Geometric Properties of Families of Galois Representations

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Geometric Properties of Families of Galois Representations

A dissertation presented

by

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Geometric Properties of Families of Galois Representations

Abstract

This thesis concerns families of Galois representations arising as étale local systems on a variety over a number field or a p-adic field.

The first part of the thesis studies families of Galois representations of number fields in the context of the relative Fontaine-Mazur conjecture. The conjecture predicts that a p-adic étale local system that satisfies the de Rham condition arises from algebraic geometry and in particular such a local system is part of a compatible system of ℓ-adic étale local systems with various primes ℓ. We show the existence of a compatible system in some cases. We also discuss the finiteness of the field generated over ℚ by Frobenius traces of a local system at closed points.

The second part of the thesis studies families of Galois representations of p-adic fields in the context of the relative p-adic Hodge theory. Sen attached to each p-adic Galois representation of a p-adic field a multiset of numbers called generalized Hodge-Tate weights. We consider a p-adic local system on a rigid analytic variety over a p-adic field and show that the multiset of generalized Hodge-Tate weights of the local system is constant. We also discuss basic properties of Hodge-Tate sheaves on a rigid analytic variety.
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Chapter 1

Introduction

1.1 The Fontaine-Mazur conjecture and geometric properties

Various cohomology theories appear in the study of algebraic varieties. For example, one can associate to a complex algebraic variety $X$ the singular cohomology and the de Rham cohomology. When $X$ is defined over $\mathbb{Q}$, one can also consider the $p$-adic étale cohomology $H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_p)$; it is a $\mathbb{Q}_p$-vector space equipped with a continuous action of the Galois group $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. This means that algebraic varieties over $\mathbb{Q}$ yield $p$-adic Galois representations

$$\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_n(\mathbb{Q}_p)$$

via the $p$-adic étale cohomology. One can then ask which $p$-adic Galois representations of $\mathbb{Q}$ arise in this way.

It is known that a Galois representation of $\mathbb{Q}$ arising from an algebraic variety enjoys the following two properties. First it is unramified at $\ell$ for almost all primes $\ell \neq p$. Namely, the
inertia subgroup at \( \ell \) has the trivial image for all but finitely many primes \( \ell \neq p \). Second its restriction to the decomposition subgroup at \( p \) is de Rham. This is a certain local condition formulated in \( p \)-adic Hodge theory. We call a Galois representation satisfying these properties geometric.

Fontaine and Mazur conjectured that the converse also holds:

**Conjecture 1** (Fontaine-Mazur conjecture). Let \( K \) be a number field, i.e., a finite extension of \( \mathbb{Q} \). An irreducible geometric Galois representation \( \rho_p : \text{Gal}(\overline{K}/K) \rightarrow \text{GL}_n(\overline{\mathbb{Q}}_p) \) is a subquotient of \( H^i_{\text{ét}}(Y, \overline{\mathbb{Q}}_p) \) up to a Tate twist for some algebraic variety \( Y \) over \( K \) and some integer \( i \).

This is a fundamental conjecture in number theory. It implies that geometric Galois representations have additional striking “geometric” properties\(^1\). Note that an algebraic variety gives rise to a compatible system of \( p \)-adic Galois representations with various \( p \) by changing the coefficient field \( \mathbb{Q}_p \) of the étale cohomology. Along this line, the Fontaine-Mazur conjecture predicts that a geometric Galois representation \( \rho_p : \text{Gal}(\overline{K}/K) \rightarrow \text{GL}_n(\overline{\mathbb{Q}}_p) \) satisfies the following geometric properties (see Conjecture 3.1.1 for the precise formulation):

- **Finiteness of Frobenius traces**: the field generated by the Frobenius traces \( \text{tr} \rho_p(\text{Frob}_v) \in \overline{\mathbb{Q}}_p \) for unramified primes \( v \) of \( K \) is finite over \( \mathbb{Q} \).

- **Existence of a compatible system**: for each prime \( p' \) there exists a \( p' \)-adic Galois representation \( \rho_{p'} \) of \( K \) that is compatible with \( \rho_p \), i.e.,

\[
\det(1 - \rho_{p'}(\text{Frob}_v)t) = \det(1 - \rho_p(\text{Frob}_v)t)
\]

\(^1\)They are also referred to as “motivic” properties.
for each unramified prime $v$ of $K$.

The goal of this thesis is to study a family of Galois representations parametrized by a variety and discuss its geometric properties.

1.2 Families of Galois representations and the relative Fontaine-Mazur conjecture

Let us consider the relative version of the Fontaine-Mazur conjecture by replacing the Galois group by the étale fundamental group of a variety and a Galois representation by an étale local system. This is a natural generalization since $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is the étale fundamental group of $\text{Spec} \mathbb{Q}$, and Galois representations of $\mathbb{Q}$ correspond to étale local systems on $\text{Spec} \mathbb{Q}$.

In this thesis, we regard an étale local system $E$ on a variety $X$ as a family of Galois representation parametrized by $X$; each $x \in X$ gives rise to a Galois representation of $k(x)$ acting on $E_x$.

The relative Fontaine-Mazur conjecture is stated as follows:

**Conjecture 1** (Relative Fontaine-Mazur conjecture). *Let $X$ be a scheme flat and of finite type over $\mathbb{Z}[p^{-1}]$ and let $E$ be an irreducible $\overline{\mathbb{Q}}_p$-local system on $X$. Assume that $E$ is de Rham over $X \otimes \mathbb{Q}_p$. Then there exists a morphism $f: Y \to X$ and an integer $i$ such that $E$ is generically a subquotient of $R^if_*\overline{\mathbb{Q}}_p$ up to a Tate twist.*

Here we work on a local system on an integral model of a variety over $\mathbb{Q}$, i.e., a scheme flat and of finite type over $\mathbb{Z}$ so that the unramified condition is already included in the set-up. We also remark that there is a notion of de Rham local systems in the relative $p$-adic Hodge theory (see Definition 4.3.1 and Remark 4.3.2).
This is a vast generalization of the Fontaine-Mazur conjecture for Galois representations, and almost no cases are known except the known cases for the usual Fontaine-Mazur conjecture for Galois representations, the abelian case, and a recent work of Snowden and Tsimerman related to elliptic curves ([49]).

Similar to the Fontaine-Mazur conjecture for Galois representations, the relative Fontaine-Mazur conjecture predicts remarkable consequences: families of Galois representations arising from de Rham local systems should satisfy geometric properties, namely, the finiteness of Frobenius traces and the existence of a compatible system (see Conjecture 3.1.2 and Conjecture 2.1.2 for the precise statements). Since one cannot expect such strong properties for general non-de Rham local systems, it gives supporting evidence of the relative Fontaine-Mazur conjecture to show that de Rham local systems satisfy these geometric properties.

In the first part of this thesis, we show the existence of a compatible system for de Rham local systems of an arbitrary rank in some cases. We also study the relation between the finiteness of Frobenius traces of Galois representations and that of local systems. In the next two sections, we explain these results in detail.

1.3 Existence of a compatible system

The first goal of this thesis is to establish the existence of a compatible system for de Rham local systems on an algebraic variety over a totally real field:

**Theorem A** (Theorem 2.1.1). Let $K$ be a totally real field which is unramified at $p$. Let $X$ be a smooth $\mathcal{O}_K[p^{-1}]$-scheme with geometrically connected fibers. Assume further that $X(\mathbb{R}) \neq \emptyset$ for every real place $K \hookrightarrow \mathbb{R}$ and that $X$ extends to an irreducible smooth $\mathcal{O}_K$-

\footnote{In Theorem 2.1.1 we use $\ell$ for the coefficient for local systems.}
scheme with nonempty fiber over each prime of $K$ above $p$. Let $E$ be a finite extension of $\mathbb{Q}$ and $\mathfrak{p}$ a prime of $E$ above $p$. Let $\mathcal{E}$ be an $E_{\mathfrak{p}}$-local system on $X$ and $\rho$ the corresponding representation of $\pi_1(X)$. Suppose that $\mathcal{E}$ satisfies the following assumptions:

(i) The polynomial $\det(1 - \text{Frob}_x t, \mathcal{E}_x)$ has coefficients in $E$ for every $x \in |X|$.

(ii) For every finite extension $L$ of $\mathbb{Q}_p$ which is unramified at $p$ and every morphism $\alpha: \text{Spec} L \to X$, $\alpha^* \rho$ is crystalline and it has distinct $\tau$-Hodge-Tate weights in the range $[0, p - 2]$ for each $\tau: L \hookrightarrow \overline{E}_p$.

(iii) $\rho$ can be equipped with a symplectic (resp. orthogonal) structure with multiplier $\mu: \pi_1(X) \to E_{\mathfrak{p}}^\times$ such that $\mu|_{\pi_1(X_K)}$ admits a factorization

$$\mu|_{\pi_1(X_K)}: \pi_1(X_K) \to \text{Gal}(\overline{K}/K) \xrightarrow{\mu_K} E_{\mathfrak{p}}^\times$$

with a totally odd (resp. totally even) character $\mu_K$.

(iv) The residual representation $\bar{\rho}|_{\pi_1(X_{[\mathfrak{p}]})}$ is absolutely irreducible.

(v) $p > 2(\text{rank } \mathcal{E} + 1)$.

Then a compatible system exists for $\mathcal{E}$. Namely, for every prime $p'$ and each prime $\mathfrak{p}'$ of $E$ above $p'$ there exists an $\overline{E}_{\mathfrak{p}'}$-local system $\mathcal{E}'$ on $X[p'^{-1}]$ such that

$$\det(1 - \text{Frob}_x t, \mathcal{E}_x) = \det(1 - \text{Frob}_x t, \mathcal{E}'_x)$$

for every closed point $x$ of $X[p'^{-1}]$.

Assumption (ii) concerns conditions coming from $p$-adic Hodge theory. Since crystalline representations are de Rham, assumption (ii) implies the de Rham assumption. Therefore,
this theorem proves a particular case of the existence of a compatible system of a de Rham local system, and hence it gives evidence of the relative Fontaine-Mazur conjecture.

Assumptions (ii)-(v) are related to the work of Barnet-Lamb, Gee, Geraghty, and Taylor (6); they proved the existence of a compatible system of Galois representations of $K$ in the case where $K$ is totally real and unramified at $p$, and the representation satisfies similar assumptions to our assumptions (ii)-(v).

Our proof of the theorem is inspired by the work of Drinfeld in the function field case; Deligne conjectured the existence of a compatible system for local systems on varieties over a finite field (see Subsection 2.1.2). L. Lafforgue established the Langlands correspondence for $\text{GL}_r$ in the function field case and proved Deligne’s conjecture in the case of curves (33). In [20], Drinfeld established a method of constructing a compatible system of local systems on a smooth variety from those on curves. By combining his method with Lafforgue’s existence result for curves, Drinfeld proved the existence of a compatible system in the case of smooth varieties.

We interpret Drinfeld’s method as “Lafforgue’s result and the Fontaine-Mazur conjecture for Galois representations imply the existence of a compatible system”. To obtain unconditional results, we improve Drinfeld’s method in the case of totally real fields and prove Theorem A using results on the existence of a compatible system in [6] and [33].

### 1.4 Finiteness of Frobenius traces

The second goal of the thesis is to discuss the relation between the finiteness of Frobenius traces of Galois representations and that of local systems:

**Theorem B** (Theorem 3.1.3). Assume the Fontaine-Mazur conjecture for Galois representations and the Generalized Riemann Hypothesis for Dedekind zeta functions. Then the
finiteness of Frobenius traces holds for every étale local system that satisfies the de Rham condition.

Deligne proved an analogous finiteness theorem of Frobenius traces in the function field case ([19]). He used Lafforgue’s result (the curve case) and the Weil conjecture (the function field analogue of the Riemann Hypothesis) as crucial inputs. Our proof is inspired by Deligne’s, which is why we need to assume the Generalized Riemann Hypothesis in Theorem B.

We prove the theorem by constructing explicitly a curve, i.e., an open subscheme of the spectrum of the ring of integers of a number field, which passes through a given point on $X$ and has small ramification and boundary.

1.5 The relative $p$-adic Hodge theory

The second part of the thesis concerns the relative $p$-adic Hodge theory. Let us start with a brief introduction to $p$-adic Hodge theory. So we turn to Galois representations of a $p$-adic field $k$, i.e., a finite extension of $\mathbb{Q}_p$.

We first remark that there are much more $p$-adic Galois representations of $k$ than $\ell$-adic Galois representations for a prime $\ell \neq p$ due to the topological structure of $\text{Gal}(\overline{k}/k)$. This makes the theory of $p$-adic Galois representations richer and more interesting.

One major goal of $p$-adic Hodge theory is to understand $p$-adic Galois representations of $k$, and it has two directions:

- the study of properties of $p$-adic Galois representations that arise from algebraic geometry;
- the study of arbitrary $p$-adic Galois representations.
In the former approach, Tate conjectured that $p$-adic Galois representations coming from varieties should satisfy the so-called Hodge-Tate condition ([50]). This conjecture was later proved by Faltings ([23]). To study further properties of such representations, Fontaine introduced the notion of de Rham representations, crystalline representations, and semistable representations. It is now known that a $p$-adic Galois representation of $k$ arising from a proper smooth variety over $k$ is de Rham and moreover it is crystalline (resp. semistable) if the variety has good (resp. semistable) reduction.

In the latter approach, Sen attached to every $p$-adic Galois representation a multiset of numbers called generalized Hodge-Tate weights ([13]).

The second part of the thesis studies a geometric family of $p$-adic Galois representations in the latter approach. To be precise, we consider an étale $\mathbb{Q}_p$-local system on a rigid analytic variety over $k$ and regard it as a geometric family of Galois representations parametrized by the variety. We prove the constancy of generalized Hodge-Tate weights in the family. Along the way, we also show basic properties of Hodge-Tate local systems. We explain these results in the next two sections, which are a shorter version of Section 4.1.

1.6 Constancy of generalized Hodge-Tate weights

The first result in the second part of the thesis is the constancy of generalized Hodge-Tate weights:

**Theorem C** (Corollary 4.4.9). Let $X$ be a geometrically connected smooth rigid analytic variety over a $p$-adic field $k$ and let $\mathcal{E}$ be a $\mathbb{Q}_p$-local system on $X$. Then the generalized Hodge-Tate weights of the $p$-adic representations $\mathcal{E}_x$ of the absolute Galois group of $k(x)$ are constant on the set of classical points $x$ of $X$.

The theorem gives one instance of the rigidity of a geometric family of Galois represen-
tations. It is worth noting that arithmetic families of Galois representations appearing in the theory of Galois deformations do not have such rigidity ([38, 45]).

To explain ideas of the proof of Theorem C as well as other results in the second part of the thesis, let us recall the work of Sen mentioned above. Let $k_1$ be the cyclotomic extension $k(\mu_{p^\infty})$ and let $K$ be the $p$-adic completion of $k_\infty$. To each Galois representation $V$ of $k$, Sen associated a $K$-vector space $\mathcal{H}(V)$ equipped with an action of $\text{Gal}(k_1/k)$. Sen then developed a theory of decompletion; he found a natural $k_\infty$-vector subspace $\mathcal{H}(V)_{\text{fin}} \subset \mathcal{H}(V)$ such that $\mathcal{H}(V)_{\text{fin}} \otimes_{k_\infty} K = \mathcal{H}(V)$. He defined a $k_\infty$-endomorphism $\phi_V$ on $\mathcal{H}(V)_{\text{fin}}$, called the Sen endomorphism of $V$, by considering the infinitesimal action of $\text{Gal}(k_\infty/k)$. The generalized Hodge-Tate weights of $V$ are defined to be eigenvalues of $\phi_V$.

Therefore, the first step toward Theorem C is to define generalizations of $\mathcal{H}(V)$ and $\phi_V$ for each $\mathbb{Q}_p$-local system. For this, we use the $p$-adic Simpson correspondence by Liu and Zhu [36]; they associated to each $\mathbb{Q}_p$-local system $\mathcal{E}$ on $X$ a vector bundle $\mathcal{H}(\mathcal{E})$ on $X_K$ equipped with a $\text{Gal}(k_\infty/k)$-action and a Higgs field, where $X_K$ is the base change of $X$ to $K$. When $X$ is a point and $\mathcal{E}$ corresponds to $V$, this agrees with $\mathcal{H}(V)$ as the notation suggests. Following Sen, we define the arithmetic Sen endomorphism $\phi_\mathcal{E}$ of $\mathcal{E}$ by decompleting $\mathcal{H}(\mathcal{E})$ and considering the infinitesimal action of $\text{Gal}(k_\infty/k)$. Then Theorem C is reduced to the following:

**Theorem D** (Theorem 4.1.2). The eigenvalues of $\phi_{\mathcal{E},x}$ for $x \in X_K$ are algebraic over $k$ and constant on $X_K$.

The key idea to obtain such constancy is to describe $\phi_V$ as the residue of a certain formal integrable connection. For this we generalize Fontaine’s decompletion theory for the de Rham period ring $B_{\text{dR}}(K)$ ([25]) to the relative setting, and combine the decompletion theory with the geometric $p$-adic Riemann-Hilbert correspondence by Liu and Zhu [36].
developing a theory of formal connections and applying it to the current set-up, we prove Theorem D.

1.7 Properties of Hodge-Tate sheaves

Sen proved that a $p$-adic Galois representation $V$ is Hodge-Tate if and only if $\phi_V$ is semisimple with integer eigenvalues ([13]). To generalize this result, we use $\phi_\mathcal{E}$ to study Hodge-Tate local systems. We define a sheaf $D_{HT}(\mathcal{E})$ on the étale site $X_{\text{ét}}$ by

$$D_{HT}(\mathcal{E}) := \nu_* (\mathcal{E} \otimes_{\mathbb{Q}_p} \mathcal{O}_{\text{HT}}),$$

where $\mathcal{O}_{\text{HT}}$ is the Hodge-Tate period sheaf on the pro-étale site $X_{\text{pro\-ét}}$ and $\nu: X_{\text{pro\-ét}} \to X_{\text{ét}}$ is the projection (see Subsection 4.3.1). A local system $\mathcal{E}$ is called Hodge-Tate if $D_{HT}(\mathcal{E})$ is a vector bundle on $X$ of rank equal to rank $\mathcal{E}$.

**Theorem E** (Theorem 4.5.5). For a $\mathbb{Q}_p$-local system $\mathcal{E}$ on $X$, the following conditions are equivalent:

(i) $\mathcal{E}$ is Hodge-Tate.

(ii) $\phi_\mathcal{E}$ is semisimple with integer eigenvalues.

Using the characterization in terms of $\phi_\mathcal{E}$, we prove the following basic property of Hodge-Tate sheaves:

**Theorem F** (Theorem 4.5.10). Let $f: X \to Y$ be a smooth proper morphism between smooth rigid analytic varieties over $k$ and let $\mathcal{E}$ be a $\mathbb{Z}_p$-local system on $X$. Assume that $R^if_*\mathcal{E}$ is a $\mathbb{Z}_p$-local system on $Y$ for each $i$. Then if $\mathcal{E}$ is a Hodge-Tate local system on $X$, $R^if_*\mathcal{E}$ is a Hodge-Tate local system on $Y$ for each $i$. 

10
Hyodo introduced the notion of Hodge-Tate local systems and proved Theorem F in the case of schemes ([29]). Deep connections between Hodge-Tate local systems and the $p$-adic Simpson correspondence can be seen in his work and are also studied by Abbes, Gros, and Tsuji in [11] and by Andreatta and Brinon in [4]. In these works, one is restricted to working with schemes, whereas we use Huber’s formalism of rigid analytic varieties and Scholze’s theory of perfectoid spaces so that our results hold for rigid analytic varieties not necessarily coming from schemes.

1.8 Organization of the thesis

Part I studies families of Galois representations of number fields. Chapter 2 discusses the existence of a compatible system of a local system. Chapter 3 discusses the finiteness of Frobenius traces of a local system.

Part II studies $p$-adic Hodge theory for families of Galois representations of $p$-adic fields. Chapter 4 discusses the constancy of generalized Hodge-Tate weights of a local system and properties of Hodge-Tate sheaves.

Each chapter has its own introduction, where we state the main theorems of the chapter and ideas of the proofs.
Part I

Around the relative Fontaine-Mazur conjecture
Chapter 2

Existence of a compatible system

2.1 Introduction

The goal of this chapter is to prove the existence of a compatible system of a de Rham local system in special cases. We work on (integral models of) varieties over totally real fields and CM fields. We note that the content of this chapter is first published in *Algebra Number Theory* ([18]).

2.1.1 Main Theorem

Let us first state the main theorem:

**Theorem 2.1.1.** Let \( \ell \) be a rational prime and \( K \) a totally real field which is unramified at \( \ell \). Let \( X \) be an irreducible smooth \( \mathcal{O}_K[\ell^{-1}] \)-scheme such that

- the generic fiber is geometrically irreducible,

- \( X_K(\mathbb{R}) \neq \emptyset \) for every real place \( K \hookrightarrow \mathbb{R} \), and
• \( X \) extends to an irreducible smooth \( \mathcal{O}_K \)-scheme with nonempty fiber over each place of \( K \) above \( \ell \).

Let \( E \) be a finite extension of \( \mathbb{Q} \) and \( \lambda \) a prime of \( E \) above \( \ell \). Let \( \mathcal{E} \) be an \( E_\lambda \)-local system on \( X \) and \( \rho \) the corresponding representation of \( \pi_1(X) \). Suppose that \( \mathcal{E} \) satisfies the following assumptions:

(i) The polynomial \( \det(1 - \text{Frob}_x t, \mathcal{E}_x) \) has coefficients in \( E \) for every closed point \( x \in X \).

(ii) For every totally real field \( L \) which is unramified at \( \ell \) and every morphism \( \alpha: \text{Spec} \, L \to X \), the \( E_\lambda \)-representation \( \alpha^* \rho \) of \( \text{Gal}(\overline{L}/L) \) is crystalline at each prime \( v \) of \( L \) above \( \ell \), and for each \( \tau: L \to \overline{E}_\lambda \) it has distinct \( \tau \)-Hodge-Tate weights in the range \([0, \ell - 2]\).

(iii) \( \rho \) can be equipped with symplectic (resp. orthogonal) structure with multiplier \( \mu: \pi_1(X) \to E_\lambda^\times \) such that \( \mu|_{\pi_1(X_K)} \) admits a factorization

\[
\mu|_{\pi_1(X_K)}: \pi_1(X_K) \to \text{Gal}(\overline{K}/K) \xrightarrow{\mu_K} E_\lambda^\times
\]

with a totally odd (resp. totally even) character \( \mu_K \) (see below for the definitions).

(iv) The residual representation \( \overline{\rho}|_{\pi_1(X[\zeta_\ell])} \) is absolutely irreducible. Here \( \zeta_\ell \) is a primitive \( \ell \)-th root of unity and \( X[\zeta_\ell] = X \otimes_{\mathcal{O}_K} \mathcal{O}_K[\zeta_\ell] \).

(v) \( \ell > 2(\text{rank } \mathcal{E} + 1) \).

Then for each rational prime \( \ell' \) and each prime \( \lambda' \) of \( E \) above \( \ell' \) there exists an \( \overline{E}_{\lambda'} \)-local system \( \mathcal{E}' \) on \( X[\ell'^{-1}] \) which is compatible with \( \mathcal{E}|_{X[\ell'^{-1}]} \). Namely,

\[
\det(1 - \text{Frob}_{x} t, \mathcal{E}'_x) = \det(1 - \text{Frob}_{x} t, \mathcal{E}_x)
\]
for every closed point $x$ in $X[\ell^{-1}]$.

Note that this theorem is the same as Theorem $[\text{A}]$ but here we use a rational prime $\ell$ for the coefficient of étale local systems.

Let us explain assumption (iii). For an $E_\lambda$-representation $\rho: \pi_1(X) \to \GL(V_\rho)$, a symplectic (resp. orthogonal) structure with multiplier is a pair $(\langle , \rangle, \mu)$ consisting of a symplectic (resp. orthogonal) pairing

$$\langle , \rangle: V_\rho \times V_\rho \to E_\lambda$$

and a continuous homomorphism $\mu: \pi_1(X) \to E_\lambda^\times$ satisfying

$$\langle \rho(g)v, \rho(g)v' \rangle = \mu(g)\langle v, v' \rangle$$

for any $g \in \pi_1(X)$ and $v, v' \in V_\rho$.

We also show a similar theorem without assuming that $K$ is unramified at $\ell$ using the potential diagonalizability assumption. See Theorem $[\text{3.1.3}]$ for this statement and Theorem $[\text{2.4.4}]$ for the corresponding statement when $K$ is CM.

These results are special cases of the following conjecture, which is a geometric consequence of the relative Fontaine-Mazur conjecture:

**Conjecture 2.1.2** (Existence of a compatible system). Let $\ell$ be a rational prime. Let $X$ be an irreducible regular scheme that is flat and of finite type over $\mathbb{Z}[\ell^{-1}]$. Let $E$ be a finite extension of $\mathbb{Q}$ and $\lambda$ a prime of $E$ above $\ell$. Let $E$ be an irreducible $E_\lambda$-local system on $X$ and $\rho$ the corresponding representation of $\pi_1(X)$. Assume that $E|_{X \otimes \Q_\ell}$ on $X \otimes \Q_\ell$ is a de Rham $E_\lambda$-local system. Then for each rational prime $\ell'$ and each prime $\lambda'$ of $E$ above $\ell'$ there exists an $\overline{E}_{\lambda'}$-local system on $X[\ell'^{-1}]$ which is compatible with $E|_{X[\ell'^{-1}]}$. 

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2.1.2 Deligne’s conjecture

Before explaining ideas of the proof of Theorem 2.1.1, let us turn to the function field analogue of geometric properties of local systems; in [17], Deligne conjectured that all the \( \mathbb{Q}_\ell \)-local systems on a variety over a finite field are mixed. A standard argument reduces this conjecture to the following:

**Conjecture 2.1.3 (Deligne).** Let \( p \) and \( \ell \) be distinct primes. Let \( X \) be a connected normal scheme of finite type over \( \mathbb{F}_p \) and \( \mathcal{E} \) an irreducible \( \mathbb{Q}_\ell \)-local system whose determinant has finite order. Then the following properties hold:

(i) \( \mathcal{E} \) is pure of weight 0.

(ii) There exists a number field \( E \subset \overline{\mathbb{Q}_\ell} \) such that the polynomial \( \det(1 - \text{Frob}_x t, \mathcal{E}_x) \) has coefficients in \( E \) for every closed point \( x \in X \).

(iii) For every non-archimedean place \( \lambda \) of \( E \) prime to \( p \), the roots of \( \det(1 - \text{Frob}_x t, \mathcal{E}_x) \) are \( \lambda \)-adic units.

(iv) For a sufficiently large \( E \) and for every non-archimedean place \( \lambda \) of \( E \) prime to \( p \), there exists an \( E_\lambda \)-local system \( \mathcal{E}_\lambda \) compatible with \( \mathcal{E} \).

Deligne’s conjecture is almost completely proved now. Let us explain the results. First the conjecture for curves is proved by L. Lafore in [33]:

**Theorem 2.1.4 ([33, Théorème VII.6]).** If \( \dim X = 1 \), then Deligne’s conjecture holds.

Lafore proved the Langlands correspondence for \( \text{GL}_r \) over function fields and obtained this theorem as a corollary. Lafore also dealt with parts (i) and (iii) of Deligne’s conjecture for general \( X \) by reducing them to the case of curves (see also [19]). However, ¹Of course, Deligne’s conjecture predates the Fontaine-Mazur conjecture.
parts (ii) and (iv) for general $X$ cannot be simply reduced to the case of curves. Deligne proved part (ii) in [19], and Drinfeld proved part (iv) for smooth varieties in [20]:

**Theorem 2.1.5** ([19, Théorème 3.1] See also Theorem 3.1.4). **Part (ii) of Deligne’s conjecture holds for an arbitrary $X$.**

**Theorem 2.1.6** ([20, Theorem 1.1]). **Part (iv) of Deligne’s conjecture holds when $X$ is smooth over $\mathbb{F}_p$.**

Deligne and Drinfeld both used La\'fargue’s results. To explain ideas of the proof of Theorem 2.1.1 we will give a brief outline of the proof of Drinfeld’s result in the next subsection.

### 2.1.3 Ideas of the proof of Theorem 2.1.1

We explain ideas of the proof of Theorem 2.1.1. We use La\'fargue’s work and the following result of Barnet-Lamb, Gee, Geraghty, and Taylor:

**Theorem 2.1.7** ([6, Theorem C]). Let $K$ be a totally real field that is unramified at $\ell$, and let $\rho: \text{Gal}(\overline{K}/K) \to \text{GL}_r(\overline{\mathbb{Q}}_\ell)$ be a continuous representation. Suppose that the following conditions are satisfied.

(i) $\rho$ is unramified at all but finitely many primes.

(ii) $\rho$ is crystalline at each prime $v$ of $K$ above $\ell$ and for each $\tau: K \hookrightarrow \overline{\mathbb{Q}}_\ell$, it has distinct $\tau$-Hodge-Tate weights in the range $[0, \ell - 2]$.

(iii) $\rho$ can be equipped with symplectic (resp. orthogonal) structure with totally odd (resp. totally even) multiplier.

(iv) The residual representation $\bar{\rho}|_{\text{Gal}(\overline{K}/K(\zeta_\ell))}$ is irreducible.
\((v) \ \ell > 2(r + 1)\).

Then \(\rho\) is part of a compatible system.

We remark that their theorem has several assumptions on Galois representations since they use potential automorphy. Correspondingly, Theorem 2.1.1 needs assumptions (ii)-(v) on local systems. We also note that the above theorem is a special case of [6, Theorem C], and the assumption on \(K\) and assumption (ii) can be relaxed using the notion of potential diagonalizability (see Theorem 2.4.1).

The main part of this chapter is devoted to constructing a compatible system of local systems on a scheme from those on curves. As we will see, this result and Theorems 2.1.4 and 2.1.7 yield Theorem 2.1.1. Our method is modeled after Drinfeld’s result (Theorem 2.1.6). So we now explain the outline of his proof.

For a given local system on a scheme, one can obtain a local system on each curve on the scheme by restriction. Conversely, Drinfeld proves that a collection of local systems on curves on a regular scheme defines a local system on the scheme if it satisfies some compatibility and tameness conditions ([20, Theorem 2.5]). See also a remark after Theorem 2.1.8. This method originates from the work of Wiesend on higher dimensional class field theory ([53, 32]).

Drinfeld uses this method to reduce part (iv) of Conjecture 2.1.3 for smooth varieties to the case when \(\dim X = 1\), where he can use Lafforgue’s result. Similarly, one can use his result to reduce the existence of a compatible system of a local system to that of Galois representations and local systems on curves over finite fields.

However, Drinfeld’s result cannot be used to reduce Theorem 2.1.1 to the results of Lafforgue and Barnet-Lamb, Gee, Geraghty, and Taylor since his theorem needs a local system on every curve on the scheme as an input. On the other hand, the results of 33
and [6] only provide compatible systems of local systems on special types of curves on the
scheme: curves over finite fields and *totally real curves*, that is, open subschemes of the
spectrum of the ring of integers of a totally real field. Thus the goal of this chapter is to
deduce Theorem 2.1.1 using the existence of compatible systems of local systems on these
types of curves.

We now explain our method. Fix a prime \( \ell \) and a finite extension \( E_\lambda \) of \( \mathbb{Q}_\ell \). Fix a
positive integer \( r \). Let \( X \) be a normal scheme of finite type over \( \text{Spec} \mathbb{Z}[\ell^{-1}] \) and let \( |X| \)
denote the set of closed points of \( X \). Each \( E_\lambda \)-local system \( \mathcal{E} \) of rank \( r \) on \( X \) defines a
polynomial-valued map

\[
f_\mathcal{E} : |X| \to E_\lambda[t]
\]

of degree \( r \) given by

\[
f_{\mathcal{E},x}(t) = \det(1 - \text{Frob}_x t, \mathcal{E}_x).
\]

Here we say that a polynomial-valued map is of degree \( r \) if its values are polynomials of
degree \( r \). Moreover, \( f_\mathcal{E} \) determines \( \mathcal{E} \) up to semisimplifications by the Chebotarev density
theorem. Conversely, we can ask whether a polynomial-valued map \( f : |X| \to E_\lambda[t] \) of
degree \( r \) arises from a local system of rank \( r \) on \( X \) in this way.

Let \( K \) be a totally real field. Let \( X \) be an irreducible smooth \( \mathcal{O}_K \)-scheme which has
geometrically irreducible generic fiber and satisfies \( X_K(\mathbb{R}) \neq \emptyset \) for every real place \( K \hookrightarrow \mathbb{R} \).
In this situation, we show the following theorem:

**Theorem 2.1.8.** A polynomial-valued map \( f \) of degree \( r \) on \( |X| \) arises from a local system
on \( X \) if and only if it satisfies the following conditions:

(i) The restriction of \( f \) to each totally real curve arises from a local system.

(ii) The restriction of \( f \) to each separated smooth curve over a finite field arises from a
local system.

We prove a similar theorem when $K$ is CM (Theorem 2.3.14).

Drinfeld’s theorem ([20, Theorem 2.5]) involves a similar equivalence, which holds for arbitrary regular schemes of finite type, although his condition (i) is required to hold for arbitrary regular curves and there is an additional tameness assumption in his condition (ii).\footnote{We do not need tameness assumption in condition (ii) in Theorem 2.1.8. This was pointed out by Drinfeld.}

If $K$ and $X$ satisfy the assumptions in Theorem 2.1.1, then we prove a variant of Theorem 2.1.8, where we require condition (i) to hold only for totally real curves which are unramified over $\ell$ (Remark 2.3.13). This variant, combined with the results by Lafforgue and Barnet-Lamb, Gee, Geraghty, and Taylor, implies Theorem 2.1.1.

One of the main ingredients for the proof of these types of theorems is an approximation theorem: One needs to find a curve passing through given points in given tangent directions and satisfying a technical condition coming from a given étale covering. To prove such a theorem Drinfeld uses the Hilbert irreducibility theorem. In our case, we need to further require that such a curve be totally real or CM. For this we use a theorem of Moret-Bailly.

### 2.1.4 Organization of this chapter

In Section 2.2, we review the theorem of Moret-Bailly and prove an approximation theorem for schemes with enough totally real curves. We show a similar theorem in the CM case. In Section 2.3, we prove Theorem 2.1.8 and its variants using the approximation theorems. Most arguments in Section 2.3 originate from Drinfeld’s paper [20]. Finally, we prove main theorems in Section 2.4.
2.1.5 Notation

A number field is a field that is finite over $\mathbb{Q}$. We call a number field a totally real field if it admits no complex place. A CM field is a totally imaginary quadratic extension of a totally real field.

For a number field $E$ and a place $\lambda$ of $E$, we denote by $\overline{E}_\lambda$ a fixed algebraic closure of $E_\lambda$.

For a scheme $X$, we denote by $|X|$ the set of closed points of $X$. We equip finite subsets of $|X|$ with the reduced scheme structure. We denote the residue field of a point $x$ of $X$ by $k(x)$. An étale covering over $X$ means a scheme which is finite and étale over $X$.

For a number field $K$ and an $O_K$-scheme $X$, $X_K$ denotes the generic fiber of $X$ regarded as a $K$-scheme. In particular, for a $K$-algebra $R$, $X_K(R)$ means $\text{Mor}_K(\text{Spec } R, X_K)$, not $\text{Mor}_\mathbb{Z}(\text{Spec } R, X_K)$. We also write $X(R)$ instead of $X_K(R)$.

For simplicity, we omit base points of fundamental groups and we often change base points implicitly.

2.2 Existence of totally real and CM curves via the theorem of Moret-Bailly

First we recall the theorem of Moret-Bailly.

**Theorem 2.2.1** (Moret-Bailly, [39, II]). Let $K$ be a number field. We consider a quadruple $(X_K, \Sigma, \{M_v\}_{v \in \Sigma}, \{\Omega_v\}_{v \in \Sigma})$ consisting of

(i) a geometrically irreducible, smooth and separated $K$-scheme $X_K$,

(ii) a finite set $\Sigma$ of places of $K$,
(iii) a finite Galois extension $M_v$ of $K_v$ for every $v \in \Sigma$, and

(iv) a nonempty $\text{Gal}(M_v/K_v)$-stable open subset $\Omega_v$ of $X_K(M_v)$ with respect to $M_v$-topology.

Then there exist a finite extension $L$ of $K$ and an $L$-rational point $x \in X_K(L)$ satisfying the following two conditions:

- For every $v \in \Sigma$, $L$ is $M_v$-split, that is, $L \otimes_K M_v \cong M_v^{[L:K]}$.

- The images of $x$ in $X_K(M_v)$ induced from embeddings $L \hookrightarrow M_v$ lie in $\Omega_v$.

Remark 2.2.2. Our formulation is slightly different from Moret-Bailly’s, but Theorem 2.2.1 is a simple consequence of [39, II, Théorème 1.3]. Namely, we can always find an integral model $f : X \to B$ of $X_K \to \text{Spec } K$ over a sufficiently small open subscheme $B$ of $\text{Spec } \mathcal{O}_K$ such that $(X \to B, \Sigma, \{M_v\}_{v \in \Sigma}, \{\Omega_v\}_{v \in \Sigma})$ is an incomplete Skolem datum (see [39, II, Définition 1.2]). Then Theorem 2.2.1 follows from [39, II, Théorème 1.3] applied to this incomplete Skolem datum.

Since the set $\Sigma$ can contain infinite places, the above theorem implies the existence of totally real or CM valued points.

Lemma 2.2.3.

(i) Let $K$ be a totally real field and $X_K$ a geometrically irreducible smooth $K$-scheme such that $X_K(\mathbb{R}) \neq \emptyset$ for every real place $K \hookrightarrow \mathbb{R}$. For any dense open subscheme $U_K$ of $X_K$, there exists a totally real extension $L$ of $K$ such that $U_K(L) \neq \emptyset$.

(ii) Let $F$ be a CM field and $Z_F$ a geometrically irreducible smooth $F$-scheme. For any dense open subscheme $V_F$ of $Z_F$, there exists a CM extension $L$ of $F$ such that $V_F(L) \neq \emptyset$.  

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Proof. In either setting, we may assume that the scheme is separated over the base field by replacing it by an open dense subscheme.

First we prove (i). For every real place $v: K \hookrightarrow \mathbb{R}$, let $U_v = U_K \cap (X_K \otimes_{K,v} \mathbb{R})(\mathbb{R})$. It follows from the assumptions and the implicit function theorem that $U_v$ is a nonempty open subset of $(X_K \otimes_{K,v} \mathbb{R})(\mathbb{R})$ with respect to real topology.

We apply the theorem of Moret-Bailly to the datum $(X_K, \{v\}, \{\mathbb{R}\}_v, \{U_v\}_v)$ to find a finite extension $L$ of $K$ and a point $x \in X_K(L)$ such that $L \otimes_K \mathbb{R} \hookrightarrow \mathbb{R}^{[L:K]}$ and the images of $x$ induced from real embeddings $L \hookrightarrow \mathbb{R}$ above $v$ lie in $U_v$. Then $L$ is totally real and $x \in U_K(L)$. Hence $U_K(L) \neq \emptyset$.

Next we prove (ii). Let $F^+$ be the maximal totally real subfield of $F$. Define $Z_{F^+}^+$ (resp. $V_{F^+}^+$) to be the Weil restriction $\operatorname{Res}_{F/F^+} Z_F$ (resp. $\operatorname{Res}_{F/F^+} V_F$). Denote the non-trivial element of $\operatorname{Gal}(F/F^+)$ by $c$ and $Z_F \otimes_{F,c} F$ by $c^* Z_F$. Then we have $Z_{F^+}^+ \otimes_{F^+} F \cong Z_F \times_F c^* Z_F$, and this scheme is geometrically irreducible over $F$. Thus $Z_{F^+}^+$ is a geometrically irreducible smooth $F^+$-scheme, and $V_{F^+}^+$ is dense and open in $Z_{F^+}^+$. Moreover, for every real place $F^+ \hookrightarrow \mathbb{R}$, we can extend it to a complex place $F \hookrightarrow \mathbb{C}$ and get $F \otimes_{F^+} \mathbb{R} \cong \mathbb{C}$. Hence we have $Z_{F^+}^+(\mathbb{R}) = Z_F(\mathbb{C}) \neq \emptyset$. Therefore we can apply (i) to the triple $(F^+, Z_{F^+}^+, V_{F^+}^+)$ and find a totally real extension $L^+$ of $F^+$ such that $V_F(L^+ \otimes_{F^+} F) = V_{F^+}^+(L^+) \neq \emptyset$. Since $L^+ \otimes_{F^+} F$ is a CM extension of $F$, this completes the proof. \square

This lemma leads to the following definitions.

**Definition 2.2.4.** A **totally real curve** is an open subscheme of the spectrum of the ring of integers of a totally real field. A **CM curve** is an open subscheme of the spectrum of the ring of integers of a CM field.

**Definition 2.2.5.** Let $K$ be a totally real field and $X$ an irreducible regular $\mathcal{O}_K$-scheme.
We say that $X$ is an $O_K$-scheme with enough totally real curves if $X$ is flat and of finite type over $O_K$ with geometrically irreducible generic fiber and $X_K(\mathbb{R}) \neq \emptyset$ for every real place $K \hookrightarrow \mathbb{R}$.

Now we introduce some notation and state our approximation theorems.

**Definition 2.2.6.** Let $g: X \to Y$ be a morphism of schemes. For $x \in X$, consider the tangent space $T_xX = \text{Hom}_{k(x)}(m_x/m_x^2, k(x))$ at $x$, where $m_x$ denotes the maximal ideal of the local ring at $x$. This contains $T_x(X_{g(x)})$, where $X_{g(x)} = X \otimes_Y k(g(x))$. A one-dimensional subspace $l$ of $T_xX$ is said to be horizontal (with respect to $g$) if $l$ does not lie in the subspace $T_x(X_{g(x)})$.

**Definition 2.2.7.** Let $X$ be a connected scheme and $Y$ a generically étale $X$-scheme. A point $x \in X(L)$ with some field $L$ is said to be inert in $Y \to X$ if for each irreducible component $Y_\alpha$ of $Y$, $(\text{Spec } L) \times_{x,X} Y_\alpha$ is nonempty and connected.

**Theorem 2.2.8.** Let $K$ be a totally real field and $X$ an irreducible smooth separated $O_K$-scheme with enough totally real curves. Consider the following data:

(i) a flat $O_K$-scheme $Y$ which is generically étale over $X$;

(ii) a finite subset $S \subseteq |X|$ such that $S \to \text{Spec } O_K$ is injective;

(iii) a one-dimensional subspace $l_s$ of $T_sX$ for every $s \in S$.

