DOCTORAL THESIS

Precision Collider Physics from Effective Field Theory

A dissertation presented by
Kai Yan
to The Department of Physics

in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the subject of Physics

Harvard University Cambridge, MA

May, 2018
In this thesis we study the factorization properties in perturbative Quantum Chromodynamics that allow separation of the physics associated with jet formation from that of hard-scattering in high-energy particle collisions. The focus of our work is to understand how factorization theorems can be applied to precision calculations in collider physics, and when they are violated at cross-section level in hardron scattering processes.

As an application of strict collinear factorization, we state and proved to all orders a factorization theorem for (soft-drop) groomed jet observables. Our calculation done at two-loop order enabled resummation of cross section matched on to fixed order results producing the first jet substructure predictions at next-to-next-to-next-to-leading-logarithmic accuracy for the LHC.

Factorization is known to be violated at amplitude level where incoming particles are collinear to outgoing ones. Through an effective field theory framework with Glauber operators, we computed the perturbative amplitude in space-like collinear limits and obtained two-loop finite terms that breaks factorization. We proved to all orders that for pure Glauber ladder graphs, all amplitude-level factorization violating effects completely cancel at cross section level for any single-scale observable. This narrows down the direction we look into the source factorization-violation in physical cross sections.
Acknowledgement

First and foremost, I would like to thank my advisor, Professor Matthew Schwartz. The projects that I worked with Matt have been a source of happiness in the past few years. Matt is a great teacher whose patience and encouragement allowed me to build up my confidence as a researcher. His knowledge and intuition have been an inspiration, where I learnt how to think about physics problems, to stay curious, aspirational, and down-to-earth. At the same time, I appreciate Matt for allowing me space and freedom to develop in my own way.

I am thankful to the time spent with my colleagues on problems where our passion lies. Hua Xing Zhu has helped and taught me immensely on Feynman diagrams and loop integrals. Our discussions have always been fruitful, thanks to the interests we share. I thank Ilya Feige for being a motivating and thoughtful collaborator on my very first project. His notes on factorization has been my favorite textbook of Soft Collinear Effective Theory. Working with Andrew Larkoski and Chris Frye was a good experience of learning how to approach to real phenomenology. I am proud of our great teamwork and I had a lot of fun debugging the programs we ran on Odyssey.

Finally, I would like to thank Professor Chong Sheng Li, as well as people I worked with in Li’s group at PKU. I have benefit a great deal from your advice, caring and support ever since 2011. Thank you for guiding me onto the right path and for being exemplary of scientific virtues.

Acknowledgements of Final Support
This thesis is based on work supported in part by the U.S. Department of Energy, under grant DE-SC0013607, and by the Office of Nuclear Physics of the U.S. Department of Energy under Contract No. DE-SC0011090.
Citations to previously published work

Parts of this dissertation cover research reported in the following articles:


物格无止境，理运有常时。
Contents

1 Introduction ................................................. 1
  1.1 Factorization: dealing with various energy scales .............. 1
  1.2 Effective theory approach .................................. 5
  1.3 Applications of factorization in precision QCD ................. 11
    1.3.1 Automating fixed-order perturbative cross sections ......... 11
    1.3.2 Resummation of jet substructur e .......................... 12
    1.3.3 Bootstrapping multi-loop scattering amplitudes .......... 12
  1.4 Factorization violation ..................................... 14
  1.5 Outline .................................................. 16

2 Removing overlapping phase-space singularities .................. 18
  2.1 Introduction ............................................. 18
  2.2 Factorization at the amplitude level .......................... 22
    2.2.1 Example subtractions .................................... 25
    2.2.2 General amplitude-level subtraction ....................... 28
  2.3 Factorization for distributions .............................. 31
    2.3.1 Factorization for thrust ................................ 32
    2.3.2 Jet broadening .......................................... 41
    2.3.3 Comparison to other approaches .......................... 44
  2.4 Conclusion ................................................ 47
  2.5 Appendix .................................................. 48
    2.5.1 Regularization schemes ................................. 48
    2.5.2 Jet Functions ........................................... 49
    2.5.3 Soft functions .......................................... 54

3 Factorization for groomed jet substructure beyond the next-to-leading logarithm ............... 59
  3.1 Introduction ............................................. 59
  3.2 Observables ............................................... 64
    3.2.1 Soft Drop Grooming Algorithm ........................... 64
    3.2.2 Energy Correlation Functions ............................ 66
  3.3 Factorization Theorem ..................................... 67
    3.3.1 Power Counting and Modes .............................. 67
3.3.2 Factorization and refactorization ........................................ 71
3.3.3 The Single Scale of the Collinear-Soft Function ...................... 75
3.4 Consequences of Factorization Theorem ................................... 79
  3.4.1 Absence of Non-Global Logarithms to All Orders .................... 79
  3.4.2 Process Independence .................................................. 81
  3.4.3 Hadronization Corrections ............................................ 82
3.5 Achieving NNLL Accuracy .................................................. 84
  3.5.1 NNLL for $\alpha = 2, \beta = 0$ ...................................... 86
  3.5.2 Reclustering with anti-$k_T$ .......................................... 94
  3.5.3 NNLL for $\alpha = 2, \beta \geq 0$ .................................... 97
3.6 Matching NNLL to Fixed Order in $e^+ e^- \rightarrow$ dijets .............. 100
  3.6.1 Matching Resummation to Fixed-Order ................................ 101
  3.6.2 Comparison to Monte Carlo .......................................... 104
3.7 Matching NNLL to Fixed Order in $pp \rightarrow Z + j$ .................. 109
  3.7.1 Resummed Cross Section in $pp \rightarrow Z + j$ ....................... 110
  3.7.2 Matching Resummation to Fixed-Order ............................... 114
  3.7.3 Comparison to Monte Carlo .......................................... 116
3.8 Conclusions .................................................................... 118
3.9 Appendix .................................................................... 120
  3.9.1 Three-Loop $\beta$-function and Cusp Anomalous Dimension ...... 120
  3.9.2 Hard Function ........................................................... 121
  3.9.3 $e^+ e^- \rightarrow q\bar{q}$ ............................................... 122
  3.9.4 $e^+ e^- \rightarrow gg$ .................................................. 123
  3.9.5 The Global Soft Function .............................................. 124
  3.9.6 Jet Functions ............................................................ 126
  3.9.7 Quark Jets ............................................................... 127
  3.9.8 Gluon Jets ............................................................... 128
  3.9.9 Collinear Jets ............................................................ 129
  3.9.10 Resummation ............................................................ 131
  3.9.11 Renormalization Group Evolution of $D_k$ ......................... 135
4 Collinear factorization violation and effective field theory ............ 142
  4.1 Introduction ................................................................ 142
  4.2 Elements of factorization and Glauber scaling ......................... 149
  4.3 Glauber containment in hard scattering ................................ 154
    4.3.1 1-loop example ....................................................... 155
    4.3.2 Spacelike example ................................................... 159
    4.3.3 General argument .................................................... 160
    4.3.4 Summary ............................................................... 163
  4.4 Isolating the Glauber contribution ....................................... 164
    4.4.1 Method of regions .................................................... 165
    4.4.2 Cut-based approach ............................................... 167
    4.4.3 Position space ....................................................... 169
4.4.4 Effective field theory Glauber operator ........................................... 171
4.5 Factorization-violation in collinear splittings ....................................... 173
  4.5.1 Strict factorization ................................................................. 175
  4.5.2 Strict factorization violation from $i\pi/\epsilon$ terms ...................... 176
  4.5.3 Strict-factorization violation from full QCD ............................... 180
4.6 Factorization violation from SCET .................................................. 184
  4.6.1 Tree-level splitting amplitudes in SCET ...................................... 185
  4.6.2 Factorization violating contributions ......................................... 187
4.7 Two-loop factorization-violation from SCET ....................................... 195
  4.7.1 Double-Glauber diagrams ....................................................... 197
4.8 Analytic properties of Glauber gluons in SCET .................................... 204
4.9 Summary and conclusions ............................................................... 210
4.10 Appendix ....................................................................................... 213
  4.10.1 Double Glauber integrals ......................................................... 213

5 Factorization Violation and Scale Invariance ........................................ 220
  5.1 Introduction ................................................................................... 220
  5.2 Factorization and Factorization Violation ......................................... 223
    5.2.1 Factorization violation ............................................................ 230
  5.3 Glauber ladder graphs ..................................................................... 234
    5.3.1 2-loop Glauber cancellation ................................................... 234
    5.3.2 All-orders Glauber cancellation ............................................... 241
  5.4 Factorization-violating effects ........................................................ 243
    5.4.1 Observable dependence ............................................................ 245
  5.5 Summary and Conclusion .................................................................. 253
Chapter 1

Introduction

1.1 Factorization: dealing with various energy scales

At hadron colliders, quarks and gluons coming from a confined state within the protons scatter at short distances, producing heavy particles, such as Higgs or W, and/or energetic quarks or gluons. As the quark or gluon moves out of the collision point it travel over long distances, radiating gluons and quark-antiquark pairs, and finally hadronize into color-neutral objects. Thus any calculation that describes quantum chromodynamics (QCD) in colliders must deal with a large range of energies from the confinement scale $\Lambda_{\text{QCD}}$ to the center-of-mass energy of the hard scattering $\mu_H$. Consequently, a given QCD amplitude might not be completely treated in the perturbative way assuming small coupling constant.

In general, the issue with evaluating multi-scale Feynman integrals arises naturally when one treats various quantities within the framework of perturbation theory. If a given Feynman integral depends on kinematic invariants and masses which essentially differ in scale, a natural idea is to expand it in ratios of small and large parameters. As a result it often turns out that the amplitude can be factorized in such a way that different
factors are responsible for contributions of different scales. According to the factorization procedure, a given cross section can be represented as a product of factors some of which can be treated only non-perturbatively but are process-independent, while others can be indeed evaluated within perturbation theory. A well-known example is the standard pdf factorization formula,

\[ \sigma(m_H) = \int dY \int \frac{d\xi_a}{\xi_a} \frac{d\xi_b}{\xi_b} f_i(\xi_a, \mu) f_j(\xi_b, \mu) H_{ij}^{\text{incl}}(\xi_a, \xi_b, m_H, \mu) + O(\frac{\Lambda_{\text{QCD}}}{m_H}) \]  

(1.1)

which justifies the use of fixed-order perturbation theory for sufficiently inclusive quantities such as the total cross section for Higgs production via gluon fusion. Such factorization theorem is at the heart of any quantitative prediction power of perturbative QCD. It lets us use perturbative calculations involving partons (quarks and gluons) to make precise predictions for experimentally measurable quantities involving color-neutral hadrons.

Moreover, in between the confinement scale and the hard-scattering scale, the process of jet production occurs, a phenomenon due to the infrared dynamics of perturbative QCD. Jets are collimated streams of particles originating from a high-energy parton produced at short distance. Many experimental analyses separate events into exclusive jet bins, using a jet algorithm to cluster the final state and then veto on jets. Exclusive jet cross sections, such as the $H + 0$-jet cross section, play a key role in channels where the Higgs cannot be reconstructed and jet binning is an effective discriminant between various Standard Model backgrounds.

These analyses often involve cuts on the final state hadronic activity, involving intermediate perturbative scales, $\Lambda_{\text{QCD}} \ll \mu_1, \mu_2, \cdots \ll \mu_H$, where the coupling constant is still small, but large logarithms of ratios of scales originating from infrared (collinear or soft) singularities destroy the convergence of perturbation theory. For cases with large logarithms between perturbative scales, there is usually a further factorization of the per-
turbative calculation into components describing different momentum regions. For example, exclusive $N$–jet cross section factorize into functions representing initial-state (I) and final-state (J) collimated particles, and soft radiations (S) off these jets,

$$\delta_{N\text{-jet}} \cong H I_a I_b J_1 \cdots \times J_N S$$  \hspace{1cm} (1.2)

A traditional way to perform such factorization procedure is to solve the problem of asymptotic expansion of Feynman integrals in the corresponding limit of momenta that is determined by the given kinematical situation. The so-called method-of-regions approach provide a universal algorithm to solve this problem. First, one divides the space of the loop momenta into various regions, e.g, collinear region around phase-space point of light-like momentum in the jet direction and soft region around the point with zero momentum. In every region, expand the integrand in a Taylor series with respect to the parameters that are considered small there and then integrate over the whole integration domain of the loop momenta. Doing so introduces UV singularities and dependence on factorization scale $\mu$, which can be treated conceptually as a Lorentz and gauge invariant cut-off separating physics associated with different scales. The leading terms of the corresponding expansion organize themselves into a product of factors depending on different physical scales ($\mu_I, \mu_S$, etc), given by Eq. (1.2). In this case, physics associated with jet formation decouples from the underlying hard process. The long-distance objects Eq. (1.2) are universal functions describing the soft and collinear physics associated with jet production. They are perturbatively calculable whose dependence on factorization scale can be used to sum the towers of large logarithms $\ln \frac{\mu_I}{\mu_H}, \ln \frac{\mu_S}{\mu_H}$ to all orders in $\alpha_s$. Thus, factorization renders the perturbative expansion convergent and reduces significantly the uncertainties in the prediction of exclusive cross sections at hadron colliders.

It is worth mentioning that the region decomposition procedure is usually performed
at fixed order, appropriate to LL or NLL resummation. Complications might arise as one proceed to higher order in perturbation theory. To achieve systematic analytic resummation of large logarithms at higher accuracy requires one to prove that the following factorization properties are preserved to all-orders:

- Universality. Are the jet functions only sensitive to the dynamics of collinear interactions, independent of the color and momenta of other non-collinear particles? Does the soft function depend only on the total charge of the collinear particles, regardless of the color evolution within the jet? If not, then the formal factorization formula becomes extremely convoluted in the color or momentum space of all partons in the final states, which will not be useful for the purpose of any realistic resummation.

- Single-scale dependence. Does the factorized component only depend on one kinematic invariant? If not, logs of ratios between different invariants within the jet or soft function is not resummed by solving its RG equation. This might be an issue in sub-regions where this type of logs become important.

