition is satisfied.

In particular, since the standard metric $\overline{g}$ on $S^{n+1}$ satisfies $\text{Ric}_{\overline{g}} = n\overline{g}$, for minimal hypersurfaces in $(S^{n+1}, \overline{g})$ we have the lower bound $\lambda_1(\Sigma) \geq n/2$.

In [129] we were able to find a real analytic deformation of the standard metric on the hemisphere $S_+^{n+1}$, $n \geq 2$, which decreases the first Laplace eigenvalue of the boundary $\partial S_+^{n+1} \simeq S^n$ whilst preserving its minimality as well as the ambient Ricci curvature bound:

**Theorem 5.4.** Let $n \geq 2$. Then there is a smooth metric $g$ on the hemisphere $S_+^{n+1}$ such that:

- The ambient Ricci curvature satisfies $\text{Ric}_g \geq ng$.
- The boundary $\Sigma = \partial S_+^{n+1}$ is minimal with respect to $g$.
- The first Laplace eigenvalue of the induced metric $\hat{g}$ on $\Sigma$ satisfies $\lambda_1(\Sigma) < n$.
- The eigenfunction $\phi_1$ corresponding to $\lambda_1(\Sigma)$ satisfies $\int_\Sigma A(\nabla\phi_1, \nabla\phi_1) > 0$, where $\nabla$ is the gradient on $\Sigma$.

Morally, the above theorem implies that in order to improve the Choi-Wang result by the necessary factor of 2 to yield Yau’s conjecture, one would have to either use a more special property of the round sphere than its positive Ricci curvature, or one would have to develop an argument which genuinely uses both sides of the minimal hypersurface $\Sigma$. 
One may also pose Yau’s conjecture for self-shrinkers, for which the natural Laplace-type operator is the drift Laplacian \( \mathcal{L} \). The corresponding conjecture is whether \( \lambda_1(\mathcal{L}) = \frac{1}{2} \), as the coordinate functions have drift eigenvalue \( \frac{1}{2} \). Recently, Ding and Xin [44] were able to extend the argument of Choi and Wang to this setting, using the lower bound for the so-called Bakry-Émery-Ricci tensor on \( \mathbb{R}^{n+1} \) with Gaussian weight.

In this thesis we verify that \( \lambda_1(\mathcal{L}) = \frac{1}{2} \) when \( \Sigma^2 \subset \mathbb{R}^3 \) is the rotationally symmetric torus self-shrinker constructed by Angenent [7], up to an estimate for that torus which is easily verified numerically. Angenent’s torus was constructed by using the rotational symmetry to reduce the self-shrinker equation to an ordinary differential equation, and then applying the shooting method to find a simple closed profile curve. Conjecturally, this is a unique rotationally symmetric self-shrinking torus. As this conjecture has not been proven, we also verify the shrinker form of Yau’s conjecture for certain other constructions of the self-shrinking torus.

6. Outline of thesis

In Chapter 2 we quickly restate some background material and set out our notations. Since several parts of this thesis depend on understanding spectra of stability operators on submanifolds, we first warm up by covering our various eigenvalue estimates on minimal hypersurfaces in Chapter 3. This includes Theorems 2.3 and Theorem 5.4 for minimal hypersurfaces in spheres, as well as our analyses of Morgan and Ritoré’s cutoff construction and of the Angenent torus.

In Chapter 4 we present the main portion of our work on the entropy of closed hypersurfaces, including the stability analysis of singular self-shrinkers and Theorems 2.1 and 2.2. We then continue to discuss self-shrinkers in Chapter 5, presenting our rigidity theorems for mean convex self-shrinkers and for graphical self-shrinkers. In particular this includes the proofs of Theorems 4.1, 4.2, and 4.3.
In Chapter 6 we describe our moving-centre monotonicity formula for minimal hypersurfaces (Theorem 3.3), and our monotone entropy functional for mean curvature flows in the round sphere. Both are used to conclude area bounds for minimal hypersurfaces.

Finally in Chapter 7 we come around full circle to the min-max construction of prescribed constant mean curvature hypersurfaces and in particular the proof of Theorem 1.1.
CHAPTER 2

Preliminaries

In this chapter we discuss notation and collect some basic background material.

1. Manifolds and submanifolds

We typically work with submanifolds $\Sigma^k$ in a smooth ambient Riemannian manifold $(M^n, g)$. If we need to distinguish geometric quantities on the submanifold from ambient quantities, we typically use $\nabla$ to denote the connection on $\Sigma$, with barred symbols $\bar{\nabla}$ denoting the corresponding objects on the ambient manifold $M$. Occasionally we may instead use $\bar{g}$ to denote a reference metric, and other decorated quantities $\tilde{g}, \bar{g}$ for certain submanifolds. We reserve $D$ for the Euclidean connection. By $B(x, r)$ or $B_r(x)$ we typically denote ambient balls of radius $r$. For convenience we set $B_r = B_r(0)$.

We let $\omega_n$ denote the volume of the Euclidean ball of radius 1 in $\mathbb{R}^n$. We set $S^n(r)$ to be the round $n$-sphere of radius $r$, and $S^n = S^n(1)$. We use $y^T$ for the projection of a vector $y$ to the tangent bundle, and $y^\perp$ for the projection to the normal bundle. In Euclidean space we typically use $x$ to denote the position vector.

We will use the sign convention for the Laplacian that makes it a negative operator. That is, for arbitrary symmetric tensors, the rough Laplacian is defined by $\Delta = g^{ij} \nabla_i \nabla_j$. It is convenient to define the Hodge Laplacian, acting on 1-forms by

\begin{equation}
\Delta_H \omega_i = \Delta \omega_i - R^j_i \omega_j
\end{equation}

and the Lichnerowicz Laplacian, defined for symmetric 2-tensors by

\begin{equation}
\Delta_L h_{ij} = \Delta h_{ij} + 2 R^l_{ki} h^k_l - R^k_i h_{jk} - R^k_j h_{ik}.
\end{equation}
We say that a function \( u \neq 0 \) is an eigenvalue of \( \Delta \) with eigenvalue \( \lambda \) if \( \Delta u = -\lambda u \).

Here we use the convention for the Riemann curvature following Chow-Lu-Ni [29]:

\[
R(e_i, e_j)e_k = R^l_{ij} e_l, \quad \text{and} \quad R_{ijkl} = g_{lm} R^m_{ijkl}.
\]

The Ricci tensor \( \text{Ric} \) is then the covariant 2-tensor with components \( R_{ij} = R^p_{pij} \). We denote by \( g^{-1} \text{Ric} \) the \((1,1)\)-Ricci tensor, and we understand the Ricci curvature bound \( \text{Ric} g \geq kg \) to mean that the endomorphism \( g^{-1} \text{Ric} g - k \) is positive semidefinite.

The second fundamental form of \( \Sigma \) in \( M \) is a symmetric bilinear form with values in the normal bundle, given by \( A(X, Y) = (\nabla_X Y)^\perp \). The mean curvature vector \( \bar{H} \) is the trace of the second fundamental form, \( \bar{H}(p) = \sum_i A(E_i, E_i) \) where \( \{E_i\} \) is an orthonormal basis for \( T_p \Sigma \).

The \( k \)-dimensional Hausdorff measure is given by considering covers by sets of small diameter,

\[
\mathcal{H}^k(S) = \lim_{\delta \to 0} \inf \left\{ \sum (\text{diam } U_i)^k \middle| \text{diam } U_i < \delta, S \subset \bigcup U_i \right\}.
\]

The Hausdorff dimension of the set \( S \) is then \( \dim \mathcal{H}(S) = \inf \{k \geq 0| \mathcal{H}^k(S) = 0\} \). For a smooth \( k \)-dimensional submanifold we have \( \text{Area}(\Sigma) = \mathcal{H}^k(\Sigma) \).

We will need the coarea formula, which states that for a proper Lipschitz function \( f \) and a locally integrable function \( u \) on a manifold \( M \), one has

\[
\int_{\{f \leq t\}} u|\nabla f| = \int_{-\infty}^t d\tau \int_{\{f = \tau\}} u.
\]

Consider a variation \( \Sigma_t \) of \( \Sigma \) generated by vector field \( X \) on \( \Sigma \). The first variation of area is given by

\[
\frac{d}{dt} \bigg|_{t=0} \text{Area}(\Sigma_t) = - \int_{\Sigma} \langle \bar{H}, X \rangle.
\]

A minimal submanifold is a critical point for the area functional; from the above formula we see that minimal submanifolds are defined by the elliptic PDE \( \bar{H} = 0 \).
A one-parameter family of submanifolds $\Sigma_t$ is a mean curvature flow if those submanifolds move by their mean curvature vector, $\frac{d\Sigma}{dt} = \vec{H}$. From the first variation formula above we see that mean curvature flow can be interpreted as the negative $L^2$ gradient flow for area.

2. Hypersurfaces

A hypersurface is a smooth ($C^k$ or $C^\infty$) embedded codimension 1 submanifold $\Sigma^{n-1}$ in the Riemannian manifold $M^n$.

We say that a hypersurface $\Sigma^n \subset \mathbb{R}^{n+1}$ has Euclidean volume growth if there exists a constant $C_V > 0$ so that $\text{Vol}(\Sigma \cap B_r(x)) \leq C_V r^n$ for any $r > 0$ and any $x \in \mathbb{R}^{n+1}$. Here, and henceforth, $B_r(x)$ denotes the open Euclidean ball of radius $r$ in $\mathbb{R}^{n+1}$ centred at $x$.

For a two-sided hypersurface $X$ with unit normal $\nu$, the second fundamental form can instead be represented by the scalar-valued form $A(X, Y) = \langle X, Y \rangle$. The mean curvature (scalar) is then $H = \text{tr} A = \text{div} \nu$; in particular our convention is such that $\vec{H} = -H \nu$. A hypersurface is said to be (mean) convex if $A$ (resp. $H$) is nonnegative.

Indeed, with this convention the round sphere in Euclidean space has positive mean curvature. Also recall that if the ambient space is orientable, then the hypersurface $\Sigma$ is two-sided if and only if it is orientable (see for instance [63, Chapter 4]).

2.1. Minimal and CMC hypersurfaces. A hypersurface $\Sigma$ with constant mean curvature (CMC) if $H \equiv c$ for some constant $c$. CMC hypersurfaces $\Sigma$ that bound a domain $\Omega$ may be characterised as critical points of area for fixed enclosed volume. This is readily seen from specialising the first variation for area to the hypersurface case, and computing first variation for volume,

$$\left. \frac{d}{dt} \right|_{t=0} \text{Vol}(\Omega_t) = \int_{\Sigma} \langle X, \nu \rangle,$$

where $\Sigma_t = \partial \Omega_t$ is generated by $X$. Of course, minimal hypersurfaces are the special case $H \equiv c = 0$, and again are unconstrained critical points of the area.
In fact, for two-sided hypersurfaces, it suffices to consider normal variations \( X = \phi \nu \). At a CMC hypersurface, the second variation of area is given by
\[
\frac{d^2}{dt^2} \left. \right|_{t=0} \text{Area}(\Sigma_t) = \int_{\Sigma} \phi L \phi,
\]
where \( L = \Delta + |A|^2 + \text{Ric}(\nu, \nu) \) is the stability or Jacobi operator on \( \Sigma \). Like with the Laplacian, our eigenvalues are defined by the equation \( Lu = -\lambda u \). If \( \Sigma \) is compact then \( L \) has discrete (Dirichlet) spectrum \( \lambda_1 < \lambda_2 \leq \cdots \), and we may define the Morse index to be the number of negative eigenvalues of \( L \).

We say that \( \Sigma \) is stable if it is stable as a critical point, that is, if the second variation is nonnegative. Equivalently, \( \lambda_1 \geq 0 \) or the Morse index is 0. We also introduce the notion of almost stability or \( \delta \)-stability, \( \delta \geq 0 \), which is simply when \( -\lambda_1 \leq \delta \).

If \( \Sigma \) is noncompact, we may define \( \lambda_1(\Sigma) \) by taking the infimum over an exhaustion \( \lambda_1(\Omega_i) \) of \( \Sigma \), although it could be \( -\infty \). If, however, \( \lambda_1(\Sigma) > -\infty \), then we immediately get the stability inequality
\[
\int_{\Sigma} |A|^2 \phi^2 \leq \int_{\Sigma} |\nabla \phi|^2 + \int_{\Sigma} (-\lambda_1 - \text{Ric}(\nu, \nu))\phi^2
\]
for any \( \phi \) with compact support in \( \Sigma \).

Minimal hypersurfaces \( \Sigma^n \subset \mathbb{S}^{n+1} \) satisfy the following Simons’ inequality:

**Lemma 2.1.** On any minimal hypersurface \( M^n \subset \mathbb{S}^{n+1} \) we have that
\[
\Delta |A|^2 = 2|\nabla A|^2 + 2n|A|^2 - 2|A|^4.
\]

Consequently,
\[
|A|\Delta |A| = |\nabla A|^2 - |\nabla |A||^2 + n|A|^2 - |A|^4
\geq \frac{2}{n} |\nabla |A||^2 + n|A|^2 - |A|^4.
\]
2.2. Entropy. We denote $\rho_{x_0,t_0}(x) = (4\pi t_0)^{-n/2} e^{-\frac{|x-x_0|^2}{4t_0}}$. The Gaussian area of $\Sigma$ centred at $x_0 \in \mathbb{R}^{n+1}$ with scale $t_0 > 0$ is then given by

$$F_{x_0,t_0}(\Sigma) = \int_{\Sigma} \rho_{x_0,t_0}.$$  

(2.4)

The normalisation is so that any multiplicity 1 hyperplane has Gaussian area $F_{x_0,t_0}(\mathbb{R}^n) = 1$. For convenience we set $\rho = \rho_{0,1}$ and $F = F_{0,1}$. The entropy introduced by Colding-Minicozzi \[37\] may be defined as the supremum over all centres and scales,

$$\Lambda(\Sigma) = \sup_{x_0 \in \mathbb{R}^{n+1}} \sup_{t_0 > 0} F_{x_0,t_0}(\Sigma).$$

(2.5)

2.3. Self-shrinkers. A self-shrinker is a hypersurface $\Sigma^n \subset \mathbb{R}^{n+1}$ that contracts homothetically under the mean curvature flow. When normalised so that they contract to the origin in unit time, self-shrinkers are characterised by the elliptic PDE $H = \frac{1}{2} \langle x, \nu \rangle$.

Self-shrinkers also satisfy a Simons-type inequality:

**Lemma 2.2 ([37], Lemma 10.8).** On any smooth orientable self-shrinker we have $L A = A$. Hence, if $|A|$ does not vanish at a point then at that point one has

$$L |A| = |A| + \frac{||\nabla A|^2 - |\nabla |A||^2}{|A|} \geq |A|.$$  

(2.6)

Similar to the minimal hypersurface case, the second variation of the $F$-functional at a self shrinker $\Sigma$ is given by

$$\left. \frac{d^2}{dt^2} \right|_{t=0} F(\Sigma_t) = -\int_{\Sigma} \phi L \phi,$$  

(2.7)

where the stability operator is

$$L = \mathcal{L} + \frac{1}{2} |A|^2.$$  

(2.8)

Here $\mathcal{L}$ is the drift Laplacian

$$\mathcal{L} = \Delta_{\Sigma} - \frac{1}{2} \langle x, \nabla \Sigma \rangle.$$
Again if $\Sigma$ is compact then $L$ has discrete (Dirichlet) spectrum $\lambda_1 < \lambda_2 \leq \cdots$, and we may define the Morse index to be the number of negative eigenvalues of $L$, and $\Sigma$ is ($L$-) stable if $\lambda_1 \geq 0$ or the Morse index is 0. Again we say that $\Sigma$ is $\delta$-stable if $-\lambda_1 \leq \delta$.

If $\Sigma$ is noncompact, we may define $\lambda_1(\Sigma)$ by taking the infimum over an exhaustion $\lambda_1(\Omega_i)$ of $\Sigma$, although it could be $-\infty$. If, however, $\lambda_1(\Sigma) > -\infty$, then we immediately get the stability inequality

\begin{equation}
\int_\Sigma |A|^2 f^2 \rho \leq \int_\Sigma |\nabla f|^2 \rho + (-1/2 - \lambda_1) \int_\Sigma f^2 \rho,
\end{equation}

for Lipschitz functions $f$ compactly supported in $\Sigma$.

**Lemma 2.3** ([37], Theorem 5.2). On any smooth orientable self-shrinker, for any constant vector $y$ we have $L \langle y, \nu \rangle = \frac{1}{2} \langle y, \nu \rangle$ and $L H = H$.

### 3. Varifolds

In this subsection we describe extensions of the previous subsection to the singular (varifold) setting.

For us a varifold will always mean an integer rectifiable (integral) varifold $V$ in a Riemannian manifold $M^{n+1}$. The reader is directed to [105] for the basic definitions for varifolds. An integral varifold $V$ is determined by its mass measure, which we denote $\mu_V$. We will always assume that the support $\text{spt} \; V := \text{spt} \; \mu_V$ is connected. We define the regular part $\text{reg} \; V$ to be the set of points $x \in \text{spt} \; V$ around which $\text{spt} \; V$ is locally a $C^2$ hypersurface; the singular set is then $\text{sing} \; V = \text{spt} \; V \setminus \text{reg} \; V$.

An integer rectifiable $k$-varifold $V$ has an approximate tangent plane $T_x V$ at $\mu_V$-almost every $x$ in $\text{spt} \; V$. We may thus define the divergence almost everywhere by

\begin{equation}
(\text{div}_V X)(x) = \text{div}_{T_x V} X(x) = \sum_{i=1}^k \langle E_i, \nabla_{E_i} X \rangle(x)
\end{equation}
where $E_i$ is an orthonormal basis for $T_x V$ and $\nabla$ is the ambient connection. The varifold $V$ is then said to have generalised mean curvature vector $\vec{H}$, if $\vec{H}$ is locally integrable and the first variation is given by

(3.2) \[ \int \text{div}_V X \, d\mu_V = - \int \langle X, \vec{H} \rangle \, d\mu_V \]

for any ambient $C^1$ vector field $X$ with compact support.

For convenience will say that a varifold $V$ is orientable if and only if $\text{reg} V$ is orientable.

For several results we will may some control on the singular set. The weakest condition we will use is the $\alpha$-structural hypothesis of Wickramasekera ([120], see also [37] Section 12): An integral varifold $V$ satisfies the $\alpha$-structural hypothesis for some $\alpha \in (0, 1)$, if no point of $\text{sing} V$ has a neighbourhood in which $\text{spt} V$ corresponds to the union of at least three embedded $C^{1,\alpha}$ hypersurfaces with boundary that meet only along their common $C^{1,\alpha}$ boundary. Note that the $\alpha$-structural hypothesis is automatically satisfied if, for instance, $\text{sing} V$ has vanishing codimension 1 Hausdorff measure.

Note that any hypersurface $\Sigma^n$ with locally bounded $n$-dimensional Hausdorff measure defines an integral varifold that we denote by $[\Sigma]$.

We say that a $k$-varifold $V$ in $\mathbb{R}^{n+1}$ has Euclidean volume growth if there exists a constant $C_V > 0$ so that $\mu_V(B_r(x)) \leq C_V r^k$ for any $r > 0$ and any $x \in \mathbb{R}^{n+1}$.

3.1. Special varifolds. We will say that $V$ splits off a line if it is invariant under translations in some direction; if this is the case then, up to a rotation of $\mathbb{R}^{n+1}$, we may write $\mu_V = \mu_{\mathbb{R}} \times \mu_{\widetilde{V}}$ as the product of a multiplicity one line with an integer rectifiable $(n - 1)$-varifold $\widetilde{V}$ in $\mathbb{R}^n$. We say that an integral varifold $V$ is a cone if it is invariant under dilations about the origin; if this is the case then the link $W = V \setminus S^n$ is indeed an integer rectifiable $(n - 1)$-varifold in $S^n$ and we write $V = C(W)$. Of course, $C(W)$ is orientable if and only if $W$ is orientable.

We say that an $n$-varifold $V$ in $\mathbb{R}^{n+1}$ is stationary (for area) if it has zero generalised mean curvature $\vec{H} = 0$. In particular the regular part must be minimal in $\mathbb{R}^{n+1}$. It is
straightforward to see that a cone $V = C(W)$ is stationary if and only if the link $W$ is stationary in $\mathbb{S}^n$. Here an integral $(n - 1)$-varifold $W$ in $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ is stationary if its generalised mean curvature in $\mathbb{R}^{n+1}$ is given by $\bar{H}(p) = -(n - 1)p$. In particular its regular part is minimal in $\mathbb{S}^n$.

We say that $V$ is $F$-stationary if it is instead stationary for the $F$-functional defined above, or alternatively with respect to the conformal metric $e^{-\frac{|x|^2}{2n}} \delta_{ij}$ on $\mathbb{R}^{n+1}$. Equivalently, its generalised mean curvature is given as before by

$$
(3.3) \quad \bar{H} = -\frac{1}{2} x^\perp.
$$

In particular the regular part must be a self-shrinking hypersurface. Also, it follows that a cone $V = C(W)$ is $F$-stationary if and only if it is stationary in $\mathbb{R}^{n+1}$.

### 3.2. Regularity theory

This following proposition follows from the regularity theory of Wickramasekera [120] for stable (that is, $\lambda_1 \geq 0$) stationary varifolds, as presented in [37] Section 12. The key is that the regularity theory goes through even if one only assumes a slightly weaker stability inequality of the form $\int_\Sigma |A|^2 \phi^2 \leq (1 + \epsilon) \int_\Sigma |\nabla \phi|^2$; see for instance [37] Proposition 12.25], which holds for small balls in any fixed Riemannian manifold. Similarly, a version of [37] Lemma 12.7 holds on any closed ambient manifold, and it holds with any lower bound $\lambda_1(\Sigma) > -\infty$ because the $\int_\Sigma \phi^2$ term in (2.9) can be controlled on small balls (by the Poincaré inequality) to be small relative to the $\int_\Sigma |\nabla \phi|^2$ term.

**Proposition 3.1.** Let $V$ be an orientable stationary $n$-varifold in $M^{n+1}$ with Euclidean volume growth and satisfying the $\alpha$-structural hypothesis for some $\alpha \in (0, \frac{1}{2})$. Suppose that $\lambda_1(V) > -\infty$. Then $V$ corresponds to an embedded, smooth (analytic if the ambient metric is analytic) hypersurface away from a closed set of singularities of Hausdorff dimension at most $n - 7$ (that is empty if $n \leq 6$ and discrete if $n = 7$.)
It will also be useful to record a connectedness lemma that follows from the varifold
maximum principle of Wickramasekera [120, Theorem 19.1] together with the proof of [70]
Theorem A(ii)].

**Lemma 3.2.** Let $V$ be a stationary integral $n$-varifold in $M^{n+1}$. Suppose $\mathcal{H}^{n-1}(\text{sing } V) = 0$. Then $\text{reg } V$ is connected if and only if $\text{spt } V$ is connected.

### 3.3. Entropy.

Note that finite entropy implies Euclidean volume growth:

**Lemma 3.3.** Let $V$ be an integral $n$-varifold in $\mathbb{R}^{n+1}$ with finite entropy $\Lambda(V) < \infty$. Then for any $x_0$ and any $r > 0$, we have

\[(3.4) \quad \mu_V(B_r(x_0)) \leq e^{\frac{1}{4}(4\pi)^{\frac{n}{2}}\Lambda(V)r^n}.\]

**Proof.** As $e^{-\frac{|x-x_0|^2}{4r^2}} \geq e^{-\frac{1}{4}}$ for any $x \in B_r(x_0)$, we have

\[(3.5) \quad \mu_V(B_r(x_0)) \leq e^{\frac{1}{4}} \int e^{-\frac{|x-x_0|^2}{4r^2}} d\mu_V(x).\]

The result follows by definition of the entropy $\Lambda(V)$.

Colding-Minicozzi showed that the entropy of a smooth self-shrinker is achieved by the $F = F_{0,1}$ functional. Ketover and Zhou [76, Lemma 10.4] extended their computation to the singular setting (in fact for more general varifolds):

**Lemma 3.4 ([76]).** Let $V$ be an $F$-stationary varifold satisfying $F(V) < \infty$. Fix $a \in \mathbb{R}$ and $y \in \mathbb{R}^{n+1}$ and set $g(s) = F_{sy,1+as^2}(V)$. Then for all $s > 0$ with $1 + as^2 > 0$ we have

\[(3.6) \quad g'(s) = -\frac{1}{2(1 + as^2)} \int \frac{[(asx + y)^{\frac{1}{2}}]s}{1 + as^2} \rho_{sy,1+as^2} d\mu_V(x) \leq 0.\]

Consequently the map $(x_0, t_0) \mapsto F_{x_0,t_0}(V)$ achieves its global maximum at $(0, 1)$, that is, $\Lambda(V) = F(V) < \infty$.

As a result we see that stationary cones have finite entropy:
Corollary 3.5. Let $V = C(W)$ be a stationary n-cone. Then $V$ has finite entropy given by $\Lambda(V) = \frac{\|W\|}{\text{Vol}(S^{n-1})}$, where $S^{n-1}$ is the totally geodesic equator of $S^n$ and $\|W\|$ is the total mass of the link $W$.

Proof. A straightforward calculation in polar coordinates gives $F_{0,1}(V) = \frac{\|W\|}{\text{Vol}(S^{n-1})} < \infty$, and the result then follows from Lemma 3.4.

We can characterise the equality case in Lemma 3.4 as follows:

Lemma 3.6. Let $V$ be an $F$-stationary varifold in $\mathbb{R}^{n+1}$.

(1) If $x^\perp = 0$ a.e. on $V$ where $x$ is the position vector, then $\Sigma$ is a stationary cone.

(2) If $y^\perp = 0$ a.e. on $V$ for some fixed vector $y$, then $\Sigma$ splits off a line.

Proof. For point (1) suppose $x^\perp = 0$ a.e. on $V$. Then the generalised mean curvature of $V$ is $\bar{H} = -\frac{1}{2}x^\perp = 0$, so $V$ is stationary for the area functional. The fact that $V$ is now a cone follows from the monotonicity formula as detailed in the proof of [105, Theorem 19.3]. We will not reproduce it here as it is similar to the proof of the second case to follow.

For point (2) suppose $y^\perp = 0$ a.e. on $V$. Without loss of generality we may assume $y = e_{n+1}$. We therefore write $x = (x', x_{n+1})$, where $x' \in \mathbb{R}^n$. By the slicing theorem, the slices $V \cap \{x_{n+1} = s\}$ are integral $(n-1)$-varifolds for almost every $s \in \mathbb{R}$.

Let $\phi : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R}^n \to \mathbb{R}$ be $C^1$, compactly supported functions. We set

$$g(s) = \int f(x')\phi(x_{n+1} + s) \, d\mu_V(x),$$

so that $g'(s) = \int f(x')\phi'(x_{n+1} + s) \, d\mu_V(x)$.

Consider the vector field $X = f(x')\phi(x_{n+1} + s)e_{n+1}$. We calculate

$$\text{div}_V X = \phi(x_{n+1} + s)\langle \nabla f, e_{n+1} \rangle + f(x)\phi'(x_{n+1} + s)\langle e_{n+1}^T, e_{n+1} \rangle$$

Since $e_{n+1} = e_{n+1}^T$ a.e. on $\Sigma$, we have that $\langle \nabla f, e_{n+1} \rangle = \langle Df, e_{n+1} \rangle = 0$, $\langle e_{n+1}^T, e_{n+1} \rangle = 1$ and $\langle x^\perp, e_{n+1} \rangle = 0$. Since $\bar{H} = -\frac{1}{2}x^\perp$, plugging into (3.2) then gives that $g'(s) \equiv 0$, hence $g(s)$ is constant in $s$. 26
Now fix $a > 0$. Using $\phi$ to approximate the characteristic function of the interval $[0, a]$, our work above shows that $\int f(x') \chi_{\{s \leq x_{n+1} \leq s+a\}} \, d\mu_V(x)$ is constant in $s$, for any compactly supported $C^1$ function $f$ on $\mathbb{R}^n$. Set $V^s = V \cap \{x_{n+1} = s\}$. For almost every $s \in \mathbb{R}$, both slices $V^s$ and $V^{s+a}$ are integer rectifiable, so using the coarea formula and differentiating gives that

\begin{equation}
\int f(x') \, d\mu_{V^s}(x) = \int f(x') \, d\mu_{V^{s+a}}(x)
\end{equation}

for all such $s$. Another application of the coarea formula then gives that $\mu_V$ is invariant under translation by $ae_{n+1}$. Since $a$ was arbitrary, this concludes the proof.

In particular, the map $(x_0, t_0) \mapsto F_{x_0, t_0}(V)$ has a strict global maximum at $(0, 1)$ for $F$-stationary varifolds $V$ that do not split off a line and are not cones. Similarly the map $x_0 \mapsto F_{x_0, 1}(V)$ has a strict global maximum at $x_0 = 0$ if $V$ does not split off a line.
CHAPTER 3

Eigenvalue estimates on minimal hypersurfaces

In this chapter we present several results by the author concerning the first eigenvalue of the Laplacian and the Jacobi operator on minimal hypersurfaces, in ambient spaces of positive Ricci or Bakry-Emery-Ricci curvature.

1. A cutoff technique for singular hypersurfaces

The work described in this section was first presented in [127]. In this section we present, in full detail, the cutoff construction first stated by Morgan and Ritoré [92]. We do so for the case of submanifolds in Euclidean space with compact support and singular set of vanishing codimension 2 measure. The other cases considered in [92] can be handled similarly; see also Remark 1.3. In this section, $B(x, r)$ denotes a Euclidean ball of radius $r$ centred at $x$.

First we need an easy bound for the number of intersections of balls of comparable radii.

**Lemma 1.1.** Let $\mathcal{B} = \{B(p_i, r_i)\}$ be a collection of balls in $\mathbb{R}^N$. Suppose that there are $\alpha, \beta \geq 1$ such that the sub-balls $\{B(p_i, r_i/\alpha)\}$ are pairwise disjoint, and that the radii are $\beta$-comparable, that is, $\sup r_i \leq \beta \inf r_i$. Then each ball in $\mathcal{B}$ intersects at most $(3\alpha\beta)^N - 1$ other balls in $\mathcal{B}$.

**Proof.** Fix a ball $B(p_i, r_i) \in \mathcal{B}$. Any ball in $\mathcal{B}$ that intersects $B(p_i, r_i)$ must be contained in the larger ball $B(p_i, r_i + 2\sup r_j)$, which has volume at most $\omega_N(3\sup r_j)^N$. Here $\omega_N$ is the volume of the unit ball in $\mathbb{R}^N$.

On the other hand, since the sub-balls $\{B(p_j, r_j/\alpha)\}$ are pairwise disjoint, each must take up a volume at least $\omega_N(\inf r_j/\alpha)^N$ in the larger ball $B(p_i, r_i + 2\sup r_j)$. So at most $\frac{(3\sup r_j)^N}{(\inf r_j/\alpha)^N} \leq (3\alpha\beta)^N$ sub-balls can fit in the larger ball, and this implies the result. \qed

We now proceed to the construction of smooth cutoff functions around the singular set.
Proposition 1.2 ([92]). Let \( \Sigma^k \) be a smooth embedded submanifold in \( \mathbb{R}^N \) with bounded mean curvature and compact closure \( \Sigma \). If \( \text{sing } \Sigma = \Sigma \setminus \Sigma \) satisfies \( \mathcal{H}^{k-2}(\text{sing } \Sigma) = 0 \), then for any \( \epsilon > 0 \) there exists a smooth function \( \varphi_\epsilon : \Sigma \to [0,1] \) supported in \( \Sigma \) such that:

1. \( \mathcal{H}^k(\{\varphi_\epsilon \neq 1\}) < \epsilon; \)
2. \( \int_\Sigma |\nabla \varphi_\epsilon|^2 < \epsilon; \)
3. \( \int_\Sigma |\Delta \varphi_\epsilon| < \epsilon. \)

Proof. Fix a smooth radial cutoff function \( \varphi : \mathbb{R}^N \to [0,1] \) such that \( \varphi = 0 \) in \( B(0,1/2) \) and \( \varphi = 1 \) outside \( B(0,1) \). The derivatives are bounded, say \( |D\varphi|^2 + |D^2\varphi| \leq C_0. \) By scaling \( \varphi \) to \( B(x,r) \), \( r \leq 1 \), we get a cutoff function satisfying \( |D\varphi|^2 + |D^2\varphi| \leq C_0 r^{-2}. \)

Since \( \Sigma \) has bounded mean curvature \( |\vec{H}| \leq C_H \), the monotonicity formula \([105]\) implies that there is a constant \( C_V \) such that \( \mathcal{H}^k(\Sigma \cap B(x,r)) \leq C_V r^k \) for any \( r \leq 1 \) and any \( x \). Moreover, on \( \Sigma \cap B_e(x) \) we will have

\[
|\Delta \varphi| \leq k|D^2\varphi| + |(\vec{H}, D\varphi)| \leq kC_0 r^{-2} + C_H \sqrt{C_0} r^{-1} \leq C_1 r^{-2}.
\]

Now let \( \epsilon > 0 \). By definition of Hausdorff measure we may cover the singular set by finitely many balls \( \{B(p_i, r_i/6)\} \) such that \( r_i \leq 1 \) and \( \sum_i r_i^{k-2} \leq \epsilon. \) Consider the enlarged cover \( \{B(p_i, r_i/2)\} \). If \( B(p_i, r_i/6) \cap B(p_j, r_j/6) \neq \emptyset \), then certainly \( B(p_j, r_j/6) \subset B(p_i, r_i/2) \), so we could discard any such \( j \). In doing so we obtain a cover \( \{B(p_i, r_i/2)\} \) such that the \( \{B(p_i, r_i/6)\} \) are pairwise disjoint. We may relabel the radii so that \( r_1 \geq r_2 \geq \cdots \), and we partition the balls into classes of comparable radii \( B_m = \{i | 2^m \leq r_i < 2^{m+1}\}. \)

Cut off on each \( B(p_i, r_i) \) by scaled cutoff functions \( \varphi_i \) as above, then set \( \varphi_\epsilon = \prod_i \varphi_i. \) Immediately we have \( \mathcal{H}^k(\{\varphi_\epsilon \neq 1\}) \leq \sum_i C_V r_i^m < C_V \epsilon. \) For properties (2) and (3) we must bound the sum of all product terms \( \int_\Sigma |\nabla \varphi_i| |\nabla \varphi_j| \). Such a term is zero if \( B(p_i, r_i) \) and \( B(p_j, r_j) \) are disjoint, otherwise we have the bound

\[
\int_\Sigma |\nabla \varphi_i| |\nabla \varphi_j| \leq \frac{C_0 C_V}{r_i r_j} \min(r_i, r_j)^k.
\]
The procedure to estimate these cross terms is as follows: We fix $j$ and consider the sum over $i \leq j$ (that is, $r_i \geq r_j$). Letting $\mathcal{B}_{m_j}$ be the class containing $j$, we will bound the number of intersections that $B(p_j, r_j)$ can have with balls $B(p_i, r_i)$ in each class $\mathcal{B}_{m_j + h}$, $h \geq 0$.

The key observation is that if $B(p_i, r_i) \cap B(p_j, r_j) \neq \emptyset$, then certainly the enlarged ball $B(p_i, r_i + r_j)$ must contain the point $p_j$. In particular all such enlarged balls must intersect each other. But for $i \in \mathcal{B}_{m_j + h}$, the radii $r_i + r_j$ are comparable to within a factor of

\begin{equation}
\frac{2^{m_j + h + 1} + r_j}{2^{m_j + h} + r_j} \leq \frac{2^{m_j + h + 1} + 2^{m_j + 1}}{2^{m_j + h} + 2^{m_j}} = 2.
\end{equation}

For any such $i$ we also have $\frac{r_j}{r_i} \leq \frac{2^{m_j + 1}}{2^{m_j + h}} = 2^{1-h}$, so in particular $\frac{r_i + r_j}{r_i} \leq 3$ and the sub-balls \{$B(p_i, \frac{r_i + r_j}{3})| i \in \mathcal{B}_{m_j + h}\}$ must be pairwise disjoint. Thus by Lemma 1.1,

\begin{equation}
\text{#}\{i \in \mathcal{B}_{m_j + h}|B(p_i, r_i) \cap B(p_j, r_j) \neq \emptyset\} \leq \text{#}\{i \in \mathcal{B}_{m_j + h}|B(p_i, r_i + r_j) \ni p_j\} \leq 108^N.
\end{equation}

Using again that $\frac{r_j}{r_i} \leq 2^{1-h}$ for $i \in \mathcal{B}_{m_j + h}$, the estimate (1.2) gives

\begin{equation}
\int_{\Sigma} |\nabla \varphi_i| |\nabla \varphi_j| \leq C_0 C_V r_j^{k-1} r_i^{-1} \leq C_0 C_V 2^{1-h} r_j^{k-2}.
\end{equation}

Then by (1.4) we have that

\begin{equation}
\sum_{i \leq j} \int_{\Sigma} |\nabla \varphi_i| |\nabla \varphi_j| \leq 108^N C_0 C_V 2^{1-h} r_j^{k-2},
\end{equation}

and summing over $h \geq 0$ we get that $\sum_{i \leq j} \int_{\Sigma} |\nabla \varphi_i| |\nabla \varphi_j| \leq 4(108^N) C_0 C_V r_j^{k-2}$.

Finally, summing over $j$ gives

\begin{equation}
\sum_{i,j} \int_{\Sigma} |\nabla \varphi_i| |\nabla \varphi_j| = 2 \sum_{j} \sum_{i \leq j} \int_{\Sigma} |\nabla \varphi_i| |\nabla \varphi_j| \leq 8(108^N) C_0 C_V \sum_j r_j^{k-2} < 8(108^N) C_0 C_V \epsilon.
\end{equation}
Since each $\varphi_j^2 \leq 1$ we conclude that

\begin{equation}
\int_\Sigma |\nabla \varphi_i|^2 \leq \sum_{i,j} \int_\Sigma |\nabla \varphi_i||\nabla \varphi_j| < 8(108^N)C_0C_V \epsilon
\end{equation}

and

\begin{equation}
\int_\Sigma |\Delta \varphi_i| \leq \sum_i \int_\Sigma |\Delta \varphi_i| + \sum_{i,j} \int_\Sigma |\nabla \varphi_i||\nabla \varphi_j| < \sum_i C_1 C_V r_i^{k-2} + 8(108^N)C_0C_V \epsilon < (C_1 + 8(108^N)C_0)C_V \epsilon.
\end{equation}

Since $\epsilon$ was arbitrary this concludes the proof. \hfill \Box

**Remark 1.3.** Morgan and Ritoré [92] state their construction also for somewhat more general assumptions, which can be dealt with as follows. If the ambient space is a regular cone, the argument proceeds with one extra cutoff around the vertex. If $k = 2$ and the surface has isolated singular points, one may use logarithmic cutoff functions (and one does not need to be concerned with the intersections). Finally, to handle the noncompact case one may proceed by covering the singular set in annuli $B_{m+1} \setminus B_{m-1}$ with balls of radius $r_{m,i} < 1$ such that $\sum_i r_{m,i}^{k-2} \leq \frac{2^{-n\epsilon}}{C_V(m)}$.

### 2. Singular minimal hypersurfaces in the round sphere

The work described in this section was first presented in [127].

In this section, we present an estimate for the first eigenvalue of the Jacobi operator of a minimal hypersurface in a round sphere $S^{n+1}$. Namely, it was known that $\lambda_1 < -2n$ for any such hypersurface which is not itself a product of at most two round spheres. Our contribution is to extend this result to the setting of singular minimal hypersurfaces.

**Theorem 2.1.** Let $V$ be a stationary integral $n$-varifold in $S^{n+1}$ with orientable regular part, and which satisfies the $\alpha$-structural hypothesis for some $\alpha \in (0, \frac{1}{2})$.

Suppose that $V$ is not totally geodesic in $S^{n+1}$. Then $\lambda_1(V) \leq -2n$, with equality if and only if $V$ is an integer multiple of a Clifford hypersurface $S^k \left(\sqrt{\frac{k}{n}}\right) \times S^l \left(\sqrt{\frac{l}{n}}\right)$, $k + l = n$. 31
Recall that $\lambda_1$ may be defined on a singular minimal hypersurface like $M = \text{reg } V$ by

$$\lambda_1(M) = \inf_{\Omega} \lambda_1(\Omega) = \inf_f \frac{\int_M (|\nabla^M f|^2 - |A|^2 f^2 - nf^2)}{\int_M f^2}. \tag{2.1}$$

Also recall the Simons’ identity and inequality:

$$\Delta |A|^2 = 2|\nabla A|^2 + 2n|A|^2 - 2|A|^4, \tag{2.2}$$

$$|A|\Delta |A| = |\nabla A|^2 - |\nabla |A||^2 + n|A|^2 - |A|^4 \geq \frac{2}{n} |\nabla |A||^2 + n|A|^2 - |A|^4. \tag{2.3}$$

2.1. Integration on singular hypersurfaces. In this subsection we present some technical results that will allow us to work on the regular part of an integral varifold with sufficiently small singular set.

2.1.1. Cutoff functions. We first detail our choice of cutoff functions that will allow us to integrate around the singular set, so long as that set is small enough.

Consider an integral $n$-varifold $V$ in $\mathbb{S}^{n+1}$. The singular set is closed and hence compact. So by definition of Hausdorff measure, if $\mathcal{H}^{n-q}(\text{sing } V) = 0$, then for any $\epsilon > 0$ we may cover the singular set by finitely many geodesic balls of $\mathbb{S}^{n+1}$, $\text{sing } V \subset \bigcup_{i=1}^{m} B_{r_i}(p_i)$, where $\sum_i r_i^{n-q} < \epsilon$ and we may assume without loss of generality that $r_i < 1$ for each $i$.

Given such a covering we take smooth cutoff functions $0 \leq \phi_{i,\epsilon} \leq 1$ on $\mathbb{S}^{n+1}$ with $\phi_{i,\epsilon} = 1$ outside $B_{2r_i}(p_i)$, $\phi_{i,\epsilon} = 0$ inside $B_{r_i}(p_i)$ and $|\nabla \phi_{i,\epsilon}| \leq \frac{2}{r_i}$ in between. We then set $\phi_\epsilon = \inf_i \phi_{i,\epsilon}$, which is Lipschitz with compact support away from $\text{sing } V$, and $|\nabla \phi_\epsilon| \leq \sup_i |\nabla \phi_{i,\epsilon}|$.

2.1.2. Integration by parts. In this subsection $M^\alpha$ will denote the regular part of an integral $n$-varifold $V$ in $\mathbb{S}^{n+1}$ with Euclidean volume growth $\mu_V(B_r(x)) \leq C_V r^n$. The main goal is to establish conditions under which integration by parts is justified on $M$. Henceforth, $L^p = L^p(M)$ and $W^{k,p} = W^{k,p}(M)$ will denote the usual function spaces on $M$. 

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Lemma 2.2. Suppose that $\mathcal{H}^{n-q}(\text{sing } V) = 0$ for some $q > 0$. Let $\phi_\epsilon$ be as in subsection 2.1.1. Then on $M = \text{reg } V$ we have the following gradient estimate:

$$(2.4) \quad \int_M |\nabla \phi_\epsilon|^q \leq 2^{n+q}C_V \epsilon.$$ 

Proof. We have

$$(2.5) \quad \int_M |\nabla \phi_\epsilon|^q \leq \sum_{i=1}^N \int_{M \cap B_{2r_i}(p_i) \setminus B_{\epsilon}(p_i)} \frac{2^q}{r_i^q} \leq 2^{n+q}C_V \sum_i r_i^{n-q} \leq 2^{n+q}C_V \epsilon.$$ 

Corollary 2.3. Assume that $\mathcal{H}^{n-q}(\text{sing } V) = 0$ for some $q$ and let $M = \text{reg } V$. Let $\phi_\epsilon$ be as in subsection 2.1.1.

(1) Suppose that $q \geq 1$ and that $f$ is $L^p$ on $M$, where $p = \frac{q}{q-1}$. Then

$$(2.6) \quad \lim_{\epsilon \to 0} \int_M ||f|| |\nabla \phi_\epsilon| = 0.$$ 

(2) Suppose that $q \geq 2$ and that $f$ is $L^p$ on $M$, where $p = \frac{2q}{q-2}$. Then

$$(2.7) \quad \lim_{\epsilon \to 0} \int_M f^2 |\nabla \phi_\epsilon|^2 = 0.$$ 

Proof. For (1), using Hölder’s inequality, we have

$$(2.8) \quad \int_M |f||\nabla \phi_\epsilon| \leq ||f||_p \left( \int_M |\nabla \phi_\epsilon|^q \right)^{\frac{1}{q}}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Similarly for (2) we have

$$(2.9) \quad \int_M f^2 |\nabla \phi_\epsilon|^2 \leq ||f||_p^2 \left( \int_M |\nabla \phi_\epsilon|^q \right)^{\frac{2}{q}}$$

where $\frac{2}{p} + \frac{2}{q} = 1.$
By supposition the $L^p$-norms of $f$ are finite, so both results now follow from Lemma 2.2.

**Lemma 2.4.** Suppose $\mathcal{H}^{n-q}(\text{sing } V) = 0$ for some $q \geq 1$. Further suppose that $u,v$ are $C^2$ functions on $M = \text{reg } V$ such that $|\nabla u|, |\nabla v|$ and $|u \Delta v|$ are $L^1$, and $|u \nabla v|$ is $L^p$, $p = \frac{q}{q-1}$. Then

$$
(2.10) \quad \int_M u \Delta v = - \int_M \langle \nabla u, \nabla v \rangle.
$$

**Proof.** If $\phi$ has compact support we may integrate by parts to get

$$
(2.11) \quad \int_M \phi u \Delta v = - \int_M \phi \langle \nabla u, \nabla v \rangle - \int_M u \langle \nabla v, \nabla \phi \rangle.
$$

Applying this to $\phi = \phi_\epsilon$, Corollary 2.3 gives that the second term on the right tends to zero as $\epsilon \to 0$, so the result follows by dominated convergence.

**2.2. Integral estimates for $|A|$.** Throughout this subsection $M^n$ will denote the regular part of an orientable stationary $n$-varifold $V$ in $S^{n+1}$, which satisfies $\lambda_1 = \lambda_1(V) > -\infty$. Recall that $V$ automatically has Euclidean volume growth, and that the stability inequality

$$
(2.12) \quad \int_M |A|^2 \phi^2 \leq \int_M |\nabla \phi|^2 + (-\lambda_1 - n) \int_M \phi^2
$$

holds for any $\phi$ with compact support in $M$. The goals of this subsection are to provide $L^2$ estimates for the second fundamental form $A$ on small balls, and to use these estimates to show that $|A|$ is $L^4$ on $M$.

**Lemma 2.5.** Suppose that $\mathcal{H}^{n-2}(\text{sing } V) = 0$ and let $M = \text{reg } V$. If $\lambda_1 > -\infty$, then $|A|$ is $L^2$ on $M$.

**Proof.** Let $\phi_\epsilon$ be as in subsection 2.1. Plugging $\phi_\epsilon$ into the stability inequality (2.12), the last term is bounded by the volume of $M$ since $\phi_\epsilon^2 \leq 1$. The gradient term is controlled by Lemma 2.2, so the result follows by Fatou’s lemma as we take $\epsilon \to 0$. 

$\square$
Lemma 2.6. Suppose that $\mathcal{H}^{n-2}(\text{sing } V) = 0$ and let $M = \text{reg } V$. Further suppose that $\lambda_1 > -\infty$. Then there exists $C = C(n, V, \lambda_1)$ so that for any $r \in (0, 2)$, $p \in S^{n+1}$, we have

\begin{equation}
\int_{M \cap B_r(p)} |A|^2 \leq Cr^{n-2}.
\end{equation}

Proof. First we fix a cutoff function $0 \leq \eta \leq 1$ so that $\eta = 1$ inside $B_r(p)$, $\eta = 0$ outside $B_{2r}(p)$ and $|\nabla \eta| \leq \frac{2}{r}$ in between. Then

\begin{equation}
\int_{M \cap B_r(p)} |A|^2 \leq \int_M |A|^2 \eta^2.
\end{equation}

Since by Lemma 2.5 $|A|$ is $L^2$, these integrals are finite and using dominated convergence we may approximate

\begin{equation}
\int_M |A|^2 \eta^2 = \lim_{\epsilon \to 0} \int_M |A|^2 \eta^2 \phi_{\epsilon}^2.
\end{equation}

Now using the stability inequality (2.12), for each $\epsilon > 0$ we have

\begin{equation}
\int_M |A|^2 \eta^2 \phi_{\epsilon}^2 \leq 2 \int_M \eta^2 |\nabla \phi_{\epsilon}|^2 + 2 \int_M \phi_{\epsilon}^2 |\nabla \eta|^2 + \alpha \int_M \eta^2 \phi_{\epsilon}^2,
\end{equation}

where we have set $\alpha = | - \lambda_1 - n|$.

Since $\eta^2 \leq 1$, by Lemma 2.2 the first term tends to zero as $\epsilon \to 0$, where we have used that $\mathcal{H}^{n-2}(\text{sing } V) = 0$. Then since also $\phi_{\epsilon}^2 \leq 1$, we have

\begin{equation}
\int_M \eta^2 \phi_{\epsilon}^2 \leq \int_M \eta^2 \leq \int_{M \cap B_{2r}(p)} 1 \leq 2^n C_V r^n,
\end{equation}

and

\begin{equation}
\int_M \phi_{\epsilon}^2 |\nabla \eta|^2 \leq \int_M |\nabla \eta|^2 \leq \int_{M \cap B_{2r}(p)} \frac{4}{r^2} \leq 2^{n+2} C_V r^{n-2},
\end{equation}

so the result follows.

Lemma 2.7. Suppose that $\mathcal{H}^{n-4}(\text{sing } V) = 0$ and let $M = \text{reg } V$. If $\lambda_1 > -\infty$, then $|A|$ is $L^4$, and $|\nabla |A||$ and $|\nabla A|^2$ are $L^2$, on $M$. 

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PROOF. We adapt the Schoen-Simon-Yau \[97\] type argument.

Set \(\alpha = | - \lambda_1 - n|\). If \(f\) has compact support in \(M\) then applying the stability inequality \[2.12\] with \(\phi = |A|f\) and using the absorbing inequality yields

\[
(2.19) \quad \int_M |A|^4 f^2 \leq (1 + a) \int_M |\nabla |A||^2 f^2 + \int_M |A|^2 ( (1 + a ^{-1} )|\nabla f|^2 + \alpha f^2) ,
\]

where \(a > 0\) is an arbitrary positive number to be chosen later.

On the other hand, multiplying the Simons’ inequality \[2.3\] by \(f^2\) (dropping the harmless \(n|A|^2\) term), integrating by parts and again using the absorbing inequality gives

\[
(2.20) \quad \int_M |A|^4 f^2 + a^{-1} \int_M |A|^2 |\nabla f|^2 \geq \left( 1 + \frac{2}{n} - a \right) \int_M |\nabla |A||^2 f^2 .
\]

Combining these gives

\[
(2.21) \quad \int_M |A|^4 f^2 \leq \frac{1 + a}{1 + \frac{2}{n} - a} \int_M |A|^4 f^2 + C \alpha \int_M |A|^2 (|\nabla f|^2 + \alpha f^2) .
\]

Choosing \(a < \frac{1}{n}\) gives that the first coefficient on the right is less than 1 and hence may be absorbed on the left, thus

\[
(2.22) \quad \int_M |A|^4 f^2 \leq C \int_M |A|^2 (f^2 + |\nabla f|^2) ,
\]

where \(C = C(n, \alpha)\).

We now apply this inequality with \(f = \phi_\epsilon\), where \(\phi_\epsilon\) is as in subsection \[2.1\] with \(q = 4\). As \(\epsilon \to 0\), the first term on the right converges to \(\int_M |A|^2\), which we already know is finite. We bound the second term as follows:

\[
(2.23) \quad \int_M |A|^2 |\nabla \phi_\epsilon|^2 \leq \sum_{i=1}^m \frac{4}{r_i^3} \int_{M \cap B_{2r_i}(p_i)\setminus B_{r_i}(p_i)} |A|^2 .
\]

Using Lemma \[2.6\] we have that

\[
(2.24) \quad \int_{M \cap B_{2r_i}(p_i)\setminus B_{r_i}(p_i)} |A|^2 \leq C' r_i^{n-2} ,
\]
where $C'$ depends on $\alpha$ and the volume bounds for $M$.

Therefore

\begin{equation}
\int_M |A|^2|\nabla \phi_\epsilon|^2 \leq 4C' \sum_i r_i^{n-4} < 4C'\epsilon.
\end{equation}

where we recall that the $r_i$ were chosen so that $\sum_i r_i^{n-4} < \epsilon$.

Taking $\epsilon \to 0$ we see that this term tends to 0, thus we have shown that indeed $|A|$ is $L^4$. With this fact in hand, it follows from (2.20) that $|\nabla |A||$ is $L^2$.

Finally, multiplying the identity (2.2) by $f^2$ and integrating by parts, we have that

\begin{equation}
\int_M f^2(|\nabla A|^2 - |A|^4) \rho \leq -\int_M 2f|A|(|\nabla f, \nabla |A||) \leq \int_M (f^2|\nabla |A||^2 + |A|^2|\nabla f|^2).
\end{equation}

Since we now know that $|\nabla |A||$ is $L^2$ and that $|A|$ is $L^4$, we again set $f = \phi_\epsilon$ and use (2.25) to control the last term; this shows that $|\nabla A|^2$ is $L^2$, as desired.

\[\Box\]

2.3. First stability eigenvalue. In this subsection we prove our main theorem that the first stability eigenvalue of a stationary $n$-varifold in $\mathbb{S}^{n+1}$ is at most $-2n$. First we need the following lemma:

\textbf{Lemma 2.8.} Let $V$ be a stationary integral $n$-varifold in $\mathbb{S}^{n+1}$, with orientable regular part. If $\mathcal{H}^{n-4}(\text{sing } V) = 0$ then we get the same $\lambda_1 = \lambda_1(V)$ by taking the infimum over Lipschitz functions $f$ on $M = \text{reg } V$ such that $f \in W^{1,2} \cap L^4$.

\textbf{Proof.} Obviously we may assume that $\lambda_1 > -\infty$. Then by Lemma 2.5 and since $f$ is $L^4$, we also have that $|A|f$ is also $L^2$. We will use the functions $f_\epsilon = f \phi_\epsilon$, which are compactly supported away from the singular set, in the definition (2.1) of $\lambda_1$.

Now since $f$ and $|A|f$ are $L^2$, dominated convergence gives that $\int_M f_\epsilon^2 \to \int_M f^2$ and $\int_M |A|^2 f_\epsilon^2 \to \int_M |A|^2 f^2$ as $\epsilon \to 0$. For the gradient term we have

\begin{equation}
\int_M |\nabla f_\epsilon|^2 = \int_M (\phi_\epsilon^2|\nabla f|^2 + 2(\nabla f, \nabla \phi_\epsilon) + f^2|\nabla \phi_\epsilon|^2).
\end{equation}
Since \( f \) is \( L^4 \), parts (1) and (2) respectively of Corollary 2.3 give that the second and third terms on the right tend to zero as \( \epsilon \to 0 \). Moreover, the first term on the right tends to \( \int_M |\nabla f|^2 \) by dominated convergence. Thus we have shown that \( \int_M |\nabla f|^2 \to \int_M |\nabla f|^2 \), and the lemma follows.

We now proceed to the proof of Theorem 2.1.

**Theorem 2.9.** Let \( V \) be a stationary integral \( n \)-varifold in \( S^{n+1} \), with orientable regular part. Suppose that \( V \) satisfies the \( \alpha \)-structural hypothesis for some \( \alpha \in (0, \frac{1}{2}) \).