Then there exist a totally real curve $C$ with fraction field $L$, a morphism $\varphi: C \to X$ and a section $\sigma: S \to C$ of $\varphi$ over $S$ such that $\varphi(\text{Spec } L)$ is inert in $Y \to X$ and $\text{Im}(T_{\sigma(s)}C \to T_sX) = l_s$ for every $s \in S$.

**Proof.** We will use the theorem of Moret-Bailly to find the desired curve. Note that we can replace $Y$ by any dominant étale $Y$-scheme.
Let $v$ denote the structure morphism $X \to \text{Spec} \mathcal{O}_K$. Take an open subscheme $U \subset X$ such that $v(U) \cap v(S) = \emptyset$ and the morphism $Y \times_X U \to U$ is finite and étale. Replacing each connected component of $Y \times_X U$ by its Galois closure, we may assume that each connected component of $Y \times_X U$ is Galois over $U$. Write $Y \times_X U = \bigsqcup_{1 \leq i \leq k} W_i$ as the disjoint union of connected components and denote by $G_i$ the Galois group of the covering $W_i \to U$. Let $H_{i_1}, \ldots, H_{i_{r_i}}$ be all the proper subgroups of $G_i$.

In the context of Theorem 2.2.1, we will choose a quadruple of the form

$$\left( X_K, \{ v_{ij} \}_{i,j}, \{ v(s) \}_{s \in S}, \{ v_{\infty,i} \}_i, \{ K_{v_{ij}} \}_{v_{ij}} \cup \{ M_s \}_{v(s)} \cup \{ R \}_{v_{\infty,i}}, \{ V_{ij} \}_{v_{ij}} \cup \{ V_s \}_{v(s)} \cup \{ V_{\infty,i} \}_{v_{\infty,i}} \right).$$

First we choose $v_{ij}$ and $V_{ij}$ ($1 \leq i \leq k, 1 \leq j \leq r_i$) which control the inverting property.

**Claim 2.2.9.** There exist a finite set $\{ v_{ij} \}_{i,j}$ of finite places of $K$ and a nonempty open subset $V_{ij}$ of $X(K_{v_{ij}})$ with respect to $K_{v_{ij}}$-topology for each $(i, j)$ such that

(i) for any finite extension $L$ of $K$ and any $L$-rational point $x \in X(L)$, $x$ is inert in $Y \to X$ if $L$ is $K_{v_{ij}}$-split and if all the images of $x$ under the induced maps $X(L) \to X(K_{v_{ij}})$ lie in $V_{ij}$, and

(ii) $v_{ij}$ are different from any element of $v(S)$ regarded as a finite place of $K$.

**Proof of Claim 2.2.9.** For each $i = 1, \ldots, k$ and $j = 1, \ldots, r_i$, let $\pi_{H_{ij}}$ denote the induced morphism $W_i/H_{ij} \to X$ and let $M_{ij}$ be the algebraic closure of $K$ in the field of rational functions of $W_i/H_{ij}$. Then we have a canonical factorization $W_i/H_{ij} \to \text{Spec} \mathcal{O}_{M_{ij}} \to \text{Spec} \mathcal{O}_K$.

If $M_{ij} = K$, then the generic fiber $(W_i/H_{ij})_K$ is geometrically integral over $K$. It follows from [17, Proposition 3.5.2] that there are infinitely many finite places $v_0$ of $K$ such that
$U(K_{v_0}) \setminus \pi_{H_{ij}}(W_i/H_{ij}(K_{v_0}))$ is a nonempty open subset of $U(K_{v_0})$. Thus choose such a finite place $v_{ij}$ and put

$$V_{ij} = U(K_{v_{ij}}) \setminus \pi_{H_{ij}}(W_i/H_{ij}(K_{v_{ij}})).$$

Next consider the case where $M_{ij} \neq K$. The Lang-Weil theorem and the Chebotarev density theorem show that there are infinitely many finite places $v_0$ of $K$ such that $U(K_{v_0}) \neq \emptyset$ and $v_0$ does not split completely in $M_{ij}$, that is, $M_{ij} \otimes_K K_{v_0} \neq K_{v_0}^{[M_{ij}:K]}$ (see [47, Propositions 3.5.1 and 3.6.1] for example). In this case, choose such a finite place $v_{ij}$ and put

$$V_{ij} = U(K_{v_{ij}}).$$

It is obvious to see that we can choose $v_{ij}$ satisfying condition (ii). We now show that these $v_{ij}$ and $V_{ij}$ satisfy condition (i). Take $L$ and $x \in X(L)$ as in condition (i). By the lemma below (Lemma 2.2.10), it suffices to prove that $x \notin \pi_{H_{ij}}(W_i/H_{ij}(L))$ for any $H_{ij}$.

When $M_{ij} = K$, this is obvious because the images of $x$ under the maps $X(L) \to X(K_{v_{ij}})$ lie in $V_{ij} = U(K_{v_{ij}}) \setminus \pi_{H_{ij}}(W_i/H_{ij}(K_{v_{ij}}))$. When $M_{ij} \neq K$, we know that $M_{ij} \otimes_K K_{v_{ij}}$ is not $K_{v_{ij}}$-split. Since $L$ is assumed to be $K_{v_{ij}}$-split, $M_{ij}$ cannot be embedded into $L$. On the other hand, we have a canonical factorization $W_i/H_{ij} \to \text{Spec} \mathcal{O}_{M_{ij}}$. Therefore $W_i/H_{ij}(L) = \emptyset$. Thus $x$ is inert in $Y \to X$ in both cases. \qed

Next we choose a finite Galois extension $M_s$ of $K_{v(s)}$ and a $\text{Gal}(M_s/K_{v(s)})$-stable nonempty open subset $V_s$ of $X(M_s)$ with respect to $M_s$-topology to make a totally real curve pass through $s$ in the tangent direction $l_s$. Here $K_{v(s)}$ denotes the completion of $K$ with respect to the finite place $v(s)$ of $K$. Let $\hat{O}_{X,s}$ denote the completed local ring of $X$ at $s \in S$. Since $\hat{O}_{X,s}$ is regular, we can find a regular one-dimensional closed subscheme
Spec $R_s \subset \text{Spec } \hat{\mathcal{O}}_{X,s}$ which is tangent to $l_s$ and satisfies $R_s \otimes_K K \neq \emptyset$ (see \cite[Lemma A.6]{20}).

It follows from the construction that $R_s$ is a complete discrete valuation ring which is finite and flat over $\mathcal{O}_{K_{v(s)}}$ and has residue field $k(s)$. Let $M'_s$ be the fraction field of $R_s$. For each $s \in S$ we first choose $M_s$ and a local homomorphism $\mathcal{O}_{X,s} \rightarrow \mathcal{O}_{M_s}$. There are two cases.

If $l_s$ is horizontal, then $R_s$ is unramified over $\mathcal{O}_{K_{v(s)}}$ and hence $M'_s$ is Galois over $K_{v(s)}$. Put $M_s := M'_s$ in this case. Then we have a natural local homomorphism $\mathcal{O}_{X,s} \rightarrow \hat{\mathcal{O}}_{X,s} \rightarrow \mathcal{O}_{M_s}$.

If $l_s$ is not horizontal, then $R_s$ is ramified over $\mathcal{O}_{K_{v(s)}}$. Let $K'_{v(s)}$ be the maximal unramified extension of $K_{v(s)}$ in $M'_s$ and $M_s$ the Galois closure of $M'_s$ over $K_{v(s)}$. Then both $K'_{v(s)}$ and $M_s$ have the same residue field $k(s)$.

We construct a local homomorphism $\hat{\mathcal{O}}_{X,s} \rightarrow \mathcal{O}_{M_s}$ in this setting. Since $X$ is smooth over $\mathcal{O}_K$, the ring $\hat{\mathcal{O}}_{X,s}$ is isomorphic to the ring of formal power series $\mathcal{O}_{K_{v(s)}}[[t_1, \ldots, t_m]]$ for some $m$ and we identify these rings.

Let $u_i \in \mathcal{O}_{M'_s}$ denote the image of $t_i$ under the homomorphism $\mathcal{O}_{K_{v(s)}}[[t_1, \ldots, t_m]] = \hat{\mathcal{O}}_{X,s} \rightarrow R_s = \mathcal{O}_{M'_s}$. Let $\pi$ (resp. $\varpi$) be a uniformizer of $\mathcal{O}_{M'_s}$ (resp. $\mathcal{O}_{M_s}$) and consider $\pi$-adic expansion $u_i = \sum_{j=0}^{\infty} a_{ij} \pi^j$. Since $\hat{\mathcal{O}}_{X,s} \rightarrow \mathcal{O}_{M'_s}$ is a local homomorphism, we have $a_{i0} = 0$ for each $i$.

Consider the differential of $\text{Spec } \mathcal{O}_{M'_s} = \text{Spec } R_s \rightarrow \text{Spec } \hat{\mathcal{O}}_{X,s}$ at the closed point. The tangent vector $\frac{\partial}{\partial \pi}$ is sent to $\sum_{i=1}^{m} a_{i1} \frac{\partial}{\partial t_i}$ under this map, and the latter spans the tangent line $l_s$.

Define a local homomorphism $\hat{\mathcal{O}}_{X,s} \rightarrow \mathcal{O}_{M_s}$ by sending $t_i$ to $\sum_{j=1}^{\infty} a_{ij} \varpi^j$. Then the image of the differential of the corresponding morphism $\text{Spec } \mathcal{O}_{M_s} \rightarrow X$ at the closed point is $l_s$ by the same computation as above.
In either case, we have chosen $M_s$ and a homomorphism $\mathcal{O}_{X,s} \to \mathcal{O}_{M_s}$. Let $\hat{s} \in X(\mathcal{O}_{M_s})$ be the point induced by the homomorphism. Note that $X(\mathcal{O}_{M_s})$ is an open subset of $X(M_s)$ by separatedness. Let $\alpha : X(\mathcal{O}_{M_s}) \to X(\mathcal{O}_{M_s}/\mathfrak{m}_{M_s}^2)$ be the reduction map, where $\mathfrak{m}_{M_s}$ denotes the maximal ideal of $\mathcal{O}_{M_s}$. Define $V'_{\hat{s}} = \alpha^{-1}(\alpha(\hat{s}))$, which is a nonempty open subset of $X(M_s)$, and put

$$V_s = \bigcup_{\sigma} \sigma(V'_{\hat{s}}),$$

where $\sigma$ runs over all the elements of $\text{Gal}(M_s/K_{v(s)})$. Since $\text{Gal}(M_s/K_{v(s)})$ acts continuously on $X(M_s)$, $V_s$ is a nonempty $\text{Gal}(M_s/K_{v(s)})$-stable open subset of $X(M_s)$.

Finally, let $v_{\infty,1}, \ldots, v_{\infty,n}$ be the real places of $K$ and put

$$V_{\infty,i} = X(\mathbb{R})$$

for each $i = 1, \ldots, n$. This is nonempty by our assumption.

It follows from the theorem of Moret-Bailly (Theorem 2.2.1) that there exist a finite extension $L$ of $K$ and an $L$-rational point $x \in X(L)$ satisfying the following properties:

(i) $L \otimes_K K_{v_{i_j}}$ is $K_{v_{i_j}}$-split and $x$ goes into $V_{i_j}$ under any embedding $L \hookrightarrow K_{v_{i_j}}$.

(ii) $L \otimes_K M_s$ is $M_s$-split and $x$ goes into $V_s$ under any embedding $L \hookrightarrow M_s$.

(iii) $L$ is totally real.

We can spread out the $L$-rational point $x : \text{Spec } L \to X$ to a morphism $\varphi : C \to X$ where $C$ is a totally real curve with fraction field $L$. By property (ii), we can choose $C$ and $\varphi$ so that all the points of $\text{Spec } \mathcal{O}_L$ above $v(S) \subset \text{Spec } \mathcal{O}_K$ are contained in $C$. Claim 2.2.9 shows that $x$ is inert in $Y \to X$. Thus it remains to prove that there exists a section $\sigma$ of $\varphi$ over $S$ such that $\text{Im}(T_{\sigma(s)}C \to T_sX) = I_s$ for every $s \in S$.
It follows from property (ii) and the definition of \( V_s \) that there exists an embedding \( L \rightarrow M_s \) such that the image of \( x \) under the associated map \( X(L) \rightarrow X(M_s) \) lies in \( V'_s \). Let \( s' \in \text{Spec} \mathcal{O}_L \) be the closed point corresponding to this embedding. Then we have \( s' \in C, k(s') = k(s) \), and \( \text{Im}(T_{s'}C \rightarrow T_sX) = l_s \). Hence we can define a desired section of \( \varphi \) over \( S \). 

\[ \square \]

**Lemma 2.2.10.** Let \( L \) be a field, \( U \) a locally noetherian connected scheme and \( \pi: W \rightarrow U \) a Galois covering with Galois group \( G \). For any subgroup \( H \subset G \), let \( \pi_H \) denote the induced morphism \( W/H \rightarrow U \). An \( L \)-valued point of \( X \) is inert in \( \pi \) if and only if it lies in \( U(L) \setminus \bigcup_{H \subset G} \pi_H(W/H(L)) \).

**Proof.** Let \( x \) denote the \( L \)-valued point. Choose a point of \( W \) above \( x \) and fix a geometric point above it. This also defines a geometric point above \( x \) and we have a homomorphism \( \pi_1(x) \rightarrow G \), where \( \pi_1(x) \) is the absolute Galois group of \( L \). Let \( H_0 \) denote the image of this homomorphism. Then \( x \) is inert in \( \pi \) if and only if the homomorphism is surjective, that is, \( H_0 = G \). On the other hand, for a subgroup \( H \subset G \), the point \( x \) lies in \( \pi_H(W/H(L)) \) if and only if \( \text{Spec} \, L \times_{x,U} W/H \rightarrow \text{Spec} \, L \) has a section, which is equivalent to the condition that some conjugate of \( H \) contains \( H_0 \). The lemma follows from these two observations. \[ \square \]

For our applications, we need a stronger variant of the theorem.

**Corollary 2.2.11.** Let \( K \) be a totally real field and \( X \) an irreducible smooth separated \( \mathcal{O}_K \)-scheme with enough totally real curves. Let \( U \) be a nonempty open subscheme of \( X \). Suppose that we are given the following data:

(i) a flat \( \mathcal{O}_K \)-scheme \( Y \) which is generically \( \text{étale} \) over \( X \);

(ii) a closed normal subgroup \( H \subset \pi_1(U) \) such that \( \pi_1(U)/H \) contains an open pro-\( \ell \) subgroup;
(iii) a finite subset $S \subset |X|$ such that $S \to \text{Spec} \mathcal{O}_K$ is injective;

(iv) a one-dimensional subspace $l_s$ of $T_sX$ for every $s \in S$.

Then there exist a totally real curve $C$ with fraction field $L$, a morphism $\varphi : C \to X$ with $\varphi^{-1}(U) \neq \emptyset$ and a section $\sigma : S \to C$ of $\varphi$ over $S$ such that

- $\varphi(\text{Spec} L)$ is inert in $Y \to X$,
- $\pi_1(\varphi^{-1}(U)) \to \pi_1(U)/H$ is surjective, and
- $\text{Im}(T_{\sigma(s)}C \to T_sX) = l_s$ for every $s \in S$.

Proof. As is shown in the proof of [21, Proposition 2.17], we can find an open normal subgroup $G_0 \subset \pi_1(U)/H$ satisfying the following property: Every closed subgroup $G \subset \pi_1(U)/H$ such that the map $G \to (\pi_1(U)/H)/G_0$ is surjective equals $\pi_1(U)/H$.

Let $Y'$ be the Galois covering of $U$ corresponding to $G_0$. Then we can apply Theorem 2.2.8 to $(Y \cup Y', S, (l_s)_{s \in S})$ and get the desired triple $(C, \varphi, \sigma)$. \qed

We have a similar approximation theorem in the CM case. The proof uses the Weil restriction and is essentially similar to the totally real case, although one has to check that the conditions are preserved under the Weil restriction.

**Theorem 2.2.12.** Let $F$ be a CM field and $Z$ an irreducible smooth separated $\mathcal{O}_F$-scheme with geometrically irreducible generic fiber. Let $U$ be a nonempty open subscheme of $Z$. Suppose that we are given the following data:

(i) a flat $\mathcal{O}_F$-scheme $W$ which is generically étale over $Z$;

(ii) a closed normal subgroup $H \subset \pi_1(U)$ such that $\pi_1(U)/H$ contains an open pro-$\ell$ subgroup;
(iii) a finite subset \( S \subset |Z| \) such that \( S \rightarrow \text{Spec} \, \mathcal{O}_{F^+} \) is injective;

(iv) a one-dimensional subspace \( l_s \) of \( T_s Z \) for every \( s \in S \).

Then there exist a CM curve \( C \) with fraction field \( L \), a morphism \( \varphi: C \rightarrow Z \) with \( \varphi^{-1}(U) \neq \emptyset \) and a section \( \sigma: S \rightarrow C \) of \( \varphi \) over \( S \) such that

- \( \varphi(\text{Spec} \, L) \) is inert in \( W \rightarrow Z \),
- \( \pi_1(\varphi^{-1}(U)) \rightarrow \pi_1(U)/H \) is surjective, and
- \( \text{Im}(T_{\sigma(s)}C \rightarrow T_s Z) = l_s \) for every \( s \in S \).

**Proof.** Let \( F^+ \) be the maximally totally real subfield of \( F \). Let \( w \) (resp. \( v \)) denote the structure morphism \( Z \rightarrow \text{Spec} \, \mathcal{O}_F \) (resp. \( Z \rightarrow \text{Spec} \, \mathcal{O}_{F^+} \)). As in the proof of Corollary 2.2.11 we may omit the datum (ii) by replacing \( W \) by another flat, generically étale \( Z \)-scheme and prove the first and third properties of the triple \((C, \varphi, \sigma)\).

Define \( Z^+ \) to be the Weil restriction \( \text{Res}_{\mathcal{O}_F/\mathcal{O}_{F^+}} Z \). Then we have \( Z^+ \otimes_{\mathcal{O}_{F^+}} \mathcal{O}_F \cong Z \times_{\mathcal{O}_F} c^* Z \), where \( c \) denotes the nontrivial element of \( \text{Gal}(F/F^+) \) and \( c^* Z \) denotes \( Z \otimes_{\mathcal{O}_{F^+}} \mathcal{O}_F \). It follows from the assumptions that \( Z^+ \) is an irreducible smooth \( \mathcal{O}_{F^+} \)-scheme with enough totally real curves. We will apply the theorem of Moret-Bailly to \( Z^+ \) with appropriate data.

We may assume that each connected component of \( W \) is a Galois cover over its image in \( Z \) by replacing \( W \) if necessary. Put \( Y = W \times_{\mathcal{O}_F} c^* W \) and regard it as an \( \mathcal{O}_{F^+} \)-scheme. Then \( Y \rightarrow Z \times_{\mathcal{O}_F} c^* Z \rightarrow Z^+ \) is flat and generically étale, and therefore satisfies the same assumptions as \( Y \rightarrow X \) in Theorem 2.2.8 and the second paragraph of its proof. Hence, as in Claim 2.2.9 in the proof of Theorem 2.2.8 there exist a finite set \( \{v_{ij}\}_{1 \leq i \leq k, 1 \leq j \leq r} \) of finite places of \( F^+ \) and a nonempty open subset \( V_{ij} \) of \( Z^+(F^+_{v_{ij}}) \) for each \((i, j)\) satisfying the following properties:
(i) For any finite extension \(L^+\) of \(F^+\) and any \(L^+\)-rational point \(z^+ \in Z^+(L^+)\), \(z^+\) is inert in \(Y \rightarrow Z^+\) if \(L^+\) is \(F_{v_{ij}}^+\)-split and if \(z^+\) lands in \(V_{ij}\) under any embedding \(L^+ \hookrightarrow F_{v_{ij}}^+\).

(ii) \(v_{ij}\) are different from any element of \(v(S)\).

Next take any \(s \in S\). We will choose a finite Galois extension \(M_s\) of \(F_{w(s)}\) and a \(\text{Gal}(M_s/F_{v(s)})\)-stable nonempty subset of \(Z^+(M_s)\) with respect to \(M_s\)-topology. Here we regard \(w(s)\) (resp. \(v(s)\)) as a finite place of \(F\) (resp. \(F^+\)). Then \(M_s\) is Galois over \(F_{v(s)}^+\) since \(w(s)\) lies above \(v(s)\) and \([F_{w(s)} : F_{v(s)}^+]\) is either 1 or 2.

As in the proof of Theorem 2.2.8 we can find a finite Galois extension \(M_s\) of \(F_{w(s)}\) with residue field \(k(s)\) and a homomorphism \(\mathcal{O}_{Z,s} \rightarrow \mathcal{O}_{M_s}\) such that the image of the differential of the corresponding morphism \(\text{Spec} \mathcal{O}_{M_s} \rightarrow Z\) at the closed point is \(l_s\). Denote by \(\hat{s} \in Z(\mathcal{O}_{M_s}) \subset Z(M_s)\) the point corresponding to this morphism.

Let \(\alpha: Z(\mathcal{O}_{M_s}) \rightarrow Z(\mathcal{O}_{M_s}/\mathfrak{m}_{M_s}^2)\) be the reduction map, where \(\mathfrak{m}_{M_s}\) denotes the maximal ideal of \(\mathcal{O}_{M_s}\). Define \(V'_s = \alpha^{-1}(\alpha(\hat{s}))\), which is a nonempty open subset of \(Z(M_s)\), and put \(V''_s = \bigcup_{\sigma} \sigma(V'_s)\) where \(\sigma\) runs over all the elements of \(\text{Gal}(M_s/F_{w(s)})\).

Denote by \(\iota\) a natural embedding \(F \hookrightarrow F_{w(s)} \hookrightarrow M_s\). We have \(\text{Hom}_{F^+}(F, M_s) = \{\iota, \iota \circ c\}\) and the \(F^+\)-homomorphism \((\iota, \iota \circ c): F \rightarrow M_s \times M_s\) induces an isomorphism \(M_s \otimes_{F^+} F \cong M_s \times M_s\) which sends \(a \otimes b\) to \((\iota(a(b)), a(\iota(c(b)))\). Hence we get identifications \(Z^+(M_s) = Z(M_s \otimes_{F^+} F) = Z(M_s) \times c^*Z(M_s)\). Here \(Z(M_s)\) denotes \(\text{Hom}_{\mathcal{O}_F}(\text{Spec} M_s, Z)\) by regarding \(\text{Spec} M_s\) as an \(F\)-scheme via \(\iota\).

Define

\[ V_s := V''_s \times c^*V''_s \subset Z(M_s) \times c^*Z(M_s) = Z^+(M_s). \]

This is a nonempty open subset of \(Z^+(M_s)\). Since \(V''_s\) is \(\text{Gal}(M_s/F_{w(s)})\)-stable and \(\text{Gal}(F/F^+) = \{\text{id}, c\}\), \(V_s\) is \(\text{Gal}(M_s/F_{v(s)})\)-stable.
Let $v_{\infty,1}, \ldots, v_{\infty,n}$ be the real places of $F^+$. Then for each $1 \leq i \leq n$ put

$$V_{\infty,i} = Z^+(\mathbb{R}) = Z(\mathbb{C})$$

via an isomorphism $F \otimes_{F^+} \mathbb{R} \cong \mathbb{C}$. This is a nonempty open set.

We apply Theorem 2.2.1 to the quadruple

$$(Z^+_F, \{v_{ij}\}_{i,j} \cup \{v(s)\}_{s \in S} \cup \{v_{\infty,i}\}_{i}, \{F^+_v\} \cup \{M_s\} \cup \{\mathbb{R}\}, \{V_{ij}\} \cup \{V_s\} \cup \{V_{\infty,i}\})$$

and find a totally real finite extension $L^+$ of $F^+$ and an $L^+$-rational point $z^+ \in Z^+(L^+)$ satisfying the following properties:

(i) $z^+$ is inert in $Y \to Z^+$.

(ii) $L^+$ is $M_s$-split and $z^+$ goes into $V_s$ via any embedding $L^+ \hookrightarrow M_s$.

Let $L$ be the CM field $L^+ \otimes_{F^+} F$ and $z \in Z(L)$ be the $L$-rational point corresponding to $z^+ \in Z^+(L^+)$. Then the morphism $z$ is equal to the composite

$$\text{pr}_Z \circ (z^+ \otimes_{F^+} F): \text{Spec} \ L \to Z^+ \otimes_{\mathcal{O}_{F^+}} \mathcal{O}_F = Z \times_{\mathcal{O}_F} c^*Z \to Z.$$

We can spread out $z: \text{Spec} \ L \to Z$ to a morphism $\varphi: C \to Z$ for some CM curve $C$ with fraction field $L$. We may assume that $C$ contains all the points of $\text{Spec} \mathcal{O}_L$ above $w(S) \subset \text{Spec} \mathcal{O}_F$. It follows from property (ii) and the definition of $V_s$ that $\varphi$ has a section $\sigma$ over $S$ such that $\text{Im}(T_{\sigma(s)}C \to T_sX) = l_s$ for every $s \in S$.

It remains to prove that $z = \varphi(\text{Spec} \ L)$ is inert in $W \to Z$. Without loss of generality, we may assume that $W_F$ is connected, and thus it suffices to show that $\text{Spec} \ L \times_{z,Z} W = \text{Spec} \ L \times_{z,Z} W_F$ is connected. Define the schemes $P$ and $Q$ such that the squares in the
following diagram are Cartesian:

\[
\begin{array}{cccccc}
P & \rightarrow & Q & \rightarrow & \text{Spec } L & \rightarrow \text{Spec } L^+ \\
\downarrow & & \downarrow & & \downarrow & \\
W_F \times_F c^*W_F & \rightarrow & W_F \times_F c^*Z_F & \rightarrow & Z_F \times_F c^*Z_F & \rightarrow Z_{F^+}^+ \\
\downarrow & & \downarrow & & \downarrow & \\
W_F & \rightarrow & Z_F.
\end{array}
\]

Since \( W_F \times_F c^*W_F = Y_{F^+} \), we have \( P \cong \text{Spec } L^+ \times_{z^+, Z_{F^+}} Y_{F^+} \). As \( Q \cong \text{Spec } L \times_{z, Z_F} W_F \), we need to show that \( Q \) is connected.

Note that \( W_F \times_F c^*Z_F \) is connected; this follows from the fact that \( c^*Z_F \) is geometrically connected over \( F \) and \( W_F \) is connected. Now take any connected component \( T \) of \( Y_{F^+} = W_F \times_F c^*W_F \). Since \( W_F \times_F c^*W_F \rightarrow W_F \times_F c^*Z_F \) is an étale covering with connected base, \( T \) surjects onto \( W_F \times_F c^*Z_F \). At the same time, the subscheme \( \text{Spec } L^+ \times_{z^+, Z_{F^+}} T \subset P \) is connected because \( z^+ \) is inert in \( Y \rightarrow Z^+ \). Since the connected scheme \( \text{Spec } L^+ \times_{z^+, Z_{F^+}} T \) surjects onto \( Q \), the latter is also connected.

\[ \square \]

\textbf{Remark 2.2.13.} In Theorem 2.2.8, Corollary 2.2.11 and Theorem 2.2.12 we assume that the scheme in question is smooth and separated. If \( S = \emptyset \), then we can replace these two assumptions by regularity. In fact, if \( S = \emptyset \), we can replace the scheme by an open subscheme, and thus reduce to the separated case. Moreover, the regularity implies that the generic fiber of the scheme is smooth. So we can apply the theorem of Moret-Bailly to our scheme. Note that the smoothness assumption was used only when \( S \neq \emptyset \) and \( l_s \) is not horizontal for some \( s \in S \).
2.3 Proofs of Theorem 2.1.8 and its variants

In this section, we prove Theorem 2.1.8 and its variants following [20]. First we set up our notation. Fix a prime $\ell$ and a finite extension $E_\lambda$ of $\mathbb{Q}_\ell$. Let $\mathcal{O}$ be the ring of integers of $E_\lambda$ and $\mathfrak{m}$ its maximal ideal.

Fix a positive integer $r$. For a normal scheme $X$ of finite type over $\text{Spec} \mathbb{Z}$, we say that two $E_\lambda$-local systems on $X$ are equivalent if they have isomorphic semisimplifications. Let $\text{LS}^{E_\lambda}(X)$ denote the set of equivalence classes of $E_\lambda$-local systems on $X$ of rank $r$, and let $\widetilde{\text{LS}}^{E_\lambda}(X)$ denote the set of maps from the set of closed points of $X$ to the set of polynomials of the form $1 + c_1 t + \cdots + c_r t^r$ with $c_i \in \mathcal{O}$ and $c_r \in \mathcal{O}^\times$. Since the coefficient field $E_\lambda$ is fixed throughout this section, we simply write $\text{LS}_r(X)$ or $\widetilde{\text{LS}}_r(X)$.

For an element $f \in \widetilde{\text{LS}}_r(X)$, we denote by $f_x(t)$ or $f(x)(t)$ the value of $f$ at $x \in X$; this is a polynomial in $t$. By the Chebotarev density theorem, we can regard $\text{LS}_r(X)$ as a subset of $\widetilde{\text{LS}}_r(X)$ by attaching to each equivalence class its Frobenius characteristic polynomials. An element $f \in \widetilde{\text{LS}}_r(X)$ is said to arise from a local system if $f \in \text{LS}_r(X)$. For another scheme $Y$ and a morphism $\alpha: Y \to X$, we have a canonical map $\alpha^*: \widetilde{\text{LS}}_r(X) \to \widetilde{\text{LS}}_r(Y)$ whose restriction to $\text{LS}_r(X)$ coincides with the pullback map of sheaves $\text{LS}_r(X) \to \text{LS}_r(Y)$. We also denote $\alpha^*(f)$ by $f|_Y$.

Let $C$ be a separated smooth curve over a finite field and $\overline{C}$ the smooth compactification of $C$. We define $\text{LS}^{\text{tame}}_r(C)$ to be the subset of $\text{LS}_r(C)$ consisting of equivalence classes of $E_\lambda$-local systems on $C$ which are tamely ramified at each point of $\overline{C} \smallsetminus C$. This condition does not depend on the choice of a local system in the equivalence class. Let $\varphi$ be a morphism $C \to X$ and $f \in \widetilde{\text{LS}}_r(X)$. When $\varphi^*(f) \in \text{LS}_r(C)$ (resp. $\varphi^*(f) \in \text{LS}^{\text{tame}}_r(C)$), we simply say that $f$ arises from a local system (resp. a tame local system) over the curve $C$. 

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To show \[20\] Theorem 2.5], which is a prototype of Theorem \[2.1.8\], Drinfeld considers a
subset \(LS'\) of \(\tilde{\mathcal{S}}_r(X)\) which contains \(LS_r(X)\) and is characterized by a group-theoretic
property. He then proves the following three statements for \(f \in \tilde{\mathcal{S}}_r(X)\), which imply \[20\]
Theorem 2.5].

- If the map \(f\) satisfies two conditions\(^3\) similar to those in Theorem \[2.1.8\], then \(f|_U \in \)
\(LS'_r(U)\) for some dense open subscheme \(U \subset X\).

- If \(U\) is regular, then \(LS'_r(U) = LS_r(U)\). In particular, the restriction \(f|_U \in \)
arises from a local system.

- If \(f|_U\) arises from a local system, then so does \(f\) under the assumptions that \(X\) is
regular and that \(f\) arises from a local system over every regular curve \(C\).

Following Drinfeld, we will introduce the group-theoretic notion of “having a kernel”
and prove similar statements, Propositions \[2.3.4\] \[2.3.10\] \[2.3.11\] and \[2.1.8\] and
its variants will be deduced from them at the end of the section.

**Definition 2.3.1.** Let \(X\) be a scheme of finite type over \(\mathbb{Z}[\ell^{-1}]\) and \(f \in \tilde{\mathcal{S}}_r(X)\). For a
nonzero ideal \(I \subset \mathcal{O}\), the map \(f\) is said to be *trivial modulo* \(I\) if it has the value congruent
to \((1 - t)^r\) modulo \(I\) at every closed point of \(X\).

When \(X\) is connected, the map \(f\) is said to have a kernel if there exists a closed normal
subgroup \(H \subset \pi_1(X)\) satisfying the following conditions:

(i) \(\pi_1(X)/H\) contains an open pro-\(\ell\) subgroup.

(ii) For every \(n \in \mathbb{N}\), there exists an open subgroup \(H_n \subset \pi_1(X)\) containing \(H\) such that
the pullback of \(f\) to \(X_n\) is trivial modulo \(m^n\). Here \(X_n\) denotes the covering of \(X\)
corresponding to \(H_n\).

\(^3\)One needs tameness assumption in the second condition (it is identical to condition (ii) of Propositions
\[2.3.4\].
When $X$ is disconnected, the map $f$ is said to have a kernel if the restriction of $f$ to each connected component of $X$ has a kernel.

Remark 2.3.2. If $f$ arises from a local system on $X$, it has a kernel. To see this, we may assume that $X$ is connected. Then the kernel of the $E_\lambda$-representation of $\pi_1(X)$ corresponding to the local system satisfies the conditions.

Remark 2.3.3. The set $\text{LS}_r(X)$ defined by Drinfeld consists of the maps $f$ which have a kernel and arise from a local system over every regular curve ([20 Definition 2.11]).

Proposition 2.3.4. Let $K$ be a totally real field. Let $X$ be an irreducible regular $\mathcal{O}_K[\ell^{-1}]$-scheme with enough totally real curves and $f \in \mathcal{L}_S(X)$. Assume that

(i) $f$ arises from a local system over every totally real curve, and

(ii) there exists a dominant étale morphism $X' \to X$ such that the pullback $f|_{X'}$ arises from a tame local system over every separated smooth curve over a finite field.

Then there exists a dense open subscheme $U \subset X$ such that $f|_U$ has a kernel.

We will first show two lemmas and then prove Proposition 2.3.4 by induction on the dimension of $X$. For this we use elementary fibrations, which we recall now.

Definition 2.3.5. A morphism of schemes $\pi : X \to S$ is called an elementary fibration if there exist an $S$-scheme $\pi : \overline{X} \to S$ and a factorization $X \to \overline{X} \xrightarrow{\pi} S$ of $\pi$ such that

(i) the morphism $X \to \overline{X}$ is an open immersion and $X$ is fiberwise dense in $\overline{\pi} : \overline{X} \to S$,

(ii) $\pi$ is a smooth and projective morphism whose geometric fibers are nonempty irreducible curves, and

(iii) the reduced closed subscheme $\overline{X} \setminus X$ is finite and étale over $S$. 

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The next lemma, which is due to Drinfeld and Wiesend, is a key to our induction argument in the proof of Proposition 2.3.4.

**Lemma 2.3.6.** Let $X$ be a scheme of finite type over $\mathbb{Z}[\ell^{-1}]$ and $f \in \widehat{\text{LS}}(X)$. Suppose that $X$ admits an elementary fibration $X \to S$ with a section $\sigma : S \to X$. Assume that

(i) $f$ arises from a tame local system over every fiber of $X \to S$, and

(ii) there exists a dense open subscheme $V \subset S$ such that $\sigma^*(f)|_V$ has a kernel.

Then there exists a dense open subscheme $U \subset X$ such that $f|_U$ has a kernel.

**Proof.** This is shown in the latter part of the proof of [20, Lemma 3.1]. For convenience of the reader, we summarize the proof below.

We may assume that $X$ is connected and normal, and that $V = S$. For every $n \in \mathbb{N}$, consider the functor which attaches to an $S$-scheme $S'$ the set of isomorphism classes of $\text{GL}_r(O/m^n)$-torsors on $X \times_S S'$ tamely ramified along $(X \setminus X) \times_S S'$ relative to $S'$ with trivialization over the section $S' \hookrightarrow X \times_S S'$. Then this functor is representable by an étale scheme $T_n$ of finite type over $S$ and the morphism $T_{n+1} \to T_n$ is finite for each $n$. By shrinking $S$, we may assume that the morphism $T_n \to S$ is finite for each $n$. We will prove that $f$ has a kernel in this situation.

Since $\sigma^*(f)$ has a kernel by assumption (ii), there exist connected étale coverings $S_n$ of $S$ such that

- the pullback of $\sigma^*(f)$ to $S_n$ is trivial modulo $m^n$, and

- for some (or any) geometric point $\bar{s}$ of $S$, the quotient of the group $\pi_1(S, \bar{s})$ by the intersection of the kernels of its actions on the fibers $(S_n)_{\bar{s}}$ where $n$ runs in $\mathbb{N}$ contains an open pro-$\ell$ subgroup.
Let $\mathcal{F}_n$ be the universal tame $\text{GL}_n(O/m^n)$-torsor over $X \times_S T_n$. Define the $X$-scheme $Y_n$ to be the Weil restriction $\text{Res}_{X \times_S T_n/X} \mathcal{F}_n$ and let $X_n$ denote $Y_n \times_S S_n$. We thus have a diagram whose squares are Cartesian

\[
\begin{array}{ccc}
X_n & \rightarrow & X \times_S S_n \\
\downarrow & & \downarrow \\
Y_n & \rightarrow & S
\end{array}
\]

and regard $X_n$ as an étale covering of $X$. Here the morphism $S_n \rightarrow X$ is the composite of $S_n \rightarrow S$ and the section $\sigma : S \rightarrow X$.

It suffices to prove the following two assertions:

(a) The pullback of $f$ to $X_n$ is trivial modulo $m^n$.

(b) For some (or any) geometric point $\bar{x}$ of $X$, the quotient of the group $\pi_1(X, \bar{x})$ by the intersection of the kernels of its actions on the fibers $(X_n)_\bar{x}$ where $n$ runs in $\mathbb{N}$ contains an open pro-$\ell$ subgroup.

In fact, if we take a Galois covering $X'_n$ of $X$ splitting the (possibly disconnected) covering $X_n$, the corresponding subgroup $H_n := \pi(X'_n, \bar{x}) \subset \pi_1(X, \bar{x})$ and the intersection $H := \bigcap_n H_n$ satisfy the conditions for the map $f$ to have a kernel.

First we prove assertion (a). Take an arbitrary closed point $x \in X_n$. Let $s \in S$ denote the image of $x$ and choose a geometric point $\bar{s}$ above $s \in S$. By assumption (i), the restriction $f|_{X_s}$ arises from an $E_\ell$-local system of rank $r$. Let $\mathcal{F}$ be a locally constant constructible sheaf of free $(O/m^n)$-modules of rank $r$ obtaining from the above local system modulo $m^n$.

Consider the $X_s$-scheme $(Y_n)_\bar{s}$. The scheme $(X \times_S T_n)_\bar{s}$ is the disjoint union of copies of $X_s$, and $(\mathcal{F}_n)_\bar{s}$ is a disjoint union of the $\text{GL}_r(O/m^n)$-torsors, each of which lies above a
copy of $X_{\bar{s}}$ in $(X \times_S T_n)_{\bar{s}}$. Since the Weil restriction and the base change commute, $(Y_n)_{\bar{s}}$ is the fiber product of the tame $\text{GL}_r(\mathcal{O}/m^n)$-torsors over $X_{\bar{s}}$. Hence $\mathcal{F}|_{(Y_n)_{\bar{s}}}$ is constant, and so is $\mathcal{F}|_{(X_n)_{\bar{s}}}$.

Now let $s' \in S_n$ be the image of $x$. By the choice of $S_n$, we have

$$(\sigma^*(f))|_{S_n}(s')(t) \equiv (1 - t)^r \mod m^n.$$  

Since we have shown that $\mathcal{F}|_{(X_n)_{\bar{s}}}$ is constant, it follows that $f|_{X_n}(x)(t) \equiv (1 - t)^r \mod m^n$.

Finally, we prove assertion (b). Let $\eta$ be the generic point of $S$. Choose a geometric point $\bar{\eta}$ above $\eta \in S$ and let $\bar{x}$ denote the geometric point above $\sigma(\eta)$ induced from $\bar{\eta}$. Let $H$ be the intersection of the kernels of actions of $\pi_1(X, \bar{x})$ on the fibers $(X_n)_{\bar{s}}$ where $n$ runs in $\mathbb{N}$. We need to show that $\pi_1(X, \bar{x})/H$ contains an open pro-$\ell$ subgroup.

Using the fact that the tame fundamental group $\pi_1^{\text{tame}}(X_{\bar{\eta}}, \bar{x})$ is topologically finitely generated, one can prove that the quotient of the group $\pi_1(X, \bar{x})$ by the intersection $H_Y$ of the kernels of its actions on the fibers $(Y_n)_{\bar{s}}, n \in \mathbb{N}$ contains an open pro-$\ell$ subgroup (see the last part of the proof of [20, Lemma 3.1]).

Let $H'_S$ be the intersection of the kernels of actions of $\pi_1(S, \bar{\eta})$ on the fibers $(S_n)_{\bar{\eta}}$ where $n$ runs in $\mathbb{N}$ and let $H_S$ be the inverse image of $H'_S$ with respect to the homomorphism $\pi_1(X, \bar{x}) \rightarrow \pi_1(S, \bar{\eta})$. By the choice of $S_n$, the group $\pi_1(S, \bar{\eta})/H'_S$ contains an open pro-$\ell$ subgroup. Since we have a surjection

$$\pi_1(X, \bar{x})/(H_Y \cap H_S) \rightarrow \pi_1(X, \bar{x})/H$$

and an injection

$$\pi_1(X, \bar{x})/(H_Y \cap H_S) \rightarrow \pi_1(X, \bar{x})/H_Y \times \pi_1(S, \bar{\eta})/H'_S,$$
the group $\pi_1(X, \bar{x})/H$ also contains an open pro-$\ell$ subgroup.

To use the above lemma, we show that there exists a chain of split fibrations ending with a totally real curve.

**Definition 2.3.7.** A sequence of schemes $X_n \to X_{n-1} \to \cdots \to X_1$ is called a chain of split fibrations if the morphism $X_{i+1} \to X_i$ is an elementary fibration which admits a section $X_i \to X_{i+1}$ for each $i = 1, \ldots, n-1$.

**Lemma 2.3.8.** Let $K$ be a totally real field and $X$ an $n$-dimensional irreducible regular $\mathcal{O}_K$-scheme with enough totally real curves. Then there exist an étale $X$-scheme $X_n$, a totally real curve $X_1$ and a chain of split fibrations $X_n \to \cdots \to X_1$.