The proof of these arguments do not follow automatically from the expansion by regions. In fact, the method-of-regions approach has a limited power in predicting the all-order factorization properties of the Feynman integrals. On the other hand, Soft-Collinear Effective Theory (SCET) provides a quantum field theory description of long-distance interactions in high energy processes. Thus it provides an ideal framework to study the structure of infrared singularities of QCD which are essential to the proof of factorization to all orders.
1.2 Effective theory approach

Soft-Collinear Effective Theory is an effective theory describing the infrared dynamics of high energy scattering processes. Multiple types of SCET fields are introduced for a given particle, distinguished by their scaling behaviours. In momentum space, they are associated with soft, collinear and glauber modes describing small fluctuations around light-like momentum, characterized by a small power-counting parameter $\lambda$. The effective theory describes the long-distance interactions mediated by these SCET modes while high-energy, off-shell modes are integrated out. Following the idea of expansion by regions, the effective lagrangian is constructed from multi-pole expansions of the full-theory lagrangian in terms of the power-counting parameter $\lambda$, thus allowing the separation of various long-distance scales.

A particular hard interaction process can be matched onto local hard operators in the effective theory.

$$\mathcal{L}_{\text{hard scatter}} = \sum_K C_K \mathcal{O}_K$$

(1.3)

Where $\mathcal{O}_K$ are built upon gauge invariant operator building blocks of collinear fields.

$$\chi_n = W_n^\dagger \xi_n, \quad B_{n,\perp} = W_n^\dagger D_{n,\perp} W_n$$

(1.4)

where $\xi_n$ and $A_n$ are collinear quark and gluon fields. $W_n$ is the collinear Wilson line built upon $\vec{n} \cdot A_n$. Furthermore, soft/collinear factorization is achieved through BPS field redefinition: $\psi \to Y_n \psi, A_n \to Y_n A_n Y_n^\dagger$, after which soft modes are decoupled from the collinear at Langragian level and they only show up as gauge invariant products of soft Wilson lines $Y_{n,i}$ emerging from the local effective operators.

Various Wilson lines that appear in the effective operators are path-ordered line inte-
ginals of gauge fields. They are defined as follows,

\[
Y_n^+(x) = P \exp \left\{ i g \int_0^\infty ds \, n \cdot A_s(x + sn) \right\}
\]

\[
W_n^+(x) = P \exp \left[ -g \frac{1}{\hat{n} \cdot \hat{p}} A_n(x) \right]
\]

Typically, the collinear Wilson lines appear from integrating out off-shell modes, while the soft Wilson lines describe dynamics of coherent soft-collinear interaction in the effective theory. They play an important role in restoring the gauge symmetry structure of the effective theory.

Factorization of different modes in SCET is achieved at Lagrangian level.

\[
\mathcal{L}_{\text{SCET}}^{(0)} = \sum_{n_i} \mathcal{L}_{n_i}(\tilde{\psi}_{n_i}, A_{n_i}) + \mathcal{L}_s(\psi_s, A_s) + \mathcal{L}_G(\{\chi_{n_i}, B_{n_i, \perp}, Y_{n_i}\}; B_{s, \perp})
\]

where hard-collinear fields interact with each other only through glauber interaction mediated by the gauge-invariant operators

\[
\mathcal{L}_G \supset \sum_{i,j} \mathcal{O}_{n_i} \frac{1}{P_{\perp}^2} \mathcal{O}_s \frac{1}{P_{\perp}^2} \mathcal{O}_{n_j}
\]

It is only relevant in some special kinematic situations where initial and final-state particles become collinear, such as forward scattering and hard-scattering processes with initial-state collinear radiations.

When there are no partons going collinear in the space-like regime, the hard-soft-collinear decoupling is completely transparent in SCET. It allows the fundamental proof of the factorization of on-shell amplitudes in a hard scattering process into hard, jet and
soft subamplitudes in terms of gauge invariant operator matrix elements.

\[ \langle \{ X_i \}; X_s | \mathcal{M} | 0 \rangle \cong C_\mathcal{O}(\{ S_{ij} \}) \frac{\langle X_1 | \bar{\chi}_{n_1} | 0 \rangle}{\langle 0 | Y_{n_1}^+ W_{n_1} | 0 \rangle} \cdots \frac{\langle X_N | \chi_{n_N} | 0 \rangle}{\langle 0 | W_{n_N}^+ Y_{n_N} | 0 \rangle} \langle X_s | Y_{n_1}^+ \cdots Y_{n_N} | 0 \rangle \] (1.8)

With the help of effective theory, progress has been achieved in understanding amplitudes and their singularities in special kinematic limits. These relate respectively to different factorization properties of gauge-theory amplitudes.

- Fixed-angle factorization theorem.

This is a special case of Eq. (1.8), where all external states are hard and non-collinear:

\[ \mathcal{M}_n(\{ p_i \}) = H_n(\{ p_i \}) S(\{ n_i \}, \{ T_i \}) \prod_{i=1}^n \frac{I_i(p_i, \bar{n}_i)}{J_i(n_i, \bar{n}_i)} \equiv H_n Z_n \] (1.9)

Where \( H_n \) is the infrared finite Wilson coefficient. \( Z_n \) is a product of soft and jet sub-amplitudes responsible for soft and collinear divergences, whose expressions can be read off from Eq. (1.8). \( I_i \)'s are collinear divergent, color singlets whose soft-collinear singularities are removed by their eikonal limit, while all the soft divergences and color evolutions between the jets are given by \( S \). The latter can be mapped onto the ultraviolet singularities of correlators of semi-infinite Wilson line operators.

From a more theoretical perspective, having a operator definitions of the singular factors provides deep insight into all-order structure of amplitudes and their iterative structure. Indeed, one of the main features of infrared singularities is that they exponentiate, and can thus be encoded into an anomalous dimension \( \Gamma_n \). More explicitly,

\[ Z_n(\{ p_i \}, \mu) = \exp \int_0^{\mu^2} d\lambda^2 \frac{\Gamma_n(\{ p_i \}, \lambda, a_s(\lambda^2))}{\lambda^2} \] (1.10)

where \( \Gamma_n(\{ p_i \}, \mu) = -\frac{1}{2} \gamma_K \sum_{i<j} \ln \frac{-s_{ij}}{\mu^2} T_i \cdot T_j + \sum_i \gamma_i \bar{n}_i + \Delta_n(\{ \rho_{ijkN} \}, \{ T_i \}) \)
**Soft factorization**

The soft factor is defined to relate an arbitrary $n$-point amplitude to its $(n+1)$-point counterpart with one extra soft emission,

$$
\mathcal{M}_{n+1}(\{p_i\};k) \equiv S(k)\mathcal{M}_n(\{p_i\})
$$

(1.11)

One important factorization property of the on-shell scattering amplitude is the universality of soft factor,

$$
S(\{k_i\}) = \frac{\langle \{k_i\} | Y_1^+ \cdots Y_N^+ | 0 \rangle}{\langle 0 | Y_1^+ \cdots Y_N^+ | 0 \rangle}
$$

(1.12)

**Collinear factorization**

In the limit where two particle become collinear, the behaviour of the amplitude depends only on the degrees of freedom (momenta, colour and helicities) of the collinear particles, and is entirely independent of the rest of the process. This strict factorization property is confirmed in SCET,

$$
Sp(p_1, p_2; P) = \frac{\langle p_1, p_2 | \chi_n | 0 \rangle}{\langle P | \chi_n | 0 \rangle}
$$

(1.13)

where $Sp$ is the standard QCD splitting amplitude:

$$
\mathcal{M}_{n+1}(p_1, p_2; \{p_i\}) \equiv Sp(p_1, p_2; P)\mathcal{M}_n(P; \{p_i\})
$$

(1.14)

The above factorization properties in the collinear and soft limits of perturbative amplitude leads to the standard all-order factorization of a wide class of event shapes, such as hemisphere jet mass spectrum in $e^+e^-$ collisions, into hard, jet and soft func-
A great advantage of the SCET approach is that with gauge invariant operator definition of the various objects, and resummation of large logarithms can be done through the renormalization group. To be more precise, for sufficiently inclusive quantities that admits factorization, each term in the factorized cross section is infrared finite. The intricate cancellation of the singularities between virtual and real corrections gives rise to logarithmic dependence on the factorization scale \( \mu \). Thus the knowledge of infrared singularities can thus be translated into all-order resummation of various towers of logarithms by solving the standard renormalization group equation. Remarkably, each factorized functions \( F(= H, J, S) \) satisfy the renormalization group equation of the following form:

\[
\frac{d}{d \ln \mu} F = \left[ \Gamma_{\text{cusp}} \ln \left( \frac{\mu^2}{\mu_F^2} \right) + \gamma_{\text{non-cusp}}^F \right] F 
\]

where \( \Gamma_{\text{cusp}} \) can be inferred from to the universal cusp anomalous dimension independent of final states. Only the non-cusp part of the anomalous dimensions of the jet and soft functions depend on the specific measurement. Resummation then reduces to the problem of obtaining the non-cusp anomalous dimension for either the jet or soft function. This standard factorization and resummation procedure has lead to precise predictions of event shape observables at colliders.

- Regge factorization

In the high-energy \((s \gg -t)\) limit, a four-point gauge theory scattering amplitude acquires a factorized structure, given by a t-channel propagator connecting two emission vertices, termed impact factors, which characterize the particles un-
dergoing the scattering. While the t-channel propagator is process independent, it is dressed by a factor of \( \alpha(t) \). At LL accuracy,

\[
\mathcal{M}_{ab \to ab}^{\text{LL}} \approx \left( \frac{s}{-t} \right)^{\alpha(t)} \mathcal{M}_{ab \to ab}^{\text{tree}}
\]

(1.17)

where \( \alpha(t) \) is the gluon regge trajectory independent of the quantum numbers of hard partons in the process.

In SCET, forward scattering is mediated by the glauber operator describing exchange of off-shell gluons with small virtuality. The high-energy logarithm arises due to rapidity difference between the collinear and soft/glauber modes. Leading rapidity logarithms comes from diagrams with one insertion of of glauber operator. At LL approximation, forward scattering amplitude factorizes into product of collinear factors representing impact factors depending on the specific scattering process, and soft factor \( S_G \) representing the gluon regge trajectory.

\[
\mathcal{M}_{ab \to ab}^{\text{LL}}(\{p_i\}) = \langle p_1 | \mathcal{O}_n^{a=q,g} | p_3 \rangle \frac{1}{P_{\perp}^2} S_G(P_{\perp}) \frac{1}{P_{\perp}^2} \langle p_2 | \mathcal{O}_n^{b=q,g} | p_4 \rangle
\]

(1.18)

where

\[
S_G(P_{\perp}) = \int d^2b_{\perp} e^{ib_{\perp} \cdot P_{\perp}} \langle 0 | O_s(b_{\perp}) \delta^2(b_{\perp}) + \sum_{f=q,g} O_s^{a,f}(b_{\perp}) O_s^{b,f}(0) | 0 \rangle
\]

(1.19)

The resummation of \( \ln \frac{s}{t} \) is achieved by through the rapidity renormalization group.

\[
\nu \frac{d}{dv} S_G = \gamma_R S_G
\]

(1.20)

The agreement between \( \gamma_R \) and \( \alpha \) has been confirmed at one-loop level.
1.3 Applications of factorization in precision QCD

A substantial part of the current research on QCD in collider physics focuses on the following closely interconnected topics: the formal calculation of amplitudes and integrals, their use in higher-order calculations and the merging of these fixed-order calculations to all order results, either analytic resummations or parton-shower simulations. Better understanding of factorization addresses several key issues in these topics which are critical to the future development of high-precision collider phenomenology.

1.3.1 Automating fixed-order perturbative cross sections

With the high experimental precision we achieve at the LHC and the improvement in techniques of experimental analyses, it is extremely important to work on higher-order QCD corrections for various production and decay processes at a fully exclusive level. Much progress in this field has been made in the past few years, especially with the development of subtraction methods (e.g. improved sector decomposition, Antenna subtraction) and phase space slicing methods (e.g. transverse momentum and N-jettiness slicing method). SCET is instrumental in the construction of phase-space slicing method such as N-jettiness subtraction. In order to achieve a point-by-point subtraction, it is crucial that the prediction of cross sections from SCET captures all the infrared singularities. The form of subtraction terms, based on factorization theorem which only holds up to a particular order in perturbation theory, needs to be refined as we proceed to higher loop level. The realization of N$^3$LO automated numerical computation might require one to resolve these theoretical subtleties in subtraction procedures.
1.3.2 Resummation of jet substructure

Jets probe extreme kinematical situations, displaying phenomenon of QCD and signatures of new physics over a wide range of energy scales. A lot of the theoretical discussion in jet physics has taken place in the context of Monte Carlo (MC) simulation studies. Nevertheless, it is important to ask whether we can do better than MC. SCET allows transparent remummation of large logarithmic enhancements in certain kinematical regimes, which increases the predictive power of theoretical calculations.

At $e^+e^-$ colliders, one of the major issues with resummation of jet substructure the non-global nature of jet clustering. Correlated soft emissions across the boundary of jet area generate $\ln R$ dependence that do not exponentiate in a simple way. The evolution of such non-global observables is governed by the non-linear equations like the Banfi-Marchesini-Smye equation, whose solution is approximated order-by-order in perturbation theory. The problem with non-global logs has been addressed by different effective theory approaches, e.g. muti-Wilson-line effective theory, dressed gluon expansion. With the development in higher-loop calculation techniques and deeper understanding of the structure of singularities of soft factors, it is still promising that one can find better theoretical control of the resummation of non-global logs.

On the other hand, it is an interesting question to think about whether non-global logs can be avoided by proposing more suitable observables to be measure at hadron colliders.

1.3.3 Bootstrapping multi-loop scattering amplitudes

Scattering amplitudes are at the heart of quantum field theory and collider physics. The evaluation of loop amplitudes in QCD or other more formal theories has seen great interest over the last decade. They are the key to achieving higher precision and exposing symmetries of the underlying theory.
Regarding the analytical computation of Feynman integrals, while some strategies such as integration-by-parts (IBP) reduction and method of differential equations (DE) have developed into mature frameworks, others remain frontiers to explore. Recent developments in modern unitarity-based methods have allowed the automated analytic and numerical computations of two-loop QCD scattering amplitudes.