Further suppose that \( V \) is not totally geodesic in \( S^{n+1} \). Then \( \lambda_1(V) \leq -2n \), with equality if and only if \( V \) is an integer multiple of a Clifford hypersurface \( S^k \left( \sqrt{\frac{k}{n}} \right) \times S^l \left( \sqrt{\frac{l}{n}} \right) \), where \( k + l = n \).

**Proof.** Set \( M = \text{reg} V \).

Obviously we may assume \( \lambda_1 > -\infty \). The regularity theory for stable minimal hypersurfaces then implies that \( \text{sing} V \) has codimension at least 7. Thus we certainly have that \( \mathcal{H}^{n-4}(\text{sing} V) = 0 \).

So by Lemma 2.7 we have that \( |A| \) is \( L^4 \) (and \( L^2 \)), and that \( |\nabla |A|| \) and \( |\nabla A| \) are \( L^2 \). Therefore, by Lemma 2.8 we may use \( |A| \) as a test function in the definition of \( \lambda_1 \), that is we have

\[
\lambda_1(M) \leq \frac{\int_M (|\nabla |A||^2 - |A|^4 - n|A|^2)}{\int_M |A|^2}.
\]

Now we wish to integrate by parts. By (2.3) and our integral estimates for \( A \) we have that \( |A| \Delta |A| \) is \( L^1 \). Using Young’s inequality we have

\[
(|A||\nabla |A||)^p \leq \frac{2}{2 - p} |A|^{\frac{2p}{2 - p}} + \frac{p}{2} |\nabla |A||^2.
\]
Again since $|A|$ is $L^4$ and $|\nabla A|$ is $L^2$, this implies that $|A| |\nabla A|$ is $L^p$ for $p = \frac{4}{3}$. Thus now we may use Lemma 2.4 together with the Simons’ inequality (2.3) to find that

$$
(2.30) \quad \int_M |\nabla A|^2 = - \int_M |A| \Delta A \leq \int_M \left( -\frac{2}{n} |\nabla A|^2 + |A|^4 - n |A|^2 \right) 
$$

$$
\leq \int_M (|A|^4 - n |A|^2),
$$

which implies that $\lambda_1(M) \leq -2n$ as claimed.

If $\lambda_1(M) = -2n$, then equality must hold in all previous inequalities. In particular for equality to hold in the last step of (2.30) we must have $\int_M |\nabla A|^2 = 0$ and $\int_M |A|^4 = n \int_M |A|^2$. Therefore $|A|$ is equal to a constant on $M$, and the constant must be $\sqrt{n}$ since it is nonzero by supposition. The Gauss equation and a theorem of Lawson [79, Theorem 1] (which does not assume completeness) then imply that $M$ is a piece of a Clifford hypersurface $M_0 = S^k \left( \sqrt{\frac{2}{n}} \right) \times S^l \left( \sqrt{\frac{1}{n}} \right)$, where $k + l = n$. The support $\text{spt} \ V$ is then contained in $M_0$, so the constancy theorem implies that $V = m [M_0]$ for some integer $m$, as desired.

\[\square\]

3. Minimal hypersurfaces in positive Ricci curvature

The work described in this section was first presented in the published article [129].

In this subsection we present a deformation of the round hemisphere which preserves minimality of the boundary as well as the Ricci curvature condition, but for which the first eigenvalue of the boundary Laplacian decreases. Specifically, we prove:

**Theorem 3.1.** Let $n \geq 2$. Then there is a smooth metric $g$ on the hemisphere $\mathbb{S}_{+}^{n+1}$ such that:

- The ambient Ricci curvature satisfies $\text{Ric}_g \geq ng$.
- The boundary $\Sigma = \partial \mathbb{S}_{+}^{n+1}$ is minimal with respect to $g$.
- The first Laplace eigenvalue of the induced metric $\widehat{g}$ on $\Sigma$ satisfies $\lambda_1(\Sigma) < n$.
- The eigenfunction $\phi_1$ corresponding to $\lambda_1(\Sigma)$ satisfies $\int_{\Sigma} A(\widehat{\nabla} \phi_1, \widehat{\nabla} \phi_1) > 0$, where $\widehat{\nabla}$ is the gradient on $\Sigma$. 

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Here, and in this section only, we use $g$ for a general ambient metric, $\hat{g}$ for the induced metric on the hypersurface $\Sigma$ and $\overline{g}$ for the standard round metric on the sphere.

It will be useful to record that the second fundamental form with respect to a conformal metric $e^f g$ is given (see [16] for example) by

$$A(\Sigma, e^f g) = e^{f/2} A(\Sigma, g) + \frac{1}{2} e^{f/2} (\partial_0 f) \hat{g},$$

and therefore the mean curvature in this conformal metric is

$$H(\Sigma, e^f g) = e^{-f} \text{tr}_{\hat{g}} A(\Sigma, e^f g) = e^{-f/2} \left( H(\Sigma, g) + \frac{n}{2} \partial_0 f \right).$$

### 3.1. Operators on symmetric tensors.

Let $\Omega^1(M)$ be the space of smooth 1-forms on $M$, and let $S^2(M)$ be the space of smooth symmetric 2-tensors on $M$.

For $\omega \in \Omega^1(M)$, its divergence $\delta_g \omega$ is the smooth function given by $\delta_g \omega = \nabla^i \omega_i$. For $h \in S^2(M)$, the $g$-trace of $h$ is given by $\text{tr}_g h = g^{ij} h_{ij}$. The divergence of $h$ is a 1-form $\delta_g h$, with components given by $(\delta_g h)_i = \nabla^i h_{ij}$. We sometimes omit the subscripts when the metric is clear from context.

We will use the sign convention for the Laplacian that makes it a negative operator. That is, for arbitrary symmetric tensors, the rough Laplacian is defined by $\Delta = g^{ij} \nabla_i \nabla_j$. It is convenient to define the Hodge Laplacian, acting on 1-forms by

$$\Delta_H \omega_i = \Delta \omega_i - R^l_i \omega_j$$

and the Lichnerowicz Laplacian, defined for symmetric 2-tensors by

$$\Delta_L h_{ij} = \Delta h_{ij} + 2 R^l_{ki,j} h^k_l - R^l_i h_{jk} - R^k_{ij} h_{ik}.$$ 

We say that $\lambda$ is an eigenvalue of $\Delta$ with eigenfunction $\phi \neq 0$ if $\tilde{\Delta} \phi = -\lambda \phi$. On the closed manifold $\Sigma$, the eigenvalues of $\tilde{\Delta}$ are well-known to be discrete and, in our sign convention,
nonnegative, so we may order the distinct eigenvalues

$$0 = \lambda_0 < \lambda_1 < \lambda_2 \cdots$$

In particular, $\lambda_0 = 0$ has multiplicity 1, and $\lambda_1$ is the least nonzero eigenvalue of $\Delta$. We will denote by $V_\lambda$ the (finite dimensional) eigenspace corresponding to $\lambda$.

### 3.2. Adjoint decompositions.

When $M$ is compact without boundary, an integration by parts shows that, up to a sign, $\delta : \Omega^1(M) \to C^\infty(M)$ is the $L^2$-adjoint of the exterior derivative $d : C^\infty(M) \to \Omega^1(M)$. Similarly, the adjoint of the divergence $\delta : \mathcal{S}^2(M) \to \Omega^1(M)$ acting on symmetric 2-tensors is given by $\delta^* = -\frac{1}{2} \mathcal{L}$, where $\mathcal{L}$ is the Lie derivative of $g$ by (the metric dual of) $\omega$,

$$\text{(3.5) } (\mathcal{L}\omega)_{ij} = \nabla_i \omega_j + \nabla_j \omega_i.$$  

Before continuing, let us record the commutators of $\delta$ and $\mathcal{L}$ with the Laplacian. The following is a theorem of Lichnerowicz (see [81], or [43]).

**Proposition 3.2.** Suppose that $g$ is a metric with parallel Ricci curvature $\text{Ric}_g$. Then

$$\text{(3.6) } \delta_g \circ \Delta_L = \Delta_H \circ \delta_g \quad \text{and} \quad \Delta_L \circ \mathcal{L} = \mathcal{L} \circ \Delta_H.$$  

Now, as in Besse [16], taking the adjoint of the map $(\omega, f) \mapsto \mathcal{L}\omega + fg$ leads to the well-known orthogonal decomposition

$$\text{(3.7) } \mathcal{S}^2(M) = (C^\infty(M)g + \mathcal{L}(\Omega^1(M))) \oplus \hat{\mathcal{S}}^2(M),$$

where $\mathcal{S}^2(M)$ is the space of transverse traceless deformations, that is, $h \in \mathcal{S}^2(M)$ satisfying $\delta_g h = 0$ and $\text{tr}_g h = 0$. 
When $M$ has nonempty boundary $\partial M = \Sigma$, a boundary term appears in the integration by parts:

$$
(3.8) \quad \int_M h^{ij}(\mathcal{L}\omega + fg)_{ij} = \int_M (f \text{tr}_g h - 2\omega^j(\delta_g h)_j) + \int_{\Sigma} \omega^j h_{0j}.
$$

Thus the orthogonal decomposition becomes (see for example [113])

$$
(3.9) \quad \mathcal{S}^2(M) = (C^\infty(M)g + \mathcal{L}(\Omega^1(M))) \oplus \mathcal{S}^2(M),
$$

where now we denote by $\mathcal{S}^2(M)$ the space of transverse traceless deformations which satisfy the additional boundary conditions $h_{0j} = h_{j0} = 0$ on $\Sigma$, for all $j$.

### 3.3. Perturbation theory.

Consider a variation $g(t)$ of the metric on $M$, with $g(0) = g$ and $g' = h$, for some symmetric 2-tensor $h \in \mathcal{S}^2(M)$. Here, and henceforth, a primed quantity will denote its first variation, that is, $(\cdot)' = \frac{d}{dt}|_{t=0} (\cdot)$. To avoid confusion with this notation for variations, derivatives of functions $F$ of a single variable will later be denoted using dots, $\dot{F}$.

Throughout, we will assume our variations are at least $C^2$ in $t$.

#### 3.3.1. Ricci curvature.

The first variation $\text{Ric}'_g = \text{Ric}'_g(h)$ of the Ricci tensor satisfies (see [16])

$$
(3.10) \quad -2\text{Ric}'_g = \Delta_L h + \nabla^2(\text{tr}_g h) - \mathcal{L}\delta_g h.
$$

Since we are interested in the Ricci curvature as compared to the metric, we are more interested in the variation $(g^{-1}\text{Ric}_g)'$; if $g$ is initially Einstein with $\text{Ric}_g = k g$, then this variation is given by

$$
(3.11) \quad (g^{-1}\text{Ric}_g)' = g^{-1}(\text{Ric}'_g - kh).
$$

#### 3.3.2. Second fundamental form.

The first variation of the second fundamental form $A = A(\Sigma, g(t))$ is not difficult to compute (see for example [85]; note that our sign conventions...
are different):

\[(3.12)\]
\[A'_{\alpha\beta} = \frac{1}{2}(-\nabla_\alpha h_{0\beta} - \nabla_\beta h_{0\alpha} + \nabla_0 h_{\alpha\beta} + h_{00} A_{\alpha\beta}).\]

Then the mean curvature \( H = H(\Sigma, g(t)) \) satisfies

\[(3.13)\]
\[H' = -h^{\alpha\beta} A_{\alpha\beta} + g^{\alpha\beta} A'_{\alpha\beta} = -\nabla^\alpha h_{0\alpha} + \frac{1}{2}(\partial_0 (\text{tr}_3 \tilde{h}) - H h_{00}),\]

where \( \tilde{h} \) is the projection of \( h \) to \( S^2(\Sigma) \). When \( \Sigma \) is totally geodesic, it is sometimes convenient to compute in the ambient space instead; in this case we have

\[(3.14)\]
\[H' = - (\delta_3 h)_0 + \frac{1}{2} \partial_0 h_{00} + \frac{1}{2} \partial_0 \text{tr}_3 h.\]

3.3.3. Laplace operator. Berger [12] computes the first variation of the Laplacian and its spectrum on a closed manifold (see also [17] and [51]). Applying Berger’s results to \( \Sigma \), we have

\[(3.15)\]
\[\hat{\Delta}' = -(\hat{h}, \hat{\nabla}^2 \phi) - (\delta_3 \hat{h}, d_\Sigma \phi) + \frac{1}{2} (d_\Sigma (\text{tr}_3 \hat{h}), d_\Sigma \phi).\]

Here \((\cdot, \cdot)\) denotes the natural scalar product induced by \( \hat{g} \). If \( \psi, \phi \) are in the same initial eigenspace \( V_\lambda \) then, after some manipulations, integration by parts yields

\[(3.16)\]
\[\langle \psi, \hat{\Delta}' \phi \rangle = -\frac{1}{2} (\hat{h}, \phi \hat{\nabla}^2 \psi + \psi \hat{\nabla}^2 \phi) + \frac{1}{2} (\delta_3 \delta_3 \hat{h}, \psi \phi) - \frac{1}{4} (\hat{\Delta}(\text{tr}_3 \hat{h}), \psi \phi).\]

Here, and for the remainder of this section, \( \langle \cdot, \cdot \rangle \) shall denote the \( L^2 \) inner product on \( \Sigma \). We write \( \hat{\Delta}'_{\lambda} = \pi_{\lambda} \circ \hat{\Delta}|_{V_\lambda} : V_\lambda \to V_\lambda \) for the projection of the restriction of \( \hat{\Delta}' \) to \( V_\lambda \). In particular, we see from \( (3.16) \) that \( \hat{\Delta}'_{\lambda} \) is symmetric.

Even under smooth variations of the metric, the Laplace eigenvalues may not evolve smoothly. However, by the Rellich-Kato perturbation theory [95, 73], under real analytic variations of the metric \( \tilde{g}(t) \), the eigenvalues and eigenfunctions of the closed manifold \( \Sigma \) do also vary analytically in \( t \). In particular, we use the following statement as in Lemma 3.15 of Berger [12]:
Lemma 3.3 (Berger). Let $(\Sigma, \tilde{g})$ be a compact Riemannian manifold. Consider a family of metrics $\tilde{g}(t)$, analytic in $t$, with $\tilde{g}(0) = \tilde{g}$. If $\lambda$ is an eigenvalue of $\Delta_{\tilde{g}(0)}$ with multiplicity $m$, then there exist families of scalars $\Lambda_i(t)$ and smooth functions $\varphi_i(t)$, for $i = 1, \cdots, m$, each depending analytically on $t$, such that:

- $\Delta_{\tilde{g}(t)} \varphi_i(t) = -\Lambda_i(t) \varphi_i(t)$ for each $i$,
- $\Lambda_i(0) = \lambda$ for each $i$, and
- the $\{\varphi_i(t)\}$ are $L^2(\Sigma, \tilde{g}(t))$-orthonormal for all $t$.

Given such families of orthonormal eigenfunctions $\varphi_i(t)$, differentiating the condition that the $\varphi_i$ are orthonormal we have that $\langle \phi_i, \Delta \phi_j \rangle = 0$ for $i \neq j$, and

$$\lambda_i' = -\langle \phi_i, \Delta' \phi_i \rangle = \langle \mathcal{h}, \phi_i \nabla^2 \phi_i \rangle + \frac{1}{2} \langle \delta \mathcal{g}, d\Sigma(\phi_i^2) \rangle - \frac{1}{4} \langle d\Sigma(\text{tr} \mathcal{h}), d\Sigma(\phi_i^2) \rangle. $$

### 3.4. Geometry of round spheres

We realise $\mathbb{S}^{n+1}$ as the unit sphere in $\mathbb{R}^{n+2}$, which has coordinate functions $x_0, \cdots, x_{n+1}$. We consider the upper hemisphere

$$\mathbb{S}^{n+1}_+ = \{x \in \mathbb{S}^{n+1} : x_{n+1} \geq 0\},$$

which has boundary given by the great sphere

$$\Sigma = \partial \mathbb{S}^{n+1}_+ = \{x \in \mathbb{S}^{n+1} : x_{n+1} = 0\}.$$

For the standard metric $\bar{g}$, the curvature tensor is given by $R_{ijkl} = \bar{g}_{il} \bar{g}_{jk} - \bar{g}_{ik} \bar{g}_{jl}$. Thus $\bar{g} = \bar{g}_{\mathbb{S}^{n+1}}$ satisfies $\text{Ric}_{\bar{g}} = n \bar{g}$, and so the variation of the Ricci tensor at $g(0) = \bar{g}$ satisfies

$$-2(\text{Ric}_{g} - nh) = \Delta_L h + \nabla^2 (\text{tr}_g h) - \mathcal{L}_h h + 2nh.$$

Moreover, the Hodge Laplacian acts as $\Delta_H \omega = \Delta \omega - n\omega$, and the Lichnerowicz Laplacian $\Delta_L$ acts simply as

$$\Delta_L h = \Delta h - 2(n+1)h + 2(\text{tr}_g h)\bar{g}. $$
Typically we choose coordinates so that the standard metric \( g \) can be written (again, with \( e_0 = \partial_r \)) as the warped product

\[
(3.22) \quad g_{S^n+1} = dr^2 + \sin^2 r g_{S^n}.
\]

In terms of the coordinate functions on \( \mathbb{R}^{n+2} \), we take \( x_{n+1} = \cos r \), so that the equator \( \Sigma \) is the level set \( r = \pi/2 \).

In these coordinates, the Christoffel symbols of \( g_{S^n+1} \) are given by

\[
\Gamma^0_{00} = \Gamma^0_{a0} = \Gamma^\gamma_{00} = 0, \quad \Gamma^0_{a\beta} = -\sin r \cos r \hat{g}_{a\beta}, \quad \Gamma^a_{a0} = \delta^a \cot r, \quad \Gamma^\gamma_{a\beta} = \hat{\gamma}^\gamma_{a\beta},
\]

where \( \hat{\gamma} \) are the Christoffel symbols for \( g_{S^n} \) and \( \delta^a \) is the Kronecker delta.

The round sphere’s warped product structure also gives a decomposition of symmetric 2-tensors \( h \in \mathcal{S}^2(M) \) as in Delay’s paper \([43]\),

\[
(3.24) \quad h = u \, dr^2 + \xi \otimes dr + dr \otimes \xi + \hat{h}.
\]

That is, if \( S(r) \subset S^{n+1} \) is the geodesic \( n \)-sphere at a fixed \( r \), then \( u \) is the smooth function on \( S(r) \) given by \( u = h_{00} \), \( \xi \) is the 1-form on \( S(r) \) given by \( \xi_a = h_{a0} \) and \( \hat{h} \) is the projection of \( h \) to \( \mathcal{S}^2(S(r)) \) so that \( \hat{h}_{a\beta} = h_{a\beta} \).

Later, we will also use the warped product form iteratively, that is, we choose a coordinate \( s \) on \( S^n \) so that the standard metric on \( S^{n+1} \) becomes

\[
(3.25) \quad g_{S^n+1} = dr^2 + \sin^2 r ds^2 + \sin^2 r \sin^2 s g_{S^{n-1}}.
\]

In terms of coordinates, recalling that earlier we took \( x_{n+1} = \cos r \) where \( e_0 = \partial_r \), we now take \( x_n = \sin r \cos s \) with \( e_1 = \partial_s \).

The eigenfunctions of the Laplacian \( \hat{\Delta} \) on \( S^n \) with the standard metric \( g_{S^n} \) are well-known; they are the restrictions of homogeneous harmonic polynomials on \( \mathbb{R}^{n+1} \), which we
have realised as the subspace $x_{n+1} = 0$ of $\mathbb{R}^{n+2}$. The eigenvalues are then given by $d(d+n-1)$ for the homogeneous harmonic polynomials of degree $d$.

In particular, the first eigenvalue of $\Delta$ is $n$ with multiplicity $n+1$, and the corresponding eigenspace $V_n$ has a canonical orthonormal basis proportional to the remaining coordinate functions, which we denote

$$
\phi_{1,i} = \frac{x_i}{\sqrt{C_n}}, \quad i = 0, \cdots, n
$$

where $C_n = \frac{1}{n+1} \gamma_n$ and $\gamma_n$ is the volume of $S^n$ with standard metric $\overline{g}$. It will be useful to compute $C_n$ and related integrals directly:

**Lemma 3.4.** For all nonnegative integers $k$, we have

$$
\int_{S^n} (1 - x_n^2)^k \, dV_{\overline{g}} = B(k + \frac{n}{2}, \frac{1}{2}) \gamma_{n-1},
$$

$$
\int_{S^n} x_n^2 (1 - x_n^2)^k \, dV_{\overline{g}} = B(k + \frac{n}{2}, \frac{3}{2}) \gamma_{n-1},
$$

where $B(x, y)$ is the beta function and again $\gamma_{n-1} = \text{Vol}(S^{n-1}, \overline{g})$.

**Proof.** In our coordinates, we have

$$
\int_{S^n} (1 - x_n^2)^k \, dV_{\overline{g}} = \gamma_{n-1} \int_0^\pi \sin^{2k} s \sin^{n-1} s \, ds.
$$

Making the substitution $y = \sin^2 s$, the right hand side is then given by

$$
2 \gamma_{n-1} \int_0^{\frac{\pi}{2}} (\sin s)^{2k+n-1} \, ds = \gamma_{n-1} \int_0^1 y^{k+\frac{n-1}{2}} - \frac{1}{2} (1 - y)^{-\frac{1}{2}} \, dy,
$$

and we recognise the last term as the desired beta integral.

Similarly, we have

$$
\int_{S^n} x_n^2 (1 - x_n^2)^k \, dV_{\overline{g}} = \gamma_{n-1} \int_0^\pi \sin^{2k} s \cos^2 s \sin^{n-1} s \, ds,
$$
and the same substitution in the right hand side gives

$$2\gamma_{n-1} \int_0^\frac{\pi}{2} (\sin s)^{2k+n-1} \cos^2 s \, ds = \gamma_{n-1} \int_0^1 y^{k+n+\frac{1}{2}}(-\frac{1}{2})(1-y)^{\frac{1}{2}} \, dy,$$

which we again recognise as the claimed beta integral.

It is well-known that the first eigenfunctions $\phi \in V_n$ satisfy the Hessian equation $\nabla^2 \phi = -\phi g_{S^n}$. Moreover, since $|x| = 1$ on the sphere, we can decompose the squares of first eigenfunctions in terms of zeroth and second degree eigenfunctions - in particular, $x^2 = \frac{1}{n+1} \left(1 + \sum_{j \neq i} (x_i^2 - x_j^2)\right)$. Since the second eigenvalue is $2(n+1)$ we then have

$$\Delta \phi^2_{1,i} = \frac{2}{C_n} - 2(n+1)\phi^2_{1,i}.$$

Also, for $i \neq j$, the product $\phi_{1,i} \phi_{1,j}$ is itself a second degree eigenfunction:

$$\Delta(\phi_{1,i} \phi_{1,j}) = -2(n+1)\phi_{1,i} \phi_{1,j}.$$

### 3.5. Deformations of the round metric.

In this subsection we fix $M$ to be $S^{n+1}$ or $S_+^{n+1}$, and consider deformations $g = g(t)$ of the standard metric $g(0) = \bar{g}$. Throughout this subsection we set $h = g'$.  

#### 3.5.1. Conformal direction. We first consider the conformal part $C^\infty(M)\bar{g} + \mathcal{L}(\Omega^1(M))$, beginning with the diffeomorphism part:

**Lemma 3.5.** Suppose $h = \mathcal{L}\omega$, for $\omega \in \Omega^1(M)$. Then:

- The Ricci curvature is preserved to first order, that is, $\text{Ric}^h = -nh = 0$.
- The variation $\Delta'$ of the Laplacian on $\Sigma$ acts on each $V_\lambda$ by $\Delta'_{\lambda} = 0$.
- The second fundamental form $A = A(\Sigma, g(t))$ varies as $A' = -\nabla^2 v - v g_{S^n}$. Consequently, $H' = -(\Delta + n) v$, where $v$ is the smooth function on $\Sigma$ given by $v = \omega_0|_\Sigma = t_\omega|_\Sigma$.

**Proof.** For the first claim, take $h = \mathcal{L}\omega$, for $\omega \in \Omega^1(M)$. Then if $X = \omega^k$ is the vector field dual to $\omega$, we have $h = \mathcal{L}_X \bar{g} = \frac{d}{dt}|_{t=0} \Phi^*_t \bar{g}$, where $\Phi_t$ is the one-parameter group of
diffeomorphisms of $M$ generated by $X$. Therefore

$$\text{(3.35)} \quad \text{Ric}'_g = \left. \frac{d}{dt} \right|_{t=0} \Phi^*_t \text{Ric}_g = \mathcal{L}_X \text{Ric}_g = n\mathcal{L}_X \tilde{g} = nh.$$  

Note that this argument is still valid on the hemisphere $S^{n+1}_+$, for example by taking an arbitrary extension of $X$ to the whole sphere $S^{n+1}$. One may also verify this claim by direct calculation.

Now since the equator $\Sigma$ is totally geodesic, we have $\widehat{h}_{\alpha\beta} = \nabla_\alpha \omega_\beta + \nabla_\beta \omega_\alpha = (\widehat{\nabla}_\omega)_{\alpha\beta}$, where $\omega$ is the projection of $\omega$ to $\Omega^1(\Sigma)$. Since the Laplace spectrum is a geometric quantity, formally this should mean that the spectrum is fixed to first order as above. More generally we deal with the quantity $\widehat{A}'$ by direct computation: By commuting indices we have $\delta_\beta \delta_\gamma \widehat{h} = 2(\widehat{A} + n - 1)\delta_\beta \omega$. Also tr$_g \widehat{h} = 2\delta_\beta \omega$. For $\psi, \phi \in V_\lambda$, then integrating by parts we can compute

$$-\frac{1}{2} \langle \widehat{\nabla} \psi, \nabla^2 \phi + \phi \nabla^2 \psi \rangle = \langle \omega, \delta_\beta (\psi \nabla^2 \phi + \phi \nabla^2 \psi) \rangle$$

$$= \langle \omega, \frac{1}{2} d\Sigma \Delta (\psi \phi) + (n - 1) d\Sigma (\psi \phi) \rangle. \quad \text{(3.36)}$$

Substituting into (3.16) and integrating by parts again then gives that $\langle \psi, \widehat{A}' \phi \rangle = 0$.

The variations of $A$ and $H$ can be essentially found in [67], but since our setting is slightly different we include the computations here for completeness. Indeed, commuting indices we compute

$$\nabla_\alpha h_{0\beta} + \nabla_\beta h_{0\alpha} - \nabla_0 h_{\alpha\beta} = \nabla_\alpha \nabla_\beta \omega_0 + \nabla_\beta \nabla_\alpha \omega_0 - R^l_{\alpha\beta\lambda} \omega_l - R^l_{0\beta\alpha} \omega_l$$

$$= \nabla_\alpha \nabla_\beta \omega_0 + \nabla_\beta \nabla_\alpha \omega_0 + 2\omega_0 \omega_{\alpha\beta}. \quad \text{(3.37)}$$

Since $\Sigma$ is totally geodesic, on $\Sigma$ we have $\nabla_\alpha \nabla_\beta \omega_0 = \nabla_\alpha \nabla_\beta v$, so equation (3.12) gives $A'_{\alpha\beta} = -\nabla_\alpha \nabla_\beta v - v \omega_{\alpha\beta}$, and finally equation (3.13) gives $H' = \tilde{g}^\alpha{}_{\beta} A'_{\alpha\beta} = -\Delta v - nv$. □

Now we consider the conformal factors:

**Lemma 3.6.** Suppose $h = f \tilde{g}$, for $f \in C^\infty(M)$. Then:

- The Ricci curvature varies as $-2(\text{Ric}'_g - nh) = (\Delta f + 2nf)\tilde{g} + (n - 1)\nabla^2 f$.  

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• The variation $\Delta'$ of the Laplacian on $\Sigma$ acts on each eigenspace $V_\lambda$ by

\begin{equation}
\langle \psi, \Delta' \phi \rangle = \lambda (f, \psi \phi) - \frac{n-2}{4} \langle \Delta f, \psi \phi \rangle,
\end{equation}

where $\psi, \phi \in V_\lambda$.

• The second fundamental form $A = A(\Sigma, g(t))$ varies as $A' = \frac{1}{2}(\partial_0 f)g_\Sigma$, and hence $H' = \frac{n}{2}\partial_0 f$.

PROOF. Take $h = f\bar{g}$, for $f \in C^\infty(M)$. Then $L\delta_\Sigma h = Ldf = 2\nabla^2 f$ and $\Delta_L h = (\Delta f)g$, so we have $-2(Ric' - nh) = (\Delta f + 2nf)\bar{g} + (n-1)\nabla^2 f$.

Now note that $\delta_\Sigma h = df$. Plugging into equation (3.12) immediately gives that $A'_{\alpha\beta} = \frac{1}{2}(\partial_0 f)g_{\alpha\beta}$. Then by (3.13) and since $\Sigma$ is totally geodesic, we have that $H' = g^{\alpha\beta}A'_{\alpha\beta} = \frac{n}{2}\partial_0 f$.

For the variation of the Laplacian, we note that $tr_\Sigma \hat{h} = nf$, $\delta_\Sigma \delta_\Sigma \hat{h} = \hat{\Delta}f$ and

\begin{equation}
\langle \hat{h}, \psi \hat{\nabla}^2 \phi \rangle = \langle f, tr_\Sigma (\psi \hat{\nabla}^2 \phi) \rangle = \langle f, \psi \hat{\Delta} \phi \rangle = -\lambda (f, \psi \phi).
\end{equation}

Similarly $\langle \hat{h}, \phi \hat{\nabla}^2 \psi \rangle = -\lambda (f, \psi \phi)$. Substituting into (3.16) gives the result.

Using (3.33) and (3.34) for the Laplacian of products of the first eigenfunctions $\phi_{1,i}$ on the standard sphere $\mathbb{S}^n$ and integrating the second term by parts, we may simplify:

COROLLARY 3.7. Let $h = f\bar{g}$ as above. Recall that the first eigenfunctions on $(\Sigma, g_\Sigma)$ are given by $\phi_{1,i} = \frac{\bar{x}_i}{\sqrt{C_n}}$. We have

\begin{equation}
\langle \phi_{1,i}, \Delta' \phi_{1,j} \rangle = -\frac{n-2}{2} \frac{\langle f, 1 \rangle}{C_n} \delta_{ij} + \frac{1}{2}(n+2)(n-1)\langle f, \phi_{1,i}\phi_{1,j} \rangle.
\end{equation}

3.5.2. Transverse direction. Now we consider the remaining variations, those with $g' = h \in \mathcal{S}^2(M)$. Recall that $\mathcal{S}^2(M)$ is the space of transverse traceless symmetric 2-tensors $h$ satisfying $tr_\Sigma h = 0$, $\delta_\Sigma h = 0$, as well as the extra boundary conditions $h_{0j} = 0$ on $\Sigma = \partial M$ in the case of the hemisphere $M = \mathbb{S}^{n+1}_+$. 

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We will use some formulae from \cite{43}, in which the action of the Lichnerowicz Laplacian on transverse traceless tensors for more general warped product spaces is computed with respect to the decomposition (3.24). The relevant formulae are, for $h \in \mathcal{S}^2(M)$ satisfying $\delta_{\mathcal{F}} h = 0$, $\text{tr}_{\mathcal{F}} h = 0$,

\begin{equation}
(3.41) \quad \Delta_L h_{00} = \partial_0^2 u + (n + 4) \cot r \partial_0 u + 2(n + 1) \left( \cot^2 r - 1 \right) u + \frac{1}{\sin^2 r} \hat{\Delta} u,
\end{equation}

\begin{equation}
(3.42) \quad \Delta_L h_{a0} = \partial_0^2 \xi_a + n \cot r \partial_0 \xi_a + \left( (n + 2) \cot^2 r + 2 - n \right) \xi_a + \frac{1}{\sin^2 r} (\hat{\Delta} \xi_a - \text{Ric}_{\mathcal{F}} \xi_a) + 2 \cot r \partial_\alpha u.
\end{equation}

Note that in these formulae $\hat{\Delta}$ and $\hat{\nabla}$ correspond to the standard metric $g_{\mathcal{F}}$, rather than the induced metric on $S(r)$, which differs in scale. In any case, we will only apply them at the boundary $\Sigma = S(\frac{\pi}{2})$, at which these metrics coincide.

First we recall that the Lichnerowicz Laplacian preserves the transverse traceless condition (see \cite{16} for example); for the round metric this is easy to verify manually:

**Lemma 3.8.** Let $M = \mathbb{S}^{n+1}$ or $M = \mathbb{S}^{n+1}_i$. Suppose that $h \in \mathcal{S}^2(M)$ satisfies $\text{tr}_{\mathcal{F}} h = 0$ and $\delta_{\mathcal{F}} h = 0$. Then $\text{tr}_{\mathcal{F}} \Delta_L h = 0$ and $\delta_{\mathcal{F}} \Delta_L h = 0$.

**Proof.** Since $\text{tr}_{\mathcal{F}} h = 0$, by equation (3.21) we have $\Delta_L h = \Delta h - 2(n + 1) h$ and hence $\text{tr}_{\mathcal{F}} \Delta_L h = \text{tr}_{\mathcal{F}} \Delta h = \Delta (\text{tr}_{\mathcal{F}} h) = 0$. Proposition 3.2 gives $\delta_{\mathcal{F}} \Delta_L h = \Delta_H \delta_{\mathcal{F}} h = 0$. \hfill $\Box$

Now we give variation formulae for deformations by transverse traceless symmetric 2-tensors. In particular we do not yet assume any boundary conditions for these formulae.

**Lemma 3.9.** Suppose that $h \in \mathcal{S}^2(M)$ satisfies $\text{tr}_{\mathcal{F}} h = 0$ and $\delta_{\mathcal{F}} h = 0$. Then:

- The Ricci curvature varies as $-2(\text{Ric}_{\mathcal{F}} - nh) = \Delta_L h + 2nh$.
- The variation $\hat{\Delta}'$ of the Laplacian on $\Sigma$ acts on the first eigenspace $V_n$ by

\begin{equation}
(3.43) \quad \langle \psi, \hat{\Delta}' \phi \rangle = \langle \psi \phi, \frac{1}{2} \partial_0^2 u - \frac{n + 3}{2} u + \frac{1}{4} \hat{\Delta} u \rangle
\end{equation}
where $\psi, \phi \in V_n$ and $u$ is as in the decomposition (3.24).

- Finally, the mean curvature $H = H(\Sigma, g(t))$ varies as $H' = \frac{1}{2} \partial_t u$.

**Proof.** The formulae for $\text{Ric}'_g$ and $H'$ follow immediately from equations (3.20) and (3.14) using the transverse traceless property.

For the variation of $\lambda$, note that since $\Sigma$ is totally geodesic, in our coordinates we have $\text{tr} \hat{h} = -h_{00} = -u$ and $(\delta_\Sigma \hat{h})_\beta = \nabla^\alpha h_{\alpha \beta} = -\nabla^0 h_{0\beta}$, on $\Sigma$. Commuting indices we have that, again on $\Sigma$,

$$
\delta_\Sigma \delta_\Sigma \hat{h} = -g^{\alpha \beta} \nabla_\alpha \nabla_0 h_{\beta 0} = -g^{\alpha \beta}(\nabla_0 \nabla_\alpha h_{\beta 0} - R^p_{\alpha 00} h_{\beta p} - R^p_{\alpha 0\beta} h_{0 p}) \\
= \nabla_0 \nabla_0 h_{00} + g^{\alpha \beta} h_{\alpha \beta} - g^{\alpha \beta} g_{\alpha \beta} u \\
= \partial_0^2 u - (n + 1)u.
$$

(3.44)

Now since $\hat{\nabla}^2 \phi = -\phi \hat{g}$, we have $\langle \hat{h}, \psi \hat{\nabla}^2 \phi \rangle = -\langle \text{tr} \hat{h}, \psi \phi \rangle = \langle u, \psi \phi \rangle$. Similarly $\langle \hat{h}, \phi \hat{\nabla}^2 \psi \rangle = \langle u, \psi \phi \rangle$. Substituting into (3.16) gives the result.

\[ \square \]

**Corollary 3.10.** Let $M = S^{n+1}_+$, with equator $\Sigma$. Suppose that $g' = h \in S^2(M)$ satisfies $\text{tr} \hat{h} = 0$, $\delta_\Sigma h = 0$ and $h_{00}|_\Sigma = 0$. If we additionally have either $\text{Ric}'_g \geq nh$ or $\text{Ric}'_g \leq nh$, then $\hat{\Delta}'_V = 0$.

**Proof.** By Lemma 3.9, the Ricci condition gives $\Delta_L h + 2nh \leq 0$ or $\Delta_L h + 2nh \geq 0$ respectively. But $\Delta_L h + 2nh$ is traceless if $h$ is transverse traceless by Lemma 3.8. Therefore if $\Delta_L h + 2nh$ has a sign, then it must in fact be zero.

We now use the explicit computation of the Lichnerowicz Laplacian. In particular, (3.41) reduces on the boundary $r = \pi/2$ to give

(3.45) \[ 0 = (\Delta_L h + 2nh)_{00} = \partial_0^2 u - 2u + \hat{\Delta} u. \]

Since $u = h_{00}$ vanishes on $\Sigma$, this implies that we also have $\partial_0^2 u|_\Sigma = 0$. Thus Lemma 3.9 implies that $\hat{\Delta}'_V = 0$ as claimed. \[ \square \]
Corollary 3.10 implies in particular that, to first order, variations \( h \in \mathcal{S}^2(S_{n+1}^n) \) cannot affect the first boundary eigenvalues whilst increasing Ricci curvature. On the whole sphere the variations \( h \in \mathcal{S}^2(S_{n+1}^n) \) the argument is simpler still:

**Proposition 3.11.** Let \( M \) be the round sphere \( S^{n+1} \) with variation \( g' = h \in \mathcal{S}^2(M) \). If \( \text{Ric}_g' \geq nh \), or if \( \text{Ric}_g' \leq nh \), then \( h = 0 \).

**Proof.** Arguing as in Corollary 3.10 the Ricci conditions both imply \( \Delta_L h + 2nh = 0 \). But for the round metric on \( S^{n+1} \), there are no nontrivial \( h \in \mathcal{S}^2(M) \) for which \( \Delta_L h + 2nh = 0 \). In fact, from Boucetta’s computation of the spectrum of the Lichnerowicz Laplacian on spheres [18, 19], it is known that the least eigenvalue of \( \Delta_L \) acting on the transverse traceless space \( \mathcal{S}^2(S_{n+1}^n) \) is \( 4(n + 1) \). \( \square \)

### 3.6. Deformations of the hemisphere with Ricci curvature bound.

In this final subsection, we consider the case of the hemisphere \( M = S^n_{n+1} \) with boundary \( \Sigma = \partial S^n_{n+1} \), with the goal of proving Theorem 3.1.

**3.6.1. An explicit ambient conformal factor.** Our first step is to construct an explicit smooth function \( f \) on \( M = S^n_{n+1} \) so that the variation \( h = f\mathcal{g} \) decreases the first Laplace eigenvalue on the boundary, whilst preserving the lower bound on Ricci curvature to first order. Specifically, this subsection contains the proof of the following:

**Proposition 3.12.** Let \( M = S^n_{n+1} \), \( \Sigma = \partial S^n_{n+1} \), for \( n \geq 2 \). There exists a smooth function \( f \in C^\infty(M) \) such that, if \(g = g(t)\) is a variation of the standard metric \( g(0) = \mathcal{g} \), with \( g' = f\mathcal{g} \), then:

- The variation of Ricci curvature satisfies \( (g^{-1}\text{Ric}_g)' \geq 0 \) on \( M \).
- The variation \( \hat{\Delta}' \) of the Laplacian acting on the first eigenspace \( V_n \) of \( (\Sigma, \mathcal{g}_S^n) \) is diagonal with respect to the basis \( \{\phi_{1,i}\} \),

\[
\hat{\Delta}'_{V_n} = \text{diag}(-\mu_1, \cdots, -\mu_n),
\]

with \( \mu_i > 0 \) for \( i < n \) and \( \mu_n < 0 \).
Moreover, $f$ may be chosen so that $\partial_0 f|_\Sigma$ is orthogonal to the space $V_n$.

Each $\mu_i$ should be formally regarded as the first variation of the eigenvalue attached to $\phi_{1,i}$. They will genuinely be the first variations under the analytic variation that we present later, in the proof of Theorem 3.1.

To construct the function $f$, we work with the coordinates $r, s$ discussed in subsection 3.4 so that the standard metric takes the form (3.25):

$$g_{S^{n+1}} = dr^2 + \sin^2 r\, ds^2 + \sin^2 r\, \sin^2 s\, g_{S^{n-1}},$$

where $0 \leq r \leq \pi/2$, $0 \leq s \leq \pi$. Recall that in our realisation of the hemisphere $M = S^{n+1}_+$, we had $\cos r = x_{n+1}|_M$, $\sin r \cos s = x_n|_M$.

We will use functions of the form

$$f = a + F(r)\psi(s),$$

where $a$ is some constant, $F$ is a polynomial in $r^2$ with

$$F(0) = 0, \quad F(\pi/2) = b > 0, \quad F'(\pi/2) = c > 0$$

and $\psi(s) = -\sin^{2k}s$ for some positive integer $k$. The restriction to even powers of $r$ and $\sin s$ ensures that $f$ is indeed smooth through the coordinate singularities $s = 0, \pi$ and especially $r = 0$.

With $f$ of this form, $\partial_0 f|_\Sigma = c\psi(s)$ may be alternatively written as a polynomial in $\cos^2 s = x_n^2|_\Sigma$. Therefore $x_i\psi$ has odd degree in $x_i$, so by symmetry $\int_{\Sigma} x_i \psi = 0$ and hence $\partial_0 f|_\Sigma$ is orthogonal to the first eigenfunctions $\phi_{1,i}$, for each $i = 0, \cdots, n$.

**Boundary Laplacian.** On the boundary $\Sigma$, our choice of $f, \psi$ restricts to

$$f|_\Sigma = a - b(1 - x_n^2)^k|_\Sigma = a - b\sin^{2k}s.$$

With this form the boundary integrals in (3.16) may be computed using Lemma 3.4.
Lemma 3.13. Consider the variation \( g = g(t) \), with \( g(0) = \bar{g} \) and \( \frac{d}{dt}|_{t=0} g = f \bar{g} \), with \( f \) of the form (3.48). Then with respect to the basis \( \{\phi_{1,i}\} \) of \( V_n \), the variation of the Laplacian acts on \( V_n \) as
\[
\Delta'_{V_n} = \text{diag}(-\mu_1, \ldots, -\mu_n),
\]
where the \( \mu_i \) are all equal for \( i < n \), and
\[
\sum_{i=0}^{n} \mu_i = -n(n+1)a + n \frac{B(k + \frac{n}{2}, \frac{1}{2})}{B(\frac{n}{2}, \frac{3}{2})} b.
\]

(3.52)

\[
\mu_n = -na - \frac{b}{2k + n + 1} \frac{B(k + \frac{n}{2}, \frac{1}{2})}{B(\frac{n}{2}, \frac{3}{2})} (k(n-2) - n).
\]

Proof. Recall that \( \phi_{1,i} = \frac{x_i}{\sqrt{C_n}} \), where \( C_n = \int_{\Sigma^n} x_i^2 \frac{dV}{\bar{g}} = B(\frac{n}{2}, \frac{3}{2}) \gamma_{n-1} \). Since \( f|_\Sigma \) can be written as the restriction of a polynomial in \( x_n \) only, for \( 0 \leq i < j \leq n \) we see that \( \phi_{1,i} \phi_{1,j} f|_\Sigma \) is an odd function with respect to \( x_i \). Therefore \( \int_{\Sigma} \phi_{1,i} \phi_{1,j} f = 0 \), so by Corollary 3.7 we thus have \( \langle \phi_{1,i}, \Delta' \phi_{1,j} \rangle = \langle \phi_{1,j}, \Delta' \phi_{1,i} \rangle = 0 \). Hence \( \Delta'_{V_n} \) is indeed diagonal.

Now since \( f = a - b(1 - x_n^2)^k \) on \( \Sigma \), using Corollary 3.7 again, together with Lemma 3.4 and the beta function identity \( B(x, y + 1) = B(x, y) \frac{y}{x+y} \), gives the formula for \( \mu_n = -\langle \phi_{1,n}, \Delta' \phi_{1,n} \rangle \).

By symmetry of \( f|_\Sigma \), it is clear that the \( \mu_i = -\langle \phi_{1,i}, \Delta' \phi_{1,i} \rangle \) are all equal for \( i < n \). Now on \( \Sigma \) we have \( \sum_{i=0}^{n} \phi_{1,i}^2 = \frac{1}{C_n} \sum_{i=0}^{n} x_i^2 = \frac{1}{C_n} \). Then by Lemma 3.6 we have \( \sum_{i=0}^{n} \mu_i = -\frac{n}{C_n} \langle f, 1 \rangle \), since the \( \Delta f \) term integrates to zero. Again using Lemma 3.4 we find that indeed
\[
\sum_{i=0}^{n} \mu_i = -n(n+1)a + n \frac{B(k + \frac{n}{2}, \frac{1}{2})}{B(\frac{n}{2}, \frac{3}{2})} b.
\]

Our main aim is to choose \( f \) so that \( \mu_n < 0 \). Observe that the second term in (3.52) may be made negative by choosing large enough \( k \), so long as \( n > 2 \). When \( n = 2 \), this term is instead always positive, and in fact increases with \( k \). For this reason, we treat these two cases separately in the subsections to follow.

Ricci curvature. Now to analyse the variation of Ricci curvature, we use our knowledge of the Christoffel symbols (3.23). Applying them iteratively, it is a straightforward computation to find the Hessian of \( f \):

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**Lemma 3.14.** Let $f \in C^\infty(M)$ be of the form (3.48). Then the Hessian $\nabla^2 f$ can be given in the block form

$$\nabla^2 f = \begin{pmatrix} \ddot{\varphi} & \dot{\psi}(\dot{\varphi} - F \cot r) & 0 \\ \dot{\psi}(\dot{\varphi} - F \cot r) & F\ddot{\psi} + \dot{\varphi} \sin r \cos r & 0 \\ 0 & 0 & \left( \begin{array}{c} F\dot{\psi} \sin s \cos s \\ + \dot{\varphi} \sin r \cos r \end{array} \right) \bar{g}_{S^{n-1}} \end{pmatrix}.$$  

In particular, we have

$$\Delta f = \ddot{\varphi} \psi + \frac{F}{\sin^2 r} \ddot{\psi} + n\dot{\varphi} \cot r + (n-1) \frac{F}{\sin^2 r} \dot{\psi} \cot s.$$  

Recall from Lemma 3.6 that with $g' = f\bar{g}$, we have $-2(\text{Ric}_{g'} - nf\bar{g}) = (\Delta f + 2nf)\bar{g} + (n-1)\nabla^2 f$. Thus the condition $(g^{-1}\text{Ric}_{g})' = g^{-1}(\text{Ric'} - nf\bar{g}) \geq 0$ amounts to showing that the endomorphism

$$\Delta f + 2nf \text{id} + (n-1)\bar{g}^{-1}\nabla^2 f$$

is negative semidefinite. By Lemma 3.14 to verify this negativity at a point $p \in M$ it is sufficient to prove the following three inequalities:

$$E_1 := \Delta f + 2nf + (n-1) \left( \frac{F}{\sin^2 r} \dot{\psi} \cot s + \dot{\varphi} \psi \cot r \right) < 0,$$

$$E_2 := \Delta f + 2nf + (n-1)\dot{\varphi} \psi \leq 0$$

$$D := (\Delta f + 2nf)^2 + (n-1)(\Delta f + 2nf)(\ddot{\varphi} \psi + \frac{F}{\sin^2 r} \dot{\psi} + \dot{\varphi} \psi \cot r) + (n-1)^2 \dot{\varphi} \psi \left( \frac{F}{\sin^2 r} \dot{\psi} + \dot{\varphi} \psi \cot r \right) - \frac{(n-1)^2}{\sin^2 r} \dot{\varphi}^2 (\dot{\varphi} - F \cot r)^2 \geq 0.$$
These correspond to the lower right block, the upper left entry, and the determinant of the upper left block respectively. The $\Delta f + 2nf$ term will be somewhat easier to handle, so we have avoided expanding it explicitly here.

The case $n \geq 3$. For $n \geq 3$, we consider $b > 0$ and then take $k > \frac{n}{n-2}$ so that the second term in (3.52) is negative. Then we will not need the scaling constant $a$, so we set $a = 0$. Lemma 3.13 then ensures that $\mu_n < 0$ and moreover that $\sum_{i=0}^{n} \mu_i > 0$, hence $\mu_i > 0$ for each $i < n$.

Our remaining strategy in this case is to choose $F(r) = r^{2m}$ for sufficiently large $m$, in order to ensure that the variation of Ricci curvature is nonnegative. Note that with this choice of $F$ we indeed have $b = (\pi/2)^{2m} > 0$ and $c = 2m(\pi/2)^{2m-1} > 0$.

To prove that the variation of Ricci curvature is nonnegative, by continuity it suffices to verify the conditions (3.56), (3.57) and (3.58) away from the coordinate singularities at $r = 0$ and $s = 0, \pi$. Thus for the remainder of this subsection, we will assume $0 < r \leq \pi/2$, $0 < s < \pi$.

For convenience, we set $L = \frac{\Delta f + 2nf}{r^{2m-2} \sin^4 s}$. A straightforward calculation shows that, with our choice of $f$,

\begin{equation}
(3.59) \quad L = -(2m)(2m - 1) - 2mnr \cot r + \frac{2kr^2}{\sin^2 r} - \frac{2k(2k + n - 2)r^2 \cot^2 s}{\sin^2 r} - 2nr^2.
\end{equation}

We will choose $m$ large enough so that $(2m - 1) > \frac{\pi^2 k}{2}$. Since $\sin r \geq \frac{2r}{\pi}$ for $r \in [0, \pi/2]$, we then have

\begin{equation}
(3.60) \quad (2m - 1) \sin^2 r \geq 2kr^2,
\end{equation}

which easily gives

\begin{equation}
(3.61) \quad L < -(2m - 1)^2 - \frac{2k(2k + n - 2)r^2 \cot^2 s}{\sin^2 r} < 0
\end{equation}

and hence $\Delta f + 2nf < 0.$
Further calculations then give

\[(3.62) \quad E_1 = \Delta f + 2nf - 2(n - 1)r^{2m} \sin^2 \frac{k \cot s}{\sin^2 r} \left( \frac{k \cot s}{\sin^2 r} + \frac{m \cot r}{r} \right) < 0, \]

\[(3.63) \quad E_2 = \Delta f + 2nf - (n - 1)2m(2m - 1)r^{2m-2} \sin^2 k s < 0. \]

Again a straightforward calculation gives

\[
\frac{D \sin^2 r}{r^{4m-4} \sin^4 k s} = L^2 \sin^2 r + 4(n - 1)^2 k^2 \cot^2 s \left( 4mr^3 \cot r - r^4 \cot^2 r \right) + 2(n - 1)L(-m(2m - 1)\sin^2 r + kr^2) - 2(n - 1)L(k(2k - 1)r^2 \cot^2 s + mr \sin r \cos r) - 4(n - 1)^2 m(2m - 1)(kr^2 - mr \sin r \cos r) - 4(n - 1)^2 km(2k + 2m - 1)r^2 \cot^2 s.
\]

\[(3.64) \quad 2L(-m(2m - 1)\sin^2 r + kr^2) > (2m - 1)^4 \sin^2 r + 2(2m - 1)^2 k(2k + n - 2)r^2 \cot^2 s. \]

Since \(\tan r \geq r\) for \(0 \leq r < \pi/2\), we have \(r \cot r \leq 1\). So in particular \(r^4 \cot^2 r \leq 4mr^3 \cot r\), and hence the first line is positive.

The remaining negative terms we must handle are the last line and the \(kr^2\) in the fourth line. We can estimate them as follows:

By \((3.60)\) and \((3.61)\), we have

\[(2m - 1)^4 \sin^2 r > 2(2m - 1)^3 kr^2 \geq 2m(2m - 1)^2 kr^2 \geq 4(n - 1)m(2m - 1)kr^2. \]
Finally, we will have

\[(3.67) \quad 2(2m-1)^2k(2k+n-2)r^2 \cot^2 s \geq 4(n-1)km(2k+2m-1)r^2 \cot^2 s\]

so long as

\[(3.68) \quad \frac{(2m-1)^2}{2m(2k+2m-1)} \geq \frac{n-1}{n+2k-2}.\]

For fixed \(k > \frac{n}{n+2} > 1\), the right hand side of (3.68) is strictly less than 1, whilst the left hand side tends to 1 as \(m \to \infty\), so this condition is satisfied for large \(m\).

With these estimates, equation (3.64) implies that \(D > 0\), and thus we have proven

**Proposition 3.15.** Let \(k > \frac{n}{n+2}\) be a positive integer. Then there is a positive integer \(m\) such that

\[(3.69) \quad 2m-1 \geq \max(2(n-1), 8k/\pi^2) \quad \text{and} \quad \frac{(2m-1)^2}{2m(2k+2m-1)} > \frac{n-1}{n+2k-2}.\]

With this choice of \(k, m\), the function \(f = -r^{2m} \sin^{2k} s\) satisfies all the properties of Proposition 3.12.

**Remark 3.16.** The above choice of \(m\) might be far from optimal. Some quick plots suggest that, for example, when \(5 \leq n \leq 11\) it is sufficient to take \(k = 2, m = 4\).

**The case \(n = 2\).** When \(n = 2\), we take \(k = 1\), so that Lemma 3.13 gives

\[(3.70) \quad \mu_n = -2a + \frac{4}{5}b, \quad \text{and} \quad \sum_{i=0}^{n} \mu_i = -6a + 4b.\]

We thus choose the scaling constant \(a = (\frac{7}{5} + \epsilon_0)b > 0\) for a small \(0 < \epsilon_0 < 4/15\), so that \(\mu_n < 0\) and \(\sum_{i=0}^{n} \mu_i > 0\), hence \(\mu_i > 0\) for \(i < n\). For our analysis we make the specific choice \(\epsilon_0 = 10^{-6}\).
Our strategy in this case is to find a particular $F$ for which the variation of Ricci curvature is still nonnegative, despite the positive scaling $a$. Specifically, we choose the function

$$F(r) = \frac{1}{C} \left( r^2 - \frac{1}{21} r^4 + \frac{4}{315} r^6 + \frac{1}{945} r^8 + \frac{74}{429925} r^{10} \right),$$

where the constant $C$ is chosen so that $b = F(\pi/2) = 1$. Note also that indeed $c = \dot{F}(\pi/2) = 1.416 \cdots > 0$.

**Remark 3.17.** The polynomial above is, up to a normalising constant, the tenth degree Taylor polynomial of the solution of $\ddot{y} + 2\dot{y} \cot r + 4y - \frac{6y}{\sin^2 r} = 0$, with boundary conditions $y(0) = 0, y(\pi/2) = 1$. This ODE arises when finding harmonic extensions of functions $z$ on $\partial S_3^3$, when $z$ is expanded in terms of spherical harmonics on $S^2$. In particular, it governs the $\lambda = 6$ (second degree) eigenspace.

The solution $y$ may be written in terms of associated Legendre functions $P^\mu_\nu$ as

$$y(r) = \frac{P^{\frac{3}{2}}_\frac{1}{2}(\cos r)}{P^{\frac{3}{2}}_\frac{1}{2}(0) \sqrt{\sin r}}.$$

A plot of the relevant quantities (3.56), (3.57) and (3.58) suggested that $y$ was also a (possibly more natural) candidate for the function $F$. However, at present the author is not aware of a proof based on the above interpretation of $y$, and the analysis was simpler with the polynomial form of $F$.