**Proof.** We prove the lemma by induction on $n = \dim X$. When $\dim X = 1$, the lemma holds by assumption. Thus we assume $\dim X \geq 2$.

By induction on $\dim X$, it suffices to prove that after replacing $K$ by a totally real field extension and $X$ by a nonempty étale $X$-scheme, there exist an irreducible regular $\mathcal{O}_K$-scheme $S$ with enough totally real curves and an elementary fibration $X \to S$ with a section $S \to X$.

It follows from Lemma 2.2.3 (i) that there exists a totally real extension $L$ of $K$ such that $X_K(L) \neq \emptyset$. Replacing $K$ by $L$ and $X$ by a nonempty open subscheme of $X \otimes_{\mathcal{O}_K} \mathcal{O}_L$ that is étale over $X$, we may further assume that the generic fiber $X_K \to \text{Spec} K$ has a section $x: \text{Spec} K \to X_K$. We also denote the image of $x$ in $X_K$ by $x$.

If $\dim X = 2$, then $X_K$ is a smooth and geometrically connected curve over $K$. Take the smooth compactification $\overline{X}_K$ of $X_K$ over $K$. Then the structure morphism $X_K \to \text{Spec} K$ has the factorization $X_K \subset \overline{X}_K \to \text{Spec} K$ and thus it is an elementary fibration with a section $x$. After shrinking $X$, we can spread it out into an elementary fibration $X \to S$ over an open subscheme $S$ of $\text{Spec} \mathcal{O}_K$ such that it admits a section $S \to X$. 

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Now assume \( \dim X \geq 3 \). We apply Artin’s theorem on elementary fibration ([5, Exposé XI, Proposition 3.3]) to the pair \((X_K, x)\), and by shrinking \( X \) if necessary we get an elementary fibration \( \pi: X_K \to S_K \) over \( K \) with a geometrically irreducible smooth \( K \)-scheme \( S_K \). Note that this theorem holds if the base field is perfect and infinite.

Since \( X_K \) is smooth over \( S_K \), there exist an open neighborhood \( V_K \) of \( x \) in \( X_K \) and an étale morphism \( \alpha: V_K \to \mathbb{A}^1_{S_K} \) such that \( \pi|_{V_K}: V_K \to S_K \) admits a factorization

\[
V_K \xrightarrow{\alpha} \mathbb{A}^1_{S_K} \to S_K.
\]

Take a section \( \tau: S_K \to \mathbb{A}^1_{S_K} \) of the projection such that \( \alpha(x) \) lies in \( \tau(S_K) \).

Consider the connected component \( S'_K \) of \( S_K \times_{\mathbb{A}^1_{S_K}} V_K \) that contains the \( K \)-rational point \( (\pi(x), x) \). This is étale over \( S_K \) and satisfies \( S'_K(K) \neq \emptyset \). Moreover, \( S'_K \) is geometrically integral over \( K \) since it is a connected regular \( K \)-scheme containing a \( K \)-rational point.

We replace \( S_K \) by \( S'_K \) and \( X_K \) by \( X_K \times_{S_K} S'_K \). By this replacement, the elementary fibration \( \pi: X_K \to S_K \) admits a section and \( S_K(K) \neq \emptyset \). After shrinking \( X \), we can spread it out into an elementary fibration \( X \to S \) with a section \( S \to X \), where \( S \) is an irreducible regular scheme which is flat and of finite type over \( \mathcal{O}_K \) with geometrically irreducible generic fiber and contains a \( K \)-rational point. The existence of a \( K \)-rational point implies that \( S \) has enough totally real curves. \( \square \)

**Proof of Proposition 2.3.4.** First note that if \( \alpha^*(f)|_{U''} \) has a kernel for a nonempty étale \( X \)-scheme \( \alpha: X'' \to X \) and a dense open subscheme \( U'' \subset X'' \), then so does \( f|_{\alpha(U'')} \).

Let \( n \) denote the dimension of \( X \). Replacing \( X \) by the image of \( X' \to X \), we may assume that \( X' \to X \) is surjective.

Take a chain of split fibrations \( X_n \to X_{n-1} \to \cdots \to X_1 \) with a totally real curve \( X_1 \) as
in Lemma 2.3.8. We regard $X_1$ as an $X$-scheme via $X_n \to X$ and the sections $X_i \to X_{i+1}$. Put $X'_1 = X' \times_X X_1$. This is a nonempty scheme. For $i = 2, \ldots, n$ we put $X'_i = X_i \times_{X_1} X'_1$ via the morphism $X_i \to X_{i-1} \to \cdots \to X_1$. Then $X'_n \to X'_{n-1} \to \cdots \to X'_1$ is a chain of split fibrations.

Since $f|_{X_1}$ lies in $\text{LS}_r(X_1)$ by assumption (i), we have $f|_{X'_1} = (f|_{X_1})|_{X'_1} \in \text{LS}_r(X'_1)$. Then we get the result for $(X'_2, f|_{X'_2})$ by Lemma 2.3.6. Repeating this argument for the chain of split fibrations $X'_n \to \cdots \to X'_2$ we get the result for $(X'_n, f|_{X'_n})$. Applying the remark at the beginning to the morphism $X'_n \to X$, we get the result for $(X, f)$. \hfill \Box

For the later use, we prove variants of Lemma 2.3.8. The proof given below is similar to that of Lemma 2.3.8, but instead of Lemma 2.2.3 we will use Corollary 2.2.11 and Theorem 2.2.12.

**Lemma 2.3.9.**

(i) With the notation as in Lemma 2.3.8, suppose that we are given a connected étale covering $Y \to X$. Then there exist an étale $X$-scheme $X_n$, a totally real curve $X_1$, and a chain of split fibrations $X_n \to \cdots \to X_1$ such that $X_1 \times_X Y$ is connected. Here $X_1 \to X$ is the composite of sections $X_{i+1} \to X_i$ and $X_n \to X$.

(ii) Let $F$ be a CM field and $Z$ an $n$-dimensional irreducible regular $O_F$-scheme with geometrically irreducible generic fiber. Let $Y \to Z$ be a connected étale covering. Then there exist an étale $Z$-scheme $Z_n$, a CM curve $Z_1$, and a chain of split fibrations $Z_n \to \cdots \to Z_1$ such that $Z_1 \times_Z Y$ is connected. Here $Z_1 \to Z$ is the composite of sections $Z_{i+1} \to Z_i$ and $Z_n \to Z$.

**Proof.** First we prove (i) by induction on $n = \dim X$. Since the claim is obvious when $\dim X = 1$, we assume $\dim X \geq 2$. 

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By induction on \( \dim X \), it suffices to prove that after replacing \( K \) by a totally real field extension, \( X \) by a nonempty étale \( X \)-scheme, and the covering \( Y \to X \) by its pullback, there exist an irreducible regular \( \mathcal{O}_K \)-scheme \( S \) with enough totally real curves and an elementary fibration \( X \to S \) with a section \( S \to X \) such that \( S \times_X Y \) is connected. The construction of such an \( S \) will be the same as that of Lemma 2.3.8.

It follows from Corollary 2.2.11 and Remark 2.2.13 that there exist a totally real extension \( L \) of \( K \) and an \( L \)-rational point \( x \in X(L) \) such that \( \Spec L \times_{x,X} Y \) is connected. Note that \( Y \otimes_{\mathcal{O}_K} \mathcal{O}_L \) is connected because \( Y \otimes_{\mathcal{O}_K} \mathcal{O}_L \to X \otimes_{\mathcal{O}_K} \mathcal{O}_L \) is an étale covering with connected base and

\[
\Spec L \times_{x,(X \otimes_{\mathcal{O}_K} \mathcal{O}_L)} (Y \otimes_{\mathcal{O}_K} \mathcal{O}_L) = \Spec L \times_{x,X} Y
\]

is connected. Thus replacing \( K \) by \( L \), \( X \) by a nonempty open subscheme of \( X \otimes_{\mathcal{O}_K} \mathcal{O}_L \) that is étale over \( X \), and \( Y \) by its pullback, we may further assume that the generic fiber \( X_K \to \Spec K \) has a section \( x : \Spec K \to X_K \) such that \( \Spec K \times_{x,X} Y \) is connected. We also denote the image of \( x \) in \( X_K \) by \( x \).

If \( \dim X = 2 \), the morphism \( X_K \to \Spec K \) is an elementary fibration with a section \( x \). After shrinking \( X \), we can spread it out into an elementary fibration \( X \to S \) over an open subscheme \( S \) of \( \Spec \mathcal{O}_K \) such that it admits a section \( S \to X \). By construction, \( S \times_X Y \) is connected.

Now assume \( \dim X \geq 3 \). We apply Artin’s theorem on elementary fibration to the pair \( (X_K, x) \), and by shrinking \( X \) if necessary we get an elementary fibration \( \pi : X_K \to S_K \) over \( K \) with a geometrically irreducible smooth \( K \)-scheme \( S_K \).

By smoothness, there exist an open neighborhood \( V_K \) of \( x \) in \( X_K \) and an étale morphism \( \alpha : V_K \to \mathbb{A}^1_{S_K} \) such that \( \pi|_{V_K} : V_K \to S_K \) admits a factorization \( V_K \xrightarrow{\alpha} \mathbb{A}^1_{S_K} \to S_K \). Take a
section \( \tau: S_K \to \mathbb{A}_{S_K}^1 \) of the projection such that \( \alpha(x) \) lies in \( \tau(S_K) \).

Consider the connected component \( S'_K \) of \( S_K \times_{\tau,\mathbb{A}_{S_K}^1} V_K \) that contains the \( K \)-rational point \((\pi(x), x)\). As is shown in Lemma \ref{lem:etale}, \( S'_K \) is étale over \( S_K \) and geometrically integral over \( K \), and \( S'_K(K) \neq \emptyset \).

The section \( \tau \) defines the morphism \( S'_K \to X_K \times_{S_K} S'_K \to X_K \). The composite of this morphism and \((\pi(x), x): \text{Spec} \, K \to S'_K \) coincides with \( x: \text{Spec} \, K \to X_K \). Since \( S'_K \times_X Y \to S'_K \) is an étale covering with connected base and

\[
\text{Spec} \, K \times_{(\pi(x), x), S'_K} (S'_K \times_X Y) = \text{Spec} \, K \times_{x, X} Y
\]

is connected, it follows that \( S'_K \times_X Y \) is connected.

We replace \( S_K \) by \( S'_K \), \( X_K \) by \( X_K \times_{S_K} S'_K \) and \( Y_K \) by \( Y_K \times_{S_K} S'_K \). By this replacement, the elementary fibration \( \pi: X_K \to S_K \) admits a section such that \( S_K(K) \neq \emptyset \) and \( S_K \times_{X_K} Y_K \) is connected. As is discussed in the last paragraph of the proof of Lemma \ref{lem:etale} after shrinking \( X \), we can spread out \( X_K \to S_K \) and \( Y_K \to S_K \) into an elementary fibration \( X \to S \) with a section \( S \to X \) and a covering \( Y \to X \), where \( S \) is an irreducible regular \( \mathcal{O}_K \)-scheme with enough totally real curves. Since \( S \times_X Y \) is connected by construction, this \( S \) works.

For part (ii), it is easy to verify that the same argument works if we apply Theorem \ref{thm:main} instead of Corollary \ref{cor:main} \( \Box \)

Next we show that if \( f \) has a kernel and arises from a local system over every totally real curve then it actually arises from a local system.

**Proposition 2.3.10.** Let \( K \) be a totally real field and \( X \) an irreducible smooth separated \( \mathcal{O}_K[\ell^{-1}] \)-scheme with enough totally real curves. Suppose that \( f \in \widehat{\mathcal{L}}_r(X) \) satisfies the following conditions:

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(i) \( f \) arises from a local system over every totally real curve.

(ii) \( f \) has a kernel.

Then \( f \in \text{LS}_r(X) \).

Proof. We follow [20, Section 4]. Since \( f \) has a kernel, we take a closed subgroup \( H \) of \( \pi_1(X) \) as in the definition of having a kernel. In particular, \( \pi_1(X)/H \) contains an open pro-\( \ell \) subgroup.

By Corollary 2.2.11, there exists a totally real curve \( C \) with a morphism \( \varphi: C \to X \) such that \( \varphi_*: \pi_1(C) \to \pi_1(X)/H \) is surjective. By assumption (i), for any such pair \( (C, \varphi) \), the pullback \( \varphi^*(f) \) arises from a semisimple representation \( \rho_C: \pi_1(C) \to \text{GL}_r(E_\lambda) \). Define

\[
H_C := \text{Ker}(\varphi_*: \pi_1(C) \to \pi_1(X)/H).
\]

Then condition (ii) in the definition of having a kernel, together with the Chebotarev density theorem, shows that \( \text{Ker}\rho_C \) contains \( H_C \). See [20, Lemma 4.1] for details. Thus we regard \( \rho_C \) as a representation

\[
\rho_C: \pi_1(X) \to \pi_1(X)/H \to \text{GL}_r(E_\lambda)
\]

of \( \pi_1(X) \). Note that \( \rho_C|_{\pi_1(C)} \) gives the original representation of \( \pi_1(C) \).

We will show that the local system on \( X \) corresponding to this representation gives \( f \). For this, we need to show that

\[
\det(1 - t\rho_C(\text{Frob}_x)) = f_x(t)
\]

for all closed points \( x \in X \). We know that this equality holds for each closed point \( x \in \varphi(C) \).
such that \( \varphi^{-1}(x) \) contains a point whose residue field is equal to \( k(x) \).

Take any closed point \( x \in X \). We will first construct a curve \( C' \) passing through \( x \) and some finitely many points on \( C \) specified below. We will then construct a local system on \( C' \) whose Frobenius polynomial at \( x \) is \( f_x(t) \), and prove that the local system on \( C' \) extends over \( X \) and the corresponding representation of \( \pi_1(X) \) coincides with \( \rho_C \).

We use a lemma by Faltings; define \( T_0 \) to be the set of closed points of \( C \) which have the same image in \( \text{Spec} \mathcal{O}_K \) as that of \( x \). By the theorem of Hermite, the Chebotarev density theorem, and the Brauer-Nesbitt theorem, there exists a finite set \( T \subset |C| \setminus T_0 \) satisfying the following properties:

(i) \( T \rightarrow \text{Spec} \mathcal{O}_K \) is injective.

(ii) For any semisimple representations \( \rho_1, \rho_2 : \pi_1(C) \rightarrow \text{GL}_r(E_\lambda) \), the equality \( \text{tr} \rho_1(\text{Frob}_y) = \text{tr} \rho_2(\text{Frob}_y) \) for all \( y \in T \) implies \( \rho_1 \cong \rho_2 \).

See [22, Satz 5] or [18, Théorème 3.1] for details.

By Corollary 2.2.11 applied to \( S = \varphi(T) \cup \{x\} \), there exists a totally real curve \( C' \) with a morphism \( \varphi' : C' \rightarrow X \) such that the map \( \varphi'_* : \pi_1(C') \rightarrow \pi_1(X)/H \) is surjective and for each \( y \in \varphi(T) \cup \{x\} \) there exists a point in \( \varphi'^{-1}(y) \) whose residue field is equal to \( k(y) \). As discussed before, this pair \( (C', \varphi') \) also defines a semisimple representation

\[
\rho_{C'} : \pi_1(X) \rightarrow \pi_1(X)/H \rightarrow \text{GL}_r(E_\lambda)
\]

such that \( \det(1 - t\rho_{C'}(\text{Frob}_y)) = f_y(t) \) for each \( y \in \varphi(T) \cup \{x\} \). Note that the surjectivity of \( \varphi_* \) implies that \( \rho_{C'}|_{\pi_1(C)} \) is semisimple.

It follows from property (ii) of \( T \) that \( \rho_C|_{\pi_1(C)} \) and \( \rho_{C'}|_{\pi_1(C)} \) are isomorphic as representations of \( \pi_1(C) \). Since the map \( \varphi_* : \pi_1(C) \rightarrow \pi_1(X)/H \) is surjective, we have \( \rho_C \cong \rho_{C'} \) as

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representations of $\pi_1(X)/H$ and thus they are also isomorphic as representations of $\pi_1(X)$. In particular, we have

$$\det(1 - t\rho_C(\text{Frob}_x)) = \det(1 - t\rho_C^r(\text{Frob}_x)) = f_x(t).$$

Hence $f$ comes from the local system on $X$ corresponding to $\rho_C$. \qed

We now prove the last proposition of our three key ingredients for Theorem 2.1.8. This concerns extendability of a local system on a dense open subset to the whole scheme. In the proof we use the Zariski-Nagata purity theorem; thus the regularity assumption for $X$ is crucial. We further need to assume that $X$ is smooth as we use Corollary 2.2.11 to find a totally real curve passing through a given point in a given tangent direction.

**Proposition 2.3.11.** Let $K$ be a totally real field and $X$ an irreducible smooth separated $O_K[\ell^{-1}]$-scheme with enough totally real curves. Suppose that $f \in \overline{\text{LS}}_r(X)$ satisfies the following conditions:

(i) $f$ arises from a local system over every totally real curve.

(ii) There exists a dense open subscheme $U \subset X$ such that $f|_U \in \text{LS}_r(U)$.

Then $f \in \text{LS}_r(X)$.

**Proof.** We follow [20, Section 5.2]. Let $\mathcal{E}_U$ be the semisimple $E_\lambda$-local system on $U$ corresponding to $f|_U$. First we show that $\mathcal{E}_U$ extends to an $E_\lambda$-local system on $X$.

Suppose the contrary. Since $X$ is regular, the Zariski-Nagata purity theorem implies that there exists an irreducible divisor $D$ of $X$ contained in $X \setminus U$ such that $\mathcal{E}_U$ is ramified along $D$. Then by a specialization argument ([20, Corollary 5.2]), we can find a closed point $x \in X \setminus U$ and a one-dimensional subspace $l \subset T_xX$ satisfying the following property:
(*) Consider a triple \((C, c, \varphi)\) consisting of a regular curve \(C\), a closed point \(c \in C\), and a morphism \(\varphi: C \to X\) such that \(\varphi(c) = x\), \(\varphi^{-1}(U) \neq \emptyset\), and \(\text{Im}(T_cC \to T_xX \otimes_{k(x)} k(c)) = l \otimes_{k(x)} k(c)\). For any such triple, the pullback of \(E_U\) to \(\varphi^{-1}(U)\) is ramified at \(c\).

Let \(H\) be the kernel of the representation \(\rho_U: \pi_1(U) \to \text{GL}_r(E_\lambda)\) corresponding to \(E_U\). The group \(\pi_1(U)/H \cong \text{Im } \rho_U\) contains an open pro-\(\ell\) subgroup because \(\text{Im } \rho_U\) is a compact open subgroup of \(\text{GL}_r(E_\lambda)\). Therefore by Corollary 2.2.11 we find a totally real curve \(C\), a closed point \(c \in C\), and a morphism \(\varphi: C \to X\) such that

- \(\varphi(c) = x\) and \(k(c) \cong k(x)\),
- \(\varphi^{-1}(U) \neq \emptyset\) and \(\varphi_*: \pi_1(\varphi^{-1}(U)) \to \pi_1(U)/H\) is surjective, and
- \(\text{Im}(T_cC \to T_xX) = l\).

Since \(\varphi_*: \pi_1(\varphi^{-1}(U)) \to \pi_1(U)/H\) is surjective, the pullback of \(E_U\) to \(\varphi^{-1}(U)\) is semisimple. Thus this \(E_\lambda\)-local system has no ramification at \(c\) by assumption (i), which contradicts property (*). Hence \(E_U\) extends to an \(E_\lambda\)-local system \(E\) on \(X\).

Let \(f'\) be the element of \(\text{LS}_r(X)\) corresponding to \(E\). We know \(f|_U = f'|_U\). Take any closed point \(x \in X\). It suffices to show that \(f(x) = f'(x)\). We can find a totally real curve \(C'\), a closed point \(c' \in C'\), and a morphism \(\varphi': C' \to X\) such that \(\varphi'(c') = x\), \(k(c') = k(x)\), and \(\varphi'^{-1}(U) \neq \emptyset\). Then

\[
\varphi'^* (f)|_{\varphi'^{-1}(U)} = (f|_U)|_{\varphi'^{-1}(U)} = (f'|_U)|_{\varphi'^{-1}(U)} = \varphi'^* (f')|_{\varphi'^{-1}(U)}.
\]

Since \(\varphi'^{-1}(U) \neq \emptyset\), the homomorphism \(\pi_1(\varphi'^{-1}(U)) \to \pi_1(C')\) is surjective and thus \(\varphi'^* (f) = \varphi'^* (f')\). In particular, \(f(x) = \varphi'^* (f)(c') = \varphi'^* (f')(c') = f'(x)\). \(\blacksquare\)
Proof of Theorem 2.1.8. First note that a polynomial-valued map \( f \) of degree \( r \) in the theorem lies in \( \widetilde{L}_S_r(X) \). One direction of the equivalence is obvious, and thus it suffices to prove that if \( f \) satisfies conditions (i) and (ii), then \( f \) lies in \( L_S_r(X) \).

First assume that \( X \) is separated. Let \( k_\lambda \) be the residue field of \( E_\lambda \) and \( N \) be the cardinality of \( \text{GL}_r(k_\lambda) \). Put \( X' := X \otimes \mathbb{Z}[N^{-1}] \). Then \( X' \to X \) is a dominant étale morphism and satisfies the following property:

The pullback \( f|_{X'} \) arises from a tame local system over every separated smooth curve over a finite field.

Thus by Proposition 2.3.4, there exists an open dense subscheme \( U \) of \( X \) such that \( f|_U \) has a kernel. Therefore \( f|_U \) lies in \( L_S_r(U) \) by Proposition 2.3.10 and we have \( f \in L_S_r(X) \) by Proposition 2.3.11.

In the general case, we consider a covering \( X = \bigcup_i U_i \) by open separated subschemes. Then we can apply the above discussion to each \( f|_{U_i} \) and obtain an \( E_\lambda \)-local system \( E_i \) on \( U_i \) that represents \( f|_{U_i} \). Since \( U_i \) is normal, we can replace \( E_i \) by its semisimplification and assume that each \( E_i \) is semisimple.

Put \( U = \bigcap_i U_i \). This is nonempty, and the restrictions \( E_i|_U \) are isomorphic to each other. Thus \( \{E_i\}_i \) glues to an \( E_\lambda \)-local system on \( X \) and this sheaf represents \( f \).

We end this section with variants of Theorem 2.1.8. Condition (i) in Theorem 2.3.12 or Remark 2.3.13 is weaker than that of Theorem 2.1.8 since they concern only totally real curves with additional properties. This weaker condition is essential to use the result of [6] in the proof of our main theorems in the next section. Theorem 2.3.14 is a variant in the CM case.

Theorem 2.3.12. Let \( K \) be a totally real field. Let \( X \) be an irreducible smooth \( \mathcal{O}_K[\ell^{-1}] \)-scheme with enough totally real curves. An element \( f \in \widetilde{L}_S(X) \) belongs to \( L_S(X) \) if and
only if it satisfies the following conditions:

(i) There exists a connected étale covering \( Y \to X \) such that \( f \) arises from a local system over every totally real curve \( C \) with the property that \( C \times_X Y \) is connected.

(ii) The restriction of \( f \) to each separated smooth curve over a finite field arises from a local system.

**Proof.** Recall that Theorem 2.1.8 is deduced from Propositions 2.3.10 and 2.3.11 and that these propositions have the same condition (i) that \( f \) arises from a local system over every totally real curve. Consider the variant statements of Propositions 2.3.4, 2.3.10, and 2.3.11 where we replace condition (i) by

\((i')\) \( f \) arises from a local system over every totally real curve \( C \) such that \( C \times_X Y \) is connected.

It suffices to prove that these variants also hold; then the theorem is deduced from them in the same way as Theorem 2.1.8.

The variant of Proposition 2.3.4 is proved in the same way as Proposition 2.3.4 if one uses Lemma 2.3.9 (i) instead of Lemma 2.3.8. For the variants of Propositions 2.3.10 and 2.3.11 the same proof works; observe that whenever one uses Corollary 2.2.11 in the proof to find a totally real curve \( C \), one can impose the additional condition that \( C \times_X Y \) is connected by adding the covering \( Y \to X \) to the input of Corollary 2.2.11.

**Remark 2.3.13.** We need another variant of Theorem 2.1.8 to prove Theorem 2.1.1: With the notation as in Theorem 2.3.12, suppose further that

- \( K \) is unramified at \( \ell \), and

- \( X \) extends to an irreducible smooth \( \mathcal{O}_K \)-scheme \( X' \) with nonempty fiber over each place of \( K \) above \( \ell \).
Then condition (i) in Theorem 2.3.12 can be replaced by

(i’) There exists a connected étale covering $Y \to X$ such that $f$ arises from a local system over every totally real curve $C$ with the properties that

- $C \times_X Y$ is connected and that
- the fraction field of $C$ is unramified at $\ell$.

This statement is proved in the same way as Theorem 2.3.12, it suffices to prove variants of Propositions 2.3.4, 2.3.10, and 2.3.11 where condition (i) in these propositions is replaced by the following condition:

The map $f$ arises from a local system over every totally real curve $C$ such that $C \times_X Y$ is connected and the fraction field of $C$ is unramified at $\ell$.

For the proof of the variant of Proposition 2.3.4, we also need to consider the variant of Lemma 2.3.9 (i) where we further require that the fraction field of $X_1$ is unramified at $\ell$.

We now explain how to prove the variants of Lemma 2.3.9 (i) and Propositions 2.3.4, 2.3.10, and 2.3.11. By the additional condition on $X$, for each place $v$ of $K$ above $\ell$, there exist a finite unramified extension $L$ of $K_v$ and a morphism $\text{Spec} \mathcal{O}_L \to X'$. We denote the image of the closed point of $\text{Spec} \mathcal{O}_L$ by $s_v$. Since $L$ is unramified over $K_v$, we can find a horizontal one-dimensional subspace $l_v$ of $T_{s_v}X'$ with respect to $X' \to \text{Spec} \mathcal{O}_K$.

If we add $\{s_v\}_v|\ell$ and $l_v$ to the input when we use Corollary 2.2.11, the fraction field of the resulting totally real curve is unramified over $K$ at each $v$, hence unramified at $\ell$. Thus we can prove the variant of Lemma 2.3.9 (i) in the same way as Lemma 2.3.9 (i), and the arguments given in Theorem 2.3.12 work for the current variants of Propositions 2.3.4, 2.3.10, and 2.3.11. Hence the statement of this remark follows.
Theorem 2.3.14. Let $F$ be a CM field. Let $Z$ be an irreducible smooth $\mathcal{O}_F[\ell^{-1}]$-scheme with geometrically irreducible generic fiber. An element $f \in \mathcal{L}_r(Z)$ belongs to $\mathcal{L}_r(Z)$ if and only if it satisfies the following conditions:

(i) There exists a connected étale covering $Y \to Z$ such that $f$ arises from a local system over every CM curve $C$ with the property that $C \times_Z Y$ is connected.

(ii) The restriction of $f$ to each separated smooth curve over a finite field arises from a local system.

Proof. We can prove variants of Propositions 2.3.4, 2.3.10, and 2.3.11 for the CM case using Theorem 2.2.12 and Lemma 2.3.9 (ii). Then the theorem is deduced from them in the same way as Theorems 2.1.8 and 2.3.12.

2.4 Proofs of the main theorems

In this section, we prove theorems on the existence of the compatible system of a local system. Theorem 2.4.3 concerns the totally real case and Theorem 2.4.4 concerns the CM case. Theorem 2.1.1 in Subsection 2.1.1 is proved after Theorem 2.4.3. Following the discussion in [20, Subsection 2.3], we deduce these main theorems from Theorems 2.3.12, 2.3.14 and theorems in [33] and [6].

2.4.1 Theorem of Barnet-Lamb, Gee, Geraghty, and Taylor

As we mentioned in Subsection 2.1.3, some of the assumptions in the main theorems come from the potential diagonalizability condition, which is introduced in [6]. We first review this notion. See [6, Section 1.4] for details.
Let $L$ be a finite extension of $\mathbb{Q}_\ell$. Let $E_\lambda$ be a finite extension of $\mathbb{Q}_\ell$. We say that an $\overline{E}_\lambda$-representation $\rho$ of $\text{Gal}(\overline{L}/L)$ is potentially diagonalizable if it is potentially crystalline and there is a finite extension $L'$ of $L$ such that $\rho|_{\text{Gal}(\overline{L}/L')} \operatorname{lies}$ on the same irreducible component of the universal crystalline lifting ring of the residual representation $\overline{\rho}|_{\text{Gal}(\overline{L}/L')}$ with fixed Hodge-Tate weights as a sum of characters lifting $\overline{\rho}|_{\text{Gal}(\overline{L}/L')}$.

There are two important examples of this notion (see [6, Lemma 1.4.3]): Ordinary representations are potentially diagonalizable. When $L$ is unramified over $\mathbb{Q}_\ell$, a crystalline representation is potentially diagonalizable if for each $\tau: L \hookrightarrow \overline{E}_\lambda$ the $\tau$-Hodge-Tate weights lie in the range $[a_\tau, a_\tau + \ell - 2]$ for some integer $a_\tau$.

With this notion, Barnet-Lamb, Gee, Geraghty, and Taylor proved the following theorem, which is a generalization of Theorem 2.1.7.

**Theorem 2.4.1** ([6, Theorem C]). Let $K$ be a totally real field and let $\rho: \text{Gal}(\overline{K}/K) \to \text{GL}_r(\overline{\mathbb{Q}}_\ell)$ be a continuous representation. Suppose that the following conditions are satisfied.

(i) $\rho$ is unramified at all but finitely many primes.

(ii) $\rho$ is potentially diagonalizable at each prime $\nu$ of $K$ above $\ell$ and for each $\tau: K \hookrightarrow \overline{\mathbb{Q}}_\ell$, it has distinct $\tau$-Hodge-Tate weights.

(iii) $\rho$ can be equipped with symplectic (resp. orthogonal) structure with totally odd (resp. totally even) multiplier.

(iv) The residual representation $\overline{\rho}|_{\text{Gal}(\overline{K}/K^\mu)}$ is irreducible.

(v) $\ell > 2(r + 1)$.

Then there exists a finite Galois totally real extension $K'$ of $K$ such that $\rho|_{\text{Gal}(\overline{K}/K')}$ is automorphic. Moreover, $\rho$ is part of a compatible system.
Remark 2.4.2. With the notation as in Theorem 2.4.1 let ℓ be a rational prime and \( \rho' : \text{Gal}(\overline{K}/K) \to \text{GL}_r(\overline{\mathbb{Q}}_\ell) \) be a continuous representation compatible with \( \rho \). Suppose \( p \) is a prime of \( K \) whose characteristic is different from \( \ell \) nor \( \ell' \). If \( \rho \) is unramified at \( p \), then \( \rho' \) is also unramified at \( p \).

To see this, first note that for each \( K \subset K'' \subset K' \) with \( K'/K'' \) solvable, \( \rho|_{\text{Gal}(\overline{K}/K'')} \) is automorphic by solvable base change. Since \( \rho|_{\text{Gal}(\overline{K}/K'')} \) is unramified at \( p'' \) for each such \( K'' \) and a prime \( p'' \) of \( K'' \) above \( p \), the automorphy and the local Langlands correspondence imply that \( \rho'|_{\text{Gal}(\overline{K}/K'')} \) is also unramified at \( p'' \). From this we see that the inertia group at \( p \) has trivial image under \( \rho' \).

2.4.2 Proof of the main theorems

Let us turn to the proof of our main theorem for the totally real case.

**Theorem 2.4.3.** Let \( \ell \) be a rational prime. Let \( K \) be a totally real field and \( X \) an irreducible smooth \( \mathcal{O}_K[\ell^{-1}] \)-scheme with enough totally real curves. Let \( E \) be a finite extension of \( \mathbb{Q} \) and \( \lambda \) a prime of \( E \) above \( \ell \). Let \( \mathcal{E} \) be an \( E_\lambda \)-local system on \( X \) and \( \rho \) the corresponding representation of \( \pi_1(X) \). Suppose that \( \mathcal{E} \) satisfies the following assumptions:

(i) The polynomial \( \det(1 - \text{Frob}_x t, \mathcal{E}_x) \) has coefficients in \( E \) for every \( x \in |X| \).

(ii) For every totally real field \( L \) and every morphism \( \alpha : \text{Spec} L \to X \), the \( E_\lambda \)-representation \( \alpha^* \rho \) of \( \text{Gal}(\overline{L}/L) \) is potentially diagonalizable at each prime \( v \) of \( L \) above \( \ell \) and for each \( \tau : L \to \overline{E}_\lambda \) it has distinct \( \tau \)-Hodge-Tate weights.

(iii) \( \rho \) can be equipped with symplectic (resp. orthogonal) structure with multiplier \( \mu : \pi_1(X) \to E_\lambda^x \) such that \( \mu|_{\pi_1(X_K)} \) admits a factorization

\[
\mu|_{\pi_1(X_K)} : \pi_1(X_K) \to \text{Gal}(\overline{K}/K) \xrightarrow{\mu_K} E_\lambda^x
\]
with a totally odd (resp. totally even) character $\mu_K$.

(iv) The residual representation $\overline{\rho}|_{\pi_1(X[\zeta_\ell])}$ is absolutely irreducible.

(v) $\ell > 2(\text{rank } \mathcal{E} + 1)$.

Then for each rational prime $\ell'$ and each prime $\lambda'$ of $E$ above $\ell'$ there exists an $\overline{E}_\lambda$-local system on $X[\ell'^{-1}]$ which is compatible with $\mathcal{E}|_{X[\ell'^{-1}]}$.

Proof. Replacing $X$ by $X[\ell'^{-1}]$, we may assume that $\ell'$ is invertible in $\mathcal{O}_X$. Let $r$ be the rank of $\mathcal{E}$. Take an arbitrary extension $M$ of $E_\lambda$ of degree $r!$. By assumption (i), we regard the map $f: x \mapsto \det(1 - \text{Frob}_x t, \mathcal{E}_x)$ as an element of $\widetilde{\text{LS}}_r^M(X)$ via the embedding $E \hookrightarrow E_\lambda \hookrightarrow M$.

We will apply Theorem 2.3.12 to $f \in \widetilde{\text{LS}}_r^M(X)$. Here we use the prime $\ell'$ and the field $M$ (we used $\ell$ and $E_\lambda$ in Section 2.3).

First we show that the map $f$ satisfies condition (i) in Theorem 2.3.12. Let $Y$ be the connected étale covering $Y \rightarrow X[\zeta_\ell]$ that corresponds to $\text{Ker} \overline{\rho}|_{\pi_1(X[\zeta_\ell])}$. We regard $Y$ as a connected étale covering over $X$ via $Y \rightarrow X[\zeta_\ell] \rightarrow X$. We will prove that this $Y \rightarrow X$ satisfies condition (i). Take any totally real curve $\varphi: C \rightarrow X$ such that $C \times_X Y$ is connected.

To show that $\varphi^*(f)$ arises from an $M$-local system on $C$, it suffices to prove that there exists an $\overline{E}_\lambda$-local system on $C$ which is compatible with $\varphi^*\mathcal{E}$; this follows from [20, Lemma 2.7]. Namely, let $\rho'_C: \pi_1(C) \rightarrow \text{GL}_r(\overline{E}_\lambda)$ denote the semisimplification of the corresponding $\overline{E}_\lambda$-representation. Since $\det(1 - t\rho'_C(\text{Frob}_x)) = f_x(t) \in E_\lambda[t]$ for every closed point $x \in C$, the character of $\rho'_C$ is defined over $E_\lambda$ by the Chebotarev density theorem. It follows from $[M : E_\lambda] = r!$ that the Brauer obstruction of $\rho'_C$ in the Brauer group $\text{Br}(E_\lambda)$ vanishes in $\text{Br}(M)$ and $\rho'_C$ can be defined over $M$. This means $\varphi^*(f) \in \text{LS}_r^M(C)$.  

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We will construct an $\overline{E}_\lambda$-local system on $C$ which is compatible with $\varphi^* \mathcal{E}$. For this we will apply Theorem 2.4.1 to the $\overline{E}_\lambda$-representation $\varphi_L^* \rho$ of $\text{Gal}(\overline{L}/L)$, where $L$ denotes the fraction field of $C$ and $\varphi_L: \text{Spec } L \to X$ denotes $\varphi|_{\text{Spec } L}$.

We need to see that the Galois representation $\varphi_L^* \rho$ satisfies the assumptions in Theorem 2.4.1. By assumptions (ii) and (v) it remains to check that

(a) $\varphi_L^* \rho$ can be equipped with symplectic (resp. orthogonal) structure with totally odd (resp. totally even) multiplier, and

(b) the residual representation $(\varphi_L^* \overline{\rho})|_{\text{Gal}(\overline{L}/L(\zeta))}$ is absolutely irreducible.

Assumption (a) follows from assumption (iii). To see (b), recall that $C \times_X Y$ is connected. Hence $C \times_X X[\zeta]$ is connected with fraction field $L(\zeta)$, and $C \times_X Y \to C \times_X X[\zeta]$ is a connected étale covering. It follows from the definition of $Y$ that $\text{Im}(\varphi_L^* \overline{\rho})|_{\text{Gal}(\overline{L}/L(\zeta))}$ coincides with $\text{Im}(\overline{\rho}|_{\text{Spec } F_p})$, and thus $(\varphi_L^* \overline{\rho})|_{\text{Gal}(\overline{L}/L(\zeta))}$ is absolutely irreducible by assumption (iv).

Hence by Theorem 2.4.1 we obtain an $\overline{E}_\lambda$-representation of $\text{Gal}(\overline{L}/L)$. By Remark 2.4.2 this representation is unramified at each closed point of $C$, and thus it gives rise to an $\overline{E}_\lambda$-local system on $C$ which is compatible with $\varphi^* \mathcal{E}$. Hence $f$ satisfies condition (i) in Theorem 2.3.12.

Next we show that $f$ satisfies condition (ii) in Theorem 2.3.12. Let $C$ be a separated smooth curve over $\mathbb{F}_p$ for some prime $p$ and denote the structure morphism $C \to \text{Spec } \mathbb{F}_p$ by $\alpha$. Let $\varphi: C \to X$ be a morphism. Note that $p$ is different from $\ell$ and $\ell'$.

We write the semisimplification of $\varphi^* \mathcal{E}$ as $\bigoplus_i \mathcal{E}_i^{\oplus r_i}$, where $\mathcal{E}_i$ are distinct irreducible $\overline{E}_\lambda$-local systems on $C$. Then there exist an irreducible $\overline{E}_\lambda$-local system $\mathcal{F}_i$ on $C$ and an $\overline{E}_\lambda$-local system $\mathcal{G}_i$ of rank 1 on $\text{Spec } \mathbb{F}_p$ such that $\mathcal{F}_i$ has determinant of finite order and $\mathcal{E}_i \cong \mathcal{F}_i \otimes \alpha^* \mathcal{G}_i$ (see [17, Section I.3] or [10, Section 0.4] for example).
By Lafforgue’s theorem (Theorem 2.1.4), for each closed point \( x \in C \), the roots of \( \det(1 - \text{Frob}_x t, \mathcal{F}_{i,x}) \) are algebraic numbers that are \( \lambda' \)-adic units. Moreover, there exists an irreducible \( \overline{E}_\lambda \)-local system \( \mathcal{F}'_i \) on \( C \) which is compatible with \( \mathcal{F}_i \).

We will show that there exists an \( \overline{E}_\lambda \)-local system \( \mathcal{G}'_i \) on \( \text{Spec} \, \mathbb{F}_p \) which is compatible with \( \mathcal{G}_i \). Note that the \( \overline{E}_\lambda \)-local system \( \mathcal{G}_i \) is determined by the value of the corresponding character of \( \text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \) at the geometric Frobenius. Denote this value by \( \beta_i \in \overline{E}_\lambda^\times \). It suffices to prove that \( \beta_i \) is an algebraic number that is a \( \lambda' \)-adic unit. Since the roots of \( \det(1 - \text{Frob}_x t, \mathcal{E}_x) \) and \( \det(1 - \text{Frob}_x t, \mathcal{F}_{i,x}) \) are all algebraic numbers, so is \( \beta_i \).

We prove that \( \beta_i \) is a \( \lambda' \)-adic unit. To see this, take a closed point \( x \) of \( C \). Then by Corollary 2.2.11 we can find a totally real curve \( C' \) and a morphism \( \varphi' : C' \to X \) such that \( \varphi(x) \in \varphi'(C') \) and \( C' \times_X Y \) is connected. As discussed before, Theorem 2.4.1 implies that there exists an \( \overline{E}_\lambda \)-local system on \( C' \) whose Frobenius characteristic polynomial map is \( \varphi'^*(f) \). Thus for each closed point \( y \in C' \) the roots of \( \varphi'^*(f)(y) \) are algebraic numbers that are \( \lambda' \)-adic units. Considering a point \( y \in \varphi'^{-1}(\varphi(x)) \), we conclude that some power of \( \beta_i \) is a \( \lambda' \)-adic unit and thus so is \( \beta_i \). Hence there exists an \( \overline{E}_\lambda \)-local system \( \mathcal{G}'_i \) on \( \text{Spec} \, \mathbb{F}_p \) which is compatible with \( \mathcal{G}_i \).

The Frobenius characteristic polynomial map associated with the semisimple \( \overline{E}_\lambda \)-local system \( \bigoplus \mathcal{F}'_i \otimes \alpha^* \mathcal{G}'_i \) is \( \varphi^*(f) \). As discussed before, this sheaf can be defined over \( M \). Thus \( f \) satisfies condition (ii) in Theorem 2.3.12.

Therefore by Theorem 2.3.12 there exists an \( M \)-local system on \( X \) which is compatible with \( \mathcal{E} \).

\[ \square \]

\textit{Proof of Theorem 2.1.1}. All the discussions in the proof of Theorem 2.4.3 also work in this setting by using Remark 2.3.13 instead of Theorem 2.3.12.

\[ \square \]

We also have a theorem for the CM case.
Theorem 2.4.4. Let $\ell$ be a rational prime, $E$ a finite extension of $\mathbb{Q}$, and $\lambda$ a prime of $E$ above $\ell$. Let $F$ be a CM field with $\zeta_\ell \notin F$ and $Z$ an irreducible smooth $\mathcal{O}_F[\ell^{-1}]$-scheme with geometrically irreducible generic fiber. Let $E$ be an $E_\lambda$-local system on $X$ and $\rho$ the corresponding representation of $\pi_1(Z)$. Suppose that $E$ satisfies the following assumptions:

(i) The polynomial $\det(1 - \text{Frob}_x t, E_{\bar{x}})$ has coefficients in $E$ for every $x \in |X|$.