The difficulty in computing Feynman diagrams increases significantly with the increase of number of kinematic invariants they depend on. Most of the calculations of higher-loop multi-point scattering amplitude are done through the bootstrap approach. The idea of bootstrapping is to propose an ansatz for the function space of the amplitude of interest, based on its analytic property. Then the calculation reduces to the problem of fixing a finite number of expansion coefficients in terms of basis functions. In some special theories such as $\mathcal{N} = 4$ super-Yang-Mills, results can be determined by constraining the ansatz by the symmetries of the underlying amplitude, which highly restricts the form of dependence on kinematic invariants. In more general cases, further insights can be gained by testing the ansatz against the known factorization properties of the gauge-theory amplitude in various kinematic limits, such as soft, collinear and Regge limit. In addition, one might try to directly compute certain aspects of amplitudes with special kinematics, such as the high-energy logarithms in (multi-)Regge limits and discontinuities across branch cuts in some kinematic invariants, and use these results as inputs to further constrain or support the bootstrapping hypothesis. These calculation are presumably simpler and can be done in effective theory framework. It is an exciting topic to study how the bootstrapping procedures can be supplemented and made more efficient by effective theory techniques.
1.4 Factorization violation

Despite the successful application of factorization for certain observables, it is merely an assumption in a lot of processes at hadron colliders. There is both theoretical evidence for non-factorization through effects like super-leading logarithms and experimental evidence, as calculations performed assuming factorization can have significant deviations from experiment. Because of factorization violation, we might lose theoretical control over observables sensitive to the underlying events or other non-perturbative effects. Since factorization limitations to the predictive power of perturbative QCD, it is imperative to understanding factorization violation and its phenomenological consequences in hadron collisions, which is the focus of this thesis. The hope is that better understanding of factorization violation can be achieved by working in two directions, we can either choose observables for which factorization violation is minimal, or find some universality in factorization-violating effects.

To find out how and when factorization violation occurs in hardron collisions, one should start with the factorization properties of the perturbative amplitude, which can be understood from first principles QCD. In particular, QCD amplitudes in certain kinematic regimes factorize into products of universal components governing their infrared behavior. One factorization property is that the color evolution of each component is independent. This independence is broken by soft quantum loop effects when external collinear partons are space-like separated. The reason for this amplitude-level factorization breaking is understood. However, whether it leads to observable consequences is somewhat obscure. Factorization breaking terms in the squared amplitude might cancel after integrating over hadronic final-state phase space, as pointed out by the Collins-Soper-Sterman argument of $q_T$—factorization in Drell-Yan. It is imperative to understand how and when cancellation fails with more general phase-space constraints, which
requires a clear picture of the behavior of the squared QCD amplitude in extreme kinematical regimes of final-state phase space.

At hadron colliders, jet physics is crucial for fundamental questions such as the validity of factorization. SCET with Glauber gluons offers a new handle on the understanding of factorization violation. As shown by our recent study, factorization violating effects in spectator-spectator interaction are related to singularities in soft Lipatov diagrams, which manifestly breaks conformal symmetry of the integrand in transverse dimensions. It remains to be seen what our observation at partonic cross-section level tells us about the non-perturbative effects from the underlying events.

A different approach to getting control over the factorization breaking effects is the attempt to suppress these effects by measuring no-global observables. By constricting the measurement to be on particles with a jet area in the central rapidity region, radiation from the initial state will be suppressed as the jet radius $R$ gets small. However, in the small $R$ limit, cross section suffer from non-global logs, the resummation of which is not well-understood. It turns out that the source of the non-global logs are correlated soft radiations close to the jet boundary. One might be able to resolve the problem by studying groomed jet observables where particles are only measured the when they get close to the jet axis. In particular, we observed that for certain groomers (modified mass drop and soft drop), factorization of jet observables can be restored after wide-angle soft radiations are removed from the jet. With a proper choice of clustering parameters and algorithms, the soft drop groomer not only suppress the contaminations from the initial state, as other grommers do, but it also gets rid of the non-global logs and other clustering effect. The groomed jet mass are only sensitive to final-state collinear dynamics, which strictly factorizes and allows systematic resummations to be achieved at the LHC.
1.5 Outline

This thesis is organized as follows.

We begin with Chaper 2 where we study phasespace integration with factorized amplitudes in effective theory. We present a regulator-independent subtraction algorithm for removing soft-collinear overlap at the amplitude level which may be useful automating computation of perturbative cross sections at hardron colliders. We then discuss how both the soft-collinear and infrared-ultraviolet overlap can be undone, thus clarifying some of the subtleties in phase-space subtractions and provide a proof of the infrared finiteness of a suitably subtracted jet function.

Chapter 3 is an application of effective theory technique in precision jet physics, where we achieved NNLL resummation of jet substructure based on the unstanding of its factorization properties. We investigate the theoretical aspects of soft-drop groomed jet mass and gave a rigorous argument of its factorization violating effects being power-suppressed by the jet mass parameter. Consequently, the groomed mass spectrum is only sensitive to final-state collinear dynamics within the jet. We propose and prove an all-orders factorization theorem for the soft drop grooming algorithm for jets on which two-point energy correlation functions are measured. Remarkably, as we will demonstrate to all orders, the normalized cross section is pile-up insensitive, free from non-global logarithms, and process independent, up to the relative fraction of quark and gluon jets in the sample. We then display our results of resummed groomed jet mass at next-to-next-to-leading logarithmic accuracy matched onto fixed-order code, which produces the first jet substructure predictions to this accuracy for the LHC.

Chaper 4 and 5 are based on our study of factorization properties of perturbative amplitudes, where we unified the SCET Glauber picture with a generalized formulation of QCD amplitudes in various collinear regimes that incorporates factorization breaking ef-
In Chapter 4, we apply the SCET formalism to computing the two-loop amplitude in $2 \rightarrow n + H(Z)$ with one final-state colored particle collinear to the initial-state particle. We establish the relationship between the Glauber contribution and factorization breaking terms in the generalized splitting amplitude due to its non-analyticity along the branch cut of hypergeometric functions. In particular, we verified that traditional SCET without Glauber operators reproduces QCD amplitudes modulo $i\pi$. Diagrams with insertions of Glauber operators generate the difference in the amplitude as we go from time-like to space-like collinear limit.

In Chapter 5, we study the kinematical regime where two pairs of particles become collinear in $2 \rightarrow 2 + H(Z)$, shedding light on the factorization violating effects due to spectator-spectator interactions in hadron scattering. We will move one step forward from fixed order calculation and proved that Glauber ladder diagrams exponentiate into a pure phase to all loop order, up to scale-invariant finite terms. We will further explain that for pure Glauber ladder graphs, all of the amplitude-level factorization violating effects completely cancel at cross section level for any single-scale observable. Finally we come to the conclusion that factorization effect comes from graphs with emissions of soft gluons off the Glauber ladders, where soft radiative corrections introduces high energy (rapidity) logs, breaking the scale invariance in transverse momentum plane.
Chapter 2

Removing overlapping phase-space singularities

2.1 Introduction

Factorization is at the heart of our ability to use perturbative quantum chromodynamics (QCD) to make theoretical predictions for scattering processes at high-energy particle colliders. It is extremely fortuitous that accurate particle distributions can be computed by convolving universal parton distribution and hadronization models with perturbative calculations of jet formation. While factorization at the non-perturbative level is hard to establish, factorization relevant to the structure and substructure of jets can be understood within perturbation theory. In particular, the radiation patterns in perturbative QCD factorize into hard, collinear and soft contributions. Moreover, subtleties in perturbative factorization (for example, related to non-global logarithms [1–7]) are a limiting factor in many ultra-precise jet-substructure calculations. Thus, there has recently been renewed interest in studying factorization, particularly in the context of Soft-Collinear Effective Theory (SCET).
A concise formulation of factorization in QCD was proposed and proven in [8] and [9], hereafter referred to as [FS1] and [FS2] respectively. These papers build upon decades of insight [10–16]. Up to color factors, the formula from [FS2] reads:

\[ \langle X_1 \cdots X_N; X_s \mid \mathcal{O} \mid 0 \rangle \cong C_{\mathcal{O}}(S_{ij}) \frac{\langle X_1 \mid \bar{\psi} W_1 \mid 0 \rangle}{\langle 0 \mid Y_1 \bar{\psi} W_1 \mid 0 \rangle} \cdots \frac{\langle X_N \mid W_N^\dagger \psi \mid 0 \rangle}{\langle 0 \mid W_N^\dagger Y_N \psi \mid 0 \rangle} \langle X_s \mid Y_1 \cdots Y_N \mid 0 \rangle \] (2.1)

In this expression, the state \( \langle X_1 \cdots X_N; X_s \rangle \) has soft particles, in \( \langle X_s \rangle \), and particles collinear to various specified directions, in \( \langle X_i \rangle \). The left-hand side is a matrix element in QCD of an operator like \( \mathcal{O} = \bar{\psi} \cdots \psi \) in this state. The right hand side is a factorized product of matrix elements, each of which involves only one collinear sector or the soft sector. The symbol \( \cong \) indicates that the two sides are identical at leading power. More precisely, if one were to compute some infrared-safe observable dominated by soft or collinear radiation, such as the sum of the jet masses \( \tau = \frac{1}{Q^2} \sum m_i^2 \), all of the terms in \( \frac{d\sigma}{d\tau} \) that are dominant as \( \tau \to 0 \) will be identical on both sides. More details can be found in Section 2.2 below and in [FS1] and [FS2].

The formula in Eq. (2.1) presupposes that the external momenta are designated as soft or collinear. If a particular momentum can be classified as soft or collinear, then the factorized formula will hold whether it is put in \( \langle X_s \rangle \) or in the appropriate \( \langle X_i \rangle \). For example, we can place all the soft-collinear momenta in the soft sector by designating any particle with energy less than some \( \Lambda \) as soft, and then draw cones of size \( R \) around each of the hard directions to distribute particles in the collinear sectors. With such hard cutoffs, one can then square the matrix elements on the right-hand side of Eq. (2.1) and perform the phase-space integrals over the appropriate measurement function to get a differential distribution. The result will agree at leading power with the distribution computed using the left-hand side of Eq. (2.1) in the limit \( R \to 0 \) and \( \Lambda \to 0 \).

There are two problems with the hard-cutoff prescription for resolving the soft-collinear
ambiguity. The first is practical: introducing an extra scale makes the relevant calculations nearly impossible. Moreover, the cutoff dependence may not exactly cancel in the factorized expression and therefore one must either take $R \to 0$ and $\Lambda \to 0$ after the calculation or live with power corrections in these cutoffs. The second is conceptual: the cutoffs violate factorization in the following sense. There will in general be leading-power dependence on the cutoff in the soft and collinear sectors separately (terms like $\frac{1}{t} \ln R$, for example) which only cancel when the sectors are combined. Thus the two sectors are not completely separated.

It would be great if we could simply perform phase-space integrals over each sector separately including all momenta. This is not as crazy as it sounds. We know that including very energetic virtual momenta in the soft or collinear sectors causes no problem, since the modification can always be compensated for in the matching coefficient ($C_O(S_{ij})$ in Eq. (2.1)). Indeed, effective theories always have different ultraviolet (UV) structure from the full theories to which they are matched. For example, in SCET, there are $\frac{1}{\varepsilon}$ UV poles at 1-loop in dimensional regularization, while in full QCD, one only ever has $\frac{1}{\varepsilon}$ poles. In fact, these double poles allow for the resummation of Sudakov double logarithms in SCET using the renormalization group. We also know that one does not have to distinguish soft from collinear momenta in loops when using Eq. (2.1): the overcounting is compensated for by the vacuum matrix elements in the denominator of this equation. Thus, we have good reason to believe that subtractions similar to the denominator factors in Eq. (2.1) can be added to this formula to allow for unrestricted phase-space integrals.

Removing the overcounting of soft and collinear momenta has been addressed in the traditional approach to factorization, for certain observables [17–19]. There, the soft limit of collinear momenta is compensated for with eikonal jet functions [20]. In SCET, the overcounting can be formally avoided by not including the zero-momentum bin in any of the collinear sectors [21]. This exclusion translates into a subtraction diagram-by-
diagram. This zero-bin subtraction is necessary in SCET because the same soft-collinear momentum region in QCD is represented by multiple fields in the effective theory (similar overcounting is present in other effective theories, such as NRQCD). In [22–24] the two prescriptions were shown to be equivalent. Alternatively, in the method-of-regions approach to SCET [25–28] the overcounting is sidestepped through careful consideration of the analytic properties of the contributions from different sectors. We briefly review these approaches and contrast them with our approach in Section 2.3.3.

The formulation of factorization in [FS1] and [FS2] and Eq. (2.1) is intermediate between traditional QCD and SCET. It provides a precise formulation of factorization purely in terms the fields in full QCD, but has a factorized form with a natural effective field theory interpretation. It is based on the observation of Freedman and Luke [16] that the unwieldy Feynman rules of SCET can be avoided and the effective Lagrangian taken simply as the direct sum of $N + 1$ copies of the QCD Lagrangian, corresponding to $N$ collinear sectors and a soft sector. The formulation in [FS1] and [FS2] can be thought of as a generalization of the Freedman-Luke proposal, equivalent but simpler at leading power, and that addresses the soft-collinear overlap of virtual momenta. In this paper, we extend the formulation so that phase-space integrations can be done without explicit cutoffs on the momenta of various sectors.

There are two main results in this paper. First, in Section 2.2, we show how the specification of which sector a gluon belongs to can be removed at the amplitude level. More precisely, suppose we have an amplitude $M(p_{1}, \ldots, p_{n}, q_{1} \cdots q_{m})$ with $n$ hard momenta and $m$ other momenta in QCD. We show how an approximation to $M$ which we call $M_{\text{sub}}$ can be derived with the property that when any of the $q_{i}$ become soft or collinear to any of the $p_{i}$, $M_{\text{sub}}$ agrees with $M$ at leading power. That is, one does not have to specify which sector the $q_{i}$ belong to – the matrix element is correct no matter what. While $M_{\text{sub}}$ is not a factorized product of matrix elements, it is the sum of factorized products of
matrix elements of fields and Wilson lines. Each term in this product is simpler than full QCD. Thus such a subtracted matrix element may be integrable analytically and therefore provide a useful basis for a subtraction scheme in fixed-order QCD.