We will verify the conditions (3.56), (3.57) and (3.58) at each point of $M$. The quantities $E_1$ and $E_2$ are manageable: A straightforward computation gives that

$$E_1 = 4a - \cos^2 s \left( \frac{6F}{\sin^2 r} \right) - \sin^2 s \left( \dot{F} + 3\dot{F} \cot r + 4F - \frac{2F}{\sin^2 r} \right),$$

$$E_2 = 4a - \cos^2 s \left( \frac{4F}{\sin^2 r} \right) - 2\sin^2 s \left( \ddot{F} + \ddot{F} \cot r + 2F - \frac{F}{\sin^2 r} \right).$$

We thus require some relatively elementary, but tedious properties of $F$, which simply state here:
Lemma 3.18. On $[0, \frac{\pi}{2}]$, the function $F(r)$ satisfies the following properties:

- $\frac{F}{\sin^2 r} \geq 0.41$,
- $\tilde{F} + 3\tilde{F} \cot r + 4F - \frac{2F}{\sin^2 r} \geq 1.9$,
- $\tilde{F} + \tilde{F} \cot r + 2F - \frac{F}{\sin^2 r} \geq 1.1$.

With our choice of $a = 0.400001$, it follows that $E_1 \leq 4a - \max(6(0.41), 1.9) < 0$ and $E_2 \leq 4a - \max(4(0.41), 2(1.1)) < 0$ for all $0 \leq r \leq \pi/2, 0 \leq s \leq \pi$.

The analysis of $D$ is significantly more complicated. Our proof relies on obtaining crude bounds for the derivative, $|\partial_r D| \leq 80, |\partial_s D| \leq 202$. The proof can then be completed by numerically calculating values of $D$ at points $(r, s)$ on a sufficiently fine grid. In particular, the function $D$ was sampled on a square grid with spacing $\delta = 10^{-4}$. Across all sampled points, the minimum value of $D$ was found to be $0.01536 \cdots$. The mean value theorem then implies that

$$D > 0.015 - 202 \frac{\delta}{\sqrt{2}} = 0.0007 \cdots > 0$$

for all $0 \leq r \leq \pi/2, 0 \leq s \leq \pi$.

3.6.2. Correcting the mean curvature. In this subsection, we complete the proof of Theorem 3.1 by first perturbing in the direction of an ambient diffeomorphism that corrects the mean curvature to first order, and then introducing a lower order conformal correction that fixes the mean curvature exactly at zero. We will verify that the resulting metric satisfies the desired properties of Theorem 3.1 (possibly after scaling).

For this subsection we fix a smooth cutoff function $\chi : [0, \pi] \to [0, 10]$ such that:

- $\chi(r) = 0$ for $r \leq \pi/3$ and $r \geq 2\pi/3$.
- $\chi(\pi/2) = 1$.

Proof of Theorem 3.1. Let $M = S^{n+1}_+$ and consider the conformal factor $f \in C^\infty(M)$ as in subsection 3.6.1. The property that $\partial_0 f|_\Sigma$ is orthogonal to $V_n = \ker(\hat{\Delta} + n)$ is crucial: It means that there exists a smooth function $v \in C^\infty(\Sigma)$ such that

$$n \hat{\Delta} v = n \partial_0 f|_\Sigma.$$

(3.73)
Recalling that $\partial_0 f|_\Sigma = -c\sin^{2k} s$ and using that $\Delta$ acts on the class of functions depending only on $s$ by $\partial_s^2 + (n-1)\cot s \partial_s$, it is easily verified that an explicit solution to equation (3.73) is given by

$$v = \frac{nc}{2(2k-1)(n+2k)} \sum_{j=0}^{k} a_j \sin^{2j} s,$$

where the coefficients satisfy $a_j = \frac{2(j+1)}{2j-1}a_{j+1}$, $a_k = 1$.

Then we may fix a (smooth) 1-form $\omega \in \Omega^1(M)$ for which $\omega_0|_\Sigma = v$. Explicitly, we may take for instance $\omega(r, \theta) = \chi(r)v(\theta)dr$, where $r$ is as in the warped product (3.22), and $\theta$ are the coordinates on $S^n$. For small $t$ consider the (analytic) variation

$$g_1(t) = \bar{g} + tf\bar{g} + t\mathcal{L}\omega.$$

For $u \in C^\infty(S^n)$, we also define a smooth extension map $\mathcal{E} : C^\infty(S^n) \to C^\infty(S^{n+1})$ by $\mathcal{E}(u)(r, \theta) = \chi(r)u(\theta)$. This extension is constructed so that $\partial_0 \mathcal{E}(u)|_\Sigma = u$.

Now consider the family of functions on $\Sigma$ defined by

$$u(t) = -\frac{2}{n}H(\Sigma, g_1(t)).$$

Since the mean curvature functional depends analytically on the metric and its derivatives, $u$ is analytic in $t$. Moreover, by Lemmas 3.5 and 3.6 we have that

$$u' = \frac{2}{n}(\Delta + n)v - \partial_0 f|_\Sigma = 0.$$

We may thus consider the real analytic family of smooth metrics

$$g(t) = e^{\mathcal{E}(u(t))}g_1(t) = e^{\mathcal{E}(u(t))}(\bar{g} + tf\bar{g} + t\mathcal{L}\omega)$$

on $M$. Note that indeed $g(0) = \bar{g}$.
By the formula (3.2) for the mean curvature under conformal change, we have

\[ H(\Sigma, g(t)) = e^{-\mathcal{E}(u(t))/2} \left( H(\Sigma, g_1(t)) + \frac{n}{2} u(t) \right) = 0. \]

Now since \( u' = 0 \), we have \( \mathcal{E}(u)' = 0 \), and therefore the first variation of the metric \( g(t) \) does not depend on \( u \). In particular, on \( M \) we have

\[ g' = g_1' = f\bar{g} + \mathcal{L}\omega. \]

Again by Lemma 3.5 the diffeomorphism part \( \mathcal{L}\omega \) does not affect the Ricci curvature nor the spectrum of \( \bar{\Delta} \) on \( \Sigma \) to first order. Thus the conclusions of Proposition 3.12 still hold, namely that the Ricci curvature on \( M \) is nondecreasing,

\[ (g(t)^{-1} \text{Ric}_{g(t)})' \geq 0, \]

and that the variation of the Laplacian on \( \Sigma \) acts on \( V_n \) by

\[ \hat{\Delta}'_{V_n} = \text{diag}(-\mu_1, \cdots, -\mu_n), \]

with respect to the basis \( \{\phi_{1,i}\} \), where

\[ \mu_i > 0 \quad \text{for} \quad i < n, \quad \text{and} \quad \mu_n < 0. \]

At this point we would like to conclude that the first eigenvalue \( \lambda_1(\Sigma) \) varies by the \( \mu_i \), but formula (3.17) will only apply if we already know that the \( \phi_{1,i} \) are initial points of some smoothly varying families of eigenfunctions of \( \hat{\Delta}_{\bar{g}(t)} \). The key is that \( \hat{\Delta}'_{V_n} \) is already diagonal in the basis \( \{\phi_{1,i}\} \):

By Lemma 3.3 there are families of \( L^2(\Sigma, \bar{g}(t)) \)-orthonormal eigenfunctions of \( \hat{\Delta}_{\bar{g}(t)} \), suggestively denoted \( \{\varphi_{1,i}(t)\}_{i=0}^n \) with corresponding eigenvalues \( \lambda_{1,i}(t) \), varying analytically in \( t \), where \( \lambda_{1,i}(0) = n \). By the discussion of subsection 3.3.3 the variation \( \hat{\Delta}'_{V_n} \) must be diagonal in the basis \( \{\varphi_{1,i}(0)\} \). But there is at most one orthonormal basis that diagonalises a matrix, up to sign and permutation, so we can indeed arrange that \( \varphi_{1,i}(0) = \phi_{1,i} \), and
hence
\begin{equation}
\lambda_{1,i} = \mu_i
\end{equation}
for each $i$. (This claim would also follow from a symmetry argument, noting that the explicit form of the variation $\tilde{g'}$ is invariant under rotations fixing $\phi_{1,n}$.)

Finally, we claim that the variation $\bigg( \int_{\Sigma} A(\tilde{\nabla} \varphi_{1,n}, \tilde{\nabla} \varphi_{1,n}) \, dV_{\tilde{g}(t)} \bigg)' > 0$. Recall that we chose coordinates so that $e_1 = \partial_s$, and also that the eigenfunction $\phi_{1,n} = \frac{\sin s}{\sqrt{c_n}} = \frac{\cos s}{\sqrt{c_n}}$ only depends on $s$. Then since $A(\Sigma, \tilde{g}) = 0$, we have
\begin{equation}
A(\tilde{\nabla} \varphi_{1,n}, \tilde{\nabla} \varphi_{1,n})' = \left( A^{\alpha \beta} \tilde{\nabla}_\alpha \varphi_{1,n} \tilde{\nabla}_\beta \varphi_{1,n} \right)' = (\partial_1 \phi_{1,n})^2 A'_{11}.
\end{equation}

Now by Lemmas 3.5 and 3.6 and since $v$ satisfies equation (3.73), we have
\begin{equation}
A'_{11} = -\partial_s^2 v - v + \frac{1}{2} \partial_0 f |_{\Sigma} = \frac{n-1}{n} (\cot s \partial_s - \partial_s^2 v).
\end{equation}
Using the explicit form (3.74) of $v$, we compute that $\cot s \partial_s v - \partial_s^2 v = \frac{n k_c}{n+2k} \sin^{2k} s$.

Since $A = A(\Sigma, \tilde{g}) = 0$, noting that $(\partial_1 \phi_{1,n})^2 = \frac{\sin^2 s}{c_n}$ and using Lemma 3.4 we indeed have
\begin{equation}
\bigg( \int_{\Sigma} A(\tilde{\nabla} \varphi_{1,n}, \tilde{\nabla} \varphi_{1,n}) \, dV_{\tilde{g}(t)} \bigg)' = \int_{\Sigma} A(\tilde{\nabla} \varphi_{1,n}, \tilde{\nabla} \varphi_{1,n})' \, dV_{\tilde{g}} = \frac{(n-1) k_c B(k+1 + \frac{n}{2}, \frac{1}{2})}{(n+2k) B\left(\frac{n}{2}, \frac{3}{2}\right)} > 0.
\end{equation}

To finish the construction, choose a small $\epsilon > 0$ so that $n \epsilon / 2 < -\lambda_{1,n}$. Since the variation $g(t)$ is analytic, we conclude that for sufficiently small $t > 0$ the smooth metric $g(t)$ on $M$ satisfies the following properties:

- The boundary $\Sigma$ remains minimal:
\begin{equation}
H(\Sigma, g(t)) = 0.
\end{equation}
• The Ricci curvature is bounded below by

\[(3.89) \quad \text{Ric}_{g(t)} \geq (1 - \epsilon t/2)ng(t).\]

• The eigenvalues of the Laplacian \(\hat{\Delta}\) on \(\Sigma\) satisfy

\[(3.90) \quad 0 < \lambda_{1,n}(\Sigma, \hat{g}(t)) < n < \lambda_{1,i}(\Sigma, \hat{g}(t)) < \cdots,\]

where \(i = 0, \cdots, n - 1\). In particular, the first nonzero eigenvalue is

\[(3.91) \quad \lambda_{1}(\Sigma, \hat{g}(t)) = \lambda_{1,n}(\Sigma, \hat{g}(t)) < (1 - \epsilon t/2)n.\]

• The second fundamental form \(A = A(\Sigma, g(t))\) satisfies

\[(3.92) \quad \int_{\Sigma} A(\nabla \phi_1, \nabla \phi_1) dV_{\hat{g}(t)} > 0,\]

where \(\phi_1 = \varphi_{1,n}(t)\) is the eigenfunction corresponding to \(\lambda_{1,n}(\Sigma, \hat{g}(t))\).

Then the scaled metric

\[(3.93) \quad g = (1 - \epsilon t/2)g(t)\]

satisfies all the desired properties, and completes the proof, of Theorem 3.1.

\[\square\]

4. Angenent’s torus

The purpose of this section is to verify that \(\lambda_1(\mathcal{L}) = \frac{1}{2}\) on the rotationally symmetric torus self-shrinker discovered by Angenent [7], where \(\mathcal{L}\) is the drift Laplacian. This torus also has a reflection symmetry in the axis of rotation. It is currently not known to be the unique rotationally symmetric self-shrinking torus (even amongst those with the reflection symmetry), although it is strongly conjectured as such. We are as yet unable to prove that \(\lambda_1 = \frac{1}{2}\) for the other tori in this class, if they exist. Note that Domingo-Juan and Miquel [45]
claimed the result for all such tori, but it is unclear whether their putative least eigenfunction \( \zeta \) indeed satisfies the necessary boundary condition \( \zeta'(L) = \zeta'(0) = 0 \).

4.1. Reduction to intersection with the shrinking sphere.

**Lemma 4.1.** Let \( \Sigma^2 \subset \mathbb{R}^3 \) be a shrinking torus of revolution, with profile curve \( \Gamma \). Suppose that \( \Gamma \) intersects the sphere of radius 2 exactly twice. Then \( \lambda_1(L_{\Sigma}) = \frac{1}{2} \).

**Proof.** We follow the outline of [45]. First, the metric on the torus is given by the warped product \( ds^2 + r(s)^2 d\theta^2 \), where \( s \) is the (Euclidean) arclength parameter on the profile \( \Gamma \), and \( \theta \in S^1 \). A complete orthonormal basis of eigenfunctions for \( L \) is given by \( \{ \psi_i \phi^\lambda_j \} \), where \( \Delta^{S^1} \psi_i = -\lambda_i \psi_i \), and \( \phi^\lambda_j \) are the eigenfunctions of the operator \( L_i = \Delta^\Gamma + \frac{1}{r} \nabla r - \frac{\lambda_i}{r^2} - \frac{1}{2}(x\nabla x + r\nabla r) \).

By Courant’s nodal domain theorem (which applies since \( L \) is the Laplace-type operator associated to the measure \( \rho \)), the eigenfunction corresponding to \( \lambda_1 \) has exactly two nodal domains. Immediately this means that \( \lambda_1 \) should come from either \( \psi_0 = 1 \) or \( \psi_1 = \cos \theta, \sin \theta \). When \( \psi_1 = \cos \theta, \sin \theta \), a direct calculation shows that \( r \) is a positive eigenfunction of the operator \( L_1 \), so the only candidates here are \( y = r \cos \theta \) and \( z = r \sin \theta \), which we already know have eigenvalue \( \frac{1}{2} \). For \( \psi_0 = 1 \), the operator \( L_0 \phi = \phi'' + \left( \frac{r'}{r} - \frac{1}{2}(rr' + xx') \right) \phi' \) is a Sturm-Liouville operator.

Sturm’s theorem for the associated eigenvalue problem with periodic boundary conditions [22] Theorem 3.1] gives that the eigenvalues of \( L_0 \) are \( 0 = \mu_0 < \mu_1 \leq \mu_2 < \mu_3 \leq \mu_4 < \cdots \), where the eigenfunctions associated to \( \mu_{2j-1}, \mu_{2j} \) have exactly \( 2j \) zeroes. Recall that (for any shrinker) \( x \) and \( |X|^2 - 2n \) are eigenfunctions of \( L \) with eigenvalues \( \frac{1}{2} \) and 1 respectively. Moreover they are rotationally symmetric, so they are eigenfunctions of \( L_0 \) with the same eigenvalues. By assumption \( |X|^2 - 2n \) vanishes exactly twice on \( \Gamma \), so it must correspond to \( \mu_2 \) and so the eigenfunction \( x \) must correspond to \( \mu_1 \). In particular the lower eigenvalue is \( \mu_1 = \frac{1}{2} \), which completes the proof. \( \square \)
4.2. Argument for Angenent’s original construction. Let us first review the construction of Angenent’s torus. The profiles of rotationally symmetric self-shrinkers are geodesics in the upper half plane \( \mathcal{H} = \{(x, r) : r > 0\} \) with the metric

\[
g_A = r^{2(n-1)} e^{-(x^2+r^2)/4} (dx^2 + dr^2).
\]

A shrinking torus then corresponds to a closed geodesic in this metric. Let \( \alpha \) be the angle the tangent vector makes with the \( x \)-axis. We write \( \gamma[x_0, r_0, \alpha_0] \) for the geodesic with initial conditions \( x_0, r_0, \alpha_0 \). Angenent argues that for large \( r_0 \), the curve \( \gamma[0, r_0, 0] \) intersects the \( r \)-axis a second time, before becoming horizontal a second time. He then sets \( R_* \) to be the infimum of all such \( r_0 \) and shows that \( \gamma_* = \gamma[0, R_*, 0] \) re-intersects the \( r \)-axis horizontally, say at \( r_* \). Angenent’s torus is then the surface obtained by rotating the profile curve \( \Gamma \) given by \( \gamma_* \) and its reflection across the \( r \)-axis. In particular, \( R_* > \sqrt{2n} \), and he also shows that \( \gamma_* \) may be written as a graph over the \( r \)-axis, say of \( f(r) \), except at the two \( r \)-intercepts. So \( r_* = \min\{r : (x, r) \in \Gamma\} \) and so \( r_* < \sqrt{2(n - 1)} \) since the torus must intersect the shrinking cylinder.

The function \( f \) satisfies the differential equation

\[
\frac{f''}{1 + (f')^2} = \left( \frac{r}{2} - \frac{n-1}{r} \right) f' - \frac{f}{2}.
\]

We will prove a slight strengthening of Drugan and Kleene’s Lemma 3 in [46] (note the sign flip):

**Lemma 4.2.** There exists \( M_1 \) with the following property:

Let \( f \) be as above with \( f(\sqrt{2(n - 1)}) < 0 \). Suppose that \( f'(r) \geq 0 \) whenever \( r \geq \sqrt{2(n - 1)} \). Then \( f(r) > 0 \) whenever \( r > M_1 \) and \( f(r) \) is defined.

When \( n = 2 \) we may take \( M_1 = \sqrt{33} - 3 + \sqrt{2} = 3.07088572 \cdots \).

**Remark 4.3.** Our contribution is a tightening of the constant \( M_1 \); Drugan-Kleene show that \( M_1 \leq 2 + \sqrt{2(n - 1)} \).
PROOF. As in Drugan-Kleene we get

\[
(4.2) \quad \frac{f'''}{1+(f')^2} = \frac{2f'(f'')^2}{(1+(f')^2)^2} \left( \frac{r}{2} - \frac{n-1}{r} \right) f'' + \frac{n-1}{r^2} f'.
\]

Set \( A = -f(\sqrt{2(n-1)}) > 0 \). Then for \( r \geq \sqrt{2(n-1)} \) we have \( f'' > 0 \) and \( f''' \geq 0 \). In fact, \( \frac{f'''}{f'} \geq \frac{2f'(f'')^2}{(1+(f')^2)^2} + \frac{r}{2} - \frac{n-1}{r} \). Integrating, we find that

\[
\begin{align*}
\frac{f''(r)}{r} & \geq \frac{f''(\sqrt{2(n-1)})}{1 + f'(\sqrt{2(n-1)})^2} \frac{1 + f'(r)^2}{1 + f'(\sqrt{2(n-1)})^2} \frac{\sqrt{2(n-1)}}{r} \exp((r^2 - 2(n-1))/4) \\
& \geq \frac{A}{2} \frac{\sqrt{2(n-1)}}{r} \exp((r^2 - 2(n-1))/4).
\end{align*}
\]

(4.3)

Taking a series expansion of the right hand side, we can then estimate

\[
(4.4) \quad f''(r) \geq \frac{A}{2} \left( 1 + \frac{n^2 - 3n + 4}{4(n-1)} (r - \sqrt{2(n-1)})^2 \right).
\]

Integrating twice we then have

\[
(4.5) \quad f(r) \geq A \left( -1 + \frac{1}{4} R^2 + \frac{n^2 - 3n + 4}{48(n-1)} R^4 \right)
\]

where we have set \( R = r - \sqrt{2(n-1)} \) for convenience.

For simplicity we now consider \( n = 2 \). Then the right hand side is positive so long as \( \frac{1}{24} R^4 + \frac{1}{4} R^2 \geq 1 \); solving this for \( r \) gives that \( f(r) \geq 0 \) for \( r \geq \sqrt{2(n-1)} + \sqrt{33} - 3 \) as claimed.

\[\square\]

We are able to prove:

PROPOSITION 4.4. Let \( n = 2 \).

Let \( \Gamma \) be the profile curve of Angenent’s torus. Suppose that the uppermost intersection of \( \Gamma \) with the \( r \)-axis occurs at \( R_* > 3.08 \). Then \( \Gamma \) intersects the semicircle \( x^2 + r^2 = 4 \) exactly twice.
Remark 4.5. The hypothesis $R_* > 3.08$ may be verified for Angenent’s torus by numerical computation: Indeed, by Angenent’s construction, it is enough to check that the geodesic starting from the point $(x, r) = (0, 3.09)$ in the direction $e_x$ becomes horizontal again before intersecting the $r$-axis again.

Proof. By the work of Kleene-Möller [77], any closed geodesic in $\mathcal{H}$ has exactly two vertical points. It will be useful to split $\Gamma$ at these vertical points, so that $\Gamma = \text{Gr}(u_1) \cup \text{Gr}(u_2)$, where $u_i$ are functions of $x$ defined on some interval $(-\beta, \beta)$, with $u_1 > u_2$. Each graph satisfies the differential equation

$$\frac{u''}{1 + (u')^2} = \frac{xu'}{2} - \frac{u}{2} + \frac{n-1}{u}.$$  

By the work of Drugan-Kleene, the curvature of each segment $u_i$ changes sign at most twice, and only changes sign twice if the segment has two minima. Since $\gamma_*$ is only horizontal when it crosses the $r$-axis, the latter cannot occur. By symmetry, if the curvature changes exactly once then the change occurs at $x = 0$. Using the differential equation for $u$, we would then have $u(0) = \sqrt{2(n-1)}$, whence $u$ is just the straight line in $\mathcal{H}$ corresponding to the shrinking cylinder. This is absurd, so we conclude that $u_1$ is a concave function and $u_2$ is a convex function. This immediately implies that $\text{Gr}(u_2)$ intersects the circle $x^2 + r^2 = 4$ at most once.

We now claim that $\text{Gr}(u_2)$ must at least cross the cylinder before becoming vertical. Suppose not, so that $\text{Gr}(u_2)$ becomes vertical at $r_1 < \sqrt{2(n-1)}$. Then by the concavity of $u_1$, the graph of $u_1$ has nonnegative slope everywhere in the second quadrant, and intersects the cylinder in the second quadrant exactly once. Switching to the representation of (the reflection of) $\gamma_*$ as a graph over the $r$-axis, say of $f(r)$, this means that $f(\sqrt{2(n-1)}) < 0$, and that $f'(r) \geq 0$ for $r \geq \sqrt{2(n-1)}$. Applying Lemma 4.2 would then give that $\text{Gr}(u_1)$ intersects the $r$-axis before $r = M_1 < 3.08$. Since we know it intersects at $R_* > 3.08$, this is a contradiction. Thus the lower segment $\text{Gr}(u_2)$ must cross the cylinder before becoming
vertical, so by concavity, if the upper segment \( \text{Gr}(u_1) \) crosses the circle \( x^2 + r^2 = 4 \) it must cross at a height \( r \geq \sqrt{2(n-1)} \).

Certainly \( \Gamma \) does not intersect the circle \( x^2 + r^2 = 4 \) along the \( r \)-axis, so by symmetry we now restrict our attention to the first quadrant. Note that the tangent to the circle at the point \((\sqrt{2}, \sqrt{2})\) intersects the \( r \)-axis at \( r = 2\sqrt{2} \). Therefore if the upper segment \( \text{Gr}(u_1) \) intersects the circle \( x^2 + r^2 = 4 \) twice in the first quadrant, then it cross twice above the line \( r = \sqrt{2(n-1)} \), so by concavity we could have \( R_* \leq 2\sqrt{2} < 3 \). Again we know that \( R_* > 3.08 \), so this cannot happen.

Thus the upper segment crosses the circle \( x^2 + r^2 = 4 \) at most once in the first quadrant. Moreover, if it does cross, then by convexity the lower segment \( \text{Gr}(u_2) \) cannot cross the circle. Since we already knew the lower segment could only cross at most once, this means that \( \Gamma \) intersects the circle at most once in the first quadrant. In fact since it must intersect by the maximum principle, we conclude that \( \Gamma \) intersects the circle exactly twice overall, as claimed. \( \square \)

4.3. Argument for Møller’s quantitative torus. Møller \cite{89} gave another construction of a self-shrinking torus of revolution. In principle, his torus should coincide with Angenent’s torus, but this has not been proven. The above analysis also allows us to verify that the first eigenvalue on Møller’s torus is \( \frac{1}{2} \):

\[ \text{Proposition 4.6 (Møller, \cite{89}). Let } \epsilon_{\text{gap}} = 10^{-3}. \text{ There exists a closed, embedded, self-shrinking torus of revolution } \Sigma^2 \subset \mathbb{R}^3 \text{ with the following properties:} \]

(1) The torus is also symmetric under reflection \( x \mapsto -x \).

(2) The profile \( \Gamma \) intersects the \( r \)-axis orthogonally at two heights \( a^+ > a^- > 0 \), where

\[ \left| a^+ - \frac{4034}{1217} \right| < \frac{5}{2} \epsilon_{\text{gap}} \text{ and } \left| a^- - \frac{7}{16} \right| < \frac{3}{98} \text{.} \]
(3) The profile $\Gamma$ intersects the sphere of radius 2 at two points $p^{\pm}$, where $\|p^{\pm} - (\pm \frac{29}{32}, \frac{41}{23})\|_{\mathbb{R}^2} \leq 5\epsilon_{\text{gap}}$. Also, the angles from the $x$-axis at these points satisfy $|\alpha - 1/10| \leq 1/30$.

That the drift eigenvalue $\lambda_1(\mathcal{L}) = \frac{1}{2}$ follows from property (3) or property (1) and the discussion above.
Entropy of closed hypersurfaces and singular self-shrinkers

The work described in this section was first presented in [128].

In this chapter we present a lower bound for the entropy of closed hypersurfaces in $\mathbb{R}^n$, as well as the classification of entropy-stable singular self-shrinkers, for any $n$. In particular, the main theorem of this chapter is

**Theorem 0.1.** Let $\Sigma$ be a smooth, closed, embedded hypersurface in $\mathbb{R}^{n+1}$. Then we have $\Lambda(\Sigma) \geq \Lambda(S^n)$, with equality if and only if $\Sigma$ is a round sphere.

In fact, the above theorem is a corollary of the following classification of singular entropy-stable shrinkers:

**Theorem 0.2.** Let $V$ be an $F$-stationary integral $n$-varifold in $\mathbb{R}^{n+1}$, which has orientable regular part and finite entropy, and satisfies the $\alpha$-structural hypothesis for some $\alpha \in (0, \frac{1}{2})$. Suppose that $V$ is not a generalised cylinder $S^k(\sqrt{2k}) \times \mathbb{R}^{n-k}$. Then $V$ is entropy-unstable.

A key step is the classification of singular mean convex shrinkers:

**Theorem 0.3.** Let $V$ be an $F$-stationary integral $n$-varifold in $\mathbb{R}^{n+1}$, with orientable regular part and finite entropy. Further suppose that $H^{n-1}(\text{sing} V) = 0$. If $H_\ast \geq 0$ on $\text{reg} V$ then either $V$ is a stationary cone, or $\text{spt} V$ is a generalised cylinder $S^k(\sqrt{2k}) \times \mathbb{R}^{n-k}$.

The concept of entropy-stability is described more precisely as follows:

**0.1. Entropy-stability.** Recall the Gaussian area functionals

$$F_{x_0,t_0}(V) = \int (4\pi t_0)^{-n/2} e^{-\frac{|x-x_0|^2}{4t_0}} \, d\mu_V,$$

and that the entropy is defined by $\Lambda(V) = \sup_{x_0,t_0} F_{x_0,t_0}(V)$. 

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A smooth self-shrinker $\Sigma$ is entropy-stable if it is a local minimum for the entropy functional amongst $C^2$ graphs over $\Sigma$. Here we make this notion precise for varifolds. We first define normal variations that are not required to be compactly supported.

**Definition 0.4.** Let $V$ be an integral $n$-varifold in a manifold $N^{n+1}$ and consider a complete Lipschitz vector field $X$ on $N$. Further suppose that $X$ vanishes on $\text{sing} V$ and is $C^2$ on $N \setminus \text{sing} V$. Writing $\{\Phi_s^X\}_{s \in (-\epsilon, \epsilon)}$ for the flow of $X$, we say that the image varifolds

$$V_s := (\Phi_s^X)_\# V$$

form a normal variation of $V$ if additionally $X(x) \perp T_x \text{reg} V_s$ for all $s$ and any $x \in \text{reg} V_s$.

This definition includes deformations by compactly supported normal graphs over an orientable regular part $\text{reg} V$, since we can construct a smooth ambient field $X$ by extending in a neighbourhood of $\text{reg} V$ away from the singular set. Similarly it includes homogenous variations of a cone $V = C(W)$ in $\mathbb{R}^{n+1}$ induced by compactly supported normal graphs over $\text{reg} W$; in this case the ambient field $X$ only fails to be smooth at the origin.

**Definition 0.5.** We say that an $F$-stationary varifold $V$ is entropy-unstable if there exists a normal variation $V_s$ of $V$ satisfying $\Lambda(V_s) < \Lambda(V)$ for $s > 0$. We say that $V$ is entropy-stable if it is not entropy-unstable.

0.1.1. **$F$-stability.** Entropy-stability is strongly related to the notion of $F$-stability of [37], which extends to the singular setting by requiring that the variation take place away from the singular set.

**Definition 0.6.** Let $V$ be an orientable $F$-stationary $n$-varifold in $\mathbb{R}^{n+1}$. We say that $V$ is $F$-unstable if there is a normal variation $V_s$ of $V$, compactly supported away from $\text{sing} V$, such that for any variations $x_s$ of $x_0 = 0$ and $t_s$ of $t_0 = 1$, we have $\partial^2 \Lambda_{|s=0}F_{x_s, t_s}(V_s) < 0$. 

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$F$-stability is no longer suited for studying the entropy when $V$ is a cone, since one may always zoom away from the compact variation. Therefore, we instead consider homogenous variations and introduce the notion of homogenous $F$-stability for stationary cones as follows:

**Definition 0.7.** Let $W$ be a stationary $(n-1)$-varifold in $\mathbb{S}^n$. We say that $W$ is homogenously $F$-unstable if there is a normal variation $W_s$ of $W$ in $\mathbb{S}^n$, compactly supported away from $\text{sing} W$, such that for any variation $x_s$ of $x_0 = 0$, we have $\partial^2_{s^1}|_{s=0} F_{x_s,1}(C(W_s)) < 0$. We say that $W$ is homogenously $F$-stable if it is not homogenously $F$-unstable.

If $V = C(W)$ is a stationary $n$-cone in $\mathbb{R}^{n+1}$ we say that $V$ is homogenously $F$-stable if and only if $W$ is homogenously $F$-stable.

The restriction $t_0 = 1$ will suffice since for any cone we have $F_{x_0,t_0}(C(W)) = F_{x_0,1}(C(W))$ by dilation invariance. Note that any stationary cone has finite entropy.

1. **Closed hypersurfaces**

Before proving Theorem 0.2, we will describe how entropy lower bounds for closed hypersurfaces and for singular self-shrinkers can be deduced from the classification of compact entropy-stable singular self-shrinkers. In particular, we will assume for this section that the following holds:

**Proposition 1.1.** Let $V$ be an $F$-stationary integral $n$-varifold in $\mathbb{R}^{n+1}$, which has orientable regular part of multiplicity 1, finite entropy and $H^{n-1}(\text{sing} V) = 0$. If $V$ is not the round sphere $\mathbb{S}^n(\sqrt{2n})$ then there is an entropy-unstable variation of $V$, which is compactly supported away from $\text{sing} V$.

Clearly Proposition 1.1 is an immediate corollary of Theorem 0.2 (see also Theorem 7.5), since a compactly supported varifold certainly cannot split off a line or be a cone. The main goal of this section will be to prove Theorem 0.1 under this assumption. The applications we present here extend the results of Bernstein-Wang to all higher dimensions, and depend crucially on their theory developed in [13].
Let $\Lambda_n = \Lambda(S^n)$ be the entropy of the round sphere. A direct computation (see [110]) shows

$$2 > \Lambda_1 > \frac{3}{2} > \Lambda_2 > \cdots > \Lambda_n > \cdots > 1.$$  

Similar to [13] we define $\mathcal{SV}_n$ to be the set of all integral $F$-stationary $n$-varifolds in $\mathbb{R}^{n+1}$ with nonempty support. We denote by $\mathcal{CSV}_n$ the subset of varifolds in $\mathcal{SV}_n$ that have compact support. For $\Lambda > 0$ we also define $\mathcal{SV}_n(\Lambda)$ to be the subset of varifolds in $\mathcal{SV}_n$ with entropy strictly less than $\Lambda$, and $\mathcal{CSV}_n(\Lambda) = \mathcal{SV}_n(\Lambda) \cap \mathcal{CSV}_n$.

### 1.1. Entropy lower bound for closed hypersurfaces.

In [35], Colding-Ilmanen-Minicozzi-White showed that the shrinking sphere $S^n(\sqrt{2}n)$ minimises entropy amongst smooth, embedded closed self-shrinkers (in fact, they showed that there is a gap to the next lowest entropy in this class). This led them to conjecture the following:

**Conjecture 1.2 ([35]).** Any smoothly embedded, closed hypersurface $\Sigma^n \subset \mathbb{R}^{n+1}$, $n \leq 6$ has entropy $\Lambda(\Sigma) \geq \Lambda_n$.

The case $n = 1$ is an easy consequence of the Gage-Hamilton-Grayson theorem, which states that any embedded closed curve contracts to a round point. Bernstein and Wang [13] settled Conjecture 1.2 for $2 \leq n \leq 6$ by leveraging their insightful observation that under a carefully chosen weak flow, the final time singularity arising from compact initial data must be collapsed in a certain sense (see [13] Definition 4.6] and [13] Definition 4.9]). In fact, they were able to prove the entropy bound for objects of weaker regularity, the compact boundary measures defined as follows (see also [13] Definition 2.10]):

**Definition 1.3.** Let $V$ be an integral $n$-varifold in $\mathbb{R}^{n+1}$. We call $V$ a compact boundary measure if there is a bounded open nonempty subset $E \subset \mathbb{R}^{n+1}$ of locally finite perimeter (that is, $\chi_E$ has locally bounded variation) such that $\text{spt} \mu_V = \partial E$ and $\mu_V = |D\chi_E|$.

In this subsection we will extend their result [13] Corollary 6.4] to all dimensions $n \geq 2$.

We first need Bernstein-Wang’s characterisation of the entropy minimiser in $\mathcal{CSV}_n(\Lambda_n)$:
Lemma 1.4 ([13], Lemma 6.1). Let \( n \geq 2 \). If for all \( 1 \leq k \leq n - 1 \), the set \( \mathcal{CSV}_k(\Lambda_n) \) is empty, then either \( \mathcal{CSV}_n(\Lambda_n) \) is also empty, or there is a \( V \in \mathcal{CSV}_n(\Lambda_n) \) satisfying:

1. \( \Lambda(V) = \inf \{ \Lambda(\mu) : \mu \in \mathcal{CSV}_n(\Lambda_n) \} \),
2. \( V \) is a compact boundary measure,
3. \( V \) is entropy stable,
4. \( \text{sing} V \) has Hausdorff dimension at most \( n - 7 \).

The following proposition is implicit in the proof of [13, Corollary 6.4]:

Proposition 1.5 ([13]). Consider \( n \geq 2 \) and let \( V \) be a compact boundary measure in \( \mathbb{R}^{n+1} \). If for all \( 2 \leq k \leq n \), the set \( \mathcal{CSV}_k(\Lambda_k) \) is empty, then \( \Lambda(V) \geq \Lambda_n \). Moreover, if equality holds then, up to translations and dilations, \( V \) is an entropy-stable member of \( \mathcal{CSV}_n \).

We are now ready to prove the main theorem of this section:

Theorem 1.6. For all \( n \geq 2 \), we have \( \mathcal{CSV}_n(\Lambda_n) = \emptyset \).

Proof. First, any \( V \in \mathcal{CSV}_1(3/2) \) must be smooth by [13, Proposition 4.2] and hence have entropy at least \( \Lambda_1 \) by [35, Theorem 0.7]. So by (1.1) we have \( \mathcal{CSV}_1(\Lambda_n) = \emptyset \) for \( n \geq 2 \).

We proceed by induction. By [13, Proposition 6.2], we already have \( \mathcal{CSV}_n(\Lambda_n) = \emptyset \) for \( 2 \leq n \leq 6 \). Now for general \( n \geq 2 \), if \( \mathcal{CSV}_k(\Lambda_k) = \emptyset \) for all \( 2 \leq k \leq n - 1 \), then using the above discussion we see that the hypotheses of Lemma 1.4 are satisfied. Thus, if \( \mathcal{CSV}_n(\Lambda_n) \) is nonempty then there is a \( V \in \mathcal{CSV}_n(\Lambda_n) \) that is entropy-stable and has singular set of codimension at least 7. Moreover, \( V \) is a compact boundary measure so its regular part is orientable (see also [13, Proposition 4.3]), and it has multiplicity 1 since it is integral with \( \Lambda(\Sigma) < \Lambda_n < 2 \). But then Proposition 1.1 gives that \( V \) must be a round sphere, so in particular \( \Lambda(V) = \Lambda_n \) which is a contradiction. \( \square \)

Corollary 1.7. Let \( n \geq 2 \). Any compact boundary measure \( V \) in \( \mathbb{R}^{n+1} \) has entropy \( \Lambda(V) \geq \Lambda(\mathbb{S}^n) \), with equality if and only if \( V \) is a round sphere.
Proof. The lower bound follows immediately from Theorem 1.6 and Proposition 1.5.

If equality holds then, up to a translation and dilation, $V \in \mathcal{CSV}_n$ and $\Lambda(V) = \Lambda_n < \frac{3}{2}$, so as above $V$ is orientable by [13, Proposition 4.3], and $\mathcal{H}^{n-2}(\text{sing } V) = 0$ by [13, Proposition 4.2]. Since $V$ must also be entropy-stable, by Proposition 1.1 it must be a round sphere. \(\square\)

Theorem 0.1 follows from Corollary 1.7 for $n \geq 2$, since any closed hypersurface separates $\mathbb{R}^{n+1}$ and hence defines a compact boundary measure. Again the case $n = 1$ follows from the Gage-Hamilton-Grayson theorem [54, 56].

1.2. Gap theorem for compact singular self-shrinkers. The main theorem of Colding-Ilmanen-Minicozzi-White [35] established that the shrinking sphere had the lowest entropy amongst (smooth) closed self-shrinkers, with a gap to the next lowest. Bernstein-Wang, using their own methods, were able to provide an independent proof of this result that in fact extended it to compact singular self-shrinkers, but only for $2 \leq n \leq 6$. In this subsection we will extend their result to all $n \geq 2$.

We will need the following proposition, which is implicit in the proof of [13, Corollary 6.5]:

**Proposition 1.8 ([13]).** Let $n \geq 2$. Assume that, for all $2 \leq k \leq n$:

- The set $\mathcal{CSV}_k(\Lambda_k)$ is empty;
- The only compact boundary measure $V \in \mathcal{CSV}_k$ with $\Lambda(V) = \Lambda_k$ is the shrinking sphere $\mathbb{S}^k(\sqrt{2k})$.

Then there exists $\epsilon_n > 0$ such that $\mathcal{CSV}_n(\Lambda_n + \epsilon_n)$ contains only the shrinking sphere $\mathbb{S}^n(\sqrt{2n})$.

Combining Proposition 1.8 with Theorem 1.6 and Corollary 1.7 then immediately yields our gap theorem for compact singular self-shrinkers in all dimensions $n \geq 2$ as follows:

**Corollary 1.9.** Let $n \geq 2$. There exists $\epsilon_n > 0$ so that $\mathcal{CSV}_n(\Lambda_n + \epsilon_n)$ contains only the shrinking sphere $\mathbb{S}^n(\sqrt{2n})$. 

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1.3. Entropy lower bound for partially collapsed self-shrinkers. We also generalise the results of Bernstein-Wang for so-called partially collapsed self-shrinkers (see \[13\] Definition 6.6]) to all dimensions \(n \geq 3\). The following is implicit in the proof of \[13\] Corollary 6.7):

**Proposition 1.10 (\[13\]).** Let \(n \geq 3\). Assume that, for all \(2 \leq k \leq n-1\):

- The set \(\mathcal{CSV}_k(\Lambda_k)\) is empty;
- The only compact boundary measure \(V \in \mathcal{CSV}_k\) with \(\Lambda(V) = \Lambda_k\) is the shrinking sphere \(S^k(\sqrt{2k})\).

Then any partially collapsed \(V \in \mathcal{SV}_n\) with noncompact support has entropy \(\Lambda(V) \geq \Lambda_{n-1}\), with equality if and only if \(V\) is the round cylinder \(S^{n-1}(\sqrt{2(n-1)}) \times \mathbb{R}\).

As before, we combine Proposition 1.10 with Theorem 1.6 and Corollary 1.7 to obtain the lower bound for all \(n \geq 3\):

**Corollary 1.11.** Let \(n \geq 3\). Any partially collapsed self-shrinker \(V \in \mathcal{SV}_n\) with noncompact support has entropy \(\Lambda(V) \geq \Lambda_{n-1}\), with equality if and only if \(V\) is the round cylinder \(S^{n-1}(\sqrt{2(n-1)}) \times \mathbb{R}\).

2. Colding-Minicozzi theory

In this section we recall some results from \[37\], which allow us to relate entropy-stability to \(F\)-stability. We will also need variation formulae for the Gaussian area functionals, as well as the regularity theory for self-shrinkers with \(\lambda_1\) bounded from below. The proofs found in \[37\] extend naturally to the varifold setting, so we will state the results in this setting.

2.1. Variations. Here we record a second variation formula for the Gaussian area of an orientable \(F\)-stationary varifold \(V\) in which the centre of the Gaussian functional may change. Specifically, in this subsection we consider normal variations \(V_s\) of \(V\), with generator \(X\) compactly supported away from \(\text{sing } V\).
If \( \text{reg } V \) is orientable, each \( \text{reg } V_s \) is still orientable with normal denoted \( \nu_s \), and the restriction of \( X \) is given by \( X|_{\text{reg } V_s} = f_s \nu_s \) for some functions \( f_s \) compactly supported in \( \text{reg } V_s \). For ease of presentation we will give the formulae using the functions \( f_s \) with the understanding that \( f_s = 0 \) off the regular part \( \text{reg } V_s \).

**Proposition 2.1** (Second variation at a critical point). Let \( V \) be an orientable \( F \)-stationary \( n \)-varifold in \( \mathbb{R}^{n+1} \) with finite entropy. Let \( V_s \) be a normal variation of \( V \) with variation field \( X \), compactly supported away from \( \text{sing } V \). Write \( X|_{\text{reg } V_s} = f_s \nu_s \), with \( f = f_0 \). Also let \( x_s \) and \( t_s \) be variations of \( x_0 = 0 \) and \( t_0 = 1 \) with \( x'_0 = y \) and \( t'_0 = a \). Then \( \partial^2_{s|s=0}(F_{x_s,t_s}(V_s)) \) is given by

\[
\int \left( -f Lf + 2faH - a^2 H^2 + f(y, \nu) - \frac{|y|^2}{2} \right) \rho \, d\mu.
\]

Here we understand the \( H^2 \) term via the generalised mean curvature, \( H^2 = |\vec{H}|^2 = \frac{1}{4} |x^\perp|^2 \).

The point is that the proofs of the first and second variation formulae, [37, Lemma 3.1] and [37, Theorem 4.1] respectively, go through essentially unchanged, since the normal variation \( V_s \) takes place away from the singular set \( \text{sing } V_s = \text{sing } V \) and the contributions of \( x_s \) and \( t_s \) just come from differentiating the weight. To specialise to a critical point as in [37, Theorem 4.14], one needs certain integral identities on self-shrinkers; these can be proven in the varifold setting by applying the divergence theorem to the appropriate (exponentially decaying) vector fields.

**2.2. Entropy stability and \( F \)-stability.** In this subsection we continue to consider normal variations \( V_s \) of an \( F \)-stationary varifold \( V \).

First, for normal variations compactly supported away from \( \text{sing } V \), the proof of [37, Theorem 0.15] goes through to give:

**Theorem 2.2.** Suppose \( V \) is an orientable \( F \)-stationary varifold with finite entropy that does not split off a line and is not a cone. If \( V \) is \( F \)-unstable then it is entropy-unstable, where the unstable variation is compactly supported away from \( \text{sing } V \).
For stationary cones $V = C(W)$ we need to consider homogenous variations, induced by a normal variation of $W$ in $S^n$ supported away from $\text{sing} W$. The following is implicit in the proof of \[37\] Theorem 0.14:

**Theorem 2.3.** Let $n \geq 3$. Suppose that $V = C(W)$ is an orientable stationary $n$-cone in $\mathbb{R}^{n+1}$ that does not split off a line. If $W$ is homogenously $F$-unstable then $V$ is entropy-unstable with respect to the induced homogenous variation.

### 2.3. Regularity of self-shrinkers with stability spectrum bounded below.

Here we record a regularity result for $F$-stationary varifolds $V$ with $\lambda_1(V) = \lambda_1(\Sigma) > -\infty$ that satisfy the $\alpha$-structural hypothesis, where $\Sigma = \text{reg} V$. The content of the following proposition is essentially contained in \[37\] Section 12 and depends on the regularity theory of Wickramasekera \[120\]; it follows from the proof of \[37\] Proposition 12.24, noting that the proof of \[37\] Lemma 12.7 goes through with any lower bound $\lambda_1(\Sigma) > -\infty$.

**Proposition 2.4.** Let $V$ be an orientable $F$-stationary $n$-varifold in $\mathbb{R}^{n+1}$ with finite entropy, satisfying the $\alpha$-structural hypothesis for some $\alpha \in (0, \frac{1}{2})$. Suppose that $\lambda_1(V) > -\infty$. Then $V$ corresponds to an embedded, analytic hypersurface away from a closed set of singularities of Hausdorff dimension at most $n - 7$ (that is empty if $n \leq 6$ and discrete if $n = 7$.)

### 3. Gaussian area functionals on cones

In this section we consider integral $(n - 1)$-varifolds $W$ in $S^n$. Specifically, we will study the Gaussian areas of their cones $V = C(W)$, which by dilation invariance satisfy

\[
F_{x_0,t_0}(C(W)) = F_{\frac{x_0}{\sqrt{t_0}},1}(C(W)).
\]

As such, it will often be enough to consider centres $x_0 \in \mathbb{R}^{n+1}$, with fixed scale $t_0 = 1$. The main goal is to provide variation formulae for the Gaussian areas $F_{x_0,t_0}(W)$ by treating them as functionals on the link $W$; note that the formulae in Section 2.1 do not apply directly since
the variations are noncompact. This will give us the means to determine the homogenous $F$-stability of a stationary cone.

Since our focus is on the link, in this section $y^T$ will refer to the projection to the (approximate) tangent space $T_pW$ at a point $p \in \text{spt} W \subset \mathbb{S}^n$, so that $y \in \mathbb{R}^{n+1}$ decomposes as

$$y = y^T + \langle y, p \rangle p + y^\perp.$$  

Here $y^\perp$ denotes the component orthogonal to $T_pW$ in $T_p\mathbb{S}^n$, which is equivalent to the component orthogonal to $T_pC(W)$ in $\mathbb{R}^{n+1}$, and is given by $y^\perp = \langle y, \nu \rangle \nu$ on the regular part.

**Lemma 3.1.** Let $W$ be an integral $(n-1)$-varifold in $\mathbb{S}^n$, and suppose that the cone $C(W)$ has finite entropy. Then we have

$$F_{x_0,1}(C(W)) = (4\pi)^{-\frac{n}{2}} e^{-|x_0|^2/4} \int K_{n-1}(\langle p, x_0 \rangle) \, d\mu_W(p),$$

where

$$K_n(t) = e^{t^2/4} I_n(t)$$

is the sequence of real analytic functions defined by the recurrence relation

$$I_n(t) = tI_{n-1}(t) + 2(n-1)I_{n-2}(t),$$

for $n \geq 2$, and

$$I_0(t) = \sqrt{\pi}(1 + \text{erf}(t/2)), \quad I_1(t) = tI_0(t) + 2 e^{-t^2/4}.$$ 

**Proof.** Set $V = C(W)$. Using polar coordinates $r > 0, p \in \mathbb{S}^n$ for $x = rp \in \mathbb{R}^{n+1}$, we have that

$$\int_{\mathbb{R}^{n+1}} e^{-\frac{|x-x_0|^2}{4}} \, d\mu_V(x) = \int_{\mathbb{S}^n} \left( \int_0^\infty e^{-\frac{|rp-x_0|^2}{4}} r^{n-1} \, dr \right) \, d\mu_W(p).$$
Completing the square we have $|rp - x_0|^2 = (r - \langle p, x_0 \rangle)^2 + |x_0|^2 - \langle p, x_0 \rangle^2$, where we have used that $|p|^2 = 1$. Setting $t = \langle p, x_0 \rangle$, it remains to compute the integrals

\[(3.8) \quad I_n(t) = \int_0^\infty e^{-\frac{(r-t)^2}{4}} r^n dr \]

for each $n$. First, for $n = 0$ by definition of the error function we have

\[(3.9) \quad I_0(t) = \int_{-t}^\infty e^{-u^2/4} du = \sqrt{\pi}(1 + \text{erf}(t/2)). \]

For $n \geq 1$ we have

\[(3.10) \quad I_n(t) = \int_0^\infty (r-t) e^{-\frac{(r-t)^2}{4}} r^{n-1} dr + t \int_0^\infty e^{-\frac{(r-t)^2}{4}} r^{n-1} dr \]

\[= -2r^{n-1} e^{-\frac{(r-t)^2}{4}} \bigg|_0^\infty + 2(n-1)I_{n-2}(t) + tI_{n-1}(t), \]

where we have used integration by parts in the second equality. For $n \geq 2$ the first term vanishes whilst for $n = 1$ it evaluates to $2e^{-t^2/4}$, which gives the result. \hfill \Box

### 3.1. Variations.

For an integral $(n-1)$-varifold $W$ in $S^n$, we consider normal variations $W_s$ of $W$ in $S^n$ generated by smooth, compactly supported vector fields $X$ on $S^n$, so that $X(p) \perp T_p \text{reg} W_s$ for any $s$ and any $p \in \text{reg} W_s$. If $W$ is orientable, then we will write $X|_{\text{reg} W_s} = \phi_s \bar{\nu}_s$. Recall that $\bar{\nu}$ and $\bar{H}$ denote the normal and mean curvature of a hypersurface $M^{n-1}$ in $S^n$, respectively.

A direct computation yields the first variation formula for the $F$-functional on cones:

**Lemma 3.2 (First variation formula).** Let $W$ be an orientable integral $(n-1)$-varifold in $S^n$. Let $W_s$ be a normal variation of $W$ in $S^n$ generated by $X$, compactly supported away from $\text{sing} W$. Write $X|_{\text{reg} W} = \phi_s \bar{\nu}_s$ with $\phi = \phi_0$. If $x_s$ is a variation of $x_0$ with $x'_0 = y$, then $\partial_{s|s=0}(F_{x_s,1}(C(W_s)))$ is given by

\[(3.11) \quad \frac{e^{-|x_0|^2/4}}{(4\pi)^{\frac{n}{2}}} \int \left( \phi \bar{H} K_{n-1}(t) - \frac{1}{2} \langle x_0, y \rangle K_{n-1}(t) + (\langle y, p \rangle + \langle x_0, \nu \rangle \phi) K'_{n-1}(t) \right) d\mu_W(p), \]

where as before we have written $t = \langle p, x_0 \rangle$ for convenience.
Lemma 3.3 (Second variation formula). Let $W$ be an orientable integral $(n-1)$-varifold in $\mathbb{S}^n$. Let $W_s$ be a normal variation of $W$ in $\mathbb{S}^n$ generated by $X$, compactly supported away from $\text{sing} W$. Write $X|_{\text{reg} W} = \phi_s \tilde{\nu}_s$ with $\phi = \phi_0$ and $\phi' = \partial_s|_{s=0}\phi_s$. Also let $x_s$ be a variation of $x_0$ with $x'_0 = y$, $x''_0 = y'$. Then $\partial_s^2|_{s=0}(F_{x_0,1}(C(W_s)))$ is given by

$$\begin{align*}
\frac{e^{-\frac{|x_0|^2}{2}}}{(4\pi)^{\frac{n}{2}}} \int \left[ - (\phi \tilde{H} - \frac{1}{2}(x_0, y))^2 K_{n-1}(t) + (\langle y, \tilde{\nu} \rangle \phi) K'_{n-1}(t) - \frac{1}{2}|y|^2 K_{n-1}(t) \\
+ \left( \phi \tilde{H} - \frac{1}{2}(x_0, y) \right)^2 K_{n-1}(t) + (\langle y, \tilde{\nu} \rangle \phi)^2 K''_{n-1}(t) \\
+ \phi' \left( \tilde{H} K_{n-1}(t) + (x_0, \tilde{\nu}) K'_{n-1}(t) \right) \\
- \frac{1}{2}(x_0, y') K_{n-1}(t) + \langle p, y' \rangle K'_{n-1}(t) \right] d\mu_W(p),
\end{align*}$$

(3.12)

where again we have written $t = \langle p, x_0 \rangle$ for convenience, and $K'_{n-1}, K''_{n-1}$ are just the usual derivatives of the single-variable function $K_{n-1}$ (as opposed to the variational derivative).

Proof. The proof is a direct calculation by differentiating the first variation formula, using that on $M = \text{reg} W$ we have $\tilde{\nu}' = -\nabla \phi$ and that $\tilde{H}'$ is given by the Jacobi operator,

$$\tilde{H}' = -\Delta_M \phi - |\tilde{A}|^2 \phi - (n-1) \phi = -\tilde{L}_M \phi$$

(3.13)

for hypersurfaces in $\mathbb{S}^n$ (see for instance [67]).

We will now specialise to the case of a critical point, but first we need some integral identities for minimal hypersurfaces in $\mathbb{S}^n$.

Lemma 3.4. If $W$ is a stationary integral $(n-1)$-varifold in $\mathbb{S}^n$ then for any fixed vector $y \in \mathbb{R}^{n+1}$ we have

$$\int \langle y, p \rangle d\mu_W(p) = 0,$$

(3.14)

$$\int |y|^2 d\mu_W = (n-1) \int \langle y, p \rangle^2 d\mu_W(p).$$

(3.15)
Proof. We apply the divergence theorem to certain ambient vector fields $X$, recalling that a stationary varifold in $S^n$ has generalised mean curvature in $\mathbb{R}^{n+1}$ given by $\tilde{H}(p) = -(n-1)p$.

For the first claim, simply take $X = y$, so that $\text{div}_W X = 0$.

For the second claim, take $X = \langle y, x \rangle y$, then we have $\text{div}_W X = \langle y^T, y \rangle = |y^T|^2$ and $\langle p, X(p) \rangle = \langle y, p \rangle^2$.