(ii) For any CM field $L$ with $\zeta_\ell \notin L$ and any morphism $\alpha: \text{Spec} L \to Z$, the $E_\lambda$-representation $\alpha^* \rho$ of $\text{Gal}(\bar{L}/L)$ satisfies the following two conditions:

(a) $\alpha^* \rho$ is potentially diagonalizable at each prime $v$ of $L$ above $\ell$ and for each $\tau: L \hookrightarrow \bar{E}_\lambda$ it has distinct $\tau$-Hodge-Tate weights.

(b) $\alpha^* \rho$ is totally odd and polarizable in the sense of [6, Section 2.1].

(iii) The residual representation $\bar{\rho}|_{\pi_1(Z[\zeta_\ell])}$ is absolutely irreducible.

(iv) $\ell > 2(\text{rank } E + 1)$.

Then for each rational prime $\ell'$ and each prime $\lambda'$ of $E$ above $\ell'$ there exists an $E_{\lambda'}$-local system on $Z[\ell'^{-1}]$ which is compatible with $E|_{Z[\ell'^{-1}]}$.

Proof. In the same way as Theorem 2.4.3, the theorem is deduced from Theorem 2.3.14 [6, Theorem 5.5.1], Lafforgue’s theorem (Theorem 2.1.4), and the following remark: If $Y \to Z[\zeta_\ell]$ denotes the connected étale covering defined by $\text{Ker} \bar{\rho}|_{\pi_1(Z[\zeta_\ell])}$ and $C$ is a CM curve with a morphism to $Z$ such that $C \times_Z Y$ is connected, then $C \times_Z Z[\zeta_\ell] = C \otimes_{\mathcal{O}_F} \mathcal{O}_F(\zeta_\ell)$ is connected. In particular, the fraction field of $C$ does not contain $\zeta_\ell$. \qed
Chapter 3

Finiteness of Frobenius traces

3.1 Introduction

This chapter studies the relation between the finiteness of Frobenius traces of Galois representations and that of local systems.

3.1.1 Main Theorem

First we recall geometric properties of Galois representations predicted by the Fontaine-Mazur conjecture:

**Conjecture 3.1.1** ([26, §4(d)]). Let \( \ell \) be a prime. Let \( K \) be a number field and let \( G_K \) denote the absolute Galois group of \( K \). For a prime \( \ell' \) we denote by \( S_{\ell'} \) the set of primes of \( K \) above \( \ell' \). Fix a finite set \( S \) of primes of \( K \) and let \( \rho \) be an irreducible continuous representation \( G_K \to \text{GL}_r(\overline{\mathbb{Q}}_\ell) \) which is unramified outside \( S \cup S_{\ell} \) and de Rham at primes of \( K \) above \( \ell \). Then the following assertions hold:

(i) For every finite prime \( v \) of \( K \) with \( v \not\in S \cup S_{\ell} \), the roots of \( \det(1 - \rho(\text{Frob}_v)t) \) are
algebraic numbers.

(ii) There exists a number field $E$ in $\overline{\mathbb{Q}}_\ell$ such that

$$\det(1 - \rho(Frob_v)t) \in E[t]$$

for every prime $v$ of $K$ with $v \not\in S \cup S_\ell$.

(iii) There exists a sufficiently large number field $E$ in $\overline{\mathbb{Q}}_\ell$ satisfying (ii) and the following condition: for every prime $\ell'$ and every embedding $\sigma: E \hookrightarrow \overline{\mathbb{Q}}_{\ell'}$, there exists a continuous representation $\rho': G_K \to \text{GL}_r(\overline{\mathbb{Q}}_\ell)$ unramified outside $S \cup S_{\ell'}$ such that

$$\det(1 - \rho'(Frob_v)t) = \sigma \det(1 - \rho(Frob_v)t)$$

for every prime $v$ of $K$ with $v \not\in S \cup S_\ell \cup S_{\ell'}$.

From the relative Fontaine-Mazur conjecture (Conjecture 3.1.1), we expect the following generalization of part (ii) of Conjecture 3.1.1:

**Conjecture 3.1.2** (Finiteness of Frobenius traces). Let $\ell$ be a prime. Let $X$ be a scheme flat and of finite type over $\mathbb{Z}[\ell^{-1}]$ and $\mathcal{E}$ a $\overline{\mathbb{Q}}_\ell$-local system on $X$. If $\mathcal{E}|_{X \otimes \overline{\mathbb{Q}}_\ell}$ is a de Rham $\overline{\mathbb{Q}}_\ell$-local system, then there exists a number field $E$ such that $\text{tr}(Frob_x, \mathcal{E}_\mathfrak{p}) \in E$ for every closed point $x$ of $X$ (and a geometric point $\mathfrak{p}$ above $x$).

Since the Fontaine-Mazur conjecture on Galois representations is still wide open, Conjecture 3.1.2 is far beyond our reach. However, it is still interesting to ask whether the Fontaine-Mazur conjecture implies Conjecture 3.1.2. In this chapter, we give a partial answer to this question. Here is the main theorem of this chapter:
Theorem 3.1.3. Assume the Fontaine-Mazur conjecture (Conjecture 3.1.1) and the Generalized Riemann Hypothesis for Dedekind zeta functions. Then Conjecture 3.1.2 holds.

The Generalized Riemann Hypothesis is only used to obtain a strong form of the effective Chebotarev theorem (Theorem 3.3.2(ii)).

This theorem is an analogue of the following finiteness theorem of Frobenius traces in the function field case by Deligne.

Theorem 3.1.4 ([19 Théorème 3.1]). Let \(\ell\) and \(p\) be distinct primes. Let \(X\) be a scheme of finite type over the finite field \(\mathbb{F}_p\) and \(E\) a \(\overline{\mathbb{Q}_\ell}\)-local system on \(X\) such that the Frobenius trace \(\text{tr}(\text{Frob}_x, E_x)\) is an algebraic number for every closed point \(x \in X\). Then there exists a finite extension \(E\) of \(\mathbb{Q}\) in \(\overline{\mathbb{Q}}\) such that \(\text{tr}(\text{Frob}_x, E_x) \in E\) for every closed point \(x \in X\).

This implies part (ii) of Deligne’s conjecture, namely, Theorem 2.1.5 in Subsection 2.1.2. As explained there, L. Lafforgue established the Langlands correspondence for \(\text{GL}_r\) over function fields and proved Deligne’s conjecture in the case of curves. Deligne used the Langlands correspondence as an input to prove Theorem 3.1.4. We remark that although Lafforgue’s result directly implies Deligne’s theorem when \(\dim X = 1\), the general case is not simply reduced to the curve case; in fact, Deligne needed the Weil conjecture and all the statements of Lafforgue’s theorem including the existence of compatible systems. This may explain why we need to assume the Generalized Riemann Hypothesis and Conjecture 3.1.1(iii) in Theorem 3.1.3.

3.1.2 Ideas of the proof of Theorem 3.1.3

Now we explain how to prove Theorem 3.1.3. In the sequel, a curve means an open subscheme of the spectrum of the ring of integers of a number field. Assume Conjecture
3.1.1 We know that for each closed point \( x \in X \) the Frobenius trace \( \text{tr}(\text{Frob}_x, \mathcal{E}_x) \) is algebraic over \( \mathbb{Q} \) by Conjecture 3.1.1(i) applied to some curve passing through \( x \).

For a positive integer \( N \) we denote by \( \mathbb{Q}(\text{tr} \mathcal{E}) \leq N \) the field generated over \( \mathbb{Q} \) by \( \text{tr}(\text{Frob}_x, \mathcal{E}_x) \) with \( \#k(x) \leq N \), where \( k(x) \) denotes the residue field of \( x \). Then \( \mathbb{Q}(\text{tr} \mathcal{E}) \leq N \) is finite over \( \mathbb{Q} \) since there are only finitely many closed points \( x \) with \( \#k(x) \leq N \).

By Conjecture 3.1.1(ii), for each curve \( C \) and each morphism \( \varphi: C \to X \), there exists a positive integer \( N \) such that \( \text{tr}(\text{Frob}_y, (\varphi^* \mathcal{E})_y) \in \mathbb{Q}(\text{tr} \varphi^* \mathcal{E}) \leq N \) for every closed point \( y \in C \). Let \( N(C, \varphi) \) be the smallest integer satisfying this property.

In this chapter, we will prove the following:

(a) The integer \( N(C, \varphi) \) is controlled by the ramification and boundary of \( C \) (Proposition 3.3.9). We note that this is the only place where we need to assume the GRH to obtain a good estimate of \( N(C, \varphi) \).

(b) For each closed point \( x \in X \), there exists a curve \( \varphi_{x^*}: C_x \to X \) passing through \( x \) with small ramification and boundary (Proposition 3.2.2).

It follows from statements (a) and (b) (in the precise form) that for each closed point \( x \in X \), there exists a curve \( \varphi_{x^*}: C_x \to X \) passing through \( x \) such that the associated integer \( N(C_x, \varphi_{x^*}) \) does not grow as fast as \( \#k(x) \).

Once this assertion is obtained, we can find a sufficiently large integer \( N_0 \) such that if \( N_0 < \#k(x) \), then \( N(C_x, \varphi_{x^*} \mathcal{E}) < \#k(x) \) and thus \( \text{tr}(\text{Frob}_x, \mathcal{E}_x) \in \mathbb{Q}(\text{tr} \mathcal{E}) \leq \#k(x)-1 \). From this we can show by induction on \( \#k(x) \) that the Frobenius trace \( \text{tr}(\text{Frob}_x, \mathcal{E}_x) \) lies in the number field \( \mathbb{Q}(\text{tr} \mathcal{E}) \leq N_0 \) for every closed point \( x \) of \( X \), which completes the proof.
3.1.3 Organization of this chapter

Section 3.2 presents part (b) (Proposition 3.2.2). As for part (a), the key inputs are a trick by Faltings appearing in his proof of the Shafarevich conjecture and the effective Chebotarev theorem (assuming the GRH). We review them in Section 3.3 and establish part (a) (Proposition 3.3.9). In Section 3.4, we combine the discussions in the previous sections and prove the main theorem.

3.1.4 Notation

A number field means a field that is finite over $\mathbb{Q}$. For a number field $E$ and a finite place $\lambda$ of $E$, we denote by $\mathcal{O}_E$ the ring of integers of $E$ and by $E_\lambda$ a fixed algebraic closure of the $\lambda$-adic completion $E_\lambda$ of $E$.

For a number field $K$, we denote by $d_K$ the absolute value of the discriminant of $K$ over $\mathbb{Q}$.

For a scheme $X$, we denote by $|X|$ the set of closed points of $X$. In particular, we identify $|\text{Spec } \mathbb{Z}|$ with the set of rational primes. We denote the residue field of a point $x$ of a scheme by $k(x)$ and $\overline{x}$ denotes a geometric point above $x$. We omit the base points of étale fundamental groups to simplify the notation.

3.2 Curves with small ramification and boundary

The purpose of this section is to prove the existence of a curve (i.e., an open subscheme of the ring of integers of a number field) with small ramification and boundary on a smooth scheme over $\mathbb{Z}$ (Proposition 3.2.2). This is statement (b) in Subsection 3.1.2.

Definition 3.2.1. Let $K$ be a number field and $C$ an open subscheme of $\text{Spec } \mathcal{O}_K$. We
define subsets $S_{K}^{\text{ram}}, S_{C}^{\text{bd}}, S_{C}$ of $|\text{Spec } \mathbb{Z}|$ as follows:

- $S_{K}^{\text{ram}}$ denotes the set of rational primes that ramify in $K$.
- $S_{C}^{\text{bd}} := u(\text{Spec } \mathcal{O}_K \setminus C)$ where $u : \text{Spec } \mathcal{O}_K \to \text{Spec } \mathbb{Z}$ is the structure morphism.
- $S_{C} := S_{K}^{\text{ram}} \cup S_{C}^{\text{bd}}$.

We consider the following situation. Let $X$ be a connected smooth scheme over $\text{Spec } \mathbb{Z}$ equipped with an étale morphism $\pi : X \to \mathbb{A}^m_{\mathbb{Z}}$. Denote by $d_{\pi}$ the degree of $\pi$ over the generic point of $\mathbb{A}^m_{\mathbb{Z}}$. Then there exists an open subscheme $U \subset \mathbb{A}^m_{\mathbb{Z}}$ such that $U \subset \text{Im } \pi$ and that $\pi^{-1}(U) \to U$ is finite and étale of degree $d_{\pi}$.

Write $\mathbb{A}^m_{\mathbb{Z}} = \text{Spec } \mathbb{Z}[t_1, \ldots, t_m]$ and let $I$ denote the definition ideal of the reduced closed subscheme $\mathbb{A}^m_{\mathbb{Z}} \setminus U$. We fix a non-zero element $f(t_1, \ldots, t_m) \in I$. Let $d_f$ be the total degree of $f$ and $B_f$ the maximum of the absolute values of the coefficients of $f$.

**Proposition 3.2.2.** With the notation as above, let $p$ be a prime and $n$ a positive integer. Then for each point $x \in X(\mathbb{F}_p^n)$ there exist a number field $K$, an open subscheme $C \subset \text{Spec } \mathcal{O}_K$ and a morphism $\varphi : C \to X$ satisfying the following conditions:

(i) There exists a point $\bar{x} \in C(\mathbb{F}_p^n)$ such that $\varphi(\bar{x}) = x$.

(ii) $[K : \mathbb{Q}] \leq d_{\pi}n$ and the following inequality holds:

$$\prod_{p' \in S_{C}} p' \leq 2^{n(d_f+m)}B_f^n(d_f+1)^d_f n!^2 n^{d_f n^2 + n} p^{d_f n^2 + 2n^2 - 2}.$$  

The rest of the section is devoted to the proof of Proposition 3.2.2. The outline of the proof is as follows: We first choose a curve of the form $\text{Spec } \mathcal{O}_M$ where $M$ is a number field using a lift of a minimal polynomial of the extension $\mathbb{F}_p \subset \mathbb{F}_p^n$. Then we construct
a morphism \( \text{Spec} \mathcal{O}_M \to \mathbb{A}_Z^m \) passing through \( \pi(x) \) using the polynomial \( f(t_1, \ldots, t_m) \in I \).

We pull back the curve by \( \pi \) and take the connected component which passes through \( x \). This is the curve \( C \). The construction of \( \text{Spec} \mathcal{O}_M \to \mathbb{A}_Z^m \) gives a control of \( S_C \) and yields the proposition.

**Proof of Proposition 3.2.2.** The point \( x \in X(\mathbb{F}_p^n) \) and the morphism \( \pi: X \to \mathbb{A}_Z^m \) define a ring homomorphism \( \overline{\varphi}: \mathbb{Z}[t_1, \ldots, t_m] \to \mathbb{F}_p^n \). Write \( \mathbb{F}_p^n = \mathbb{F}_p[t]/(\overline{g}(t)) \) with a monic irreducible polynomial \( \overline{g}(t) \in \mathbb{F}_p[t] \) of degree \( n \).

Choose a polynomial \( g(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0 \in \mathbb{Z}[t] \) such that \( g \) is a lift of \( \overline{g} \) and \( 0 \leq a_i \leq p-1 \) for each \( i = 0, \ldots, n-1 \). Denote \( \mathbb{Q}[t]/(g(t)) \) by \( M \). Then the \( \mathbb{Q} \)-algebra \( M \) is a field extension of \( \mathbb{Q} \) of degree \( n \).

**Lemma 3.2.3.**

(i) If \( \alpha \in \mathbb{C} \) is a root of \( g(t) \), then \( |\alpha| \leq np \).

(ii) Let \( d_M \) denote the absolute value of the discriminant of \( M \) over \( \mathbb{Q} \). Then \( d_M \leq (n!)^2 n^np^{2n-2} \). In particular, \( \prod_{p' \in \mathcal{S}_M} p' \leq (n!)^2 n^n p^{2n-2} \).

**Proof.** Note that \( |a_i| \leq p \) for each \( i \). Then (i) follows from the following inequalities:

\[
|\alpha^n| = |a_{n-1}\alpha^{n-1} + \cdots + a_1 \alpha + a_0| \\
\leq |a_{n-1}| |\alpha|^{n-1} + \cdots + |a_1| |\alpha| + |a_0| \\
\leq np \max \{1, |\alpha|^{n-1}\}.
\]

As for (ii), it suffices to show the first inequality since the second follows from the first and the obvious inequality \( \prod_{p' \in \mathcal{S}_M} p' \leq d_M \). Let \( \text{disc} g \in \mathbb{Z} \) be the discriminant of the polynomial \( g(t) \). Since \( g(t) \) is monic, \( \mathbb{Z}[t]/(g(t)) \) is an order of \( M \) contained in \( \mathcal{O}_M \).
Comparing the discriminants of the orders yields $d_M \leq |\text{disc}g|$. Thus it suffices to prove $|\text{disc}g| \leq (n!)^2n^np^{2n-2}$.

Note that the discriminant $\text{disc}g$ is a polynomial in $a_0, \ldots, a_{n-1}$ of degree $2n - 2$ with integer coefficients. Moreover, $\text{disc}g$ consists of at most $(n!)^2$ terms and the absolute value of each coefficient is at most $n^n$; this follows from the relation between $\text{disc}g$ and the resultant of $g(t)$ and its derivative $g'(t)$. Since $|a_i| \leq p$ for every $i$,

$$|\text{disc}g| \leq (n!)^2n^n(\max_i |a_i|)^{2n-2} \leq (n!)^2n^np^{2n-2}.$$  

We fix a root $\alpha$ of $g(t)$ in $\mathbb{C}$. This gives an embedding $M \hookrightarrow \mathbb{C}$ and we regard $\alpha$ as an element of $\mathcal{O}_M$. By construction, there exists a unique prime $\mathfrak{p}$ of $M$ above $p$ and it satisfies $\mathcal{O}_M/\mathfrak{p} \cong \mathbb{F}_{p^n}$. Recall that we have fixed a non-zero polynomial $f(t_1, \ldots, t_m)$ of degree $d_f$ that vanishes on $\mathbb{A}^m_\mathbb{Z} \setminus U$.

**Lemma 3.2.4.** There exist $m$ elements $\beta_1, \ldots, \beta_m \in \mathcal{O}_M$ satisfying the following conditions:

- Each $\beta_j$ is of the form $\sum_{0 \leq i \leq n-1} b_i \alpha^i$ ($b_i \in \mathbb{Z}$) with $0 \leq b_i \leq (d_f + 1)p - 1$.
- The ring homomorphism $\mathbb{Z}[t_1, \ldots, t_m] \to \mathcal{O}_M$ sending $t_i$ to $\beta_i$ is a lift of $\overline{\varphi} : \mathbb{Z}[t_1, \ldots, t_m] \to \mathbb{F}_{p^n} = \mathcal{O}_M/\mathfrak{p}$.
- $f(\beta_1, \ldots, \beta_m) \neq 0$.

**Proof.** The lemma follows from an elementary fact: If $f(t_1, \ldots, t_m) \in \mathbb{Z}[t_1, \ldots, t_m]$ is a non-zero polynomial of total degree at most $d_f$, then for any subsets $S_1, \ldots, S_m$ of $\mathbb{C}$
of cardinality $d_f + 1$, there exists an $m$-tuple $(\beta_1, \ldots, \beta_m) \in S_1 \times \cdots \times S_m$ such that $f(\beta_1, \ldots, \beta_m) \neq 0$.

The fact is easily proved by induction on $m$ and the verification is left to the reader. Since $\mathcal{O}_M/p = \mathbb{F}_p[t]/(\overline{g}(t))$ is generated over $\mathbb{F}_p$ by $\alpha \mod p$, each $\varphi(t_j) \in \mathcal{O}_M/p$ has $(d_f+1)$ distinct lifts in the set

$$\left\{ \sum_{0 \leq i \leq n-1} b_i \alpha^i \in \mathcal{O}_M \mid 0 \leq b_i \leq (d_f+1)p - 1 \right\}.$$ 

Thus the fact implies the lemma.

We take $\beta_i$ as in Lemma 3.2.4 and denote by $\varphi_M$ the induced morphism $\text{Spec} \mathcal{O}_M \to \mathbb{A}_Z^m$. By construction, the prime $p$ defines a point $x_M \in (\text{Spec} \mathcal{O}_M)(\mathbb{F}_p)$ and it satisfies $\varphi_M(x_M) = x$. We also have $\text{Im} \varphi_M \subsetneq \mathbb{A}_Z^m \setminus U$ since $f(\beta_1, \ldots, \beta_m) \neq 0$. Let $C_M \subset \text{Spec} \mathcal{O}_M$ denote the open subscheme $\varphi_M^{-1}(U) = \text{Spec} \mathcal{O}_M \times \mathbb{A}_Z^m U$.

We give an estimate of the boundary of $C_M$. Recall that $B_f$ denotes the maximum of the absolute values of the coefficients of $f(t_1, \ldots, t_m)$.

**Lemma 3.2.5.**

$$\prod_{p' \in S_{bd}^C_M} p' \leq 2^n(d_f+m)B_f^n(d_f+1)^dtn^d d_f n^2 p^d n^2.$$ 

**Proof.** Note that the complement $\text{Spec} \mathcal{O}_M \setminus C_M = \text{Spec} \mathcal{O}_M \times \mathbb{A}_Z^m (\mathbb{A}_Z^m \setminus U)$ is contained in $\text{Spec} \mathcal{O}_M/(f(\beta_1, \ldots, \beta_m))$ as underlying topological spaces. Therefore $\prod_{p' \in S_{bd}^C_M} p' \leq |N_{M/Q} f(\beta_1, \ldots, \beta_m)|$, where $N_{M/Q} : M \to \mathbb{Q}$ is the norm map. We will estimate the norm of $f(\beta_1, \ldots, \beta_m)$.

By Lemma 3.2.3(i), $|\sigma \alpha| \leq np$ for each $\sigma \in \text{Hom}_\mathbb{Q}(M, \mathbb{C})$. Hence

$$|\sigma \beta_i| \leq n((d_f + 1)p - 1)(np)^{n-1} \leq (d_f + 1)n^np^n$$
for each \( i = 1, \ldots, m \) (see the first condition in Lemma 3.2.4). As \( f(t_1, \ldots, t_m) \) is a polynomial in \( m \) variables of total degree \( d_f \), the number of the terms of \( f(t_1, \ldots, t_m) \) is at most

\[
\sum_{0 \leq i \leq d_f} \binom{i + m - 1}{i} = \binom{d_f + m}{d_f} \leq 2^{d_f + m}.
\]

Hence \( |\sigma f(\beta_1, \ldots, \beta_m)| \leq 2^{d_f + m} B_f \left( (d_f + 1)n^np^n \right)^{d_f} \) and

\[
|N_{M/\mathbb{Q}} f(\beta_1, \ldots, \beta_m)| = \prod_{\sigma \in \text{Hom}_{\mathbb{Q}}(M, C)} |\sigma f(\beta_1, \ldots, \beta_m)|
\leq \left( 2^{(d_f + m)} B_f (d_f + 1)n^{d_f n^2 p^{d_f n^2}} \right)^n
= 2^{n(d_f + m)} B_f^n (d_f + 1)n^{d_f n^2 p^{d_f n^2}}.
\]

Finally we construct a curve \( \varphi : C \to X \) satisfying the conditions in Proposition 3.2.2. We have two morphisms \( \pi : X \to \mathbb{A}_Z^m \) and \( \varphi_M : \text{Spec} \mathcal{O}_M \to \mathbb{A}_Z^m \) with \( \pi(x) = \varphi_M(x_M) \). Therefore there exists a point \( \bar{x} \in \left( X \times_{\mathbb{A}_Z^m} \text{Spec} \mathcal{O}_M \right) \left( \mathbb{F}_{p^n} \right) \) mapping to \( x \) and \( x_M \) under the projections. Define \( C \) to be the connected component of \( X \times_{\mathbb{A}_Z^m} \text{Spec} \mathcal{O}_M \) containing \( \bar{x} \). Since \( \pi \) is étale, \( C \) is an open subscheme of the ring of integers of a number field. Let \( K \) be the fraction field of \( C \) and let \( \varphi \) denote the morphism \( C \to X \).

It remains to check that \( \varphi : C \to X \) satisfies (i) and (ii) in Proposition 3.2.2. By construction, (i) holds. Since \( \pi \) is étale of generic degree \( d_\pi \), we have \( [K : M] \leq d_\pi \) and thus \( [K : \mathbb{Q}] \leq d_\pi n \).

We estimate \( \prod_{p' \in S_C} p' \). As \( \pi : X \to \mathbb{A}_Z^m \) is finite and étale over \( U \), so is \( C \to \text{Spec} \mathcal{O}_M \) over \( C_M \). This implies that \( S_{C_M}^{\text{bd}} \subset S_{C_M}^{\text{bd}} \) and \( S_{C_M}^{\text{ram}} \subset S_{M}^{\text{ram}} \cup S_{C_M}^{\text{bd}} \). In particular, we have
$S_C \subset S_{M}^{\text{ram}} \cup S_{C_M}^{\text{bd}}$, and Lemmas 3.2.3(ii) and 3.2.5 yield

$$
\prod_{p' \in S_C} p' \leq \prod_{p' \in S_{M}^{\text{ram}}} p' \prod_{p' \in S_{C_M}^{\text{bd}}} p' 
$$

$$
\leq ((n!)^2 n^p 2^{2n-2}) (2^n (d_f+m) B_f^n (d_f+1)^d_f n^d_f n^2 p^d_f n^2)
$$

$$
= 2^n (d_f+m) B_f^n (d_f+1)^d_f n^2 (n!)^2 n^d_f n^2 + n^d_f n^2 + 2n-2.
$$

This completes the proof of Proposition 3.2.2.

\[\square\]

### 3.3 Faltings’ trick and the effective Chebotarev theorem

In this section, we first review Faltings’ trick and the effective Chebotarev theorem. Then we will prove statement (a) in Subsection 3.1.2 (Proposition 3.3.9).

Fix a rational prime $\ell$ and a finite extension $E_\lambda$ of $\mathbb{Q}_\ell$. We denote by $q$ the cardinality of the residue field of $E_\lambda$. Let $r$ be a positive integer.

Faltings used the following lemma in the proof of the Shafarevich conjecture ([22]).

**Lemma 3.3.1.** Let $K$ be a number field, $C$ an open subscheme of $\text{Spec} \mathcal{O}_K$, and $Y \subset |C|$ a subset. Assume that for every finite Galois extension $L$ of $K$ that is unramified over $C$ and has degree at most $q^{2n^2}$, every conjugacy class of $\text{Gal}(L/K)$ is the image of the Frobenius $\text{Frob}_y$ under $\pi_1(C) \to \text{Gal}(L/K)$ for some $y \in Y$. Then for any two semisimple continuous representations $\rho_1, \rho_2 : \pi_1(C) \to \text{GL}_r(E_\lambda)$, $\rho_1 \cong \rho_2$ if $\text{tr} \rho_1(\text{Frob}_y) = \text{tr} \rho_2(\text{Frob}_y)$ for every $y \in Y$.

**Proof.** This follows from the Chebotarev density theorem and the Brauer-Nesbitt theorem; see the proof of [22 Satz 5] or [18 Théorème 3.1].

\[\square\]

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Theorem 3.3.2 (the effective Chebotarev theorem).

(i) There exists a positive constant $A_0$ such that for every number field $K$, every finite Galois extension $L$ of $K$, and every conjugacy class $c \in \text{Gal}(L/K)$ there exists a prime $p$ of $K$ which is unramified in $L$ and satisfies $\text{Frob}_p = c$ and

$$N_{K/Q} p \leq 2d_L^{A_0}.$$  

Here $N_{K/Q} p$ is the absolute norm of $p$ and $d_L$ is the absolute value of the discriminant of $L$ over $Q$.

(ii) If the GRH holds for the Dedekind zeta function $\zeta_L(s)$, then there exists a positive constant $A$ such that (i) is valid with a sharper estimate

$$N_{K/Q} p \leq A(\log d_L)^2.$$  

Remark 3.3.3. Theorem 3.3.2(i) is unconditional, but the inequality is not strong enough for our purpose. This is why we assume the GRH in Theorem 3.1.3.

Remark 3.3.4. The inequality given in [35, Corollary 1.2] is of the form

$$N_{K/Q} p \leq A(\log d_L)^2(\log \log d_L)^4,$$

and Theorem 3.3.2(ii) is mentioned with a sketch of a proof at the end of the paper (see also Introduction of [34]). In our setting, the above weaker estimate suffices; we use the inequality in Theorem 3.3.2(ii) only to simplify the growth rate argument. For details, see Lemma 3.4.4 and Remark 3.4.5 where we compare the estimates in Proposition 3.2.2 and Theorem 3.3.2(ii).
The goal of the rest of this section is to prove statement (a) in Subsection 3.1.2 (Proposition 3.3.9). We first recall basic facts in algebraic number theory.

**Lemma 3.3.5.** For a number field $M$, we have

$$d_M \leq \prod_{p \in S_{\text{ram}}^M} p^{[M:Q](1 + \log_p[M:Q])}.$$  

**Proof.** This is standard. For example, it follows from Remark 1 after Proposition 13 in [46, III-6].

**Lemma 3.3.6.** Let $m$ be a positive integer and let $a$ be a nonzero integer. Denote a primitive $m$-th root of unity by $\zeta_m$. Then $\mathbb{Q}(\sqrt[m]{a}, \zeta_m)$ is a finite Galois extension of $\mathbb{Q}$ of degree at most $m(m-1)$. Moreover, if a rational prime $p$ ramifies in $\mathbb{Q}(\sqrt[m]{a}, \zeta_m)$, then $p$ divides $ma$.

**Proof.** The proof is easy and left to the reader.

**Lemma 3.3.7.** Let $K$ be a number field and $C$ an open subscheme of $\text{Spec} \mathcal{O}_K$. Let $L$ be a finite Galois extension of $K$ that is unramified over $C$ and has degree at most $q^{2r^2}$. Set $m := [K:Q] + 1$ and $a := \prod_{p \in S_C^\text{ram}} p$, and define $L' := L(\sqrt[m]{a}, \zeta_m)$. Then $L'$ is a finite Galois extension of $K$ of the degree at most $[K:Q][[K:Q] + 1]q^{2r^2}$, and the primes in $\text{Spec} \mathcal{O}_K \setminus C$ ramify in $L'$. Moreover, the following inequality holds:

$$d_{L'} \leq \left( ([K:Q] + 1) \prod_{p \in S_C} p \right)^{( [K:Q] + 1)^3 q^{2r^2} (1 + 3 \log_2 ([K:Q] + 1) + 2r \log_2 q) }.$$  

**Proof.** Since $L'$ is the composite field of $L$ and $\mathbb{Q}(\sqrt[m]{a}, \zeta_m)$ over $\mathbb{Q}$, Lemma 3.3.6 implies that $L'$ is a finite Galois extension of $K$ of the degree at most $[K:Q][[K:Q] + 1]q^{2r^2}$. If we denote by $V([K:Q] + 1)$ the set of rational primes dividing $[K:Q] + 1$, then we have
\[ S_{Q(\sqrt{a},\zeta_m)} = S_C \cup V([K : \mathbb{Q}] + 1). \] It follows from this that

\[ S_{L'}^{\text{ram}} = S_L^{\text{ram}} \cup S_{Q(\sqrt{a},\zeta_m)} \subset S_C \cup V([K : \mathbb{Q}] + 1). \]

Take \( p \in \text{Spec} \mathcal{O}_K \setminus C \) and let \( p \) be the rational prime below \( p \). Then the absolute ramification index of \( p \) is at most \( [K : \mathbb{Q}] \). On the other hand, the absolute ramification index of a prime of \( \mathbb{Q}(\sqrt{a},\zeta_m) \) above \( p \) is at least \( m = [K : \mathbb{Q}] + 1 \) as \( p \) divides \( a \) exactly once. Therefore the absolute ramification index of a prime of \( L' \) above \( p \) is also at least \( m = [K : \mathbb{Q}] + 1 \), and thus \( p \) ramifies in \( L' \).

Finally, the estimate of \( d_{L'} \) follows from Lemma \[3.3.5\] and the above discussions on \([L' : K]\) and \( S^{\text{ram}}_{L'} \). Namely, we have

\[
d_{L'} \leq \prod_{p \in S_{L'}^{\text{ram}}} p^{[L' : \mathbb{Q}](1+\log_p[L' : \mathbb{Q}] )} \leq \left( \prod_{p \in S^{\text{ram}}_{L'}} p \right)^{[L' : \mathbb{Q}](1+\log_2[L' : \mathbb{Q}])} \leq \left( ([K : \mathbb{Q}] + 1) \prod_{p \in S_C} p \right)^{([K : \mathbb{Q}] + 1)^3 q^{2r^2} (1 + 3 \log_2([K : \mathbb{Q}] + 1) + 2r^2 \log_2 q)}.
\]

\[ \square \]

**Proposition 3.3.8.** Let \( K \) be a number field and \( C \) an open subscheme of \( \text{Spec} \mathcal{O}_K \). Assume the GRH and let \( A \) be the constant in Theorem \[3.3.2\] (ii). Set

\[ D(C) = \left( ([K : \mathbb{Q}] + 1) \prod_{p \in S_C} p \right)^{([K : \mathbb{Q}] + 1)^3 q^{2r^2} (1 + 3 \log_2([K : \mathbb{Q}] + 1) + 2r^2 \log_2 q)}, \]

and

\[ Y = \{ y \in C \mid \#k(y) \leq A(\log D(C))^2 \}. \]

Then for any two semisimple continuous representations \( \rho_1, \rho_2 : \pi_1(C) \to \text{GL}_r(E_\lambda), \rho_1 \cong \rho_2 \)
if $\text{tr} \rho_1(\text{Frob}_y) = \text{tr} \rho_2(\text{Frob}_y)$ for every $y \in Y$.

Proof. It suffices to prove that $Y$ satisfies the assumption in Lemma 3.3.1. Let $L$ be a finite Galois extension of $K$ that is unramified over $C$ and has degree at most $q^{2r^2}$. Set $m := [K : \mathbb{Q}] + 1$ and $a := \prod_{p \in \mathcal{O}_K} p$ and define $L' := L(\sqrt[m]{a}, \zeta_m)$. By Lemma 3.3.7, $L'$ is a finite Galois extension of $K$ and $d_{L'} \leq D(C)$.

By Theorem 3.3.2 (ii), for every conjugacy class $c \in \text{Gal}(L'/K)$ there exists a prime $p$ of $K$ which is unramified in $L'$ and satisfies $\text{Frob}_p = c$ and

$$\#k(p) = N_{K/\mathbb{Q}}p \leq A(\log d_{L'})^2 \leq A(\log D(C))^2.$$  

Moreover, the primes in $\text{Spec} \mathcal{O}_K \setminus C$ ramify in $L'$, and thus we have $p \in C$. Since $\text{Gal}(L'/K)$ is a quotient of $\text{Gal}(L'/K)$, we see that $Y$ satisfies the assumption in Lemma 3.3.1. □

Proposition 3.3.9. Assume Conjecture 3.1.1 and the GRH. Let $A$ be the constant in Theorem 3.3.2 (ii). Let $K$ be a number field, $C$ an open subscheme of $\text{Spec} \mathcal{O}_K$ and let $\mathcal{F}$ be an $E_\lambda$-local system on $C$ that is de Rham above $\ell$. Denote by $\mathbb{Q}(\text{tr} \mathcal{F})_{\leq A(\log D(C))^2}$ the field generated over $\mathbb{Q}$ by $\text{tr}(\text{Frob}_y, \mathcal{F}_y)$ for all $y \in C$ with $\#k(y) \leq A(\log D(C))^2$. Then for each closed point $y$ of $C$,

$$\text{tr}(\text{Frob}_y, \mathcal{F}_y) \in \mathbb{Q}(\text{tr} \mathcal{F})_{\leq A(\log D(C))^2}.$$  

Remark 3.3.10. With the notation as in Subsection 3.1.2, applying Proposition 3.3.9 to $\mathcal{F} = \varphi^* \mathcal{E}$ yields $N(C, \varphi) \leq A(\log D(C))^2$. This is the precise formulation of statement (a) in Subsection 3.1.2 and it is also an analogue of [19, Proposition 2.10].

Proof. Taking semisimplification if necessary, we may assume that $\mathcal{F}$ is semisimple. Set $E_0 = \mathbb{Q}(\text{tr} \mathcal{F})_{\leq A(\log D(C))^2}$. By Conjecture 3.1.1, $E_0$ is a number field and there exists a
sufficiently large Galois extension $E$ of $\mathbb{Q}$ in $E_\lambda$ containing $E_0$ such that the following conditions hold:

- The polynomial $\det(1 - \text{Frob}_y t, \mathcal{F}_y)$ has coefficients in $E$ for every closed point $y \in C$.
- For every embedding $\sigma : E \hookrightarrow E_\lambda$ there exists an $E_\lambda$-local system $\sigma \mathcal{F}$ on $C$ satisfying

  $$\det(1 - \text{Frob}_y t, (\sigma \mathcal{F})_y) = \sigma \det(1 - \text{Frob}_y t, \mathcal{F}_y)$$

  for every closed point $y \in C$.

Take any element $\sigma \in \text{Gal}(E/E_0)$. Then $\sigma$ defines an embedding $E \xrightarrow{\sigma} E \hookrightarrow E_\lambda$ and we get a local system $\sigma \mathcal{F}$ as in the second property above. Taking semisimplification if necessary, we may further assume that $\sigma \mathcal{F}$ is semisimple. We will prove that $\mathcal{F} \cong \sigma \mathcal{F}$.

Consider the set

$$Y = \{y \in C \mid \# k(y) \leq A(\log D(C))^2 \}.$$  

By the definition of $E_0$, we know that for each $y \in Y$, $\text{tr} (\text{Frob}_y, \mathcal{F}_y) \in E_0$ and thus

$$\text{tr}(\text{Frob}_y, \mathcal{F}_y) = \sigma \text{tr}(\text{Frob}_y, \mathcal{F}_y) = \text{tr}(\text{Frob}_y, (\sigma \mathcal{F})_y).$$

Consider the semisimple continuous representations of $\pi_1(C)$ corresponding to $\mathcal{F}$ and $\sigma \mathcal{F}$. Then they are isomorphic by Proposition 3.3.8. Hence $\mathcal{F} \cong \sigma \mathcal{F}$. Therefore for every closed point $y \in C$,

$$\text{tr}(\text{Frob}_y, \mathcal{F}_y) \in E^{\text{Gal}(E/E_0)} = E_0.$$  

\[\square\]
3.4 Finiteness of Frobenius traces

In this section, we prove Theorem 3.1.3. More precisely, we show the following theorem.

**Theorem 3.4.1.** Let \( \ell \) be a prime and \( E_\lambda \) a finite extension of \( \mathbb{Q}_\ell \). Let \( X \) be a flat \( \mathbb{Z}[\ell^{-1}] \)-scheme of finite type and \( \mathcal{E} \) an \( E_\lambda \)-local system on \( X \) that is de Rham over \( X \otimes \mathbb{Q}_\ell \). Assume Conjecture 3.1.1 and the GRH. Then there exists a number field \( E \subset \overline{E}_\lambda \) such that \( \text{tr}(\text{Frob}_x, \mathcal{E}_x) \in E \) for every closed point \( x \in X \).

Note that Conjecture 3.1.2 and Theorem 3.1.3 are stated for \( \overline{\mathbb{Q}}_\ell \)-local systems. However, since any \( \overline{\mathbb{Q}}_\ell \)-local system comes from an \( E_\lambda \)-local system for some finite extension \( E_\lambda \) of \( \mathbb{Q}_\ell \), Theorem 3.1.3 follows from Theorem 3.4.1.

**Remark 3.4.2.** This theorem implies that there exists a number field \( E \subset E_\lambda \) such that \( \det(1 - \text{Frob}_x t, \mathcal{E}_x) \in E[t] \) for every closed point \( x \in X \). For this apply the theorem to irreducible constituents of \( \bigwedge^k \mathcal{E} \).

In the sequel we prove Theorem 3.4.1. We fix \( \ell \) and \( E_\lambda \). By extending \( E_\lambda \) if necessary, we may assume that \( E_\lambda \) is Galois over \( \mathbb{Q}_\ell \). We denote by \( q \) the cardinality of the residue field of \( E_\lambda \). Let \( r \) denote the rank of \( \mathcal{E} \).

**Definition 3.4.3.** For a \( \mathbb{Z} \)-scheme \( Z \) of finite type and an \( E_\lambda \)-local system \( \mathcal{F} \) on \( Z \), we define \( \mathbb{Q}(\text{tr} \mathcal{F}) \) to be the field generated over \( \mathbb{Q} \) by \( \text{tr}(\text{Frob}_z, \mathcal{F}_z) \) for all \( z \in |Z| \). For a positive integer \( N \), we denote by \( \mathbb{Q}(\text{tr} \mathcal{F})_{\leq N} \) the field generated over \( \mathbb{Q} \) by \( \text{tr}(\text{Frob}_z, \mathcal{F}_z) \) for all \( z \in |Z| \) with \( \#k(z) \leq N \).

Note that Conjecture 3.1.1 implies that \( \mathbb{Q}(\text{tr} \mathcal{E}) \) is algebraic over \( \mathbb{Q} \); in fact, for every closed point \( x \in X \) there exist a number field \( K \), an open subscheme \( C \subset \text{Spec} \mathcal{O}_K \), a point \( \tilde{x} \in C(k(x)) \), and a morphism \( \varphi : C \to X \) with \( \varphi(\tilde{x}) = x \). Since the category of
de Rham representations of the Galois group of an \(\ell\)-adic field is stable under subquotients, each constituent of \(\varphi^*E\) is de Rham above \(\ell\). Hence Conjecture \textit{3.1.1(i)} implies that \(\text{tr}(\text{Frob}_x, E_x) = \text{tr}(\text{Frob}_y, \varphi^*E_y)\) is an algebraic number.

Next we reduce to the case where \(X\) is connected and smooth over \(\mathbb{Z}\). Note that if \(X\) is covered by finitely many locally closed subschemes \(X_i\), then \(\mathbb{Q}(E) = \bigcup_i \mathbb{Q}(E|_{X_i})\) and thus it suffices to prove Theorem \textit{3.4.1} for each \(X_i\).