The second result, in Section 2.3, is a derivation of how at the amplitude-squared level factorization can be preserved and phase-space cutoffs removed for certain inclusive event shapes. Although the result of this section agrees with the eikonal-jet function subtraction method of traditional QCD (which is itself equivalent to SCET), we believe our derivation elucidates some subtleties and makes the procedure more systematic. In addition, we present explicit 1-loop formulas for various relevant soft and jet functions, with and without cutoffs and with different regulators. These formulas demonstrate which objects are infrared safe, cutoff-dependent, and well-defined. Some calculational details are relegated to the appendix. Section 2.3.3 contrasts our approach with previous approaches. We conclude in Section 2.4.

2.2 Factorization at the amplitude level

We begin by quickly reviewing the notation and main results of [FS1] and [FS2]. These papers showed that factorization holds for massless particles whose momenta are either soft or collinear to one of \( N \) directions \( n_j^\mu \). States with particles of these momenta are written as \( |X_1 \cdots X_N; X_s\rangle \). The hard scale (such as the center-of-mass energy) is denoted as \( Q \) and scaling parameters \( \lambda_j \) are defined each collinear sector and \( \lambda_s \) for the soft sector. Momenta in each collinear sector scale as

\[
\langle X_j | = \langle \ldots, q_j, \ldots | \implies \frac{1}{Q} (n_j \cdot q_j, \bar{n}_j \cdot q_j, q_j^\perp) \sim (\lambda_j^2, 1, \lambda_j)
\]  

(2.2)
and momenta in the soft sector scale like

$$\langle X_s | = \langle \ldots, k_s, \ldots | \implies \frac{1}{Q} (n_j \cdot k_s, n_j \cdot k_s, k_s^+) \sim (\lambda_s, \lambda_s, \lambda_s), \ \forall j$$

(2.3)

For simplicity, assume the scattering process under consideration is the decay of a heavy particle mediated by an operator $O$ in QED (to avoid cumbersome color indices of QCD). Then, the factorization formula takes the form of Eq. (2.1):

$$\langle X_1 \ldots X_N; X_s | O | 0 \rangle \simeq C_O(S_{ij}) \frac{\langle X_1 | \bar{\psi} W_1 | 0 \rangle}{\langle 0 | Y_1^\dagger W_1 | 0 \rangle} \ldots \frac{\langle X_N | W_N^\dagger \psi | 0 \rangle}{\langle 0 | W_N^\dagger Y_N | 0 \rangle} \langle X_s | Y_1^\dagger \ldots Y_N | 0 \rangle$$

Here $C_O(S_{ij})$ is a finite function of the large products of the net momentum in each jet, $S_{ij} = P_i \cdot P_j$; it does not depend on the small power-counting parameters, $\lambda_j$ of $\lambda_s$. The Wilson lines, $W_j^\dagger$ and $Y_j^\dagger$, are defined in QCD as follows:

$$Y_j^\dagger = P\left\{ \exp \left[ ig \int_0^\infty ds n_j \cdot A(x + s n_j) e^{-is} \right] \right\}$$

(2.4)

and

$$W_j^\dagger = P\left\{ \exp \left[ ig \int_0^\infty ds t_j \cdot A(x + s t_j) e^{-is} \right] \right\}$$

(2.5)

where $t_j^\mu$ are some lightlike directions assumed not collinear to their associated $n_j^\mu$. The $P\{ \}$ denotes path ordering; in QED the path ordering is trivial and the electromagnetic charge is $e = -g$. Eq. (2.1) is an equality at leading power in all of $\lambda_j$ and $\lambda_s$ separately. For many applications, such as for thrust, one takes $\lambda_j^2 = \lambda_s$ for all $j$; in the SCET literature, this power counting is referred to as SCET$_I$ [29]. For recoil sensitive observables like jet broadening, one takes $\lambda_j = \lambda_s$ as in SCET$_{II}$ [30]. The factorization in Eq. (2.1) holds for any relative scaling.

The important physics contained in Eq. (2.1) is that each factor on the right-hand side
represents a different factorized sector: the Wilson coefficient, \( C_\mathcal{O}(S_{ij}) \), represents all of the hard physics and must be IR-insensitive. Each collinear sector is represented by the ratio \( \langle X_j | W_j^\dagger \psi | 0 \rangle / \langle 0 | W_j^\dagger Y_j | 0 \rangle \) and contains only \( n_j \)-collinear IR divergences. Finally, the soft sector is fully described by the matrix element, \( \langle X_s | Y_1^\dagger \cdots Y_N | 0 \rangle \), which contains all of the soft divergences of the full amplitude on the left-hand side of Eq. (2.1).

One attractive feature of Eq. (2.1) is that each matrix element is constructed out of full-theory operators and evaluated using the full-theory Lagrangian; there are no additional subtractions/prescriptions needed, just simple QCD/QED Feynman rules. Moreover, the power counting is a consequence only of the scaling of the external momenta in the states \( \langle X_1 \cdots X_N; X_s | \mathcal{O} | 0 \rangle \). An obvious fact with important repercussions is that Eq. (2.1) is not valid when any of the momenta in a given sector does not obey the scaling that is associated with that sector. Consequently, one cannot, for example, integrate over the entire phase space of one of the external momenta in Eq. (2.1) because it would enter the scaling regime of other sectors.

Therefore, when calculating cross sections by squaring Eq. (2.1) one can either integrate over the phase space \( d\Pi_{X_j} \) with cutoffs in the integrals restricting each integral to be within the collinear region, or one can try to extend the integrations to the entire phase space and perform a subtraction that gets rid of the errors that we introduced by extending \( d\Pi_{X_j} \) to the entire phase space. Introducing cutoffs to integrals is incredibly tedious and produces new scales in the effective theory that obscure factorization (as shown explicitly in Section 2.3). The subtraction procedure is the only reasonable way forward. We next discuss subtractions at the amplitude level, and discuss subtractions at the cross section level in Section 2.3.
2.2.1 Example subtractions

Consider the case of a $q\bar{q}g$ final state, with quark momenta $p_1^\mu$ and $p_2^\mu$ in different directions and the gluon momentum $q^\mu$. Suppose we want to integrate over the gluon momenta inclusively. We can do so using Eq. (2.1) if when $q \parallel p_1$ we use

$$\mathcal{M}_1(p_1, p_2, q) \equiv \frac{\langle p_1 | \bar{\psi} W_1 | 0 \rangle \langle p_2 | W_2^\dagger \psi | 0 \rangle}{\langle 0 | Y_1^\dagger W_1 | 0 \rangle \langle 0 | W_2^\dagger Y_2 | 0 \rangle} \langle 0 | Y_1^\dagger Y_2 | 0 \rangle , \tag{2.6}$$

if $q \parallel p_2$, we use

$$\mathcal{M}_2(p_1, p_2, q) \equiv \frac{\langle p_1 | \bar{\psi} W_1 | 0 \rangle \langle p_2 | W_2^\dagger \psi | 0 \rangle}{\langle 0 | Y_1^\dagger W_1 | 0 \rangle \langle 0 | W_2^\dagger Y_2 | 0 \rangle} \langle 0 | Y_1^\dagger Y_2 | 0 \rangle , \tag{2.7}$$

and if $q$ is soft, we use

$$\mathcal{M}_s(p_1, p_2, q) \equiv \frac{\langle p_1 | \bar{\psi} W_1 | 0 \rangle \langle p_2 | W_2^\dagger \psi | 0 \rangle}{\langle 0 | Y_1^\dagger W_1 | 0 \rangle \langle 0 | W_2^\dagger Y_2 | 0 \rangle} \langle q | Y_1^\dagger Y_2 | 0 \rangle \tag{2.8}$$

However we split up the integration regions (say with a soft energy cutoff $\Lambda$ and cone radius $R$) the dependence on the split (on $\Lambda$ and $R$) will drop out at leading power when all three contributions are added. Nevertheless, it would be nice to have an expression that we could simply integrate over $q$ without ever introducing $\Lambda$ and $R$ in the first place.

To proceed, we first examine the consequences of soft-collinear factorization for the operator $\mathcal{O}_{\bar{\psi}W} = \bar{\psi} W_1$ (rather than a local QCD operator like $\bar{\psi} \psi$). The all-orders proof of factorization in [FS2] applies to $\mathcal{O}_{\bar{\psi}W}$. In particular, if we have a state with momenta $p_1 \cdots p_n$ all of which are collinear to each other as well as momenta $q_1 \cdots q_m$ all of which are soft, then

$$\langle p_1 \cdot \cdot \cdot p_n; q_1 \cdot \cdot \cdot q_m | \bar{\psi} W_1 | 0 \rangle \simeq C_{\bar{\psi}W} \frac{\langle q_1 \cdot \cdot \cdot q_m | Y_1^\dagger W_1 | 0 \rangle}{\langle 0 | Y_1^\dagger W_1 | 0 \rangle} \frac{\langle p_1 \cdot \cdot \cdot p_n | \bar{\psi} W_1 | 0 \rangle}{\langle 0 | Y_1^\dagger W_1 | 0 \rangle} \tag{2.9}$$
for some $\bar{C}_W$. To determine $\bar{C}_W$, we note that $\bar{C}_W$ does not depend on how the momentum in the collinear and soft sectors are distributed; this equation holds for any $n > 0$ and any $m \geq 0$. In particular, if we take $m = 0$ then the two sides are identical (and agree at leading power) if and only if $\bar{C}_W = 1$. Thus we must have $\bar{C}_W = 1$ for any states.

As a special case, Eq. (2.9) implies that for one collinear and one soft momentum

$$
\frac{\langle p_1; q | \bar{\psi} W_1 | 0 \rangle}{\langle 0 | Y_1^\dagger W_1 | 0 \rangle} \overset{q \text{ soft}}{=} \frac{\langle p_1 | \bar{\psi} W_1 | 0 \rangle \langle q | Y_1^\dagger W_1 | 0 \rangle}{\langle 0 | Y_1^\dagger W_1 | 0 \rangle \langle 0 | Y_1^\dagger W_1 | 0 \rangle}
$$

(2.10)

Similarly, applying the general factorization formula to $O = Y_1^\dagger Y_2$, we get

$$
\langle q | Y_1^\dagger Y_2 | 0 \rangle \overset{q \parallel p_1}{=} \frac{\langle q | Y_1^\dagger W_1 | 0 \rangle}{\langle 0 | Y_1^\dagger W_1 | 0 \rangle} \times \langle 0 | Y_1^\dagger Y_2 | 0 \rangle
$$

(2.11)

In this case one can see that the Wilson coefficient is 1 to all orders by using the proof in [FS2] that the factorization theorem is independent of the collinear Wilson-line direction, $t_1$, and then choosing $t_1^\parallel = n_2^\parallel$, so that $W_1 = Y_2$.

With these results, we can now analyze the following all-loop-order subtracted matrix element:
If we take $q$ soft, then neither of the first two lines contribute by Eq. (2.10), and the result is given by the third line which is the correct leading power matrix element $M_s$. When $q \parallel p_1$, then neither term in the second line is IR sensitive and the subtraction term in the first line (which is IR-sensitive) is canceled by collinear limit of the third line, using Eq. (2.11). Thus, only the first term on the first line contributes at leading power in this limit, in agreement with $M_1$. The analogous argument works for the $q \parallel p_2$ limit. We conclude that $M_{\text{sub}}(p_1, p_2, q)$ agrees with full QCD at leading power for any $q$. Thus, we can integrate $M_{\text{sub}}$ over phase space without splitting the soft and collinear sectors.

To be explicit, we can evaluate Eq. (2.12) in perturbation theory. At tree-level,

$$M(p_1, p_2, q) = \text{tree} \left\{ \frac{-gq}{2p_1 \cdot q} + \frac{gt_1 \cdot e_q}{t_1 \cdot q} - \left( \frac{-gn_1 \cdot e_q}{n_1 \cdot q} + \frac{gt_1 \cdot e_q}{t_1 \cdot q} \right) \right\} v(p_2)$$

$$+ \left\{ \frac{g}{2p_2 \cdot q} + \frac{gt_2 \cdot e_q}{t_2 \cdot q} - \left( \frac{-gn_2 \cdot e_q}{n_2 \cdot q} + \frac{gt_2 \cdot e_q}{t_2 \cdot q} \right) \right\} v(p_2)$$

$$+ \left\{ \frac{-gn_1 \cdot e_q}{n_1 \cdot q} + \frac{gn_2 \cdot e_q}{n_2 \cdot q} \right\} v(p_2)$$

where each term in round brackets corresponds to one of the matrix elements containing the gluon, and thereby each satisfies the Ward identity separately. From the explicit expression in Eq. (2.13) it is easy to check that each soft and collinear limit works out exactly as stated in the paragraph after Eq. (2.12). It can also be seen that the $t_j$ dependent terms
cancel out completely as do the soft terms containing $n_j$ at this order, leaving:

$$\mathcal{M}_{\text{sub}}(p_1, p_2, q) = \text{tree}\left( \bar{u}(p_1) \left( \frac{-g \epsilon_q (p_1 + q)}{2p_1 \cdot q} + \frac{g(p_2 + q) \epsilon_q}{2p_2 \cdot q} \right) v(p_2) \right) = \mathcal{M}(p_1, p_2, q) \quad (2.14)$$

So the full matrix element of QED is reproduced exactly in this case. Of course, for more complex calculations we expect $\mathcal{M}$ to only reproduce the full-theory matrix element at leading power, rather than be exactly equal to it.