Proposition 3.5 (Second variation at a critical point). Let $W$ be an orientable stationary integrable $(n-1)$-varifold in $S^n$. Let $W_s$ be a normal variation of $W$ in $S^n$ generated by $X$, compactly supported away from $\text{sing} W$. Write $X|_{\text{reg} W} = \phi, \nu$ with $\phi = \phi_0$. Also let $x_s$ be a variation of $x_0 = 0$ with $x'_0 = y$. Then $\partial^2_s|_{s=0}(F_{x_0, 1}(C(W_s)))$ is given by

$$
\frac{1}{2} \pi^{-\frac{n}{2}} \Gamma \left( \frac{n}{2} \right) \int \left( -\phi \tilde{L} \phi + 2 \frac{\Gamma(1+n)}{\Gamma(\frac{n}{2})} \phi \langle y, \nu \rangle - \frac{1}{2} |y|^2 \right) d\mu_W(p)
$$

(3.16)

Proof. Using the recurrence for $K_{n-1}$ one may verify the special values $K_{n-1}(0) = 2^{n-1} \Gamma(\frac{n}{2})$, $K'_{n-1}(0) = 2^{n-1} \Gamma(1+n, 2)$ and $K''_{n-1}(0) = 2^{n-2} n \Gamma(\frac{n}{2})$. Plugging $x_0 = 0$ and $\tilde{H} = 0$ into Lemma 3.3 we get that $\partial^2_s|_{s=0}(F_{x_0, 1}(C(W_s)))$ is given by

$$
\frac{1}{2} \pi^{-\frac{n}{2}} \Gamma \left( \frac{n}{2} \right) \int \left[ -\phi \tilde{L} \phi + 2 \frac{\Gamma(1+n)}{\Gamma(\frac{n}{2})} \phi \langle y, \nu \rangle - \frac{1}{2} |y|^2 
+ \frac{n}{2} \langle y, p \rangle^2 + \frac{\Gamma(1+n)}{\Gamma(\frac{n}{2})} \langle y', p \rangle \right] d\mu_W(p),
$$

(3.17)

where $y' = x''_0$. Using Lemma 3.4 to handle the last three terms completes the proof, recalling that according to the decomposition (3.2) we have $|y|^2 = |y^T|^2 + \langle y, p \rangle^2 + |y'|^2$.

Remark 3.6. If $V = C(W)$ is a stationary cone then, working in polar coordinates $r = |x|$ on the regular part $\Sigma = \text{reg} C(W)$, the stability operator $L_\Sigma$ has the decomposition

$$
(3.18) \quad L f = r^{-2} \Delta_M f + \frac{n-1}{r} \partial_r f + \partial_r^2 f - \frac{r}{2} \partial_r f + \frac{|A|^2}{r^2} f + \frac{1}{2} f = r^{-2}(\tilde{L}_M - (n-1) + L_1)f,
$$
where

\[ (3.19) \quad L_1 = r^2 \partial_r^2 + (n - 1)r \partial_r - \frac{r^3}{2} \partial_r + \frac{r^2}{2}. \]

Noting that \( L_1 r = (n - 1)r \), and using the evaluation of the special integrals \( I_n(0) \), it follows that the integral over the cone \( C(W) \)

\[ (3.20) \quad \int \left( -fLf + f \langle y, \nu \rangle - \frac{|y|^2}{2} \right) \rho \, d\mu_{C(W)} \]

coincides with (3.16) if we set \( f(x) = r\phi(\frac{x}{r}) \). This shows in particular that the second variation formula Proposition 2.1 is valid for homogenous variations of a stationary cone.

We record the following estimate for the coefficient of the middle term of (3.16).

**Lemma 3.7.** For any integer \( n \geq 2 \) we have

\[ (3.21) \quad \frac{\Gamma(\frac{1+n}{2})^2}{\Gamma(\frac{3}{2})^2} < n - 1. \]

**Proof.** Let \( A_n = \frac{1}{n-1} \frac{\Gamma(\frac{1+n}{2})^2}{\Gamma(\frac{3}{2})^2} \). By the functional equation for the gamma function, we have for all \( n > 3 \) that \( A_n = \frac{(n-1)(n-3)}{(n-2)^2} A_{n-2} < A_{n-2} \), so the lemma follows from checking that \( A_2 = \frac{\Gamma(3/2)^2}{\Gamma(1)^2} = \frac{\pi}{4} < 1 \) and \( A_3 = \frac{1}{2} \frac{\Gamma(2)^2}{\Gamma(3/2)^2} = \frac{2}{\pi} < 1. \) \( \square \)

4. Integration on singular hypersurfaces

In this section we present some technical results that will allow us to work on the regular part of an integral varifold with small enough singular set.

4.1. Cutoff functions. Given an integral \( n \)-varifold \( V \) in \( \mathbb{R}^{n+1} \) satisfying \( \mathcal{H}^{n-q}(\text{sing } V) = 0, \ q \geq 0 \), we describe here our choice of cutoff functions (on \( \mathbb{R}^{n+1} \)) that will allow us to integrate around the singular set.

For any fixed \( R > 4 \) and \( \epsilon > 0 \), since the singular set is closed, using the definition of Hausdorff measure we may cover the compact set \( \text{sing } V \cap B_R \) by finitely many Euclidean
balls,

\[(4.1) \quad \text{sing} \, V \cap B_R \subset \bigcup_{i=1}^{m} B_{r_i}(p_i), \quad \text{where} \quad \sum_i r_i^{n-q} < \epsilon,\]

and of course we may assume without loss of generality that \(r_i < 1\) for each \(i\). This covering depends on \(q, R\) and \(\epsilon\), but we will suppress this dependence in the notation.

Given such a covering, we may take smooth cutoff functions \(0 \leq \phi_i \leq 1\) such that \(\phi_i = 1\) outside \(B_{3r_i}(p_i)\) and \(\phi_i = 0\) inside \(B_{2r_i}(p_i)\), with \(|D\phi_i| \leq \frac{2}{r_i}\) in between. We will also need to cut off on large balls so we fix a cutoff function \(0 \leq \eta_R \leq 1\) such that \(\eta_R = 1\) inside \(B_{R-3}\) and \(\eta_R = 0\) outside \(B_{R-2}\), with \(|D\eta_R| \leq 2\) in between. Then, we combine these cutoffs by setting \(\phi_{R,\epsilon} = \inf_i (\phi_i, \eta_R) \leq 1\), which is Lipschitz with compact support in \(B_{R-1} \setminus \bigcup_{i=1}^{m} B_{2r_i}(p_i)\), and satisfies \(|D\phi_{R,\epsilon}| \leq \sup_i (|D\phi_i|, |D\eta_R|)\).

We will also need cutoff functions on annuli by smooth functions \(0 \leq \psi_i \leq \frac{2}{r_i}\) satisfying \(\psi_i = \frac{2}{r_i}\) inside \(B_{3r_i}(p_i) \setminus B_{2r_i}(p_i)\) and \(\psi_i = 0\) outside \(B_{4r_i}(p_i) \setminus B_{r_i}(p_i)\), with \(|D\psi_i| \leq \frac{4}{r_i^2}\) in between. We also take \(0 \leq \xi_R \leq 2\) such that \(\xi_R = 2\) inside \(B_{R-2} \setminus B_{R-3}\) and \(\xi_R = 0\) outside \(B_{R-1} \setminus B_{R-4}\), with \(|D\xi_R| \leq 4\) in between. We combine these by setting \(\psi_{R,\epsilon} = \sup_i (\psi_i, \xi_R)\), which is Lipschitz and satisfies \(|D\psi_{R,\epsilon}| \leq \sup_i (|D\psi_i|, |D\xi_R|)\). In particular, we have

\[(4.2) \quad |D\phi_{R,\epsilon}| \leq \psi_{R,\epsilon}.\]

We will reduce the dependence to the single parameter \(R\) by choosing \(\epsilon = \epsilon(R)\) such that \(\lim_{R \to \infty} \epsilon(R) = 0\). In this setting we write more compactly \(\phi_R = \phi_{R,\epsilon}, \psi_R = \psi_{R,\epsilon}\).

**4.2. Integration.** We will conduct our analysis in the weighted \(L^p\) spaces introduced in [37]. We say that a function \(f\) is weighted \(L^p\) on a hypersurface \(\Sigma\) if it is \(L^p\) with respect to the measure \(\rho \, d\mu_\Sigma\). That is, for \(p \in (0, \infty)\) we say \(f\) is weighted \(L^p\) if \(\|f\|_p := \left(\int_\Sigma |f|^p \rho\right)^{\frac{1}{p}} < \infty\), and for \(p = \infty\) we require \(\|f\|_\infty = \sup_\Sigma |f| < \infty\). The weighted \(W^{k,p}\) spaces are defined analogously. The goal of this subsection is to establish conditions under which integration by parts is justified in these spaces.

Recall that the operator \(\mathcal{L}\) is symmetric with respect to the weight \(\rho\):
Lemma 4.1 ([37], Lemma 3.8). If $\Sigma \subset \mathbb{R}^{n+1}$ is any hypersurface, $u$ is a $C^1$ function with compact support in $\Sigma$ and $v$ is a $C^2$ function, then

\begin{equation}
\int_{\Sigma} u(\mathcal{L}v)\rho = -\int_{\Sigma} \langle \nabla v, \nabla u \rangle \rho.
\end{equation}

In the remainder of this subsection $\Sigma^n$ will denote the regular part of an $n$-varifold $V$ in $\mathbb{R}^{n+1}$ with Euclidean volume growth. The exponential decay of the weight $\rho = (4\pi)^{-n/2} e^{-|x|^2/4}$ then gives that any function on $\Sigma$ of polynomial growth in $|x|$ is automatically weighted $L^p$ for any $p \in (0, \infty)$.

Lemma 4.2. Let $q > 0$ and suppose that $\mathcal{H}^{n-q}(\text{sing} V) = 0$. Let $\Sigma = \text{reg} V$, and take $\phi_R = \phi_{R, \epsilon}$ as in Section 4.1. Then we have the following gradient estimate for $\phi_R$:

\begin{equation}
\int_{\Sigma} |\nabla \phi_R|^q \rho \leq 2^q C_V (R^n e^{-\frac{(R-3)^2}{4}} + 3^n \epsilon),
\end{equation}

where $C_V$ is the volume growth constant. In particular $\lim_{R \to \infty} \int_{\Sigma} |\nabla \phi_R|^q \rho = 0$.

Proof. We have

\begin{equation}
\int_{\Sigma} |\nabla \phi_R|^q \rho \leq \int_{\Sigma \cap B_{R-2} \setminus B_{R-3}} 2^q \rho + \sum_{i=1}^{m} \int_{\Sigma \cap B_{3r_i}(p_i) \setminus B_{2r_i}(p_i)} \frac{2^q}{r_i^q} \leq 2^q C_V \left( R^n e^{-\frac{(R-3)^2}{4}} + 3^n \sum_i r_i^{-q} \right) \leq 2^q C_V (R^n e^{-\frac{(R-3)^2}{4}} + 3^n \epsilon).
\end{equation}

The limit follows since we choose $\epsilon$ such that $\lim_{R \to \infty} \epsilon(R) = 0$.

Corollary 4.3. Assume $\mathcal{H}^{n-q}(\text{sing} V) = 0$ for some $q$. Let $\Sigma = \text{reg} V$ and $\phi_R$ be as above.

(1) Suppose that $q \geq 1$ and that $f$ is weighted $L^p$, $p = \frac{q}{q-1}$. Then

\begin{equation}
\lim_{R \to \infty} \int_{\Sigma} |f| |\nabla \phi_R| \rho = 0.
\end{equation}
(2) Suppose that \(q \geq 2\) and that \(f\) is weighted \(L^p\), \(p = \frac{2q}{q-2}\). Then

\[
(4.7) \quad \lim_{R \to \infty} \int_\Sigma f^2 |\nabla \phi_R|^2 \rho = 0.
\]

Note again that here we allow \(p = \infty\).

**Proof.** For (1), using Hölder’s inequality, we have

\[
(4.8) \quad \int_\Sigma |f||\nabla \phi_R| \rho \leq \|f\|_p \left( \int_\Sigma |\nabla \phi_R|^q \rho \right)^{\frac{1}{q}}
\]

where \(\frac{1}{p} + \frac{1}{q} = 1\).

Similarly for (2) we have

\[
(4.9) \quad \int_\Sigma |f|^2 |\nabla \phi_R|^2 \rho \leq \|f\|^2_p \left( \int_\Sigma |\nabla \phi_R|^q \rho \right)^{\frac{2}{q}}
\]

where \(\frac{2}{p} + \frac{2}{q} = 1\).

By supposition the weighted \(L^p\)-norms of \(f\) are finite, so both results now follow from Lemma 4.2.

**Lemma 4.4.** Suppose that \(\mathcal{H}^{n-q}(\text{sing} \, V) = 0\) for some \(q \geq 1\). Further suppose that \(u, v\) are \(C^2\) functions on \(\Sigma = \text{reg} \, V\) such that \(|\nabla u|\,|\nabla v|\) and \(|u\nabla v|\) are weighted \(L^1\), and \(|u\nabla v|\) is weighted \(L^p\), \(p = \frac{q}{q-1}\). Then

\[
(4.10) \quad \int_\Sigma (u \nabla v) \rho = - \int_\Sigma \langle \nabla u, \nabla v \rangle \rho.
\]

**Proof.** If \(\phi\) has compact support we may use Lemma 4.1 to get

\[
(4.11) \quad \int_\Sigma \phi (u \nabla v) \rho = - \int_\Sigma \phi \langle \nabla u, \nabla v \rangle \rho - \int_\Sigma \langle \nabla v, \nabla \phi \rangle \rho.
\]

Applying this to \(\phi = \phi_R\), Corollary 4.3 gives that the second term on the right tends to zero as \(R \to \infty\), so the result follows by dominated convergence.

\[\square\]
In practice we will refer to both Lemma 4.1 and Lemma 4.4 simply as integration by parts.

5. Stability of singular self-shrinkers

Throughout this section $\Sigma^n$ will denote an orientable self-shrinker in $\mathbb{R}^{n+1}$ with Euclidean volume growth $\text{Vol}(\Sigma \cap B_r(x)) \leq C V r^n$. The main goals of this section are to understand the first stability eigenvalue of $\Sigma$ and to construct $F$-unstable variations when it is low enough.

Frequently we will take $\Sigma$ to be the regular part of an $F$-stationary varifold $V$ with finite entropy (which necessarily has Euclidean volume growth), and the results will depend on the size of the singular set. In several cases the assumptions on $\text{sing} V$ may be weakened using the regularity theory Proposition 2.4, but we state the stronger hypotheses to clarify the degree of regularity required.

5.1. Stability spectrum of $\Sigma$. Recall that the first stability eigenvalue of the stability operator

$$L = \Delta_\Sigma - \frac{1}{2} \langle x, \nabla \Sigma \rangle + |A|^2 + \frac{1}{2}$$

on a self-shrinker $\Sigma$ is defined by

$$\lambda_1(\Sigma) = \inf_\Omega \lambda_1(\Omega) = \inf_f \frac{\int_\Sigma (|\nabla f|^2 - |A|^2 f^2 - \frac{1}{2} f^2) \rho}{\int_\Sigma f^2 \rho},$$

where the infimum is taken over functions compactly supported in $\Sigma$, and could potentially be $-\infty$. Also recall that if indeed $\lambda_1 = \lambda_1(\Sigma) > -\infty$, then we have the stability inequality

$$\int_\Sigma |A|^2 f^2 \rho \leq \int_\Sigma |\nabla f|^2 \rho + \left( -\frac{1}{2} - \lambda_1 \right) \int_\Sigma f^2 \rho,$$

for Lipschitz functions $f$ compactly supported in $\Sigma$.

**Lemma 5.1.** Suppose that $u > 0$ is a $C^2$ function on $\Sigma$ with $Lu = -\lambda u$. Then $\lambda_1(\Sigma) \geq \lambda$. 88
Moreover, if \( f \) is Lipschitz with compact support in \( \Sigma \), then
\[
\int_{\Omega} f^2(|A|^2 + |\nabla \log u|^2) \rho \leq \int_{\Omega} (4|\nabla f|^2 - 2\lambda f^2) \rho.
\]

**Proof.** Since \( u > 0 \), the function \( \log u \) is well-defined on \( \Sigma \) and we can compute that
\[
\mathcal{L} \log u = -\lambda - \frac{1}{2} - |A|^2 - |\nabla \log u|^2.
\]
Since \( f \) has compact support in \( \Sigma \), then integrating \( f^2 \mathcal{L} \log u \) by parts we have that
\[
\int_{\Sigma} \left( \lambda + \frac{1}{2} + |A|^2 + |\nabla \log u|^2 \right) f^2 \rho = \int_{\Sigma} \langle \nabla f^2, \nabla \log u \rangle \rho.
\]
Using the absorbing inequality \( |\langle \nabla f^2, \nabla \log u \rangle| \leq |\nabla f|^2 + f^2 |\nabla \log u|^2 \) we get that
\[
\int_{\Sigma} \left( \lambda + \frac{1}{2} + |A|^2 \right) f^2 \rho \leq \int_{\Sigma} |\nabla f|^2 \rho
\]
and hence
\[
\frac{\int_{\Sigma}(|\nabla f|^2 - |A|^2 f^2 - \frac{1}{2} f^2) \rho}{\int_{\Sigma} f^2 \rho} \geq \lambda.
\]
Since this holds for any \( f \) with compact support in \( \Sigma \), we conclude that \( \lambda_1(\Sigma) \geq \lambda \) as claimed.

If we instead absorb using \( |\langle \nabla f^2, \nabla \log u \rangle| \leq 2|\nabla f|^2 + \frac{1}{2} f^2 |\nabla \log u|^2 \) we get that
\[
\int_{\Sigma} \left( \lambda + \frac{1}{2} + |A|^2 + \frac{1}{2} |\nabla \log u|^2 \right) f^2 \rho \leq 2 \int_{\Sigma} |\nabla f|^2 \rho,
\]
which implies the bound (5.4).

We will frequently apply Lemma 5.1 to subdomains \( \Omega \) of the regular part of an \( F \)-stationary varifold as well as to the regular part itself.

5.1.1. **Weighted integral estimates.**

**Lemma 5.2.** Let \( V \) be an orientable \( F \)-stationary \( n \)-varifold in \( \mathbb{R}^{n+1} \) with finite entropy and \( \mathcal{H}^{n-q}(\text{sing } V) = 0 \) for some \( q \geq 2 \). Suppose that \( u > 0 \) is a \( C^2 \) function on \( \Sigma = \text{reg } V \)
with \( Lu = -\lambda u \). Then if \( \phi \) is weighted \( W^{1,2} \) and weighted \( L^p \), \( p = \frac{2q}{q-2} \). Then

\[
\int_\Omega \phi^2(|A|^2 + |\nabla \log u|^2) \rho \leq \int_\Omega (8|\nabla \phi|^2 - 2\lambda \phi^2) \rho.
\]

**Proof.** We take \( f = \phi_R \phi \), where \( \phi_R \) is as in Section 4. Applying Lemma 5.1 we get that

\[
\int_\Sigma \phi_R^2 \phi^2(|A|^2 + |\nabla \log u|^2) \rho \leq \int_\Sigma (8\phi^2|\nabla \phi_R|^2 + 8\phi_R^2 |\nabla \phi|^2 - 2\lambda \phi_R^2 \phi^2) \rho.
\]

As \( R \to \infty \), the second and third terms on the right converge since \( \phi \) is weighted \( W^{1,2} \), and Corollary 4.3 implies that the first term on the right term tends to zero, whence Fatou’s lemma gives the result.

For any integer \( k \geq 0 \), the function \(|x|^{2k}\) is a polynomial in \( x \), so by the Euclidean volume growth it is of course weighted \( W^{1,p} \) for any \( p \in (0, \infty) \). Thus we immediately get:

**Corollary 5.3.** Let \( V \) be an orientable \( F \)-stationary \( n \)-varifold in \( \mathbb{R}^{n+1} \) with finite entropy and \( \mathcal{H}^{n-q}(\text{sing } V) = 0 \) for some \( q > 2 \). Suppose that \( u > 0 \) is a \( C^2 \) function that satisfies \( Lu = -\lambda u \) on \( \Sigma = \text{reg } V \). Then \(|A||x|^k\) and \(|x|^k|\nabla \log u|\) are weighted \( L^2 \) for any \( k \geq 0 \).

We now record the main quantitative \( L^2 \) estimates for \(|A|\) and \(|\nabla \log u|\) that will be essential both for constructing unstable variations when \( \lambda_1 < -1 \), and for classifying mean convex self-shrinkers. It is crucial that the estimate holds for positive eigenfunctions \( u \) defined only on a subdomain \( \Omega \).

**Lemma 5.4.** Let \( V \) be an orientable \( F \)-stationary \( n \)-varifold in \( \mathbb{R}^{n+1} \) with finite entropy and \( \mathcal{H}^{n-4}(\text{sing } V) = 0 \). Let \( \phi_R = \phi_{R,\epsilon} \) be as in Section 4.1 and consider a domain \( \Omega \subset \Sigma = \text{reg } V \) such that \( \text{spt}(\phi_R|_{\Omega}) \subset \subset \Omega \). If \( u \) is a positive \( C^2 \) function on \( \Omega \) satisfying \( Lu = -\lambda u \), then

\[
\int_\Omega (|A|^2 + |\nabla \log u|^2) \phi_R^2 |\nabla \phi_R|^2 \rho \leq (256 + 8|\lambda|) C_V(R^n e^{-(R-4)^2} + 4^n \epsilon).
\]
Proof. Recall that we cover the singular set by $\text{sing} V \cap B_R \subset \bigcup_{i=1}^{m} B_{r_i}(p_i)$, where $\sum_{i=1}^{m} r_i^{n-4} < \epsilon$ and without loss of generality $r_i < 1$ for each $i$.

The key is to replace $|\nabla \phi_R|$ by the annular bump function $\psi_R = \psi_{R, \epsilon} \geq |\nabla \phi_R|$, which has better regularity properties:

\begin{equation}
\int_{\Omega} (|A|^2 + |\nabla \log u|^2) \phi_R^2 |\nabla \phi_R|^2 \rho \leq \int_{\Omega} (|A|^2 + |\nabla \log u|^2) \psi_R^2 \rho.
\end{equation}

In particular, the function $f = (\phi_R \psi_R)|_\Omega$ is Lipschitz with compact support in $\Omega$, and we may now apply Lemma 5.1 on $\Omega$ which yields

\begin{equation}
\int_{\Omega} (|A|^2 + |\nabla \log u|^2) \phi_R^2 \psi_R^2 \rho \leq \int_{\Omega} (8\psi_R^2 |\nabla \phi_R|^2 + 8\phi_R^2 |\nabla \psi_R|^2 + 2|\phi_R \psi_R|^2) \rho.
\end{equation}

We may bound the first term on the right in (5.14) by

\begin{equation}
\int_{\Sigma} \psi_R^2 |\nabla \phi_R|^2 \rho \leq \int_{\Sigma} \psi_R^4 \rho \leq \int_{\Sigma \cap B_{R-1} \setminus B_{R-4}} 16\rho + \sum_{i=1}^{m} \int_{\Sigma \cap B_{4r_i}(p_i) \setminus B_{r_i}(p_i)} \frac{16}{r_i^4}
\end{equation}

\begin{align*}
&\leq 16C_V \left( R^n e^{-\frac{(R-4)^2}{4}} + 4^n \sum_i r_i^{n-4} \right) \\
&\leq 16C_V (R^n e^{-\frac{(R-4)^2}{4}} + 4^n \epsilon).
\end{align*}

Since $\phi_R \leq 1$ the second term on the right in (5.14) is bounded by

\begin{equation}
\int_{\Sigma} \phi_R^2 |\nabla \psi_R|^2 \rho \leq \int_{\Sigma \cap B_{R-1} \setminus B_{R-4}} 16\rho + \sum_{i=1}^{m} \int_{\Sigma \cap B_{4r_i}(p_i) \setminus B_{r_i}(p_i)} \frac{16}{r_i^4}
\end{equation}

\begin{align*}
&\leq 16C_V \left( R^n e^{-\frac{(R-4)^2}{4}} + 4^n \sum_i r_i^{n-4} \right) \\
&\leq 16C_V (R^n e^{-\frac{(R-4)^2}{4}} + 4^n \epsilon),
\end{align*}
and since \( r_i < 1 \) the last term is bounded by

\[
\int_{\Sigma} \phi_R^2 \psi_R^2 \rho \leq \int_{\Sigma \cap B_{R-4}} 4 \rho + m \int_{\Sigma \cap B_{4r_i}(p_i) \setminus B_{r_i}(p_i)} \frac{4}{r_i^2} \\
\leq 4C_V \left( R^n e^{-\frac{(R-4)^2}{4}} + 4^n \sum_{i=1}^m \rho_i^{n-2} \right) \\
\leq 4C_V (R^n e^{-\frac{(R-4)^2}{4}} + 4^n \epsilon).
\]

Combining these estimates gives the result as claimed. \( \square \)

5.1.2. Bottom of the spectrum.

**Lemma 5.5.** Let \( \Sigma^n \) be a connected, orientable self-shrinker with \( \lambda_1 = \lambda_1(\Sigma) > -\infty \). Then there is a positive \( C^2 \) function on \( \Sigma \) with \( Lu = -\lambda_1 u \).

Moreover, suppose that \( \Sigma \) is the regular part of an \( F \)-stationary \( n \)-varifold \( V \) with finite entropy and \( H^{n-q}(\text{sing } V) = 0 \) for some \( q \geq 2 \). If \( v \) is a \( C^2 \) function on \( \Sigma \) with \( Lv = -\lambda_1 v \), which is weighted \( W^{1,2} \) and weighted \( L^p \), \( p = \frac{2q}{q-2} \), then \( v = cu \) for some \( c \in \mathbb{R} \).

**Proof.** For the existence of \( u \) we proceed as in [37]: Fix \( p \in \Sigma \) and consider an exhaustion \( p \in \Omega_1 \subset \Omega_2 \subset \cdots \) of \( \Sigma = \bigcup_i \Omega_i \). For each \( i \) there is a positive Dirichlet eigenfunction \( Lu_i = -\lambda_1(\Omega_i) u_i \) on \( \Omega_i \), and we may normalise so that \( u_i(p) = 1 \). Since \( \lambda_1(\Omega_i) \) decreases monotonically to \( \lambda_1 > -\infty \), the Harnack inequality gives \( 1 \leq \sup u_i \leq C \inf u_i \leq C \), where \( C = C(\Omega_i, \lambda_1) \). Elliptic theory gives uniform \( C^{2,\alpha} \) bounds on the \( u_i \) on each compact set, so we get a subsequence converging uniformly in \( C^2 \) to a nonnegative solution of \( Lu = -\lambda_1 u \) on \( \Sigma \) with \( u(p) = 1 \). The Harnack inequality again implies that \( u \) is positive on \( \Sigma \).

For the uniqueness, by the assumptions on \( v \), Lemma 5.2 gives that \( |A|v \) and \( v \|\nabla \log u\| \) are weighted \( L^2 \). By expansion this implies that \( vL v \) and \( v^2 \mathcal{L} \log u \) are weighted \( L^1 \), and since \( v \) is weighted \( W^{1,2} \) we see that \( |\nabla v|^2 |\nabla \log u| \leq |\nabla v|^2 + v^2 |\nabla \log u|^2 \) is weighted \( L^1 \). Moreover since \( \frac{1}{2} + \frac{1}{p} = \frac{q-1}{q} \), Hölder’s inequality gives that \( \|v \nabla v\|_{\frac{q}{q-1}} \leq \|\nabla v\|_2 \|v\|_p < \infty \) and \( \|v^2 \nabla \log u\|_{\frac{q}{q-1}} \leq \|v \nabla \log u\|_2 \|v\|_p < \infty \).
Lemma 4.4 now allows us to integrate by parts to get

\[(5.18) \int \langle \nabla v^2, \nabla \log u \rangle \rho = - \int v^2 \mathcal{L} \log u \rho = \int v^2 (\lambda_1 + |A|^2 + \frac{1}{2} + |\nabla \log u|^2) \rho. \]

and

\[(5.19) \int |\nabla v|^2 \rho = - \int v \mathcal{L} \rho = \int v^2 (\lambda_1 + |A|^2 + \frac{1}{2}) \rho. \]

Rearranging we find that

\[(5.20) \int |v \nabla \log u - \nabla v|^2 \rho = 0, \]

hence \(v \nabla \log u - \nabla v = 0\) and \(\frac{2}{u}\) is constant on \(\Sigma\).

\[\square\]

**Lemma 5.6.** Let \(V\) be an orientable \(F\)-stationary \(n\)-varifold in \(\mathbb{R}^{n+1}\) with finite entropy and \(H^{n-q}(\text{sing} V) = 0\) for some \(q \geq 2\). Then on \(\Sigma = \text{reg} V\) we get the same \(\lambda_1(\Sigma)\) by taking the infimum over Lipschitz functions \(f\) on \(\Sigma\) that are weighted \(W^{1,2}\) and \(L^p\), \(p = \frac{2q}{q-2}\).

**Proof.** Obviously we may assume that \(\lambda_1 = \lambda_1(\Sigma) > -\infty\). By using the global eigenfunction produced by Lemma 5.5 in Lemma 5.2 we have that \(|A|f\) is weighted \(L^2\). Let \(\phi_R\) be as in Section 4. We will use the test functions \(f_R = f\phi_R\) in the definition of \(\lambda_1\).

Now since \(f\) and \(|A|f\) are weighted \(L^2\), dominated convergence gives that \(\int \phi_R^2 f_R^2 \rho \to \int f^2 \rho\) and \(\int |A|^2 f_R^2 \rho \to \int |A|^2 f^2 \rho\) as \(R \to \infty\). For the gradient term we have

\[(5.21) \int |\nabla f_R|^2 \rho = \int (\phi_R^2 |\nabla f|^2 + 2(\nabla f, \nabla \phi_R) + f^2 |\nabla \phi_R|^2) \rho. \]

The second and third terms on the right tend to zero as \(R \to \infty\), by parts (1) and (2) of Corollary 4.3 respectively. Moreover, the first term tends to \(\int |\nabla f|^2 \rho\) by dominated convergence. Thus we have shown that \(\int |\nabla f_R|^2 \rho \to \int |\nabla f|^2 \rho\), and the lemma follows. \(\square\)

**Proposition 5.7.** Let \(V\) be an orientable \(F\)-stationary \(n\)-varifold in \(\mathbb{R}^{n+1}\) with finite entropy and \(H^{n-q}(\text{sing} V) = 0\) for some \(q \geq 2\). Suppose that \(v \neq 0\) is a \(C^2\) function on
\( \Sigma = \text{reg } V \) satisfying \(Lv = -\lambda v\), which is weighted \( W^{1,2} \) and weighted \( L^p \), \( p = \frac{2q}{q-2} \). Then \( \lambda_1(\Sigma) \leq \lambda \).

**Proof.** Obviously we may assume \( \lambda_1 = \lambda_1(\Sigma) > -\infty \).

Using the positive eigenfunction produced by Lemma 5.5 as in the proof of that lemma, we have by Lemma 5.2 that \(|A|v\) is weighted \( L^2 \), and hence that \( vLv \) is weighted \( L^1 \). Again since \( \frac{1}{2} + \frac{1}{p} = \frac{q-1}{q} \) we have that \( \|v \nabla v\|_{\frac{q}{q-1}} \leq \|\nabla v\|_2 \|v\|_p < \infty \), so by Lemma 5.6 we may use \( v \) as a test function in the definition of \( \lambda_1 \), and moreover Lemma 4.4 allows us to integrate by parts:

\[
(5.22) \quad \int_{\Sigma} |\nabla v|^2 \rho = \int_{\Sigma} v^2 \left( \frac{1}{2} + \lambda + |A|^2 \right) \rho.
\]

This implies that \( \lambda_1 \leq \lambda \) as claimed.

**Corollary 5.8.** Let \( V \) be an orientable \( F \)-stationary \( n \)-varifold in \( \mathbb{R}^{n+1} \) with finite entropy and \( \mathcal{H}^{n-q}(\text{sing } V) = 0 \) for some \( q > 2 \). Then \( \lambda_1(V) \leq -\frac{1}{2} \), with equality if and only if \( \text{spt } V \) is a hyperplane.

**Proof.** Clearly we may assume \( \lambda_1 > -\infty \), and we assume without loss of generality that \( \Sigma = \text{reg } V \) is connected. Fix a point \( p \in \Sigma \) and set \( v(x) = \langle \nu(p), \nu(x) \rangle \). Then \( |v| \leq 1 \) is bounded, and using the positive eigenfunction from Lemma 5.5 for Corollary 3.3 we see that \( |\nabla v| \leq |A| \) is weighted \( L^2 \). The upper bound for \( \lambda_1 \) then follows from Proposition 5.7 since \( Lv = \frac{1}{2}v \). Moreover, if equality holds then since \( L\langle y, \nu \rangle = \frac{1}{2}\langle y, \nu \rangle \) for any fixed \( y \), the uniqueness in Lemma 5.5 implies that \( \nu \) is constant on \( \Sigma \). The constancy theorem then implies that \( \text{spt } V \) is a hyperplane.

**Corollary 5.9.** Let \( V \) be an orientable \( F \)-stationary \( n \)-varifold in \( \mathbb{R}^{n+1} \) with finite entropy and \( \mathcal{H}^{n-q}(\text{sing } V) = 0 \) for some \( q > 2 \). If \( H \) is not identically zero on \( \Sigma = \text{reg } V \), then we have \( \lambda_1(\Sigma) \leq -1 \), with equality if and only if \( H \) does not change sign on \( \Sigma \).
Proof. From the self-shrinker equation $H = \frac{1}{2} \langle x, \nu \rangle$ we see that $|H| \leq |x|$ is weighted $L^p$ for any $p \in (0, \infty)$. Moreover, differentiating the self-shrinker equation leads to $|\nabla H| \leq |A||x|$.

Now clearly we may assume $\lambda_1 > -\infty$, and we may assume without loss of generality that $\Sigma = \text{reg} V$ is connected. Then using the positive eigenfunction $u$ of Lemma 5.5 for Corollary 5.3 implies that $|\nabla H| \leq |A||x|$ is weighted $L^2$. The result follows from Proposition 5.7 since $LH = H$. If equality holds, the uniqueness of Lemma 5.5 implies that $H = cu$ does not change sign.

5.2. Constructing unstable variations. Here we construct $F$-unstable variations when the first stability eigenvalue $\lambda_1$ is small. We first consider the easy case when $\lambda_1 < -\frac{3}{2}$ which does not require any assumptions on the singular set. The proof is essentially the same as in [37, Lemma 12.4], but we include it here for completeness.

Proposition 5.10. Let $V$ be an orientable $F$-stationary n-varifold in $\mathbb{R}^{n+1}$ with finite entropy and regular part $\Sigma = \text{reg} V$. If $\lambda_1(\Sigma) < -\frac{3}{2}$, then there exists a domain $\Omega \subset\subset \Sigma$ such that if $u$ is a Dirichlet eigenfunction for $\lambda_1(\Omega)$, then for any $a \in \mathbb{R}$ and any $y \in \mathbb{R}^{n+1}$ we have

\[(5.23)\quad \int_{\Omega} \left( -uL\bar{u} + 2ua\bar{H} - a^2\bar{H}^2 + u\langle y, \nu \rangle - \frac{(y, \nu)^2}{2} \right) \rho < 0.\]

Consequently, $V$ is $F$-unstable.

Proof. Since $\lambda_1(\Sigma) < -\frac{3}{2}$ we may choose a domain $\Omega \subset\subset \Sigma$ so that $\lambda_1(\Omega) < -\frac{3}{2}$. Then completing the square, the left hand side above is given by

\[(5.24)\quad \int_{\Omega} \left( \left( \frac{3}{2} + \lambda_1(\Omega) \right) u^2 - (u - aH)^2 - \frac{1}{2} (u - \langle y, \nu \rangle)^2 \right) \rho < 0.\]

so we are done by the second variation formula Proposition 2.1.
We now construct $F$-unstable variations when $\lambda_1 < -1$. The key, as in [37, Section 9.2], is to quantify an “almost orthogonality” between the first eigenfunction and the eigenfunction $H$, but our analysis of the cross term differs significantly - instead of estimating boundary terms arising from integration by parts, we use our chosen cutoff functions adapted to sufficiently large domains to estimate the cross term directly. To do so, we require that the singular set is small enough that we may use the previous results of this section.

**Proposition 5.11.** Let $V$ be an orientable $F$-stationary $n$-varifold in $\mathbb{R}^{n+1}$ with finite entropy and regular part $\Sigma = \text{reg } V$. Suppose that $\mathcal{H}^{n-4}(\text{sing } V) = 0$. If $\lambda_1(\Sigma) < -1$, then there exists a domain $\Omega \subset \subset \Sigma$ such that if $u$ is a Dirichlet eigenfunction for $\lambda_1(\Omega)$, then for any $a \in \mathbb{R}$ and any $y \in \mathbb{R}^{n+1}$ we have

\[
\int_{\Omega} \left( -uLu + 2uaH - a^2H^2 + u\langle y, \nu \rangle - \frac{\langle y, \nu \rangle^2}{2} \right) \rho < 0.
\]

Consequently, $V$ is $F$-unstable.

**Proof.** As before we can absorb the cross term $u\langle y, \nu \rangle$ using $-\frac{1}{2}u^2$ and $-\frac{1}{2}\langle y, \nu \rangle^2$, so the left hand side is bounded above by

\[
\int_{\Omega} \left( \left( \frac{1}{2} + \lambda_1(\Omega) \right) u^2 + 2uaH - a^2H^2 \right) \rho.
\]

If $H$ is identically zero on $\Sigma$ then we are done, so henceforth we assume this is not the case.

By Proposition 5.10 we may assume $-\frac{3}{2} \leq \lambda_1(\Sigma) < -1$. Also we may assume without loss of generality that $\Sigma$ is connected. We now claim that we can find a domain $\Omega \subset \subset \Sigma$ with $\lambda_1(\Omega) < -1$ and for which the cross term can be absorbed by:

\[
\left( \int_{\Omega} uH \rho \right)^2 \leq \frac{1}{2} \left( \int_{\Omega} H^2 \rho \right) \left( \int_{\Omega} u^2 \rho \right).
\]
Given the claim, the proof proceeds by again completing the square: Using (5.27) to bound the cross term, the expression (5.26) is bounded above by

\[(1 + \lambda_1(\Omega)) \left( \int_{\Omega} u^2 \rho \right) - \left( \frac{1}{\sqrt{2}} \left( \int_{\Omega} u^2 \rho \right)^{\frac{1}{2}} - |a| \left( \int_{\Omega} H^2 \rho \right)^{\frac{1}{2}} \right)^2 < 0,\]

which is strictly negative since \( \lambda_1(\Omega) < -1 \). This implies that \( V \) is \( F \)-unstable by the second variation formula Proposition 2.1.

To prove the claim, we let \( R > 4 \), set \( \epsilon = R^{-3} \) and cover the singular set as in Section 4.1:

\[\text{sing} V \cap B_R \subset \bigcup_{i=1}^n B_{r_i}(p_i), \text{ with } \sum_i r_i^{n-4} < \epsilon \text{ and } r_i < 1 \text{ for each } i.\]

Now we let \( \phi_R = \phi_{R,\epsilon} \) be as in Section 4.1 and take a domain \( \Omega = \Omega_R \) such that

\[(5.29) \quad \text{spt}(\phi_R|_{\Omega_R}) \subset \subset \Omega_R \subset \subset \Sigma \cap B_R.\]

Then the \( \Omega = \Omega_R \) must exhaust \( \Sigma \) as \( R \to \infty \), so by domain monotonicity of the first eigenvalue there exists a \( \delta_0 > 0 \) such that

\[(5.30) \quad \lambda_1(\Omega) \leq -1 - \delta_0 \]

for any \( R \) sufficiently large.

To prove (5.27), we may assume without loss of generality that \( \Omega \) is connected and that \( u > 0 \) on \( \Omega \). Indeed, let \( \{\Omega^i\}_{i=1}^N \) be the connected components of \( \Omega \), and order them so that \( \lambda_1(\Omega) = \lambda_1(\Omega^1) \leq \lambda_1(\Omega^2) \leq \cdots \leq \lambda_1(\Omega^N) \). Then the first Dirichlet eigenfunction \( u = u_\Omega \) of \( L \) on \( \Omega \) may be taken to be zero on \( \Omega^i, i > 1 \), and equal to \( u_{\Omega^1} \) on \( \Omega^1 \), where \( u_{\Omega^1} \) is the first Dirichlet eigenfunction of \( L \) on the connected domain \( \Omega^1 \) and may therefore be taken to be positive. The estimate (5.27) then follows from the corresponding estimate on \( \Omega^1 \).
To achieve the estimate, we give ourselves some room using the cutoff function $\phi_R^2 \leq 1$,

\begin{align}
\int_\Omega uH\rho &= \left| \int_\Omega (uH\phi_R^2 \rho + uH(1-\phi_R^2)\rho) \right| \\
&\leq \left| \int_\Omega uH\phi_R^2 \rho \right| + \int_{\Omega \setminus \text{spt}(1-\phi_R^2)} |uH|\rho \\
&\leq \left| \int_\Omega uH\phi_R^2 \rho \right| + \left( \int_\Omega u^2 \rho \right)^{\frac{1}{2}} \left( \int_{\Omega \setminus \text{spt}(1-\phi_R^2)} H^2 \rho \right)^{\frac{1}{2}}.
\end{align}

We can crudely estimate using $|H| \leq |x| \leq R$ on $B_R$ that

\begin{align}
\int_{\Omega \setminus \text{spt}(1-\phi_R^2)} H^2 \rho &\leq R^2 \left( \int_{\Omega \setminus B_R \setminus B_{R-3}} \rho + \sum_{i=1}^m \int_{\Omega \setminus B_{r_i}(p_i)} \rho \right) \\
&\leq C_V R^2 \left( R^n e^{-\frac{(R-3)^2}{4}} + \sum_{i=1}^m 3^n C_V r_i^n \right) \\
&\leq C_V (R^{n+2} e^{-\frac{(R-3)^2}{4}} + 3^n R^2 \epsilon),
\end{align}

where $C_V$ is the volume growth constant, and we have used that the $r_i < 1$.

For the other term, we note that

\begin{align}
H\mathcal{L}u - u\mathcal{L}H &= HLu - uLH = (-\lambda_1(\Omega) - 1)uH
\end{align}

on $\Omega$. Setting

\begin{align}
\alpha &= -\lambda_1(\Omega) - 1 \in [\delta_0, \frac{1}{2}],
\end{align}

we then have

\begin{align}
\int_\Omega uH\phi_R^2 \rho &= \frac{1}{\alpha} \int_\Omega \phi_R^2 (H\mathcal{L}u - u\mathcal{L}H) \rho \\
&= \frac{2}{\alpha} \int_\Omega \phi_R (\nabla \phi_R, u\nabla H - H\nabla u) \rho,
\end{align}

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where we integrated by parts for the second equality. Therefore

\begin{equation}
\left| \int_{\Omega} u H \phi_R^2 \rho \right| \leq \frac{2}{\alpha} \int_{\Omega} \phi_R |\nabla \phi_R| (|u \nabla H| + |H \nabla u|) \rho.
\end{equation}

We estimate the gradient terms as follows: First, Cauchy-Schwarz gives

\begin{equation}
\int_{\Omega} \phi_R |\nabla \phi_R| |u \nabla H| \rho \leq \left( \int_{\Omega} u^2 \rho \right)^{\frac{1}{2}} \left( \int_{\Omega} \phi_R^2 |\nabla \phi_R|^2 |\nabla H|^2 \rho \right)^{\frac{1}{2}}.
\end{equation}

Using $|\nabla H| \leq |A||x| \leq |A|R$ on $B_R$, we have

\begin{equation}
\int_{\Omega} \phi_R^2 |\nabla \phi_R|^2 |\nabla H|^2 \rho \leq R^2 \int_{\Omega} \phi_R^2 |\nabla \phi_R|^2 |A|^2 \rho.
\end{equation}

For the second gradient term, since $u > 0$ on $\Omega$, we may use Cauchy-Schwarz to get

\begin{equation}
\int_{\Omega} \phi_R |\nabla \phi_R| |H \nabla u| \rho \leq \left( \int_{\Omega} u^2 \rho \right)^{\frac{1}{2}} \left( \int_{\Omega} H^2 \phi_R^2 |\nabla \phi_R|^2 \frac{\nabla u^2}{u^2} \rho \right)^{\frac{1}{2}}.
\end{equation}

Again using $|H| \leq |x| \leq R$ on $B_R$, we have

\begin{equation}
\int_{\Omega} H^2 \phi_R^2 |\nabla \phi_R|^2 \frac{\nabla u^2}{u^2} \rho \leq R^2 \int_{\Omega} \phi_R^2 |\nabla \phi_R|^2 \frac{\nabla u^2}{u^2} \rho.
\end{equation}

But now by Lemma 5.4, since $|\lambda_1(\Omega)| \leq |\lambda_1(\Sigma)| \leq \frac{3}{2}$, we have

\begin{equation}
\int_{\Omega} (|A|^2 + |\nabla \log u|^2) \phi_R^2 |\nabla \phi_R|^2 \rho \leq 268 C_V (R^n e^{-\frac{(R-4)^2}{4}} + 4^n \epsilon).
\end{equation}

Putting all our estimates into (5.31), using that $\alpha \geq \delta_0$, we obtain that

\begin{equation}
\frac{\int_{\Omega} u H \rho}{(\int_{\Omega} u^2 \rho)^{\frac{1}{2}}} \leq C \left( R^{n+2} e^{-\frac{(R-4)^2}{4}} + 4^n R^2 \epsilon \right)^{\frac{1}{2}},
\end{equation}

where $C = \left( 1 + \frac{2\sqrt{205}}{\delta_0} \right) \sqrt{C_V}$ does not depend on $R$. Since we chose $\epsilon = R^{-3}$, the right hand side tends to zero as $R \to \infty$. This shows that we can make $\frac{\int_{\Omega} u H \rho}{(\int_{\Omega} u^2 \rho)^{\frac{1}{2}}}$ as small as we like by choosing $R$ large. But since $H$ is not identically zero, and since the $\Omega_R$ form an exhaustion of $\Sigma$, we see that $\int_{\Omega} H^2 \rho$ has a uniform positive lower bound $\delta_1^2$ for sufficiently
large $R$. Choosing $R$ large enough so that 
\[ \frac{|f_{\Omega} u H \rho|}{(f_{\Omega} w^2 \rho)^{\frac{1}{2}}} < \frac{1}{\sqrt{2}} \delta_1 \]
will satisfy the condition (5.27). Together with (5.30) this establishes the claim and thus concludes the proof.

\[ \square \]

Finally, we briefly record the construction of $F$-unstable variations of stationary cones.

**Proposition 5.12.** Let $V$ be an orientable stationary $n$-cone in $\mathbb{R}^{n+1}$ so that $H = 0$ on $\Sigma = \text{reg} V$. If $\lambda_1(\Sigma) < -\frac{1}{2}$, then there exists a domain $\Omega \subset \subset \Sigma$ such that if $u$ is a Dirichlet eigenfunction for $\lambda_1(\Omega)$, then for any $y \in \mathbb{R}^{n+1}$ we have

\[ (5.43) \quad \int_{\Omega} \left( -u Lu + u \langle y, \nu \rangle - \frac{\langle y, \nu \rangle^2}{2} \right) \rho < 0. \]

Consequently, $V$ is $F$-unstable.

**Proof.** Since $\lambda_1(\Sigma) < -\frac{1}{2}$ we may choose a domain $\Omega \subset \subset \Sigma$ so that $\lambda_1(\Omega) < -\frac{1}{2}$. Completing the square, the left hand side is bounded above by $(\frac{1}{2} + \lambda_1(\Omega)) \int_{\Omega} u^2 \rho < 0$, which implies that $\Sigma$ is $F$-unstable by the second variation formula Proposition 2.1 since $H = 0$ on $\Sigma$.

\[ \square \]

6. Mean convex singular self-shrinkers

Throughout this section $\Sigma$ denotes the regular part of an orientable $F$-stationary $n$-varifold $V$ in $\mathbb{R}^{n+1}$ with Euclidean volume growth. The goal is to extend the classification of mean convex self-shrinkers due to Huisken [65] and Colding-Minicozzi [37] to the singular setting.

By Lemma 5.1 if $H > 0$ on $\Sigma$, then $\lambda_1(\Sigma) \geq -1$, so again some of the hypotheses on the singular set in this section may be weakened using the regularity theory Proposition 2.4. We continue to state the results with the stronger hypotheses to clarify the dependence on the size of the singular set. We will need the following Simons-type inequality for self-shrinkers:
Lemma 6.1 ([37], Lemma 10.8). On any smooth orientable self-shrinker we have $LA = A$. Hence, if $|A|$ does not vanish at a point then at that point one has

\begin{equation}
L|A| = |A| + \frac{|\nabla A|^2 - |\nabla |A||^2}{|A|} \geq |A|.
\end{equation}

We now adapt the Schoen-Simon-Yau [97] argument to improve our control on $|A|$.

Lemma 6.2. Suppose that $\mathcal{H}^{n-4}(\text{sing } V) = 0$. If $H > 0$ on $\Sigma = \text{reg } V$ then $|A|$ is weighted $L^4$ and $|\nabla |A||$, $|\nabla A|$ are weighted $L^2$.

Proof. First, for $\eta$ with compact support in $\Sigma$, integrating $|A|^2\eta^2\log H$ by parts as in Lemma 5.1 and using the absorbing inequality (twice) gives

\begin{equation}
\int_{\Sigma} |A|^4\eta^2 \rho \leq (1 + a) \int_{\Sigma} |\nabla |A||^2\eta^2 \rho + \int_{\Sigma} |A|^2 \left(1 + a^{-1} |\nabla \eta|^2 + \frac{1}{2}\eta^2\right) \rho,
\end{equation}

where $a$ is an arbitrary positive number to be chosen later.

Second, it follows from the Simons-type inequality (6.1) and Colding-Minicozzi’s Kato inequality ([37] Lemma 10.2) that

\begin{equation}
\int_{\Sigma} |A|^4\eta^2 \rho + \int_{\Sigma} \left(\frac{2n}{n + 1} |\nabla H|^2\eta^2 + a^{-1}|A|^2|\nabla \eta|^2\right) \rho \geq \left(1 + \frac{2}{n + 1} - a\right) \int_{\Sigma} |\nabla |A||^2\eta^2 \rho.
\end{equation}

Combining (6.2) and (6.3) then gives

\begin{equation}
\int_{\Sigma} |A|^4\eta^2 \rho \leq \frac{1 + a}{1 + \frac{2}{n + 1} - a} \int_{\Sigma} |A|^4\eta^2 \rho + C_{n,a} \int_{\Sigma} (|\nabla H|^2\eta^2 + |A|^2\eta^2 + |A|^2|\nabla \eta|^2) \rho.
\end{equation}

Choosing $a < \frac{1}{n+1}$ will give that the first coefficient on the right is less than 1 and thus may be absorbed on the left, therefore

\begin{equation}
\int_{\Sigma} |A|^4\eta^2 \rho \leq C \int_{\Sigma} (|\nabla H|^2\eta^2 + |A|^2\eta^2 + |A|^2|\nabla \eta|^2) \rho,
\end{equation}

where $C = C(n)$.

Let $\phi_R = \phi_{R,\epsilon}$ be as in Section 4.1. We will apply (6.5) with $\eta = \phi_R^2$. 

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As in Corollary 5.9 using Corollary 5.3 with the positive eigenfunction $H$ shows that $|A|$ and $|\nabla H|$ are weighted $L^2$. Therefore as $R \to \infty$, the first and second terms on the right will converge to the finite integrals $\int_{\Sigma} |\nabla H|^2 \rho$ and $\int_{\Sigma} |\nabla A|^2 \rho$ respectively. To bound the last term in (6.5) we use Lemma 5.4 with the globally defined eigenfunction $H$, which gives

\begin{equation}
\int_{\Sigma} |A|^2 |\nabla \eta|^2 \rho = 4 \int_{\Sigma} |A|^2 \phi_R^2 |
\nabla \phi_R|^2 \rho \leq 1056 C_V \left(R^\alpha e^{-\frac{(R-\epsilon)^2}{4}} + 4^\alpha \epsilon \right).
\end{equation}

Choosing $\epsilon = R^{-1}$ and taking $R \to \infty$ we see that this term tends to 0, thus we have shown that indeed $|A|$ is weighted $L^4$ by Fatou’s lemma. With this fact in hand, it follows from (6.3) that $|\nabla |A||$ is weighted $L^2$.

Finally, multiplying the identity $\mathcal{L}|A|^2 = 2|\nabla A|^2 + |A|^2 - 2|A|^4$ by $\frac{1}{2} \eta^2$ and integrating by parts, we have

\begin{equation}
\int_{\Sigma} \eta^2 (|\nabla A|^2 - |A|^4) \rho \leq - \int_{\Sigma} 2 \eta |A| \langle \nabla \eta, \nabla |A| \rangle \rho \leq \int_{\Sigma} (\eta^2 |\nabla |A||^2 + |A|^2 |\nabla \eta|^2) \rho.
\end{equation}

Since we now know that $|\nabla |A||$ is weighted $L^2$ and that $|A|$ is weighted $L^4$, we again set $\eta = \phi_R^2$ and use (6.6) to handle the last term; this shows that $|\nabla A|^2$ is weighted $L^2$, as desired.

\begin{lemma}
Suppose that $\mathcal{H}^{n-4}(\text{sing} V) = 0$. If $H > 0$ on $\Sigma = \text{reg} V$, then $|A|/H$ is constant and hence $|\nabla A|^2 = |\nabla |A||^2$ on $\Sigma$.
\end{lemma}

\begin{proof}
We assume without loss of generality that $\Sigma$ is connected.

We wish to integrate $|A|^2 \mathcal{L} \log H$ and $|A| \mathcal{L} |A|$ by parts. So we check:

First, since $|A|$ is weighted $W^{1,2}$ and $L^4$ by the above lemma, using Lemma 5.2 with $H > 0$ gives that $|A||\nabla \log H|$ is weighted $L^2$. Using Young’s inequality we then have

\begin{equation}
(|A|^2 |\nabla \log H|)^p = |A|^p (|A||\nabla \log H|)^p \leq \frac{2}{2} |A|^\frac{2p}{2} + \frac{p}{2} |A|^2 |\nabla \log H|^2.
\end{equation}

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Since $|A|$ was weighted $L^4$ this shows that $|A|^2|\nabla \log H|$ is weighted $L^p$ for $p = \frac{4}{3}$. Since $L \log H = \frac{1}{2} - |A|^2 - |\nabla \log H|^2$, we see that $|A|^2|L \log H|$ is weighted $L^1$. Also

$$(6.9) \quad |\nabla |A|^2| |\nabla \log H| = 2|A||\nabla |A|| |\nabla \log H| \leq |A|^2|\nabla \log H|^2 + |\nabla |A||^2$$

is weighted $L^1$ since $|\nabla |A||$ was weighted $L^2$. By Lemma 4.4 we may now integrate $|A|^2L \log H$ by parts to find that

$$(6.10) \quad \int_\Sigma \langle |A|^2, \nabla \log H \rangle \rho = \int_\Sigma |A|^2(|A|^2 - \frac{1}{2} + |\nabla \log H|^2)\rho.$$ 

Now using the Simon’s equality we have that

$$(6.11) \quad |A|L|A| = \frac{1}{2}|A|^2 - |A|^4 + |\nabla A|^2 - |\nabla |A||^2$$

is weighted $L^1$. We already know that $|\nabla |A||$ is weighted $L^2$, and as above we have that

$$(6.12) \quad (|A||\nabla |A||)^p \leq \frac{2 - p}{2}|A|^\frac{2p}{p} + \frac{p}{2}|\nabla |A||^2.$$ 

Again since $|A|$ is weighted $L^4$ this gives that $|A||\nabla |A||$ is weighted $L^p$ for $p = \frac{4}{3}$, so we may use Lemma 4.4 to get that

$$(6.13) \quad \int_\Sigma |\nabla |A||^2 \rho = -\int_\Sigma |A|L|A| \rho \leq \int_\Sigma (|A|^4 - \frac{1}{2}|A|^2)\rho.$$ 

Subtracting (6.10) from (6.13) and rearranging we get

$$(6.14) \quad 0 \geq \int_\Sigma ||A|\nabla \log H - \nabla |A||^2 \rho,$$

which implies that $|A|\nabla \log H = \nabla |A|$ and hence $|A|/H$ is constant on $\Sigma$. The final statement follows again from the Simons inequality (6.1) since equality now must hold in the previous inequalities. \[\Box\]

We are now ready to present the proof of Theorem 0.3.
Theorem 6.4. Let $V$ be an orientable $F$-stationary $n$-varifold in $\mathbb{R}^{n+1}$ with finite entropy, and suppose that $\mathcal{H}^{n-1}(\text{sing} V) = 0$. If $H \geq 0$ on $\text{reg} V$ then either $V$ is a stationary cone, or $\text{spt} V$ is a generalised cylinder $\mathbb{S}^k(\sqrt{2k}) \times \mathbb{R}^{n-k}$.