Since there are at most finitely many irreducible components of \(X\), we may assume that \(X\) is irreducible. We may further assume that \(X\) is reduced, hence integral. Consider the generic fiber of \(X \to \text{Spec} \mathbb{Z}\). It admits a stratification by finitely many locally closed smooth subschemes over \(\mathbb{Q}\), and this defines a stratification of the whole \(X\) by finitely many locally closed subschemes with smooth generic fibers. Thus we may assume that \(X\) has smooth generic fiber.

Then there exists an open subscheme \(U \subset X\) such that \(U\) is smooth over \(\mathbb{Z}\) and contains the generic fiber of \(X\). Note that the complement \(X \setminus U\) is fibered over finitely many closed points of \(\text{Spec} \mathbb{Z}\). Applying Theorem \textit{3.1.4} to each of these fibers, we conclude that \(\mathbb{Q}(\text{tr} E|_{X \setminus U})\) is finite over \(\mathbb{Q}\). Thus we may replace \(X\) by \(U\) and assume that \(X\) is connected and smooth over \(\mathbb{Z}\).

Now that \(X \to \text{Spec} \mathbb{Z}\) is smooth, we may shrink \(X\) and assume that there is an étale morphism \(\pi : X \to \mathbb{A}^m_{\mathbb{Z}}\). This is the situation considered in Section \textit{3.2}. We use the same notation as in Section \textit{3.2}, which we now recall.

Denote by \(d_\pi\) the degree of \(\pi\) over the generic point of \(\mathbb{A}^m_{\mathbb{Z}}\). Let \(U\) be an open subscheme of \(\mathbb{A}^m_{\mathbb{Z}}\) such that \(U \subset \text{Im} \pi\) and that \(\pi^{-1}(U) \to U\) is finite and étale of degree \(d_\pi\). Write \(\mathbb{A}^m_{\mathbb{Z}} = \text{Spec} \mathbb{Z}[t_1, \ldots, t_m]\) and let \(I\) denote the definition ideal of the reduced closed subscheme \(\mathbb{A}^m_{\mathbb{Z}} \setminus U\). We fix a non-zero element \(f(t_1, \ldots, t_m) \in I\). Let \(d_f\) be the total degree of \(f\) and \(B_f\) the maximum of the absolute values of the coefficients of \(f\).
We also recall the notation from Section 3.3. For a number field \( K \) and an open subscheme \( C \) of \( \text{Spec} \mathcal{O}_K \), we set
\[
D(C) = \left( ([K : \mathbb{Q}] + 1) \prod_{p \in S_C} p \right)^{([K : \mathbb{Q}] + 1)^3 q^{2r^2} (1 + 3 \log_2([K : \mathbb{Q}] + 1) + 2r^2 \log_2 q)}.
\]

**Lemma 3.4.4.** There exists a positive integer \( N_0 \) satisfying the following: Let \( p \) be a prime, \( n \) a positive integer, and \( x \in X(\mathbb{F}_p^n) \). If \( p^n > N_0 \), then for every number field \( K \) and every curve \( \varphi: C \to X \) with fraction field \( K \) satisfying condition (ii) in Proposition 3.2.2, the following inequality holds:
\[
A(\log D(C))^2 < p^n.
\]

**Proof.** Let \( p \) be a prime, \( n \) a positive integer, and \( x \in X(\mathbb{F}_p^n) \). Take any number field \( K \) and curve \( \varphi: C \to X \) satisfying the conditions in Proposition 3.2.2. Condition (ii) in Proposition 3.2.2 gives 
\[
[K : \mathbb{Q}] \leq d_\pi n \quad \text{and thus}
\]
\[
\log D(C) \leq (d_\pi n + 1)^3 q^{2r^2} (1 + 3 \log_2(d_\pi n + 1) + 2r^2 \log_2 q)
\]
\[
\times \log \left( (d_\pi n + 1)^{2^n d_f^n m} B^n_f (d_f + 1)^{d_f n} (n!)^2 n^{d_f n^2 + n p^{d_f n^2 + 2n^2}} \right).
\]

Note that \( q, r, d_\pi, m, d_f, \) and \( B_f \) are constants that do not depend on \( p, n, \) or \( x \). We denote by \( M(p, n) \) the latter quantity in the above inequality. Then the lemma reduces to the following claim:

**Claim.** There exists a positive integer \( N_0 \) such that for every prime \( p \) and every positive integer \( n \) with \( p^n > N_0 \), the inequality
\[
A(M(p, n))^2 \leq p^n
\]
holds.

We prove the claim. Ignoring the constants $q$, $r$, $d_x$, $m$, $d_f$, and $B_f$, we have

$$M(p, n) = O(\log n(\log n + \log p)n^5) \quad \text{as } p^n \to \infty,$$

and thus

$$A(M(p, n))^2 = O((\log n)^2(\log n + \log p)^2n^{10}) \quad \text{as } p^n \to \infty.$$

If we set $N = p^n$, then $p \leq N$ and $n \leq \log_2 N$. Hence we have

$$(\log n)^2(\log n + \log p)^2n^{10} \leq (\log_2 N)^2(\log_2 N + \log N)^2(\log_2 N)^{10}.$$

Since the growth rate of the right hand side is $o(N)$ as $N \to \infty$, the claim follows.

Remark 3.4.5. As the proof shows, one can replace the inequality in Lemma 3.4.4 by

$$A(\log D(C))^2(\log \log D(C))^4 < p^n.$$

However, it cannot be replaced by an inequality of the form

$$2(D(C))^{A_0} < p^n.$$

So the unconditional estimate in Theorem 3.3.2(i) does not work.

We complete the proof of Theorem 3.4.1. Let $N_0$ be the positive integer in Lemma 3.4.4. We will prove that

$$\text{tr}(\text{Frob}_x, \mathcal{E}_x) \in \mathbb{Q}(\text{tr} \mathcal{E})_{\leq N_0}$$

for every closed point $x \in X$. For this, by induction, it suffices to show that for every prime
p, a positive integer \( n \) with \( p^n > N_0 \), and a point \( x \in X(\mathbb{F}_{p^n}) \), the trace \( \text{tr}(\text{Frob}_x, \mathcal{E}_x) \) lies in \( \mathbb{Q}(\text{tr} \mathcal{E})_{\leq p^n-1} \). We fix such \( p, n, \) and \( x \).

Take \( \varphi : C \to X \) and \( \bar{x} \in C(\mathbb{F}_{p^n}) \) as in Proposition 3.2.2. Then \( \text{tr}(\text{Frob}_x, \mathcal{E}_x) = \text{tr}(\text{Frob}_{\bar{x}}, (\varphi^* \mathcal{E})_{\bar{x}}) \), and Proposition 3.3.9 implies

\[
\text{tr}(\text{Frob}_x, \mathcal{E}_x) \in \mathbb{Q}(\text{tr} \varphi^* \mathcal{E})_{\leq A(\log D(C))^2}.
\]

On the other hand, Lemma 3.4.4 implies \( A(\log D(C))^2 < p^n \) and thus \( \text{tr}(\text{Frob}_x, \mathcal{E}_x) \in \mathbb{Q}(\text{tr} \mathcal{E})_{\leq p^n-1} \). This finishes the proof of Theorem 3.4.1.
Part II

The relative $p$-adic Hodge theory
Chapter 4

Constancy of generalized Hodge-Tate weights

4.1 Introduction

This chapter studies the relative $p$-adic Hodge theory. Namely, we discuss properties of $\mathbb{Q}_p$-local systems on a rigid analytic variety over a $p$-adic field. We start with explaining the main results of this chapter and ideas of the proofs, which is an expanded version of Sections 1.6 and 1.7.

4.1.1 Constancy of generalized Hodge-Tate weights

In the celebrated paper [50], Tate studied the Galois cohomology of $p$-adic fields and obtained the so-called Hodge-Tate decomposition of the Tate module of a $p$-divisible group with good reduction. The paper has been influential in the developments of $p$-adic Hodge theory, and one of the earliest progresses was done by Sen. In [43], he attached to each $p$-adic Galois representation of a $p$-adic field $k$ a multiset of numbers that are algebraic...
over $k$. These numbers are called generalized Hodge-Tate weights, and they serve as one of the basic invariants in $p$-adic Hodge theory, especially for the study of Galois representations that may not be Hodge-Tate (e.g. Galois representations attached to finite slope overconvergent modular forms).

In this chapter, we study how generalized Hodge-Tate weights vary in a geometric family of Galois representations. To be precise, we consider an étale $\mathbb{Q}_p$-local system on a rigid analytic varieties over $k$ and regard it as a family of Galois representations of residue fields of its classical points. Here is one of the main theorems of this chapter.

**Theorem 4.1.1** (Corollary 4.4.9). Let $X$ be a geometrically connected smooth rigid analytic variety over $k$ and let $\mathbb{L}$ be a $\mathbb{Q}_p$-local system on $X$. Then the generalized Hodge-Tate weights of the $p$-adic Galois representations $\mathbb{L}_x$ of $k(x)$ are constant on the set of classical points $x$ of $X$.

The theorem gives one instance of the rigidity of a geometric family of Galois representations. It is worth noting that arithmetic families of Galois representations do not have such rigidity; consider a representation of the absolute Galois group of $k$ with coefficients in some $\mathbb{Q}_p$-affinoid algebra. One can associate to each maximal ideal a Galois representation of $k$. In such a situation, the generalized Hodge-Tate weights vary over the maximal ideals.

### 4.1.2 Ideas of the proof of Theorem 4.1.1 and other results

To explain ideas of the proof of Theorem 4.1.1 as well as other results of this chapter, let us recall the work of Sen mentioned above. For each $p$-adic Galois representation $V$ of $k$, we set

$$\mathcal{H}(V) := (V \otimes_{\mathbb{Q}_p} \mathbb{C}_p)^{\text{Gal}(\overline{k}/k_{\infty})},$$

83
where $\mathbb{C}_p$ is the $p$-adic completion of $\kbar$ and $k_\infty := k(\mu_{p^{\infty}})$ is the cyclotomic extension of $k$. This is a vector space over the $p$-adic completion $K$ of $k_\infty$ equipped with a continuous semilinear action of $\text{Gal}(k_\infty/k)$ and satisfies $\dim_K H(V) = \dim_{\mathbb{Q}_p} V$. Sen developed a theory of decompletion; he found a natural $k_\infty$-vector subspace $H(V)_{\text{fin}} \subset H(V)$ that is stable under $\text{Gal}(k_\infty/k)$-action and satisfies

$$H(V)_{\text{fin}} \otimes_{k_\infty} K = H(V).$$

He then defined a $k_\infty$-endomorphism $\phi_V$ on $H(V)_{\text{fin}}$, called the Sen endomorphism of $V$, by considering the infinitesimal action of $\text{Gal}(k_\infty/k)$. The generalized Hodge-Tate weights are defined to be eigenvalues of $\phi_V$.

Therefore, the first step toward Theorem 4.1.1 is to define generalizations of $H(V)$ and $\phi_V$ for each $\mathbb{Q}_p$-local system. For this, we use the $p$-adic Simpson correspondence by Liu and Zhu [36]; based on recent developments in relative $p$-adic Hodge theory by Kedlaya-Liu and Scholze, Liu and Zhu associated to each $\mathbb{Q}_p$-local system $L$ on $X$ a vector bundle $H(\mathbb{L})$ of the same rank on $X_K$ equipped with a $\text{Gal}(k_\infty/k)$-action and a Higgs field, where $X_K$ is the base change of $X$ to $K$. When $X$ is a point and $\mathbb{L}$ corresponds to $V$, this agrees with $H(V)$ as the notation suggests. Following Sen, we will define the arithmetic Sen endomorphism $\phi_L$ of $\mathbb{L}$ by decompleting $H(\mathbb{L})$ and considering the infinitesimal action of $\text{Gal}(k_\infty/k)$. Then Theorem 4.1.1 is reduced to the following:

**Theorem 4.1.2** (Theorem 4.4.8). The eigenvalues of $\phi_{L,x}$ for $x \in X_K$ are algebraic over $k$ and constant on $X_K$.

Before discussing ideas of the proof, let us mention consequences of Theorem 4.1.2. Sen proved that a $p$-adic Galois representation $V$ is Hodge-Tate if and only if $\phi_V$ is semisimple with integer eigenvalues. In the same way, we use $\phi_L$ to study Hodge-Tate sheaves. We
define a sheaf $D_{HT}(\mathbb{L})$ on the étale site $X_{\text{ét}}$ by

$$D_{HT}(\mathbb{L}) := \nu_*(\mathbb{L} \otimes_{\mathbb{Q}_p} \mathcal{O}_{B_{HT}}),$$

where $\mathcal{O}_{B_{HT}}$ is the Hodge-Tate period sheaf on the pro-étale site $X_{\text{pro} \text{ ét}}$ and $\nu: X_{\text{pro} \text{ ét}} \to X_{\text{ét}}$ is the projection (see Section 4.5). A $\mathbb{Q}_p$-local system $\mathbb{L}$ is called Hodge-Tate if $D_{HT}(\mathbb{L})$ is a vector bundle on $X$ of rank equal to rank $\mathbb{L}$.

**Theorem 4.1.3 (Theorem 4.5.5).** The following conditions are equivalent for a $\mathbb{Q}_p$-local system $\mathbb{L}$ on $X$:

(i) $\mathbb{L}$ is Hodge-Tate.

(ii) $\phi_\mathbb{L}$ is semisimple with integer eigenvalues.

The study of the Sen endomorphism for a geometric family was initiated by Brinon as a generalization of Sen’s theory to the case of non-perfect residue fields ([12]). Tsuji obtained Theorem 4.1.3 in the case of schemes with semistable reduction ([52]).

Using this characterization, we prove the following basic property of Hodge-Tate sheaves:

**Theorem 4.1.4 (Theorem 4.5.10).** Let $f: X \to Y$ be a smooth proper morphism between smooth rigid analytic varieties over $k$ and let $\mathbb{L}$ be a $\mathbb{Z}_p$-local system on $X_{\text{ét}}$. Assume that $R^if_*\mathbb{L}$ is a $\mathbb{Z}_p$-local system on $Y_{\text{ét}}$ for each $i$. Then if $\mathbb{L}$ is a Hodge-Tate sheaf on $X_{\text{ét}}$, $R^if_*\mathbb{L}$ is a Hodge-Tate sheaf on $Y_{\text{ét}}$.

Hyodo introduced the notion of Hodge-Tate sheaves and proved Theorem 4.1.4 in the case of schemes ([29]). Links between Hodge-Tate sheaves and the $p$-adic Simpson correspondence can be seen in his work and were also studied by Abbes-Gros-Tsuji ([1]) and Tsuji ([51]). In fact, they undertook a systematic development of the $p$-adic Simpson correspondence started by Faltings ([21]) and their focus is much broader than ours.
and Brinon also studied Higgs modules and Sen endomorphisms in a different setting ([4]). In these works, one is restricted to working with schemes or log schemes, whereas we work with rigid analytic varieties.

We now turn to the proof of Theorem 4.1.2. The key idea to obtain such constancy is to describe $\phi_L$ as the residue of a certain formal integrable connection. Such an idea occurs in the work [4] of Andreatta and Brinon. Roughly speaking, they associated to $L$ a formal connection over some pro-étale cover of $X_K$ when $X$ is an affine scheme admitting invertible coordinates. In our case, we want to work over $X_K$, and thus we use the geometric $p$-adic Riemann-Hilbert correspondence by Liu and Zhu [36] and Fontaine’s decompletion theory for the de Rham period ring $B_{dR}(K)$ in the relative setting.

Liu and Zhu associated to each $\mathbb{Q}_p$-local system $L$ on $X$ a locally free $\mathcal{O}_X \hat{\otimes} B_{dR}(K)$-module $\mathcal{RH}(L)$ equipped with an integrable connection

$$\nabla : \mathcal{RH}(L) \to \mathcal{RH}(L) \otimes \Omega_X^1$$

and a $\text{Gal}(k_\infty/k)$-action (see Subsection 4.4.1 for the notation). To regard $\phi_L$ as a residue, we also need a connection in the arithmetic direction $B_{dR}(K)$. For this we use Fontaine’s decompletion theory [25]; recall the natural inclusion $k_\infty((t)) \subset B_{dR}(K)$ where $t$ is the $p$-adic analogue of the complex period $2\pi i$. Fontaine extended the work of Sen and developed a decompletion theory for $B_{dR}(K)$-representations of $\text{Gal}(k_\infty/k)$. We generalize Fontaine’s decompletion theory to the relative setting, i.e., that for $\mathcal{O}_X \hat{\otimes} B_{dR}(K)$-modules (Theorem 4.2.5 and Proposition 4.2.24), which yields a submodule $\mathcal{RH}(L)_{\text{fin}} \subset \mathcal{RH}(L)$ and an endomorphism $\phi_{dR,L}$ on $\mathcal{RH}(L)_{\text{fin}}$ satisfying

$$\phi_{dR,L}(t^n v) = nt^n v + t^n \phi_{dR,L}(v)$$
and $\text{gr}^0 \phi_{\text{dR},L} = \phi_L$. Informally, this means that we have an integrable connection

$$
\nabla + \frac{\phi_{\text{dR},L}}{t} \otimes dt : \mathcal{R}\mathcal{H}(\mathbb{L}) \to \mathcal{R}\mathcal{H}(\mathbb{L}) \otimes \left((\mathcal{O}_X \otimes B_{\text{dR}}(K)) \otimes \Omega^1_X + (\mathcal{O}_X \otimes B_{\text{dR}}(K)) \otimes dt \right)
$$

over $X \otimes B_{\text{dR}}(K)$ whose residue along $t = 0$ coincides with the arithmetic Sen endomorphism $\phi_L$. We develop a theory of formal connections to analyze our connection and prove Theorem 4.1.2.

### 4.1.3 Rigidity of Hodge-Tate sheaves and the relative $p$-adic monodromy conjecture

Finally, let us mention two more results in this chapter. The first result is a rigidity of Hodge-Tate local systems of rank at most two.

**Theorem 4.1.5** (Theorem 4.5.12). Let $X$ be a geometrically connected smooth rigid analytic variety over $k$ and let $\mathbb{L}$ be a $\mathbb{Q}_p$-local system on $X_{\text{ét}}$. Assume that $\text{rank} \mathbb{L}$ is at most two. If $\mathbb{L}_x$ is a Hodge-Tate representation at a classical point $x \in X$, then $\mathbb{L}$ is a Hodge-Tate sheaf. In particular, $\mathbb{L}_y$ is a Hodge-Tate representation at every classical point $y \in X$.

Liu and Zhu proved such a rigidity for de Rham local systems ([36, Theorem 1.3]). We do not know whether a similar statement holds for Hodge-Tate local systems of higher rank.

The second result concerns the relative $p$-adic monodromy conjecture for de Rham local systems; the conjecture states that a de Rham local system on $X$ becomes semistable at every classical point after a finite étale extension of $X$ (cf. [30, §0.8], [36, Remark 1.4]). This is a relative version of the $p$-adic monodromy theorem proved by Berger ([8]), and it is a major open problem in relative $p$-adic Hodge theory. We work on the case of de
Rham local systems with a single Hodge-Tate weight, in which case the result follows from a theorem of Sen (Theorem 4.5.13).

**Theorem 4.1.6** (Theorem 4.5.15). Let $X$ be a smooth rigid analytic variety over $k$ and let $L$ be a $\mathbb{Z}_p$-local system on $X_{\text{ét}}$. Assume that $L$ is a Hodge-Tate sheaf with a single Hodge-Tate weight. Then there exists a finite étale cover $f: Y \to X$ such that $(f^*L)_{\eta}$ is semistable at every classical point $y$ of $Y$.

This is the simplest case of the relative $p$-adic monodromy conjecture. In [15], Colmez gave a proof of the $p$-adic monodromy theorem for de Rham Galois representations using Sen’s theorem mentioned above. It is an interesting question whether one can adapt Colmez’s strategy to the relative setting using Theorem 4.1.6.

**4.1.4 Organization of this chapter**

Section 4.2 presents Sen-Fontaine’s decompletion theory in the relative setting. In Section 4.3, we start with a brief review of Scholze’s approach to the relative $p$-adic Hodge theory. Then we explain the $p$-adic Simpson correspondence by Liu and Zhu, and define the arithmetic Sen endomorphism $\phi_L$. Section 4.4 discusses a Fontaine-type decompletion for the geometric $p$-adic Riemann-Hilbert correspondence by Liu and Zhu, and develops a theory of formal connections. Combining them together we prove Theorem 4.1.1. Section 4.5 presents applications of the study of the arithmetic Sen endomorphism including basic properties of Hodge-Tate sheaves, a rigidity of Hodge-Tate sheaves, and the relative $p$-adic monodromy conjecture.
4.1.5 Conventions

We will use Huber’s adic spaces as our language for non-archimedean analytic geometry. In particular, a rigid analytic variety over \( \mathbb{Q}_p \) will refer to a quasi-separated adic space that is locally of finite type over \( \text{Spa}(\mathbb{Q}_p, \mathbb{Z}_p) \) ([27, §4], [28, 1.11.1]).

We will use Scholze’s theory of perfectoid spaces and pro-étale site. For the pro-étale site, we will use the one introduced in [41, 42].

4.2 Sen-Fontaine’s decompletion theory for an arithmetic family

4.2.1 Set-up

Let \( k \) be a finite field extension of \( \mathbb{Q}_p \). We set \( k_m := k(\mu_p^m) \) and \( k_\infty := \varprojlim_m k_m \). Let \( K \) denote the \( p \)-adic completion of \( k_\infty \). We set \( \Gamma_k := \text{Gal}(k_\infty/k) \). Then \( \Gamma_k \) is identified with an open subgroup of \( \mathbb{Z}_p^\times \) via the cyclotomic character \( \chi: \Gamma_k \to \mathbb{Z}_p^\times \) and it acts continuously on \( K \).

Let \( L^+_{\text{dR}} \) (resp. \( L_{\text{dR}} \)) denote the de Rham period ring \( B^+_{\text{dR}}(K) \) (resp. \( B_{\text{dR}}(K) \)) introduced by Fontaine. We fix a compatible sequence of \( p \)-power roots of unity \( (\zeta_p^n) \) and set \( t := \log[\varepsilon] \) where \( \varepsilon = (1, \zeta_p, \zeta_p^2, \ldots) \in \mathcal{O}_K^\times \). Then \( \Gamma_k \) acts on \( t \) via the cyclotomic character and the \( \mathbb{Z}_p \)-submodule \( \mathbb{Z}_p t \subset L^+_{\text{dR}} \) does not depend on the choice of \( (\zeta_p^n) \). Note that \( L^+_{\text{dR}} \) is a discrete valuation ring with residue field \( K \), fraction field \( L_{\text{dR}} \), and a uniformizer \( t \), and that \( k_\infty[[t]] \) is embedded into \( L^+_{\text{dR}} \).

We now recall the Sen-Fontaine’s decompletion theory ([43, Theorem 3], [25, Théorème 3.6]).
Theorem 4.2.1.

(i) (Sen) Let $V$ be a $K$-representation of $\Gamma_k$. Denote by $V_{\text{fin}}$ the union of finite-dimensional $k$-vector subspaces of $V$ that are stable under the action of $\Gamma_k$. Then the natural map

$$V_{\text{fin}} \otimes_{k_{\infty}} K \to V$$

is an isomorphism.

(ii) (Fontaine) Let $V$ be an $L_{dR}^+$-representation of $\Gamma_k$ and set

$$V_{\text{fin}} := \lim_{\leftarrow n}(V/t^nV)_{\text{fin}},$$

where $(V/t^nV)_{\text{fin}}$ is defined to be the union of finite-dimensional $k$-vector subspaces of $V/t^nV$ that are stable under the action of $\Gamma_k$. Then the natural map

$$V_{\text{fin}} \otimes_{k_{\infty}[[t]]} L_{dR}^+$$

is an isomorphism.

Using this theorem, Sen defined the so-called Sen endomorphism $\phi_V$ on $V_{\text{fin}}$ for a $K$-representation $V$ of $\Gamma_k$ (cf. [13, Theorem 4]), and Fontaine defined a formal connection on $V_{\text{fin}}$ for a $L_{dR}^+$-representation $V$ of $\Gamma_k$ (cf. [25, Proposition 3.7]).

We now turn to the relative setting. Let $A$ be a Tate $k$-algebra that is reduced and topologically of finite type over $k$. It is equipped with the supremum norm and we use this norm when we regard $A$ as a Banach $k$-algebra. We further assume that $(A, A^o)$ is smooth
over \((k, \mathcal{O}_k)\). We set

\[ A_{k_m} := A \hat{\otimes}_k k_m, \quad A_\infty := \lim_{m} A_{k_m}, \quad \text{and} \quad A_K := A \hat{\otimes}_k K. \]

Here we use a slightly heavy notation \(A_{k_m}\) to reserve \(A_m\) for a different ring in a later section. Since \(A, k_m\) and \(K\) are all complete Tate \(k\)-algebras, the completed tensor product is well-defined (or one can use Banach \(k\)-algebra structures). Note that \(A_{k_m}\) (resp. \(A_K\)) is a complete Tate \(k_m\)-algebra (resp. \(K\)-algebra), that \(A_\infty\) is a Tate \(k_\infty\)-algebra and that \(A_K\) is the completion of \(A_\infty\).

We introduce the relative versions of \(k_\infty[[t]]\), \(L^+_{dR}\) and \(L_{dR}\) over \(A\). We set

\[ A_\infty[[t]] := \lim_n A_\infty[t]/(t^n), \]

and equip \(A_\infty[[t]]\) with the inverse limit topology of Tate \(k_\infty\)-algebras \(A_\infty[t]/(t^n)\). We also set

\[ A \hat{\otimes} L^+_{dR} := \lim_n A \hat{\otimes}_k L^+_{dR}/(t^n), \]

and equip \(A \hat{\otimes} L^+_{dR}\) with the inverse limit topology. We finally set

\[ A \hat{\otimes} L_{dR} = (A \hat{\otimes} L^+_{dR})[t^{-1}] \]

and equip \(A \hat{\otimes} L_{dR}\) with the inductive limit topology. Note that \(\Gamma_k\) acts continuously on these rings (cf. \([7\) Appendix]).

**Definition 4.2.2.** An \(A \hat{\otimes} L^+_{dR}\)-representation of \(\Gamma_k\) is an \(A \hat{\otimes} L^+_{dR}\)-module \(V\) that is isomorphic to either \((A \hat{\otimes} L^+_{dR})^r\) or \((A \hat{\otimes} L^+_{dR}/(t^n))^r\) for some \(r\) and \(n\), equipped with a continuous \(A \hat{\otimes} L^+_{dR}\)-semilinear action of \(\Gamma_k\). We denote the category of \(A \hat{\otimes} L^+_{dR}\)-representations of \(\Gamma_k\)
by $\text{Rep}_k(A \hat{\otimes} L_{\text{dR}}^+)$. An $A \hat{\otimes} L_{\text{dR}}^+$-representation of $\Gamma_k$ that is annihilated by $t$ is also called an $A_K$-representation of $\Gamma_k$.

If $V$ is isomorphic to either $(A \hat{\otimes} L_{\text{dR}}^+)^r$ or $(A \hat{\otimes} L_{\text{dR}}^+/ (t^n))^r$ then $V$ admits a topology by taking a basis and the topology is independent of the choice of the basis. Thus the continuity condition of the action of $\Gamma_k$ makes sense. Note that if $V$ is an $A \hat{\otimes} L_{\text{dR}}^+$-representation of $\Gamma_k$, then so are $t^nV$ and $V/t^nV$.

We are going to discuss the relative version of Sen-Fontaine’s theory. Namely, we will work on $A_K$-representations of $\Gamma_k$ and $A \hat{\otimes} L_{\text{dR}}^+$-representations of $\Gamma_k$. Note that Sen’s theory in the relative setting is established by Sen himself ([44, 45]) and that Fontaine’s decompletion theory in the relative setting is established by Berger-Colmez and Bellovin for representations which come from $A$-representations of $\text{Gal}(\overline{k}/k)$ via the theory of $(\varphi, \Gamma)$-modules ([9, 7]). Since we need a Fontaine-type decompletion theory for arbitrary $A \hat{\otimes} L_{\text{dR}}^+$-representations of $\Gamma_k$, we give detailed arguments; we will discuss the decompletion theory in the next subsection, and define Sen’s endomorphism and Fontaine’s connection in Subsection 4.2.3.

We end this subsection with establishing basic properties of the rings we have introduced.

**Proposition 4.2.3.**

(i) For each $n \geq 1$, $A \hat{\otimes}_k L_{\text{dR}}^+/(t^n)$ is Noetherian and faithfully flat over $A_\infty[t]/(t^n)$.

(ii) $A \hat{\otimes}_K L_{\text{dR}}^+$ is a $t$-adically complete flat $L_{\text{dR}}^+$-algebra with $(A \hat{\otimes}_K L_{\text{dR}}^+)/(t^n) = A \hat{\otimes}_k L_{\text{dR}}^+/(t^n)$.

**Proof.** For (i), the first assertion is proved in [10, Lemma 13.3]. We prove that $A \hat{\otimes}_k L_{\text{dR}}^+/(t^n)$ is faithfully flat over $A_\infty[t]/(t^n)$.

First we deal with the case $n = 1$, i.e., faithful flatness of $A_K$ over $A_\infty$. The proof is similar to that of [10, Lemme 5.9]. Recall $A_\infty = \varprojlim_n A_{k_m}$. Since $k_m$ and $K$ are both complete...
valuation fields, $A_K = A_{k_m} \hat{\otimes}_{k_m} K$ is faithfully flat over $A_{k_m}$ (e.g. use \cite{11} Proposition 2.1.7/8 and Theorem 2.8.2/2).

We prove that $A_K$ is flat over $A_1$. For this it suffices to show that for any finitely generated ideal $I \subset A_\infty$, the map $I \otimes_{A_\infty} A_K \to A_K$ is injective. Take such an ideal $I$. As $I$ is finitely generated, there exist a positive integer $m$ and a finitely generated ideal $I_m \subset A_{k_m}$ such that $I = \text{Im}(I_m \otimes_{A_{k_m}} A_\infty \to A_\infty)$. Since $A_K$ is flat over $A_{k_m}$, the map $I_m \otimes_{A_{k_m}} A_K \to A_K$ is injective. On the other hand, this map factors as $I_m \otimes_{A_{k_m}} A_K \to I \otimes_{A_\infty} A_K \to A_K$ and the first map is surjective by the choice of $I_m$. Hence the second map $I \otimes_{A_\infty} A_K \to A_K$ is injective.

For faithful flatness, it remains to prove that the map $\text{Spec } A_K \to \text{Spec } A_\infty$ is surjective. Assume the contrary and take a prime ideal $\mathfrak{P} \in \text{Spec } A_\infty$ that is not in the image of the map. Set $p = \mathfrak{P} \cap A \in \text{Spec } A$. Note that the prime ideals of $A_\infty$ above $p$ are conjugate to each other by the action of $\Gamma_k$. From this we see that no prime ideal of $A_\infty$ above $p$ is in the image of $\text{Spec } A_K \to \text{Spec } A_\infty$. Hence $p$ does not lie in the image of $\text{Spec } A_K \to \text{Spec } A$, which contradicts that $A_K$ is faithfully flat over $A$.

Next we deal with the general $n$. By the local flatness criterion (\cite{37} Theorem 22.3) applied to the nilpotent ideal $(t) \subset A_\infty[t]/(t^n)$, the flatness follows from the case $n = 1$. Moreover, since $\text{Spec } A_K \to \text{Spec } A_\infty$ is surjective, so is $\text{Spec } A \hat{\otimes}_k L^+_{\text{fl}}/(t^n) \to \text{Spec } A_\infty[t]/(t^n)$. Hence $A \hat{\otimes}_k L^+_{\text{fl}}/(t^n)$ is faithfully flat over $A_\infty[t]/(t^n)$.

Assertion (ii) is proved in \cite{10} Lemma 13.4. Note that the proof of \cite{10} Lemma 13.4 works in our setting since we assume the smoothness of $A$. \hfill $\Box$
4.2.2 Sen-Fontaine’s decompletion theory in the relative setting

**Definition 4.2.4.** For an $A \otimes \hat{L}_{dR}^+$-representation $V$ of $\Gamma_k$, we define the subspace $V_{\text{fin}}$ as follows:

- If $V$ is annihilated by $t^n$ for some $n \geq 1$, then $V_{\text{fin}}$ is defined to be the union of finitely generated $A$-submodules of $V$ that are stable under the action of $\Gamma_k$.
- In general, define
  $$V_{\text{fin}} := \lim_{n \to \infty} (V/t^n V)_{\text{fin}}.$$  

If $V$ is killed by $t^n$, then $V_{\text{fin}}$ is an $A_\infty[t]/(t^n)$-module. In general, $V_{\text{fin}}$ is an $A_\infty[[t]]$-module equipped with a semilinear action of $\Gamma_k$.

The following theorem is the main goal of this subsection.

**Theorem 4.2.5.** For an $A \otimes \hat{L}_{dR}^+$-representation $V$ of $\Gamma_k$ that is finite free of rank $r$ over $A \otimes \hat{L}_{dR}^+$, the $A_\infty[[t]]$-module $V_{\text{fin}}$ is finite free of rank $r$. Moreover, the natural map

$$V_{\text{fin}} \otimes_{A_\infty[[t]]} (A \otimes \hat{L}_{dR}^+) \to V$$

is an isomorphism, and $V_{\text{fin}}/t^n V_{\text{fin}}$ is isomorphic to $(V/t^n V)_{\text{fin}}$ for each $n \geq 1$.

The key tool in the proof is the Sen method, which is axiomatized in [9, §3]. We review parts of the Tate-Sen conditions that are used in our proofs. For a thorough treatment, we refer the reader to *loc. cit.*

Consider Tate’s normalized trace map

$$R_{k,m} = R_m : K \to k_m.$$
On $k_{m+m'} \subset K$, this map is defined as

$$[k_{m+m'} : k_m]^{-1} \text{tr}_{k_{m+m'}/k_m} : k_{m+m'} \to k_m,$$

and it extends continuously to $R_{k,m} : K \to k_m$. We denote the kernel $\text{Ker} R_{k,m}$ by $X_m$. The map $R_{k,m}$ extends $A$-linearly to the map $R_{A,m} : A_K \to A_k$. Fix a real number $c_3 > 1$. By work of Tate and Sen (Proposition 3.1.4 and Proposition 4.1.1), $G_0 = \Gamma_k, \tilde{\Lambda} = A_K, R_m$, and the valuation $\text{val}$ on $A_K$ satisfy the Tate-Sen axioms in \cite{S} §3 for any fixed positive numbers $c_1$ and $c_2$.

In particular, $X_{A,m} := A \hat{\otimes}_k X_m$ is the kernel of $R_{A,m}$, and we have topological splitting $A_K = A_k m \oplus X_{A,m}$. For $\gamma \in \Gamma_k$, let $m(\gamma) \in \mathbb{Z}$ be the valuation of $\chi(\gamma) - 1 \in \mathbb{Z}_p$. Then there exists a positive integer $m(k)$ such that for each $m \geq m(k)$ and $\gamma \in \Gamma_k$ with $m(\gamma) \leq m$, $\gamma - 1$ is invertible on $X_{A,m}$ and

$$\text{val}((\gamma - 1)^{-1} a) \geq \text{val}(a) - c_3$$

for each $a \in A_K$.

Finally, for each matrix $U = (a_{ij}) \in M_r(A_K)$, we set $\text{val} U := \min_{i,j} \text{val} a_{ij}$.

**Proposition 4.2.6.** Each finitely generated $A$-submodule of $A_K$ that is stable under the action of an open subgroup of $\Gamma_k$ is contained in $A_\infty$.

**Proof.** We follow the proof of \cite{S} Proposition 3]. By \cite{S} Corollaire 2.1.4], there exist complete discrete valuation fields $E_1, \ldots, E_s$ and an isometric embedding $A \hookrightarrow \prod_{i=1}^{s} E_i$. Then extending the scalar yields an isometric embedding

$$A_K = A_k m \oplus X_{A,m} \hookrightarrow \prod_{i=1}^{s} E_i \hat{\otimes}_k K = \prod_{i=1}^{s} (E_i \hat{\otimes}_k k_m \oplus E_i \hat{\otimes}_k X_m)$$

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preserving the topological splittings.

Let $\Gamma_k'$ be an open subgroup of $\Gamma_k$ and $W$ a finitely generated $A$-submodule of $A_k$ that is stable under the action of $\Gamma_k'$. Let $W_i$ be the finite-dimensional $E_i$-vector subspace of $E_i \hat{\otimes}_k K$ generated by the image of $W$ under the map $A_k \rightarrow \prod_{i=1}^s E_i \hat{\otimes}_k K \rightarrow E_i \hat{\otimes}_k K$. To prove that $W$ is contained in $A_\infty = \bigcup_m A_{k_m}$, it suffices to prove that for each $i$, there exists a large integer $m$ such that $W_i$ is contained in $E_i \hat{\otimes}_k k_m$.

Replacing $\Gamma_k'$ by a smaller open subgroup if necessary, we may assume that there exists a topological generator $\gamma$ of $\Gamma_k'$. Replacing $E_i$ by a finite field extension, we may also assume that all the eigenvalues of the $E_i$-endomorphism $\gamma$ on $W_i$ lie in $E_i$.

Let $w \in W_i$ be an eigenvector for $\gamma$ and let $\lambda \in E$ be its eigenvalue. Note that $\Gamma_k'$ acts continuously on $W_i$. When $j$ goes to infinity, $\gamma^j$ approaches 1 and thus $\lambda^j$ approaches 1. This implies that $\lambda$ is a principal unit, i.e. $|\lambda - 1|_{E_i} < 1$.

**Lemma 4.2.7.** The eigenvalue $\lambda$ is a $p$-power root of unity.

**Proof.** We follow the proof of [50, Proposition 7(c)]. Assume the contrary. We will prove that $\gamma - \lambda : E_i \hat{\otimes}_k K \rightarrow E_i \hat{\otimes}_k K$ is bijective, which would contradict that the nonzero element $w \in W_i \subset E_i \hat{\otimes}_k K$ satisfies $(\gamma - \lambda)w = 0$.

Let $m$ be the integer such that $k_m$ is the fixed subfield of $k_\infty$ by $\gamma$. Consider the map $\gamma - 1 : E_i \hat{\otimes}_k K \rightarrow E_i \hat{\otimes}_k K$. This map preserves the decomposition $E_i \hat{\otimes}_k K = E_i \hat{\otimes}_k k_m \oplus E_i \hat{\otimes}_k X_m$. Moreover, it is zero on $E_i \hat{\otimes}_k k_m$ and bijective on $E_i \hat{\otimes}_k X_m$ with continuous inverse. Denote the inverse by $\rho$. Then $\rho$ is a bounded $E_i \hat{\otimes}_k k_m$-linear operator with operator norm at most $p^{e^3}$. Since $\lambda \in E_i$ and $\lambda \neq 1$, the map $\gamma - \lambda$ is bijective on $E_i \hat{\otimes}_k k_m$. So it suffices to prove that $\gamma - \lambda$ is bijective on $E_i \hat{\otimes}_k X_m$. 

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As operators on $E_i \hat{\otimes}_k X_m$, we have
\[
(\gamma - \lambda)\rho = ((\gamma - 1) - (\lambda - 1))\rho = 1 - (\lambda - 1)\rho.
\]
Thus if $|\lambda - 1|_{E_i p^{c_3}} < 1$, then $1 - (\lambda - 1)\rho$ has an inverse on $E_i \hat{\otimes}_k X_m$ given by a geometric series and thus $\gamma - \lambda$ admits a continuous inverse on $E_i \hat{\otimes}_k X_m$. If $|\lambda - 1|_{E_i p^{c_3}} \geq 1$, first take a large integer $j$ with $|\lambda^{p^j} - 1|_{E_i p^{c_3}} < 1$. Then we can prove that $\gamma^{p^j} - \lambda^{p^j}$ has a bounded inverse on $E_i \hat{\otimes}_k X_m$. Hence so does $\gamma - \lambda$.

We continue the proof of the proposition. Since each eigenvalue of $\gamma$ on $W_i$ is a $p$-power root of unity, we replace $\gamma$ by a higher $p$-power and may assume that $\gamma$ acts on $W_i$ unipotently. Thus $\gamma - 1$ acts on $W_i$ nilpotently.

Let $m$ be the integer such that $k_m$ is the fixed subfield of $k_\infty$ by $\gamma$. Then the map $\gamma - 1: E_i \hat{\otimes}_k K \to E_i \hat{\otimes}_k K$ is zero on $E_i \hat{\otimes}_k k_m$ and bijective on $E_i \hat{\otimes}_k X_m$. This implies that the nilpotent endomorphism $\gamma - 1$ on $W_i$ is actually zero and thus $W_i$ is contained in $E_i \hat{\otimes}_k k_m$.

\textbf{Example 4.2.8.} For the trivial $A_K$-representation $V = A_K$ of $\Gamma_k$, we have $V_{\text{fin}} = A_\infty$ by Proposition 4.2.6.

The following theorem describes $V_{\text{fin}}$ for a general $A_K$-representation $V$ of $\Gamma_k$, and it was first proved by Sen ([11], [15]).

\textbf{Theorem 4.2.9.} For an $A_K$-representation $V$ of $\Gamma_k$, the $A_\infty$-module $V_{\text{fin}}$ is finite free. Moreover, the natural map
\[
V_{\text{fin}} \otimes_{A_\infty} A_K \to V
\]
is an isomorphism.
Proof. First we prove the following lemma.

**Lemma 4.2.10.** There exist an $A_K$-basis $v_1, \ldots, v_r \in V$ and a large positive integer $m$ such that the transformation matrix of $\gamma$ with respect to this basis has entries in $A_{km}$ for each $\gamma \in \Gamma_k$.

Proof. This follows from the Tate-Sen method for $\Gamma_k$-representations in the relative setting. By [14, Lemme 3.18], $V$ has a $\Gamma_k$-stable $A_K$-lattice. Note that [14, Lemme 3.18] only concerns reduced affinoid algebras over a finite extension of $\mathbb{Q}_p$ but the same proof works for $A_K$ since one can apply Raynaud’s theory to $A_K$.

By [9, Corollaire 3.2.4], there exist an $A_K$-basis $v_1, \ldots, v_r \in V$, a large positive integer $m$, and an open subgroup $\Gamma'_k$ of $\Gamma_k$ such that the transformation matrix of $\gamma$ with respect to this basis has entries in $A_{km}$ for each $\gamma \in \Gamma'_k$. By increasing $m$ if necessary, we may also assume that $\Gamma'_k$ acts trivially on $A_{km}$.