### 2.2.2 General amplitude-level subtraction

The generalization of Eq. (2.12) for arbitrary collinear and soft sectors is

$$\langle X_1 \cdots X_N; X_s; q | \mathcal{O} | 0 \rangle \approx_{\text{ir}} \langle X_1 | \bar{\psi} W_1 | 0 \rangle \cdots \langle X_N | W_N^T \psi | 0 \rangle \langle X_s, q | Y_1^T \cdots Y_N | 0 \rangle + \sum_{i=1}^{N} \langle X_i | \bar{\psi} W_i | 0 \rangle \cdots \left\{ \frac{\langle X_i, q | W_i^T \psi \rangle | 0 \rangle}{\langle 0 | W_i^T Y_i | 0 \rangle} \right\}_{\text{soft sub}} \cdots \langle X_N | W_N^T \psi | 0 \rangle \langle X_s | Y_1^T \cdots Y_N | 0 \rangle \quad (2.15)$$

where the $\left\{ \right\}_{\text{soft sub}}$ notation means the operator matrix element corresponding to having subtracted the $q \rightarrow \text{soft limit}$. To be explicit, we can use the notation $S(q)$ as in [FS2] for the leading order contribution in the $q \rightarrow \text{soft limit}$. Then

$$\left\{ \frac{\langle X_i, q | W_i^T \psi \rangle | 0 \rangle}{\langle 0 | W_i^T Y_i | 0 \rangle} \right\}_{\text{soft sub}} = \frac{\langle X_i, q | W_i^T \psi \rangle | 0 \rangle}{\langle 0 | W_i^T Y_i | 0 \rangle} - \frac{\langle X_i, q | W_i^T \psi \rangle | 0 \rangle}{\langle 0 | W_i^T Y_i | 0 \rangle} S(q)$$

$$= \frac{\langle X_i, q | W_i^T \psi \rangle | 0 \rangle}{\langle 0 | W_i^T Y_i | 0 \rangle} - \frac{\langle X_i | W_i^T \psi \rangle | 0 \rangle \langle q | W_i^T Y_i | 0 \rangle}{\langle 0 | W_i^T Y_i | 0 \rangle^2} \quad (2.16)$$

This subtracted quantity is exactly the same as what was used in Eq. (2.12) and vanishes at leading power in the $q \rightarrow \text{soft limit}$ by Eq. (2.10). Eq. (2.15) is a sum of factorized expressions which agrees at leading power with full QCD in any soft or collinear limit of $q$. 28
To generalize to multiple gluons or quarks with momenta \( q_i \), the analogous formula is easiest to define recursively. For example, adding a second gluon to Eq. (2.15), we can either place it in the soft matrix element, or in a collinear matrix element. If it is in the collinear matrix element, we must subtract off the soft limit. Thus we get a sum of terms:

\[
\langle X_1 \cdots X_N; X_i; q_1, q_2 | \mathcal{O} | 0 \rangle \cong \langle X_1 | \hat{\psi} W_1 | 0 \rangle \cdots \langle X_N | W_N^t \psi | 0 \rangle \langle X_s, q_1, q_2 | Y_1^t \cdots Y_N | 0 \rangle \\
+ \sum_{i=1}^{N} \left\{ \frac{\langle X_i, q_1 | W_i^t \psi | 0 \rangle}{\langle 0 | W_i^t Y_i | 0 \rangle} \right\} q_1_{\text{soft}} \langle X_s, q_2 | Y_1^t \cdots Y_N | 0 \rangle \\
+ \sum_{i=1}^{N} \left\{ \frac{\langle X_i, q_2 | W_i^t \psi | 0 \rangle}{\langle 0 | W_i^t Y_i | 0 \rangle} \right\} q_2_{\text{soft}} \langle X_s, q_1 | Y_1^t \cdots Y_N | 0 \rangle \\
+ \sum_{i,j=1}^{N} \left\{ \frac{\langle X_i, q_1 | W_i^t \psi | 0 \rangle}{\langle 0 | W_i^t Y_i | 0 \rangle} \right\} q_{1_{\text{soft}}} \sum_{\text{sub}} \left\{ \frac{\langle X_j, q_2 | W_j^t \psi | 0 \rangle}{\langle 0 | W_j^t Y_j | 0 \rangle} \right\} q_{2_{\text{soft}}} \langle X_s | Y_1^t \cdots Y_N | 0 \rangle (2.17)
\]

where \( \cdots \) represent the other collinear matrix elements which do not contain any \( q_i \)’s. In the last line, when \( i = j \) the soft subtraction must be done iteratively to ensure that the subtraction mitigates the soft enhancement in any order of limits of \( q_1 \) and \( q_2 \) going soft.

That is,

\[
\frac{\langle X_j, q_1, q_2 | W_j^t \psi | 0 \rangle}{\langle 0 | W_j^t Y_j | 0 \rangle} \equiv \frac{\langle X_j, q_1, q_2 | W_j^t \psi | 0 \rangle}{\langle 0 | W_j^t Y_j | 0 \rangle} - \left( \frac{\langle X_j, q_1, q_2 | W_j^t \psi | 0 \rangle}{\langle 0 | W_j^t Y_j | 0 \rangle} S(q_1, q_2) \right) \\
- \left( \frac{\langle X_j, q_1, q_2 | W_j^t \psi | 0 \rangle}{\langle 0 | W_j^t Y_j | 0 \rangle} - \left( \frac{\langle X_j, q_1, q_2 | W_j^t \psi | 0 \rangle}{\langle 0 | W_j^t Y_j | 0 \rangle} \right) S(q_1, q_2) \right) S(q_1) \\
- \left( \frac{\langle X_j, q_1, q_2 | W_j^t \psi | 0 \rangle}{\langle 0 | W_j^t Y_j | 0 \rangle} - \left( \frac{\langle X_j, q_1, q_2 | W_j^t \psi | 0 \rangle}{\langle 0 | W_j^t Y_j | 0 \rangle} \right) S(q_1, q_2) \right) S(q_2) \right) (2.18)
\]

where \( S(q_1, q_2) \) means taking the leading-power expression in the \( q_1, q_2 \to \text{soft limit simultaneously and, therefore, does not drop } q_1 \) with respect to \( q_2 \) or vice-versa. Note that, as always, we can write the soft limits in terms of amplitudes with Wilson lines using the
factorization theorem of Eq. (2.1). For example,
\[
\frac{\langle X_j, q_1, q_2 | W^+_j \psi | 0 \rangle}{\langle 0 | W^+_j Y_j | 0 \rangle} S(q_1, q_2) = \frac{\langle X_j | W^+_j \psi | 0 \rangle}{\langle 0 | W^+_j Y_j | 0 \rangle} \frac{\langle q_1, q_2 | W^+_j Y_j | 0 \rangle}{\langle 0 | W^+_j Y_j | 0 \rangle}
\]
(2.19)
where we know that the Wilson coefficient will always be 1 to all orders by the argument given after Eq. (2.9).

That these subtractions will always work follows using the arguments of [FS2]. In particular, the “coloring algorithm” in Section 6 of that paper is exactly the recursive soft subtraction procedure indicated by Eqs. (2.12), (2.15)–(2.18). As with the algorithm in [FS2], the soft limit of any subset of the $q$’s in the \( \{ q_1 \ldots q_m \} \) matrix elements are power suppressed, and they should correspondingly be colored blue. With this knowledge, it is easy to check that Eq. (2.17) agrees in the IR: when $q_1, q_2 \to$ soft, all of the \( \{ q_i \} \) matrix elements are power suppressed and only the top line survives, which gives the correct answer. When $q_1 \parallel p_j$ and $q_2 \to$ soft, say, the bottom two lines are power suppressed and the \( S(q_1) \)-subtracted term cancels with the top line, leaving only the one term that matches the full-factorized formula in this limit. Similarly, all other limits can be simply checked. The pattern of subtractions with more than two gluons follows exactly as with the coloring algorithm stated in generality in [FS2].

The procedure outlined in this section produces amplitudes which can be computed as a sum of factorized terms. These amplitudes, which are a new result, reproduce all of the leading-power IR-sensitive limits of the full-QCD amplitudes, at all-loop order. Each factor in each term in the sum involves matrix elements of fields and Wilson lines that are universal and simpler than the factors in the full QCD amplitude. Given these properties, an interesting application of the matrix elements derived in this section might be towards subtraction procedures for QCD calculations at NNLO or beyond. One application of subtraction methods is to split an amplitude into a universal IR-sensitive piece
that is simple enough to integrate analytically and a piece that is IR-finite which could be integrated numerically [31–35]. The amplitudes presented in this section could be a candidate for such a procedure at any order in perturbation theory and for any number of external particles.

### 2.3 Factorization for distributions

Despite having many strengths, amplitudes as in Eq. (2.15), are no longer factorized: they cannot be written as a single product of terms with the same external states (in this case the collinear sectors and the soft sector are tangled). When the amplitudes are squared, the interference effects between various terms in the sum contribute at leading power, so they must all be included. Thus, while one can integrate over the momenta $q_j$ without overcounting the infrared-sensitive region, the separation between soft and collinear contributions is no longer manifest. Moreover, it is not clear how the large logarithms associated with the leading-power IR sensitivity can be resummed using such amplitudes.

Fortunately, for certain observables, one can perform subtractions differently so that factorization is preserved at the cross-section level. In this section, we discuss a class of factorizing observables. Namely, we discuss observables whose measurement function, that is, the mapping from the final-state momenta to the observable, is linear in the soft and collinear momenta. These observables include many $e^+e^-$ event shapes, such as thrust [36–39], angularities [20, 40, 41] heavy jet mass [42], the $C$ parameter [43–45] and jet broadening [46–50]. Many hadron collider observables are also in this class [51], such Drell-Yan near threshold [52], deep inelastic scattering as $x \to 1$ [53], direct photon production [54, 55], $W/Z +$ jet [56–58], jet mass [59, 60], or $t\bar{t}$ production near the hadronic threshold [61, 62] as well as $N$-(sub)jettiness [63–65].

Factorization at the cross-section level for observables in this class has been under-
stood already by traditional QCD and by Soft-Collinear Effective Theory (see above references). The overcounting of soft and collinear integration regions is also well-understood in both approaches, and the two approaches have already been shown to be equivalent [22–24]. Unfortunately, it is challenging to extract from the literature which aspects of the removal of overcounting have been rigorously proven (in either approach) and which aspects are simply assumed. Moreover, the overcounting in phase-space integrals has not been addressed at all in the effective field theory formulation with full-theory fields [16], [FS1],[FS2]. The goal of this section is to give a self-contained proof that the overcounting induced by removing phase-space cutoffs can be completely compensated for. We thereby demonstrate a form of factorization that holds exactly at leading power at the cross-section level with no phase space cutoffs.

### 2.3.1 Factorization for thrust

For concreteness and simplicity, we begin our discussion with thrust, the paradigmatic observable whose distribution factorizes. Thrust, $T$, is defined as [36]

$$T \equiv \frac{\sum_j |\vec{p}_j \cdot \vec{n}|}{\sum_j |\vec{p}_j|}$$ (2.20)

where $\vec{n}$ is the thrust axis, defined to maximize $T$. The region where factorization holds is where $\tau = 1 - T \ll 1$. Then

$$\tau \approx 1 - \sum_j \frac{1}{Q} |\vec{p}_j \cdot \vec{n}| = \frac{1}{2Q} \sum_j \Omega_\tau(p_j)$$ (2.21)

where $Q$ is the center of mass energy and $\Omega_\tau(p)$ is the measurement function for thrust:

$$\Omega_\tau(p) = p^- \theta(p^+ - p^-) + p^+ \theta(p^- - p^+)$$ (2.22)
where \( p^+ = n \cdot p \) and \( p^- = \bar{n} \cdot p \).

Note that \( \tau \) has the property that it is linear in the momenta: each particle momentum contributes additively to thrust, independent of the other momenta in the final state \( \langle X \rangle \). In particular, if we decompose \( \langle X \rangle \) into soft, collinear and hard momenta, then we can compute the contribution to thrust from each sector separately and just add the results. In other words, linearity implies

\[
\delta \left( \tau - \frac{1}{2Q} \Omega_{\tau}(p_X) \right) = \delta \left( \tau - \frac{1}{2Q} p_s - \frac{1}{2Q} p_1 - \frac{1}{2Q} p_2 - \frac{1}{2Q} p_h \right)
\]

where \( p_s \) is the sum of \( \Omega_{\tau}(k) \) over the soft momenta, \( p_1 \) and \( p_2 \) the sum over collinear momenta in each direction and \( p_h \) the sum over the remaining momenta. Writing the argument of the \( \delta \)-function as a sum lets us turn products of matrix elements into convolutions.

To be concrete let us place the momenta into sectors using hard cuts: we draw cones of angular size \( R \) around the \( \bar{n} \) and \( -\bar{n} \) collinear directions and a ball of size \( \Lambda \) around the origin; anything in the cones but not the ball is collinear, \( \langle X_j \rangle \), anything in the ball but not the cones is soft, \( \langle X_s \rangle \). For later convenience, we include the soft-collinear radiation, which is in both the ball and the a cone, in the collinear sector (we could equally well have put it in the soft sector). Anything not in the cone or ball is called hard, \( \langle X_H \rangle \). This breakdown of phase space is shown in Fig 2.1.

Consider the thrust distribution in full QCD mediated by the operator \( \mathcal{O} = \bar{\psi} \gamma^\mu \psi \). That is,

\[
\frac{d\sigma}{d\tau} = \sum_X \int d\Pi_X \langle X | \mathcal{O} | 0 \rangle^2 \delta \left( \tau - \frac{1}{2Q} \Omega_{\tau}(p_X) \right)
\]

where the sum is over all possible final states \( \langle X \rangle \) and the normalization and momentum-conserving \( \delta \)-function are left implicit.