Proof. Since $\mathcal{H}^{n-1}(\text{sing} V) = 0$, we may assume that $\Sigma = \text{reg} V$ is connected. Then since $LH = H$, by the Harnack inequality we must either have $H > 0$ or $H \equiv 0$ on $\Sigma$. If $H \equiv 0$ on $\Sigma$ then in particular $x^+ = 0$ almost everywhere on $V$, so $V$ must be a stationary cone.

Otherwise, we have $H > 0$ on $\Sigma$. By Lemma 5.1, we then have $\lambda_1(\Sigma) \geq -1$ so by the regularity theory Theorem 2.4 we may assume that $\text{sing} V$ has codimension at least 7.

Now by Lemma 6.3, we have that $|A|/H$ is constant and $|\nabla A|^2 = |\nabla|A||^2$ on $\Sigma$. The remainder of the proof of [37, Theorem 0.17] goes through to prove that either $\nabla A \equiv 0$ on $\Sigma$, or there are constant vectors $e_2, \ldots, e_n \in \mathbb{R}^{n+1}$ that that are tangent at every point of $\Sigma$.

If $\nabla A \equiv 0$ on $\Sigma$, then [79, Theorem 4] (which does not assume completeness) implies that $\Sigma$ is a piece of a generalised cylinder $\Sigma_0 = \mathbb{S}^k(\sqrt{2k}) \times \mathbb{R}^{n-k}$. Then $\text{spt} V$ is contained in $\Sigma_0$, so by the constancy theorem we must have $\text{spt} V = \Sigma_0$.

On the other hand, if $e_2, \ldots, e_n \in \mathbb{R}^{n+1}$ are constant vectors tangent at every point of $\Sigma$, then $V$ must split those lines, so we have that $\mu_V = \mu_{\mathbb{R}^{n-1}} \times \mu_{\tilde{V}}$, where $\tilde{V}$ is an orientable $F$-stationary 1-varifold in $\mathbb{R}^2$. Since the singular set had codimension at least 7, certainly $\tilde{V}$ and hence $V$ must in fact correspond to smooth complete embedded hypersurfaces. By the result of [37, Theorem 0.17] or the remainder of its proof, we conclude that in this case $\text{spt} V$ must be a cylinder $\mathbb{S}^1(\sqrt{2}) \times \mathbb{R}^{n-1}$. \qed

7. Classification of stable self-shrinkers

In this section we classify $F$-stable and entropy-stable singular self-shrinkers. It will be convenient to include a quick lemma verifying that there are no nontrivial stationary cones in low dimensions which satisfy the $\alpha$-structural hypothesis.
Lemma 7.1. Let \( n \leq 2 \) and suppose that \( V = C(W) \) be a stationary \( n \)-cone in \( \mathbb{R}^{n+1} \). If \( V \) satisfies the \( \alpha \)-structural hypothesis for some \( \alpha \in (0,1) \), then \( \text{spt} \ V \) must be a hyperplane.

Proof. If \( n = 1 \), then the \( \alpha \)-structural hypothesis implies that any tangent cone to \( \text{spt} \ V \) consists of at most two rays, for which the only stationary configuration is a straight line. This shows that \( V \) is an integer multiple of a smooth cone, hence of a line.

If \( n = 2 \), by dilation invariance the link must also satisfy the \( \alpha \)-structural hypothesis. The above argument then shows that the link \( W \) is smooth. But the only smooth closed geodesics in \( S^2 \) are the great circles, so \( V \) must be a multiple of a plane. \( \square \)

7.1. \( F \)-stable self-shrinkers. First we classify \( F \)-stable self-shrinkers.

Theorem 7.2. Let \( V \) be an orientable \( F \)-stationary \( n \)-varifold in \( \mathbb{R}^{n+1} \) with finite entropy, that satisfies the \( \alpha \)-structural hypothesis for some \( \alpha \in (0, \frac{1}{2}) \). If \( V \) is \( F \)-stable then \( \text{spt} \ V \) must be a hyperplane \( \mathbb{R}^n \) or a shrinking sphere \( S^n(\sqrt{2}n) \).

Proof. Set \( \Sigma = \text{reg} \ V \). By Proposition 5.10, we may assume that \( \lambda_1(V) = \lambda_1(\Sigma) \geq -\frac{3}{2} \). As such, by the regularity theory Proposition 2.4 and the corresponding varifold maximum principle, we may assume that \( \text{sing} \ V \) has codimension at least 7 and hence that \( \Sigma \) is connected. Since \( LH = H \), the Harnack inequality gives three cases for the sign of \( H \):

Case 1: \( H \equiv 0 \) on \( \Sigma \). If \( \text{spt} \ V \) is not a hyperplane \( \mathbb{R}^n \), then Corollary 5.8 gives that \( \lambda_1(V) < -\frac{1}{2} \). But then Proposition 5.12 shows that \( V \) is \( F \)-unstable.

Case 2: \( H \) does not vanish on \( \Sigma \). In this case by Theorem 6.4 we know that \( \text{spt} \ V \) must be a generalised cylinder \( S^k(\sqrt{2}k) \times \mathbb{R}^{n-k} \), \( k > 0 \). Colding-Minicozzi showed in [37, Theorem 0.16] that of these only the \( k = n \) case is \( F \)-stable.

Case 3: \( H \) changes sign on \( \Sigma \). In this final case, Corollary 5.9 gives that \( \lambda_1(\Sigma) < -1 \). Then Proposition 5.11 provides an \( F \)-unstable variation. \( \square \)

We also need to classify homogenously \( F \)-stable stationary cones:
Theorem 7.3. Let $V = C(W)$ be an orientable stationary $n$-cone in $\mathbb{R}^{n+1}$, that satisfies the $\alpha$-structural hypothesis for some $\alpha \in (0, \frac{1}{2})$. If $V$ is homogenously $F$-stable, then $\text{spt} V$ must be a hyperplane.

Proof. By Lemma 7.1, we may assume $n \geq 3$. Suppose that $W$ is not totally geodesic. We will show that $V = C(W)$ is homogenously $F$-unstable. Indeed, let $M = \text{reg} W$ and consider a domain $\Omega \subset \subset M$. Let $u$ be a Dirichlet eigenfunction for the Jacobi operator $L$ on $\Omega$, so that $Lu = -\kappa_1(\Omega)u$. We would like to use $u$ as our normal variation of $M$ in $\mathbb{S}^n$.

By the second variation formula for the $F$-functional on cones, Proposition 3.5, it suffices to ensure that

\begin{equation}
\int_M \left( \kappa_1(\Omega)u^2 + 2 \frac{\Gamma(\frac{1+n}{2})}{\Gamma(\frac{n}{2})} u \langle y, \vec{\nu} \rangle - \frac{1}{2} \langle y, \vec{\nu} \rangle^2 \right) < 0
\end{equation}

for any $y \in \mathbb{R}^{n+1}$. Completing the square we have that

\begin{equation}
- \frac{1}{2} \langle y, \vec{\nu} \rangle^2 + 2 \frac{\Gamma(\frac{1+n}{2})}{\Gamma(\frac{n}{2})} u \langle y, \vec{\nu} \rangle \leq 2 \frac{\Gamma(\frac{1+n}{2})^2}{\Gamma(\frac{n}{2})^2} u^2.
\end{equation}

But now $M$ is not totally geodesic and $n \geq 3$, so by Chapter 3, Theorem 2.1 (see also [127]) and Lemma 3.7 respectively give that

\begin{equation}
\kappa_1(M) \leq -2(n-1) < -2 \frac{\Gamma(\frac{1+n}{2})^2}{\Gamma(\frac{n}{2})^2}.
\end{equation}

This implies the existence of the desired domain $\Omega$ and thus concludes the proof.

Alternatively, having verified that the second variation formula Proposition 2.1 is valid for homogenous variations (see Remark 3.6), we may use it directly. Setting $f(x) = |x|u(x/|x|)$ and $\Sigma = \text{reg} V$, as in the proof of [37] Theorem 0.14] it suffices to ensure that

\begin{equation}
\int_\Sigma \left( -fLf + f(y, \nu) - \frac{1}{2} \langle y, \nu \rangle^2 \right) \rho = \int_\Sigma \left( \kappa_1(\Omega)u^2 + |x|u \langle y, \nu \rangle - \frac{1}{2} \langle y, \nu \rangle^2 \right) \rho < 0.
\end{equation}

Estimating $2|x|u \langle y, \nu \rangle \leq \langle y, \nu \rangle^2 + |x|^2u^2$, we may bound the left hand side from above by

\begin{equation}
\int_\Sigma \left( \kappa_1(\Omega)u^2 + \frac{1}{2} |x|^2u^2 \right) \rho = \int_\Sigma \left( \kappa_1(\Omega) + n \right) u^2 \rho,
\end{equation}

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where we have used the fact that \( \int_0^\infty r^{n+1} e^{-\frac{r^2}{2}} \, dr = \frac{n}{2} \int_0^\infty r^{n-1} e^{-\frac{r^2}{2}} \, dr \). Again the fact that \( \kappa_1(M) \leq -2(n-1) < -n \) completes the proof.

\[ \square \]

**Remark 7.4.** Similarly to Lemma 3.7, using that \( \lim_{n \to \infty} \frac{\Gamma(\frac{1+n}{2})\sqrt{2}}{\Gamma(\frac{3}{2}) n^{n/2}} = 1 \) one may verify that \( n - 1 < \frac{2\Gamma(1+n/2)^2}{\Gamma(3/2) n^{n/2}} < n \) for all \( n \). The upper bound confirms that working on the link is slightly sharper than absorbing on the cone as in (7.5). The lower bound ensures that the computation above (correctly) does not apply to the totally geodesic (planar) case.

### 7.2. Entropy-stable self-shrinkers

Finally we are ready to classify entropy-stable self-shrinkers.

**Theorem 7.5.** Let \( V \) be an orientable \( F \)-stationary \( n \)-varifold in \( \mathbb{R}^{n+1} \) with finite entropy, that satisfies the \( \alpha \)-structural hypothesis for some \( \alpha \in (0, \frac{1}{2}) \). Assume that \( V \) is not a cone.

If \( \text{spt} V \) is not a generalised cylinder \( S^k(\sqrt{2}k) \times \mathbb{R}^{n-k} \), then \( V \) is entropy-unstable. Furthermore, if \( V \) does not split off a line and if \( \text{spt} V \) is not the shrinking sphere \( S^n(\sqrt{2n}) \), then the unstable variation can be taken to have compact support away from \( \text{sing} V \).

**Proof.** First suppose that \( V \) does not split off a line. If \( V \) is \( F \)-stable then the classification of singular \( F \)-stable self-shrinkers Theorem 7.2 gives that \( \text{spt} V \) must be a hyperplane \( \mathbb{R}^n \) or the shrinking sphere \( S^n(\sqrt{2n}) \). On the other hand, if \( V \) is \( F \)-unstable then by Theorem 2.2 it is entropy-unstable with respect to compactly supported variations.

Now suppose that \( \mu_V = \mu_{\mathbb{R}^{n-k}} \times \mu_{\tilde{V}} \), where \( \tilde{V} \) is an orientable \( F \)-stationary \( k \)-varifold in \( \mathbb{R}^{k+1} \) that does not split off a line. Then \( \Lambda(V) = \Lambda(\tilde{V}) \). But by the above, if \( \tilde{V} \) is not spherical then it is entropy-unstable, and the induced (translation-invariant) variation of \( V \) will also be entropy-unstable. \( \square \)

**Theorem 7.6.** Let \( V = C(W) \) be an orientable stationary \( n \)-cone in \( \mathbb{R}^{n+1} \), that satisfies the \( \alpha \)-structural hypothesis for some \( \alpha \in (0, \frac{1}{2}) \). If \( \text{spt} V \) is not a hyperplane \( \mathbb{R}^n \), then \( V \) is entropy-unstable under a homogenous variation induced by variation of the link \( W \) away from its singular set.
Proof. By Lemma 7.1 we may assume $n \geq 3$. If $V$ is homogenously $F$-stable, then by Theorem 7.3, $\text{spt} \ V$ must then be a hyperplane $\mathbb{R}^n$. On the other hand, if $V$ is homogenously $F$-unstable, then by Theorem 2.3 it is entropy-unstable under the corresponding homogenous variation. \hfill $\Box$

Remark 7.7. It may be useful contextually to recall that any dilation-invariant or translation-invariant self-shrinker is entropy-stable amongst compactly supported variations, since we may shift the Gaussian centre away from the variation. Therefore, the natural variations to consider, as we have above, are those with the same symmetries as the original self-shrinker.

One may note in particular that even the area-minimising non-flat cones are entropy-unstable when we allow the class of homogenous variations. On the one hand this makes sense since the area-minimising condition is only with respect to local perturbations, and there are certainly area-decreasing perturbations if again one allows homogenous variations. On the other hand, this suggests that the entropy functional may be limited in its ability to detect the dynamical stability of stationary cones under the mean curvature flow.

Finally, Theorem 0.2 is simply the combination of Theorems 7.5 and 7.6. We also observe:

Remark 7.8. Here we have considered deformations by certain ambient vector fields; in particular for higher multiplicity varifolds this does not allow the sheets to come apart. It is easy to verify that two distinct parallel planes together have entropy strictly less than 2, and similarly two distinct concentric spheres together have entropy strictly less than twice that of a single sphere. Thus, if the sheets are allowed to separate, it follows that higher multiplicity cylinders are also entropy-unstable in that sense.
CHAPTER 5

Mean curvature flow self-shrinkers

1. Rigidity of mean convex self-shrinkers

The work described in this section was first presented in the published article [58] and is joint with Dr Qiang Guang.

In this section, we describe strong rigidity theorems for the cylinder amongst mean convex self-shrinkers. For lower dimensions we have the following result, which removes the second fundamental form assumption from [35] for dimensions $n \leq 6$.

**Theorem 1.1.** Given $n \leq 6$ and $\lambda_0$, there exists $R = R(n, \lambda_0)$ so that if $\Sigma^n \subset \mathbb{R}^{n+1}$ is a smoothly embedded self-shrinker with entropy $\lambda(\Sigma) \leq \lambda_0$ which satisfies

\[ \langle H \rangle \geq 0 \text{ on } B_R \cap \Sigma, \]

then $\Sigma$ is a generalised cylinder $\mathbb{S}^k(\sqrt{2k}) \times \mathbb{R}^{n-k}$ for some $0 \leq k \leq n$.

For the remaining dimensions, we have a similar theorem assuming a positive lower bound for the mean curvature:

**Theorem 1.2.** Given $n$, $\lambda_0$ and $\delta > 0$, there exists $R = R(n, \lambda_0, \delta)$ so that if $\Sigma^n \subset \mathbb{R}^{n+1}$ is a smoothly embedded self-shrinker with entropy $\lambda(\Sigma) \leq \lambda_0$ which satisfies

\[ \langle H \rangle \geq \delta \text{ on } B_R \cap \Sigma, \]

then $\Sigma$ is a generalised cylinder $\mathbb{S}^k(\sqrt{2k}) \times \mathbb{R}^{n-k}$ for some $1 \leq k \leq n$.

The key technical tool is the following curvature estimate for uniformly mean convex shrinkers:

**Theorem 1.3.** Given $n$ and $\delta > 0$, there exists $C = C(n, \delta)$ so that for any smooth properly embedded self-shrinker $\Sigma^n \subset \mathbb{R}^{n+1}$ which satisfies
(*) $H \geq \delta$ on $B_R \cap \Sigma$ for $R > 2$,

we have

(1.1) $|A|(x) \leq \frac{CR}{R - |x|} H(x)$, for all $x \in B_{R-1} \cap \Sigma$.

1.1. Curvature estimates for strictly mean convex shrinkers. This subsection is devoted to proving Theorem 1.3. The proof requires some modifications of Ecker-Huisken’s interior estimates for mean curvature flow [49] (see also [47]), in which the authors derive curvature estimates using the maximum principle under the assumption that the flow is locally graphical. For our estimates, the mean convexity will replace the local graphical assumption - in particular, the key ingredient is the identity $LH = H$ that holds on all shrinkers.

First, in order to apply the maximum principle, we describe the choice of cutoff functions and detail the relevant computations:

Fix $n$ and $\delta > 0$. Let $\Sigma \subset \mathbb{R}^{n+1}$ be a self-shrinker which satisfies

(*) $H \geq \delta$ on $B_R \cap \Sigma$ for $R > 2$.

Set $v = 1/H$ and $v_0 = 1/\delta$, then we have $v \leq v_0$ on $B_R \cap \Sigma$. Recall that $LH = H$ on any self-shrinker. Hence, $v$ satisfies the equation

(1.2) $\Delta v = 2 \frac{\nabla H |^2}{H^3} - \frac{\Delta H}{H^2} = \frac{1}{2} \langle x, \nabla v \rangle + 2 \frac{\nabla v |^2}{v} + \left( |A|^2 - \frac{1}{2} \right) v$.

We now fix the function

(1.3) $h(y) = \frac{y}{1 - ky}$,

where $k = (2v_0^2)^{-1}$. Simple computations give that

(1.4) $h'(y) = \frac{1}{(1 - ky)^2}$ and $h''(y) = \frac{2k}{(1 - ky)^3}$.
For convenience, in what follows we will abuse notation slightly and write $h = h(v^2)$, $h' = h'(v^2)$ and so on.

Let $f = |A|^2h$. Then we have

$$\Delta f = h\Delta |A|^2 + |A|^2\Delta h + 2\langle \nabla |A|^2, \nabla h \rangle. \quad (1.5)$$

Note that

$$\nabla h = h'\nabla v^2 = 2h'v\nabla v, \quad \text{and} \quad \Delta h = h'\Delta v^2 + h''|\nabla v|^2. \quad (1.6)$$

Combining this with the Simons inequality for self-shrinkers gives that

$$\Delta f = h\left(2|\nabla A|^2 + (1 - 2|A|^2)|A|^2 + \frac{1}{2}\langle x, \nabla |A|^2 \rangle\right) + |A|^2\left(h''|\nabla v|^2 + h'\Delta v^2\right)$$

$$+ 2\langle \nabla |A|^2, \nabla h \rangle. \quad (1.7)$$

Now we estimate the right hand side of the equation $(1.7)$. First, we have

$$2\langle \nabla |A|^2, \nabla h \rangle = \frac{\langle \nabla h, \nabla f \rangle}{h} - |A|^2\frac{|\nabla h|^2}{h} + 4h'|A|v\langle \nabla |A|, \nabla v \rangle. \quad (1.8)$$

Using the absorbing inequality gives that

$$4h'|A|v\langle \nabla |A|, \nabla v \rangle \leq \frac{2(h')^2|A|^2v^2|\nabla v|^2}{h} + 2h|\nabla |A||^2. \quad (1.9)$$

This implies

$$2\langle \nabla |A|^2, \nabla h \rangle \geq \frac{\langle \nabla h, \nabla f \rangle}{h} - 6\frac{(h')^2|A|^2v^2|\nabla v|^2}{h} - 2h|\nabla |A||^2. \quad (1.10)$$

We also have that

$$h''|\nabla v|^2 + h'\Delta v^2 = 4h''v^2|\nabla v|^2 + h'\left[v\langle x, \nabla v \rangle + 4|\nabla v|^2 + (2|A|^2 - 1)v^2 + 2|\nabla v|^2\right] \quad (1.11)$$

and

$$\frac{1}{2}\langle x, \nabla f \rangle = \frac{h}{2}\langle x, \nabla |A|^2 \rangle + \frac{|A|^2}{2}\langle x, \nabla h \rangle = \frac{h}{2}\langle x, \nabla |A|^2 \rangle + |A|^2h'v\langle x, \nabla v \rangle. \quad (1.12)$$
Therefore, we obtain that

\[
\Delta f \geq \frac{\langle \nabla h, \nabla f \rangle}{h} + \frac{1}{2} \langle x, \nabla f \rangle + (1 - 2|A|^2)|A|^2 h + 2h'v^2|A|^4 - h'v^2|A|^2
\]

\[+ \left[ 4h''v^2 + 6\left( h' - \frac{(h')^2v^2}{h} \right) \right]|A|^2|\nabla v|^2. \tag{1.13}
\]

Now by the choice of \( h \) (compare (1.4)), we have

\[
h - h'v^2 = -kh^2, \tag{1.14}
\]

and

\[
4h''v^2 + 6\left( h' - \frac{(h')^2v^2}{h} \right) = \frac{2k}{(1 - kv^2)^2}h. \tag{1.15}
\]

Inserting these inequalities into (1.13) implies that

\[
\Delta f \geq \frac{\langle \nabla h, \nabla f \rangle}{h} + \frac{1}{2} \langle x, \nabla f \rangle - f + 2kf^2 + \frac{2k|\nabla v|^2}{(1 - kv^2)^2} f. \tag{1.16}
\]

Here we used that \( h'v^2|A|^2 \leq 2f \). We will set

\[
a = \frac{\nabla h}{h} \quad \text{and} \quad d = \frac{2k|\nabla v|^2}{(1 - kv^2)^2}. \tag{1.17}
\]

**Lemma 1.4.** Let \( x_0 \in \mathbb{R}^{n+1} \) and \( \rho > 0 \), and set \( \phi(x) = (\mu(x))^3 \), where \( (\mu(x))_+ = \max(\mu(x), 0) \) and \( \mu(x) = \rho^2 - |x - x_0|^2 \). If \( \Sigma^n \) is a shrinker, then on \( B_{\rho}(x_0) \cap \Sigma \) we have

\[
\Delta \phi = 24\mu|(x - x_0)^T|^2 - 6n\mu^2 + 6\mu^2H(x - x_0, \nu). \tag{1.18}
\]

In particular, we have the estimate

\[
|\Delta \phi(x)| \leq 24\mu\rho^2 + 6n\mu^2 + 3\mu^2\rho|x| \leq (24 + 6n)\rho^4 + 3\rho^3|x|. \tag{1.19}
\]
Proof. Since $\nabla \phi = -3(\rho^2 - |x - x_0|^2)^2 \nabla |x - x_0|^2 = -6\mu^2(x - x_0)^T$, we have

$$
\Delta \phi = -6 \text{div}(\mu^2(x - x_0)^T)
$$

(1.20)

$$
= -6 \left[ 2\mu \langle \nabla \mu, (x - x_0)^T \rangle + \mu^2 \left( n - \langle x - x_0, \nu \rangle H \right) \right]
$$

$$
= 24\mu |(x - x_0)^T|^2 - 6n\mu^2 + 6\mu^2 H\langle x - x_0, \nu \rangle.
$$

The second claim follows easily from the shrinker equation and the fact that $\mu \leq \rho^2$. $\square$

We are now ready to prove our main curvature estimate.

1.1.1. Proof of Theorem 1.3. Now fix a point $x_0 \in B_{R-1} \cap \Sigma$ and set $\rho = R - |x_0|$. Let $\phi$ be the function defined in Lemma 1.4. We will work on $B_{\rho}(x_0) \cap \Sigma$. Using (1.16) gives that

$$
\Delta(\phi f) = \phi \Delta f + f \Delta \phi + 2\langle \nabla \phi, \nabla f \rangle
$$

(1.21)

$$
\geq \phi \left[ \langle a + \frac{x}{2}, \nabla f \rangle - f + 2k f^2 + df \right] + f \Delta \phi + 2\langle \nabla \phi, \nabla f \rangle.
$$

Note that

(1.22)

$$
\langle a, \nabla(\phi f) \rangle = \phi \langle a, \nabla f \rangle + f \langle a, \nabla \phi \rangle
$$

and

(1.23)

$$
\langle \nabla \phi, \nabla(\phi f) \rangle = f|\nabla \phi|^2 + \phi \langle \nabla \phi, \nabla f \rangle.
$$

This implies

$$
\Delta(\phi f) \geq \langle a + \frac{x}{2}, \nabla(\phi f) \rangle - \langle a + \frac{x}{2}, \nabla \phi \rangle f + \phi \left[ (d - 1)f + 2k f^2 \right]
$$

(1.24)

$$
+ f \Delta \phi + \frac{2}{\phi} \langle \nabla \phi, \nabla(\phi f) \rangle - 2\frac{|\nabla \phi|^2}{\phi} f.
$$

Now we set $F(x) = \phi(x)f(x)$ and consider its maximum on $B_{\rho}(x_0) \cap \Sigma$. Since $F$ vanishes on $\partial B_{\rho}(x_0) \cap \Sigma$, $F$ achieves its maximum at some point $y_0 \in B_{\rho}(x_0) \cap \Sigma$. At the point $y_0$, we have

(1.25)

$$
\nabla F(y_0) = 0 \text{ and } \Delta F(y_0) \leq 0.
$$
In the following, we will work at the point $y_0$. By (1.24) and $f(y_0) > 0$, we have

\begin{equation}
\langle a + \frac{y_0}{2}, \nabla \phi \rangle + 2\frac{|\nabla \phi|^2}{\phi} \geq \phi(d - 1) + 2k\phi + \Delta \phi.
\end{equation}

Note that

\begin{equation}
|a|^2 = 4 \left( \frac{h'}{h} \right)^2 v^2 |\nabla v|^2 = \frac{2}{kv^2} d \leq \frac{|y_0|^2}{2k} d.
\end{equation}

This yields that

\begin{equation}
\langle a, \nabla \phi \rangle \leq (d + 1)\phi + \frac{|a|^2}{4(d + 1)} \frac{|\nabla \phi|^2}{\phi} \leq (d + 1)\phi + \frac{|y_0|^2}{8k} \frac{|\nabla \phi|^2}{\phi}.
\end{equation}

Combining (1.28) with (1.26) gives that

\begin{equation}
2k\phi f \leq -\Delta \phi + 2\phi + \left( 2 + \frac{|y_0|^2}{8k} \right) \frac{|\nabla \phi|^2}{\phi} + \frac{|y_0|}{2} |\nabla \phi|.
\end{equation}

By the definition of $\phi$, we have

\begin{equation}
\phi \leq \rho^6, \quad |\nabla \phi| \leq 6\rho^5 \quad \text{and} \quad \frac{|\nabla \phi|^2}{\phi} \leq 36\rho^4.
\end{equation}

Combining this with Lemma 1.4, $|y_0| \leq R$ and (1.29) yields that

\begin{equation}
F(y_0) = \phi(y_0)f(y_0) \leq C(\rho^6 + R\rho^5 + R^2\rho^4),
\end{equation}

where $C$ is a constant depending on $n$ and $\delta$.

Since $F$ achieves its maximum at $y_0$, we have $F(x_0) \leq F(y_0)$. This implies

\begin{equation}
\frac{\rho^6 |A|^2(x_0)}{H^2(x_0) - k} = F(x_0) \leq F(y_0) \leq C(\rho^6 + R\rho^5 + R^2\rho^4).
\end{equation}

In particular, we have

\begin{equation}
|A|(x_0) \leq C \left( 1 + \frac{R}{\rho} \right) H(x_0).
\end{equation}

Since $x_0$ is an arbitrary point in $B_{R-1} \cap \Sigma$, this completes the proof of Theorem 1.3.
1.2. Rigidity theorems for mean convex shrinkers. In this subsection we prove Theorems 1.1 and 1.2 by adapting the iteration and improvement scheme used to prove \([31]\) Theorem 0.1. For convenience of the reader, we briefly outline this scheme here; recall that the two key ingredients are the so-called iterative step \([31]\) Proposition 2.1] and the improvement step \([31]\) Proposition 2.2] (compare Proposition 1.6 below). In the iterative step, it is shown that if a self-shrinker is almost cylindrical (quantified by \(H\) and \(|A|\)) on a large scale, then it is still close to a cylinder on a larger scale, albeit with some loss in the estimates. It is important here that the scale extends by a fixed multiplicative factor.

**Proposition 1.5.** (Iteration; \([31]\) Proposition 2.1]}) Given \(\lambda_0 < 2\) and \(n\), there exist positive constants \(R_0, \delta_0, C_0\) and \(\theta\) so that if \(\Sigma^n \subset \mathbb{R}^{n+1}\) is a shrinker with \(\lambda(\Sigma) \leq \lambda_0\), \(R \geq R_0\), and

- \(B_R \cap \Sigma\) is smooth with \(H \geq 1/4\) and \(|A| \leq 2\),

then \(B_{(1+\theta)R} \cap \Sigma\) is smooth with \(H \geq \delta_0\) and \(|A| \leq C_0\).

On the other hand, in the improvement step, it is shown that if a shrinker is close to a cylinder on some scale, then the estimates can be improved so long as we decrease the scale by a fixed amount. We will show that the initial closeness in the improvement step only needs to be quantified by \(H\) — using our curvature estimate Theorem 1.3 and a compactness result of shrinkers, we can show that the bounded curvature assumption in the improvement step (Proposition 2.2 of \([31]\)) can be removed, which in turn implies Theorem 1.1. Our improvement step is stated as follows:

**Proposition 1.6.** (Improvement) Given \(n\) and \(\lambda_0\), let \(\delta_0 \in (0, 1/4)\) be given by Proposition 1.5. Then there exists \(R = R(n, \lambda_0)\) so that if \(\Sigma^n \subset \mathbb{R}^{n+1}\) is a shrinker with \(\lambda(\Sigma) \leq \lambda_0\) and

- \(H \geq \delta_0 \) on \(B_R \cap \Sigma\),

then \(H \geq 1/4\) and \(|A| \leq 2\) on \(B_{R-4} \cap \Sigma\).
The main argument in the improvement step is to control the derivatives of the tensor $\tau = A/H$. These estimates are shown to decay exponentially as $R^\alpha e^{-R/4}$ for some $\alpha$, allowing one to extend good cylindrical estimates from a fixed scale $5\sqrt{2n}$ to almost the whole ball of radius $R$. For us, instead of assuming $|A| \leq C$ for some constant $C$ as in [31], our curvature estimates give that $|A| \leq CR$ for shrinkers with positive mean curvature $H$ in $B_R$. In the proof of Proposition 1.6 we show that this is still enough to control the derivatives of $\tau$, possibly with a worse exponent $\alpha$ of $R$. But the exponential factor still decays much faster than any polynomial factor, so the polynomial factor can be eventually absorbed into the exponential factor as long as we choose $R$ sufficiently large. The remaining details of our proof will be deferred to subsection 1.2.2.

To complete the iteration and improvement scheme, we first apply Proposition 1.6, then apply Proposition 1.5 and repeat the process. The multiplicative factor extends the scale by more than the fixed decrease if $R$ is large enough, so we get strict mean convexity on all of $\Sigma$, which must therefore be a cylinder by the classification of mean convex shrinkers. Thus we have:

**Proposition 1.7.** Given $n$ and $\lambda_0 < 2$, let $\delta_0 \in (0,1/4)$ be given by Proposition 1.5. Then there exists $R = R(n, \lambda_0)$ so that if $\Sigma^n \subset \mathbb{R}^{n+1}$ is a shrinker with entropy $\lambda(\Sigma) \leq \lambda_0$ which satisfies

- $H \geq \delta_0$ on $B_R \cap \Sigma$,

then $\Sigma$ is a generalized cylinder $\mathbb{S}^k \times \mathbb{R}^{n-k}$ for some $1 \leq k \leq n$.

We also need the following compactness theorem for self-shrinkers which plays an important role in our argument:

**Lemma 1.8 (Compactness).** Let $\Sigma_i \subset \mathbb{R}^{n+1}$ be a sequence of shrinkers with $\lambda(\Sigma_i) \leq \lambda_0$ and

\begin{equation}
|A|(x) \leq C(1 + |x|) \quad \text{on} \quad B_i \cap \Sigma_i.
\end{equation}
Then there exists a subsequence \( \Sigma'_i \) that converges smoothly and with multiplicity one to a complete embedded shrinker \( \Sigma \) with

\[(1.35) \quad |A|(x) \leq C(1 + |x|) \quad \text{and} \quad \lim_{i \to \infty} \lambda(\Sigma'_i) = \lambda(\Sigma).\]

**Proof.** The key is that the a priori bound on \(|A|\) is uniform on compact subsets. Thus, as in Lemma 2.7 in [31], for any \( R \) we may obtain smooth convergence in \( B_R \) by covering with a finite number of balls. Passing to a diagonal argument gives the overall smooth convergence to a smooth, complete, embedded shrinker \( \Sigma \) with \( \lambda(\Sigma) \leq \lambda_0 \). Again arguing as in [31], if multiplicity is greater than one then the limit \( \Sigma \) must be \( L \)-stable. But there are no such shrinkers with polynomial volume growth (see Theorem 0.5 in [32]), so the multiplicity must be one. \( \square \)

Now we are ready to prove Theorem 1.2.

**Proof of Theorem 1.2.** Since we assumed \( H \geq \delta \) on \( B_R \cap \Sigma \), the curvature estimate Theorem 1.3 gives in particular that \(|A| \leq C \lambda \leq \frac{C}{2} \lambda \) on \( B_{R/2} \cap \Sigma \). Applying the compactness Lemma 1.8 we get that \( \Sigma \) is smoothly close to \( S^k \times \mathbb{R}^{n-k} \) in \( B_{R/2} \). Thus for \( R \) sufficiently large we may assume \( \lambda(S^k) \leq \lambda_0 < 2 \), and \( H \geq \delta_0 \) on \( B_{R/2} \cap \Sigma \). The result then follows from Proposition 1.7. \( \square \)

1.2.1. **Proof of Theorem 1.1.** For the proof of Theorem 1.1, we will need the following curvature estimate from subsection 3 in [59] (see in particular Theorem 0.4 and Remark 3.6 therein). The key fact was that \( LH = H \) on any shrinker \( \Sigma \), which implies an almost-stability inequality for \( \Sigma \) if the eigenfunction \( H \) is positive.

**Lemma 1.9.** Given \( n \leq 6 \) and \( \alpha > 0 \), there exists \( C = C(n, \alpha) \) so that if \( \Sigma^n \subset \mathbb{R}^{n+1} \) is a shrinker with \( \lambda(\Sigma) \leq \alpha \) and \( H > 0 \) on \( B_R \cap \Sigma \) for some \( R > 2 \), then on \( B_{R-1} \cap \Sigma \) we have

\[(1.36) \quad |A| \leq C(1 + |x|).\]
Remark 1.10. Note that for properly embedded self-shrinkers with finite genus in $\mathbb{R}^3$, Song [108] (see also [115]) gave the linear growth of the second fundamental form.

Now we give the proof of Theorem 1.1 via Proposition 1.7.

Proof of Theorem 1.1 using Proposition 1.7. First, the Harnack inequality gives that either $H \equiv 0$ or $H > 0$. If $H \equiv 0$ in $B_R$, then $\Sigma$ is a hyperplane in $B_R$. Thus by the rigidity of the hyperplane, $\Sigma$ must be a hyperplane $\mathbb{R}^n$ if $R$ is sufficiently large.

Next, we assume $H > 0$ in $B_R$. Lemma 1.9 then gives a curvature estimate on $B_{R-1} \cap \Sigma$. By the compactness of Lemma 1.8 we can assume that $\Sigma$ is smoothly close to $S^k \times \mathbb{R}^{n-k}$ in $B_{R_1}$ for some $k \geq 0$, where $R_1$ can be taken as large as we wish. If $k = 0$, then again the rigidity of the hyperplane means that $\Sigma$ must be a hyperplane, although this is a contradiction since in this case we assume $H > 0$ on $B_R \cap \Sigma$. So $k \geq 1$, and consequently $H$ is approximately $\sqrt{k/2}$ on $B_{R_1} \cap \Sigma$, then Theorem 1.1 follows directly from Proposition 1.7. \qed

Remark 1.11. In the above proofs of Theorems 1.1 and 1.2, the smooth closeness (obtained via compactness) also implies a bound for $|A|$ on a large ball, so at that point we could also appeal directly to Theorem 0.1 in [31]. The compactness Lemma 1.8 can also give a shorter proof of our main rigidity theorem for graphical shrinkers in [59], but in both cases we feel that the more effective proofs given may provide a more complete understanding.

1.2.2. Proof of the improvement step. In this subsubsection, we prove Proposition 1.6 by sketching the necessary modifications of the proof of Proposition 2.2 in [31].

As discussed earlier, the central argument is the very tight estimate on the tensor $\tau = A/H$, that decays exponentially in $R$. Thus, our main modification is the following lemma, which removes the curvature bound of Corollary 4.12 in [31] by accepting a slightly larger power of $R$, although we still have the exponential decay.

Lemma 1.12. Given $n$, $\lambda_0$ and $\delta > 0$, there exists a constant $C_\tau > 0$ such that if $\lambda(\Sigma) \leq \lambda_0$, $R \geq 2$, and
\[ B_{R+1} \cap \Sigma \text{ is smooth with } H \geq \delta > 0, \]
then
\[ \sup_{B_{R-2} \cap \Sigma} |\nabla \tau|^2 + R^{-4} |\nabla^2 \tau|^2 \leq C_{\tau} R^{3n+4} e^{-R/4}. \]  
\[ (1.37) \]

**Proof.** First, Theorem 1.3 gives there exists a constant \( C = C(n, \delta) \) such that \(|A| \leq CRH\) in \( B_R \). Hence, Proposition 4.8 in [31] with \( s = 1/2 \) implies that
\[ \int_{B_{R-1/2} \cap \Sigma} |\nabla \tau|^2 e^{-|x|^2/4} \leq C R^{n+4} e^{-(R-1/2)^2/4}. \]  
\[ (1.38) \]
Since \( e^{-|x|^2/4} \geq e^{-R^2/4} \) on \( B_{R-1} \), it follows that
\[ \int_{B_{R-1} \cap \Sigma} |\nabla \tau|^2 \leq C R^{n+4} e^{-R^2/4}. \]  
\[ (1.39) \]
This gives the desired integral decay on \( \nabla \tau \). We will combine this with elliptic theory to get the pointwise bounds. The key is that \( \tau \) satisfies the elliptic equation \( L_{H^2} \tau = 0 \) (see Proposition 4.5 in [31]), that is,
\[ \Delta \tau - \frac{1}{2} \langle x, \nabla \tau \rangle + \langle \nabla \log H^2, \nabla \tau \rangle = 0. \]
\[ (1.40) \]
Note that we have
\[ |\nabla \log H^2| = \frac{2|\nabla H|}{H} \leq \frac{|A||x|}{H} \leq CR|x|, \]  
\[ (1.41) \]
where we used that \(|\nabla H| \leq \frac{1}{2}|A||x| \) and \(|A| \leq CRH \).

Therefore, the two first order terms in the equation (1.40) come from \( x^T \) in \( L \) and \( \nabla \log H^2 \); both grow at most quadratically. Now we can apply elliptic theory on balls of radius \( 1/R^2 \) to get for any \( p \in B_{R-2} \cap \Sigma \) that
\[ (|\nabla \tau|^2 + R^{-4} |\nabla^2 \tau|^2) (p) \leq C R^{2n} \int_{B_{1/R^2} (p) \cap \Sigma} |\nabla \tau|^2. \]  
\[ (1.42) \]
Combining this with the integral bounds (1.39) gives the lemma. \( \square \)

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Now we sketch the proof of Proposition 1.6.

Fix $n$, $\lambda_0 > 0$, and $\delta_0 > 0$. Let $R > 0$ and assume that $\Sigma$ is a self-shrinker in $\mathbb{R}^{n+1}$, $\lambda(\Sigma) \leq \lambda_0$ and $H \geq \delta_0$ on $\Sigma \cap B_R$. By Lemma 1.12 the tensor $\tau = A/H$ satisfies

\begin{equation}
|\nabla \tau| + |\nabla^2 \tau| \leq \epsilon_\tau \quad \text{on } B_{R-2} \cap \Sigma,
\end{equation}

where

$$\epsilon_\tau^2 := C R^{3n+8} e^{-R/4}$$

and the constant $C$ depends only on $n$, $\delta_0$ and $\lambda_0$. As in [31], the key point is that $\epsilon_\tau$ can still be made small for large $R$, due to the decaying exponential factor.

Now fix small $\epsilon_0 > 0$, to be chosen as needed, but depending only on $n$. Combining the compactness of Lemma 1.8 with the classification of mean convex shrinkers [37], there exists a constant $R_1 = R_1(n, \lambda_0, \delta_0, \epsilon_0)$ so that if $R \geq R_1$, then $B_{3\sqrt{n}} \cap \Sigma$ is $C^2$ $\epsilon_0$-close to a cylinder $\mathbb{S}^k \times \mathbb{R}^{n-k}$ for some $1 \leq k \leq n$. The remainder of Proposition 1.6 follows from the proof of Proposition 2.2 in [31].

2. Rigidity of graphical self-shrinkers

The work described in this section was first presented in the published articles [59] and [58] and is joint with Dr Qiang Guang.

The main theorem of this section is the following strong rigidity theorem for self-shrinkers that are almost stable on large balls:

**Theorem 2.1.** Given $n \leq 6$ and $\lambda_0$, there exists $R = R(n, \lambda_0)$ so that if $\Sigma^n \subset \mathbb{R}^{n+1}$ is a smooth complete self-shrinker with entropy $\lambda(\Sigma) \leq \lambda_0$ satisfying

\begin{enumerate}
\item[(i)] $\Sigma$ is $1/2$-stable in $B_R$,
\end{enumerate}

then $\Sigma$ is a hyperplane.
By Lemma 2.6 the class of $\frac{1}{2}$-stable self-shrinkers includes the class of self-shrinkers for which each component is graphical (not necessarily over the same hyperplane), so the above theorem includes a strong rigidity for graphical self-shrinkers.

A crucial step in our argument was to obtain the following curvature estimate:

**Theorem 2.2.** Given $2 \leq n \leq 6$ and $\lambda_0$, there exists $C = C(n, \lambda_0)$ so that for any self-shrinker $\Sigma^n \subset \mathbb{R}^{n+1}$ with entropy $\lambda(\Sigma) \leq \lambda_0$ satisfying

- $\Sigma$ is $\frac{1}{2}$-stable in $B_R$ for $R > 2$,

we have

\[
\left| A \right|(x) \leq C(1 + \left| x \right|), \quad \text{for all } x \in B_{R-1} \cap \Sigma.
\]

The $\frac{1}{2}$-stability here may be replaced by $\delta$-stability for a fixed $\delta$; in particular, this includes the strictly mean convex case $H > 0$.

The methods developed in [59] and presented here can also be applied to certain other classes of hypersurfaces, for instance to the class of translators - translating solitons of the mean curvature flow.

The curvature estimates above rely on the Schoen-Simon [96] regularity theory for stable minimal hypersurfaces, and apply for $n \leq 6$. On the other hand, we have the following strong rigidity theorem and curvature estimate in all dimensions, if we assume a positive lower bound of $w = \langle V, \nu \rangle$.

**Theorem 2.3.** Given $n$, $\lambda_0$ and $\delta > 0$, there exists $R = R(n, \lambda_0, \delta)$ so that if $\Sigma^n \subset \mathbb{R}^{n+1}$ is a smoothly embedded self-shrinker with entropy $\lambda(\Sigma) \leq \lambda_0$ satisfying

- $w = \langle V, \nu \rangle \geq \delta$ on $B_R \cap \Sigma$ for some constant unit vector $V$,

then $\Sigma$ is a hyperplane.

**Theorem 2.4.** Given $n$ and $\delta > 0$, there exists $C = C(n, \delta)$ so that for any smooth properly embedded self-shrinker $\Sigma^n \subset \mathbb{R}^{n+1}$ which satisfies

- $w = \langle V, \nu \rangle \geq \delta$ on $B_R \cap \Sigma$ for some constant unit vector $V$ and $R > 2$,
we have

\[(2.2) \quad |A| \leq C, \quad \text{on } B_{R/2} \cap \Sigma.\]

Theorem 2.4 is essentially a corollary of Theorem 3.1 in \textsuperscript{49}, and the proof is similar to Theorem 1.3 — the essential component being that \(Lw = \frac{1}{2}w\). Combining Theorem 2.4 and some ingredients from \textsuperscript{59}, we can now prove Theorem 2.3.

**Proof of Theorem 2.3** Given \(n, \lambda_0\) and \(\delta\), Theorem 2.4 gives a curvature bound \(C\). Since \(\Sigma\) is graphical and satisfies a curvature bound, Theorem 2.2 in \textsuperscript{59} allows us to make \(|A|\) as small as we want by choosing \(R\) sufficiently large. In particular, we can choose \(R\) such that \(|A|^2 \leq 1/4\) on \(B_{R/2} \cap \Sigma\). Now Theorem 2.3 follows directly from the compactness of Lemma 1.8, Brakke’s Theorem \textsuperscript{20} (see also \textsuperscript{118}) and the fact that any complete shrinker with \(|A|^2 < 1/2\) is a hyperplane (see \textsuperscript{22}). \(\Box\)

### 2.1. \(\delta\)-stability

In \textsuperscript{58} we introduced the notion of \(\delta\)-stability for self-shrinkers and proved a stability type inequality for graphical self-shrinkers.

**Definition 2.5.** Given \(\delta\), we will say that a self-shrinker \(\Sigma\) is \(\delta\)-stable in a domain \(\Omega\) if

\[(2.3) \quad \int_{\Sigma} (-\phi L\phi)\rho + \delta \int_{\Sigma} \phi^2 \rho \geq 0\]

for any compactly supported function \(\phi\) in \(\Omega\).

Note that integrating by parts gives that (2.3) is equivalent to

\[(2.4) \quad \int_{\Sigma} \left( |A|^2 + \frac{1}{2} - \delta \right) \phi^2 \rho \leq \int_{\Sigma} |\nabla \phi|^2 \rho.\]

In particular, when \(\delta = 0\), our 0-stability is just the \(L\)-stability defined in \textsuperscript{32}. The next lemma shows that if a self-shrinker is graphical in \(B_R\), then it is \(\frac{1}{2}\)-stable in \(B_R\). The proof is essentially same as Lemma 2.1 in \textsuperscript{114}. For convenience of the reader, we also include a proof here.
Lemma 2.6. Suppose $\Sigma^n$ is self-shrinker which is graphical in $B_R$, then it is $\frac{1}{2}$-stable in $B_R$, i.e., for any compactly supported function $\phi$ in $B_R$, we have

$$(2.5) \quad \int_{\Sigma} |A|^2 \phi^2 \rho \leq \int_{\Sigma} |\nabla \phi|^2 \rho.$$ 

Proof. Since $\Sigma$ is graphical in $B_R$, we can find a constant vector $v$ such that $w(x) = \langle v, \nu(x) \rangle$ is positive on $B_R \cap \Sigma$. But we know that $w$ is an eigenfunction of $L$ for any self-shrinker, in fact $Lw = \frac{1}{2}w$. Hence, the function $h = \log w$ is well-defined, and it follows that $h$ satisfies the equation

$$(2.6) \quad Lh = -|\nabla h|^2 - |A|^2.$$ 

For any compactly supported function $\phi$ in $B_R$, multiplying by $\phi^2 \rho$ on both sides of (2.6) and integrating by parts, we then have

$$(2.7) \quad \int_{\Sigma} (|A|^2 + |\nabla h|^2) \phi^2 \rho = - \int_{\Sigma} \langle \phi^2 Lh \rangle \rho = \int_{\Sigma} 2\phi \langle \nabla \phi, \nabla h \rangle \rho.$$ 

Combining this with the inequality $2\phi \langle \nabla \phi, \nabla h \rangle \leq \phi^2 |\nabla h|^2 + |\nabla \phi|^2$ gives

$$(2.8) \quad \int_{\Sigma} |A|^2 \phi^2 \rho \leq \int_{\Sigma} |\nabla \phi|^2 \rho.$$ 

Remark 2.7. Similar to the above and to the proof of Lemma 9.15 in [37], we can prove that if $\Sigma$ is a self-shrinker and $\psi$ is a nontrivial function on $\Sigma$ with $L\psi = \mu \psi$, and $\psi > 0$ on $\Sigma \cap B_R$, then $\Sigma$ is $\mu$-stable on $B_R$.

In particular, a self-shrinker with $H > 0$ on $B_R \cap \Sigma$ is 1-stable in $B_R$. Obviously, 0-stable is stronger than $\frac{1}{2}$-stable and 1-stable.

Remark 2.8. By performing the integration by parts on each connected component $\Sigma_i$ of $\Sigma \cap B_R$ separately, we see that the conclusion of Lemma 2.6 holds under the weaker
assumption that each \( \Sigma_i \) is graphical, that is, if there are some \( v_i \) so that \( w_i = \langle v_i, \nu \rangle > 0 \) on each \( \Sigma_i \).

In this section, we refer to the inequality (2.5) as the stability inequality.

### 2.2. Key ingredients and proof of the main theorem

In this subsection, we will prove the main theorem, that is, Theorem 2.1. The key ingredients are the pointwise curvature estimate in Theorem 2.2 and a rapidly decaying integral curvature estimate.

#### 2.2.1. Initial curvature estimates

First, we use the rapid decay of the weight \( \rho = e^{-|x|^2/4} \) to show that shrinkers which are \( \frac{1}{2} \)-stable on large balls \( B_R \) satisfy an integral curvature estimate that decays exponentially in \( R \).

**Proposition 2.9.** Given \( n \) and \( \lambda_0 \), there exists \( C = C(n, \lambda_0) \) so that if \( \Sigma^n \subset \mathbb{R}^{n+1} \) is a self-shrinker with entropy \( \lambda(\Sigma) \leq \lambda_0 \), which is \( \frac{1}{2} \)-stable in \( B_R \) for some \( R > 1 \), then we have

\[
\int_{B_{R-1} \cap \Sigma} |A|^2 \leq CR^n e^{-\frac{R}{4}}.
\]

**Proof.** Let \( a > 0 \). We may choose a smooth cutoff function \( \phi \) so that \( \phi \equiv 1 \) on \( B_{R-a} \), \( \phi \equiv 0 \) outside \( B_R \) and \( |\nabla \phi| \leq \frac{2}{a} \). From the stability inequality (2.5), since \( |\nabla \phi| \) is supported in \( B_R \setminus B_{R-a} \), we get

\[
\int_{B_{R-a} \cap \Sigma} |A|^2 \rho \leq \frac{4}{a^2} e^{-\frac{1}{4}(R-a)^2} \text{Vol}(B_R \cap \Sigma) \leq \frac{C}{a^2} R^n e^{-\frac{1}{4}(R-a)^2},
\]

where we have used the Euclidean volume growth for the second inequality.

Therefore

\[
\int_{B_{R-2a} \cap \Sigma} |A|^2 \leq e^{\frac{1}{4}(R-2a)^2} \int_{B_{R-a} \cap \Sigma} |A|^2 \rho \leq \frac{C'}{a^2} R^n e^{-\frac{a}{2} R}.
\]

Taking \( a = \frac{1}{2} \) gives the result. \( \Box \)

#### 2.2.2. Improvement of curvature estimates

Using the pointwise curvature estimate Theorem 2.2 together with the very tight integral estimate Proposition 2.9 we are now able to
improve the pointwise estimate and show that the curvature is in fact uniformly small, so long as \( R \) is sufficiently large.

**Theorem 2.10.** Given \( n, \lambda_0, C \) and \( \delta > 0 \), there exists \( R_0 = R_0(n, \lambda, C, \delta) \) such that if \( R \geq R_0 > 2 \) and \( \Sigma^n \subset \mathbb{R}^{n+1} \) is a self-shrinker with entropy \( \lambda(\Sigma) \leq \lambda_0 \), which is \( \frac{1}{2} \)-stable in \( B_R \) and satisfies

\[
|A|(x) \leq C(1 + |x|), \quad \text{for all } x \in B_{R-1} \cap \Sigma,
\]

then in fact

\[
|A|(x) \leq \delta, \quad \text{for all } x \in B_{R-2} \cap \Sigma.
\]

**Proof.** We will use the Simons-type inequality for self-shrinkers, which gives

\[
\Delta|A|^2 = \frac{1}{2} \langle x, \nabla|A|^2 \rangle + 2 \left( \frac{1}{2} - |A|^2 \right) |A|^2 + 2 |\nabla A|^2
\]

\[
\geq -\frac{1}{8} |x|^2 |A|^2 - 2 |\nabla A|^2 + 2 \left( \frac{1}{2} - |A|^2 \right) |A|^2 + 2 |\nabla A|^2
\]

\[
\geq -\frac{1}{8} |x|^2 |A|^2 + |A|^2 - 2 |A|^4.
\]

The assumed curvature estimate (2.12) allows us to estimate part of the \( |A|^4 \) term, turning the above into a linear differential inequality. Specifically, for \( x \in B_{R-1} \cap \Sigma \) we will have

\[
\Delta|A|^2 \geq -\frac{R^2}{8} |A|^2 - 2CR^2 |A|^2 = -C'R^2 |A|^2.
\]

This will allow us to use the mean value inequality as follows. Fix \( x_0 \in B_{R-2} \cap \Sigma \) and set

\[
g(s) = s^{-n} \int_{B_s(x_0) \cap \Sigma} |A|^2.
\]

Using a mean value inequality for general hypersurfaces (See Lemma 1.18 in [36]), we have the following inequality:

\[
g'(s) \geq \frac{1}{2s^{n+1}} \int_{B_s(x_0) \cap \Sigma} \left( s^2 - |x - x_0|^2 \right) \Delta|A|^2 - \frac{1}{s^{n+1}} \int_{B_s(x_0) \cap \Sigma} |A|^2 \langle x - x_0, H \nu \rangle.
\]
By the shrinker equation $H = \frac{1}{2} (x, \nu)$ we have $|H| \leq \frac{1}{2} |x| \leq \frac{R}{2}$, so together with (2.15) we then obtain

$$g'(s) \geq -C' R^2 s^{1-n} \int_{B_s(x_0) \cap \Sigma} |A|^2 - \frac{R}{2} s^{-n} \int_{B_s(x_0) \cap \Sigma} |A|^2 = -C' R^2 s g(s) - \frac{R}{2} g(s).$$

Therefore the quantity

$$g(s) \exp \left( C' R^2 s^2 + \frac{R}{2} s \right)$$

is nondecreasing in $s$. Applying this monotonicity at scale $s = R^{-1} \leq 1$ and using the integral estimate Proposition 2.9 gives

$$|A|^2(x_0) \leq \frac{e^{C' + \frac{1}{2}}}{{\omega_n} R^n} \int_{B_\frac{s}{R} (x_0) \cap \Sigma} |A|^2 \leq C'' R^n \int_{B_{R} \cap \Sigma} |A|^2 \leq C R^{2n} e^{\frac{R}{4}}.$$ 

Here $\omega_n$ is the volume of the unit ball in $\mathbb{R}^n$.