For each $\gamma \in \Gamma_k$, we denote by $U_\gamma \in \text{GL}_r(A_K)$ the transformation matrix of $\gamma$ with respect to $v_1, \ldots, v_r$. Note that $U_{\gamma \gamma'} = U_\gamma (U_{\gamma'})$ for $\gamma, \gamma' \in \Gamma_k$.

Take a set $\{\gamma_1, \ldots, \gamma_s\}$ of coset representatives of $\Gamma_k/\Gamma'_k$ and let $W$ be the finitely generated $A_{km}$-submodule of $A_K$ generated by the entries of $U_{\gamma_1}, \ldots, U_{\gamma_s}$. Since $U_{\gamma_1 \gamma'} = U_{\gamma_1} (U_{\gamma'})$ for $\gamma' \in \Gamma'_k$ and $\gamma_1 (U_{\gamma'})$ has entries in $A_{km}$ by our construction, it follows that $W$ is independent of the choice of the representatives $\gamma_1, \ldots, \gamma_s$. Moreover, we have $\gamma'(U_{\gamma_i}) = U_{\gamma'}^{-1} U_{\gamma', \gamma_i}$ for $\gamma' \in \Gamma'_k$. From this we see that $W$ is stable under the action of $\Gamma'_k$.

Proposition 4.2.6 implies that $W \subset A_{\infty}$, namely, $U_{\gamma_1}, \ldots, U_{\gamma_s} \in \text{GL}_r(A_{\infty})$. Thus if we increase $m$ so that $U_{\gamma_1}, \ldots, U_{\gamma_s} \in \text{GL}_r(A_{km})$, then $U_{\gamma} \in \text{GL}_r(A_{km})$ for any $\gamma \in \Gamma_k$.

We keep the notation in the proof of the lemma. From the lemma, we see that $\bigoplus_{i=1}^{r} A_{\infty} v_i \subset V_{\text{fin}}$. So it suffices to prove that this is an equality.

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Take any $v \in V_{\text{fin}}$. Let $W_v$ be the $A_{km}$-submodule of $A_K$ generated by the coordinates of $\gamma v$ with respect to the basis $v_1, \ldots, v_r$ where $\gamma$ runs over all elements of $\Gamma_k$. Since $v \in V_{\text{fin}}$, this is a finitely generated $A_{km}$-module.

Write $v = \sum_{i=1}^r a_i v_i$ with $a_i \in A_K$ and denote the column vector of the $a_i$ by $\bar{a}$. Then it is easy to see that $W_v$ is generated by the entries of $U_{\gamma \gamma}(\bar{a})$ ($\gamma \in \Gamma_k$). Since $U_{\gamma \gamma} = U_{\gamma' \gamma'}(U_{\gamma})$ for $\gamma, \gamma' \in \Gamma_k$, we compute

$$\gamma'(U_{\gamma \gamma}(\bar{a})) = U_{\gamma' \gamma}(\gamma'(\gamma))(\bar{a}).$$

From this we see that $W_v$ is stable under the action of $\Gamma_k$.

By Proposition 4.2.6 we have $W_v \subset A_\infty$. In particular, $a_1, \ldots, a_r \in A_\infty$ and thus $v \in \bigoplus_{i=1}^r A_\infty v_i$. \qed

**Proposition 4.2.11.** Let $V$ be an $A \widehat{\otimes} L_{\text{dr}}^+$-representation of $\Gamma_k$. If $V$ is finite free of rank $r$ over $A \widehat{\otimes} L_{\text{dr}}^+/(t^n)$, then $V_{\text{fin}}$ is finite free of rank $r$ over $A_\infty[t]/(t^n)$. Moreover, the natural map

$$V_{\text{fin}} \otimes_{A_\infty[[t]]} (A \widehat{\otimes} L_{\text{dr}}^+) \to V$$

is an isomorphism.

**Proof.** We prove this proposition by induction on $n$. When $n = 1$, this is Theorem 4.2.9. So we assume $n > 1$.

Set $V' := t^{n-1}V$ and $V'' := V/V'$. They are $A \widehat{\otimes} L_{\text{dr}}^+$-representations of $\Gamma_k$ and $V''$ is finite free of rank $r$ over $A \widehat{\otimes} L_{\text{dr}}^+/(t^{n-1})$. By induction hypothesis, $V''_{\text{fin}}$ is finite free of rank $r$ over $A_\infty[t]/(t^{n-1})$ and $V''_{\text{fin}} \otimes_{A_\infty[[t]]} (A \widehat{\otimes} L_{\text{dr}}^+/(t^{n-1})) \cong V''$.

Take lifts $v_1, \ldots, v_r$ of a basis of $V''_{\text{fin}}$ to $V$. Then $v_1, \ldots, v_r$ form an $A_\infty[t]/(t^n)$-basis of $V$. We will prove that after a suitable modification of $v_1, \ldots, v_r$ the transformation matrix
of $\gamma$ on $V$ with respect to the new basis has entries in $A_\infty[t]/(t^n)$ for every $\gamma \in \Gamma_k$.

Suppose that we are given an element $\gamma$ of $\Gamma_k$. For each $1 \leq j \leq r$, write $\gamma v_j = \sum_{i=1}^r a_{ij} v_i$ with $a_{ij} \in \hat{A} \otimes L^+_{\text{dr}}/(t^n)$. Then the $r \times r$ matrix $T := (a_{ij})$ is invertible since it is so modulo $t^{n-1}$. By the property of $V_\text{fin}^n$, we can write

$$a_{ij} = a_{ij}^0 + t^{n-1} a_{ij}^1, \quad a_{ij}^0 \in A_\infty[t]/(t^n), \quad a_{ij}^1 \in A_K = A \hat{\otimes} L^+_{\text{dr}}/(t).$$

Set $U := (a_{ij}^0 \text{ mod } t) \in M_r(A_\infty)$. This is invertible. In fact, $U$ is the transformation matrix of $\gamma$ acting on $V/tV$ with respect to the basis $(v_i \text{ mod } t)$.

Since $\Gamma_k$ acts continuously on $V/tV$, $\text{val}(U - 1) > c_3$ and $m(\gamma) > \max\{c_3, m(k)\}$ for some $\gamma \neq 1$ close to 1. From now on, we fix such $\gamma$.

**Claim 4.2.12.** There exists an element in $\text{GL}_r(A \hat{\otimes} L^+_{\text{dr}}/(t^n))$ of the form $1 + t^{n-1} M$ with $M \in M_r(A_K)$ such that the $r \times r$ matrix

$$(1 + t^{n-1} M)^{-1} T \gamma (1 + t^{n-1} M)$$

lies in $\text{GL}_r(A_\infty[t]/(t^n))$.

**Proof.** Noting that every element in $A \hat{\otimes} L^+_{\text{dr}}/(t^n)$ is annihilated by $t^n$, we compute

$$(1 + t^{n-1} M)^{-1} T \gamma (1 + t^{n-1} M) = (1 - t^{n-1} M) T(1 + \chi(\gamma)^{n-1} t^{n-1} \gamma(M))$$

$$= T - t^{n-1}(M T - \chi(\gamma)^{n-1} T \gamma(M))$$

$$- t^{2(n-1)} \chi(\gamma)^{n-1} M T \gamma(M)$$

$$= T - t^{n-1}(M U - \chi(\gamma)^{n-1} U \gamma(M)).$$

Since $T = (a_{ij}^0) + t^{n-1}(a_{ij}^1)$ with $(a_{ij}^0) \in \text{GL}_r(A_\infty[t]/(t^n))$, it suffices to find $M \in M_r(A_K)$
such that
\[
(a_{ij}^1) - (MU - \chi(\gamma)^{n-1}U\gamma(M)) \in M_r(A_\infty).
\]

We will apply Lemma 4.2.13 below to $U$, $U' = U^{-1}$ and $s = n-1$. Take $m \geq m(\gamma)$ large enough so that $U$ and $U^{-1}$ lie in $\text{GL}_r(A_{km})$. Recall the normalized trace map $R_{A,m} : A_K \to A_{km}$ with the kernel $X_{A,m}$. Since $R_{A,m}$ is $A_{km}$-linear, we see that $(1 - R_{A,m}(a_{ij}))U^{-1} \in M_r(X_{A,m})$. Therefore, by Lemma 4.2.13 there exists $M_0 \in M_r(X_{A,m})$ such that
\[
((1 - R_{A,m}(a_{ij}))U^{-1} = M_0 - \chi(\gamma)^{n-1}U\gamma(M_0)U^{-1}.
\]

From this we have
\[
(a_{ij}^1) - (M_0U - \chi(\gamma)^{n-1}U\gamma(M_0)) = R_{A,m}(a_{ij}) \in M_r(A_{km}),
\]

and the matrix $1 + t^{n-1}M_0$ satisfies the condition of the lemma.

We continue the proof of the proposition. We replace the basis $v_1, \ldots, v_r$ by the one corresponding to the matrix $1 + t^{n-1}M$ in the lemma. Then the transformation matrix of our fixed $\gamma$ with respect to the new $v_1, \ldots, v_r$ has entries in $A_{km}[t]/(t^n)$. Thus for each $1 \leq i \leq r$, the $\gamma^{v_i}$-orbit of $v_i$ is contained in a finitely generated $A_{km}[t]/(t^n)$-submodule of $V$ that is stable under $\gamma^{v_i}$. Since $\gamma^{v_i}$ is of finite index in $\Gamma_k$, the $\Gamma_k$-orbit of $v_i$ is also contained in a finitely generated $A_{km}[t]/(t^n)$-submodule of $V$ that is stable under $\Gamma_k$. This means that $v_1, \ldots, v_r \in V_{\text{fin}}$. Hence $\bigoplus_{i=1}^r A_\infty[t]/(t^n)v_i \subset V_{\text{fin}}$.

It remains to prove that $\bigoplus_{i=1}^r A_\infty[t]/(t^n)v_i = V_{\text{fin}}$. Since $A_\infty[t]/(t^n) \to A \widehat{\otimes} L^+_{\text{dr}}/(t^n)$ is faithfully flat and $V = \bigoplus_{i=1}^r A \widehat{\otimes} L^+_{\text{dr}}/(t^n)v_i$, it is enough to show that the natural map $V_{\text{fin}} \otimes A_\infty[t]/(t^n) A \widehat{\otimes} L^+_{\text{dr}}/(t^n) \to V$ is injective. Note that $V_{\text{fin}} \otimes A_\infty[[t]] A \widehat{\otimes} L^+_{\text{dr}} = V_{\text{fin}} \otimes A_\infty[[t]] A \widehat{\otimes} L^+_{\text{dr}}$. 

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Recall the exact sequence \(0 \to V' \to V \to V'' \to 0\). From this we have an exact sequence \(0 \to V_{\text{fin}} \to V_{\text{fin}} \to V_{\text{fin}}'' \to 0\), and it yields the following commutative diagram with exact rows

\[
\begin{array}{cccccc}
0 & \longrightarrow & V_{\text{fin}} & \otimes & (A \hat{\otimes} L_{dR}^+) & \longrightarrow & V_{\text{fin}} & \otimes & (A \hat{\otimes} L_{dR}^+) & \longrightarrow & V_{\text{fin}}'' & \otimes & (A \hat{\otimes} L_{dR}^+) & \longrightarrow & 0,
\end{array}
\]

where the tensor products in the first row are taken over \(A_{\infty[[t]]}\). By induction hypothesis, the first and the third vertical maps are isomorphisms. Hence the second vertical map is injective and this completes the proof. 

The following lemma is used in the proof of Proposition 4.2.11.

**Lemma 4.2.13.** Let \(s\) be a positive integer. Let \(U, U'\) be elements in \(M_r(A_{\infty})\) satisfying \(\text{val}(U-1) > c_3\) and \(\text{val}(U'-1) > c_3\). Take a positive integer \(m\) such that \(m > \max\{m(k), c_3\}\) and \(U, U' \in M_r(A_{k_m})\). Then for any \(\gamma \in \Gamma_k\) with \(c_3 < m(\gamma) \leq m\), the map

\[
f: M_r(A_K) \to M_r(A_K), \quad M \mapsto M - \chi(\gamma)^*U\gamma(M)U'
\]

is bijective on the subset \(M_r(X_{A,m})\) consisting of the \(r \times r\) matrices with entries in the kernel \(X_{A,m}\) of \(R_{A,m}: A_K \to A_{k_m}\).

**Proof.** The proof of [13, Lemma 15.3.9] works in our setting. For the convenience of the reader, we reproduce their proof here.

We first check that \(f\) restricts to an endomorphism on \(M_r(X_{A,m})\). This follows from the fact that the map \(R_{A,m}\) is \(A_{k_m}\)-linear and \(\Gamma_k\)-equivariant and thus \(X_{A,m}\) is an \(A_{k_m}\)-module stable under the action of \(\Gamma_k\).
We define a map $h: M_r(A_K) \to M_r(A_K)$ by

$$h(N) := N - \chi(\gamma)^* UNU'$$

$$= (N - \chi(\gamma)^* N) + \chi(\gamma)^* ((N - UN) + UN(1 - U')).$$

Then the same argument as above shows that $h$ restricts to an endomorphism on $M_r(X_{A,m})$. We also have $f(M) = (1 - \gamma)M + h(\gamma M)$.

Recall that the map $1 - \gamma: M_r(X_{A,m}) \to M_r(X_{A,m})$ admits a continuous inverse with the operator norm at most $p^{c_3}$. We denote this inverse by $\rho$. Since $(f \circ \rho - \text{id})M = h(\gamma \rho(M))$, it suffices to prove that the operator norm of $h$ is less than $p^{-c_3}$; this would imply that the operator norm of $h \circ \gamma \circ \rho$ is less than 1. Thus $f \circ \rho$ admits a continuous inverse given by a geometric series and hence $f$ is bijective on $M_r(X_{A,m})$.

By the second expression of $h$, we have

$$\text{val}(h(N)) \geq \min\{\text{val}((1 - \chi(\gamma)^*)N), \text{val}((U - 1)N), \text{val}(UN(1 - U'))\}$$

$$\geq \min\{\text{val}((1 - \chi(\gamma))N), \text{val}((U - 1)N), \text{val}(N(1 - U'))\}.$$

From this we have

$$\text{val}(h(N)) \geq \text{val}(N) + \delta$$

where $\delta := \min\{m(\gamma), \text{val}(U - 1), \text{val}(U' - 1)\}$. Thus the operator norm of $h$ is at most $p^{-\delta}$. Since $\delta > c_3$ by assumption, this completes the proof.

**Proof of Theorem 4.2.5.** For each $n \geq 1$, put $V_n := V/t^nV$. This is an $A \hat{\otimes} L_{\text{dR}}^+$-representation of $\Gamma_k$ that is finite free of rank $r$ over $A \hat{\otimes} L_{\text{dR}}^+/t^n$. Thus by Proposition 4.2.11, $(V_n)_{\text{fin}}$ is finite free of rank $r$ over $A_{\infty}[t]/(t^n)$, and $(V_n)_{\text{fin}} \otimes_{A_{\infty}[t]} (A \hat{\otimes} L_{\text{dR}}^+) \to V_n$ is an isomorphism.

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By definition, we have \( V_{\text{fin}} = \lim_{\longleftarrow n} (V_n)_{\text{fin}} \). Since the natural map \( V_{n+1} \to V_n \) is surjective, so is the map \((V_{n+1})_{\text{fin}} \to (V_n)_{\text{fin}}\) by the faithfully flatness of \( A_\infty[t]/(t^{n+1}) \to A_\hat\otimes L^+_dR/(t^{n+1})\). Thus lifting a basis of \((V_n)_{\text{fin}}\) gives a basis of \( V_{\text{fin}} \) and we see that \( V_{\text{fin}} \) is finite free of rank \( r \) over \( A_\infty[[t]] \). The remaining assertions also follow from this.

**Proposition 4.2.14.** For an \( A_\hat\otimes L^+_dR \)-representation \( V \) of \( \Gamma_k \) that is finite free of rank \( r \) over \( A_\hat\otimes L^+_dR \), the \( A_\infty[[t]] \)-module \( V_{\text{fin}} \) is the union of finitely generated \( A_\infty[[t]] \)-submodules of \( V \) that are stable under the action of \( \Gamma_k \). In particular, the natural inclusion

\[
(V_{\text{fin}})^{\Gamma_k} \hookrightarrow V^{\Gamma_k}
\]

is an isomorphism.

**Proof.** Let \( V'_{\text{fin}} \) denote the union of finitely generated \( A_\infty[[t]] \)-submodules of \( V \) that are stable under the action of \( \Gamma_k \). Then \( V_{\text{fin}} \subseteq V'_{\text{fin}} \) by Theorem 4.2.5. So it remains to prove the opposite inclusion. For this it suffices to prove \( V'_{\text{fin}}/t^n V'_{\text{fin}} \subseteq V_{\text{fin}}/t^n V_{\text{fin}} \) for each \( n \geq 1 \). Since \( V_{\text{fin}}/t^n V_{\text{fin}} = (V/t^n V)_{\text{fin}} \) by Theorem 4.2.5, the desired inclusion follows from the definition of \((V/t^n V)_{\text{fin}}\) noting \( A_\infty[t]/(t^n) = \bigcup_m A_{k_m}[t]/(t^n) \). The second assertion follows from the first. \( \square \)

**Example 4.2.15.** For the trivial \( A_\hat\otimes L^+_dR \)-representation \( V = A_\hat\otimes L^+_dR \) of \( \Gamma_k \), we have \( V_{\text{fin}} = A_\infty[[t]] \).

Finally, we discuss topologies on \( V_{\text{fin}} \) and the continuity of the action of \( \Gamma_k \).

**Lemma 4.2.16.** Let \( W \) be a finite free \( A_\infty[[t]]/(t^n) \)-module equipped with an action of \( \Gamma_k \). Then \( \Gamma_k \)-action is continuous with respect to the topology on \( W \) induced from the product topology on \( A_\infty[[t]]/(t^n) \cong A^n_\infty \) if and only if it is continuous with respect to the topology on \( W \) induced from the subspace topology on \( A_\infty[[t]]/(t^n) \subseteq A_\hat\otimes L^+_dR/(t^n) \).
Proof. For each of the two topologies on $W$, the continuity of $\Gamma_k$ implies that there exist an $A_\infty[[t]]/(t^n)$-basis $w_1, \ldots, w_r$ of $W$ and a large positive integer $m$ such that $W_m := \bigoplus_{i=1}^r A_{km}[[t]]/(t^n)w_i$ is stable under $\Gamma_k$ and its action on $W_m$ is continuous with respect to the induced topology $W_m \subset W$. Conversely, if the $\Gamma_k$-action on $W_m$ is continuous with respect to the induced topology $W_m \subset W$ for such $\Gamma_k$-stable $A_{km}[[t]]/(t^n)$-submodule $W_m$ with $W_m \otimes_{A_{km}[[t]]/(t^n)} A_\infty[[t]]/(t^n) = W$, the $\Gamma_k$-action on $W$ is continuous.

The subspace topology on $A_{km}[[t]]/(t^n)$ from $A \otimes L_{dR}^+/ (t^n)$ coincides with the product topology on $A_{km}[[t]]/(t^n) \cong A_{km}$. From this we find that the continuity conditions on the action of $\Gamma_k$ on $W_m$ with respect to the two topologies coincide. Hence the two continuity properties of the action of $\Gamma_k$ on $W$ are equivalent.

\begin{definition}
Let $V$ be an $A \otimes L_{dR}^+$-representation of $\Gamma_k$.

- If $V$ is finite free over $A \otimes L_{dR}^+/ (t^n)$ for some $n \geq 1$, we equip $V_{\text{fin}}$ with the topology acquired from topologizing $A_\infty[[t]]/(t^n)$ with the product topology of the $p$-adic topology on $A_\infty$. Then $\Gamma_k$ acts continuously on $V_{\text{fin}}$ by Lemma 4.2.16.

- If $V$ is finite free over $A \otimes L_{dR}^+$, we equip $V_{\text{fin}}$ with the inverse limit topology via $V_{\text{fin}} = \varprojlim_n (V/t^nV)_{\text{fin}}$. Then $\Gamma_k$ acts continuously on $V_{\text{fin}}$.
\end{definition}

\begin{definition}
An $A_\infty[[t]]$-representation of $\Gamma_k$ is an $A_\infty[[t]]$-module $W$ that is isomorphic to either $(A_\infty[[t]])^r$ or $(A_\infty[[t]]/(t^n))^r$ for some $r$ and $n$, equipped with a continuous $A_\infty[[t]]$-semilinear action of $\Gamma_k$ (here the topology on $W$ is acquired from the $p$-adic topology on $A_\infty$ by considering the product topology and the inverse limit topology as before).

We denote the category of $A_\infty[[t]]$-representations of $\Gamma_k$ by $\text{Rep}_{\Gamma_k}(A_\infty[[t]])$. An $A_\infty[[t]]$-representation of $\Gamma_k$ that is annihilated by $t$ is also called an $A_\infty$-representation of $\Gamma_k$.
\end{definition}
Theorem 4.2.19. The decompletion functor

\[ \text{Rep}_{\Gamma_k}(A \hat{\otimes} L^+_{\text{dR}}) \rightarrow \text{Rep}_{\Gamma_k}(A_\infty[[t]]), \quad V \mapsto V_{\text{fin}} \]

is an equivalence of categories. A quasi-inverse is given by \( W \mapsto W \otimes_{A_\infty[[t]]} (A \hat{\otimes} L^+_{\text{dR}}). \)

Proof. By Theorem 4.2.5, Proposition 4.2.11, and Lemma 4.2.16, the functor is well-defined and essentially surjective. The full faithfulness follows from Proposition 4.2.14. \( \square \)

4.2.3 Sen’s endomorphism and Fontaine’s connection in the relative setting

Proposition 4.2.20. Let \( W \) be an \( A_\infty \)-representation of \( \Gamma_k \). Then there exists a unique \( A_\infty \)-linear map \( \phi_W: W \rightarrow W \) satisfying the following property: For any \( w \in W \), there exists an open subgroup \( \Gamma_{k,w} \) of \( \Gamma_k \) such that

\[ \gamma w = \exp(\log(\chi(\gamma))\phi_W)(w) \]

for each \( \gamma \in \Gamma_{k,w} \). Here \( \log \) (resp. \( \exp \)) is the \( p \)-adic logarithm (resp. exponential). Moreover, \( \phi_W \) is \( \Gamma_k \)-equivariant and functorial with respect to \( W \).

Remark 4.2.21. The proposition says that the endomorphism \( \phi_W \) is computed as

\[ \phi_W(w) = \lim_{\gamma \rightarrow 1} \frac{\gamma w - w}{\log \chi(\gamma)} \]

for \( w \in W \).

Proof. This is standard; arguments in [13, Theorem 4] also work in our setting. See also [55, §2], [41, Proposition 4], [25, Proposition 2.5], and [13, §15.1]. \( \square \)
The following lemma is also proved by standard arguments.

**Lemma 4.2.22.** Let $W_1$ and $W_2$ be $A$-representations of $\Gamma_k$. Then we have the following equalities:

- $\phi_{W_1 \oplus W_2} = \phi_{W_1} \oplus \phi_{W_2}$ on $W_1 \oplus W_2$.
- $\phi_{W_1 \otimes W_2} = \phi_{W_1} \otimes \text{id}_{W_2} + \text{id}_{W_1} \otimes \phi_{W_2}$ on $W_1 \otimes W_2$.
- $\phi_{\text{Hom}(W_1, W_2)}(f) = \phi_{W_2} \circ f - f \circ \phi_{W_1}$ for $f \in \text{Hom}(W_1, W_2)$.

**Definition 4.2.23.** Let $V$ be an $A_K$-representation of $\Gamma_k$. We denote by $\phi_V$ the $A_K$-linear endomorphism $\phi_{V_{\text{fin}}} \otimes \text{id}_{A_K}$ on $V = V_{\text{fin}} \otimes A_{\infty} A_K$.

**Proposition 4.2.24.** Let $W$ be an $A_{\infty}[[t]]$-representation of $\Gamma_k$. Then there exists a unique $A_{\infty}$-linear map $\phi_{\text{dR},W}: W \to W$ satisfying the following property: For each $n \in \mathbb{N}$ and $w \in W$, there exists an open subgroup $\Gamma_{k,n,w}$ of $\Gamma_k$ such that

$$\gamma w \equiv \exp(\log(\chi(\gamma)) \phi_{\text{dR},W})(w) \pmod{t^n W}$$

for $\gamma \in \Gamma_{k,n,w}$.

**Proof.** Note that $W/t^n W$ is an $A_{\infty}$-representation of $\Gamma_k$. So the proposition follows from Proposition 4.2.20. 

**Definition 4.2.25.** Set $A_{\infty}((t)) := A_{\infty}[[t]][t^{-1}]$. We denote by $\partial_t$ the $A_{\infty}$-linear endomorphism

$$A_{\infty}((t)) \to A_{\infty}((t)), \quad \sum_{j \gg -\infty} a_j t^j \mapsto \sum_{j \gg -\infty} j a_j t^{j-1}.$$ 

The restriction of $\partial_t$ to $A_{\infty}[[t]]$ is also denoted by $\partial_t$. 

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Proposition 4.2.26. For an $A_\infty[[t]]$-representation $W$ of $\Gamma_k$, the endomorphism $\phi_{\text{DR},W} : W \to W$ satisfies

$$\phi_{\text{DR},W}(\alpha w) = t\partial_t(\alpha)w + \alpha \phi_{\text{DR},W}(w)$$

for every $\alpha \in A_\infty[[t]]$ and $w \in W$.

Proof. By the characterizing property of $\phi_{\text{DR},W}$, we may assume that $W$ is annihilated by some power of $t$. In this case, it is enough to check the equality for $\alpha = t^j$ by $A_\infty$-linearity of $\phi_{\text{DR},W}$. By induction on $j$, we may further assume that $\alpha = t$.

So we need to show $\phi_{\text{DR},W}(tw) = tw + t\phi_{\text{DR},W}(w)$. This follows from

$$\phi_{\text{DR},W}(tw) = \lim_{\gamma \to 1} \frac{\gamma(tw) - tw}{\log \chi(\gamma)}$$

$$= \lim_{\gamma \to 1} \frac{\chi(\gamma) - 1}{\log \chi(\gamma)} t\gamma(w) + t \lim_{\gamma \to 1} \frac{\gamma(w) - w}{\log \chi(\gamma)}$$

$$= tw + t\phi_{\text{DR},W}(w).$$

\[\square\]

Lemma-Definition 4.2.27. Let $W$ be a finite free $A_\infty[[t]]$-representation of $\Gamma_k$. Then $W[t^{-1}] := W \otimes_{A_\infty[[t]]} A_\infty((t))$ is a finite free $A_\infty((t))$-representation of $\Gamma_k$ with $\Gamma_k$-stable decreasing filtration $\text{Fil}^j W[t^{-1}] := t^j W$. Moreover, the $A_\infty$-linear endomorphism

$$\phi_{\text{DR},W[t^{-1}]} : W[t^{-1}] \to W[t^{-1}]$$

sending $w \in \text{Fil}^j W[t^{-1}]$ to

$$\phi_{\text{DR},W[t^{-1}]|^j}(w) := jw + t^j \phi_{\text{DR},W}(t^{-j}w)$$

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is well-defined and satisfies $\phi_{dR,W[t^{-1}]}|_{W} = \phi_{dR,W}$.

Proof. This follows from Proposition 4.2.26.

Definition 4.2.28. Let $V$ be a finite free filtered $A\widehat{\otimes} L_{dR}$-representation of $\Gamma_k$. Define

$$V_{\text{fin}} := (\text{Fil}^0 V)^{\text{fin}}[t^{-1}].$$

Then $V_{\text{fin}}$ is a finite free $A_\infty((t))$-representation of $\Gamma_k$ equipped with $\Gamma_k$-stable decreasing filtration and $\phi_{dR,V_{\text{fin}}}$. Since $\phi_{dR,V_{\text{fin}}}$ preserves the filtration, it defines an $A_\infty$-linear endomorphism on $\text{gr}^0 V_{\text{fin}}$, which we denote by $\text{Res}_{\text{Fil}^0 V_{\text{fin}}} \phi_{dR,V_{\text{fin}}}$, it follows from definition that

$$\text{Res}_{\text{Fil}^0 V_{\text{fin}}} \phi_{dR,V_{\text{fin}}} = \phi_{\text{gr}^0 V}$$

as endomorphisms on the finite free $A_\infty$-module $\text{gr}^0(V_{\text{fin}}) = (\text{gr}^0 V)_{\text{fin}}$.

4.3 The arithmetic Sen endomorphism of a $p$-adic local system

From this section, we study the relative $p$-adic Hodge theory in geometric families. Let $k$ be a finite field extension of $\mathbb{Q}_p$ and let $X$ be an $n$-dimensional smooth rigid analytic variety over $\text{Spa}(k, \mathcal{O}_k)$. Let $K$ be the $p$-adic completion of $k_\infty := \bigcup_n k(\mu_{p^n})$ and let $X_K$ denote the base change of $X$ to $\text{Spa}(K, \mathcal{O}_K)$. We denote by $\Gamma_k$ the Galois group $\text{Gal}(k_\infty/k)$.

Based on the recent progresses on the relative $p$-adic Hodge theory [30, 31, 40, 41], Liu and Zhu attached to an étale $\mathbb{Q}_p$-local system $\mathbb{L}$ a nilpotent Higgs bundle $\mathcal{H}(\mathbb{L})$ on $X_K$ equipped with $\Gamma_k$-action ([30]). Our goal is to define an endomorphism $\phi_{\mathbb{L}}$ on $\mathcal{H}(\mathbb{L})$ by decompleting $\Gamma_k$-action. The endomorphism $\phi_{\mathbb{L}}$, which we will call the arithmetic Sen
endomorphism, is a natural generalization of the Sen endomorphism of a \( p \)-adic Galois representation of \( k \).

### 4.3.1 Brief review of Scholze’s approach to the relative \( p \)-adic Hodge theory

First let us briefly recall the sites and sheaves that we use. Let \( X_{\text{pro\'{e}t}} \) be the pro-étale site on \( X \) in the sense of \([41, 42]\). The pro-étale site is equipped with a natural projection to the étale site on \( X \)

\[ \nu: X_{\text{pro\'{e}t}} \to X_{\text{\acute{e}t}}. \]

Let \( \nu': X_{\text{pro\'{e}t}}/X_K \to (X_K)_{\text{\acute{e}t}} \) be the restriction of \( \nu \) and we identify \( X_{\text{pro\'{e}t}}/X_K \) with \( (X_K)_{\text{pro\'{e}t}} \) (see a discussion before Proposition 6.10 in \([41]\)).

We denote by \( \hat{\mathbb{Z}}_p \) (resp. \( \hat{\mathbb{Q}}_p \)) the constant sheaf on \( X_{\text{pro\'{e}t}} \) associated to \( \mathbb{Z}_p \) (resp. \( \mathbb{Q}_p \)).

For a \( \mathbb{Z}_p \)-local system \( L \) (resp. \( \mathbb{Q}_p \)-local system) on \( X_{\text{\acute{e}t}} \), let \( \hat{L} \) denote the \( \hat{\mathbb{Z}}_p \)-module (resp. \( \hat{\mathbb{Q}}_p \)-module) on \( X_{\text{pro\'{e}t}} \) associated to \( L \) (see \([41, \S 8.2]\)).

We define sheaves on \( X_{\text{pro\'{e}t}} \) as follows. We set

\[ O_X^+ := \nu^* O_{X_{\text{\acute{e}t}}}^+, \quad O_X := \nu^* O_{X_{\text{\acute{e}t}}}, \quad \hat{O}_X := \lim_{\substack{\to \cr n}} O_{X/p^n}[p^{-1}]. \]

We also set \( \Omega_X^1 = \nu^* \Omega_{X_{\text{\acute{e}t}}}^1 \) and we denote its \( i \)-th exterior power by \( \Omega_X^i \). Moreover, Scholze introduced the de Rham period sheaves \( B_{\text{dR}}^+, B_{\text{dR}}, \mathcal{O}B_{\text{dR}}^+ \) and \( \mathcal{O}B_{\text{dR}} \) on \( X_{\text{pro\'{e}t}} \) in \([41, \S 6]\) and \([42]\). The structural de Rham sheaf \( \mathcal{O}B_{\text{dR}} \) has the following properties: it is a sheaf of \( \mathcal{O}_X \)-algebras equipped with a decreasing filtration \( \text{Fil}^\bullet \mathcal{O}B_{\text{dR}} \) and an integrable connection

\[ \nabla: \mathcal{O}B_{\text{dR}} \to \mathcal{O}B_{\text{dR}} \otimes_{\mathcal{O}_X} \Omega_X^1. \]
satisfying the Griffiths transversality. Since $X$ is assumed to be smooth of dimension $n$, this gives rise to the following exact sequence of sheaves on $X_{\text{pro\acute{e}t}}$:

$$0 \to \mathcal{B}_{dR} \to \mathcal{O} \mathcal{B}_{dR} \xrightarrow{\nabla} \mathcal{O} \mathcal{B}_{dR} \otimes_{\mathcal{O}_X} \Omega^1_X \xrightarrow{\nabla} \cdots \mathcal{O} \mathcal{B}_{dR} \otimes_{\mathcal{O}_X} \Omega^n_X \to 0.$$

Let us recall the definition of the de Rham local systems.

**Definition 4.3.1** ([\textit{[36]} Definition 8.3]). A $\mathbb{Z}_p$-local system $L$ on $X_{\text{\acute{e}t}}$ is said to be \textit{de Rham} if the $\mathbb{B}^+_{dR}$-local system $\mathcal{M} := \hat{L} \otimes_{\hat{\mathbb{Z}_p}} \mathbb{B}^+_{dR}$ is associated with some filtered module with integrable connection $(\mathcal{E}, \nabla, \text{Fil}^*)$. This means that there exists a filtered module with integrable connection $(\mathcal{E}, \nabla, \text{Fil}^*)$ such that there is an isomorphism of sheaves on $X_{\text{pro\acute{e}t}}$

$$\mathcal{M} \otimes_{\mathcal{B}_{dR}^+} \mathcal{O} \mathcal{B}_{dR} \cong \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O} \mathcal{B}_{dR}$$

compatible with filtrations and connections.

**Remark 4.3.2.** Liu and Zhu proved that in the case where $X$ is geometrically connected, if there exists a classical point $x$ such that $L_x$ is a de Rham Galois representation of $k(x)$, then $L$ is a de Rham local system ([\textit{36]} Theorem 1.5 (iii)]).

Finally, we set $\mathcal{O} \mathcal{C} := \text{gr}^0 \mathcal{O} \mathcal{B}_{dR}$. Taking the associated graded connection of $\nabla$ on $\mathcal{O} \mathcal{B}_{dR}$ equips $\mathcal{O} \mathcal{C}$ with a Higgs field

$$\text{gr}^0 \nabla: \mathcal{O} \mathcal{C} \to \mathcal{O} \mathcal{C} \otimes_{\mathcal{O}_X} \Omega^1_X(-1),$$

where $(-1)$ stands for the $(-1)$st Tate twist.
4.3.2 Review of the $p$-adic Simpson correspondence à la Liu and Zhu

We review the formulation of the $p$-adic Simpson correspondence by Liu and Zhu. Let $L$ be a $\mathbb{Q}_p$-local system on $X_{\text{ét}}$ of rank $r$. We define

$$\mathcal{H}(L) = \nu'_*(\hat{L} \otimes_{\hat{\mathbb{Q}}_p} \mathcal{O}_C).$$

Then Liu and Zhu proved the following theorem.

**Theorem 4.3.3** (Rough form of [36, Theorem 2.1]). $\mathcal{H}(L)$ is a vector bundle on $X_K$ of rank $r$ equipped with a nilpotent Higgs field $\vartheta_L$ and a semilinear action of $\Gamma_K$. The functor $\mathcal{H}$ is a tensor functor from the category of $\mathbb{Q}_p$-local systems on $X_{\text{ét}}$ to the category of nilpotent Higgs bundles on $X_K$. Moreover, $\mathcal{H}$ is compatible with pullback and (under some conditions) smooth proper pushforward.

**Remark 4.3.4.** For our purpose, we use the $p$-adic Simpson correspondence formulated by Liu and Zhu as their output is a Higgs bundle over $X_K$ with a $\Gamma_k$-action. See [24] and [1] for the $p$-adic Simpson correspondence by Faltings and Abbes-Gros-Tsuji in a more general setting, and see [3, 4] for the one over a pro-étale cover of $X_K$ by Andreatta and Brinon.

To define the arithmetic Sen endomorphism on $\mathcal{H}(L)$ and discuss its properties, let us recall Liu and Zhu’s arguments in the proof of Theorem 4.3.3.

We follow the notation on base changes of adic spaces and rings in [36]. We denote by $\mathbb{T}^n$ the $n$-dimensional rigid analytic torus

$$\text{Spa}(k\langle T_1^\pm, \ldots, T_n^\pm \rangle, \mathcal{O}_k\langle T_1^\pm, \ldots, T_n^\pm \rangle).$$

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For $m \geq 0$, we set

$$T_m^n = \text{Spa}(k_m \langle T_1^{\pm 1/p^m}, \ldots, T_n^{\pm 1/p^m} \rangle, \mathcal{O}_{k_m} \langle T_1^{\pm 1/p^m}, \ldots, T_n^{\pm 1/p^m} \rangle).$$

The denoted by $\tilde{T}_m^n$ the affinoid perfectoid $\lim_{\leftarrow m} T_m^n$ in $X_{\text{pro}\acute{e}t}$.

To study properties of $H(L)$, we introduce the following base $\mathcal{B}$ for $(X_K)_{\acute{e}t}$: objects of $\mathcal{B}$ are the étale maps to $X_K$ that are the base changes of standard étale morphisms $Y \to X_{k'}$ defined over some finite extension $k'$ of $k$ in $K$ where $Y$ is affinoid admitting a toric chart after some finite extension of $k'$. Recall that an étale morphism between adic spaces is called standard étale if it is a composite of rational localizations and finite étale morphisms and that a toric chart means a standard étale morphism to $\mathbb{T}^n$. Morphisms of $\mathcal{B}$ are the base changes of étale morphisms over some finite extension of $k$ in $K$. We equip $\mathcal{B}$ with the induced topology from $(X_K)_{\acute{e}t}$. Then the associated topoi $(X_K)_{\acute{e}t}$ and $\mathcal{B}^\sim$ are equivalent ([36, Lemma 2.5]).

When $Y = \text{Spa}(B, B^+)$ admits a toric chart over $k$, we use the following notation: we set

$$Y_m = \text{Spa}(B_m, B_m^+) := Y \times_{\mathbb{T}^n_m} T_m^n.$$

Then $\tilde{Y}_\infty := Y \times_{\mathbb{T}^n} \tilde{T}_\infty^n$ is the affinoid perfectoid in $Y_{\text{pro}\acute{e}t}$ represented by the relative toric tower $(Y_n)$. We denote by $(\hat{B}_\infty, \hat{B}_\infty^+)$ the perfectoid affinoid completed direct limit of the $(B_m, B_m^+)$’s and set $\hat{Y}_\infty := \text{Spa}(\hat{B}_\infty, \hat{B}_\infty^+)$, the affinoid perfectoid space associated to $Y_\infty$. We also set $B_{k_m} = B \otimes_k k_m$ as in Subsection 4.2.1. When $Y$ admits a toric chart over a finite extension of $k$ in $K$, we similarly define these objects using the rigid analytic torus over the field.

Let $Y_{K,m} := \text{Spa}(B_{K,m}, B_{K,m}^+)$ be the base change of $Y_m$ from $k_m$ to $K$ and let $\hat{Y}_{K,\infty}$ be the affinoid perfectoid represented by the toric tower $(Y_{K,m})$. We denote the associated
affinoid perfectoid space by $\hat{Y}_{K,\infty} = \text{Spa}(\hat{B}_{K,m}, \hat{B}_{K,m}^+)$. The cover $\hat{Y}_{K,\infty}/Y$ is Galois. We denote its Galois group by $\Gamma$. Then $\Gamma$ fits into a splitting exact sequence

$$1 \rightarrow \Gamma_{\text{geom}} \rightarrow \Gamma \rightarrow \Gamma_k \rightarrow 1.$$ 

To prove Theorem 4.3.3, Liu and Zhu gave a simple description of

$$\mathcal{H}(\mathbb{L})(Y_K) = H^0(X_{\text{pro\'{e}t}}/Y_K, \hat{\mathbb{L}} \otimes \mathcal{O}_X)$$

for $(Y = \text{Spa}(B, B^+) \rightarrow X_{k'}) \in \mathcal{B}$, which we recall now.

**Proposition 4.3.5** ([36, Proposition 2.8]). Put $\mathcal{M} = \hat{\mathbb{L}} \hat{\otimes}_{\hat{\mathbb{Q}}_p} \hat{\mathcal{O}}_X$. Then there exists a unique finite projective $B_K$-submodule $M_K(Y)$ of $\mathcal{M}(\hat{Y}_{K,\infty})$, which is stable under $\Gamma$, such that

(i) $M_K(Y) \otimes_{B_K} \hat{B}_{K,\infty} = \mathcal{M}(\hat{Y}_{K,\infty})$;

(ii) The $B_K$-linear representation of $\Gamma_{\text{geom}}$ on $M_K(Y)$ is unipotent;

In addition, the module $M_K(Y)$ has the following properties:

(P1) There exist some positive integer $j_0$ and some finite projective $B_{k_{j_0}}$-submodule $M(Y)$ of $M_K(Y)$ stable under $\Gamma$ such that $M(Y) \otimes_{B_{k_{j_0}}} B_K = M_K(Y)$. Moreover, the construction of $M(Y)$ is compatible with base change along standard étale morphisms.

(P2) The natural map

$$M_K(Y)^\Gamma_{\text{geom}} \rightarrow \mathcal{M}(\hat{Y}_{K,\infty})^{\Gamma_{\text{geom}}}$$

is an isomorphism.