When \( \tau \) is small, only states of the form \( \langle X \rangle = \langle X_1 \rangle \langle X_2 \rangle \langle X_s \rangle \) contribute at leading
power in $\tau$. With the hard phase-space cuts in place, the factorization formula at the amplitude level, Eq. (2.1) along with Eq. (2.23), immediately generates a factorization formula for the thrust distribution:

$$\frac{d\sigma}{d\tau} \approx H \times S^{\Lambda R} \otimes J^{R_1} \otimes J^{R_2}$$  \hspace{1cm} (2.25)

Here $H = |C|^2$ refers to the hard function (the square of the Wilson coefficient in the factorization formula). $J^{R_1}$ and $J^{R_2}$ are jet functions with restricted phase-space integrals:

$$J^{R_1}(\tau) = \sum_{X_1} \int d\Pi_{X_1} \left| \frac{\langle X_1 | \tilde{\psi} W_1 | 0 \rangle}{\langle 0 | Y_1^+ W_1 | 0 \rangle} \right|^2 \delta(\tau - p^+_{X_1})$$  \hspace{1cm} (2.26)

with $X_1$ the set of states all of whose momenta are within an angular distance $R$ of the $n^{th}$ direction. Note that we have modified the measurement function from $\Omega_\tau(p)$ in Eq. (2.22) to simply $p^+$. This is allowed since all the momenta in the cone necessarily have $p^- > p^+$ so the step functions in Eq. (2.22) can be evaluated explicitly. Analogously, the jet function, $J^{R_2}$, will have the measurement function replaced by $p^-$. Lastly, $S^{\Lambda R}$ is the phase-space
restricted soft function

\[ S^{\Lambda R}(\tau) = \sum_{X_s} \int d\Pi_{X_s} \left| \langle X_s | Y_1^\dagger \cdots Y_N | 0 \rangle \right|^2 \delta \left( \tau - \frac{1}{2Q} \Omega_\tau(p_{X_s}) \right) \]  

(2.27)

Here the states have momenta which are not collinear, that is, they are an angular distance greater than \( R \) from all jets, and they have energy less than \( \Lambda \).

The equivalence in Eq. (2.25) holds at leading power in \( \tau \) only if \( R \) and \( \Lambda \) are small enough so that the collinear radiation is collinear and the soft radiation is soft. More precisely, it holds at leading power in \( R \) and \( \Lambda \), meaning that the two sides may differ by terms of order \( R \) or order \( \Lambda \) which vanish as \( R \to 0 \) and \( \Lambda \to 0 \). Since the operators entering the soft and jet function are different, we do not expect the \( R \) dependence to cancel exactly between them; the factorization theorem only guarantees that it vanish at leading power.

We suspect it may be pedagogically useful to examine explicit expressions for \( S^{\Lambda R} \) and \( J^R \). To distinguish UV divergences from IR divergences we include an off-shellness regulator \( \omega \) for the IR (see Eq. (2.50) in the Appendix) and analytically continue to \( d = 4 - 2\epsilon \) dimensions for the UV. Some intermediate steps are relegated to Section 2.5.2 in the Appendix. For the unrenormalized jet function at finite \( R \), we find

\[ J^R_j(\tau) \cong \delta(\tau) + \frac{\alpha_s C_F}{2\pi} \left( \frac{\mu^2}{Q^2} \right)^\epsilon \left\{ \delta(\tau) \left( \frac{2}{\epsilon^2} + \frac{3}{2} + \frac{7}{2} + \frac{\pi^2}{6} \right) + \delta(\tau) \left( -\frac{2}{\epsilon} \ln \omega - 2 \ln \omega \ln R + 2 \ln^2 \omega + \mathcal{O}(R) \right) - \left( \frac{3}{2} - 2 \ln R \right) \left[ \frac{1}{\tau} \right]_+ - 2 \left[ \ln \frac{\tau}{\tau} \right]_+ \right\} \]  

(2.28)

This expression includes both the real and virtual contributions to \( \langle X_1 | \bar{\psi} W_1 | 0 \rangle \) and the purely virtual contributions to \( \langle 0 | Y_1^\dagger W_1 | 0 \rangle \) in Eq. (2.26). Note that it has \( \frac{1}{\epsilon} \) UV poles, which come from the virtual graphs. It also has an overlapping UV-IR singularity (the
\[ \ln \omega \text{ term on the second line}. \] This singularity, which cannot be removed through local counterterms, comes from loops involving the Wilson lines which go to infinite energy collinear to one of the Wilson line directions. The \( \log R \) dependence in Eq. (2.28) comes from the soft-collinear region of the restricted phase-space integral. Indeed, it cannot come from the collinear-but-not soft region, since at arbitrarily small \( \tau \), the radiation is forced arbitrarily close to the jet axis and must be a finite distance from the cone boundary. In the soft-collinear region, the radiation can be soft but an angular distance \( R \) from the axis, so there can be \( R \) dependence at leading power in \( \tau \). That the \( \ln R \) dependence comes from only the soft-collinear region is to be expected if it is to be completely canceled by the soft function.

The soft function outside the cones with finite \( \Lambda \) is

\[
S^{\Lambda R}(\tau) = \delta(\tau) + C_F \frac{\alpha_s}{\pi} \left( \frac{\mu^2}{Q^2} \right)^\epsilon \left\{ \delta(\tau) \left( -\frac{1}{\epsilon^2} - \frac{7\pi^2}{12} + \frac{2}{\epsilon} \ln \phi + 2 \ln \psi \ln R - 2 \ln^2 \phi + \mathcal{O}(R) \right) \right.
- \left[ \frac{2}{\tau} \ln R \right]_+ \theta \left( \Lambda - \frac{\tau}{R} \right)
+ C_F \frac{\alpha_s}{\pi} \left( \frac{\mu^2}{Q^2} \right)^\epsilon \left\{ \delta(\tau) \left[ -\frac{1}{\epsilon^2} - \frac{\pi^2}{4} - 2 \left( -\frac{1}{\epsilon} \ln \phi + \ln \psi \ln \frac{\Lambda}{Q} + \frac{1}{2} \ln^2 \phi \right) \right] - \left[ \frac{2}{\tau} \ln \frac{\tau Q}{\Lambda} \right]_+ \right\} \theta \left( \frac{\tau}{R} - \Lambda \right)
\]

(2.29)

This function also has an incomplete cancellation between the real and virtual contributions. In particular, the virtual includes the soft-collinear region which is excluded from the real emission. Note that the \( \Lambda \) dependence is entirely subleading power in \( \tau \): for small \( \tau \), the \( \theta \) function in the third line in Eq. (2.29) vanishes and the other \( \theta \) function evaluates to unity. The \( R \) dependence is not subleading power as \( \tau \to 0 \). There are also \( \mathcal{O}(R) \) terms not shown here but written out in Eq. (2.82). We will come back to the cancellation of the \( R \) dependence among the two jet functions and the soft function shortly.
Convolving Eq. (2.28) for each jet with Eq. (2.29) we get

\[ S^\Lambda_R \otimes J^{R_1} \otimes J^{R_2} \]

\[ \cong \delta(\tau) + C_F \frac{\alpha_s}{\pi} \left( \frac{\mu^2}{Q^2} \right)^\varepsilon \left\{ \delta(\tau) \left( \frac{1}{\varepsilon^2} + \frac{3}{2\varepsilon} + \frac{7}{2} - \frac{5\pi^2}{12} + \mathcal{O}(R) \right) - \frac{3}{2} \right\} \frac{1}{\tau} + 2 \left[ \ln \frac{\tau}{r} \right]_+ \]

Note that the $\Lambda$ dependence has dropped out completely, and the $R$ dependence which is singular as $R \to 0$ has also dropped out. It is not hard to verify that this result agrees with the full-theory result for thrust at leading power, up to the coefficient of $\delta(\tau)$ which is corrected by the hard function.

While the factorization formula for thrust in Eq. (2.25) works, it has numerous flaws. On the practical side, it is difficult to use because of the phase-space cuts. On the conceptual side, the cuts introduce additional scales into the soft and jet functions which frustrate factorization and resummation. The most serious flaw, however, is that the jet and soft functions are not individually infrared safe: they each have infrared divergences which cancel only when combined, as we saw with the explicit example above. These divergences come from an incomplete cancellation between the real-emission graphs, which have phase-space restrictions, and the virtual graphs, which do not. We could attempt to put phase-space cuts on the virtual graphs as well. However, it is more logical to try to remove the phase-space cuts from the real-emission contributions to the jet and soft functions, since this would simplify their calculation and removes the spurious scales.

First we remove $\Lambda$. This is quite simple. Only the soft function depends on $\Lambda$. By our definition, $S^\Lambda_R$ in Eq. (2.25) only integrates over the soft-but-not-collinear region of phase space. The phase-space region outside of the cones but with energy $\Lambda < E < \infty$ does not contribute at all at leading power in $\tau$. So we can simply define a new soft function by
including also this region:

\[ S^\mathcal{R} \equiv S^{\Lambda \mathcal{R}} \]  

(2.31)

where \( S^\mathcal{R} = S^{(\Lambda=\infty)\mathcal{R}} \) has no cutoff on energy in the soft function. This equivalence can be verified at order \( \alpha_s \) in Eq. (2.85), where the entire \( \Lambda \) dependence is subleading power in \( \tau \), as observed above. Explicitly,

\[
S^\mathcal{R} (\tau) = \delta (\tau) + C_F \frac{\alpha_s}{\pi} \left( \frac{\mu^2}{Q^2} \right)^\epsilon \left\{ \delta (\tau) \left[ - \frac{1}{\epsilon^2} - \frac{7\pi^2}{12} \right] - 2 \delta (\tau) \left[ - \frac{1}{\epsilon} \ln \omega - \ln \omega \ln R + \ln^2 \omega + \mathcal{O} (R) \right] - \left[ \frac{2}{\tau} \ln R \right]_+ \right\} \]  

(2.32)

Note that \( S^\mathcal{R} \) is identical to the coefficient of \( \theta (\Lambda - \frac{\epsilon}{R}) \) in Eq. (2.85). One might have imagined that taking \( \Lambda \to \infty \) would introduce new UV poles. However, radiation in \( \mathcal{R} \), say in the right hemisphere, at a given \( \tau \) must have \( k^+ = Q \tau \) and \( k^- < \frac{1}{R} k^+ = Q \frac{\tau}{R} \), so the energy \( E = \frac{1}{2} (k^+ + k^-) < \frac{Q \tau}{2} (1 + \frac{1}{R}) \) of all radiation contributing to \( S^\mathcal{R} \) at fixed \( \tau \) is in fact bounded from above so there are no new UV divergences. Thus, we now have

\[
\frac{d\sigma}{d\tau} \approx H \times S^\mathcal{R} \otimes f^{R_1} \otimes f^{R_2} \]  

(2.33)

with no \( \Lambda \) dependence on either side. Keep in mind that this equivalence is still valid only as \( R \to 0 \): there are power corrections in \( R \) on the right-hand side.

Removing the \( R \) dependence is more subtle, since the \( R \) dependence in both the soft and jet functions is relevant at leading power in \( \tau \) and since the dependence on \( R \) in both functions is singular as \( R \to 0 \). To remove it, we need a subtraction. To construct the subtraction, first recall that the general amplitude-level factorization proof in [FS2] applies to any operator, including one composed of Wilson lines. In particular, collinear
factorization for a Wilson-line operator implies

$$S \cong S^R \otimes J^R_{\text{eik}} \otimes J^R_{\text{eik}}$$

where $S = S^{\infty}$ has no angular or energy restriction and the **eikonal jet function** is defined as

$$J_{\text{eik}}^R_{X_j}(\tau) = \sum_{X_j} \int d\Pi_{X_j} \left| \frac{\langle X_1 | Y_{\dagger} W_1 | 0 \rangle}{\langle 0 | Y_{\dagger} W_1 | 0 \rangle} \right|^2 \delta(\tau - p_{X_j}^+ )$$

The eikonal jet function differs from the jet function in Eq. (2.26) in that $Y_{\dagger}$ replaces the field $\bar{\psi}$. Note that the measurement function in the eikonal jet function is the power-expanded version, $\delta(p - p^+)$, rather than $\Omega_\tau(p)$. This is consistent with Eq. (2.34) since the phase space in the eikonal jet function is restricted to be in a cone.

Explicitly, to order $\alpha_s$, we find

$$J_{\text{eik}}^R_{X_j}(\tau) = \delta(k) + \frac{\alpha_s C_F}{2\pi} \left( \frac{\mu^2}{Q^2} \right)^{\epsilon} \left\{ \delta(\tau) \left[ \frac{2\pi^2}{3} - \frac{2}{\epsilon} \ln \omega - 2 \ln \omega \ln R + 2 \ln^2 \omega + \mathcal{O}(R) \right] \right. + \left. \left( \frac{2}{\epsilon} + 2 \ln R \right) \left[ \frac{1}{\tau} \right] + 4 \left[ \frac{\ln \tau}{\tau} \right] \right\}$$

Comparing Eq. (2.36) to Eq. (2.28), we see that the $\omega$ and $R$ dependence in $J_{\text{eik}}^R_{X_j}$ is the same as that in $J^R_{X_j}$. This is expected, since the only IR-sensitive difference between the two is in the collinear-but-not-soft region of $\langle X_1 | Y_{\dagger} W_1 | 0 \rangle$ and $\langle X_1 | \bar{\psi} W_1 | 0 \rangle$. In this region, there is a complete cancellation of real and virtual graph for both functions, hence both are IR-finite. Note also that there are no $\frac{1}{\epsilon^2}$ poles in the eikonal jet function. These double UV poles in the regular jet function come from virtual graphs. In the eikonal jet function, the virtual graphs in the numerator and denominator of Eq. (2.35) are identical and hence cancel in the ratio to order $\alpha_s$. The lack of $\frac{1}{\epsilon}$ poles also implies that there are no Sudakov double logs in the eikonal jet function.
Now, if we convolve both sides of Eq. (2.25) with the eikonal jet functions and use Eq. (2.31) and Eq. (2.34), we get

\[
\frac{d\sigma}{d\tau} \otimes J_{\text{eik}}^{R_1} \otimes J_{\text{eik}}^{R_2} \approx H \times S \otimes J^{R_1} \otimes J^{R_2}
\]  

(2.37)

At this point, no object in this leading-power equivalence depends on \(\Lambda\) and the \(R\) dependence on both sides is only in the jet functions and eikonal jet functions. We still must have \(R\) small though, since there are power corrections in \(R\) on both sides.