The exponential factor decays faster than any polynomial factor, so taking $R$ large enough gives the desired result.

\[\square\]

**Remark 2.11.** Note that for $n \leq 6$, the curvature hypothesis (2.12) is automatically satisfied by Theorem 2.2. Moreover, it is evident from the proof of Proposition 2.9 that the conclusions of Theorem 2.10 hold with a somewhat weaker pointwise curvature estimate $|A|(x) \leq f(|x|)$, so long as this estimate is uniform in $\Sigma$ and $f = O(e^{aR})$ for some $a > 0$. However, the main difficulty, even in the graphical case, was to obtain any initial pointwise curvature estimate.

2.2.3. **Proof of the main theorem.** To finish the proof of Theorem 2.1, we will make use of the following smooth version of the compactness theorem for self-shrinkers in [31].

**Lemma 2.12 ([31]).** Let $\Sigma_i \subset \mathbb{R}^{n+1}$ be a sequence of smooth self-shrinkers with $\lambda(\Sigma_i) \leq \lambda_0$ and

$$|A| \leq C \text{ on } B_{R_i} \cap \Sigma_i,$$ 

and

$$|A|^2 \leq C' R^2 s g(s) - \frac{R}{2} g(s).$$

Therefore the quantity

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Here $\omega_n$ is the volume of the unit ball in $\mathbb{R}^n$.

The exponential factor decays faster than any polynomial factor, so taking $R$ large enough gives the desired result. 

\[\square\]
where $R_i \to \infty$. Then there exists a subsequence $\Sigma'_i$ that converges smoothly and with multiplicity one to a complete embedded self-shrinker $\Sigma$ with \(|A| \leq C\) and

\begin{equation}
\lim_{i \to \infty} \lambda(\Sigma'_i) = \lambda(\Sigma).
\end{equation}

**Proof of Theorem 2.1** Given $n \leq 6$ and $\lambda_0$, Theorem 2.2 gives a constant $C = C(n, \lambda_0)$ that controls the linear growth of $|A|$. Moreover, Theorem 2.10 enables us to make $|A|$ as small as we want. In particular, we can choose $R_0$ such that if $\Sigma^n$ is graphical in $B_R$ for $R \geq R_0$, then $|A| \leq 1/2$ for all $x \in B_{R-2} \cap \Sigma$.

Now we claim that there exists a constant $R \geq R_0$ such that Theorem 2.1 holds. Otherwise, there is a sequence of smooth, complete, non-flat self-shrinkers $\Sigma_i \subset \mathbb{R}^{n+1}$ ($\Sigma_i \neq \mathbb{R}^n$) with $\lambda(\Sigma_i) \leq \lambda_0$ and $\Sigma_i$ is graphical in $B_{R_i}$ for $R_i \to \infty$ and $R_i \geq R_0$. Theorem 2.10 gives that $|A| \leq 1/2$ on $B_{R_i-2} \cap \Sigma_i$. Applying the compactness of Lemma 2.12, there is a subsequence $\Sigma'_i$ that converges smoothly and with multiplicity one to a complete embedded self-shrinker $\Sigma$ with $|A| \leq 1/2$ and

\begin{equation}
\lim_{i \to \infty} \lambda(\Sigma'_i) = \lambda(\Sigma).
\end{equation}

Recall that any smooth complete self-shrinker with $|A|^2 < 1/2$ is a hyperplane, so in particular the limit $\Sigma$ must be a hyperplane. By the smooth convergence or the convergence in entropy, we know by Brakke’s theorem that for sufficiently large $i$, the self-shrinker $\Sigma_i$ must be a hyperplane. This provides the desired contradiction, completing the proof. \(\square\)

### 2.3. Curvature estimates for almost stable self-shrinkers.

In this subsection we will prove the curvature estimate Theorem 2.2 for self-shrinkers $\Sigma^n$, which satisfy the stability inequality (2.5). We give a self-contained proof for the cases $n \leq 5$ based on the Schoen-Simon-Yau [97] curvature estimates for stable minimal hypersurfaces. We will also sketch how the result follows for $n \leq 6$ from the arguments of Schoen-Simon [96].
2.3.1. Schoen-Simon-Yau type estimates. First, we prove a small energy curvature estimate of Choi-Schoen [27] type, allowing us to obtain pointwise estimates on $|A|$ from suitable integral estimates.

**Theorem 2.13.** There exists $\varepsilon = \varepsilon(n) > 0$ so that if $\Sigma \subset \mathbb{R}^{n+1}$ is self-shrinker, properly embedded in $B_{r_0}(x_0)$ for some $r_0 \leq \theta = \min\{1, |x_0|^{-1}\}$, which satisfies

$$
\int_{B_{r_0}(x_0) \cap \Sigma} |A|^n < \varepsilon,
$$

then for all $0 < \sigma \leq r_0$ and $y \in B_{r_0-\sigma}(x_0) \cap \Sigma$,

$$
\sigma^2 |A|^2(y) \leq 1.
$$

**Proof.** We will follow the Choi-Schoen type argument and argue by contradiction.

Consider the function $f$ defined on $B_{r_0}(x_0) \cap \Sigma$ by $f(x) = (r_0 - r(x))^2 |A|^2(x)$, where $r(x) = |x - x_0|$. This function vanishes on $\partial B_{r_0}(x_0)$, so it achieves its maximum at some $y_0 \in B_{r_0}(x_0) \cap \Sigma$. If $f(y_0) \leq 1$, then we are done. So we assume $f(y_0) \geq 1$ and will show that this leads to a contradiction for $\varepsilon$ sufficiently small.

Choose $\sigma > 0$ so that

$$
\sigma^2 |A|^2(y_0) = \frac{1}{4}.
$$

Then $f(y_0) \geq 1$ implies that $2\sigma \leq r_0 - r(y_0)$. Using this bound for $\sigma$ we see that $B_{\sigma}(y_0) \subset B_{r_0}(x_0)$, and the triangle inequality gives that for $x \in B_{\sigma}(y_0)$, we have

$$
\frac{1}{2} \leq \frac{r_0 - r(x)}{r_0 - r(y_0)} \leq \frac{3}{2}.
$$

Combining [2.26] with the fact that $f$ achieves its maximum at $y_0$, we get that

$$
(r_0 - r(y_0))^2 \sup_{B_{\sigma}(y_0) \cap \Sigma} |A|^2 \leq 4 \sup_{B_{\sigma}(y_0) \cap \Sigma} f = 4f(y_0) = 4(r_0 - r(y_0))^2 |A|^2(y_0).
$$
This gives the following estimate

\[(2.28) \quad \sup_{B_{\sigma}(y_0) \cap \Sigma} |A|^2 \leq 4|A|^2(y_0) = \frac{1}{\sigma^2}.\]

By the Simons-type inequality (2.14), it follows from (2.28) that on \(B_{\sigma}(y_0) \cap \Sigma\),

\[(2.29) \quad \Delta|A|^2 \geq -\frac{1}{8}|x|^2|A|^2 - \frac{2}{\sigma^2}|A|^2.\]

A simple computation then gives

\[(2.30) \quad \Delta|A|^n \geq -\frac{n}{2} \left(\frac{1}{8}|x|^2 + \frac{2}{\sigma^2}\right)|A|^n.\]

Next, for \(0 < s \leq \sigma\), we define the function \(g(s)\) by

\[g(s) = \frac{1}{s^n} \int_{\Sigma_{y_0,s}} |A|^n,\]

where we use \(\Sigma_{y_0,s}\) to denote \(B_s(y_0) \cap \Sigma\).

Again using the mean value inequality (2.17), we also have

\[(2.31) \quad g'(s) \geq \frac{1}{2s^{n+1}} \int_{\Sigma_{y_0,s}} (s^2 - |x - y_0|^2) \Delta|A|^n - \frac{1}{s^{n+1}} \int_{\Sigma_{y_0,s}} |A|^n \langle x - y_0, H \nu \rangle.\]

Combining this with (2.30) gives

\[(2.32) \quad g'(s) \geq -\frac{1}{2s^{n+1}} \int_{\Sigma_{y_0,s}} \left(\frac{n}{2} |x|^2 + \frac{2}{\sigma^2}\right)|A|^n - \frac{1}{s^n} \frac{\sqrt{n}}{\sigma} \int_{\Sigma_{y_0,s}} |A|^n \]

\[\geq -\frac{n}{4} \left(\frac{|x| + 1}{8} + \frac{2}{\sigma^2}\right) sg(s) - \frac{\sqrt{n}}{\sigma} g(s),\]

where we use that \(|x| \leq (|x_0| + 1)\) and \(|H| \leq \sqrt{n}|A| \leq \frac{\sqrt{n}}{\sigma}\) for \(x \in B_{\sigma}(y_0) \cap \Sigma\).

It follows that the function

\[(2.33) \quad h(t) = g(t) \exp \left\{ \left(\frac{n}{4} \frac{(|x| + 1)^2}{8} + \frac{2}{\sigma^2}\right) \frac{t^2}{2} + \frac{\sqrt{n}}{\sigma} t \right\}\]

is non-decreasing for \(0 < t \leq \sigma\).
Applying at \( t = \sigma \), since \( \sigma \leq r_0 \leq \min\{1, |x_0|^{-1}\} \), we get

\[
\frac{\omega_n}{2^n \sigma^n} = \omega_n |A|^n(y_0) = h(0) \leq h(\sigma) \leq e^{\frac{2n}{\sigma}} + \frac{1}{\sigma^n} \int_{B_\sigma(y_0) \cap \Sigma} |A|^n \leq \frac{e^{2n}}{\sigma^n} \varepsilon,
\]

where \( \omega_n \) is the volume of the unit ball in \( \mathbb{R}^n \).

This gives a contradiction for sufficiently small \( \varepsilon \).

Theorem 2.13 holds in any dimension, but of course the hypotheses require an \( L^n \) bound on \( |A| \), and at face value the stability inequality (2.5) only provides an \( L^2 \) bound. So we now adapt the arguments of Schoen-Simon-Yau [97] to improve our control on \( |A| \).

**Theorem 2.14.** Suppose that \( \Sigma^n \subset \mathbb{R}^{n+1} \) is a smooth self-shrinker which is \( \frac{1}{2} \)-stable in \( B_R \), and let \( \phi \) be a smooth function with compact support in \( B_R \). Then for all \( q \in [0, \sqrt{2/(n+1)}] \), we have

\[
\int_{\Sigma} |A|^{4+2q} \phi^2 e^{-\frac{|x|^2}{4}} \leq C \int_{\Sigma} |A|^{2+2q} |\nabla \phi|^2 e^{-\frac{|x|^2}{4}} + C R^2 \int_{\Sigma} |A|^{2+2q} \phi^2 e^{-\frac{|x|^2}{4}},
\]

for some \( C = C(n, q) \).

**Proof.** We apply the stability inequality (2.5) with test function \( |A|^{1+q} \phi \). This gives

\[
\int_{\Sigma} |A|^{4+2q} \phi^2 \rho \leq \int_{\Sigma} |A|^{2+2q} |\nabla \phi|^2 \rho + (1 + q)^2 \int_{\Sigma} |A|^{2q} \phi^2 |\nabla |A||^2 \rho
\]

\[
+ 2(1 + q) \int_{\Sigma} |A|^{1+2q} \phi \langle \nabla \phi, \nabla |A| \rangle \rho.
\]

From the Simons-type identity for self-shrinkers, we have

\[
|A| |\mathcal{L} |A| = \frac{1}{2} |A|^2 - |A|^4 + |\nabla A|^2 - |\nabla |A||^2.
\]

Now we use the following inequality for general hypersurfaces from [37] Lemma 10.2]

\[
\left(1 + \frac{2}{n + 1}\right) |\nabla |A||^2 \leq |\nabla A|^2 + \frac{2n}{n + 1} |\nabla H|^2.
\]
The self-shrinker equation implies that $|\nabla H| \leq \frac{1}{2} |x| |A| \leq \frac{R}{2} |A|$ for all $x \in B_R(0) \cap \Sigma$.

Therefore,

\[(2.39) \quad |A| |\mathcal{L} A| \geq -|A|^4 + \frac{2}{n+1} |\nabla |A||^2 - \frac{n}{n+1} \frac{R^2}{2} |A|^2.\]

Multiplying by $|A|^{2q} \phi^2 \rho$ and integrating by parts, we get

\[
\begin{align*}
\int_\Sigma |A|^{1+2q} \phi^2 \mathcal{L} |A| \rho &= - \int_\Sigma \langle \nabla |A|, \nabla (|A|^{1+2q} \phi^2) \rangle \rho \\
&= - \int_\Sigma |A|^{1+2q} \langle \nabla |A|, \nabla \phi^2 \rangle \rho - (1 + 2q) \int_\Sigma |A|^{2q} \phi^2 |\nabla |A||^2 \rho \\
&\geq \int_\Sigma \left( \frac{2}{n+1} |\nabla |A||^2 - \frac{nR^2}{2(n+1)} |A|^2 - |A|^4 \right) |A|^{2q} \phi^2 \rho.
\end{align*}
\]

This implies

\[
\begin{align*}
\frac{2}{n+1} \int_\Sigma |\nabla |A||^2 |A|^{2q} \phi^2 \rho &\leq \int_\Sigma |A|^{2+2q} \phi^2 \rho + \frac{nR^2}{2(n+1)} \int_\Sigma |A|^{2+2q} \phi^2 \rho \\
&- 2 \int_\Sigma |A|^{1+2q} \phi \langle \nabla \phi, \nabla |A| \rangle \rho - (1 + 2q) \int_\Sigma |A|^{2q} \phi^2 |\nabla |A||^2 \rho.
\end{align*}
\]

Combining (2.36) and (2.41) gives

\[
\begin{align*}
\frac{2}{n+1} \int_\Sigma |\nabla |A||^2 |A|^{2q} \phi^2 \rho &\leq \int_\Sigma |A|^{2+2q} |\nabla \phi|^2 \rho + q^2 \int_\Sigma |A|^{2q} |\nabla |A||^2 \phi^2 \rho \\
&+ 2q \int_\Sigma |A|^{1+2q} \phi \langle \nabla \phi, \nabla |A| \rangle \rho + \frac{nR^2}{2(n+1)} \int_\Sigma |A|^{2+2q} \phi^2 \rho.
\end{align*}
\]

Using the absorbing inequality, we get that $2 |A| \phi \langle \nabla \phi, \nabla |A| \rangle \leq \frac{1}{a} |A|^2 |\nabla \phi|^2 + a |\nabla |A||^2 \phi^2$.

Therefore, we get

\[
\begin{align*}
\left( \frac{2}{n+1} - q^2 - aq \right) \int_\Sigma |\nabla |A||^2 |A|^{2q} \phi^2 \rho &\leq \left( 1 + \frac{q}{a} \right) \int_\Sigma |A|^{2+2q} |\nabla \phi|^2 \rho \\
&+ \frac{nR^2}{2(n+1)} \int_\Sigma |A|^{2+2q} \phi^2 \rho.
\end{align*}
\]

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Applying the Cauchy-Schwarz inequality to absorb the cross-term in (2.36) and then substituting (2.43) gives

\[
\int_\Sigma |A|^{4+2q}\phi^2 \rho \leq \left(2 + \frac{2(1 + q)^2(1 + \frac{q}{a})}{\frac{n}{n+1} - q^2 - aq}\right) \int_\Sigma |A|^{2+2q}|
abla \phi|^2 \rho \\
+ \frac{n}{n+1}(1 + q)^2 R^2 \int_\Sigma |A|^{2+2q}\phi^2 \rho.
\]

Thus so long as \(q^2 < \frac{2}{n+1}\), there exists a constant \(C\) depending on \(n\) and \(q\) such that

\[
\int_\Sigma |A|^{4+2q}\phi^2 \rho \leq C \int_\Sigma |A|^{2+2q}|
abla \phi|^2 \rho + CR^2 \int_\Sigma |A|^{2+2q}\phi^2 \rho.
\]

Remark 2.15. In particular, if we set \(q = 0\), then

\[
\int_\Sigma |A|^4\phi^2 \rho \leq C \int_\Sigma |A|^2|
abla \phi|^2 \rho + CR^2 \int_\Sigma |A|^2\phi^2 \rho.
\]

It will also be useful to record that (for \(n \leq 6\)) we may set \(q = 1/2\) to obtain

\[
\int_\Sigma |A|^5\phi^2 \rho \leq C \int_\Sigma |A|^3|
abla \phi|^2 \rho + CR^2 \int_\Sigma |A|^3\phi^2 \rho.
\]

We now record a lemma that estimates a ‘scale-invariant energy’ \(r^{p-n} \int_{B_r(x) \cap \Sigma} |A|^p\). This lemma will be convenient for obtaining our curvature estimates for low dimensions, but in fact holds for all \(n \geq 2\).

Lemma 2.16. Given \(n \geq 2\), \(\lambda_0 > 0\) and \(2 \leq p \leq 4\), there exists \(C = C(n, p, \lambda_0)\) so that if \(\Sigma^n \subset \mathbb{R}^{n+1}\) is a self-shrinker with \(\lambda(\Sigma) \leq \lambda_0\) and \(\Sigma\) is \(1/2\)-stable in \(B_R\) for some \(R > 2\), then for all \(x_0 \in B_{R-1} \cap \Sigma\) and \(r \leq \frac{1}{2}\theta\), where \(\theta = \min\{1, |x_0|^{-1}\}\), we have

\[
\int_{B_r(x_0) \cap \Sigma} |A|^p \leq C r^{n-p}.
\]
Proof. Fix $x_0 \in B_{R-1} \cap \Sigma$, and set $r(x) = |x - x_0|$. Also let $r_0 \leq \theta$. First note that with this choice of $r_0$, we have

$$(2.49) \quad \frac{\sup_{B_{r_0}(x_0)} \rho}{\inf_{B_{r_0}(x_0)} \rho} \leq e.$$ 

This is clear if $|x_0| \leq 1$, since in this case any $x \in B_{r_0}(x_0)$ satisfies $|x| \leq |x_0| + r_0 \leq 2$. On the other hand, if $|x_0| \geq 1$ then $r_0 \leq |x_0|$, so $e^{-\frac{1}{4}(|x_0| + r_0)^2} \leq e^{-\frac{1}{4}|x|^2} \leq e^{-\frac{1}{4}(|x_0| - r_0)^2}$ for $x \in B_{r_0}(x_0)$, and hence

$$(2.50) \quad \frac{\sup_{B_{r_0}(x_0)} \rho}{\inf_{B_{r_0}(x_0)} \rho} \leq e^{|x_0|r_0} \leq e.$$ 

Now we may fix a smooth cutoff function $\phi$ with $\phi = 1$ if $r \leq r_0$, $\phi = 0$ if $r > 2r_0$, and such that $|\nabla \phi| \leq \frac{2}{r_0}$.

From the stability inequality (2.5) and the above discussion we have

$$(2.51) \quad \int_{B_{r_0}(x_0) \cap \Sigma} |A|^2 \leq \frac{1}{\inf_{B_{r_0}(x_0)} \rho} \int_{\Sigma} |A|^2 \phi^2 \rho \leq \frac{\sup_{B_{r_0}(x_0)} \rho}{\inf_{B_{r_0}(x_0)} \rho} \int_{\Sigma} |\nabla \phi|^2$$ 

$$\leq \frac{4e}{r_0^2} \text{Vol}(B_{2r_0}(x_0) \cap \Sigma) \leq C r_0^{n-2},$$

since $r_0 \leq \frac{1}{|x_0|}$. Again we have used that volume growth is controlled by entropy for the last inequality. This completes the proof for $p = 2$.

Now further suppose $r_0 \leq \frac{1}{2} \theta$. Arguing as above using Remark 2.15 ($q = 0$), we obtain

$$(2.52) \quad \int_{B_{r_0}(x_0) \cap \Sigma} |A|^4 \leq C \left( \frac{4}{r_0^2} + (|x_0| + 1)^2 \right) \int_{B_{2r_0}(x_0) \cap \Sigma} |A|^2.$$ 

We may now apply (2.51) at radius $2r_0$ to conclude that

$$(2.53) \quad \int_{B_{r_0}(x_0) \cap \Sigma} |A|^4 \leq C \left( \frac{4}{r_0^2} + (|x_0| + 1)^2 \right) r_0^{n-2} \leq C' r_0^{n-4},$$

where we have again used that $r_0 \leq \min(1, |x_0|^{-1})$. This completes the proof for $p = 4$.

For $2 < p < 4$, we obtain the desired result by interpolating (2.51) and (2.52) using Hölder’s inequality. 

□
We are now ready to give, in the case $n \leq 5$, the proof of the curvature estimate Theorem 2.2.

**Proof of Theorem 2.2 for $n \leq 5$.** Fix $p \in B_{R-1} \cap \Sigma$, and set $r(x) = |x - p|$. Also let $r_0 \leq \frac{1}{4} \min \{1, |p|^{-1} \}$.

\begin{equation}
\frac{\sup_{B_{r_0}(p)} \rho}{\inf_{B_{r_0}(p)} \rho} \leq e.
\end{equation}

The goal is to show that $\Sigma$ has small energy at a uniform scale $\delta r_0$, in the sense that

\[ \int_{B_{\delta r_0}(p) \cap \Sigma} |A|^n < \varepsilon, \]

where $\varepsilon$ is as in Theorem 2.13 and $\delta$ depends only on $n$ and $\lambda_0$. From Theorem 2.13 we would then conclude that

\begin{equation}
|A|(p) \leq \frac{1}{\delta r_0} \leq C(1 + |p|)
\end{equation}

as claimed.

To achieve this, we will use a logarithmic cutoff function. In particular, we fix a large integer $k$ to be determined later, and define a cutoff function $\eta$ by

\begin{equation}
\eta = \begin{cases} 
1 & \text{if } r \leq e^{-k}r_0, \\
\frac{\log(r_0) - \log(r)}{k} & \text{if } e^{-k}r_0 < r \leq r_0, \\
0 & \text{if } r > r_0.
\end{cases}
\end{equation}

Note that $|\nabla \eta| \leq \frac{1}{kr}$ and $|\nabla \eta|$ is supported in the annulus between $e^{-k}r_0$ and $r_0$. Using (2.54) as in the proof of Lemma 2.16, we obtain from the stability inequality (2.5) that

\begin{equation}
\int_{B_{e^{-k}r_0}(p) \cap \Sigma} |A|^2 \eta^2 \leq \frac{\sup_{B_{r_0}(p)} \rho}{\inf_{B_{r_0}(p)} \rho} \int_{\Sigma} |\nabla \eta|^2 \leq \frac{e}{k^2} \int_{(B_{r_0}(p) \setminus B_{e^{-k}r_0}(p)) \cap \Sigma} \frac{1}{r^2}.
\end{equation}
Since \( n \geq 2 \) we can use the usual trick as well as the volume estimate in terms of entropy to bound the integral:

\[
\int_{(B_{r_0}(p) \setminus B_{e-r_0}(p)) \cap \sum} \frac{1}{r^2} = \sum_{l=0}^{k-1} \int_{(B_{e-lr_0}(p) \setminus B_{e-l-1r_0}(p)) \cap \sum} \frac{1}{r^2} \leq C \sum_{l=0}^{k-1} e^{2(l+1)r_0^{-2}} e^{-nlr_0^n} \\
\leq C' r_0^{n-2} \sum_{l=0}^{k-1} e^{-(n-2)l} \leq C'' kr_0^{n-2}.
\]

Thus

\[
\int_{B_{e-kr_0}(p) \cap \sum} |A|^2 \leq \int_{B_{e-kr_0}(p) \cap \sum} |A|^2 \eta^2 \leq \frac{C}{k} r_0^{n-2}.
\]

This completes the proof for \( n = 2 \), by taking \( k \) sufficiently large.

Now using the Cauchy-Schwarz inequality and that \( B_{e-kr_0}(p) \subset B_{r_0}(p) \), we have

\[
\int_{B_{e-kr_0}(p) \cap \sum} |A|^3 \eta^2 \leq \left( \int_{B_{e-kr_0}(p) \cap \sum} |A|^2 \eta^2 \right)^{\frac{1}{2}} \left( \int_{B_{r_0}(p) \cap \sum} |A|^4 \eta^2 \right)^{\frac{1}{2}}.
\]

Using (2.57) for the \( |A|^2 \) factor and Lemma 2.16 for the \( |A|^4 \) factor we conclude that

\[
\int_{B_{e-kr_0}(p) \cap \sum} |A|^3 \eta^2 \leq \int_{B_{e-kr_0}(p) \cap \sum} |A|^3 \eta^2 \leq \frac{C}{k} r_0^{n-3}.
\]

Thus, again by taking \( k \) sufficiently large, the result follows for \( n = 3 \).

Consider the remaining cases \( n = 4, 5 \). Using (2.54) to estimate the weight \( \rho \) as before, and applying Remark 2.15 we obtain

\[
\int_{\sum} |A|^n \eta^2 \leq C \left( \int_{\sum} |A|^{n-2} |\nabla \eta|^2 + (|p| + 1)^2 \int_{\sum} |A|^{n-2} \eta^2 \right).
\]

By (2.58) or (2.61) respectively, the second term can be estimated using that \( r_0 \leq \min(1, |p|^{-1}) \),

\[
(|p| + 1)^2 \int_{\sum} |A|^{n-2} \eta^2 \leq \frac{C}{k} (|p| + 1)^2 r_0^{n-(n-2)} \leq \frac{C'}{k}.
\]
To handle the first term we use the logarithmic trick again:

$$\int_{\Sigma} |A|^{n-2} |\nabla \eta|^2 \leq \frac{1}{k^2} \sum_{l=0}^{k-1} \int_{B_{e^{-l_0}} \setminus B_{e^{-l-1_0}}} |A|^{n-2} \frac{1}{r^2}$$

(2.64)

$$\leq C \frac{k-1}{k^2} \sum_{l=0}^{k-1} e^{2l+1} r_0^{-2} e^{n-(n-2)l} r_0^{n-(n-2)} = \frac{C'}{k}.$$  

Here we have used Lemma 2.16 to get to the second line. Thus

(2.65)

$$\int_{B_{e^{-k_0}}(p) \cap \Sigma} |A|^n \leq \frac{C}{k}$$

in the remaining cases $n = 4, 5$, and the proof is complete.

\[\square\]

**Remark 2.17.** When $n = 2$, we can prove that (small) geodesic balls of $\frac{1}{2}$-stable self-shrinkers in $\mathbb{R}^3$ with small radius have at most quadratic area growth, without assuming an entropy bound. Consequently, we do not need to assume an entropy bound in Theorem 2.2.

The key here is to use the Gauss-Bonnet theorem, together with a technical lemma due to Shiohama-Tanaka ([104, 103]; see [52] for a more directly applicable statement), to estimate the area of geodesic balls $B_{r_0}^{\Sigma}(x)$ by a curvature integral:

(2.66)

$$\text{Area}(B_{r_0}^{\Sigma}(x)) - \pi r_0^2 \leq \frac{1}{4} \int_{B_{r_0}^{\Sigma}(x)} |A|^2 (r_0 - r^2).$$

This bound holds for general surfaces and any $r_0$. If $\Sigma$ is a $\frac{1}{2}$-stable self-shrinker in $\mathbb{R}^3$, applying the stability inequality to the right hand side at the right scale $r_0 \leq \theta = \min(1, |x|^{-1})$ yields the local area bound

(2.67)

$$\text{Area}(B_{r_0}^{\Sigma}(x)) \leq \frac{4\pi}{4-e} r_0^2.$$  

To obtain the desired curvature estimate, we then note that the intrinsic analogue of Theorem 2.13 - that is, with extrinsic balls $B_{r_0}(x_0)$ replaced by geodesic balls $B_{r_0}^{\Sigma}(x_0)$ - also holds, due to a chord-arc bound for general surfaces (see Lemma 2.4 of [36]). The resulting
intrinsic small energy hypothesis can be satisfied using the same method as in Theorem 2.2,
using a logarithmic cutoff function defined in terms of intrinsic distances instead of extrinsic
distances.

2.3.2. Schoen-Simon theory. To obtain the desired curvature estimate Theorem 2.2 in the
case \( n = 6 \), we need to apply the Schoen-Simon theory \([96]\), with a few minor modifications
that we will outline here. Of course, this argument will indeed apply for all \( 2 \leq n \leq 6 \).

The key point is to apply the theory at the optimal scale \( \theta = \min(1, |x_0|^{-1}) \) (see for
instance subsection 12 of \([37]\)). At this scale, the conformal metric \( g_{ij} = e^{-\frac{|x|^2}{4n}} \delta_{ij} \) is uniformly
Euclidean in the sense that the volume form \( \rho = e^{-\frac{|x|^2}{4n}} \) satisfies the familiar estimate

\[
\sup_{B_\theta(x_0)} \rho \leq e, \quad \inf_{B_\theta(x_0)} \rho \leq e,
\]

with similar uniform estimates for its derivatives.

In particular, consider a smooth self-shrinker \( \Sigma^n \subset \mathbb{R}^{n+1} \), which has entropy \( \lambda(\Sigma) \leq \lambda_0 \)
and is \( \frac{1}{2} \)-stable in \( B_R, R > 1 \). For \( x_0 \in B_{R-1} \cap \Sigma \), we wish to apply the Schoen-Simon theory
to the (renormalised) \( F \)-functional

\[
e^{-\frac{|x|^2}{2n}} \int_\Sigma e^{-\frac{|x|^2}{2n}}.
\]

By the uniform estimates mentioned above, it may be verified that this functional satisfies
all the conditions in subsection 1 of \([96]\) with parameters depending only on \( n \). Also, the
entropy bound gives the required density bound. The main issue with this approach is that
\( \Sigma \) may not be stable for the \( F \) functional. At the scale \( \theta \), however, the \( \frac{1}{2} \)-stability inequality
\((2.5)\) gives that

\[
\int_\Sigma |A|^2 \rho^2 \leq \frac{\sup_{B_\theta(x_0)} \rho}{\inf_{B_\theta(x_0)} \rho} \int_\Sigma |
abla \phi|^2 \leq e \int_\Sigma |
abla \phi|^2,
\]

for compactly supported functions \( \phi \) in \( B_\theta(x_0) \). All the arguments of \([96]\) then go through
with this slightly weaker stability inequality, at the cost of carrying around the universal
constant \( e \).
In particular, the conclusions of theorem 3 of [96] hold, so since $\Sigma$ is smooth we conclude that

\begin{equation}
|A|(x_0) \leq C\theta^{-1} \leq C(1 + |x_0|),
\end{equation}

where $C = C(n, \lambda_0)$ as desired.

**Remark 2.18.** Indeed, the Schoen-Simon argument goes through assuming only a $\delta$-stability inequality (2.4) at the optimal scale $\theta = \min(1, |x_0|^{-1})$, for $\delta$ depending only on $n$. (That is, assuming a uniform lower bound for the least eigenvalue of $L$ on balls $B_{\theta}(x_0)$.)

In particular, the curvature estimate also applies to self-shrinkers with bounded entropy and positive mean curvature $H > 0$ in $B_R$, $R > 1$. In this strictly mean convex setting, the curvature estimate was already known to Colding-Ilmanen-Minicozzi.
CHAPTER 6

Monotonicity formulae and consequences

1. Moving-centre monotonicity formulae

The work described in this section was first presented in the published article [130].

In this section we present a sharp ‘moving-centre’ monotonicity formula for minimal submanifolds. As a corollary, one can recover a bound for the area of minimal submanifolds in a Euclidean ball that was conjectured by Alexander, Hoffman and Osserman [2] and first proven by Brendle and Hung [21] using a clever, but somewhat geometrically mysterious, choice of vector field. The methods developed in [130] and presented here can also be used to prove moving-centre monotonicity formulae for stationary $p$-harmonic maps, mean curvature flow and the harmonic map heat flow.

The classical monotonicity formula for minimal submanifolds (see for instance [105], or [36]) states that:

**Proposition 1.1.** Let $\Sigma^k$ be a minimal submanifold in the ball $B^n_r \subset \mathbb{R}^n$ with $\partial \Sigma \subset \partial B^n_r$.

Then for $0 < r < \bar{r}$ one has

$$
\frac{d}{dr} \left( r^{-k} |\Sigma \cap B_r| \right) = r^{-k-1} \int_{\Sigma \cap \partial B_r} \frac{|x^\perp|^2}{|x|^2}.
$$

Equivalently, for $0 < r < t < \bar{r}$, we have

$$
t^{-k} |\Sigma \cap B_t| - r^{-k} |\Sigma \cap B_r| = \int_{\Sigma \cap B_t \setminus B_r} \frac{|x^\perp|^2}{|x|^{k+2}}.
$$

In particular, the area ratio $r^{-k} |\Sigma \cap B_r|$ is non-decreasing in $r$, and is constant if and only if $\Sigma$ is a cone (with vertex at 0).
In order to state our moving-centre monotonicity formula, we first define a family of extrinsic balls on which to view the submanifold. Here $y$ will denote a point in the unit ball.

**Definition 1.2.** Fix $y \in B_1^n$. For $s \geq 0$, denote the ball

$$E_s = B^n ((1 - s)y, r(s)) \subset \mathbb{R}^n,$$

where

$$r(s) = \sqrt{s(1 - |y|^2) + s^2|y|^2}. \tag{1.4}$$

Note that the $E_s$, $s \geq 0$ foliate the half-space defined by $\langle x, y \rangle < \frac{1 + |y|^2}{2}$.

These balls may also be realised as sub-level sets,

$$E_s = \{0 \leq f < s\}, \tag{1.5}$$

where explicitly

$$f(x) = \frac{|x - y|^2}{1 - 2\langle x, y \rangle + |y|^2} = \frac{|x - y|^2}{1 - |x|^2 + |x - y|^2}. \tag{1.6}$$

The moving-centre monotonicity formula is then as follows:

**Theorem 1.3.** Let $\Sigma^k$ be a minimal submanifold in $E_{\bar{s}} \subset \mathbb{R}^n$ with $\partial \Sigma \subset \partial E_{\bar{s}}$ for some $\bar{s}$. Then for $0 < s < \bar{s}$, we have that

$$\frac{d}{ds} \left( s^{-\frac{k}{2}} |\Sigma \cap E_s| \right) = \frac{s^{-\frac{k+2}{2}}}{2} \int_{\Sigma \cap \partial E_s} \frac{|(x - y)\perp^2 + s^2|y^T|^2}{|(x - y + sy)^T|}. \tag{1.7}$$

Equivalently, for $0 < s < t < \bar{s}$, we have

$$t^{-\frac{k}{2}} |\Sigma \cap E_t| - s^{-\frac{k}{2}} |\Sigma \cap E_s| = \int_{\Sigma \cap E_t \setminus E_s} f^{-\frac{k}{2}} \left( \frac{|(x - y)\perp^2 + f^2|y^T|^2}{|x - y|^2} \right). \tag{1.8}$$

In particular, the quantity

$$s^{-\frac{k}{2}} |\Sigma \cap E_s| \tag{1.9}$$
is nondecreasing, and is constant if and only if \( \Sigma \) is a flat disk orthogonal to \( y \).

**Remark 1.4.** Our proof will only require the coarea and first variation formulae, so it can be seen that Theorem 1.3 also holds for stationary varifolds \( \Sigma \), except that in the equality case one must allow for cones with vertex at \( y \) that are orthogonal to \( y \).

Taking \( y = 0 \) of course recovers the classical monotonicity formula for minimal submanifolds.

It may be helpful to note that if \( \Sigma_0 \) is indeed a flat \( k \)-plane orthogonal to \( y \), then any \( x \in \Sigma_0 \) satisfies \(|x|^2 = |y|^2 + |x - y|^2\) and hence \( \Sigma_0 \cap E_s \) is a flat \( k \)-disk of radius \( \sqrt{s(1 - |y|^2)} \) as expected.

**Corollary 1.5.** Let \( y \) be a minimal submanifold in the unit ball \( B^n_1 \subset \mathbb{R}^n \) with \( \partial \Sigma \subset \partial B^n_1 \) and \( y \in \Sigma \). Then \( |\Sigma| \geq |B^n_1|(1 - |y|^2)^{\frac{k}{2}} \), with equality if and only if \( \Sigma \) is a flat disk orthogonal to \( y \).

Corollary 1.5 was first proven in full generality by Brendle and Hung [21], using a carefully chosen vector field that we will return to later in this section.

**Proof of Corollary 1.5 using Theorem 1.3.** As \( s \searrow 0 \), the balls \( E_s \) are asymptotic to the balls \( B(y, \sqrt{s(1 - |y|^2)}) \). So the limit \( (1 - |y|^2)^{-\frac{k}{2}} \lim_{s \to 0} s^{-\frac{k}{2}} |\Sigma \cap E_s| \) is equal to the density of \( \Sigma \) at \( y \), which is at least 1 since \( y \in \Sigma \). On the other hand \( E_1 = B(0, 1) \), and thus comparing \( s = 0 \) to \( s = 1 \) using Theorem 1.3 immediately yields Corollary 1.5 \( \square \)

We now turn to the proof of Theorem 1.3, for which we first calculate the gradient of \( f \):

**Lemma 1.6.** Let \( r(s), E_s \) and the function \( f \) with \( \partial E_s = \{ f = s \} \) be as in Definition 1.2. Then wherever \( f > 0 \), we have that

\[
\frac{Df}{2f} = \frac{x - y + fy}{|x - y|^2}.
\]

**Proof.** One may verify this using the explicit formula for \( f \), but it is more illuminating to proceed using only the characterisation by level sets together with the choice of \( r(s) \).
Indeed, let $\rho(s) = r(s)^2 = s(1 - |y|^2) + s^2|y|^2$. By construction, the level sets of $f$ are the spheres $\partial E_s$ with centre $(1 - s)y$ and radius $r(s)$. So the function $f$ satisfies

$$|x - (1 - f)y|^2 = \rho(f)$$

or in somewhat expanded form

$$|x - y|^2 = -2f\langle x - y, y \rangle + \rho(f) - f^2|y|^2.$$ 

Moreover $Df$ must be proportional to $x - (1 - f)y$, so implicitly differentiating (1.11), we find that

$$\frac{1}{2}Df = \frac{x - y + fy}{\rho'(f) - 2\langle x - y, y \rangle - 2f|y|^2}.$$ 

On the other hand, we note that $\rho$ satisfies the differential equation

$$s(\rho'(s) - 2s|y|^2) = \rho(s) - s^2|y|^2.$$ 

Therefore using (1.12) yields

$$\frac{1}{2}Df = \frac{f(x - y + fy)}{|x - y|^2}.$$ 

\[\square\]

**Proof of Theorem 1.3.** The outward unit normal $\nu$ to $\Sigma \cap \{f = s\}$ considered as the boundary of $\Sigma \cap \{f < s\}$ is given by $\nu = \frac{\nabla f}{|\nabla f|}$. For fixed $s$ we let $X_s$ be a vector field with $\text{div}_\Sigma X_s \equiv k$, to be chosen later. By the divergence theorem we would then have

$$|\Sigma \cap E_s| = \frac{1}{k} \int_{\Sigma \cap \{f = s\}} \langle X_s, \frac{\nabla f}{|\nabla f|} \rangle.$$ 

On the other hand, by the coarea formula we have

$$|\Sigma \cap E_s| = \int_0^s d\tau \int_{\Sigma \cap \{f = \tau\}} \frac{1}{|\nabla f|}.$$
This allows us to compute the derivative of $|\Sigma \cap E_s|$ as an integral over $\Sigma \cap \partial E_s$, using Lemma 1.6:

\begin{equation}
\frac{d}{ds} \left( s^{-\frac{k}{2}} |\Sigma \cap E_s| \right) = s^{-\frac{k+2}{2}} \int_{\Sigma \cap \{f=s\}} \frac{1}{|\nabla f|} \left( s - \frac{1}{2} \langle X_s, \nabla f \rangle \right) \tag{1.18}
\end{equation}

\begin{align*}
&= s^{-\frac{k+2}{2}} \int_{\Sigma \cap \{f=s\}} \frac{s}{|\nabla f|} \left( 1 - \frac{1}{2} \langle X_s, \nabla f \rangle \right) \\
&= s^{-\frac{k}{2}} \int_{\Sigma \cap \{f=s\}} \frac{1}{|\nabla f|} \left( 1 - \frac{\langle X_s, (x - y + sy)^T \rangle}{|x - y|^2} \right) 
\end{align*}

Choosing $X_s = x - y - sy$, we indeed have $\text{div}_\Sigma X_s = k$ since $s$ is fixed, and moreover

\begin{equation}
\langle X_s, (x - y + sy)^T \rangle = |(x - y)^T|^2 - s^2 |y^T|^2. \tag{1.19}
\end{equation}

Thus

\begin{equation}
\frac{d}{ds} \left( s^{-\frac{k}{2}} |\Sigma \cap E_s| \right) = s^{-\frac{k}{2}} \int_{\Sigma \cap \{f=s\}} \frac{1}{|\nabla f|} \left( \frac{|(x - y)^T|^2 + s^2 |y^T|^2}{|x - y|^2} \right). \tag{1.20}
\end{equation}

Using Lemma 1.6 to replace $|\nabla f|$ yields the differential form (1.7), whilst integrating (1.20) using the coarea formula a second time gives the integral form (1.8). It is clear from either formulation that $s^{-\frac{k}{2}} |\Sigma \cap E_s|$ is constant if and only if $(x - y)^T \equiv 0$ and $y^T \equiv 0$ on $\Sigma$. The first condition implies that $\Sigma$ is a cone with vertex at $y$ (hence a plane, if $\Sigma$ is smooth), and the second implies that $\Sigma$ is orthogonal to $y$.

**Remark 1.7.** As with the classical monotonicity formula, one still obtains an almost-monotonicity if one assumes only an $L^p$ bound for the mean curvature $\tilde{H}$, by following the proof above and bounding the vector field $X_s$ on the set $E_s$ to handle the extra term.

For instance, if the mean curvature is bounded by $|\tilde{H}| \leq C_H$, then using the bound

\begin{equation}
|X_s| \leq 2s|y| + \sqrt{s(1 - |y|^2) + s^2 |y|^2} \leq 3s|y| + \sqrt{s(1 - |y|^2)} \tag{1.21}
\end{equation}
on \( \{0 \leq f \leq s\} \), one still obtains a monotone quantity after multiplying by the integrating factor \( \exp(kC_H \mu) \), where

\[
\mu = \frac{1}{2} \int \left( 3|y| + \sqrt{\frac{1-|y|^2}{s}} \right) \, ds = \frac{3}{2} s|y| + \sqrt{s(1-|y|^2)}.
\]

For completeness, we now give another proof of Theorem 1.3 that is instead motivated by, and utilises, the work of Brendle-Hung [21]. By using the divergence theorem (twice), this proof recovers the integral formulation (1.8).

**Alternative proof of Theorem 1.3.** With \( f \) defined as above, the vector field used by Brendle and Hung may be written as

\[
W = -\frac{1}{k}(f^{-\frac{1}{2}} - 1)(x - y) + F(f)y,
\]

\[
F(t) := \begin{cases} 
\frac{1}{k-2}(t^{\frac{2-k}{2}} - 1) & , k > 2 \\
-\frac{1}{2} \log t & , k = 2.
\end{cases}
\]

Setting \( W_0 := \frac{1}{k}(x - y) - W \), the computations of Brendle and Hung [21] yield that

\[
\text{div}_\Sigma W_0 = 1 - \text{div}_\Sigma W = \frac{f^{-\frac{1}{k}}(x - y)^{\frac{1}{2}}|2 + f^{-\frac{k}{4}}|y^T|^2}{|x - y|^2}
\]

On the other hand, for any \( 0 < s < \bar{s} \), when restricted to \( \partial E_s = \{ f = s \} \) we have \( W_0 = \frac{1}{k}s^{-\frac{1}{k}}(x - y) - F(s)y \). So applying the divergence theorem we find that

\[
\int_{\Sigma \cap \{ f = s \}} \langle W_0, \nu \rangle = \int_{\Sigma \cap \{ f = s \}} \langle \frac{1}{k}s^{-\frac{1}{k}}(x - y) - F(s)y, \nu \rangle = s^{-\frac{1}{k}}|\Sigma \cap \{ f \leq s \}|,
\]

since \( \text{div}_\Sigma x = k \) and \( \text{div}_\Sigma y = 0 \).
Then applying the divergence theorem a second time, for any $0 < s < t < \tilde{s}$ we indeed have

$$
t^{-\frac{1}{2}}|\Sigma \cap \{f < t\}| - s^{-\frac{1}{2}}|\Sigma \cap \{f < s\}| = \int_{\Sigma \cap \{f=t\}} \langle W_0, \nu \rangle - \int_{\Sigma \cap \{f=s\}} \langle W_0, \nu \rangle
\]

$$
\]

\begin{equation}
\int_{\Sigma \cap \{s<f<t\}} \text{div}_\Sigma W_0
\end{equation}

$$
\end{equation}

\begin{equation}
\int_{\Sigma \cap \{s<f<t\}} \frac{f^{-\frac{1}{2}}|(x-y)|^2 + f^{-\frac{k-4}{2}}|y^T|^2}{|x-y|^2}
\end{equation}

\begin{equation}
= \int_{\Sigma \cap \{s<f<t\}} \frac{f^{-\frac{1}{2}}|(x-y)|^2 + f^{-\frac{k-4}{2}}|y^T|^2}{|x-y|^2}
\end{equation}

\begin{equation}
\cdot
\end{equation}

2. Entropy of hypersurfaces in the round sphere

In this section we observe new monotonicity formulae for mean curvature flows $\Sigma^n_t$ in the round sphere $S^{n+k}$. As a corollary we are able to prove a gap theorem for the volume of minimal hypersurfaces $\Sigma^n$ in $S^{n+1}$ which are not isotopic to the equator.

For a submanifold $\Sigma^n \subset S^{n+k} \subset \mathbb{R}^{n+k+1}$, the mean curvature vector is given by

$$
\tilde{H}_\Sigma^{S^{n+k+1}} = \tilde{H}_\Sigma^{S^{n+k}} - n\tilde{x}.
\end{equation}

Therefore, running the mean curvature flow from $\Sigma$ in $S^{n+k}$ is equivalent to running the mean curvature flow in $\mathbb{R}^{n+k+1}$ and dilating back to the unit sphere, at least before the latter flow collapses to the origin.

Specifically, let $\{\Sigma^n_t\}_{t \in [0,T]}$ be a smooth mean curvature flow in $S^{n+k}$. Then

\begin{equation}
\hat{\Sigma}_s := e^{-nt} \Sigma_{1-e^{-2nt}}, \quad s := \frac{1 - e^{-2nt}}{2n} \in \left[\frac{1}{2n}, \frac{1 - e^{-2nT}}{2n}\right],
\end{equation}

defines a smooth mean curvature flow in $\mathbb{R}^{n+k+1}$. In particular the flow $\hat{\Sigma}_s$ satisfies Huisken’s monotonicity formula. Translating back to the original flow $\Sigma_t$ in the sphere yields the following monotonicity theorem:
Theorem 2.1. Let \( \{\Sigma^t\}_{t \in [0,T]} \) be a smooth mean curvature flow in \( S^{n+k} \). Further take any \( x_0 \in \mathbb{R}^{n+k+1} \) and any \( t_0 > \frac{e^{-2nT}}{2n} \), and set \( y(t) = e^{nt} x_0 \) and \( a(t) = e^{2nt} t_0 + \frac{1}{2n} \). Then so long as \( a(t) > 0 \), we have

\[
\frac{d}{dt} F_{y(t),a(t)}(\Sigma_t) = -\int_{\Sigma_t} \left( \frac{(2nt_0 e^{2nt} + \langle x, y(t) \rangle)^2}{4a(t)^2} + \left| H_{\Sigma_t}^{S^{n+k}} - \frac{y(t)^1}{2a(t)} \right|^2 \right) \rho_{y(t),a(t)}. \tag{2.2}
\]

In particular, the entropy \( \Lambda(\Sigma_t) \) (defined by considering \( \Sigma^n_t \) as a submanifold in \( \mathbb{R}^{n+k+1} \)) is non-decreasing in \( t \).

Now let \( \Sigma^n_0 \) be a minimal hypersurface in \( S^{n+1} \). Then \( \sqrt{2n} \Sigma_0 \) is a self-shrinker in \( \mathbb{R}^{n+2} \), and its entropy is given by

\[
\Lambda(\Sigma_0) = F_{0,1}(\sqrt{2n} \Sigma_0) = \frac{e^{-n/2}}{(4\pi)^{n/2}} (2n)^{n/2} \text{Vol}(\Sigma_0).
\]

If we perturb \( \Sigma_0 \) by the first eigenfunction of its Jacobi operator, we obtain a family \( \Sigma^{(e)} \) of mean convex hypersurfaces in \( S^{n+1} \). Running the mean curvature flow in \( S^{n+1} \) thus either runs into a cylindrical singularity of type \( S^k \times \mathbb{R}^{n-k} \), \( k < n \), or contracts to a round point and is thus isotopic to the round \( S^n \). The entropy of the cylinder is given by

\[
\Lambda(S^k \times \mathbb{R}^{n-k}) = \Lambda(S^k) = \frac{e^{-k/2}}{(4\pi)^{k/2}} (2k)^{k/2} \text{Vol}(S^k),
\]

so since \( \Lambda^1 \) is decreasing along the flow we have the following corollary:

**Corollary 2.2.** Let \( \Sigma^n_0 \) be a minimal hypersurface in \( S^{n+1} \). If \( \Sigma_0 \) is not isotopic to the round \( S^n \), then

\[
\frac{\text{Vol}(\Sigma_0)}{\text{Vol}(S^n)} \geq \frac{\Lambda(S^{n-1})}{\Lambda(S^n)}. \tag{2.3}
\]

We should compare our results to two existing gap theorems for volumes of minimal hypersurfaces in \( S^{n+1} \). In [25], Cheng, Li and Yau proved a gap theorem of the form \( \frac{\text{Vol}(\Sigma^n_0)}{\text{Vol}(S^n)} \geq 1 + \epsilon(n,k) \) for non-geodesic minimal submanifolds in \( S^{n+k} \) for any codimension \( k \) and any
topology. However, following their proof, one sees that the constant $\epsilon(n, k)$ is very small and indeed their $\epsilon(n, 1) \sim \exp(-n^{n/2}\Gamma(n/2, 1))$ whereas \( \frac{\Lambda(S^{n-1})}{A(S^n)} - 1 \sim n^{-2} \).

On the other hand, Ilmanen and White [71] proved a much better bound for hypersurfaces $\Sigma_0 \subset S^{n+1}$ under the assumptions that:

(i) The cone $C(\Sigma_0)$ is area-minimising in $\mathbb{R}^{n+2}$;
(ii) At least one of the components of $S^{n+1} - \Sigma_0$ is non-contractible.

Under these assumptions they showed that $\frac{\text{Vol}(\Sigma_0)}{\text{Vol}(S^n)} > \sqrt{2}$.

Our result thus yields a significantly larger gap than that of Cheng-Li-Yau under an extra topological assumption; on the other hand our gap is much smaller than that of Ilmanen-White, but they require a stronger topological assumption and the more serious assumption that the cone is area-minimising.
CHAPTER 7

Min-max construction for prescribed mean curvature problems

The work described in this chapter was first presented in [126] and is joint with Prof. Xin Zhou.

The main theorem of this chapter is the construction, via a min-max approach, of non-trivial closed CMC hypersurfaces of any prescribed mean curvature, in any smooth closed Riemannian manifold $M^{n+1}$ of dimension at most seven. (The dimensional restriction arises from, and matches with, the regularity theory for minimal hypersurfaces [97, 96].)

**Theorem 0.1.** Let $M^{n+1}$ be a smooth, closed Riemannian manifold of dimension $3 \leq n + 1 \leq 7$. Given any $c \in \mathbb{R}$, there exists a nontrivial, smooth, closed, almost embedded hypersurface $\Sigma^n$ of constant mean curvature $c$.

Here we say that an immersed hypersurface $\Sigma$ is almost embedded if $\Sigma$ locally decomposes into smoothly embedded components that (pairwise) lie to one side of each other. That is, the sheets may touch but not cross; the example of touching spheres shows that this regularity is optimal.

**Remark 0.2.** For $c \neq 0$, our min-max procedure converges to the constructed hypersurface $\Sigma$ with multiplicity 1. This is a stark and surprising contrast to the minimal ($c = 0$) case, for which the min-max multiplicity 1 conjecture is a fundamental open problem [87].

**Remark 0.3.** The theory we have developed and presented here for CMC hypersurfaces is in fact strong enough to also produce hypersurfaces $\Sigma$ of certain prescribed mean curvatures $h$; that is, given an ambient function $h : M \to \mathbb{R}$, the hypersurface $\Sigma$ should satisfy $H_\Sigma = h|_\Sigma$. The min-max procedure goes through relatively unchanged for positive prescription functions.
In an ongoing work with Prof. Xin Zhou we are developing the theory to handle certain prescription functions without assuming a sign condition.

1. Notation

In this section, we collect some additional notation. We refer to [105] and [94, §2.1] for further background in geometric measure theory.

Let \((M^{n+1}, g)\) denote a closed, oriented, smooth Riemannian manifold of dimension \(3 \leq (n + 1) \leq 7\). Assume that \((M, g)\) is embedded in some \(\mathbb{R}^L, \ L \in \mathbb{N}\). \(B_r(p), \widetilde{B}_r(p)\) denote respectively the Euclidean ball of \(\mathbb{R}^L\) or the geodesic ball of \((M, g)\). We denote by \(\mathcal{H}^k\) the \(k\)-dimensional Hausdorff measure; \(I_k(M)\) the space of \(k\)-dimensional integral currents in \(\mathbb{R}^L\) with support in \(M\); \(\mathcal{Z}_k(M)\) the space of integral currents \(T \in I_k(M)\) with \(\partial T = 0\); \(\mathcal{V}_k(M)\) the closure, in the weak topology, of the space of \(k\)-dimensional rectifiable varifolds in \(\mathbb{R}^L\) with support in \(M\); \(G_k(M)\) the Grassmannian bundle of un-oriented \(k\)-planes over \(M\); \(\mathcal{F}\) and \(\mathcal{M}\) respectively the flat norm \([105, \S31]\) and mass norm \([105, 26.4]\) on \(I_k(M)\); \(\mathcal{F}\) the varifold \(\mathcal{F}\)-metric on \(\mathcal{V}_k(M)\) and currents \(\mathcal{F}\)-metric on \(I_k(M)\), \([94, 2.1(19)(20)]\); \(C(M)\) or \(C(U)\) the space of sets \(\Omega \subset M\) or \(\Omega \subset U \subset M\) with finite perimeter (Caccioppoli sets), \([105, \S14],[55, \S1.6]\); and \(\mathcal{X}(M)\) or \(\mathcal{X}(U)\) the space of smooth vector fields in \(M\) or supported in \(U\).

We also utilise the following definitions:

(a) Given \(T \in I_k(M)\), \(|T|\) and \(\|T\|\) denote respectively the integral varifold and Radon measure in \(M\) associated with \(T\);

(b) Given \(c > 0\), a varifold \(V \in \mathcal{V}_k(M)\) is said to have \(c\)-bounded first variation in an open subset \(U \subset M\), if

\[
|\delta V(X)| \leq c \int_M |X| d\mu_V, \text{ for any } X \in \mathcal{X}(U);
\]

here the first variation of \(V\) along \(X\) is \(\delta V(X) = \int_{G_k(M)} \text{div}_S X(x) dV(x, S)\), \([105, \S39]\);

(c) \(U_r(V)\) denotes the ball in \(\mathcal{V}_k(M)\) under \(\mathcal{F}\)-metric with centre \(V \in \mathcal{V}_k(M)\) and radius \(r > 0\);
(d) Given \( p \in \text{spt} \|V\| \), \( \text{VarTan}(V, p) \) denotes the space of tangent varifolds of \( V \) at \( p \) \[105\];

(e) Given a smooth, immersed, closed, orientable hypersurface \( \Sigma \) in \( M \), or a set \( \Omega \in \mathcal{C}(M) \) with finite perimeter, \([\Sigma]\), \([\Omega]\) denote the corresponding integral currents with the natural orientation, and \([\Sigma], [\Omega]\) denote the corresponding integer-multiplicity varifolds;

(f) \( \partial \Omega \) denotes the (reduced)-boundary of \([\Omega]\) as an integral current, and \( \nu_{\partial \Omega} \) denotes the outward pointing unit normal of \( \partial \Omega \), \[105\] 14.2.