Once this proposition is proved, we can describe $\mathcal{H}(\mathbb{L})(Y_K)$ in terms of $M_K(Y)$ as follows: the vanishing theorem on affinoid perfectoid spaces ([36, Proposition 7.13]) implies
the degeneration of the Cartan-Leray spectral sequence to the Galois cover $\{\tilde{Y}_K, \infty \to Y_K\}$ with Galois group $\Gamma_{\text{geom}}$, and thus we have

$$H^i(\Gamma_{\text{geom}}, \mathcal{M}(\tilde{Y}_K, \infty)) \xrightarrow{\cong} H^i(X_{\text{pro\acute{e}t}}/Y_K, \mathcal{M}),$$

$$H^i(\Gamma_{\text{geom}}, (\mathcal{M} \otimes \mathcal{O}_C)(\tilde{Y}_K, \infty)) \xrightarrow{\cong} H^i(X_{\text{pro\acute{e}t}}/Y_K, \mathcal{M} \otimes \mathcal{O}_C).$$

Moreover, we know that $\mathcal{O}_C|_{\tilde{Y}_K, \infty} \cong (\hat{\mathcal{O}_X}|_{\tilde{Y}_K, \infty})[V_1, \ldots, V_n]$, where $V_i = t^{-1} \log([T_i^p]/T_i)$ for a fixed compatible sequence of $p$-power roots of the coordinate $T_i = (T_i, T_i^{1/p}, \ldots)$. It follows from these results and a simple argument on the direct limit of sheaves on $X_{\text{pro\acute{e}t}}$ that the natural $\Gamma_k$-equivariant map

$$(M_K(Y)[V_1, \ldots, V_n])_{\Gamma_{\text{geom}}} \to \mathcal{H}(\mathbb{L})(Y_K)$$

is an isomorphism. A simple computation shows that the map $M_K(Y)[V_1, \ldots, V_n] \to M_K(Y)$ sending $V_i$ to 0 induces a $\Gamma_k$-equivariant isomorphism

$$(M_K(Y)[V_1, \ldots, V_n])_{\Gamma_{\text{geom}}} \xrightarrow{\cong} M_K(Y).$$

Thus we have a $\Gamma_k$-equivariant isomorphism

$$\mathcal{H}(\mathbb{L})(Y_K) \cong M_K(Y).$$
The above discussion is summarized in the following commutative diagram:

\[
\begin{align*}
M_K(Y) \otimes_{B_K} \hat{B}_{K,\infty} & \to (M_K(Y) \otimes_{B_K} \hat{B}_{K,\infty})[V_1, \ldots, V_n] \\
M_K(Y) & \to M_K(Y)[V_1, \ldots, V_n] \xrightarrow{V_i=0} M_K(Y) \\
M_K(Y)^\Gamma_{\text{geom}} & \to (M_K(Y)[V_1, \ldots, V_n])^\Gamma_{\text{geom}} \cong M_K(Y).
\end{align*}
\]

Finally, we recall the Higgs field \( \vartheta_L \). This is defined to be

\[
\vartheta_L := \nu_*(\text{gr} \nabla : \hat{\mathbb{L}} \otimes \mathcal{O}\mathcal{C} \to \hat{\mathbb{L}} \otimes \mathcal{O}\mathcal{C} \otimes \Omega^1_X(-1))
\]

under the identification \( \nu_*(\hat{\mathbb{L}} \otimes \mathcal{O}\mathcal{C} \otimes \Omega^1_X(-1)) \cong \mathcal{H}(\mathbb{L}) \otimes \Omega^1_{X/k}(-1) \). Here \( \mathcal{H}(\mathbb{L}) \otimes \Omega^1_{X/k}(-1) \) denotes the \( \mathcal{O}_{X_K} \)-module \( \mathcal{H}(\mathbb{L}) \otimes_{\mathcal{O}_X} \Omega^1_{X/k}(-1) = \mathcal{H}(\mathbb{L}) \otimes_{\mathcal{O}_{X_K}} \Omega^1_{X_K/k}(-1) \) equipped with a natural \( \Gamma_k \)-action.

We have another description under the isomorphism \( \mathcal{H}(\mathbb{L})(Y_K) \cong M_K(Y) \), which proves \( \vartheta_L \) is nilpotent. Namely, let \( \rho_{\text{geom}} \) denote the action of \( \Gamma_{\text{geom}} \) on \( M_K(Y) \) and let \( \chi_i : \Gamma_{\text{geom}} \cong \mathbb{Z}_p(1)^n \to \mathbb{Z}_p(1) \) denote the composite of the natural identification and projection to the \( i \)-th component. We can take the logarithm of \( \rho_{\text{geom}} \) on \( M(Y) \subset M_K(Y) \) since the action is unipotent. Suppose the logarithm is written as

\[
\log \rho_{\text{geom}} = \sum_{i=1}^n \vartheta_i \otimes \chi_i \otimes t^{-1},
\]

where \( \vartheta_i \in \text{End}(M(Y)) \). Then \( \vartheta_i \) can be regarded as an endomorphism on \( M_K(Y) \) by
extension of scalars and we define
\[
\vartheta_{M_K(Y)} := \sum_{i=1}^{n} \vartheta_i \otimes d \log T_i \otimes t^{-1} = \sum_{i=1}^{n} \vartheta_i \otimes \frac{dT_i}{T_i} \otimes t^{-1} \in \text{End}(M_K(Y)) \otimes_B \Omega^1_{B/k'}(-1). \tag{4.3.1}
\]

We can check \(\vartheta_{M_K(Y)} \wedge \vartheta_{M_K(Y)} = 0\) and this defines a Higgs field on \(M_K(Y)\). It turns out that \(\vartheta_L(Y_K) = \vartheta_{M_K(Y)}\) under the \(\Gamma_k\)-equivariant isomorphism \(H(L)(Y_K) \cong M_K(Y)\). See [36 §2] for the detail.

### 4.3.3 Definition and properties of the arithmetic Sen endomorphism

We will define the arithmetic Sen endomorphism \(\phi_L \in \text{End} \mathcal{H}(\mathbb{L})\). Let \(\mathcal{B}_L\) be the refinement of the base \(\mathcal{B}\) for \((X_K)_{\text{et}}\) whose objects consist of \((Y = \text{Spa}(B, B^+) \to X_{k'}) \in \mathcal{B}\) such that \(H(L)(Y_K)\) is a finite free \(B_K\)-module.

For \((Y = \text{Spa}(B, B^+) \to X_{k'}) \in \mathcal{B}_L\), \(H(L)(Y_K)\) is a \(B_K\)-representation of \(\Gamma_{k'} := \text{Gal}(K/k')\) in the sense of Definition 4.2.2. Thus Proposition 4.2.20 and Definition 4.2.23 equip \(H(L)(Y_K)\) with the \(B_K\)-linear endomorphism

\[
\phi_{H(L)(Y_K)} : H(L)(Y_K) \to H(L)(Y_K).
\]

**Lemma-Definition 4.3.6.** The assignment of endomorphisms

\[
\mathcal{B}_L \ni (Y = \text{Spa}(B, B^+) \to X_{k'}) \mapsto \phi_{H(L)(Y_K)} \in \text{End}_{B_K} H(L)(Y_K)
\]

defines an endomorphism \(\phi_L\) of the vector bundle \(H_L\) on \((X_K)_{\text{et}}\). We call \(\phi_L\) the arithmetic Sen endomorphism of \(\mathbb{L}\).
Proof. We need to check the compatibility of \( \phi_{L,Y_K} \) via the pullback \( Y'' \to Y_K \) for \( Y = \text{Spa}(B, B^+) \to X_{k'} \), \( Y'' = \text{Spa}(B'', B''^+) \to X_{k''} \in B_L \). For this it suffices to prove that

\[
\mathcal{H}(L)(Y_K)_{\text{fin}} \otimes_{B_\infty} B''_\infty \cong \mathcal{H}(L)(Y''_K)_{\text{fin}}
\]

as \( B''_\infty \)-representation of \( \text{Gal}(k_\infty/k''_\infty) \), where \( B_\infty \) and \( B''_\infty \) are defined as in Subsection 4.2.1.

Since \( \mathcal{H}(L) \) is a vector bundle on \( X_K \), we have the natural isomorphisms

\[
\left( \mathcal{H}(L)(Y_K)_{\text{fin}} \otimes_{B_\infty} B''_\infty \right) \otimes_{B''_\infty} B''_K \cong \left( \mathcal{H}(L)(Y_K)_{\text{fin}} \otimes_{B_\infty} B_K \right) \otimes_{B_K} B''_K
\]

\[
\cong \mathcal{H}(L)(Y_K) \otimes_{B_K} B''_K \cong \mathcal{H}(L)(Y''_K).
\]

On the other hand, we see from definition \( \mathcal{H}(L)(Y_K)_{\text{fin}} \otimes_{B_\infty} B''_\infty \subset \mathcal{H}(L)(Y''_K)_{\text{fin}} \). Hence the lemma follows from the faithful flatness of \( B''_\infty \to B''_K \).

Proposition 4.3.7. The following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{H}(L) & \xrightarrow{\varphi_L} & \mathcal{H}(L) \otimes \Omega^1_{X/k}(-1) \\
\phi_L \downarrow & & \phi_L \otimes \text{id} - \text{id} \otimes \text{id} \\
\mathcal{H}(L) & \xrightarrow{\varphi_L} & \mathcal{H}(L) \otimes \Omega^1_{X/k}(-1).
\end{array}
\]

In particular, the endomorphisms \( \phi_L \otimes \text{id} - i(\text{id} \otimes \text{id}) \) on \( \mathcal{H}(L) \otimes \Omega^i_{X/k}(-1) \) give rise to an endomorphism on the complex of \( \mathcal{O}_{X_K} \)-modules on \( X_K \)

\[
\mathcal{H}(L) \xrightarrow{\varphi_L} \mathcal{H}(L) \otimes \Omega^1_{X/k}(-1) \xrightarrow{\varphi_L} \mathcal{H}(L) \otimes \Omega^2_{X/k}(-2) \longrightarrow \cdots
\]

induced by the Higgs field.

Proof. It is enough to check the commutativity of the diagram evaluated at \( Y_K \) for each
(Y = Spa(B, B⁺) → Xₖ) ∈ Bₚ. In this setting, we can use the identification

\[(H(L)(Y_K), varphi_L(Y_K), phi_L(Y_K)) \cong (M_K(Y), varphi_{M_K}(Y), phi_{M_K}(Y)).\]

So it suffices to show the commutativity of the diagram

\[
\begin{array}{ccc}
M_K(Y) \quad & \phi_{M_K}(Y) & M_K(Y) \\
\downarrow \phi_{M_K}(Y) & & \downarrow \phi_{M_K}(Y) \\
M_K(Y) \quad & \phi_{M_K}(Y) & M_K(Y)
\end{array}
\]

\[
\otimes_B \Omega^1_{B/k'}(-1) \quad \otimes_B \Omega^1_{B/k'}(-1).
\]

Moreover, since \(M(Y) \otimes_{B_{k'}} B_K = M_K(Y)\), we only need to check the commutativity on \(M(Y) \subset M_K(Y)\).

We use the notation in [4.3.1]. Then we have

\[
\vartheta_{M_K}(Y) \circ \phi_{M_K}(Y) = \sum_{i=1}^n (\vartheta_i \circ \phi_{M_K}(Y)) \otimes \frac{dT_i}{T_i} \otimes t^{-1}, \quad \text{and}
\]

\[
(\phi_{M_K}(Y) \otimes \text{id} - \text{id} \otimes \text{id}) \circ \vartheta_{M_K}(Y) = \sum_{i=1}^n (\phi_{M_K}(Y) \circ \vartheta_i - \vartheta_i) \otimes \frac{dT_i}{T_i} \otimes t^{-1}.
\]

Thus we need to show that \([\phi_{M_K}(Y), \vartheta_i] = \vartheta_i\) for each \(i\). To see this, take a topological generator \(\gamma_i\) of the \(i\)-th component of \(\Gamma_{\text{geom}} \cong \mathbb{Z}_p(1)^n\). Let \(\rho_{\text{geom}}\) denote the action of \(\Gamma_{\text{geom}}\) on \(M_K(Y)\) and write \(\log \rho_{\text{geom}} = \sum_{i=1}^n \vartheta_i \otimes \chi_i \otimes t^{-1}\) as before. Since \(\gamma \gamma_i \gamma^{-1} = \gamma_i^{\chi(\gamma)}\) for \(\gamma \in \Gamma_k\), we have

\[
\gamma (\log \rho_{\text{geom}}(\gamma_i)) \gamma^{-1} = \log \rho_{\text{geom}}(\gamma \gamma_i \gamma^{-1}) = \chi(\gamma) \log \rho_{\text{geom}}(\gamma_i).
\]

Hence \(\gamma \vartheta_i = \chi(\gamma) \vartheta_i \gamma\) for \(\gamma \in \Gamma_k\).
For $m \in M(Y)$, we compute

$$
\phi_{MK(Y)} \vartheta_i m = \lim_{j \to \infty} \frac{1}{\log \chi(\gamma)} \frac{\gamma^p \vartheta_i m - \vartheta_i m}{p^j}
= \lim_{j \to \infty} \frac{1}{\log \chi(\gamma)} (\chi(\gamma)^{p^j} - 1)\vartheta_i \gamma^{p^j} m + \vartheta_i (\gamma^{p^j} m - m)
= \vartheta_i m + \vartheta_i \phi_{MK(Y)} m.
$$

Hence $[\phi_{MK(Y)}, \vartheta_i] = \vartheta_i$. \qed

Remark 4.3.8. Brinon generalized Sen’s theory to the case of $p$-adic fields with imperfect residue fields in \cite{12}. Analogues of $\phi_L$ and $\vartheta_i$ have already appeared in his work.

We discuss properties of the arithmetic Sen endomorphism along the lines of Theorem 4.3.3 (i.e., \cite[Theorem 2.1]{36}).

**Theorem 4.3.9.**

(i) There are canonical isomorphisms

$$(H(\mathbb{L}_1 \otimes \mathbb{L}_2), \vartheta_{\mathbb{L}_1 \otimes \mathbb{L}_2}, \phi_{\mathbb{L}_1 \otimes \mathbb{L}_2}) \cong (H(\mathbb{L}_1) \otimes H(\mathbb{L}_2), \vartheta_{\mathbb{L}_1} \otimes \text{id} + \text{id} \otimes \vartheta_{\mathbb{L}_2}, \phi_{\mathbb{L}_1} \otimes \text{id} + \text{id} \otimes \phi_{\mathbb{L}_2})$$

and

$$(H(\mathbb{L}_1), \vartheta_{\mathbb{L}_1}, \phi_{\mathbb{L}_1}) \cong (H(\mathbb{L}_1)^\vee, (\vartheta_{\mathbb{L}_1})^\vee, (\phi_{\mathbb{L}_1})^\vee).$$

(ii) Let $f : Y \to X$ be a morphism between smooth rigid analytic varieties over $k$ and $\mathbb{L}$ be a $\mathbb{Q}_p$-local system on $X_{\text{\acute{e}t}}$. Then there is a canonical isomorphism

$$f^*(H(\mathbb{L}), \vartheta_{\mathbb{L}}, \phi_{\mathbb{L}}) \cong (f^*H(\mathbb{L}), \vartheta_{f^*\mathbb{L}}, \phi_{f^*\mathbb{L}}).$$
Proof. Part (i) follows from [36, Theorem 2.1(iv)] and Lemma 4.2.22. Part (ii) follows from
[36, Theorem 2.1(iii)] and Proposition 4.2.20 (functoriality of $\phi_W$).

By construction, we also have the following for the case of points.

**Proposition 4.3.10.** If $X$ is a point, then $\phi_L$ coincides with the Sen endomorphism at-
tached to the Galois representation $\mathbb{L}$.

For pushforwards, we have the following theorem (the notation is explained after the
statement).

**Theorem 4.3.11.** Let $f: X \to Y$ be a smooth proper morphism between smooth rigid
analytic varieties over $k$ and let $\mathbb{L}$ be a $\mathbb{Z}_p$-local system on $X_{\text{et}}$. Assume that $R^i_{f_*}\mathbb{L}$ is a
$\mathbb{Z}_p$-local system on $Y_{\text{et}}$ for every $i$. Then we have

$$(H(R^i_{f_*}\mathbb{L}), \vartheta(R^i_{f_*}\mathbb{L})) \cong R^i f_{\text{Higgs}*}(H(\mathbb{L}) \otimes \Omega^i_{X/Y}(\cdot), \overline{\vartheta}_L).$$

Moreover, under this isomorphism, we have

$$\phi_{R^i_{f_*}\mathbb{L}} = R^i f_{K,\text{et}*}(\phi_L \otimes \text{id} - \bullet (\text{id} \otimes \text{id})).$$

Let us explain the notation in the theorem. Recall the complex of $\mathcal{O}_{X_K}$-modules

$$\mathcal{H}(\mathbb{L}) \xrightarrow{\partial_1} \mathcal{H}(\mathbb{L}) \otimes \Omega^1_{X/k}(-1) \xrightarrow{\partial_2} \mathcal{H}(\mathbb{L}) \otimes \Omega^2_{X/k}(-2) \longrightarrow \cdots.$$  

This has an $\mathcal{O}_{X}$-linear endomorphism $\phi_L \otimes \text{id} - \bullet (\text{id} \otimes \text{id})$ by Proposition 4.3.7. The complex
yields a complex of $\mathcal{O}_{X_K}$-modules

$$\mathcal{H}(\mathbb{L}) \xrightarrow{\partial_1} \mathcal{H}(\mathbb{L}) \otimes \Omega^1_{X/Y}(-1) \xrightarrow{\partial_2} \mathcal{H}(\mathbb{L}) \otimes \Omega^2_{X/Y}(-2) \longrightarrow \cdots.$$
by composing with the projection $\Omega^i_{X/k} \to \Omega^i_{X/Y}$. The new complex has an induced $\mathcal{O}_X$-linear endomorphism, which we still denote by $\phi \otimes \text{id} - \bullet (\text{id} \otimes \text{id})$.

We denote by $f_K: X_K \to Y_K$ the base change of $f$. Then $R^i f_{\text{Higgs},*}$ is the $i$-th derived pushforward of the complex with the Higgs field. In particular, $R^i f_{\text{Higgs},*}(\mathcal{H}(\mathbb{L}) \otimes \Omega^\bullet_{X/Y}(-\bullet), \bar{\theta}_L)$ is the $\mathcal{O}_{X,K,\text{ét}}$-module $R^i f_{K,\text{ét},*}(\mathcal{H}(\mathbb{L}) \otimes \Omega^\bullet_{X/Y}(-\bullet))$ together with a Higgs field.

Proof. The first part is [36, Theorem 2.1(v)] (see Theorem 4.3.3). So we will prove the statement on arithmetic Sen endomorphisms.

Since the statement is local on $Y$, we may assume that $Y$ is an affinoid $\text{Spa}(A, A^+)$ and that $\mathcal{H}(R^i f_* \mathbb{L})$ is a globally free vector bundle on $Y_K$. So $\mathcal{H}(R^i f_* \mathbb{L})$ is associated to a finite free $A_K$-module (say $V$). Then $V$ is an $A_K$-representation of $\Gamma_k$ and the endomorphism $\phi_{R^i f_* \mathbb{L}}$ is associated to $\phi_V$.

Since $X$ is quasi-compact, there exists a finite affinoid open cover $X = \bigcup_{i \in I} U^{(i)}$ with $U^{(i)} = \text{Spa}(B^{(i)}, B^{(i),+})$ such that $\mathcal{H}(\mathbb{L})|_{U^{(i)}_{K}}$ is a globally finite free vector bundle for each $i$. So $\mathcal{H}(\mathbb{L})|_{U^{(i)}_{K}}$ gives rise to a $B^{(i)}_K$-representation of $\Gamma_k$ and the latter is defined over $B^{(i)}_K$ for a sufficiently large $m$ (cf. the proof of Theorem 4.2.9). Since the same holds for $\mathcal{H}(\mathbb{L})|_{U^{(i)}_{K} \cap U^{(j)}_{K}}$, there exists a large integer $m$ such that the complex of $\mathcal{O}_{X,K}$-modules

$$
\mathcal{H}(\mathbb{L}) \xrightarrow{\bar{\theta}_L} \mathcal{H}(\mathbb{L}) \otimes \Omega^1_{X/Y}(-1) \xrightarrow{\bar{\theta}_L} \mathcal{H}(\mathbb{L}) \otimes \Omega^2_{X/Y}(-2) \longrightarrow \cdots
$$

with the $\Gamma_k$-action and the endomorphism $\phi \otimes \text{id} - \bullet (\text{id} \otimes \text{id})$ descends to a complex of $\mathcal{O}_{X_{km}}$-modules

$$
\mathcal{H}(\mathbb{L})_{km} \xrightarrow{\bar{\theta}_L} \mathcal{H}(\mathbb{L})_{km} \otimes \Omega^1_{X/Y}(-1) \xrightarrow{\bar{\theta}_L} \mathcal{H}(\mathbb{L})_{km} \otimes \Omega^2_{X/Y}(-2) \longrightarrow \cdots
$$

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on $X_{km}$ equipped with a $\Gamma$-action and an endomorphism $\phi_L \otimes \text{id} - \bullet (\text{id} \otimes \text{id})$ such that $\mathcal{H}(L)_{km}|_{U^{(i)}_{km}}$ is a globally finite free vector bundle for each $i$. We denote by $\mathcal{F}^\bullet$ the complex on $X_{km}$ and by $\phi^\bullet$ the descended endomorphism.

Let $f_{km} : X_{km} \to Y_{km}$ denote the base change of $f$. Set

$$\mathcal{H}_{Y,km} := R^if_{km,\text{ét}}\mathcal{F}^\bullet.$$ 

Since $f_{km}$ is proper, this is a coherent $\mathcal{O}_{Y_{km}}$-module by Kiehl’s finiteness theorem\footnote{For a coherent $\mathcal{O}_{km}$-module $\mathcal{F}$, we have $(R^if_{km,\text{ét}}\mathcal{F})_{\text{ét}} = R^if_{km,\text{ét}}\mathcal{F}_{\text{ét}}$ (H Proposition 9.2). So we simply write $\mathcal{F}$ for the sheaf $\mathcal{F}_{\text{ét}}$ on $X_{km,\text{ét}}$.}. We have

$$\mathcal{H}_{Y,km}|_{Y_K} = (R^if_{km,\text{ét}}\mathcal{F}^\bullet)|_{Y_K} = R^if_{K,\text{ét}}(\mathcal{F}^\bullet|_{X_K}) = R^if_{K,\text{ét}}(\mathcal{H}(L) \otimes \Omega^*_{X/Y}(-\bullet)),$$

and this is isomorphic to $\mathcal{H}(R^if_*L)$ by the first assertion. Thus (after increasing $m$) $\mathcal{H}_{Y,km}$ is globally finite free and associated to a finite free $A_{km}$-module (say $V_{km}$) with a $\Gamma_K$-action satisfying $V_{km} \otimes_{A_{km}} A_K = V$. By construction, $V_{km}$ is contained in $V_{\text{fin}}$ and the $A_K$-linear endomorphism $\phi_V$ on $V$ is uniquely characterized by the following property: for each $v \in V_{km}$, there exists an open subgroup $\Gamma'_k \subset \Gamma_k$ such that

$$\exp(\log \chi(\gamma)\phi_V)v = \gamma v$$

for all $\gamma \in \Gamma'_k$.

We will show that $R^if_{K,\text{ét},*}(\phi_L \otimes \text{id} - \bullet (\text{id} \otimes \text{id}))$ defines an $A_K$-linear endomorphism on $V$ with the same property. To see this, we compute $V_{km}$ via the Čech-to-derived functor.
spectral sequence. Note that

\[ V_{km} = \Gamma(Y_{km,\text{ét}}, \mathcal{H}_{Y_{km}}) = R^i \Gamma(X_{km,\text{ét}}, \mathcal{F}^*) \]

by definition.

Let us briefly recall the Čech-to-derived functor spectral sequence. Set \( U := \{ U_{j}^{(i)} \}_{i \in I} \).

For \( i_0, \ldots, i_a \in I \), we denote by \( U_{km}^{(i_0 \cdots i_a)} \) the affinoid open \( U_{km}^{(i_0)} \cap \cdots \cap U_{km}^{(i_a)} \). Consider the Čech double complex \( \check{C}^*(U, \mathcal{F}^*) \) associated to the complex \( \mathcal{F}^* \); this is defined by

\[ \check{C}^a(U, \mathcal{F}^b) := \prod_{i_0, \ldots, i_a \in I} \mathcal{F}^b(U_{km}^{(i_0 \cdots i_a)}). \]

Let \( H^b \) be the \( b \)-th right derived functor of the forgetful functor from the category of abelian sheaves on \( X_{km,\text{ét}} \) to the category of abelian presheaves on \( X_{km,\text{ét}} \); for an abelian sheaf \( \mathcal{G} \), \( H^b(\mathcal{G}) \) associates to \( (U \to X_{km}) \) the abelian group \( H^b(U, \mathcal{G}) \). Then the Čech-to-derived functor spectral sequence is a spectral sequence with

\[ E_2^{a,b} = H^a(\text{Tot}(\check{C}^*(U, H^b(\mathcal{F}^*)))) \]

converging to \( R^{a+b} \Gamma(X_{km,\text{ét}}, \mathcal{F}^*) \). Moreover, this is functorial in \( \mathcal{F}^* \).

In our case, \( \mathcal{F}^* \) consists of coherent \( \mathcal{O}_{km} \)-modules and \( U_{km}^{(i_0 \cdots i_a)} \) are all affinoid. So \( H^b(\mathcal{F}^c)(U_{km}^{(i_0 \cdots i_a)}) = 0 \) for each \( b > 0 \) and any \( a \) and \( c \) by Kiehl’s theorem. Thus the spectral sequence yields an isomorphism

\[ H^i(\text{Tot}(\check{C}^*(U, \mathcal{F}^*))) \xrightarrow{\cong} R^i \Gamma(X_{km,\text{ét}}, \mathcal{F}^*) = V_{km}. \]

Moreover, this isomorphism is \( \Gamma_k \)-equivariant as the construction is functorial in \( \mathcal{F}^* \).
Let us unwind the definition of $\check{C}^a(\mathcal{U}, \mathcal{F}^b)$:

$$\check{C}^a(\mathcal{U}, \mathcal{F}^b) = \prod_{i_0, \ldots, i_a \in I} \Gamma(\mathcal{U}_{k_m}^{(i_0 \ldots i_a)}, \mathcal{H}(\mathbb{L})_{k_m} \otimes \Omega^{b}_{X/Y}(-b)).$$

Set

$$W^{(i_0 \ldots i_a), b} := \Gamma(U^{(i_0 \ldots i_a)}_K, \mathcal{H}(\mathbb{L}) \otimes \Omega^{b}_{X/Y}(-b))$$

and

$$W_{k_m}^{(i_0 \ldots i_a), b} := \Gamma(U_{k_m}^{(i_0 \ldots i_a)}, \mathcal{H}(\mathbb{L})_{k_m} \otimes \Omega^{b}_{X/Y}(-b)).$$

They have a natural $\Gamma_k$-action, and $W^{(i_0 \ldots i_a), b}_{k_m}$ is contained in $(W^{(i_0 \ldots i_a), b})_{\text{fin}}$. In particular, the restriction of $\phi_{W^{(i_0 \ldots i_a), b}}$ to $W^{(i_0 \ldots i_a), b}_{k_m}$ satisfies the following property: for each $w \in W^{(i_0 \ldots i_a), b}_{k_m}$, there exists an open subgroup $\Gamma'_k \subset \Gamma_k$ such that

$$\exp(\log(\chi(\gamma))\phi_{W^{(i_0 \ldots i_a), b}})w = \gamma w$$

for all $\gamma \in \Gamma'_k$.

It follows from our construction that under the isomorphism $H^i(\text{Tot}(\check{C}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))) \cong R^i\Gamma(X_{k_m, \text{ét}}, \mathcal{F}^\bullet) = V_{k_m}$, the endomorphism $R^i f_{K, \text{ét}, *}(\phi_{\mathbb{L}} \otimes \text{id} - \bullet(\text{id} \otimes \text{id}))|_{V_{k_m}} = R^i f_{k_m, \text{ét}, *}\phi^\bullet$ corresponds to

$$H^i(\text{Tot}(\prod_{i_0, \ldots, i_a \in I} \phi_{W^{(i_0 \ldots i_a), b}})).$$

Since differentials in the complex $\text{Tot}(\check{C}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))$ are all $\Gamma_k$-equivariant, we see that

$$H^i(\text{Tot}(\prod_{i_0, \ldots, i_a \in I} \phi_{W^{(i_0 \ldots i_a), b}}))$$

satisfies the above-mentioned characterizing property of $\phi_V$. This completes the proof. "$\square"
4.4 Constancy of generalized Hodge-Tate weights

In this section, we prove the multiset of eigenvalues of \( \phi_L \) is constant on \( X_K \) (Theorem 4.4.8). For this we give a description of \( \phi_L \) as the residue of a formal connection in Subsection 4.4.1. Then the constancy is proved by the theory of formal connections developed in Subsection 4.4.2.

4.4.1 The decompletion of the geometric Riemann-Hilbert correspondence

We review the geometric Riemann-Hilbert correspondence by Liu and Zhu and discuss its decompletion.

Keep the notation in Section 4.3. Let \( L \) be a \( \mathbb{Q}_p \)-local system on \( X_{\text{ét}} \) of rank \( r \). Following \cite{36}, we define

\[
\mathcal{RH}(L) = \nu'_*(\hat{L} \otimes_{\hat{\mathbb{Q}}_p} \mathcal{O}_{\text{dR}}).
\]

In order to state their theorem, let us recall a ringed space \( \mathcal{X} \) introduced in \cite{36}, §3.1. Let \( L^+_{\text{dR}} \) denote the de Rham period ring \( \mathbb{H}^+_{\text{dR}}(K, \mathcal{O}_K) \) as before (\cite{36} uses \( B^+_{\text{dR}} \) but we prefer to use \( L^+_{\text{dR}} \)). Define a sheaf \( \mathcal{O}_X \hat{\otimes}(L^+_{\text{dR}}/t^i) \) on \( X_{K, \text{ét}} \) by assigning

\[
(Y = \text{Spa}(B, B+) \to X_{k'}) \in \mathcal{B} \mapsto B \hat{\otimes}_{k'}(L^+_{\text{dR}}/t^i).
\]

This defines a sheaf by the Tate acyclicity theorem. We also set

\[
\mathcal{O}_X \hat{\otimes} L^+_{\text{dR}} = \lim_{\to i} \mathcal{O}_X \hat{\otimes}(L^+_{\text{dR}}/t^i)
\]

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and

\[ \mathcal{O}_X \otimes L_{\text{dR}} = (\mathcal{O}_X \otimes L_{\text{dR}}^+)[t^{-1}] . \]

We denote the ringed space \((X_K, \mathcal{O}_X \otimes L_{\text{dR}})\) by \(\mathcal{X}\). We have a natural base change functor \(\mathcal{E} \mapsto \mathcal{E} \otimes L_{\text{dR}}\) from the category of vector bundles on \(X\) to the category of vector bundles on \(\mathcal{X}\). We set

\[ \Omega^1_{\mathcal{X}/L_{\text{dR}}} = \Omega^1_{X/k} \otimes L_{\text{dR}}. \]

**Theorem 4.4.1** (see [36, Theorem 3.8]).

(i) \(\mathcal{R}\mathcal{H}(\mathbb{L})\) is a filtered vector bundle on \(\mathcal{X}\) of rank \(r\) equipped with an integrable connection

\[ \mathcal{R}\mathcal{H}(\mathbb{L}) \xrightarrow{\nabla_{\mathbb{L}}} \mathcal{R}\mathcal{H}(\mathbb{L}) \otimes_{\mathcal{O}_{\mathcal{X}}} \Omega_{\mathcal{X}/L_{\text{dR}}} \]

that satisfies the Griffiths transversality. Moreover, \(\text{Gal}(K/k)\) acts on \(\mathcal{R}\mathcal{H}(\mathbb{L})\) semilinearly, and the action preserves the filtration and commutes with \(\nabla_{\mathbb{L}}\).

(ii) There is a canonical isomorphism

\[ (\text{gr}^0 \mathcal{R}\mathcal{H}(\mathbb{L}), \text{gr}^0(\nabla_{\mathbb{L}})) \cong (\mathcal{H}(\mathbb{L}), \partial_{\mathbb{L}}). \]

We want to consider a decompletion of \(\mathcal{R}\mathcal{H}(\mathbb{L})\). Here we only develop an ad hoc local theory that is sufficient for our purpose.

Take \((Y = \text{Spa}(B, B^+) \to X_{k'}) \in \mathcal{B}_L\) and consider \(\text{Fil}^0 \mathcal{R}\mathcal{H}(\mathbb{L})(Y_K)\). This is a \(B \otimes_{k'} L^+_{\text{dR}}\)-representation of \(\text{Gal}(K/k')\), and \(\mathcal{R}\mathcal{H}(\mathbb{L})(Y_K) = (\text{Fil}^0 \mathcal{R}\mathcal{H}(\mathbb{L})(Y_K))[t^{-1}]\). Moreover, since \(\text{gr}^0 \mathcal{R}\mathcal{H}(\mathbb{L})(Y_K)\) is a finite free \(B_K\)-module by the definition of \(\mathcal{B}_L\), the \(B \otimes_{k'} L^+_{\text{dR}}\)-module \(\text{Fil}^0 \mathcal{R}\mathcal{H}(\mathbb{L})(Y_K)\) is also finite free.

Definition 4.2.28 yields the \(B_\infty((t))\)-module \(\mathcal{R}\mathcal{H}(\mathbb{L})(Y_K)_{\text{fin}}\) and the \(B_\infty\)-linear endomor-
Phism

$$\phi_{dR, RH(L)(Y_K)_\text{fin}} : \mathcal{RH}(L)(Y_K)_\text{fin} \to \mathcal{RH}(L)(Y_K)_\text{fin}.$$ 

For simplicity, we denote $$\phi_{dR, RH(L)(Y_K)_\text{fin}}$$ by $$\phi_{dR, L, Y_K}$$. It satisfies

$$\phi_{dR, L, Y_K}(\alpha m) = t\partial_t(\alpha)m + \alpha \phi_{dR, L, Y_K}(m)$$

for every $$\alpha \in B_\infty((t))$$ and $$m \in \mathcal{RH}(L)(Y_K)_\text{fin}$$. Note that $$\nabla_L$$ is Gal($$K/k$$)-equivariant. Hence under the identification

$$\mathcal{RH}(L) \otimes_{\mathcal{O}_X} \Omega^1_{X/L, dR} \cong \mathcal{RH}(L) \otimes_{\mathcal{O}_X} \Omega^1_{X/k},$$

we have

$$\nabla_{L, Y_K}(\mathcal{RH}(L)(Y_K)_\text{fin}) \subset (\mathcal{RH}(L)(Y_K) \otimes \Omega^1_{B/k'})_\text{fin} = \mathcal{RH}(L)(Y_K)_\text{fin} \otimes \Omega^1_{B/k'}.$$ 

**Proposition 4.4.2.** The following diagram commutes:

$$\begin{array}{ccc}
\mathcal{RH}(L)(Y_K)_\text{fin} & \xrightarrow{\nabla_{L, Y_K}} & \mathcal{RH}(L)(Y_K)_\text{fin} \\
\phi_{dR, L, Y_K} & & \phi_{dR, L, Y_K} \\
\mathcal{RH}(L)(Y_K)_\text{fin} & \xrightarrow{\nabla_{L, Y_K}} & \mathcal{RH}(L)(Y_K)_\text{fin} \\
\phi_{dR, L, Y_K} & & \phi_{dR, L, Y_K} \otimes \text{id}
\end{array}$$

Moreover, we have

$$\text{Res}_{\text{Fil}^0} \mathcal{RH}(L)(Y_K)_\text{fin} \phi_{dR, L, Y_K} = \phi_{L, Y_K}.$$ 

**Proof.** The commutativity of the diagram follows from the fact that $$\nabla_L$$ is Gal($$K/k$$)-equivariant. The second assertion is a consequence of Theorem $4.4.1$ (ii) (cf. Definition $4.2.28$). 

$$\boxempty$$

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Remark 4.4.3. In [H], Andreatta and Brinon developed a Fontaine-type decompletion theory in the relative setting. Roughly speaking, they associated to a local system on $X$ a formal connection over the pro-étale cover $\tilde{X}_{K,\infty}$ over $X_K$ when $X$ is an affine scheme admitting a toric chart.

4.4.2 Theory of formal connections

To study $\phi_{dR,L,Y_K}$ in the previous subsection, we develop a theory of formal connections. We work on the following general setting: let $R$ be an integral domain of characteristic 0 (e.g. $R = B_\infty$ in the previous subsection) and fix an algebraic closure of the fraction field of $R$. Consider the ring of Laurent series $R((t))$ and define the $R$-linear derivation $d_0: R((t)) \to R((t))$ by

$$d_0\left(\sum_{j \in \mathbb{Z}} a_j t^j\right) = \sum_{j \in \mathbb{Z}} j a_j t^{j-1}.$$ 

Let $M$ be a finite free $R((t))$-module of rank $r$ and let $D_0: M \to M$ be an $R$-linear map which satisfies the Leibniz rule

$$D_0(\alpha m) = \alpha D_0(m) + d_0(\alpha)m \quad (\alpha \in R((t)), m \in M).$$

Definition 4.4.4. A $tD_0$-stable lattice of $M$ is a finite free $R[[t]]$-submodule $\Lambda$ of $M$ that satisfies

$$\Lambda \otimes_{R[[t]]} R((t)) = M \quad \text{and} \quad tD_0(\Lambda) \subset \Lambda.$$ 

For a $tD_0$-stable lattice $\Lambda$ of $M$, we have $tD_0(t\Lambda) \subset t\Lambda$ by the Leibniz rule. Thus $tD_0: \Lambda \to \Lambda$ induces an $R$-linear endomorphism on $\Lambda/t\Lambda$. We denote this endomorphism by $\text{Res}_{\Lambda}D_0$. Since $\Lambda/t\Lambda$ is a finite free $R$-module of rank $r$, the endomorphism $\text{Res}_{\Lambda}D_0$ has $r$ eigenvalues (counted with multiplicity) in the algebraic closure of the fraction field of $R$. 

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The following is known for $tD_0$-stable lattices.

**Theorem 4.4.5.** Assume that $R$ is an algebraically closed field.

(i) There exists a finite subset $A$ of $R$ such that the submodule

$$\Lambda_A := \bigoplus_{\alpha \in A} \text{Ker}(tD_0 - \alpha)^r \otimes_R R[[t]]$$

is a $tD_0$-stable lattice of $M$. In particular, the eigenvalues of $\text{Res}_A D_0$ lie in $A$.

(ii) For any $tD_0$-stable lattices $\Lambda$ and $\Lambda'$ of $M$, the eigenvalues of $\text{Res}_\Lambda D_0$ and those of $\text{Res}_{\Lambda'} D_0$ differ by integers. Namely, for each eigenvalue $\alpha$ of $\text{Res}_\Lambda D_0$, there exists an eigenvalue $\alpha'$ of $\text{Res}_{\Lambda'} D_0$ such that $\alpha - \alpha' \in \mathbb{Z}$.

See [21, III.8 and V. Lemma 2.4] and [2, Proposition 3.2.2] for details.

We now turn to the following multivariable situation: Let $R$ be an integral domain of characteristic 0 as before. Suppose that $R$ is equipped with pairwise commuting derivations $d_1, \ldots, d_n$; this means that for each $i = 1, \ldots, n$, the map $d_i: R \to R$ is additive and satisfies the Leibniz rule

$$d_i(ab) = d_i(a)b + ad_i(b) \quad (a, b \in R),$$

and $d_i \circ d_j = d_j \circ d_i$ for each $i$ and $j$. Since $R$ is an integral domain of characteristic 0, the derivations $d_1, \ldots, d_n$ extend uniquely over the algebraic closure of the fraction field of $R$.

Consider the ring of Laurent series $R((t))$ and define the $R$-linear derivation $d_0: R((t)) \to R((t))$ by

$$d_0 \left( \sum_{j \in \mathbb{Z}} a_j t^j \right) = \sum_{j \in \mathbb{Z}} ja_j t^{j-1}.$$
For each $i = 1, \ldots, n$, we extend $d_i : R \to R$ to an additive map $d_i : R((t)) \to R((t))$ by

$$d_i \left( \sum_{j \in \mathbb{Z}} a_j t^j \right) = \sum_{j \in \mathbb{Z}} d_i(a_j) t^j.$$

Then endomorphisms $d_0, d_1, \ldots, d_n$ on $R((t))$ commute with each other. Moreover, $d_1, \ldots, d_n$ commute with $td_0$.

Let $M$ be a finite free $R((t))$-module of rank $r$ together with pairwise commuting additive endomorphisms $D_0, D_1, \ldots, D_n : M \to M$ satisfying the Leibniz rules

$$D_i(\alpha m) = \alpha D_i(m) + d_i(\alpha) m \quad (\alpha \in R((t)), m \in M, 0 \leq i \leq n).$$

Note that $D_0$ is $R$-linear and $D_1, \ldots, D_n$ commute with $tD_0$.

The following proposition is the key to the constancy of generalized Hodge-Tate weights.

**Proposition 4.4.6.** With the notation as above, let $\Lambda$ be a $tD_0$-stable lattice of $M$. Then each eigenvalue $\alpha$ of $\text{Res}_\Lambda D_0$ in the algebraic closure of the fraction field of $R$ satisfies

$$d_1(\alpha) = \cdots = d_n(\alpha) = 0.$$

**Proof.** By extending scalars from $R$ to the algebraic closure of its fraction field, we may assume that $R$ is an algebraically closed field. By Theorem 4.4.5 (i), there exists a finite subset $\mathcal{A}$ of $R$ such that the submodule

$$\Lambda_{\mathcal{A}} := \bigoplus_{\alpha \in \mathcal{A}} \ker(tD_0 - \alpha)^r \otimes_R R[[t]]$$

is a $tD_0$-stable lattice of $M$.

By Theorem 4.4.5 (ii), the eigenvalues of $\text{Res}_\Lambda D_0$ and those of $\text{Res}_{\Lambda_{\mathcal{A}}} D_0$ differ by inte-
gers. Since every integer \( a \) satisfies \( d_1(a) = \cdots = d_n(a) = 0 \), it suffices to treat the case where \( \Lambda = \Lambda_A \).

**Lemma 4.4.7.** The finite free \( R[[t]] \)-submodule \( \Lambda_A \) is stable under \( D_1, \ldots, D_n \).

Note that Lemma 4.4.7 says that the connection \( (\Lambda_A, D_0, \ldots, D_n) \) is regular singular along \( t = 0 \). In this case, Proposition 4.4.6 is easy to prove. In fact, this is an algebraic analogue of the following fact: let \( X \) be the complex affine space \( \mathbb{A}^n_C \) and \( D \) the divisor \( \{0\} \times \mathbb{A}^{n-1}_C \). Consider a vector bundle \( \Lambda \) on \( X \) and an integrable connection \( \Gamma \) on \( \Lambda|_{X \setminus D} \) that admits a logarithmic pole along \( D \). Let \( T \) be the monodromy transformation of \( \Lambda|_{X \setminus D} \) defined by the positive generator of \( \pi_1(X \setminus D) = \mathbb{Z} \). Then \( T \) extended to an automorphism \( \tilde{T} \) of \( \Lambda \) and satisfies

\[
\tilde{T}|_D = \exp(-2\pi i \text{Res}_D \Lambda).
\]

See [16, Proposition 3.11].