Finally, we want to remove the \(R\)-dependence completely. Let us call a jet function with no restriction on \(R\) an inclusive jet function and denote it by \(J^I\). Removing the \(R\) introduces additional unphysical singularities collinear to the Wilson-line direction \(t_j\) which are not regulated with the off-shellness regulator. We must introduced another regulator for these singularities, so we use the \(\Delta\)-regulator \([56], \delta_j\), as shown in Eq. (2.51). To order \(\alpha_s\) we find for the inclusive jet function

\[
J^I(\tau) = \delta(\tau) + \frac{\alpha_s C_F}{2\pi} \left( \frac{\mu^2}{Q^2} \right)^\varepsilon \left\{ \delta(\tau) \left( \frac{2}{\varepsilon^2} + \frac{3}{2\varepsilon} + \frac{7}{2} - \frac{\pi^2}{6} \right) + \delta(\tau) \left( -\frac{2}{\varepsilon} \ln \omega + 2 \ln \omega \ln \delta_j + \ln^2 \omega \right) - \left( 2 \ln \delta_j + \frac{3}{2} \right) \left[ \frac{1}{\tau} \right]_+ \right\} 
\]

(2.38)

Similarly, for the inclusive eikonal jet function we find

\[
J_{\text{eik}}^I(\tau) = \delta(\tau) + \frac{\alpha_s C_F}{2\pi} \left( \frac{\mu^2}{Q^2} \right)^\varepsilon \left\{ \delta(\tau) \left( \frac{\pi^2}{3} - \frac{2}{\varepsilon} \ln \omega + 2 \ln \omega \ln \delta_j + \ln^2 \omega \right) + \left( \frac{2}{\varepsilon} - 2 \ln \delta_j \right) \left[ \frac{1}{\tau} \right]_+ - 2 \left[ \frac{\ln \tau}{\tau} \right]_+ \right\} 
\]

(2.39)

Note that the \(\delta_j\) dependence associated with the Wilson line direction is identical in the two inclusive jet functions.

Next, note that since \(\overline{R}_j\) does not contain the jet direction the only leading-power con-
tributions to the jet function from this region are soft.\footnote{One might be concerned about collinear singularities associated with the Wilson-line direction $t_j^\mu$. However, since the measurement function forces $p \cdot n = \tau$, at small $\tau$ radiation cannot be collinear to both $t_j^\mu$ and the jet direction $n^\mu$. Thus, radiation collinear to $t_j^\mu$ cannot contribute at leading power.} Thus, we can apply the general amplitude-level factorization theorem to the operator $\hat{\psi} W_j$ to get

$$J^j \simeq J^{R_j} \otimes J^{K_j}_{\text{eik}}$$  \hspace{1cm} (2.40)

This equation can be verified at 1-loop by comparing Eq. (2.38) with the combination of Eqs. (2.28), (2.36) and (2.39). Similarly,

$$J^j_{\text{eik}} \simeq J^{R_j}_{\text{eik}} \otimes J^{K_j}_{\text{eik}}$$  \hspace{1cm} (2.41)

Therefore, convolving both sides of Eq. (2.37) with $J^{R_j}_{\text{eik}}$ and $J^{R_j}_{\text{eik}}$ gives

$$\frac{d\sigma}{d\tau} \otimes J^{1}_{\text{eik}} \otimes J^{2}_{\text{eik}} \simeq H \times S \otimes J^1 \otimes J^2$$  \hspace{1cm} (2.42)

In this final form, all the dependence on $\Lambda$ or $R$ has been explicitly removed.

Finally, we want to isolate $\frac{d\sigma}{d\tau}$ from Eq. (2.42). To do this, we use that convolutions map to products in Laplace space. Taking the Laplace transform, Eq. (2.42) translates to

$$\int d\tau \frac{d\sigma}{d\tau} e^{-\nu \tau} \simeq \frac{H \tilde{S}(v) \tilde{J}^1(v) \tilde{J}^2(v)}{\tilde{J}^1_{\text{eik}}(v) \tilde{J}^2_{\text{eik}}(v)}$$  \hspace{1cm} (2.43)

This form is in agreement with previously known expressions in the literature [20,22].

### 2.3.2 Jet broadening

The above discussion shows how the phase-space cutoffs separating collinear and soft radiation as well as the UV phase-space cutoff can be removed in the factorization for-
mula for a particular observable (thrust). The derivation easily generalizes to many other observables. The key general property that was used is that the vanishing limit of the observable forces the phase space into the $N$-jet configuration at leading power. This allows the factorization theorem in Eq. (2.1) to be used to factorize the matrix-element squared in the full distribution. It also ensures that the dependence on the phase-space cutoffs is power suppressed, once the eikonal jet functions are included. For observables whose measurement function is not linear in each sector, the integrals will not be a simple convolution.

For a marginally different example, consider jet broadening [46–50]. (Total) jet broadening acting on a state $|X\rangle$ with particles of momenta $p_j^\mu$ has the eigenvalue

$$b(X) = \frac{1}{2Q} \sum_j \Omega_b(p_j), \quad \Omega_b(p) = |\vec{p}_\perp|$$

(2.44)

where $\vec{p}_\perp$ are the components of the 3-momenta of the particles transverse to the thrust axis.

In the SCET literature, jet broadening is considered a SCET_{II} observable because soft emissions which are hard enough to recoil against collinear emissions contribute to jet broadening at leading power, while they are subleading power for thrust. More explicitly, for thrust only the small component of momentum $p^+ = n \cdot p$ contributes (for particles going in the $n$ hemisphere). Thus collinear momenta, with $(p^-, p^+, p_\perp) \sim Q(1, \lambda, \lambda^2)$ and soft momenta with $p \sim Q\lambda^2$ contribute at the same order. Soft momenta scaling like $p \sim Q\lambda$ give a power-suppressed contribution to thrust. For jet broadening, $p_\perp$ is measured. So collinear momenta contribute $p_\perp \sim Q\lambda$ and therefore soft momenta scaling like $p \sim Q\lambda$ are relevant at leading power making jet broadening a SCET_{II} observable.

From the point of view of the factorization as set up in [FS1] and [FS2], the soft scaling is unrelated to the collinear scaling. That is, the amplitude-level factorization for-
mula, Eq. (2.1) holds for any relationship between the soft-scaling parameter, $\lambda_s$, and the collinear-scaling parameter, $\lambda_c$. SCET$_I$ corresponds to $\lambda_s = \lambda_c^2$ and SCET$_{II}$ to $\lambda = \lambda_c$. The relevant implication of the soft and collinear momenta having $p_\perp$ components of the same order is that configurations where soft particles recoil against collinear particles must be accounted for in the factorization theorem. The result is that the factorization formula has the form

$$\frac{d\sigma}{db} = H \int db_s db_1 db_2 d^2p^+_1 d^2p^+_2 J^1(b_1, \bar{p}^+_1) f^2(b_2, \bar{p}^+_2) S(b_s, -\bar{p}^+_1, -\bar{p}^+_2) \delta(b - b_s - b_1 - b_2)$$

(2.45)

We can write this heuristically as

$$\frac{d\sigma}{db} \approx H \times J^1 \otimes f^2 \otimes S$$

(2.46)

with the understanding that $\otimes$ for jet broadening refers to the double convolution in Eq. (2.45).

With phase-space restrictions, these jet functions are given by

$$J^{R_1}(b, \bar{p}^\perp) = \sum_{X_1} d\Pi_{X_1} \left| \frac{\left< X_1 | \bar{\psi} W_1 | 0 \right>}{\left< 0 | Y_1^\dagger W_1 | 0 \right>} \right|^2 \delta \left( b - b(X_1) \right) \delta \left( Q - p^+_X \right) \delta (\bar{p}^\perp - \bar{p}^\perp_{X_1})$$

(2.47)

and the soft function by

$$S^{^R}(b, \bar{p}^1, \bar{p}^2) = \sum_{X_s} d\Pi_{X_s} \left| \frac{\left< X_s | Y_1^\dagger Y_2 | 0 \right>}{\left< 0 | Y_1^\dagger Y_2 | 0 \right>} \right|^2 \delta \left( b - b(X_s) \right) \left[ \delta (\bar{p}^1_{X_s} - \bar{p}^1_{X_s}) + \delta (\bar{p}^2_{X_s} - \bar{p}^2_{X_s}) \right]$$

(2.48)

where $\bar{p}^1_{X_s}$ is the net $\perp$ momenta in the left hemisphere and $\bar{p}^2_{X_s}$ is the net $\perp$ momenta in the right hemisphere. As with thrust, these phase-space restricted functions will have overlapping UV-IR divergences and unwieldy dependence on the cutoffs $R$ and $\Lambda$. How-
ever, as with thrust, we can convolve both sides of Eq. (2.46) with eikonal jet functions to get a factorization formula with only objects with no phase space cutoffs. The result in Laplace space is

\[
\int db \frac{d\sigma}{db} e^{-vb} \cong H \int d^2x_L^+ d^2x_R^+ \frac{\bar{f}(x_L^+, v) \bar{f}(x_R^+, v) \bar{S}(x_L^+, x_R^+, v)}{\bar{f}_{eik}(x_L^+, v) \bar{f}_{eik}(x_R^+, v)}
\]

(2.49)

Here, \( v \) is the Laplace-conjugate variable to \( b \) and \( x_1^+ \) and \( x_2^+ \) are the Laplace conjugate variables to \( p_1^+ \) and \( p_2^+ \) respectively.

### 2.3.3 Comparison to other approaches

We have seen how phase-space cutoffs can be removed for certain inclusive observables if the double counting in the soft-collinear region is removed with an eikonal jet function. In this section, we would like to emphasize some conceptual differences in our derivation and previous ones and contrast with the literature.

First of all, it is easy to compare our results to those in traditional QCD, where the eikonal jet function first appeared. The final factorization formulas are identical. One difference is that in the early literature the eikonal jet functions were subtracted from the soft function rather than the jet function. From the point of view of the final formula, there is no difference. However, conceptually our analysis makes it clear that the eikonal jet function should be subtracted from the jet function rather than the soft function. Indeed, the soft function is by itself infrared finite while the naive inclusive jet function (without the subtraction) is not. As shown explicitly in Eq. (2.38), the infrared divergences do not cancel between the real and virtual graphs for the jet function. With the subtraction the jet function is a well-defined and infrared safe object. The fact that the subtraction is more naturally applied in the collinear sector was also appreciated in [22].

The comparison to SCET is perhaps more illuminating than the comparison to tra-
ditional QCD. In the early days of SCET, calculations were mostly done in dimensional regularization (DR) and the overlapping of soft and collinear phase-space regions were not much discussed. In retrospect, it is easy to see why the correct answers result in DR without a subtraction: the eikonal jet functions, as in Eq. (2.35), give scaleless integrals in DR and thus formally vanish and can be ignored.

It is natural to be somewhat uncomfortable with setting scaleless but IR and UV divergent integrals to zero in DR. The mathematical justification notwithstanding, it is dangerous from a practical point of view if one hopes to extract an anomalous dimension from the poles in the jet and soft functions at $d = 4$. The only way it will work is if the object one computes is infrared finite. For infrared finite objects, all the poles are by definition UV. So setting $\frac{1}{\epsilon_{\text{UV}}} - \frac{1}{\epsilon_{\text{IR}}} = 0$ has no effect. As we have shown, the subtracted inclusive jet function is IR finite, so practically, one can ignore the subtraction in DR. Morally, though, to do this one must be able to show that the jet function is IR finite. Without the subtraction it is not. In this respect, the success of SCET in pure DR was somewhat accidental.

The missing subtractions were understood in the classic paper on zero-bin subtraction by Manohar and Stewart [21]. These authors showed the the proper derivation of the effective Lagrangian for SCET involves binning the momenta into collinear momenta in different directions and soft momenta. The zero bin in each collinear sector should be formally excluded. In [21] it was shown that this exclusion amounts to the subtraction diagram-by-diagram of the soft-limit of the collinear momenta. In [22–24], this subtraction procedure was shown to be equivalent to the eikonal jet function subtraction of traditional QCD.

A somewhat different perspective comes from the method-of-regions multipole-expansion approach to SCET [25–27], as recently reviewed in [28]. In this approach, the soft-collinear subtraction and the extension of the soft integrals to $\Lambda = \infty$ are not discussed or needed. The basis of the argument is that the non-analytic dependence on the observable in each
region is independent of the possible phase-space cutoffs. Since all the physics is in this non-analytic dependence, one can remove the cutoffs without consequence. For more details, see [25, 28].
2.4 Conclusion

In this paper we have presented two new results. First, we have given a recursive formula for constructing amplitudes which agree with full QCD at leading power to all-loop order in any soft and collinear limit. The subtracted amplitudes we describe are matrix elements of fields and Wilson lines. Unlike with the amplitude-level factorization formula in [FS2], one does not have to specify whether the particles are soft or collinear ahead of time: the subtracted matrix elements will be correct in any limit. Although the amplitudes appear simpler than in full QCD (for example, the only interference effects from different directions involve gluons emitted off Wilson lines), it remains to be seen whether they can be integrated simply to provide a productive subtraction scheme. In our derivation of this formula, extensive use was made of the proof of factorization in [FS2].

Second, we showed how phase-space cutoffs can be removed when integrating a factorized amplitude squared against the measurement function for certain inclusive observables. Removing the cutoffs does two things: it overcounts the soft-collinear region and adds UV divergences to the phase-space integrals. These two effects can be compensated for by integrating the full QCD distribution against an eikonal jet function. This convolution can be easily disentangled, at least for thrust, jet broadening and angularities. This extends the results of amplitude-level factorization from [FS1] and [FS2] to the level of observables.

In our presentation, we have included explicit 1-loop expressions for soft and jet functions with cutoffs and for the eikonal jet function in a regularization scheme which separates UV from IR. These expressions confirm generally the qualitative analyses that we have presented of the UV and IR structure of the integrals.

Although our final factorization formulas are not new, we believe our derivation is systematic and rigorous. We hope that the step-by-step procedure we have presented
will be useful in future studies of factorization, where subtleties abound.

2.5 Appendix

In this appendix, we report the explicit computations of many of the cut-off dependent objects that appear in Section 2.3.1. All expressions are computed at one-loop order in QCD for the observable thrust.

2.5.1 Regularization schemes

We will use dimensional regularization to control the UV divergences: we analytically continue to \( d = 4 - 2\epsilon \) dimensions, with \( \epsilon > 0 \).