In this chapter, we are interested in the following weighted area functional defined on \( \mathcal{C}(M) \). Given \( c > 0 \), define the \( \mathcal{A}^c \)-functional on \( \mathcal{C}(M) \) as

\[
\mathcal{A}^c(\Omega) = \mathcal{H}^n(\partial \Omega) - c\mathcal{H}^{n+1}(\Omega).
\]

The first variation formula for \( \mathcal{A}^c \) along \( X \in \mathfrak{X}(M) \) is (see \[105\] 16.2)

\[
\delta \mathcal{A}^c|_\Omega(X) = \int_{\partial \Omega} \text{div}_{\partial \Omega} X \, d\mu_{\partial \Omega} - c \int_{\partial \Omega} X \cdot \nu \, d\mu_{\partial \Omega},
\]

where \( \nu = \nu_{\partial \Omega} \) is the outward unit normal on \( \partial \Omega \).

When the boundary \( \partial \Omega = \Sigma \) is a smooth immersed hypersurface, we have

\[
\text{div}_\Sigma X = H X \cdot \nu,
\]

where \( H \) is the mean curvature of \( \Sigma \) with respect to \( \nu \); if \( \Omega \) is a critical point of \( \mathcal{A}^c \), then \( [1.2] \) directly implies that \( \Sigma = \partial \Omega \) has constant mean curvature \( c \) with respect to the outward unit normal \( \nu \). In this case, we can calculate the second variation formula for \( \mathcal{A}^c \) along normal vector fields \( X \in \mathfrak{X}(M) \) such that \( X = \varphi \nu \) along \( \partial \Omega = \Sigma \) where \( \varphi \in C^\infty(\Sigma) \), \[8\] Proposition 2.5],

\[
\delta^2 \mathcal{A}^c|_\Omega(X, X) = \mathcal{H}^\Sigma(\varphi, \varphi) = \int_{\Sigma} (|\nabla \varphi|^2 - (\text{Ric}^M(\nu, \nu) + |A^{\Sigma}|^2) \varphi^2) \, d\mu_{\Sigma}.
\]

In the above formula, \( \nabla \varphi \) is the gradient of \( \varphi \) on \( \Sigma \); \( \text{Ric}^M \) is the Ricci curvature of \( M \); \( A^{\Sigma} \) is the second fundamental form of \( \Sigma \).
2. CMC hypersurfaces

In this section, we collect some preliminary results. First, we study the compactness properties of stable CMC hypersurfaces. In particular, we describe the structure of the touching sets which appear naturally when one takes the limit of embedded stable CMC hypersurfaces. We also present a maximum principle for varifolds with bounded first variation, a regularity result for boundaries that minimise the $A^c$-functional, and a result on isoperimetric profile for small volumes.

2.1. Compactness of stable CMC hypersurfaces.

**Definition 2.1.** Let $\Sigma$ be a smooth, immersed, two-sided hypersurface with unit normal vector $\nu$, and $U \subset M$ an open subset. We say that $\Sigma$ is a *stable* $c$-hypersurface in $U$ if

- the mean curvature $H$ of $\Sigma \cap U$ with respect to $\nu$ equals to $c$; and
- $II_\Sigma(\varphi, \varphi) \geq 0$ for all $\varphi \in C^\infty(\Sigma)$ with $\text{spt} \varphi \subset \Sigma \cap U$, where $II_\Sigma$ is as in (1.3).

**Definition 2.2.** Let $\Sigma_i$, $i = 1, 2$, be connected embedded hypersurfaces in a connected open subset $U \subset M$, with $\partial \Sigma_i \cap U = \emptyset$ and unit normals $\nu_i$. We say that $\Sigma_2$ *lies on one side of* $\Sigma_1$ if $\Sigma_1$ divides $U$ into two connected components $U_1 \cup U_2 = U \setminus \Sigma_1$, where $\nu_1$ points into $U_1$, and either:

- $\Sigma_2 \subset \text{Clos}(U_1)$, which we write as $\Sigma_1 \leq \Sigma_2$ or that $\Sigma_2$ lies on the positive side of $\Sigma_1$; or
- $\Sigma_2 \subset \text{Clos}(U_2)$, which we write as $\Sigma_1 \geq \Sigma_2$ or that $\Sigma_2$ lies on the negative side of $\Sigma_1$.

**Definition 2.3 (Almost embedding).** Let $U \subset M^{n+1}$ be an open subset, and $\Sigma^n$ be a smooth $n$-dimensional manifold. A smooth immersion $\phi : \Sigma \to U$ is said to be an *almost embedding* if at any point $p \in \phi(\Sigma)$ where $\Sigma$ fails to be embedded, there is a small neighbourhood $W \subset U$ of $p$, such that

- $\Sigma \cap \phi^{-1}(W)$ is a disjoint union of connected components $\bigcup_{i=1}^l \Sigma_i$;
• $\phi(\Sigma_i)$ is an embedding for each $i = 1, \cdots, l$;

• for each $i$, any other component $\phi(\Sigma_j), j \neq i$, lies on one side of $\phi(\Sigma_i)$ in $W$.

We will simply denote $\phi(\Sigma)$ by $\Sigma$ and denote $\phi(\Sigma_i)$ by $\Sigma_i$. The subset of points in $\Sigma$ where $\Sigma$ fails to be embedded will be called the *touching set*, and denoted by $\mathcal{S}(\Sigma)$. We will call $\Sigma \setminus \mathcal{S}(\Sigma)$ the regular set, and denote it by $\mathcal{R}(\Sigma)$.

**Remark 2.4.** From the definition, the collection of components $\\{\Sigma_i\}$ meet tangentially along $\mathcal{S}(\Sigma)$.

**Definition 2.5 (Almost embedded $c$-boundary).** 

1. An almost embedded hypersurface $\Sigma \subset U$ is said to be a *boundary* if there is an open subset $\Omega \in \mathcal{C}(U)$, such that $\Sigma$ is equal to the boundary $\partial \Omega$ (in $U$) in the sense of currents;

2. The *outer unit normal* $\nu_\Sigma$ of $\Sigma$ is the choice of the unit normal of $\Sigma$ which points outside of $\Omega$ along the regular part $\mathcal{R}(\Sigma)$;

3. $\Sigma$ is called a *stable $c$-boundary* if $\Sigma$ is a boundary as well as a stable immersed $c$-hypersurface.

We have the following variant of the famous Schoen-Simon-Yau (for $2 \leq n \leq 5$) [97] and Schoen-Simon ($n = 6$) [96] curvature estimates.

**Theorem 2.6 (Curvature estimates for stable $c$-hypersurfaces).** Let $2 \leq n \leq 6$, and $U \subset M$ be an open subset. If $\Sigma \subset U$ is a smooth, immersed (almost embedded when $n = 6$), two-sided, stable $c$-hypersurface in $U$ with $\partial \Sigma \cap U = \emptyset$, and $\text{area}(\Sigma) \leq C$, then there exists $C_1$ depending only on $n, M, c, C$, such that

$$|A^c|^2(x) \leq \frac{C_1}{\text{dist}_M^2(x, \partial U)}$$

for all $x \in \Sigma$.

Moreover if $\Sigma_k \subset U$ is a sequence of smooth, immersed (almost embedded when $n = 6$), two-sided, stable $c$-hypersurfaces in $U$ with $\partial \Sigma_k \cap U = \emptyset$ and $\sup_k \text{area}(\Sigma_k) < \infty$, then up to a subsequence, $\Sigma_k$ converges locally smoothly (possibly with multiplicity) to some stable $c$-hypersurface $\Sigma_\infty$ in $U$. 

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Proof. The compactness statement follows in the standard way from the curvature estimates. The curvature estimates follow from standard blowup arguments together with the Bernstein Theorem [97, Theorem 2] and [96, Theorem 3], the key being that the blowup will be a stable minimal hypersurface, and when \( n = 6 \), the blowup of a sequence of almost embedded \( c \)-hypersurfaces will be embedded by the classical maximum principle for embedded minimal hypersurfaces (c.f. [36]).

We need the following maximum principle.

Lemma 2.7 (Maximum principle for embedded \( c \)-hypersurfaces). Given a connected open subset \( U \subset M \), let \( \Sigma_i \subset U \) be two connected embedded hypersurfaces with \( \partial \Sigma_i \cap U = \emptyset \) for \( i = 1, 2 \). Suppose that the mean curvature of each \( \Sigma_i \) is a given constant \( c > 0 \) with respect to the respective unit normal \( \nu_i \). Assume that \( \Sigma_2 \) lies on one side of \( \Sigma_1 \). Then we have the following:

(i) If there exists \( p \in \Sigma_1 \cap \Sigma_2 \) such that \( \nu_1(p) = \nu_2(p) \), then \( \Sigma_1 = \Sigma_2 \);

(ii) Suppose \( \Sigma_2 \) lies on the negative side of \( \Sigma_1 \). Then if \( \Sigma_1 \cap \Sigma_2 \neq \emptyset \), we must have \( \nu_1(p) = \nu_2(p) \) for any \( p \in \Sigma_1 \cap \Sigma_2 \), and hence \( \Sigma_1 = \Sigma_2 \). In particular, either \( \Sigma_1 \cap \Sigma_2 = \emptyset \) or \( \Sigma_1 = \Sigma_2 \).

Proof. This follows directly from the classical maximum principle as follows.

Consider \( p \in \Sigma_1 \cap \Sigma_2 \). Since \( \Sigma_2 \) lies on one side of \( \Sigma_1 \), the tangent planes must coincide at any point of their intersection. So without loss of generality we may assume that \( U \) is a small ball around \( p \) for which \( \Sigma_1, \Sigma_2 \) may be written as graphs \( u_1, u_2 \) in the \( \nu_1 \)-direction over the tangent plane \( T_p \Sigma_1 = T_p \Sigma_2 \).

Let \( u = u_1 - u_2 \), then a standard computation shows that \( u \) satisfies a linear elliptic equation of the form: \( Lu = 0 \), if \( \nu_2(p) = \nu_1(p) \); or \( Lu = 2c \) if \( \nu_2(p) = -\nu_1(p) \). Here \( L \) is a positive elliptic operator with smooth coefficients. Moreover, if \( \Sigma_2 \subset \text{Clos}(U_1) \) then \( u \leq 0 \); if \( \Sigma_2 \subset \text{Clos}(U_2) \) then \( u \geq 0 \). Both items then follow from the maximum principle for nonpositive (or nonnegative) functions.
**Lemma 2.8.** Let $\Omega$ be a domain in $\mathbb{R}^m$ and suppose that $u$ is a classical solution on $\Omega$ of a linear inhomogenous elliptic PDE with smooth coefficients:

\begin{equation}
Lu = a^{ij} D_{ij} u + b^j D_j u + qu = f, \tag{2.1}
\end{equation}

where $f$ has no zeroes on $\Omega$. Then the zero set $\{u = 0\}$ is contained in a countable union of connected, embedded $(m - 1)$-dimensional submanifolds.

**Proof.** Let $K$ be a compact subset of $\Omega$.

First, the implicit function theorem implies that the zero set is smooth away from the critical set. In particular, for any $\epsilon > 0$ the compact set $\{u = 0, |Du| \geq \epsilon\} \cap K$ is contained in the union of finitely many connected, smoothly embedded $(m - 1)$-dimensional submanifolds.

Now consider $x \in \{u = 0, Du = 0\}$. Then we have $a^{ij}(x) D_{ij} u(x) = f(x) \neq 0$, so by ellipticity, the Hessian $D^2 u$ must have rank at least 1. Thus for some $j$, the gradient $D(D_j u) \neq 0$, so again by the implicit function theorem there is an $r > 0$ such that $B_r(x) \cap \{D_j u = 0\}$ is an embedded $(m - 1)$-dimensional submanifold, which clearly contains $B_r(x) \cap \{u = 0\} \cap \{Du = 0\}$. It follows that the compact set $\{u = 0, Du = 0\} \cap K$ is also contained in a finite union of connected, embedded $(m - 1)$-dimensional submanifolds.

Taking $\epsilon = 1/j \to 0$ and $K = K_j$, where $K_j$ is an exhaustion of $\Omega$, then completes the proof. \qed

**Proposition 2.9** (Touching sets for almost embedded $c$-hypersurface). If the metric on $U^{n+1}$ is smooth, then for any almost embedded hypersurface $\Sigma^n \subset U$ of constant mean curvature $c$, the touching set $S(\Sigma)$ is contained in a countable union of connected, embedded $(n - 1)$-dimensional submanifolds.

In particular, the regular set $R(\Sigma)$ is open and dense in $\Sigma$.

**Proof.** Let $p \in S(\Sigma)$. As in the proof of Lemma 2.7, there is a small neighbourhood $W$ of $p$ so that the image $\Sigma \cap W$ decomposes as graphs $\{u_i\}_{i=1}^k$, ordered by height, over
the common tangent plane $T_p \Sigma$. In fact by the conclusions of that lemma, after possibly shrinking $W$ there will be exactly two distinct graphs $u_1 \leq u_2$, for which the difference $u = u_1 - u_2$ satisfies $Lu = 2c$. The zero set of $u$ corresponds to the touching set $\mathcal{S}(\Sigma) \cap W$. The proposition then follows from the previous lemma. 

Remark 2.10. In the case that the metric on $M$ is real analytic, we have the stronger statement that the touching set is a finite union of real analytic subvarieties $\bigcup_{k=0}^{n-1} S^k$ of respective dimension $k$. This follows from [78, Theorem 5.2.3], since in this setting the operator $L$ will have analytic coefficients, and hence the solution $u$ is also real analytic.

Theorem 2.11 (Compactness theorem for almost embedded stable $c$-hypersurfaces). Let $2 \leq n \leq 6$. Suppose $\Sigma_k \subset U$ is a sequence of smooth, almost embedded, two-sided, stable $c_k$-hypersurfaces in $U$, with $\sup_k \text{area}(\Sigma_k) < \infty$ and $\sup_k c_k < \infty$. Then the following hold:

(i) if $\inf c_k > 0$, then up to a subsequence, $\{\Sigma_k\}$ converges locally smoothly (with multiplicity) to some almost embedded stable $c$-hypersurface $\Sigma_\infty$ in $U$;

(ii) if additionally $\{\Sigma_k\}$ are all boundaries, then $\Sigma_\infty$ is also a boundary, and the density of $\Sigma_\infty$ is 1 along $R(\Sigma_\infty)$ and 2 along $S(\Sigma_\infty)$;

(iii) if $c_k \to 0$, then up to a subsequence, $\{\Sigma_k\}$ converges locally smoothly (with multiplicity) to some smooth embedded stable minimal hypersurface $\Sigma_\infty$ in $U$.

Remark 2.12. We learned that Bellettini-Wickramasekera [10] also have similar compactness results for stable CMC varifolds.

Proof of Theorem 2.11. Case (i) follows straightforwardly from Theorem 2.6, the almost embedded assumption, together with the maximum principle Lemma 2.7.

Now we prove Case (ii). Denote $\Sigma_k = \partial \Omega_k$ for some $\Omega_k \in \mathcal{C}(U)$. By standard compactness [105, Theorem 6.3], a subsequence of $\partial \Omega_k$ converges weakly as currents to some $\partial \Omega_\infty$ with $\Omega_\infty \in \mathcal{C}(U)$. We claim that $\partial \Omega_\infty = \Sigma_\infty$ as varifold. To show this, we only need to check that the density of $\Sigma_\infty$ along $R(\Sigma_\infty)$ is one, and then by Lemma 2.7 and Proposition 2.9, the density of $\Sigma_\infty$ along the touching set $\mathcal{S}(\Sigma_\infty)$ is automatically two. 

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To show that the density along $R(\Sigma_\infty)$ is 1, take an arbitrary point $p \in R(\Sigma_\infty)$. If the density at $p$ is larger than 1, then by the locally smooth convergence of $\Sigma_k$ to $\Sigma_\infty$, there is a neighbourhood $\tilde{B}_p \subset U$ of $p$, such that for $k$ large enough $\Sigma_k \cap \tilde{B}_p$ has a graphical decomposition as $\cup_{i=1}^{l_k} \Sigma^i_k$ with $l_k \geq 2$. Moreover, by Lemma 2.7 we have $\Sigma^1_k < \Sigma^2_k < \cdots < \Sigma^{l_k}_k$, and the outward unit normals $\nu^i_k$ of $\Sigma^i_k$ all point to the same direction. With out loss of generality, we may assume $l_k = 2$ and omit the sub-index $k$. Then $\tilde{B}_p \setminus (\Sigma^1 \cup \Sigma^2)$ has three connected components $U_0, U_1, U_2$ with, counting orientation, $(\partial U_0) \llcorner \tilde{B}_p = \Sigma^1$, $(\partial U_1) \llcorner \tilde{B}_p = \Sigma^2 - \Sigma^1$, and $(\partial U_2) \llcorner \tilde{B}_p = -\Sigma^2$.

On the other hand, for each $i$ the Constancy Theorem [105] Theorem 26.27] applied to $\Omega_k \llcorner U_i$ implies that $\Omega_k \llcorner U_i$ is identical to either $\emptyset$ or $U_i$. That is, $\Omega_k \llcorner \tilde{B}_p = \sum_{i=0}^{2} a_i U_i$, where each $a_i = 0, 1$. It is then easy to see that any choice of the $a_i$ will contradict the fact that, counting orientation, $\partial (\Omega_k \llcorner \tilde{B}_p) \llcorner \tilde{B}_p = \Sigma_k \cap \tilde{B}_p = \Sigma^1 + \Sigma^2$.

Case (iii) follows directly from Theorem 2.6, the almost embedded assumption, and the classical maximum principle for embedded minimal hypersurfaces (c.f. [36]).

\begin{proof}
2.2. Maximum principle for varifolds with $c$-bounded first variation. We will need the following maximum principle which is essentially due to White [119, Theorem 5].

\begin{proposition}
(Maximum principle for varifolds with $c$-bounded first variation)
Suppose $V \in \mathcal{V}_c(M)$ has $c$-bounded first variation in an open subset $U \subset M$. Let $K \subset U$ be an open subset with compact closure in $U$, such that $\text{spt}(\|V\|) \subset K$, and

(i) $\partial K$ is smoothly embedded in $M$,

(ii) the mean curvature of $\partial K$ with respect to the outward pointing normal is greater than $c$.

Then $\text{spt}(\|V\|) \cap \partial K = \emptyset$.

\end{proposition}

\end{proof}

2.3. Regularity for boundaries which minimise the $\mathcal{A}^c$ functional. The following result about regularity of boundaries which minimise the $\mathcal{A}^c$ functional can be found in [91].
Theorem 2.14. Given $\Omega \in \mathcal{C}(M)$, $p \in \text{spt} \|\partial \Omega\|$, and some small $r > 0$, suppose that $\Omega \subseteq \tilde{B}_r(p)$ minimises the $\mathcal{A}^c$-functional: that is, for any other $\Lambda \in \mathcal{C}(M)$ with $\text{spt} \|\Lambda - \Omega\| \subset \tilde{B}_r(p)$, we have $\mathcal{A}^c(\Lambda) \geq \mathcal{A}^c(\Omega)$. Then except for a set of Hausdorff dimension at most $n-7$, $\partial \Omega \subseteq \tilde{B}_r(p)$ is a smooth and embedded hypersurface, and is real analytic if the ambient metric on $M$ is real analytic.

Proof. Since $\Omega \subseteq \tilde{B}_r(p)$ minimises the $\mathcal{A}^c$-functional, for all $\Lambda \in \mathcal{C}(M)$ as in the supposition, we have

$$\mathcal{H}^n(\partial \Lambda) - \mathcal{H}^n(\partial \Omega) \geq -c|\mathcal{H}^{n+1}(\Lambda) - \mathcal{H}^{n+1}(\Omega)|.$$ 

This is precisely condition [91, 3.1(1)]. The regularity then follows from [91, Corollary 3.7, 3.8].

2.4. Isoperimetric profiles for small volume. We have the following lower bound for the isoperimetric profiles for small volumes, which is a consequence of the fact that the isoperimetric profile is asymptotically Euclidean for small volumes [11] (see also [93, Theorem 3]). Note that although it was only stated for domains with smooth boundary, the result indeed holds for any $\Omega \in \mathcal{C}(M)$ by using the regularity theory for isoperimetric domains (c.f. Theorem 2.14).

Theorem 2.15. There exists constants $C_0 > 0$ and $V_0 > 0$ depending only on $M$ such that

$$\text{area}(\partial \Omega) \geq C_0 \text{Vol}(\Omega)^{\frac{n}{n+1}}, \text{ whenever } \Omega \in \mathcal{C}(M) \text{ and } \text{Vol}(\Omega) \leq V_0.$$ 

2.5. Good replacement property and regularity. Here we record the notions of good replacements and the good replacement property. Recall the following definitions by Colding-De Lellis [34]. Consider two open subsets $W \subset U \subset M^{n+1}$.
**Definition 2.16.** [34] Definition 6.1. Let $V \in V_n(U)$ be stationary in $U$. A stationary varifold $V' \in V_n(U)$ is said to be a *replacement for $V$ in $W$* if

$$V' \mathbin{\mathcal{L}} (U \setminus W) = V \mathbin{\mathcal{L}} (U \setminus W), \quad \|V'\|(U) = \|V\|(U),$$

and $V' \mathbin{\mathcal{L}} W$ is an embedded stable minimal hypersurface $\Sigma$ with $\partial \Sigma \subset \partial W$.

**Definition 2.17.** [34] Definition 6.2. Let $V \in V_n(U)$ be stationary in $U$. $V$ is said to have the *good replacement property* in $W$ if

(a) there is a positive function $r : W \to \mathbb{R}$ such that for every annulus $A_{s,t}(x) \cap M \subset W$ with $0 < s < t < r(x)$, there is a replacement $V'$ for $V$ in $A_{s,t}(x) \cap M$;

(b) the replacement $V'$ has a replacement $V''$ in every annulus $A_{s,t}(y) \cap M \subset W$ with $0 < s < t < r(y)$;

(c) $V''$ has a replacement $V'''$ in every annulus $A_{s,t}(z) \cap M \subset W$ with $0 < s < t < r(z)$.

Note that our formulations are local compared to those in [34]. Indeed, the proofs of [34] Proposition 6.3] and [42] Theorem 2.8] are purely local, so the following proposition still holds:

**Proposition 2.18.** [34] Proposition 6.3], [42] Proposition 2.8]. When $2 \leq n \leq 6$, if $V \in V_n(U)$ has the good replacement property in $W$, then $V \mathbin{\mathcal{L}} W$ is an integer multiple of some smooth embedded minimal hypersurface $\Sigma$.

### 3. The $c$-Min-max construction

In this section, we present the setups of the min-max construction mainly followed Pitts [94]. We also prove the existence of a non-trivial sweepout with positive $\mathcal{A}_c$-min-max value.

#### 3.1. Homotopy sequences

We will introduce the min-max construction using the scheme developed by Almgren and Pitts [3, 4, 94].

**Definition 3.1.** (cell complex.)

1. Denote $I = [0, 1]$, $I_0 = \partial I = I \setminus (0, 1)$;
(2) For \( j \in \mathbb{N} \), \( I(1, j) \) is the cell complex of \( I \), whose 1-cells are all intervals of form \([\frac{i}{2^j}, \frac{i+1}{2^j}]\), and 0-cells are all points \([\frac{i}{2^j}]\);

(3) For \( p = 0, 1 \), \( \alpha \in I(1, j) \) is a \( p \)-cell if \( \dim(\alpha) = p \). 0-cell is also called a vertex;

(4) \( I(1, j)_p \) denotes the set of all \( p \)-cells in \( I(1, j) \), and \( I_0(1, j)_0 \) denotes the set \{[0], [1]\};

(5) Given a 1-cell \( \alpha \in I(1, j)_1 \), and \( k \in \mathbb{N} \), \( \alpha(k) \) denotes the 1-dimensional sub-complex of \( I(1, j+k) \) formed by all cells contained in \( \alpha \). For \( q = 0, 1 \), \( \alpha(k)_q \) and \( \alpha_0(k)_q \) denote respectively the set of all \( q \)-cells of \( I(1, j+k) \) contained in \( \alpha \), or in the boundary of \( \alpha \);

(6) The boundary homeomorphism \( \partial : I(1, j) \to I(1, j) \) is given by \( \partial[a, b] = [b] - [a] \) if \([a, b] \in I(1, j)_1 \), and \( \partial[a] = 0 \) if \([a] \in I(1, j)_0 \);

(7) The distance function \( d : I(1, j)_0 \times I(1, j)_0 \to \mathbb{N} \) is defined as \( d(x, y) = 3^j|x - y| \);

(8) The map \( n(i, j) : I(1, i)_0 \to I(1, j)_0 \) is defined as: \( n(i, j)(x) \in I(1, j)_0 \) is the unique element of \( I(1, j)_0 \), such that \( d(x, n(i, j)(x)) = \inf \{ d(x, y) : y \in I(1, j)_0 \} \).

Consider a map to the space of Caccioppoli sets: \( \phi : I(1, j)_0 \to \mathcal{C}(M) \). The fineness of \( \phi \) is defined as:

\[
(\phi) = \sup \left\{ \frac{\mathbf{M}(\partial \phi(x) - \partial \phi(y))}{d(x, y)} : x, y \in I(1, j)_0, x \neq y \right\}.
\]

Similarly we can define the fineness of \( \phi \) with respect to the \( \mathcal{F} \)-norm and \( \mathcal{F} \)-metric. We use \( \phi : I(1, j)_0 \to (\mathcal{C}(M), \{0\}) \) to denote a map such that \( \phi(I(1, j)_0) \subset \mathcal{C}(M) \) and \( \partial \phi|_{I_0(1, j)_0} = 0 \), i.e. \( \phi([0]), \phi([1]) = \emptyset \) or \( M \).

**Definition 3.2.** Given \( \delta > 0 \) and \( \phi_i : I(1, k_i)_0 \to (\mathcal{C}(M), \{0\}) \), \( i = 0, 1 \), we say \( \phi_1 \) is 1-homotopic to \( \phi_2 \) in \( (\mathcal{C}(M), \{0\}) \) with fineness \( \delta \), if \( \exists k_3 \in \mathbb{N} \), \( k_3 \geq \max\{k_1, k_2\} \), and

\[
\psi : I(1, k_3)_0 \times I(1, k_3)_0 \to \mathcal{C}(M),
\]

such that

* \( (\psi) \leq \delta; \)
• $\psi([i], x) = \phi_i(n(k_3, k_i)(x)), i = 0, 1$;
• $\partial\psi(I(1, k_3)_0 \times I_0(1, k_3)_0) = 0$.

**Definition 3.3.** A $(1, M)$-homotopy sequence of mappings into $(\mathcal{C}(M), \{0\})$ is a sequence of mappings $\{\phi_i\}_{i \in \mathbb{N}}$,

$$\phi_i : I(1, k_i)_0 \to (\mathcal{C}(M), \{0\}),$$

such that $\phi_i$ is 1-homotopic to $\phi_{i+1}$ in $(\mathcal{C}(M), \{0\})$ with fineness $\delta_i$, and

• $\lim_{i \to \infty} \delta_i = 0$;
• $\sup_i \{M(\partial\phi_i(x)) : x \in I(1, k_i)_0\} < +\infty$.

**Remark 3.4.** Note that the second condition implies that $\sup_i \left\{ \mathcal{A}^c(\phi_i(x)) : x \in I(1, k_i)_0 \right\} < +\infty$.

**Definition 3.5.** Given two $(1, M)$-homotopy sequences of mappings $S_1 = \{\phi^1_i\}_{i \in \mathbb{N}}$ and $S_2 = \{\phi^2_i\}_{i \in \mathbb{N}}$ into $(\mathcal{C}(M), \{0\})$, $S_1$ is homotopic to $S_2$ if $\exists \{\delta_i\}_{i \in \mathbb{N}}$, such that

• $\phi^1_i$ is 1-homotopic to $\phi^2_i$ in $(\mathcal{C}(M), \{0\})$ with fineness $\delta_i$;
• $\lim_{i \to \infty} \delta_i = 0$.

It is easy to see that the relation “is homotopic to” is an equivalence relation on the space of $(1, M)$-homotopy sequences of mappings into $(\mathcal{C}(M), \{0\})$. An equivalence class is a $(1, M)$-homotopy class of mappings into $(\mathcal{C}(M), \{0\})$. Denote the set of all equivalence classes by $\pi^\#_1(\mathcal{C}(M, M), \{0\})$.

**3.2. Min-max construction.**

**Definition 3.6.** (Min-max definition) Given $\Pi \in \pi^\#_1(\mathcal{C}(M, M), \{0\})$, define $L^\ell : \Pi \to \mathbb{R}^+$ as a function given by:

$$L^\ell(S) = L^\ell(\{\phi_i\}_{i \in \mathbb{N}}) = \limsup_{i \to \infty} \max_i \mathcal{A}^c(\phi_i(x)) : x \text{ lies in the domain of } \phi_i \}.$$
The $\mathcal{A}^c$-min-max value of $\Pi$ is defined as

\[
\mathbf{L}^c(\Pi) = \inf \{ \mathbf{L}^c(S) : S \in \Pi \}.
\]

A sequence $S = \{\phi_i\} \in \Pi$ is called a critical sequence if $\mathbf{L}^c(S) = \mathbf{L}^c(\Pi)$.

Given a critical sequence $S$, then

\[
K(S) = \{ V = \lim_{j \to \infty} |\partial \phi_i(x_j)| : x_j \text{ lies in the domain of } \phi_i \}
\]

is a compact subset of $\mathcal{V}_n(M^{n+1})$. The critical set of $S$ is the subset $C(S) \subset K(S)$ defined by

\[
C(S) = \{ V = \lim_{j \to \infty} |\partial \phi_i(x_j)| : \text{ with } \lim_{j \to \infty} \mathcal{A}^c(\phi_i(x_j)) = \mathbf{L}^c(S) \}.
\]

Note that by \[94, 4.1(4)]\, we immediately have:

**Lemma 3.7.** Given any $\Pi \in \pi_1^\#(\mathcal{C}(M, M), \{0\})$, there exists a critical sequence $S \in \Pi$.

The main theorem of this chapter may be stated precisely as follows:

**Theorem 3.8.** Let $2 \leq n \leq 6$. Given a smooth closed Riemannian manifold $M^{n+1}$ and $c > 0$, there exists $\Pi \in \pi_1^\#(\mathcal{C}(M, M), \{0\})$ and a critical sequence $S \in \Pi$ such that:

- $\mathbf{L}^c(\Pi) = \mathbf{L}^c(S) > 0$;
- There exists an element of $C(S)$ induced by a nontrivial, smooth, almost embedded, closed hypersurface $\Sigma^n \subset M$ of constant mean curvature $c$ with multiplicity one.

**Proof of Theorem 3.8.** This follows from combining Theorem 3.9, Theorem 5.6 and Theorem 6.1.

3.3. Existence of nontrivial sweepouts.

**Theorem 3.9.** There exists $\Pi \in \pi_1^\#(\mathcal{C}(M, M), \{0\})$, such that for any $c > 0$, we have $\mathbf{L}^c(\Pi) > 0$.  

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Remark 3.10. Let us first describe a heuristic argument using smooth sweepouts which will help to reveal the key idea. Let $C_0 > 0$ and $V_0 > 0$ to be the constants in Theorem 2.15 and fix $0 < V \leq V_0$ such that $V \frac{c}{C_0} > 2c/C_0$. Note that $V$ only depends on $c, C_0, V_0$.

Then for any $\Omega$ with Vol($\Omega$) = $V$, we have

(3.3) $\mathcal{A}^c(\Omega) \geq C_0 V^{\frac{n}{n+1}} - cV > cV > 0.$

Now consider any smooth 1-parameter family $\{\Omega_x : x \in [0, 1]\}$ satisfying $\Omega_0 = \emptyset$ and $\Omega_1 = M$. Since $\{\Omega_x\}$ sweeps out $M$, there must exist some $x_0 \in (0, 1)$ such that Vol($\Omega_{x_0}$) = $V$, whence max$_{x \in [0, 1]} A^c(\Omega_x) \geq cV > 0$. Since this holds for any sweepout we then have $L^c(\Pi) \geq cV > 0$.

Proof of Theorem 3.9. Take a Morse function $\phi : M \to [0, 1]$, and consider the sub-level sets $\Phi : [0, 1] \to \mathcal{C}(M)$, given by $\Phi(t) = \{x \in M : \phi(x) < t\}$. By the interpolation theorem of the first author [125, Theorem 5.1], $\Phi$ can be discretised to a $(1, M)$-homotopy sequence $S = \{\bar{\phi}_i\}$ where $\bar{\phi}_i : I(1, k_i) \to (\mathcal{C}(M), \{0\})$. Moreover, under Almgren’s isomorphism $F_A$ [3, §3.2] (see also [125, §4.2]), $\bar{\phi}_i$ is mapped to the fundamental class in $H_{n+1}(M)$ for $i$ large, i.e. $F_A(\bar{\phi}_i) = [[M]]$. Consider $\Pi = \{S\}$, then by [3, Theorem 7.1] for any $S = \{\phi_i\} \in \Pi$, we have $F_A(\phi_i) = [[M]]$ for $i$ large. In particular, this means that $\sum_{j \in I(1, k_i)} Q_j = M$ as currents; here for given $j$ and $\alpha_j = [\frac{i-1}{3^k}, \frac{i}{3^k}] = [x_{j-1}, x_j], Q_j \in I_{n+1}(M)$ is the isoperimetric choice (c.f. [3, 1.14]) of $\partial \phi_i(x_j) - \partial \phi_i(x_{j-1})$, i.e.

$$M(Q_j) = F(\partial \phi_i(x_j) - \partial \phi_i(x_{j-1})), \text{ and } \partial Q_\alpha = \partial \phi_i(x_j) - \partial \phi_i(x_{j-1}).$$

Denote $\Omega_l = \sum_{j=1}^l Q_j$. Then we have

$$\partial \Omega_l = \partial \phi_i(x_l) - \partial \phi_i(x_0) = \partial \phi_i(x_l),$$

and this implies, by the Constancy Theorem [105, 26.27], that $\Omega_l$ is a Caccioppoli set (possibly with a negative orientation) when $M(\Omega_l) < Vol(M)$. Note that $M(Q_j) < (\phi_i) = \delta_i$, hence by continuity there exists some $l_c \in \mathbb{N}, l_c < 3^{k_i}$, such that $M(\Omega_{l_c}) \in [V - \delta_i, V + \delta_i]$.
where $V$ is as in Remark 3.10 and $\phi_i(x_{k_i}) = \Omega_{k_i}$. Then the same argument as in the remark above gives a uniform positive lower bound for $A^c(\phi_i(x_{k_i}))$, and this finishes the proof. □

4. Tightening

In this section, we construct the tightening map adapted to the $\mathcal{A}^c$ functional and prove that after applying the tightening map to a critical sequence, every element in the critical set has uniformly bounded first variation. Our approach is adapted from those in [34 §4] and [94 §4.3].

4.1. Annular decomposition. Given $L > 0$, consider the set of varifolds in $\mathcal{V}_n(M)$ with $2L$-bounded mass: $A^L = \{V \in \mathcal{V}_n(M) : \|V\|(M) \leq 2L\}$. Denote

$$A^c_\infty = \{V \in A^L : |\delta V(X)| \leq c \int |X|d\mu_V, \text{ for any } X \in \mathfrak{X}(M)\}.$$

Consider the concentric annuli around $A_\infty$ under the $F$-metric, i.e.

$$A_j = \{V \in A^L : \frac{1}{2^j} \leq F(V, A^c_\infty) \leq \frac{1}{2^{j-1}}\}, \quad j \in \mathbb{N}.$$

Since $c$-bounded first variation is a closed condition, we have

**Lemma 4.1.** $A^c_\infty$ is a compact subset of $A^L$ under the $F$-metric.

It is easy to show (by contradiction, for instance) that for any varifold in $A_j$, we can find a vector field satisfying the following condition.

**Lemma 4.2.** For any $V \in A_j$, there exists $X_V \in \mathfrak{X}(M)$, such that

$$\|X_V\|_{C^1(M)} \leq 1, \quad \delta V(X_V) - c \int_M |X_V|d\mu_V \leq -c_j < 0,$$

where $c_j$ depends only on $j$.

4.2. A map from $A^L$ to the space of vector fields. In this part, we will construct a map $X : A^L \to \mathfrak{X}(M)$, which is continuous with respect to the $C^1$ topology on $\mathfrak{X}(M)$.
Given \( V \in A_j \), let \( X_V \) be given in Lemma 4.2. Since \( \text{div}_S X_V \) is Lipschitz on \( G_n(M) \) for fixed \( X_V \), the map

\[
W \mapsto \delta W(X_V) - c \int_M |X_V|d\mu_W = \int_{G_n(M)} \text{div}_S(X_V) dW(x,S) - c \int_M |X_V|d\mu_W
\]

is continuous with respect to the \( F \)-metric. Therefore for any \( V \in A_j \), there exists \( 0 < r_V < \frac{1}{2^{|j|}} \), such that for any \( W \in U_{r_V}(V) \), i.e. \( F(W,V) < r_V \),

\[
\delta W(X_V) - c \int |X_V|d\mu_W \leq \frac{1}{2} \left( \delta V(X_V) - c \int |X_V|d\mu_V \right) \leq -\frac{1}{2} c_j < 0.
\]

Now \( \{U_{r_V/2}(V) : V \in A_j\} \) is an open covering of \( A_j \). By the compactness of \( A_j \), we can find finitely many balls \( \{U_{r_j,i}(V_{j,i}) : V_{j,i} \in A_j, 1 \leq i \leq q_j\} \), where \( r_{j,i} = r_{V_{j,i}} \), such that

(i) The balls \( U_{r_j,i/2}(V_{j,i}) \) with half radii cover \( A_j \);

(ii) The balls \( U_{r_j,i}(V_{j,i}) \) are disjoint from \( A_k \) if \( |j - k| \geq 2 \).

In the following, we denote \( U_{r_{j,i}}(V_{j,i}), U_{r_{j,i}/2}(V_{j,i}) \) and \( X_{V_{j,i}} \) by \( U_{j,i}, \tilde{U}_{j,i} \) and \( X_{j,i} \) respectively.

Now we can construct a partition of unity \( \{\varphi_{j,i} : j \in \mathbb{N}, 1 \leq i \leq q_j\} \) subordinate to the covering \( \{\tilde{U}_{j,i}, 1 \leq i \leq q_j, j \in \mathbb{N}\} \) by

\[
\varphi_{j,i}(V) = \sum_{p \in \mathbb{N}, 1 \leq q \leq q_p} \psi_{j,i}(V),
\]

where \( \psi_{j,i}(V) = F(V, A^c_l \setminus \tilde{U}_{j,i}) \).

The map \( X : A^L \to \mathcal{X}(M) \) is defined by

\[
X(V) = F(V, A^c_{\infty}) \sum_{j \in \mathbb{N}, 1 \leq i \leq q_j} \varphi_{j,i}(V) X_{j,i}.
\]

The following lemma is a straightforward consequence of the construction.

**Lemma 4.3.** The map \( X : V \to X(V) \) is continuous with respect to the \( C^1 \) topology on \( \mathcal{X}(M) \).

**4.3. A map from \( A^L \) to the space of isotopies.** In this part, we will associate each \( V \in A \) with an isotopy of \( M \) in a continuous manner. The isotopy will be generated by the
vector field $X(V)$. In particular, given $V \in A^L$, we use $\Phi_V : \mathbb{R}^+ \times M \to M$ to denote the one parameter group of diffeomorphisms generated by $X(V)$.

Given $\Omega \in \mathcal{C}(M)$ with $\partial \Omega \in A^L$, we will deform $\Omega$ by $\Phi_{|\partial \Omega|}(t)$ to get a 1-parameter family of sets of finite perimeter $\Omega_t = \Phi_{|\partial \Omega|}(t)(\Omega)$, and we will show that the $A^c$ functional of $\Omega_t$ for some $t > 0$ can be deformed down by a fixed amount depending only on $F(|\partial \Omega|, A^c_{\infty})$.

In fact, given $V \in A_j$, let $\rho(V)$ be the smallest radii of the balls $U_{k,i}$ which contain $V$. As there are only finitely many balls $U_{k,i}$ which intersect $A_j$ nontrivially, we know that $\rho(V) \geq r_j > 0$, where $r_j$ depends only on $j$; moreover, by construction the sub-index $k$ of these $U_{k,i}$ can only be $j-1$, $j$, or $j+1$. Then by (4.2) and (4.3), we have for any $W \in U_{\rho(V)}(V)$ that

$$\delta W(X(V)) - c \int |X(V)| d\mu_W \leq -\frac{1}{2j+1} \min\{c_{j-1}, c_j, c_{j+1}\}.$$ 

Therefore we can find two continuous functions $g : \mathbb{R}^+ \to \mathbb{R}^+$ and $\rho : \mathbb{R}^+ \to \mathbb{R}^+$, such that $\rho(0) = 0$ and

$$\delta W(X(V)) - c \int |X(V)| d\mu_W \leq -g(F(V, A^c_{\infty})), \quad \text{if } F(W, V) \leq \rho(F(V, A^c_{\infty})).$$

In particular, by (1.2),

(4.4) \hspace{1cm} \delta A^c|_{\Omega}(X(V)) \leq -g(F(V, A^c_{\infty})), \quad \text{if } \Omega \in \mathcal{C}(M), F(|\partial \Omega|, V) \leq \rho(F(V, A^c_{\infty})).$

Next, we will construct a continuous time function $T : [0, \infty) \to [0, \infty)$, such that

(i) $\lim_{t \to 0} T(t) = 0$, and $T(t) > 0$ if $t \neq 0$;

(ii) For any $V \in A^L$, denote $\gamma = F(V, A^c_{\infty})$; then $V_t = (\Phi_V(t))_\# V \in U_{\rho(\gamma)}(V)$ for all $0 \leq t \leq T(\gamma)$.

In fact, given $V \in A_j$, and $\rho = \rho(F(V, A^c_{\infty})) > 0$, there exists $T_V > 0$, such that $V_t \in U_{\rho}(V)$ for all $0 \leq t \leq T_V$. Moreover, by the compactness of $A_j$ and the continuity of $\Phi_V(t)_\# V$ in $V$ and $t$, we may choose $T_V$ such that $T_V \geq T_j > 0$ for all $V \in A_j$, where $T_j$ depends only on $j$. Interpolating between the $T_j$ yields the desired continuous function $T$ depending only on $F(V, A^c_{\infty})$. 

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In summary, given \( V \in A^c \setminus A^c_\infty \), denote \( \gamma = F(V, A^c_\infty) > 0 \),
\[
\Psi_V(t, \cdot) = \Phi_V(T(\gamma)t, \cdot), \quad \text{for } t \in [0, 1],
\]
and \( L : \mathbb{R}^+ \to \mathbb{R}^+ \), with \( L(\gamma) = T(\gamma)g(\gamma) \); then \( L(0) = 0 \) and \( L(\gamma) > 0 \) if \( \gamma > 0 \). We can deform \( V \) through a continuous family \( \{ V_t = (\Psi_V(t)) : t \in [0, 1] \} \subset U_{\rho(\gamma)}(V) \), such that

\begin{enumerate}[(i)]
  
  \item The map \((t, V) \to V_t\) is continuous under the \( F \)-metric;
  
  \item Using (4.4), when \( V = \partial \Omega, \Omega \in \mathcal{C}(M), \gamma = F(\partial \Omega, A^c_\infty) > 0 \), we have

\[
\mathcal{A}^c(\Omega_1) - \mathcal{A}^c(\Omega) \leq \int_0^{T(\gamma)} [\delta \mathcal{A}^c|_{\Omega_1}](X(\partial \Omega)) \, dt \leq -T(\gamma)g(\gamma)
\]
\[
= -L(\gamma) < 0.
\]
\end{enumerate}

Finally note that the flow \( \Psi_V(t, \cdot) \) is generated by the vector field
\[
(4.6) \quad \tilde{X}(V) = T(\gamma)X(V).
\]

### 4.4. Deforming sweepouts by the tightening map.

Applying our tightening map constructed above in place of [94, §4.3] to a critical sequence provided by Lemma 3.7, we can deduce the following result.

**Proposition 4.4 (Tightening).** Let \( \Pi \in \pi^*_1(\mathcal{C}(M, M), \{0\}) \), and assume \( L^c(\Pi) > 0 \). For any critical sequence \( S^* \) for \( \Pi \), there exists another critical sequence \( S \) for \( \Pi \) such that \( C(S) \subset C(S^*) \) and each \( V \in C(S) \) has \( c \)-bounded first variation.

**Proof.** Take \( S^* = \{ \phi^*_i \} \), where \( \phi^*_i : I(1, k_i)_0 \to (\mathcal{C}(M), \{0\}) \), and \( \phi^*_i \) is \( 1 \)-homotopic to \( \phi^*_{i+1} \) in \( (\mathcal{C}(M), \{0\}) \) with fineness \( \delta_i \searrow 0 \). Let \( \Xi_i : I(1, k_i) \times [0, 1] \to \mathcal{C}(M) \) be defined as
\[
\Xi_i(x, t) = \Psi_{|_{\partial \phi^*_i(x)}}(t)(\phi^*_i(x)).
\]

Denote \( \phi^*_i(\cdot) = \Xi_i(\cdot, t) \). Heuristically, one would like to set \( \phi_i = \phi^*_i \) as the desired sequence, but since the isotopies \( \Psi_{|_{\partial \phi^*_i(x)}} \) depend on \( x \), the fineness of \( \{ \phi^*_i \} \) could be large even if \( \phi^*_i \) is small. Thus we need to interpolate \( \phi^*_i \) to get the desired \( \phi_i \), but we need to make sure
the values of \( \phi_i \) after interpolation are \( F \)-close to those of \( \phi_1^i \). Similar difficulties appeared in the same way in [86 §15]. The authors in [86] used a discrete-to-continuous interpolation argument. Unfortunately we cannot adapt their argument, since their constructions involve currents which may not be boundaries of Caccioppoli sets. Instead, we develop another interpolation method in Claim 2. Before that, we pause to prove:

**Claim 1:** if \( \lim_{i \to \infty} A^c(\phi_1^i(x_i)) = L^c(\Pi) \), then (up to relabelling) there is a subsequence \( \{\phi_1^i(x_i)\} \) converging (as varifolds) to a varifold in \( C(S^*) \) of \( c \)-bounded first variation.

**Proof of Claim 1:** By (4.5),

\[
(4.7) \quad A^c(\phi_1^i(x_i)) - A^c(\phi_1^*(x_i)) = -L(\gamma_i),
\]

where \( \gamma_i = F(|\partial \phi_1^*(x_i)|, A_{\infty}^c) \). Therefore,

\[
L^c(\Pi) = \lim A^c(\phi_1^i(x_i)) = \lim A^c(\phi_1^*(x_i)) - L(\lim \gamma_i) \leq L^c(\Pi) - L(\lim \gamma_i),
\]

so actually we must have \( \lim \gamma_i = 0 \) and this implies that \( \lim |\partial \phi_1^*(x_i)| \in A_{\infty}^c \). Moreover, by our construction of the tightening map, each \( |\partial \phi_1^i(x_i)| \) had to be \( \rho(\gamma_i) \)-close to \( |\partial \phi_1^*(x_i)| \) under the \( F \)-metric, therefore

\[
\lim |\partial \phi_1^i(x_i)| = \lim |\partial \phi_1^*(x_i)| \in A_{\infty}^c \cap C(S^*),
\]

and this finishes the proof of the claim.

**Claim 2:** there exist integers \( l_i > k_i \) and maps \( \phi_i : I(1,l_i)_0 \to (C(M), \{0\}) \) for each \( i \), such that \( S = \{\phi_i\} \) is homotopic to \( S^* \), and

(a) \( \phi_1^i = \phi_i \circ n(l_i, k_i) \) on \( I(1,k_i)_0 \);

(b) \( (\phi_i) \to 0, \) as \( i \to \infty \);

(c) \( A^c(\phi_i(x)) - \max \{A^c(\phi_1^i(y)) : \alpha \in I(1,k_i)_1, x, y \in \alpha \} \to 0, \) uniformly in \( x \in I(1,l_i)_0 \) as \( i \to \infty \).

(d) \( \max \{F(\partial \phi_i(x), \partial \phi_1^i(y)) : \alpha \in I(1,k_i)_1, x, y \in \alpha \} \to 0, \) as \( i \to \infty \).
Proof of Claim 2: The idea is to extend \( \phi_1^i \) to a piecewise continuous (with respect to the \( F \)-metric) map on \( I \) and then apply the discretisation result in [125, Theorem 5.1]. The technical details may be found in [126, Appendix B].

In particular, \( S \) is a valid sequence in \( \Pi \), and we now check that it satisfies the requirements of the proposition. First, property (c) and the fact that \( S^* \) is a critical sequence directly imply that \( S \) is also a critical sequence. It remains to show that every element in \( C(S) \) must lie in \( C(S^*) \) and have \( c \)-bounded first variation. Given \( V \in C(S) \), one can find a subsequence (without relabelling) \( \{ \phi_i(\pi_i) : \pi_i \in I(1, l_i)_o \} \subset C(M) \), such that \( V = \lim |\partial \phi_i(\pi_i)| \) as varifolds, and

\[
\lim A^c(\phi_i(\pi_i)) = L^c(\Pi).
\]

We will need to first consider \( \phi_i(x_i) = \phi_i^1(x_i) \), where \( x_i \) is the nearest point to \( \pi_i \) in \( I(1, k_i)_o \). By (c) and (d), we have \( \lim A^c(\phi_i^1(x_i)) = L^c(\Pi) \) and also \( \lim |\partial \phi_i(\pi_i)| = \lim |\partial \phi_i^1(x_i)| \) as varifolds. Then by Claim 1, we conclude that \( V \in A_{\infty}^c \cap C(S^*) \). This completes the proof.

5. \( c \)-Almost minimising

In this section, we introduce the notion of \( c \)-almost minimising varifolds, and prove the existence of such a varifold from min-max construction. We prove the existence of a \( c \)-replacement for any \( c \)-almost minimising varifold. Using this property, we show that every blowup of such varifold is regular. As an easy consequence, the tangent cones of such varifolds are always integer multiples of planes.

**Definition 5.1** (\( c \)-almost minimising varifolds). Let \( \nu \) be the \( F \), \( M \)-norms or the \( F \)-metric. For any given \( \epsilon, \delta > 0 \) and an open subset \( U \subset M \), we define \( A_n^c(U; \epsilon, \delta; \nu) \) to be the set of all \( \Omega \in C(M) \) such that if \( \Omega = \Omega_0, \Omega_1, \Omega_2, \ldots, \Omega_m \in C(M) \) is a sequence with:

(i) \( \text{spt}(\Omega_i - \Omega) \subset U \);
(ii) \( \nu(\partial \Omega_{i+1}, \partial \Omega_i) \leq \delta \);
(iii) \( A^c(\Omega_i) \leq A^c(\Omega) + \delta \), for \( i = 1, \ldots, m \),

\( A_{\infty}^c \cap C(S^*) \).
then $\mathcal{A}^c(\Omega_m) \geq \mathcal{A}^c(\Omega) - \epsilon$.

We say that a varifold $V \in \mathcal{V}_n(M)$ is $c$-almost minimising in $U$ if there exist sequences $\epsilon_i \to 0$, $\delta_i \to 0$, and $\Omega_i \in \mathcal{A}^c_n(U; \epsilon_i, \delta_i; \mathcal{F})$, such that $F(|\partial \Omega_i|, V) \leq \epsilon_i$.

The following simple fact says that $c$-almost minimising implies $c$-bounded first variation.

**Lemma 5.2.** Let $V \in \mathcal{V}_n(M)$ be $c$-almost minimising in $U$, then $V$ has $c$-bounded first variation in $U$.

**Proof.** Suppose by contradiction that $V$ does not have $c$-bounded first variation, then there exist $\epsilon_0 > 0$ and a smooth vector field $X \in \mathfrak{X}(U)$ compactly supported in $U$, such that

$$\left| \int_{G_n(M)} \text{div}_S X(x) dV(x, S) \right| \geq (c + \epsilon_0) \int_M |X| d\mu_V > 0.$$  

By changing the sign of $X$ if necessary, we have

$$\int_{G_n(M)} \text{div}_S X(x) dV(x, S) \leq -(c + \epsilon_0) \int_M |X| d\mu_V.$$  

By continuity, we can find $\epsilon_1 > 0$ small enough depending only on $\epsilon_0, V, X$, such that if $\Omega \in \mathcal{C}(M)$ with $F(|\partial \Omega|, V) < 2\epsilon_1$, then

$$\delta \mathcal{A}^c|\Omega(X) \leq \int_{\partial \Omega} \text{div}_{\partial \Omega} X d\mu_{\partial \Omega} + c \int_{\partial \Omega} |X| d\mu_{\partial \Omega} \leq -\frac{\epsilon_0}{2} \int_M |X| d\mu_V < 0.$$  

If $F(|\partial \Omega|, V) < \epsilon_1$, then by deforming $\Omega$ along the 1-parameter flow $\{\Phi^X(t) : t \in [0, \tau]\}$ of $X$ for a uniform short time $\tau > 0$, we can obtain a 1-parameter family $\{\Omega_t \in \mathcal{C}(M) : t \in [0, \tau]\}$, such that $t \to \partial \Omega_t$ is continuous under the $\mathcal{F}$-topology, with $\text{spt}(\Omega_t - \Omega) \subset U$, $F(|\partial \Omega_t|, V) < 2\epsilon_1$ and $\mathcal{A}^c(\Omega_t) \leq \mathcal{A}^c(\Omega_0) = \mathcal{A}^c(\Omega)$ for all $t \in [0, \tau)$, but with $\mathcal{A}^c(\Omega_t) \leq \mathcal{A}^c(\Omega) - \epsilon_2$ for some $\epsilon_2 > 0$ depending only on $\epsilon_0, \epsilon_1, V, X$.

Summarising the above, given any $\epsilon < \min\{\epsilon_1, \epsilon_2\}$ and $\delta > 0$, if $\Omega \in \mathcal{C}(M)$ and $F(|\partial \Omega|, V) < \epsilon$, then $\Omega \notin \mathcal{A}^c_n(U; \epsilon, \delta; \mathcal{F})$; this contradicts the $c$-almost minimising property of $V$.  

$\square$
We will need the following equivalence result among several almost minimising concepts using the three different topology. In particular, we can actually use the $M$-norm instead at the expense of shrinking the open subset $U \subset M$.

**Proposition 5.3.** Given $V \in \mathcal{V}_n(M)$, then the following statements satisfy $(a) \implies (b) \implies (c) \implies (d)$:

- $(a)$ $V$ is $c$-almost minimising in $U$;
- $(b)$ For any $\epsilon > 0$, there exists $\delta > 0$ and $\Omega \in \mathcal{A}_n^c(U; \epsilon, \delta; F)$ such that $F(V, |\partial \Omega|) < \epsilon$;
- $(c)$ For any $\epsilon > 0$, there exists $\delta > 0$ and $\Omega \in \mathcal{A}_n^c(U; \epsilon, \delta; M)$ such that $F(V, |\partial \Omega|) < \epsilon$;
- $(d)$ $V$ is $c$-almost minimising in $W$ for any relatively open subset $W \subset U$.