**Proof of Lemma 4.4.7.** This is [2, Lemma 3.3.2]. For the convenience of the reader, we reproduce the proof here. Fix \( 1 \leq i \leq n \) and \( \alpha \in \mathcal{A} \). It is enough to show that for each \( 0 \leq j \leq r \),

\[
D_i \ker(tD_0 - \alpha)^j \subset \ker(tD_0 - \alpha)^{j+1}.
\]

We prove this inclusion by induction on \( j \). The assertion is trivial when \( j = 0 \). Assume \( j > 0 \) and take \( m \in \ker(tD_0 - \alpha)^j \). Then \( (tD_0 - \alpha)m \in \ker(tD_0 - \alpha)^{j-1} \), and thus \( D_i(tD_0 - \alpha)m \in \ker(tD_0 - \alpha)^j \) by the induction hypothesis. We need to show \( (tD_0 - \alpha)^{j+1}D_i m = 0 \). Since \( D_i \) commutes with \( tD_0 \) and satisfies \( D_i(\alpha m) = \alpha D_i(m) + d_i(\alpha)m \), we have

\[
(tD_0 - \alpha)D_i m = D_i(tD_0 - \alpha)m + d_i(\alpha)m.
\]

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Therefore

\[(tD_0 - \alpha)^{j+1}D_im = (tD_0 - \alpha)^j(tD_0 - \alpha)D_im\]

\[= (tD_0 - \alpha)^jD_i(tD_0 - \alpha)m + (tD_0 - \alpha)^jd_i(\alpha)m\]

\[= (tD_0 - \alpha)^jD_i(tD_0 - \alpha)m + d_i(\alpha)(tD_0 - \alpha)^jm.\]

For the third equality, note that \(d_i(\alpha) \in R \) and \(D_0 \) is \(R\)-linear. Since \(D_i(tD_0 - \alpha)m \in \text{Ker}(tD_0 - \alpha)^j\) and \(m \in \text{Ker}(tD_0 - \alpha)^j\), the last sum is zero. \(\square\)

We continue the proof of Proposition 4.4.6. Fix an \(R[[t]]\)-basis of \(\Lambda_A\) and identify \(\Lambda_A\) with \(R[[t]]^r\). Note that \(R[[t]]^r\) has natural differentials \(d_0, d_1, \ldots, d_n : R[[t]]^r \to R[[t]]^r\). Consider the map

\[t(D_0 - d_0) : R[[t]]^r \to R[[t]]^r.\]

This is \(R[[t]]\)-linear. We denote the corresponding \(r \times r\) matrix by \(C_0 \in M_r(R[[t]])\).

Fix \(1 \leq i \leq n\). By Lemma 4.4.7 the map \(D_i\) gives an endomorphism on \(R[[t]]^r\) that satisfies the Leibniz rule and thus \(D_i - d_i\) is an \(R[[t]]\)-linear endomorphism on \(R[[t]]^r\). We denote the corresponding \(r \times r\) matrix by \(C_i \in M_r(R[[t]])\).

We have \([tD_0, D_i] = 0\) and \([td_0, d_i] = 0\) in \(\text{End}(R[[t]]^r)\). Plugging \(tD_0 = td_0 + C_0\) and \(D_i = d_i + C_i\) into \([tD_0, D_i] = 0\) yields

\[[C_0, C_i] = d_iC_0 - td_0C_i,\]  \hspace{1cm} (4.4.1)

where \(d_0\) and \(d_i\) are differentials acting on the matrices entrywise.

Consider the surjection \(R[[t]] \to R\) evaluating \(t\) by 0. We denote the image of \(C_0\) (resp. \(C_i\)) in \(M_r(R)\) by \(\overline{C}_0\) (resp. \(\overline{C}_i\)). By construction \(\overline{C}_0\) is the matrix corresponding
to Res_{\Lambda^d} D_0. Thus it suffices to show that each eigenvalue of $C_0$ is killed by $d_i$. This is standard. Namely, from (4.4.1), we have

$$[C_0, C_i] = d_i C_0.$$ 

This implies that

$$d_i(C_0^2) = C_0 d_i(C_0) + d_i(C_0) C_0 = C_0 [C_0, C_i] + [C_0, C_i] C_0 = [C_0^2, C_i].$$

Similarly, for each $j \in \mathbb{N}$,

$$d_i(C_0^j) = [C_0, C_i].$$

In particular, we get

$$d_i(\text{tr}(C_0^j)) = 0.$$ 

This implies that each eigenvalue of $C_0$ is killed by $d_i$. 

4.4.3 Constancy of generalized Hodge-Tate weights

Here is the key theorem of this chapter.

**Theorem 4.4.8.** Let $k$ be a finite extension of $\mathbb{Q}_p$. Let $X$ be a smooth rigid analytic variety over $k$ and $\mathbb{L}$ a $\mathbb{Q}_p$-local system on $X_{\acute{e}t}$. Consider the arithmetic Sen endomorphism $\phi_{\mathbb{L}} \in \text{End}(\mathcal{H}(\mathbb{L})).$ Then eigenvalues of $\phi_{\mathbb{L}, x} \in \text{End}(\mathcal{H}(\mathbb{L})_x)$ for $x \in X_K$ are algebraic over $k$ and constant on each connected component of $X_K$.

We call these eigenvalues *generalized Hodge-Tate weights* of $\mathbb{L}$.

**Proof.** Since $\phi_{\mathbb{L}}$ is an endomorphism on the vector bundle $\mathcal{H}(\mathbb{L})$ on $X_K$, it suffices to prove
the statement étale locally on $X$. Thus we may assume that $X$ is an affinoid $\text{Spa}(B, B^+)$ which admits a toric chart $X_{k'} \to \mathbb{T}^n_{k'}$ over some finite extension $k'$ of $k$ in $K$.

Take $(Y = \text{Spa}(B, B^+) \to X_{k'}) \in \mathcal{B}_L$. We may assume that $B_\infty$ is connected, hence an integral domain. Note that $Y$ admits a toric chart

$$Y_{k''} \to \mathbb{T}^n_{k''} = \text{Spa}(k''(T_1^\pm, \ldots, T_n^\pm), \mathcal{O}_{k''}(T_1^\pm, \ldots, T_n^\pm))$$

after base change to a finite extension $k''$ of $k'$ in $K$. Then the derivations $\frac{\partial}{\partial T_1}, \ldots, \frac{\partial}{\partial T_n}$ on $k''(T_1^\pm, \ldots, T_n^\pm)$ extends over $B_\infty$. We also denote the extensions by $\frac{\partial}{\partial T_1}, \ldots, \frac{\partial}{\partial T_n}$.

We set

$$R = B_{k_\infty}, \quad d_0 = \partial_t, \quad \text{and} \quad d_i = \frac{\partial}{\partial T_i} \quad (1 \leq i \leq n).$$

Consider $R((t))$-module

$$M = \mathcal{R}\mathcal{H}(\mathbb{L})(Y_K)_{\text{fin}}$$

equipped with endomorphisms

$$D_0 = t^{-1} \phi_{dR, L, Y_K}, \quad \text{and} \quad D_i = (\nabla_{L, Y_K}) \frac{\partial}{\partial T_i} \quad (1 \leq i \leq n).$$

By Proposition 4.4.2, they satisfy the assumptions in the previous subsection.

Consider the $R[[t]]$-submodule of $M$

$$\Lambda = (\text{Fil}^0 \mathcal{R}\mathcal{H}(\mathbb{L})(Y_K))_{\text{fin}}.$$ 

Then $\Lambda$ is $tD_0$-stable, and $\text{Res}_\Lambda D_0$ is $\phi_{dR, L, Y_K}$. Thus by Proposition 4.4.6, each eigenvalue $\alpha$
of $\text{Res}_A D_0$ in an algebraic closure $L$ of $\text{Frac} R$ satisfies

$$d_1(\alpha) = \cdots = d_n(\alpha) = 0.$$ 

On the other hand, we can check that

$$L^{d_1=\cdots=d_0}=0 = \left(\frac{\text{Frac} k'\langle T_1^{\pm}, \ldots, T_n^{\pm} \rangle}{\partial \over \partial T_1} = \cdots = \partial \over \partial T_n} = 0\right) = \bar{k}.$$ 

Therefore the eigenvalues of $\phi_{L,Y_K}$ are algebraic over $k$ and constant on $Y_K$. 

**Corollary 4.4.9.** Let $k$ be a finite extension of $\mathbb{Q}_p$. Let $X$ be a geometrically connected smooth rigid analytic variety over $k$ and $L$ a $\mathbb{Q}_p$-local system on $X$. Then the multiset of generalized Hodge-Tate weights of the $p$-adic representations $L_{\tau}$ of $\text{Gal}(\overline{k(x)}/k(x))$ does not depend on the choice of a classical point $x$ of $X$.

In particular, if $L_{\tau}$ is presque Hodge-Tate for one classical point $x$ of $X$ (i.e., generalized Hodge-Tate weights are all integers), $L_{\tau}$ is presque Hodge-Tate for every classical point $y$ of $X$.

**Proof.** This follows from Theorem 4.4.8. 

### 4.5 Applications and related topics

We study properties of Hodge-Tate sheaves using the arithmetic Sen endomorphism. We keep the notation in Section 4.3.

Consider the Hodge-Tate period sheaf on $X_{\text{pro\acute{e}t}}$

$$\mathcal{O}_{\mathcal{B}_{\text{HT}}} := \text{gr}^\bullet \mathcal{O}_{\mathcal{B}_{\text{dR}}} = \bigoplus_{j \in \mathbb{Z}} \mathcal{O} C(j).$$

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For a $\mathbb{Q}_p$-local system $\mathbb{L}$ on $X_{\text{ét}}$, we define a sheaf $D_{HT}(\mathbb{L})$ on $X_{\text{ét}}$ by

$$D_{HT}(\mathbb{L}) := \nu_*(\hat{\mathbb{L}} \otimes_{\hat{\mathbb{Q}_p}} \mathcal{O}_{\mathbb{B}_{HT}}).$$

**Proposition 4.5.1.** The sheaf $D_{HT}(\mathbb{L})$ is a coherent $\mathcal{O}_{X_{\text{ét}}}$-module. Moreover, for every affinoid $Y \subset X_{\text{ét}},$

$$\Gamma(Y, D_{HT}(\mathbb{L})) = \bigoplus_{j \in \mathbb{Z}} H^0(\Gamma_k, \mathcal{H}(\mathbb{L})(j)).$$

**Proof.** This follows from the proof of [36, Theorem 3.9 (i)].

**Remark 4.5.2.** In [31, Theorem 8.6.2 (a)], Kedlaya and Liu proved this statement for pseudocoherent modules over a pro-coherent analytic field.

We are going to study the relation between $D_{HT}(\mathbb{L})$ and $\phi_{\mathbb{L}} \in \text{End} \mathcal{H}(\mathbb{L})$. For each $j \in \mathbb{Z}$, we set

$$\mathcal{H}(\mathbb{L})^{\phi_{\mathbb{L}} = j} := \text{Ker}(\phi_{\mathbb{L}} - j \text{id}: \mathcal{H}(\mathbb{L}) \to \mathcal{H}(\mathbb{L})).$$

This is a coherent $\mathcal{O}_{X_{K,\text{ét}}}$-module. We denote by $D_{HT}(\mathbb{L})|_{X_K}$ the coherent $\mathcal{O}_{X_{K,\text{ét}}}$-module associated to the pullback of $D_{HT}(\mathbb{L})$ on $X$ to $X_K$ as coherent sheaves.

**Proposition 4.5.3.** Let $\mathbb{L}$ be a $\mathbb{Q}_p$-local system of rank $r$ on $X_{\text{ét}}$. Assume that $\mathbb{L}$ satisfies one of the following conditions:

(i) $\mathcal{H}(\mathbb{L})^{\phi_{\mathbb{L}} = j}$ is a vector bundle on $X_{K,\text{ét}}$ for each $j \in \mathbb{Z}$.

(ii) $D_{HT}(\mathbb{L})$ is a vector bundle of rank $r$ on $X_{\text{ét}}$.

Then we have

$$D_{HT}(\mathbb{L})|_{X_{K,\text{ét}}} \cong \bigoplus_{j \in \mathbb{Z}} \mathcal{H}(\mathbb{L}(j))^{\phi_{\mathbb{L}(j)} = 0}.$$
Moreover, this is isomorphic to $\bigoplus_{j \in \mathbb{Z}} \mathcal{H}(\mathbb{L})^{\phi_{\mathbb{L}}=j}$. In particular, $D_{HT}(\mathbb{L})$ is a vector bundle on $X_{\acute{e}t}$ and $\bigoplus_{j \in \mathbb{Z}} \mathcal{H}(\mathbb{L})^{\phi_{\mathbb{L}}=j}$ is a vector bundle on $X_{K,\acute{e}t}$.

Proof. The statement is local. So it suffices to prove that for each affinoid $Y = \text{Spa}(B, B^+)$ $\in X_{\acute{e}t}$ such that $\mathcal{H}(\mathbb{L})|_{Y_K}$ is associated to a finite free $B_K$-module (say $V$), we have

$$\Gamma(Y, D_{HT}(\mathbb{L})) \otimes_B B_K \cong \bigoplus_{j \in \mathbb{Z}} V(j)^{\phi_{\mathbb{L}}(j)=0}.$$ 

Note $\Gamma(Y, D_{HT}(\mathbb{L})) = \bigoplus_{j \in \mathbb{Z}} (V(j))^\Gamma_k$. Moreover, it follows from the Tate-Sen method ([36, Lemma 3.10]) that

$$(V^{\phi_{\mathbb{L}}=0})^{\Gamma_k} \cong (V(j))^{\Gamma_k}.$$ 

Lemma 4.5.4.

(i) $V^{\phi_{\mathbb{L}}=0} \otimes_{B_{\infty}} B_K \cong V^{\phi_{\mathbb{L}}=0}$.

(ii) The natural map

$$(V^{\Gamma_k}_{\text{fin}}) \otimes_B B_{\infty} \rightarrow V^{\phi_{\mathbb{L}}=0}_{\text{fin}}$$

is injective.

Proof. Part (i) follows from the flatness of $B_{\infty} \rightarrow B_K$ and $V_{\text{fin}} \otimes_{B_{\infty}} B_K \cong V$.

We prove part (ii). By the definition of $\phi_{\mathbb{L}}$, the natural map

$$(V^{\Gamma_k}_{\text{fin}}) \otimes_B B_{\infty} \rightarrow V_{\text{fin}}$$

factors through $V^{\phi_{\mathbb{L}}=0}_{\text{fin}}$. So we show that the above map is injective.

We denote the total fraction ring of $B$ (resp. $B_{\infty}$) by $\text{Frac} B$ (resp. $\text{Frac} B_{\infty}$). We first
claim that the natural map
\[ V_{\text{fin}}^{\Gamma_k} \to V_{\text{fin}}^{\Gamma_k} \otimes_B \text{Frac } B \]
is injective. To see this, note that \( V_{\text{fin}} \) is a finite free \( B_\infty \)-module. Hence the composite
\[ V_{\text{fin}} \to V_{\text{fin}} \otimes_B \text{Frac } B = V_{\text{fin}} \otimes_{B_\infty} (B_\infty \otimes_B \text{Frac } B) \to V_{\text{fin}} \otimes_{B_\infty} \text{Frac } B_\infty \]
is injective, and thus so is the first map. Since the composite
\[ V_{\text{fin}}^{\Gamma_k} \to V_{\text{fin}}^{\Gamma_k} \otimes_B \text{Frac } B \to V_{\text{fin}} \otimes_B \text{Frac } B \]
coincides with the composite of injective maps \( V_{\text{fin}}^{\Gamma_k} \to V_{\text{fin}} \) and \( V_{\text{fin}} \to V_{\text{fin}} \otimes_B \text{Frac } B \), the map \( V_{\text{fin}}^{\Gamma_k} \to V_{\text{fin}}^{\Gamma_k} \otimes_B \text{Frac } B \) is also injective.

By the above claim, it suffices to show the injectivity of the natural map
\[ (V_{\text{fin}}^{\Gamma_k} \otimes_B \text{Frac } B) \otimes_{\text{Frac } B} \text{Frac } B_\infty \to V_{\text{fin}} \otimes_{B_\infty} \text{Frac } B_\infty. \]

Now that \( \text{Frac } B \) and \( \text{Frac } B_\infty \) are products of fields, this follows from standard arguments; we may assume that \( \text{Frac } B \) is a field. Replacing \( k \) by an algebraic closure in \( \text{Frac } B \), we may further assume that \( \text{Frac } B_\infty \) is also a field. Note that \( \text{Frac } B_\infty = (\text{Frac } B) \otimes_k k_\infty \) and thus \( (\text{Frac } B_\infty)^{\Gamma_k} = \text{Frac } B \).

Assume the contrary. Take \( v_1, \ldots, v_a \in V_{\text{fin}}^{\Gamma_k} \otimes_B \text{Frac } B \) and \( b_1, \ldots, b_a \in \text{Frac } B_\infty \) such that \( v_1, \ldots, v_a \) are linearly independent over \( \text{Frac } B \) and \( b_1 v_1 + \cdots + b_a v_a = 0 \). We may assume that they are of minimum length. Then \( b_1 \neq 0 \) and thus replacing \( b_i \) by \( b_i^{-1} b_i \) we may further assume \( b_1 = 1 \). It is obvious when \( a = 1 \), so we assume \( a > 1 \). Take any \( \gamma \in \Gamma_k \). As \( v_1, \ldots, v_a \in V_{\text{fin}}^{\Gamma_k} \otimes_B \text{Frac } B \), we have \( v_1 + \gamma(b_2)v_2 + \cdots + \gamma(b_a)v_a = 0 \) and thus
\((\gamma(b_2) - b_2)v_2 + \cdots + (\gamma(b_a) - b_a)v_a = 0\). By the minimality, we have \(\gamma(b_i) = b_i\) for each \(2 \leq i \leq a\) and \(\gamma \in \Gamma_k\). Therefore we have \(b_i \in \operatorname{Frac} B\) for all \(i\), which contradicts the linear independence of \(v_1, \ldots, v_a\) over \(\operatorname{Frac} B\).

We continue the proof of Proposition 4.5.3. By Lemma 4.5.4 and discussions above, it is enough to show \(V \kappa \otimes B \cong V^{\phi_V = 0}\) assuming either condition (i) or (ii). In fact, the Tate twist of this isomorphism implies \((V(j))^{\kappa} \otimes B \cong (V(j))^{\phi_V (j) = 0}\), and a choice of a generator of \(O_{X_K, c}(j)\) yields \(\mathcal{H}(\mathbb{L}(j))^{\phi_L (j) = 0} \cong \mathcal{H}(\mathbb{L})^{\phi_L = -j}\).

We show that condition (ii) implies condition (i). For each \(j \in \mathbb{Z}\), let \(V_{\text{fin}}^{(j)}\) denote the generalized eigenspace of \(\phi_V\) on \(V_{\text{fin}}\) with eigenvalue \(j\). By the constancy of \(\phi_V\), \(V_{\text{fin}}^{(j)}\) is a direct summand of \(V_{\text{fin}}\) and thus a finite projective \(B_\infty\)-module. By Lemma 4.5.4(ii), we have injective \(B_\infty\)-linear maps

\[
(V(j)_{\text{fin}})^{\kappa} \otimes B \cong (V(j)_{\text{fin}})^{\phi_V (j) = 0} \cong V_{\text{fin}}^{\phi_V = -j} \hookrightarrow V_{\text{fin}}^{(-j)}
\]

for each \(j \in \mathbb{Z}\). From this we obtain

\[
\text{rank } D_{HT}(\mathbb{L}) = \sum_{j \in \mathbb{Z}} \text{rank}_B (V_{\text{fin}}(j))^{\kappa} \leq \sum_{j \in \mathbb{Z}} \text{rank}_B V_{\text{fin}}^{(-j)} \leq \text{rank } \mathcal{H}(\mathbb{L}) = r.
\]

Hence it follows from condition (ii) that \((V_{\text{fin}}(j))^{\kappa} \otimes B \cong V_{\text{fin}}^{(-j)}\) are finite projective \(B_\infty\)-modules of the same rank. This implies \(V_{\text{fin}}^{\phi_V = -j} = V_{\text{fin}}^{(-j)}\). So \(V_{\text{fin}}^{\phi_V = -j}\) is a finite projective \(B_\infty\)-module for every \(j \in \mathbb{Z}\) and thus \(\mathcal{H}(\mathbb{L})\) satisfies condition (i).

From now on, we assume that \(\mathcal{H}(\mathbb{L})\) satisfies condition (i). By condition (i) and Lemma 4.5.4(i), \(V_{\text{fin}}^{\phi_V = 0}\) is finite projective over \(B_\infty\). So shrinking \(Y\) if necessary, we may assume that \(V_{\text{fin}}^{\phi_V = 0}\) is finite free over \(B_\infty\). Note that we only concern the \(B_\infty\)-representation \(V_{\text{fin}}\) of \(\Gamma_k\) and we have \((V_{\text{fin}}^{\phi_V = 0})^{\kappa} = V_{\text{fin}}^{\kappa}\). Thus replacing \(V_{\text{fin}}\) by the subrepresentation
\( V_\text{fin}^{\phi_V = 0} \), we may further assume \( \phi_V = 0 \) on \( V_\text{fin} \). Under this assumption, it remains to prove
\[
V_\text{fin}^{\Gamma_k} \otimes_B B_\infty \cong V_\text{fin}.
\]

Fix a \( B_\infty \)-basis \( v_1, \ldots, v_r \) of \( V_\text{fin} \). Then there exists a large positive integer \( m \) such that for each \( \gamma \in \Gamma_k \) the matrix of \( \gamma \) with respect to \((v_i)\) has entries in \( \text{GL}_r(B_{k_m}) \). Since \( \phi_V = 0 \), by increasing \( m \) if necessary, we may further assume that \( \gamma v_i = v_i \) for each \( 1 \leq i \leq r \) and \( \gamma \in \Gamma'_k := \text{Gal}(k_\infty/k_m) \subset \Gamma_k \). Set \( V_{k_m} := \bigoplus_{1 \leq i \leq r} B_{k_m} v_i \). This is a \( B_{k_m} \)-representation of \( \Gamma_k/\Gamma'_k = \text{Gal}(k_m/k) \) and satisfies \( V_\text{fin} = V_{k_m} \otimes_{B_{k_m}} B_\infty \).

It follows from [9, Proposition 2.2.1] that \((V_{k_m})^{\Gamma_k/\Gamma'_k}\) is a finite projective \( B \)-module and that \((V_{k_m})^{\Gamma_k/\Gamma'_k} \otimes_B B_{k_m} \cong V_{k_m} \). As \( V_\text{fin} = (V_{k_m})^{\Gamma_k/\Gamma'_k} \), this yields
\[
V_\text{fin}^{\Gamma_k} \otimes_B B_\infty \cong V_\text{fin}.
\]

\[\blacksquare\]

**Theorem 4.5.5.** Let \( \mathbb{L} \) be a \( \mathbb{Q}_p \)-local system of rank \( r \) on \( X_{\text{ét}} \). Then the following conditions are equivalent:

(i) \( D_{\text{HT}}(\mathbb{L}) \) is a vector bundle of rank \( r \) on \( X_{\text{ét}} \).

(ii) \( \nu^* D_{\text{HT}}(\mathbb{L}) \otimes_{\mathcal{O}_X} \mathcal{O}_{\text{BHT}} \cong \tilde{\mathbb{L}} \otimes_{\mathbb{Q}_p} \mathcal{O}_{\text{BHT}} \).

(iii) \( \phi_{\mathbb{L}} \) is a semisimple endomorphism on \( \mathcal{H}(\mathbb{L}) \) with integer eigenvalues.

(iv) There exist integers \( j_1 < \ldots < j_a \) such that if we set \( F(s) := \prod_{1 \leq i \leq a} (s - j_i) \in \mathbb{Z}[s] \), then
\[
F(\phi_{\mathbb{L}}) = 0
\]
as endomorphism of \( \mathcal{H}(\mathbb{L}) \).
Definition 4.5.6. A $\mathbb{Q}_p$-local system on $X_{\text{ét}}$ is a *Hodge-Tate sheaf* if it satisfies the equivalent conditions in Theorem 4.5.5.

Remark 4.5.7. Tsuji obtained Theorem 4.5.5 in the case of semistable schemes ([52, Theorem 9.1]). He also gave a characterization of Hodge-Tate local systems in terms of restrictions to divisors. See loc. cit. for the detail.

Proof of Theorem 4.5.5. The equivalence of (iii) and (iv) is clear, and (iii) implies (i) by Proposition 4.5.3. Conversely, assume condition (i). Thus $D_{HT}(L)|_{X_K}$ is a vector bundle of rank $r$ on $X_{K,\text{ét}}$. By Proposition 4.5.3, it is also isomorphic to $\bigoplus_{j \in \mathbb{Z}} H(L)^{\phi_L=j}$. Thus $\bigoplus_{j \in \mathbb{Z}} H(L)^{\phi_L=j} = H(L)$, and there exist integers $j_1 < \ldots < j_a$ such that $\bigoplus_{1 \leq i \leq a} H(L)^{\phi_L=j_i} = H(L)$. So $F(s) := \prod_{1 \leq i \leq a} (s - j_i)$ satisfies $F(\phi_L) = 0$, which is condition (iv).

Next we show that condition (iv) implies (ii). Obviously, there is a natural morphism

$$\nu^* D_{HT}(L) \otimes_{\mathcal{O}_X} \mathcal{O}_{B_{HT}} \to \hat{L} \otimes_{\mathbb{Q}_p} \mathcal{O}_{B_{HT}} \quad (4.5.1)$$

on $X_{\text{pro"et}}$ and we will prove that this is an isomorphism. It is enough to check this on $X_{\text{pro"et}}/X_K \cong X_{K,\text{pro"et}}$. Recall a canonical isomorphism in [36, Theorem 2.1(ii)]:

$$\nu^* H(L) \otimes_{\mathcal{O}_X} \mathcal{O}_{C}|_{X_{K,\text{pro"et}}} \cong (\hat{L} \otimes_{\mathbb{Q}_p} \mathcal{O}_{C})|_{X_{K,\text{pro"et}}}.$$
Then the restriction of the morphism (4.5.1) to $X_{K,\text{pro\acute e\textbf{t}}}$ is obtained as

\[ (\nu^* D_{\text{HT}}(\mathbb{L}) \otimes_{O_X} O_{\text{BHT}})|_{X_{K,\text{pro\acute e\textbf{t}}}} \cong \nu^* D_{\text{HT}}(\mathbb{L})|_{X_{K,\text{pro\acute e\textbf{t}}}} \otimes_{O_{X_K}} O_{\text{BHT}}|_{X_{K,\text{pro\acute e\textbf{t}}}} \]

\[ \cong \nu^*(D_{\text{HT}}(\mathbb{L})|_{X_K} \otimes \bigoplus_{j \in \mathbb{Z}} O_{X_K}(j)) |_{X_{K,\text{pro\acute e\textbf{t}}}} \]

\[ \cong \nu^*(D_{\text{HT}}(\mathbb{L})|_{X_K} \otimes \bigoplus_{j \in \mathbb{Z}} O_{X_K}(j)) \otimes_{O_{X_K}} O_{C}|_{X_{K,\text{pro\acute e\textbf{t}}}}. \]

\[ \rightarrow \nu^*(\bigoplus_{j \in \mathbb{Z}} \mathcal{H}(\mathbb{L})(j)) \otimes_{O_{X_K}} O_{C}|_{X_{K,\text{pro\acute e\textbf{t}}}}. \]

\[ \cong \bigoplus_{j \in \mathbb{Z}} \hat{L} \otimes_{\hat{Q}_p} O_{C}(j)|_{X_{K,\text{pro\acute e\textbf{t}}}} \cong (\hat{L} \otimes_{\hat{Q}_p} O_{\text{BHT}})|_{X_{K,\text{pro\acute e\textbf{t}}}}. \]

This can be checked by considering affinoid perfectoids represented by the toric tower, and the verification is left to the reader. It follows from condition (iii) and Proposition 4.5.3 that

\[ \bigoplus_{j \in \mathbb{Z}} D_{\text{HT}}(\mathbb{L})|_{X_K}(j) \cong \bigoplus_{j \in \mathbb{Z}} \mathcal{H}(\mathbb{L})(j). \]

Hence

\[ (\nu^* D_{\text{HT}}(\mathbb{L}) \otimes_{O_X} O_{\text{BHT}})|_{X_{K,\text{pro\acute e\textbf{t}}}} \cong (\hat{L} \otimes_{\hat{Q}_p} O_{\text{BHT}})|_{X_{K,\text{pro\acute e\textbf{t}}}}. \]

Finally we show that (ii) implies (i). By condition (ii), we have

\[ \nu'(\nu^* D_{\text{HT}}(\mathbb{L}) \otimes_{O_X} O_{\text{BHT}})|_{X_{K,\text{pro\acute e\textbf{t}}}} \cong \nu'(\hat{L} \otimes_{\hat{Q}_p} O_{\text{BHT}})|_{X_{K,\text{pro\acute e\textbf{t}}}}. \]

On the other hand, it is easy to check

\[ \nu'(\nu^* D_{\text{HT}}(\mathbb{L}) \otimes_{O_X} O_{\text{BHT}})|_{X_{K,\text{pro\acute e\textbf{t}}}} \cong \bigoplus_{j \in \mathbb{Z}} D_{\text{HT}}(\mathbb{L})|_{X_K}(j). \]
Since \( \nu^i_{\mathcal{K}}((\mathbb{L} \otimes \mathbb{O}_{\mathcal{B}_{\mathcal{HT}}})|_{X_{K,\text{proet}}}) = \bigoplus_{j \in \mathbb{Z}} \mathcal{H}(\mathbb{L}(j)) \), we have

\[
\bigoplus_{j \in \mathbb{Z}} D_{\mathcal{HT}}(\mathbb{L})|_{X_k}(j) \cong \bigoplus_{j \in \mathbb{Z}} \mathcal{H}(\mathbb{L}(j)).
\]

In particular, \( D_{\mathcal{HT}}(\mathbb{L})|_{X_k} \) is a vector bundle on \( X_{K,\text{et}} \), and thus \( D_{\mathcal{HT}}(\mathbb{L}) \) is a vector bundle on \( X_{\text{et}} \). Moreover, condition (ii) implies \( \text{rank } D_{\mathcal{HT}}(\mathbb{L}) = r \). \( \square \)

**Example 4.5.8.** Suppose that there exists a Zariski dense subset \( T \subset X \) consisting of classical rigid points with residue field finite over \( k \) such that the restriction of \( \mathbb{L} \) to each \( x \in T \) defines a Hodge-Tate representation. Then \( \mathbb{L} \) is a Hodge-Tate sheaf by Theorem 4.4.8 and Theorem 4.5.5(iii). See [31, Theorem 8.6.6] for a generalization of this remark.

**Corollary 4.5.9.**

(i) Hodge-Tate sheaves are stable under taking dual, tensor product, and subquotients.

(ii) Let \( f: Y \to X \) be a morphism between smooth rigid analytic varieties over \( k \). If \( \mathbb{L} \) is a Hodge-Tate sheaf on \( X_{\text{et}} \), then \( f^*\mathbb{L} \) is a Hodge-Tate sheaf on \( Y_{\text{et}} \).

**Proof.** This follows from Proposition 4.2.20, Lemma 4.2.22 and Theorem 4.5.5(iii). \( \square \)

We next turn to the pushforward of Hodge-Tate sheaves.

**Theorem 4.5.10.** Let \( f: X \to Y \) be a smooth proper morphism between smooth rigid analytic varieties over \( k \) of relative dimension \( m \) and let \( \mathbb{L} \) be a \( \mathbb{Z}_p \)-local system on \( X_{\text{et}} \). Assume that \( R^i f_* \mathbb{L} \) is a \( \mathbb{Z}_p \)-local system on \( Y_{\text{et}} \) for each \( i \).

(i) If \( \alpha \in \overline{k} \) is a generalized Hodge-Tate for \( R^i f_* \mathbb{L} \), then \( \alpha \) is of the form \( \beta - j \) with a generalized Hodge-Tate weight \( \beta \) of \( \mathbb{L} \) and an integer \( j \in [0, m] \).
(ii) If $\mathbb{L}$ is a Hodge-Tate sheaf on $X_{\text{ét}}$ then $R^i f_* \mathbb{L}$ is a Hodge-Tate sheaf on $Y_{\text{ét}}$.

Remark 4.5.11. Theorem 4.5.10(ii) is proved by Hyodo ([29, §3, Corollary]) when $f : X \to Y$ and $\mathbb{L}$ are analytifications of corresponding algebraic objects.

Proof. Let $f_K : X_K \to Y_K$ denote the base change of $f$ over $K$.

Part (i) easily follows from Theorem 4.3.11. In fact, we have the isomorphism

$$\mathcal{H}(R^i f_* \mathbb{L}) \cong R^i f_{K,\text{ét},*} (\mathcal{H}(\mathbb{L}) \otimes \Omega^\bullet_{X/Y}(-\bullet)), $$

and under this identification $\phi_{R^i f_* \mathbb{L}}$ corresponds to $R^i f_{K,\text{ét},*} (\phi_{\mathbb{L}} \otimes \text{id} - \bullet (\text{id} \otimes \text{id}))$. Consider the spectral sequence with

$$E_1^{a,b} = R^b f_{K,\text{ét},*} \mathcal{H}(\mathbb{L}) \otimes \Omega^a_{X/Y}(-a)$$

converging to $\mathcal{H}(R^{a+b} f_* \mathbb{L})$. Then the endomorphism $R^b f_{K,\text{ét},*} ((\phi_{\mathbb{L}} - a) \otimes \text{id})$ on $E_1^{a,b}$ converges to $\phi_{R^{a+b} f_* \mathbb{L}}$, and this implies part (i).

For part (ii), we need arguments similar to the proof of Theorem 4.3.11. We may assume that $Y$ is affinoid. Take a finite affinoid covering $\mathcal{U} = \{U^{(i)}_K\}$ of $X_K$. Let $\mathcal{F}^\bullet$ denote the complex of $\mathcal{O}_{X_K}$-modules

$$\mathcal{H}(\mathbb{L}) \xrightarrow{\varphi_1} \mathcal{H}(\mathbb{L}) \otimes \Omega^1_{X/Y}(-1) \xrightarrow{\varphi_1} \mathcal{H}(\mathbb{L}) \otimes \Omega^2_{X/Y}(-2) \to \cdots$$

on $X_K$ equipped with the natural $\Gamma_K$-action and the endomorphism $\phi_{\mathcal{F}^\bullet} = \phi_{\mathbb{L}} \otimes \text{id} - \bullet (\text{id} \otimes \text{id}).$

\footnotetext{2This means that the $\mathbb{Q}_p$-local system $\mathbb{L} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is a Hodge-Tate sheaf on $X_{\text{ét}}$.}
Recall also the Čech-to-derived functor spectral sequence with

$$E_2^{a,b} = H^a(\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, H^b(\mathcal{F}^\bullet))))$$

converging to $R^{a+b}\Gamma(X_{K,\text{ét}}, \mathcal{F}^\bullet)$. This spectral sequence degenerates at $E_2$ and yields

$$H^i(\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet))) \cong R^i\Gamma(X_{K,\text{ét}}, \mathcal{F}^\bullet) = \Gamma(Y_K, \mathcal{H}(R^i f_* \mathbb{L})).$$

Note that both source and target in (4.5.2) have arithmetic Sen endomorphisms and they are compatible under the isomorphism.

Since $\mathbb{L}$ is a Hodge-Tate sheaf, there exist integers $j_1 < \ldots < j_a$ such that $F(\phi_{\mathbb{L}}) = 0$ with $F(s) := \prod_{1 \leq i \leq a} (s - j_i)$. Set $J := \{ j_1 - m, j_1 - m + 1, \ldots, j_a - 1, j_a \}$. This is a finite subset of $\mathbb{Z}$. We set $G(s) := \prod_{j \in J} (s - j) \in \mathbb{Z}[s]$. For each $0 \leq j \leq m$, the endomorphism $\phi_{\mathbb{L}} \otimes \text{id} - j(\text{id} \otimes \text{id})$ on $\mathcal{H}(\mathbb{L}) \otimes \Omega^j_{X/Y}(-j)$ satisfies

$$G(\phi_{\mathbb{L}} \otimes \text{id} - j(\text{id} \otimes \text{id})) = 0.$$ 

This implies $G(\phi_{\mathcal{F}^\bullet}) = 0$, and thus $G(\text{Tot}(\check{\mathcal{C}}^\bullet(\mathcal{U}, \phi_{\mathcal{F}^\bullet}))) = 0$. Therefore (4.5.2) yields

$$G(\phi_{R^i f_* \mathbb{L}}) = 0.$$

Hence $R^i f_* \mathbb{L}$ is a Hodge-Tate sheaf on $Y_{\text{ét}}$. 

We now turn to a rigidity of Hodge-Tate representations. Let us first recall Liu and Zhu's rigidity result for de Rham representations ([36, Theorem 1.3]): let $X$ be a geometrically connected smooth rigid analytic variety over $k$ and let $\mathbb{L}$ be a $\mathbb{Q}_p$-local system on $X_{\text{ét}}$. If $\mathbb{L}_{\mathcal{F}}$ is a de Rham representation at a classical point $x \in X$, then $\mathbb{L}$ is a de Rham sheaf. In
particular, \( L_y \) is a de Rham representation at every classical point \( y \in X \).

The same result holds for Hodge-Tate local systems of rank at most 2. We do not know whether this is true for Hodge-Tate local systems of higher rank.

**Theorem 4.5.12.** Let \( k \) be a finite extension of \( \mathbb{Q}_p \). Let \( X \) be a geometrically connected smooth rigid analytic variety over \( k \) and let \( L \) be a \( \mathbb{Q}_p \)-local system on \( X_{\text{ét}} \). Assume that \( \text{rank} L \) is at most two. If \( L_x \) is a Hodge-Tate representation at a classical point \( x \in X \), then \( L \) is a Hodge-Tate sheaf. In particular, \( L_y \) is a Hodge-Tate representation at every classical point \( y \in X \).

Before the proof, let us recall a remarkable theorem by Sen on Hodge-Tate representations of weights 0.

**Theorem 4.5.13 ([43 §Corollary]).** Let \( k \) be a finite extension of \( \mathbb{Q}_p \) and let \( \rho: G_k \to \text{GL}_r(\mathbb{Q}_p) \) be a continuous representation of the absolute Galois group of \( k \). Then \( \rho \) is a Hodge-Tate representations with all the Hodge-Tate weights zero if and only if \( \rho \) is potentially unramified, i.e., the image of the inertia subgroup of \( k \) is finite.

Note that \( \rho \) being a Hodge-Tate representations with all the Hodge-Tate weights zero is equivalent to the Sen endomorphism of \( \rho \) being zero. Since potentially unramified representations are de Rham and de Rham representations are stable under Tate twists, Theorem 4.5.13 implies that a Hodge-Tate representation with a single weight is necessarily de Rham.

**Proof of Theorem 4.5.12.** We check condition (iii) in Theorem 4.5.5. By Theorem 4.4.8 and assumption, all the eigenvalues of \( \phi_L \) are integers. So the statement is obvious either when \( \text{rank} L = 1 \) or when \( \text{rank} L = 2 \) and two eigenvalues are distinct integers.
Assume that rank $L = 2$ and two eigenvalues are the same integer. Then $L$ is de Rham by Theorem 4.5.13 and thus $L$ is de Rham by “Principle B” for de Rham sheaves (Theorem 1.3). In particular, $L$ is a Hodge-Tate sheaf.

Remark 4.5.14. The proof shows that Theorem 4.5.12 holds for $L$ of an arbitrary rank if one of the following conditions holds:

(i) $L_x$ is a Hodge-Tate representation with a single weight at a classical point $x \in X$.

(ii) $L_x$ is a Hodge-Tate representation with rank $L$ distinct weights at a classical point $x \in X$.

We end with another application of Sen’s theorem in the relative setting.

Theorem 4.5.15. Let $k$ be a finite extension of $\mathbb{Q}_p$. Let $X$ be a smooth rigid analytic variety over $k$ and let $L$ be a $\mathbb{Z}_p$-local system on $X_{\text{ét}}$. Assume that $L$ is a Hodge-Tate sheaf with a single Hodge-Tate weight. Then there exists a finite étale cover $f : Y \to X$ such that $(f^* L)_y$ is semistable at every classical point $y$ of $Y$.

Proof. Since semistable representations are stable under Tate twists, we may assume that $L$ is a Hodge-Tate sheaf with all the weights zero. Let $\mathbb{L}$ denote the $\mathbb{Z}/p^2$-local system $L/p^2 L$ on $X_{\text{ét}}$. Then there exists a finite étale cover $f : Y \to X$ such that $f^* \mathbb{L}$ is trivial on $Y_{\text{ét}}$. We will prove that this $Y$ works.

Let $y$ be a classical point of $Y$. We denote by $k'$ the residue field of $y$. Let $\rho : G_{k'} \to \text{GL}(V)$ be the Galois representation of $k'$ corresponding to the stalk $V := (f^* \mathbb{L})_y$ at a geometric point $\overline{y}$ above $y$. By assumption, $\rho$ is a Hodge-Tate representation with all the weights zero, and thus it is potentially unramified by Theorem 4.5.13. Hence if we denote the inertia group of $k'$ by $I_{k'}$, $\rho(I_{k'})$ is finite.
By construction, the mod $p^2$ representation

$$G_k \xrightarrow{\rho} \text{GL}(V) \rightarrow \text{GL}(V/p^2V)$$

is trivial. On the other hand, $\text{Ker}(\text{GL}(V) \rightarrow \text{GL}(V/p^2V))$ does not contain elements of finite order except the identity. Thus we see that $\rho(I_k)$ is trivial and hence $\rho$ is an unramified representation. In particular, $\rho$ is semistable. \qed

Remark 4.5.16. As mentioned in Subsection 4.1.3, it is an interesting question whether one can extend Colmez’s strategy ([15]) to prove the relative $p$-adic monodromy conjecture using Theorem 4.5.15.
Bibliography