**Remark 5.4.** The proof was originally due to Pitts [94, Theorem 3.9]. In our context, we work with boundaries instead of general integral currents. Furthermore, in Definition 5.1(iii), we use the $A^c$ functional instead of the mass $M$.

**Proof.** It is easy to see $(a) \implies (b) \implies (c)$. The last implication $(c) \implies (d)$ is an interpolation process which was originally established in Pitts [94, Proposition 3.8] using integral cycles. The corresponding interpolation process using boundaries of Caccioppoli sets was obtained by the first author in [125, Proposition 5.3]. The detailed description of this process may be found in [126, Appendix A].

**Definition 5.5.** A varifold $V \in \mathcal{V}_n(M)$ is said to be $c$-almost minimising in small annuli if for each $p \in M$, there exists $r_{am}(p) > 0$ such that $V$ is $c$-almost minimising in $A_{s,r}(p) \cap M$ for all $0 < s < r \leq r_{am}(p)$, where $A_{s,r}(p) = B_r(p) \setminus B_s(p)$.

**Theorem 5.6 (Existence of $c$-almost minimising varifold).** Let $\Pi \in \pi_1^c(\mathcal{C}(M, M), \{0\})$, and assume that $L^c(\Pi) > 0$. There exists a nontrivial $V \in \mathcal{V}_n(M)$, such that

- $(i)$ $V \in C(S)$ for some critical sequence $S$ of $\Pi$;
- $(ii)$ $V$ has $c$-bounded first variation;
- $(iii)$ $V$ is $c$-almost minimising in small annuli.

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Proof. First we can pick a critical sequence $S$ of II which has been pulled-tight by Proposition 4.4, so that every $V \in C(S)$ has $c$-bounded first variation. Suppose for the sake of contradiction that for each $V \in C(S)$, there exists a $p \in M$, such that there are arbitrarily small annuli centred at $p$ on which $V$ is not $c$-almost minimising. Then by Proposition 5.3, $V$ is also not $c$-almost minimising with respect to the mass norm on these annuli (i.e. $\nu = M$).

Specifically, for any $\tilde{r} > 0$, there exists $r, s > 0$ with $\tilde{r} > r + 2s > r - 2s > 0$, and $\epsilon > 0$, such that for any $\delta > 0$, and $\Omega \in C(M)$ with $\mathbf{F}(|\partial \Omega|, V) < \epsilon$, then $\Omega \notin \mathcal{A}_m^c(A_{r - 2s, r + 2s}(p) \cap M; \epsilon, \delta; M)$. Now using the same argument as in [94, 4.10] by changing the mass functional $M$ to the $A^c$-functional, one can construct a new 1-homotopic sequence $\tilde{S}$ which is homotopic to $S$, and $\mathbf{L}^c(\tilde{S}) < \mathbf{L}^c(S)$; but this contradicts the criticality of $S$. \hfill \Box

Now we formulate and solve a natural constrained minimisation problem which will be used in the construction of $c$-replacements.

**Lemma 5.7** (A constrained minimisation problem). Given $\epsilon, \delta > 0$, $U \subset M$ and any $\Omega \in \mathcal{A}_n^c(U; \epsilon, \delta; \mathcal{F})$, fix a compact subset $K \subset U$. Let $C_\Omega$ be the set of all $\Lambda \in C(M)$ such that there exists a sequence $\Omega = \Omega_0, \Omega_1, \cdots, \Omega_m = \Lambda$ in $C(M)$ satisfying:

(a) $\text{spt}(\Omega_i - \Omega) \subset K$;

(b) $\mathcal{F}(\partial \Omega_i - \partial \Omega_{i+1}) \leq \delta$;

(c) $\mathcal{A}^c(\Omega_i) \leq \mathcal{A}^c(\Omega) + \delta$, for $i = 1, \cdots, m$.

Then there exists $\Omega^* \in C(M)$ such that:

(i) $\Omega^* \in C_\Omega$, and

\[ \mathcal{A}^c(\Omega^*) = \inf \{ \mathcal{A}^c(\Lambda) : \Lambda \in C_\Omega \}, \]

(ii) $\Omega^*$ is locally $\mathcal{A}^c$-minimising in $\text{int}(K)$,

(iii) $\Omega^* \in \mathcal{A}_n^c(U; \epsilon, \delta; \mathcal{F})$.

**Proof.** Proof of (i): Take any minimising sequence $\{\Lambda_j\} \subset C_\Omega$, i.e.

\[ \lim_{j \to \infty} \mathcal{A}^c(\Lambda_j) = \inf \{ \mathcal{A}^c(\Lambda) : \Lambda \in C_\Omega \}. \]
Notice that \( \text{spt}(\Lambda_j - \Omega) \subset K \) and \( A^c(\Lambda_j) \leq A^c(\Omega) + \delta \) for all \( j \). By standard compactness \([105]\) Theorem 6.3, after passing to a subsequence, \( \partial \Lambda_j \) converges weakly to some \( \partial \Omega^* \) with \( \Omega^* \in \mathcal{C}(M) \) and \( \text{spt}(\Omega^* - \Omega) \subset K \). We will show that \( \Omega^* \) is our desired minimiser. Since \( \partial \Lambda_j \) converges weakly to \( \partial \Omega^* \), we have that \( H^n(\partial \Omega^*) \leq \lim_{j \to \infty} H^n(\partial \Lambda_j) \) and \( H^{n+1}(\Omega^*) = \lim_{j \to \infty} H^{n+1}(\Lambda_j) \). Therefore,

\[
(5.1) \quad A^c(\Omega^*) \leq \inf \{ A^c(\Lambda) : \Lambda \in \mathcal{C}_\Omega \}.
\]

It remains to show that \( \Omega^* \in \mathcal{C}_\Omega \). For \( j \) sufficiently large, we have \( \mathcal{F}(\partial \Lambda_j - \partial \Omega^*) < \delta \). Since \( \Lambda_j \in \mathcal{C}_\Omega \), there exists a sequence \( \Omega = \Omega_0, \Omega_1, \cdots, \Omega_m = \Lambda_j \) in \( \mathcal{C}(M) \) satisfying conditions (a-c) above. Consider now the sequence \( \Omega = \Omega_0, \Omega_1, \cdots, \Omega_m = \Lambda_j, \Omega_{m+1} = \Omega^* \) in \( \mathcal{C}(M) \); it trivially satisfies conditions (a) and (b). Moreover, using (5.1), we also have

\[
A^c(\Omega^*) \leq A^c(\Lambda_j) \leq A^c(\Omega) + \delta.
\]

Therefore, \( \Omega^* \in \mathcal{C}_\Omega \) and hence (i) has been proved.

Proof of (ii): For \( p \in \text{int}(K) \), we claim that there exists a small \( \tilde{B}_r(p) \subset \text{int}(K) \) such that

\[
(5.2) \quad A^c(\Omega^*) \leq A^c(\Lambda),
\]

for any \( \Lambda \in \mathcal{C}(M) \) with \( \text{spt}(\Lambda - \Omega^*) \subset \tilde{B}_r(p) \). To establish (5.2), first choose \( r > 0 \) small so that \( c \cdot \text{Vol}(\tilde{B}_r(p)) < \delta/4 \) and \( M(\partial \Omega^* \cap \tilde{B}_r(p)) < \delta/4 \) (this is possible since \( \partial \Omega^* \) is rectifiable). Suppose (5.2) were false, then there exists \( \Omega' \in \mathcal{C}(M) \) with \( \text{spt}(\Omega' - \Omega^*) \subset \tilde{B}_r(p) \) such that \( A^c(\Omega') < A^c(\Omega^*) \). We will show that \( \Omega' \in \mathcal{C}_\Omega \), which contradicts that \( \Omega^* \) is a minimiser from part (i).

To see that \( \Omega' \in \mathcal{C}_\Omega \), take a sequence \( \Omega = \Omega_0, \Omega_1, \cdots, \Omega_m = \Omega^* \) in \( \mathcal{C}(M) \) satisfying (a-c) above, and append \( \Omega_{m+1} = \Omega' \) to the sequence. Since \( \text{spt}(\Omega^* - \Omega) \subset K \) and \( \text{spt}(\Omega' - \Omega^*) \subset K \), we have \( \text{spt}(\Omega' - \Omega) \subset K \). By the facts that \( \text{spt}(\Omega' - \Omega^*) \subset \tilde{B}_r(p) \) and \( A^c(\Omega') < A^c(\Omega^*) \), we
have
\[ M(\partial \Omega' \setminus \bar{B}_r(p)) \leq M(\partial \Omega' \setminus \bar{B}_r(p)) + c[\mathcal{H}^{n+1}(\Omega' \cap \bar{B}_r(p)) + \mathcal{H}^{n+1}(\Omega' \cap \bar{B}_r(p))] \]

\[ \leq M(\partial \Omega' \setminus \bar{B}_r(p)) + 2c \text{Vol}(\bar{B}_r(p)); \]

hence
\[ M(\partial \Omega' - \partial \Omega^*) \leq M(\partial \Omega' \setminus \bar{B}_r(p)) + M(\partial \Omega^* \setminus \bar{B}_r(p)) < \delta. \]

So \( F(\partial \Omega' - \partial \Omega^*) \leq M(\partial \Omega' - \partial \Omega^*) < \delta. \) Finally note \( \mathcal{A}^c(\Omega') < \mathcal{A}^c(\Omega^*) \leq \mathcal{A}^c(\Omega) + \delta. \) Therefore \( \Omega' \in \mathcal{C}_\Omega, \) and this proves part (ii).

**Proof of (iii):** Suppose that the claim is false. Then by Definition 5.1 there exists a sequence \( \Omega^* = \Omega_0^*, \Omega_1^*, \ldots, \Omega_\ell^* \) in \( \mathcal{C}(M) \) satisfying
- \( \text{spt}(\Omega_i^* - \Omega^*) \subseteq U; \)
- \( F(\partial \Omega_i^* - \partial \Omega_{i+1}^*) \leq \delta; \)
- \( \mathcal{A}^c(\Omega_i^*) \leq \mathcal{A}^c(\Omega^*) + \delta, \) for \( i = 1, \ldots, \ell, \)

but \( \mathcal{A}^c(\Omega_\ell^*) < \mathcal{A}^c(\Omega^*) - \varepsilon. \)

Since \( \Omega^* \in \mathcal{C}_\Omega \) by part (i), there exists a sequence \( \Omega = \Omega_0, \Omega_1, \ldots, \Omega_m = \Omega^* \) satisfying conditions (a-c) above. Then the sequence \( \Omega = \Omega_0, \Omega_1, \ldots, \Omega_m, \Omega_1^*, \ldots, \Omega_\ell^* \) in \( \mathcal{C}(M) \) still satisfies those conditions (a-c), since \( \mathcal{A}^c(\Omega^*) \leq \mathcal{A}^c(\Omega) \) implies that \( \mathcal{A}^c(\Omega_i^*) \leq \mathcal{A}^c(\Omega) + \delta. \) Therefore \( \Omega \in \mathcal{A}^c(U; \varepsilon, \delta; \mathcal{F}) \) implies that \( \mathcal{A}^c(\Omega_\ell^*) \geq \mathcal{A}^c(\Omega) - \varepsilon \geq \mathcal{A}^c(\Omega^*) - \varepsilon, \) which is a contradiction. This proves part (iii). \( \square \)

**Proposition 5.8 (Existence and properties of replacements).** Let \( V \in \mathcal{V}_n(M) \) be \( c \)-almost minimising in an open set \( U \subset M \) and \( K \subset U \) be a compact subset, then there exists \( V^* \in \mathcal{V}_n(M), \) called a \( c \)-replacement of \( V \) in \( K \) such that

(i) \( V \mathcal{L}(M \setminus K) = V^* \mathcal{L}(M \setminus K); \)

(ii) \( -c \text{Vol}(K) \leq \|V\|(M) - \|V^*\|(M) \leq c \text{Vol}(K); \)

(iii) \( V^* \) is \( c \)-almost minimising in \( U; \)

(iv) moreover, \( V^* = \lim_{i \to \infty} |\partial \Omega_i^*| \) as varifolds for some \( \Omega_i^* \in \mathcal{C}(M) \) such that \( \Omega_i^* \in \mathcal{A}^c(U; \varepsilon_i, \delta_i; \mathcal{F}) \) with \( \varepsilon_i, \delta_i \to 0; \) furthermore \( \Omega_i^* \) locally minimises \( \mathcal{A}^c \) in \( \text{int}(K); \)
(v) if $V$ has $c$-bounded first variation in $M$, then so does $V^\star$.

**Proof.** Let $V \in \mathcal{V}_n(M)$ be $c$-almost minimising in $U$. By definition there exists a sequence $\Omega_i \in \mathcal{A}_n^c(U; \epsilon_i, \delta_i; \mathcal{F})$ with $\epsilon_i, \delta_i \to 0$ such that $V$ is the varifold limit of $|\partial \Omega_i|$. By Lemma 5.7 we can construct a $c$-minimiser $\Omega_i^\star \in \mathcal{C}_\Omega_i$ for each $i$. Since $\mathcal{M}(\partial \Omega_i^\star)$ is uniformly bounded, by compactness there exists a subsequence $\partial \Omega_i^\star$ converging as varifolds to some $V^\star \in \mathcal{V}_n(M)$. We claim that $V^\star$ satisfies items (i)-(v) in Proposition 5.8 and thus is our desired $c$-replacement.

- First, by part (i) of Lemma 5.7 we have $\Omega_i^\star \in \mathcal{C}_\Omega_i$ and thus $\text{spt}(\Omega_i^\star - \Omega_i) \subset K$. Hence the varifold limits satisfy $V^\star \mathcal{L} (M \setminus K) = V \mathcal{L} (M \setminus K)$.
- Second, as $\Omega_i \in \mathcal{A}_n^c(U; \epsilon_i, \delta_i; \mathcal{F})$ and $\Omega_i^\star \in \mathcal{C}_\Omega_i$, we have
  $$\mathcal{A}^c(\Omega_i) - \epsilon_i \leq \mathcal{A}^c(\Omega_i^\star) \leq \mathcal{A}^c(\Omega_i)$$
  thus by (1.1),
  $$\mathcal{M}(\partial \Omega_i) - c\mathcal{H}^{n+1}(\Omega_i) - \epsilon_i \leq \mathcal{M}(\partial \Omega_i^\star) - c\mathcal{H}^{n+1}(\Omega_i^\star) \leq \mathcal{M}(\partial \Omega_i) - c\mathcal{H}^{n+1}(\Omega_i).$$
  Note that $|\mathcal{H}^{n+1}(\Omega_i) - \mathcal{H}^{n+1}(\Omega_i^\star)| \leq \text{Vol}(K)$; taking $i \to \infty$, we have $-c \text{Vol}(K) \leq \|V\|(M) - \|V^\star\|(M) \leq c\text{vol}(K)$.
- Since each $\Omega_i^\star \in \mathcal{A}_n^c(U; \epsilon_i, \delta_i; \mathcal{F})$ by Lemma 5.7(iii), by definition $V^\star$ is $c$-almost minimising in $U$.
- (iv) follows from Lemma 5.7(ii).
- Finally by (iii) and Lemma 5.2 $V^\star$ has $c$-bounded first variation in $U$. By (i) and a standard cutoff trick it is easy to show that $V^\star$ has $c$-bounded first variation in $M$ whenever $V$ does.

\[\square\]

**Lemma 5.9 (Regularity of $c$-replacement).** Let $2 \leq n \leq 6$. Under the same hypotheses as Proposition 5.8, if $\Sigma = \text{spt} \|V^\star\| \cap \text{int}(K)$, then

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(1) \( \Sigma \) is a smooth, almost embedded, stable \( c \)-boundary;

(2) the density of \( V^* \) is 1 along \( R(\Sigma) \) and 2 along \( S(\Sigma) \);

(3) the restriction of the \( c \)-replacement \( V^*|_{\text{int}(K)} = \Sigma \).

**Proof.** By the regularity for local minimisers of the \( A^c \) functional (Theorem 2.14), we know that each \( \partial \Omega_i^c \) is a smooth, embedded, stable \( c \)-boundary in \( \text{int}(K) \) by Proposition 5.8(iv). The lemma then follows from the compactness Theorem 2.11.
subsequences we have
\[ V' = \lim_{i \to \infty} V'_{i}, \] for some varifold \( V' \) in \( T_{p}M \).

Moreover, we can deduce the following for \( V' \):

- Proposition 5.8(i) \( \implies \) \( V'_{i} \subset (W \setminus \overline{A_{n}}) \Rightarrow V_{i} \subset (W \setminus \overline{A_{n}}) \), hence
  \[ V' \subset (W \setminus \overline{A_{n}}) = V \subset (W \setminus \overline{A_{n}}); \]

- Proposition 5.8(ii) \( \implies \) \(- c \text{Vol}(\overline{A_{n_{i}}}) \cdot r_{i}^{-n} \leq \|V'_{i}\|(W) - \|V_{i}\|(W) \leq c \text{Vol}(\overline{A_{n_{i}}}) \cdot r_{i}^{-n}.\)

Since the outer radius of \( \overline{A_{n_{i}}} \) is at most \( r_{i} \), for large \( i \) there exists some \( C_{0} > 0 \) depending only on \( M \) such that \( \text{Vol}(\overline{A_{n_{i}}}) \leq C_{0}r_{i}^{n+1} \). This implies that
  \[ \|V'\|(W) = \|V\|(W); \]

- Proposition 5.8(iii) \( \implies \) \( V'_{i} \) has \( c \)-bounded first variation in \( U \), hence \( V' \) is stationary in \( W \);

- By Lemma 5.9 the restriction \( V'_{i} \subset A_{n_{i}} \) is a smooth, almost embedded, stable \( c \)-boundary \( \Sigma_{i}^{*} \). Consider the rescalings: \( \overline{\Sigma_{i}} = \eta_{p, r_{i}}(\Sigma_{i}^{*}) \subset A_{n} \). By Proposition 5.8(ii) and the monotonicity formula [105 40.2], \( \overline{\Sigma_{i}} \) have uniformly bounded mass.

This together with the compactness Theorem 2.11(iii) implies that \( V' \subset A_{n} \) is an embedded stable minimal hypersurface.

Therefore, \( V' \) is a good replacement of \( V \) in \( A_{n} \). By Proposition 5.8(iii), each \( V'_{i} \) is still \( c \)-almost minimising in \( U \). Hence for any other annulus \( A_{n'} \subset W \) of outer radius \( \leq 1 \), we can repeat the above process and produce a good replacement \( V'' \) of \( V' \) in \( A_{n'} \). In fact, we may repeat this process any finite number of times. In particular, \( V \) satisfies the good replacement property (Definition 2.17) in \( W \), and this completes the proof. \( \square \)

**Proposition 5.11 (Tangent cones are planes).** Let \( 2 \leq n \leq 6 \). Suppose \( V \in \mathcal{V}_{n}(M) \) has \( c \)-bounded first variation in \( M \) and is \( c \)-almost minimising in small annuli. Then \( V \) is
integer rectifiable. Moreover, for any $C \in \text{VarTan}(V, p)$ with $p \in \text{spt} \|V\|$, 

$$C = \Theta^n(\|V\|, p)S$$

for some $n$-plane $S \subset T_pM$ where $\Theta^n(\|V\|, p) \in \mathbb{N}$.

**Proof.** Let $r_i \to 0$ be a sequence such that $C$ is the varifold limit:

$$C = \lim_{i \to \infty} (\eta_{p, r_i})_\# V.$$ 

First we know $C$ is stationary in $T_pM$. Since $V$ is $c$-almost minimising in small annuli centred at $p$, by the same argument as in Lemma 5.10 we can show that $C$ satisfies the good replacement property (Definition 2.17) in $T_pM$. (Note that Definition 2.17 only requires the existence of good replacements in small annuli.) Therefore, by Proposition 2.18 $C$ is an integer multiple of some embedded minimal hypersurface of $T_pM$, and moreover, it is a cone by [105, 19.3]. In particular $C$ is smooth and hence $\text{spt} \|C\|$ must be a plane. This finishes the proof. 

6. Regularity for $c$-min-max varifold 

In this section, we prove the regularity of our min-max varifolds. In particular we prove that every varifold which has $c$-bounded variation and is $c$-almost minimising in small annuli is a smooth, closed, almost embedded, CMC hypersurface with multiplicity one.

**Theorem 6.1 (Main regularity).** Let $2 \leq n \leq 6$, and $(M^{n+1}, g)$ be an $(n+1)$-dimensional smooth, closed Riemannian manifold. Suppose $V \in \mathcal{V}_n(M)$ is a varifold which

1. has $c$-bounded first variation in $M$ and
2. is $c$-almost minimising in small annuli,

then $V$ is induced by $\Sigma$, where

1. $\Sigma$ is a closed, almost embedded $c$-hypersurface (possibly disconnected); 
2. the density of $V$ is exactly 1 at the regular set $\mathcal{R}(\Sigma)$ and 2 at the touching set $\mathcal{S}(\Sigma)$.
Proof. The conclusion is purely local, so we only need to prove the regularity of $V$ near an arbitrary point $p \in \text{spt} \|V\|$. Fix a $p \in \text{spt} \|V\|$, then there exists $0 < r_0 < r_{am}(p)$ such that for any $r < r_0$, the mean curvature $H$ of $\partial B_r(p) \cap M$ in $M$ is greater than $c$. Here $r_{am}(p)$ is as in Definition 5.5

In particular, if $r < r_0$ and $W \in \mathbb{V}_n(M)$ has $c$-bounded first variation in $B_r(p) \cap M$ and $W \neq 0$ in $B_r(p)$, then by the maximum principle (Proposition 2.13)

(6.1) \[ \emptyset \neq \text{spt} \|W\| \cap \partial B_r(p) = \text{Clos} (\text{spt} \|W\| \setminus \text{Clos}(B_r(p))) \cap \partial B_r(p). \]

Note that in the second equality we need a localised version of Proposition 2.13 which holds true by the remark after [119, Theorem 2].

We will show that $V \subseteq B_{r_0}(p)$ is an almost embedded hypersurface of constant mean curvature $c$ with density equal to 2 along its touching set. The argument consists of five steps:

Step 1: Constructing successive $c$-replacements $V^*$ and $V^{**}$ on two overlapping concentric annuli.

Step 2: Gluing the $c$-replacements smoothly (as immersed hypersurfaces) on the overlap.

Step 3: Extending the $c$-replacements down to the point $p$ to get a $c$-‘replacement’ $\tilde{V}$ on the punctured ball.

Step 4: Showing that the singularity of $\tilde{V}$ at $p$ is removable, so that $\tilde{V}$ is regular.

Step 5: $V$ coincides with the almost embedded hypersurface $\tilde{V}$ on a small neighbourhood of $p$.

We now proceed to the proof.

Step 1. We first describe the construction of $c$-replacements on overlapping annuli; a key property will be that the replacements are also boundaries in the chosen annulus (see Claim 1).
Fix $0 < s < t < r_0$. By the choice of $r_0$, we can apply Proposition 5.8 to $V$ to obtain a $c$-replacement $V^*$ in $K = \text{Clos}(A_{s,t}(p) \cap M)$. By (6.1) and Lemma 5.9, the restriction
\[ \Sigma_1 = V^* \mathbb{L}(A_{s,t}(p) \cap M) \]
is a nontrivial, smooth, almost embedded, stable $c$-boundary with outer unit normal $\nu_1$.

By Proposition 2.9 the touching set $\mathcal{S}(\Sigma_1)$ is contained in a countable union of $(n - 1)$-dimensional connected submanifolds $\bigcup S_1^{(k)}$. Since a countable union of sets of measure zero still has measure zero, it follows from Sard’s theorem that we may choose $s_2 \in (s, t)$ such that $\partial(B_{s_2}(p) \cap M)$ intersects $\Sigma_1$ and all the $S_1^{(k)}$ transversally.

Given any $s_1 \in (0, s)$, by Proposition 5.8(iii), we can apply Proposition 5.8 again to get a $c$-replacement $V^{**}$ of $V^*$ in $K = \text{Clos}(A_{s_1,s_2}(p) \cap M)$. By (6.1) and Lemma 5.9 again, the restriction
\[ \Sigma_2 = V^{**} \mathbb{L}(A_{s_1,s_2}(p) \cap M) \]
is also a nontrivial, smooth, almost embedded, stable $c$-boundary with outer unit normal $\nu_2$.

Note that by Proposition 5.8(v), both $V^*$ and $V^{**}$ have $c$-bounded first variation.

We can choose the second $c$-replacement $V^{**}$ so that it satisfies:

**Claim 1:** there exists a set $\Omega^{**} \in \mathcal{C}(M)$, such that

a) $\Sigma_1 \cap A_{s_2,t}(p)$ and $\Sigma_2$ are the boundaries of $\Omega^{**}$ in $A_{s_2,t}(p)$ and $A_{s_1,s_2}(p)$ respectively;

b) $\nu_1, \nu_2$ coincide with the outer unit normal of $\Omega^{**}$ in $A_{s_2,t}(p)$ and $A_{s_1,s_2}(p)$ respectively;

c) if $\|V^{**}\|_{\partial B_{s_2}(p)} = 0$, then $V^{**}$ is identical to $|\partial \Omega^{**}|$ in $A_{s_1,s_2}(p) \cap M$.

**Proof of Claim 1:** Fix $0 < \tau < s_1$, then by Proposition 5.8(iv), $V^* = \lim_{i \to \infty} |\partial \Omega_i^*|$ as varifolds for some $\Omega_i^* \in \mathcal{C}(M)$, where $\Omega_i^* \in \mathcal{C}(A_{\tau,r_0}(p) \cap M; \epsilon_i, \delta_i; \mathcal{F})$ with $\epsilon_i, \delta_i \to 0$. The regularity of the $\partial \Omega_i^*$ as in the proof of Lemma 5.9 together with the compactness Theorem 2.11 imply that $\partial \Omega_i^* \mathbb{L} A_{s,t}(p)$ converges locally smoothly to $\Sigma_1$ in $A_{s,t}(p) \cap M$.

Applying Lemma 5.7 for each $\Omega_i^*$ in $K = \text{Clos}(A_{s_1,s_2}(p) \cap M)$, we can construct new $c$-minimisers $\Omega_i^{**}$, such that
• $\text{spt}(\Omega_i^* - \Omega_i^{**}) \subset K = \text{Clos}(A_{s_1,s_2}(p) \cap M)$;
• $\Omega_i^{**}$ is locally $\mathcal{A}$-minimising in $A_{s_1,s_2}(p) \cap M$;
• $V^{**} = \lim |\partial \Omega_i^{**}|$ as varifolds;
• $\partial \Omega_i^{**} \cap A_{s_1,s_2}(p)$ converges locally smoothly to $\Sigma_2$ (again as in the proof of Lemma 5.9).

By the weak compactness [105] Theorem 6.3, up to a subsequence, $\partial \Omega_i^{**}$ converges weakly as currents to some $\partial \Omega^{**}$ with $\Omega^{**} \in \mathcal{C}(M)$. Claims (a) and (b) follow from the locally smooth convergence. The weak convergence implies that $\|\partial \Omega^{**}(A_{s_1,t}(p))\| \leq \|V^{**}(A_{s_1,t}(p))\|$. If $\|V^{**}(\partial B_{s_2}(p))\| = 0$, then together with the locally smooth convergence, we have $\|\partial \Omega^{**}(A_{s_1,t}(p))\| = \|V^{**}(A_{s_1,t}(p))\|$; moreover, $V^{**} \cap (A_{s_1,t}(p) \cap M) = |\partial \Omega^{**}| \cap (A_{s_1,t}(p) \cap M)$ by [94] 2.1(18)(f). This confirms (c). \hfill \Box

**Step 2.** We now show that $\Sigma_1$ and $\Sigma_2$ glue smoothly (as immersed hypersurfaces) across $\partial(B_{s_2}(p) \cap M)$. Indeed, define the intersection set

\begin{equation}
(6.2) \quad \Gamma = \Sigma_1 \cap \partial(B_{s_2}(p) \cap M), \quad S(\Gamma) = \Gamma \cap S(\Sigma_1).
\end{equation}

Then by transversality, $\Gamma$ is an almost embedded hypersurface in $\partial(B_{s_2}(p) \cap M)$, and $S(\Gamma)$ is its touching set. Notice that

\begin{equation}
(6.3) \quad S(\Gamma) \text{ is closed, and } R(\Gamma) = \Gamma \setminus S(\Gamma) \text{ is open in } \Gamma.
\end{equation}

It follows from the maximum principle that

$$\text{Clos}(\Sigma_2) \cap \partial(B_{s_2}(p) \cap M) \subset \Gamma.$$  

Indeed, (6.1) implies that any $y \in \text{Clos}(\Sigma_2) \cap \partial(B_{s_2}(p) \cap M)$ is also a limit point of $\text{spt} \|V^{**}\|$ from the outer side of $\partial B_{s_2}(p)$, on which $V^{**}$ coincides with $\Sigma_1$. In fact, with a little more work we have
Claim 2: \( \text{Clos}(\Sigma_2) \cap \partial(B_{s_2}(p) \cap M) = \Gamma \), and then \( \Sigma_1 \) glues together continuously with \( \Sigma_2 \).

Proof of Claim 2: By Proposition 5.8(i), we have

\[
V^* = V^{**} = \Sigma_1, \quad \text{in } A_{s_2,t}(p) \cap M.
\]

Given any \( x \in \Gamma \), using (6.4), Proposition 5.11 and the fact that \( \Sigma_1 \) meets \( \partial(B_{s_2}(p) \cap M) \) transversally, we have

\[
\text{VarTan}(V^{**}, x) = \{ \Theta^n(\|V^*\|, x) | T_x \Sigma_1 | \}.
\]

This implies that \( x \) is a limit point of \( \text{spt} \|V^{**}\| \) from inside of \( \partial B_{s_2}(p) \), and thus completes the proof of the claim.

As a direct corollary of (6.5), Theorem 3.2(2) and Claim 1(c), we have

\[
\|V^{**}\|(\partial B_{s_2}(p)) = 0, \quad \text{and hence } V^{**} = |\partial \Omega^{**}| \text{ in } A_{s_1,t}(p) \cap M.
\]

Furthermore, we will show that \( \Sigma_1 \) glues with \( \Sigma_2 \) in \( C^1 \), i.e. the tangent spaces of \( \Sigma_1 \) and \( \Sigma_2 \) agree along \( \Gamma \), with matching normals. Take an arbitrary \( q \in \Gamma \). We will need to divide to two sub-cases:

Sub-case (A): \( q \) is a regular point of \( \Sigma_1 \), i.e. \( q \in \mathcal{R}(\Gamma) \).

First we have the following.

Claim 3(A): Fix \( x \in \mathcal{R}(\Gamma) \), for any sequence of \( x_i \to x \) with \( x_i \in \mathcal{R}(\Gamma) \) and \( r_i \to 0 \), we have

\[
\lim_{i \to \infty} (\eta_{x_i,r_i})_{\#} V^{**} = T_x \Sigma_1 \quad \text{as varifolds}.
\]
Proof of Claim 3(A): By the weak compactness of Radon measures, after passing to a subsequence,
\[
\lim_{i \to \infty} (\nu_{x_i, r_i})_{x} V^{**} = C \in \mathcal{V}_n(T_x M).
\]

By Lemma 5.10, \( C \) is a regular, proper, complete minimal hypersurface in \( T_x M \). By (6.4), \( C \) coincides with \( T_x \Sigma_1 \) on a half space of \( T_x M \). The classical maximum principle implies that \( C \supset T_x \Sigma_1 \). It then follows from the half space theorem for minimal hypersurfaces [64, Theorem 3] that there are no other connected components of \( C \). Thus \( C = T_x \Sigma_1 \) and the proof is complete.

Since \( \{(\nu_{x_i, r_i})_{x} V^{**} : i \in \mathbb{N}\} \) have uniformly bounded first variation, a standard argument using the monotonicity formula implies that
\[
\text{spt} \|(\nu_{x_i, r_i})_{x} V^{**}\| \to T_x \Sigma_1 \text{ in the Hausdorff topology.}
\]

To show that \( \Sigma_1 \) and \( \Sigma_2 \) glue together along \( \Gamma \) in \( C^1 \) near \( q \). We need to show that:

Claim 4(A): For each \( x \in \mathcal{R}(\Gamma) \), we have
\[
\lim_{z \to x, z \in \Sigma_2} \nu_2(z) = \nu_1(x).
\]

Moreover, the convergence is uniform in \( x \) on compact subsets of \( \Gamma \) near \( q \).

Proof of Claim 4(A): The uniformity follows from the fact that \( \nu_1 \) is continuous on \( \Gamma \), so we only need to establish the convergence to \( \nu_1 \).

So consider a sequence \( z_i \in \Sigma_2 \) converging to some \( x \in \mathcal{R}(\Gamma) \). Since \( x \) is a regular point of \( \Sigma_1 \), by Claim 3(A) and the upper semi-continuity of density function for varifolds with bounded first variation [105, 17.8], we know that \( z_i \) is also a regular point of \( \Sigma_2 \) for \( i \) large enough.

Take \( x_i \in \Gamma \) to be the nearest point projection (in \( \mathbb{R}^L \)) of \( z_i \) to \( \Gamma \) and \( r_i = |z_i - x_i| \). Note that \( x_i \to x \in \mathcal{R}(\Gamma) \) and \( r_i \to 0 \), so we are in the situation of Claim 3(A). Note that
\( \Sigma_2 \cap B_{\epsilon_2}(z_i) \) is an embedded, stable \( c \)-hypersurface in \( M \), so by Theorem 2.11 a subsequence of the blow-ups \( \eta_{x_i,r_i}(\Sigma_2 \cap B_{\epsilon_2}(z_i)) \) converges smoothly to a smooth, embedded, stable, minimal hypersurface \( \Sigma_\infty \) contained in a half-space of \( T_x M \).

On the other hand, Claim 3(A) and (6.8) imply that \( \eta_{x_i,r_i}(\Sigma_2 \cap B_{\epsilon_2}(z_i)) \) converges in the Hausdorff topology to a domain in \( T_x \Sigma_1 \). Therefore, we have \( \Sigma_\infty \subset T_x \Sigma_1 \). The smooth convergence then implies that \( \nu_2(z_i) \) converges to one of the unit normals \( \pm \nu_1(x) \) of \( T_x \Sigma_1 \). By Claim 1 and (6.6), we know that
\[
V^{**} = |\partial \Omega^{**}| \text{ in } A_{s_{1,t}(p)} \cap M \text{ and } \|\partial \Omega^{**}(\partial B_{s_{2}(p)})\| = 0,
\]
therefore the limit of the \( \nu_2(z_i) \) must be \( \nu_1(x) \), so Claim 4(A) is proved.

Thus we have proven that near any regular point \( q \in R(\Gamma) \), \( \Sigma_1 \) and \( \Sigma_2 \) glue together along \( \Gamma \) as a \( C^1 \) hypersurface with matching outer unit normals \( \nu_1, \nu_2 \). The higher regularity follows from a standard elliptic PDE argument. More precisely, \( \Sigma_1 \) and \( \Sigma_2 \) can be written as graphs of some functions \( u_1, u_2 \) over \( T_q \Sigma_1 \) respectively. Since the mean curvatures of both \( \Sigma_1 \) and \( \Sigma_2 \) are identical to \( c > 0 \) with respect to some unit normals pointing to the same side of \( T_q \Sigma_1 \), they satisfy the same mean curvature type elliptic PDE with in-homogenous term equal to \( c \). The higher regularity follows from the elliptic regularity of this PDE. This finishes **Sub-case (A)**.

At this point we have proven that \( \Sigma_2 \) glues smoothly with \( \Sigma_1 \cap A_{s_{2,t}(p)} \) along \( R(\Gamma) \). Moreover, by the unique continuation for elliptic PDE, we know that \( \Sigma_2 \) is identical to \( \Sigma_1 \) in a neighbourhood of \( R(\Gamma) \) in \( A_{s_{2,t}(p)} \cap M \). We will need to show that the smooth gluing extends to the touching set \( S(\Gamma) \).

**Sub-case (B):** \( q \) is a touching point of \( \Sigma_1 \), i.e. \( q \in S(\Gamma) \subset S(\Sigma_1) \).

By Lemma 5.9 in some small neighbourhood \( U \subset M \) of \( q \), \( \Sigma_1 \cap U \) is the union of two connected, embedded \( c \)-hypersurfaces \( \Sigma_{1,1} \cup \Sigma_{1,2} \) with unit normals \( \nu_{1,1} \) and \( \nu_{1,2} \), such that \( \Sigma_{1,2} \) lies on one side of \( \Sigma_{1,1} \) and they touch tangentially at \( S(\Sigma_1) \cap U = \Sigma_{1,1} \cap \Sigma_{1,2} \). By Lemma 2.7, \( \nu_{1,1} = -\nu_{1,2} \) along the touching set \( S(\Sigma_1) \cap U \). Denote \( \Gamma \cap \Sigma_{1,1} = \Gamma_1 \) and \( \Gamma \cap \Sigma_{1,2} = \Gamma_2 \),
then as embedded submanifolds of \( \partial(B_{s_2}(p) \cap U) \), \( \Gamma_2 \) lies on one-side of \( \Gamma_1 \) and they touch tangentially along \( \mathcal{S}(\Gamma) \cap U \).

**Claim 3(B):** Fix \( x \in \mathcal{S}(\Gamma) \) and denote \( P_x = T_x \Sigma_{1,1} = T_x \Sigma_{1,2} \). For any sequence of \( x_i \to x \) with \( x_i \in \Gamma \) and \( r_i \to 0 \), up to a subsequence we have

\[
\lim_{i \to \infty} (\eta_{x_i,r_i})_* V^{**} = \begin{cases} 
   P_x + \tau_v P_x & \text{for Type I convergence} \\
   2P_x & \text{for Type II convergence}
\end{cases},
\]

where \( \tau_w \) denotes translation by a vector \( w \), and \( v \in (P_x)^\perp \) is a vector in \( T_x M \) orthogonal to \( P_x \) (\( v \) may be \( \infty \), in which case \( \tau_v P \) is understood to be the empty set). The two convergence scenarios are:

- **Type I:** \( \liminf_{i \to \infty} \text{dist}_{\mathcal{R}L}(x_i, \mathcal{S}(\Gamma))/r_i = \infty \),
- **Type II:** \( \liminf_{i \to \infty} \text{dist}_{\mathcal{R}L}(x_i, \mathcal{S}(\Gamma))/r_i < \infty \).

**Proof of Claim 3(B):** First we determine the blowup limit \( C' = \lim_{i \to \infty} (\eta_{x_i,r_i})(\Sigma_1) \).

In Type I convergence, for any \( R > 0, \Gamma \cap B_{r_i R}(x_i) \subset \mathcal{R}(\Gamma) \) for \( i \) large enough. Up to a subsequence, we can assume that all \( x_i \) belong to \( \Gamma_1 \), then \( (\eta_{x_i,r_i})(\Sigma_{1,1}) \) converges locally smoothly to \( P_x \). Let \( x_i' \) be the nearest point projection of \( x_i \) to \( \Sigma_{1,2} \), and let \( v = \lim_{i \to \infty} \frac{x_i' - x_i}{r_i} \) (up to taking a subsequence), which maybe \( \infty \). If \( v \) is finite, then \( (\eta_{x_i,r_i})(\Sigma_{1,2}) \) converges locally smoothly to \( P_x + v \); if \( v \) is infinite, then \( (\eta_{x_i,r_i})(\Sigma_{1,2}) \) disappears in the limit. So in this case \( C' = P_x + \tau_v P_x \). In the Type II scenario, the touching set does not disappear in the limit and we have \( C' = \lim(\eta_{x_i,r_i})(\Sigma_1) = 2P_x \).

Now let \( C = \lim_{i \to \infty} (\eta_{x_i,r_i})_* V^{**} \) be the varifold limit as in [6.7]. By Lemma 5.10, \( C \) is a regular, proper, complete minimal hypersurface in \( T_x M \). Again \( C \) must coincide with \( C' = \lim_{i \to \infty} (\eta_{x_i,r_i})(\Sigma_1) \) on a halfspace of \( T_x M \). Since \( C' \) consists of one or two parallel planes, the classical maximum principle implies that \( C \) contains these planes, and again the halfspace theorem [6.4, Theorem 3] rules out any other components of \( C \). Thus \( C \) is identical to \( C' \), which completes the proof.
By Claim 3(B) and the same argument in Claim 4(A), we know that

\begin{equation}
\lim_{z \to x, z \in \Sigma_2} [T_z \Sigma_2] = [T_x \Sigma_1],
\end{equation}

where \([T_z \Sigma_2]\) and \([T_z \Sigma_1]\) denote the un-oriented tangent planes of \(\Sigma_2\) and \(\Sigma_1\) (without counting multiplicity) respectively. Moreover, the convergence is uniform in \(x\) on compact subsets of \(\mathcal{S}(\Gamma)\) near \(q\). Therefore using the regularity of \(\Sigma_2\) in Lemma 5.9 near \(q\) the hypersurface \(\Sigma_2\) can be written as a set of graphs \(\{\Sigma_{2,i} : i = 1 \cdots l\}\) over the half space \([T_q(\Sigma_1 \cap B_{s_2}(p))]\).

Indeed, take \(\rho\) small so that \(B_\rho(q) \cap M\) may be identified with the tangent space \(T_qM\), and for \(z \in \Sigma_2 \cap B_\rho(q)\) let \(C_\epsilon(z)\) denote (the image of) the cylinder in \(T_qM\) with axis perpendicular to \(T_q \Sigma_1\) and radius \(\epsilon\). For small enough \(\rho, \epsilon\), by almost-embeddedness and the uniform convergence of tangent planes, \(\Sigma_2 \cap C_\epsilon(z) \cap B_\rho(q)\) decomposes as a finite number of ordered graphs over \(T_q \Sigma_1 \cap C_\epsilon(z)\), which have uniformly bounded gradient \(\delta \ll 1\). The uniform gradient bound, together with unique continuation for CMC hypersurfaces imply that this graphical decomposition may be extended all the way to the boundary \(\Gamma\), preserving the ordering and the gradient bound.

Now since \(\Sigma_2\) glues smoothly with \(\Sigma_1\) along \(\mathcal{R}(\Gamma)\), and since \(\mathcal{R}(\Gamma)\) is an open and dense subset of \(\Gamma\), we know that the set \(\{\Sigma_{2,i} : i = 1 \cdots l\}\) consists of exactly two elements: one of them, denoted by \(\Sigma_{2,1}\), glues smoothly with \(\Sigma_{1,1}\) along \(\Gamma_1 \setminus \mathcal{S}(\Gamma)\); the other one, denoted by \(\Sigma_{2,2}\), glues smoothly with \(\Sigma_{1,2}\) along \(\Gamma_2 \setminus \mathcal{S}(\Gamma)\). Now (6.9) implies that the pairs \((\Sigma_{1,1}, \Sigma_{2,1})\) and \((\Sigma_{1,2}, \Sigma_{2,2})\) glue together in \(C^1\) near \(q\) respectively. Again higher regularity follows from the elliptic PDE argument as in Sub-case (A). This finishes **Sub-case (B)**.

**Step 3.** We now wish to extend the replacements, via unique continuation, all the way to the centre \(p\).

Henceforth we denote \(V^{**}\) by \(V^{**}_{s_1}\) and \(\Sigma_2\) by \(\Sigma_{s_1}\) to indicate the dependence on \(s_1\). By varying \(s_1 \in (0, s)\), we obtain a family of nontrivial, smooth, almost embedded, stable \(c\)-boundaries \(\{\Sigma_{s_1} \subset A_{s_1,s_2}(p) \cap M\}\). Since unique continuation holds for immersed CMC hypersurfaces, by Step 2 we have \(\Sigma_{s_1} = \Sigma_1\) in \(A_{s_1,s_2}(p)\), and moreover, for any \(s'_1 < s_1 < s\),
we have \( \Sigma_{s_1'} = \Sigma_{s_1} \) in \( A_{s_1,s_2}(p) \). Hence
\[
\Sigma := \bigcup_{0 < s_1 < s} \Sigma_{s_1}
\]
is a nontrivial, smooth, almost embedded, stable \( c \)-hypersurface in \( (B_{s_2}(p) \setminus \{p\}) \cap M \). Also
\[
V^{**}_{s_1} = \Sigma, \text{ in } A_{s_1,s_2}(p), \quad V^{**} = V^* \text{ in } A_{s,s_2}(p),
\]
and for any \( s'_1 < s_1 < s, V^{**}_{s'_1} = V^{**}_{s_1} \text{ in } A_{s_1,s_2}(p) \).

By Proposition 5.8, \( V^{**}_{s_1} \) has \( c \)-bounded first variation and uniformly bounded mass for all \( 0 < s_1 < s \), so the monotonicity formula [105 40.2] implies that \( \|V^{**}_{s_1}\|(B_r(p)) \leq Cr^n \) for some uniform \( C > 0 \). Therefore as \( s_1 \to 0 \), the family \( V^{**}_{s_1} \) will converge to a varifold \( \tilde{V} \in V_n(M) \), i.e.
\[
\tilde{V} = \lim_{s_1 \to 0} V^{**}_{s_1}, \quad \text{such that}
\]
\[
(6.10) \quad \tilde{V} = \begin{cases} 
\Sigma & \text{in } (B_{s_2}(p) \setminus \{p\}) \cap M \\
V^* & \text{in } M \setminus B_s(p)
\end{cases}, \quad \text{and } \|\tilde{V}\|\{(p)\} = 0.
\]

Since \( p \in \text{spt } \|V^{**}_{s_1}\| \), by the upper semi-continuity of density function for varifolds with bounded first variation [105 17.8], we know that \( p \in \text{spt } \|\tilde{V}\| \).

**Step 4.** We now determine the regularity of \( \tilde{V} \) at \( p \).

First, since \( \tilde{V} \) is the varifold limit of a sequence \( V^{**}_{s_1} \) which all have \( c \)-bounded first variation, we know that \( \tilde{V} \) also has \( c \)-bounded first variation. Second, \( \tilde{V} \), when restricted to any small annulus \( A_{r,2r}(p) \cap M \), already coincides with a smooth, almost embedded, stable \( c \)-boundary \( \Sigma \). Using these two ingredients, we can use a blowup argument as in the proofs of Lemma 5.10 and Proposition 5.11 (without the need for replacements) to show that every tangent varifold of \( \tilde{V} \) at \( p \) is an integer multiple of some \( n \)-plane, i.e. for any
$C \in \text{VarTan}(V, p)$,

$$C = \Theta^n(\|\tilde{V}\|, p)|S|$$ for some $n$-plane $S \subset T_pM$ where $\Theta^n(\|\tilde{V}\|, p) \in \mathbb{N}$.

Now the removability of the singularity of $\tilde{V}$ at $p$ (as an almost embedded hypersurface) follows similarly to [94, Theorem 7.12]. We include the details for completeness. We can assume that

$$\Theta^n(\|\tilde{V}\|, p) = m$$

for some $m \in \mathbb{N}$. Since $\Sigma$ is stable in a punctured ball of $p$, by Theorem [2.11] for any sequence $r_i \to 0$,

$$\eta_{p, r_i}(\Sigma) \to m \cdot S$$

locally smoothly in $\mathbb{R}^L \setminus \{0\}$ for some $n$-plane $S \subset T_pM$. However, $S$ may depend on the sequence $r_i$. By the convergence and the regularity of $\Sigma$, there exists $\sigma_0 > 0$ small enough, such that for any $0 < \sigma \leq \sigma_0$, $\Sigma$ has an $m$-sheeted, ordered (in the sense of Definition [2.2]), graphical decomposition in $A_{\sigma/2, \sigma}(p)$:

$$(6.11) \quad \tilde{V} \pitchfork_{A_{\sigma/2, \sigma}(p)} = \sum_{i=1}^{m} |\Sigma_i(\sigma)|.$$ Here $\Sigma_i(\sigma)$ is a graph over $A_{\sigma/2, \sigma}(p) \cap S$ for some $n$-plane $S \subset T_pM$.

Since (6.11) holds for all $\sigma$, by continuity of $\Sigma$ we can continue each $\Sigma_i(\sigma_0)$ to $(B_{\sigma_0}(p) \setminus \{p\}) \cap M$, and we denote the continuation by $\Sigma_i$. Since each piece $\Sigma_i$ has constant mean curvature $c$, by a standard extension argument (c.f. the proof in [62, Theorem 4.1]), each $\Sigma_i$ can be extended as a varifold with $c$-bounded first variation in $B_{\sigma_0}(p) \cap M$. Given $C_i \in \text{VarTan}(\Sigma_i, p)$, to see that $C_i$ has multiplicity one, first notice that

$$(6.12) \quad \Theta^n(\|C_i\|, p) \geq 1,$$

since each $\Sigma_i$ is $c$-stable and thus its re-scalings converge with multiplicity to a smooth, embedded, stable, minimal hypersurface by Theorem [2.11]. If equality does not hold for
some $i$ in (6.12), this will derive a contradiction since

$$
\tilde{V} \subset B_{\sigma_0}(p) = \sum_{i=1}^{m} |\Sigma_i|.
$$

Therefore, each $\Sigma_i$ has $c$-bounded first variation in $B_{\sigma_0}(p) \cap M$ and $\Theta^p(\|\Sigma_i\|, p) = 1$; by the Allard regularity theorem [105 Theorem 24.2] and elliptic regularity, $\Sigma_i$ extends as a smooth, embedded $c$-hypersurface across $p$. Finally, by the maximum principle (Lemma 2.7), we have $m = 1$ or $m = 2$ and this shows that $\tilde{V}$ extends as an almost embedded $c$-hypersurface across $p$.

**Step 5.** Finally, to complete the proof we show that $V$ coincides with $\tilde{V}$ on a small ball about $p$.

We will need the following simple corollary of the first variation formula.

**Lemma 6.2.** For small enough $r$ the set

$$
T^Y_p = \left\{ y \in \text{spt} \|V\| \cap (B_r(p) \setminus \{p\}) : \ Var\text{Tan}(V, y) \text{ consists of an integer multiple of an } \ n \text{-plane transverse to } \partial(B_{\text{dist}(y,p)}(p) \cap M) \right\}
$$

is a dense subset of $\text{spt} \|V\| \cap B_r(p)$.

**Proof.** Assume to the contrary that there exists $y \in \text{spt} \|V\| \cap (B_r(p) \setminus \{p\})$, and some some neighbourhood $U$ of $p$ in $B_r(p) \cap M$ such that the tangent plane $S_z$ of $V$ at each point $z \in U \cap \text{spt} \|V\|$ is tangential to $\partial(B_{\text{dist}(z,p)}(p) \cap M)$. Since $M$ is embedded in $\mathbb{R}^L$, the family $\{\partial(B_\rho(p) \cap M) \cap M : 0 < \rho < r\}$ forms a smooth foliation of $(B_r(p) \cap M) \setminus \{p\}$ for small enough $r$. Let $\nu$ be the outer unit normal of this smooth foliation. By our choice of $r_0$, the mean curvature on the foliation is

$$
\text{div}_{T^y_p(\partial(B_{\text{dist}(y,p)}(p) \cap M))} \nu > c.
$$
Consider the vector field $X = \varphi \cdot \nu$, where $\varphi \geq 0$ is a smooth cutoff function supported in $U$. Plugging $X$ into the first variation formula, we get

$$
\delta V(X) = \int \text{div}_z (\varphi \nu) d\|V\|(z) = \int \varphi \cdot \text{div}_z (\nu) d\|V\|(z)
$$

$$
> c \cdot \int \varphi d\|V\|(z) = c \int |X| d\|V\|(z).
$$

In the second equality we have used the fact that $S_z$ is tangential to the foliation and hence perpendicular to $\nu$. This contradicts that $V$ has $c$-bounded first variation and thus finishes the proof.

\[\square\]

**Claim 5:** For small enough $r$, $\text{spt} \|V\| = \Sigma$ in the punctured ball $(B_r(p) \setminus \{p\}) \cap M$.

**Proof of Claim 5:** We first prove that $\text{Tr}^Y_{\rho} \subset \Sigma$, which combined with Lemma 6.2 will imply that $\text{spt} \|V\| \cap (B_r(p) \setminus \{p\}) \subset \Sigma$. Fix $y \in \text{Tr}^Y_{\rho} \cap (B_r(p) \setminus \{p\})$, and let $\rho = \text{dist}(y, p)$. Consider $V^*_{\rho}$. By transversality we have $y \in \text{Clos}(\text{spt} \|V\| \cap B_\rho(p))$. On the other hand, $V^*_{\rho} = V^* = V$ inside $B_\rho(p)$, so by (6.1) we have

$$
\text{Clos}(\text{spt} \|V\| \cap B_\rho(p)) \cap \partial B_\rho(p) = \text{Clos}(\text{spt} \|V^*_{\rho}\| \cap B_\rho(p)) \cap \partial B_\rho(p).
$$

Since $\text{spt} \|V^*_{\rho}\| = \Sigma$ on $A_{\rho, s_2}(p)$, we therefore have $y \in \Sigma$.

Next we show the reverse inclusion $\Sigma \subset \text{spt} \|V\|$. Since $\Sigma$ extends across $p$ as an almost embedded hypersurface, we know that $T_y \Sigma$ is transverse to $\partial(B_{\text{dist}(y, p)} \cap M)$ for all $y \in \Sigma \cap B_r(p)$ for small enough $r$. Let $\rho$ and $V^*_{\rho}$ be as above, then $y \in \text{spt} \|V^*_{\rho}\|$. By Proposition 5.11, $\text{VarTan}(V^*_{\rho}, y) = \{\Theta^\rho(\|V^*_{\rho}\|, y)|T_y \Sigma]\}$. By the transversality, we then have $y \in \text{Clos}(\text{spt} \|V^*_{\rho}\| \cap B_\rho(p))$, so since $V^*_{\rho} = V$ inside $B_\rho(p)$ we conclude that $y \in \text{Clos}(\text{spt} \|V\| \cap B_\rho(p)) \subset \text{spt} \|V\|$ as desired.

\[\square\]

Note that we do not have the Constancy Theorem (c.f. [105, 41.1]) for varifolds with bounded first variation. In order to show that $V$ coincides with $\Sigma$ near $p$, our strategy is to show that $V = \tilde{V}$ as varifolds in a neighbourhood of $p$. By the transversality argument as
above, we can first show that the densities of $V$ and $\tilde{V}$ are identical along $\Sigma \cap (B_r(p) \setminus \{p\})$, where $r$ is chosen as in Claim 5.

**Claim 6:** $\Theta^n(\|V\|, \cdot) = \Theta^n(\|\tilde{V}\|, \cdot)$ on $\Sigma \cap B_r(p) \setminus \{p\}$.

**Proof of Claim 6:** Let $y \in \Sigma$ and $\rho = \text{dist}(y, p) < r$ be as above. Then since $V_{\rho}^{**} = V$ inside $B_\rho(p)$, by transversality and Proposition 5.11, we have $\text{VarTan}(V, y) = \text{VarTan}(V_{\rho}^{**}, y)$. But $V_{\rho}^{**} = \tilde{V}$ on $A_{\rho, \rho_2}(p)$, so we must have $\text{VarTan}(V_{\rho}^{**}, y) = \{\Theta^n(\|\tilde{V}\|, y) | T_y \Sigma]\}$. Thus $\Theta^n(\|V\|, y) = \Theta^n(\|\tilde{V}\|, y)$.

Combining Claims 5 and 6 yields that $V = \tilde{V}$ on $B_r(p) \setminus M$. This finishes the proof of Step 5, and hence also completes the proof of the main Theorem 6.1.
Bibliography


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