To regulate the IR divergences, we take the outgoing external fermion lines to have an offshellness of \( Q^2 \omega \). Consequently, the propagator in \( Y_j \) will look like

\[
\frac{n_j^\mu}{n_j \cdot k} \rightarrow \frac{p_j^\mu}{p_j \cdot k + \frac{Q^2 \omega}{2}}
\]

(2.50)

It is helpful when using an offshellness also to slightly modify the measurement function for thrust, so that virtual graphs still contribute only at \( \tau = 0 \). We can do this by replacing \( \delta(\tau - \frac{1}{Q}p^-) \rightarrow \delta(\tau - \frac{1}{Q}p^- + \omega) \).

In scaleless integrals involving Wilson lines, the offshellness may not completely control all the IR divergences. Thus, in addition we use the \( \Delta \) regulator [66] for the collinear Wilson line, so that the propagator in \( W_j \) will be shifted by \( \frac{\Delta}{t_j \cdot p_j} \). More precisely, we define the dimensionless parameter \( \delta_j \equiv \Delta / (t_j \cdot p_j) \) and shift the eikonal propagators as

\[
\frac{1}{t_j \cdot k} \rightarrow \frac{1}{t_j \cdot k + \frac{\Delta}{t_j \cdot p_j}} \equiv \frac{1}{t_j \cdot k + \delta_j (t_j \cdot p_j)}
\]

(2.51)
With these two IR regulators, all the $1/\varepsilon$ poles in the following expressions correspond to UV divergences only.

For simplicity, in the following sections we choose the direction of the collinear Wilson line, $W_1$ to be $t_1^\mu = n_1^\mu = (1, -\vec{n})$ and that of $W_2$ to be $t_2^\mu = n_2^\mu = (1, \vec{n})$, where $\vec{n}$ is the thrust axis. Then, in light-cone coordinates

$$k^\mu = \frac{1}{2} k^- n_1^\mu + \frac{1}{2} k^+ n_2^\mu + k_\perp^\mu = (k^-, k^+, k_\perp) \quad (2.52)$$

While it is not impossible to do the integrals for generic choices of $t_1^\mu$ and $t_2^\mu$, the phase space integrals inside the cone become significantly more complicated.

### 2.5.2 Jet Functions

We will denote a jet function restricted to a cone of size $R$ by $J^R$. In our notation, $R$ is a rapidity-type cone angle: a particle is in the $n$-cone if $k^+ < R k^-$. $R$ is related to the cone opening angle $\theta$ by $R = \tan^2 \frac{\theta}{2}$.

At order $\alpha_s$, the virtual contributions to the jet functions are of course independent of phase space restrictions and therefore the same for the restricted or unrestricted jet functions. The purely virtual contributions come from evaluating

$$\left[J(\tau)\right]_{\text{virt.}} = \int \frac{dp^+}{2\pi} \left| \frac{\langle p | \bar{\psi} W_j | 0 \rangle}{\langle 0 | Y_j^+ W_j | 0 \rangle} \right|^2 2\pi \delta \left( Q p^+ - Q^2 \omega \right) \theta(p^+) \delta \left( \tau - \frac{p^+}{Q} + \omega \right)$$

$$= \delta(\tau) \frac{1}{Q} \left| \frac{\langle p | \bar{\psi} W_j | 0 \rangle}{\langle 0 | Y_j^+ W_j | 0 \rangle} \right|^2 = \delta(\tau) + \mathcal{O}(\alpha_s) \quad (2.53)$$

where the sum over spins is implicit.

There are 3 virtual graphs contributing to $\langle p_j | \bar{\psi} W_j | 0 \rangle$ at 1-loop. The self-energy graph on the Wilson line exactly vanishes; the vertex correction and quark self-energy graph
sum to

$$\frac{\langle p | \bar{\psi} W_j | 0 \rangle_{\text{virt.}}}{\langle p | \bar{\psi} W_j | 0 \rangle_{\text{tree}}} = \delta_s^2 \mu^{2 \epsilon} C_F \int \frac{d^d k}{(2\pi)^d} \left( \frac{2(p^- - k^-)}{k^+ + p^- \delta_j} + \frac{d - 2p^+ - k^+}{p^+} \right) \frac{1}{k^2} \frac{1}{(p^- - k^-)(p^+ - k^+) - k^2_\perp}$$

$$= -\frac{\alpha_s C_F}{4\pi} \left( \frac{4\pi \mu^2}{Q^2} \right)^\epsilon (-\omega)^{-\epsilon} \Gamma(\epsilon) \frac{\Gamma(1 - \epsilon) \Gamma(2 - \epsilon)}{\Gamma(2 - 2\epsilon)}$$

$$\times \left[ -(3 + \epsilon) + 4 \left( 1 + \frac{1}{\delta_j} \right) \right]$$

$$= -\frac{\alpha_s C_F}{4\pi} \left( \frac{\bar{\mu}^2}{Q^2} \right)^\epsilon \left[ \frac{3}{2\epsilon} - \frac{3}{2\epsilon} \ln \omega + \frac{2}{\epsilon} \ln \delta_j - 2 \ln \omega \ln \delta_j - \ln^2 \delta_j + \frac{7}{2} - \frac{2\pi^2}{3} \right]$$

(2.55)

In the last step, we expanded in $\delta_j$ and $\omega$ and dropped the $O(\delta_j)$ and $O(\omega)$ terms.

To one-loop, the denominator factor in the jet function evaluates to

$$\bar{Z}_j = \langle 0 | \bar{\psi} W_j | 0 \rangle = 1 + \delta_s^2 \mu^{2 \epsilon} C_F \int \frac{d^d k}{(2\pi)^d} 2 \left( \frac{1}{-k^+ + Q \omega k^- + Q \delta_j} \right) \frac{1}{k^2}$$

$$= -\frac{\alpha_s C_F}{4\pi} \left( \frac{4\pi \mu^2}{Q^2} \right)^\epsilon (-\omega)^{-\epsilon} \Gamma(\epsilon) \left[ 2 \delta_j^{-\epsilon} \Gamma(\epsilon)(1 - \epsilon) \right]$$

$$= -\frac{\alpha_s C_F}{4\pi} \left( \frac{\bar{\mu}^2}{Q^2} \right)^\epsilon \left[ -\frac{2}{\epsilon^2} + \frac{2}{\epsilon} \ln \omega - \ln^2 \omega + \frac{2}{\epsilon} \ln \delta_j - 2 \ln \omega \ln \delta_j - \ln^2 \delta_j - \frac{\pi^2}{2} \right]$$

(2.56)

We note here that the $\delta_j$ dependence is identical in Eqs. (2.55) and (2.56). Thus virtual
IR divergences introduced by the $W_j$, which are regulated with the $\Delta$-regulator, cancel in
$[J(\tau)]_{\text{virt.}}$ to one loop. Explicitly,

$$\left[ J(\tau) \right]_{\text{virt.}} = \delta(\tau) - \delta(\tau) \frac{\alpha_s C_F}{2\pi} \left( \frac{\bar{\mu}^2}{Q^2} \right)^\epsilon \left[ \frac{2}{\epsilon^2} + \frac{3}{2\epsilon} - \frac{2}{\epsilon} \ln \omega - \frac{3}{2} \ln \omega + \ln^2 \omega + \frac{7}{2} - \frac{\pi^2}{6} \right]$$

(2.57)

Now let’s look at the real-emission diagrams. With emissions restricted to a cone of
size $R$, the real-emission contributions to the jet function at order $\alpha_s$ are
\[
\left[ J^R_j(\tau) \right]_{\text{real}} = 2g_s^2 \mu^2 C_F \left( \frac{d}{2\pi} \right) \delta(p^- - Q) \\
\times \int \frac{d^d k}{(2\pi)^d} \frac{1}{p^+} \left( \frac{2(p^- - k^-)}{k^+ + p^-} + \frac{d - 2}{2} \frac{p^+ - k^+}{p^+} \right) 2\pi \delta(k^+ k^- - k_{+}^2) \theta(k^+) \theta(k^-) \\
\times 2\pi \delta((p^+ - k^+)(p^- - k^-) - k_{+}^2 - Q^2 \omega) \theta(p^+ - k^+) \theta(p^- - k^-) \delta(\tau - p^+ / Q + \omega) \\
\times \theta \left( \frac{k^+}{k^+ - R} \right) \theta \left( \frac{p^- - k^+}{p^+ - k^+ - R} \right)
\]

(2.58)

It is helpful to write the result as

\[
\left[ J^R_j(\tau) \right]_{\text{real}} = \frac{\alpha_s C_F}{2\pi} \frac{1}{\Gamma(1 - \varepsilon)} \left( \frac{4\pi \mu^2}{Q^2} \right)^\varepsilon \frac{1}{\tau + \omega} \frac{1}{\tau^\varepsilon} J^R_j(\tau)
\]

(2.59)

and we find

\[
J^R_j(\tau) = \int_{\tau/R}^1 dx \left[ \frac{2(1 - x)}{x + \delta_j} + (1 - \varepsilon)x \right] \left[ \frac{1}{x(1 - x)} \right]^{\varepsilon}
= \mathcal{I} - \frac{2(1 + \delta_j)}{(1 - \varepsilon)\delta_j} \left( \frac{\tau}{R} \right)^{1-\varepsilon} F_1 \left( 1 - \varepsilon, \varepsilon, 1; 2 - \varepsilon; \tau \frac{R'}{R} - \frac{\tau}{R} \delta_j \right) \\
+ \frac{1}{2} \left[ \frac{\tau}{R} \left( 1 - \frac{\tau}{R} \right)^{-1-\varepsilon} + \frac{1}{2}(3 + \varepsilon) B \left( \frac{R'}{R}; 1 - \varepsilon, 1 - \varepsilon \right) \right]
= \left( -\frac{3}{2} - 2 \ln \left( \frac{\tau}{R} + \delta_j \right) + \mathcal{O}(\tau) \right) + \mathcal{O}(\varepsilon)
\]

(2.60)

where \( F_1(a, \beta, \beta'; \gamma; x, y) \) is the Appell hypergeometric function and \( B(z; a, b) \) is the Incomplete Beta Function. These real emission graphs are of course UV finite, so we can simply set \( \varepsilon = 0 \).

To expand the result for small \( \omega \), we can use that in the limit \( \omega \to 0 \)

\[
\frac{1}{\tau + \omega} J(\tau) = \delta(\tau) \left[ \int_0^1 \frac{d\tau'}{\tau' + \omega} J(\tau') \right] + \left[ \frac{J(\tau)}{\tau} \right]_+
\]

(2.61)

so that
\[
\left[ J^R_j(\tau) \right]_{\text{real}} = \frac{\alpha_s C_F}{2\pi} \left\{ \delta(\tau) \left[ \frac{3}{2} \ln \omega - 2 \ln \omega \ln R + \ln^2 \omega + \mathcal{O}(R) \right] + \left[ \frac{1}{\tau} \left( -2 \ln \frac{\tau}{R} - \frac{3}{2} \right) \right]_+ \right\} 
\]

(2.62)

Eq. (2.62) shows that the jet function in cone does not have \( \log \delta_j \) singularity. Note, however, that this jet function does have double logs of \( \omega \), coming from soft-collinear region; these will cancel against the virtual soft-collinear singularity in Eq. (2.56). The \( R \) dependent terms \( \log \omega \log R \) come from soft emissions at the cone edge. These do not cancel against any other contribution to \( J^R \), but will cancel against either \( S^R \) of the eikonal jet function \( J^R_{\text{eik}} \).

Next, consider the inclusive jet function, \( J^I \). This is defined identically to \( J^R \), but without the phase space restriction. The virtual contributions are the same. The real emission contributions at order \( \alpha_s \) are the same without the phase space restriction on the last line of Eq. (2.58). Using the same notation as in Eq. (2.59), we find

\[
I^j = \int_0^1 dx \left[ \frac{2(1-x)}{x+\delta_j} + (1-\varepsilon)x \right] \left[ \frac{1}{x(1-x)} \right]^{\varepsilon} 
= \frac{4^{1+\varepsilon}\sqrt{\pi\Gamma(1-\varepsilon)}}{\delta_j \Gamma(\frac{3}{2}-\varepsilon)} \left[ -(3+\varepsilon)\delta_j + 4(1+\delta_j) \right] \frac{1}{\Gamma(1-\varepsilon, 2-2\varepsilon, -\frac{1}{\delta_j})} 
= -\frac{3}{2} + 2(1+\delta_j) \ln(1+\frac{1}{\delta_j}) 
\]

(2.63)

(2.64)

We have set \( \varepsilon = 0 \) on the last line since the real emission contribution to the inclusive jet function, like with \( J^R \), is UV finite. Using Eq. (2.61) we then find

\[
\left[ J^I(\tau) \right]_{\text{real}} = \frac{\alpha_s C_F}{2\pi} \left\{ \delta(\tau) \left[ \frac{3}{2} \ln \omega + 2 \ln \omega \ln \delta_j \right] + \left[ \frac{1}{\tau} \left( -2 \ln \delta_j - \frac{3}{2} \right) \right]_+ \right\} 
\]

(2.65)

The phase space integral in the inclusive jet function contains single log of \( \omega \), indicating pure collinear singularity. It also contains a double IR singularity, \( \log \omega \log \delta_j \), coming
from the soft-collinear region.

Next, we consider the eikonal jet function, with the field $\bar{\psi}$ replaced by a soft Wilson line. The virtual contributions are given by matrix elements similar to Eq. (2.53):

$$
\left[ J_{\text{eik}}(\tau) \right]_{\text{virt.}} = \int \frac{dp^+}{2\pi} \frac{\langle 0 | Y_1 W_1 | 0 \rangle^2}{\langle 0 | Y_1^+ W_1 | 0 \rangle} 2\pi \delta \left( p^+ - Q \omega \right) \theta(p^+) \delta(\tau - \frac{p^+}{Q} + \omega) \quad (2.66)
$$

$$
= \delta(\tau) \quad (2.67)
$$

Since there is an exact cancellation between numerator and denominator, the purely virtual contribution is $\delta(\tau)$ to all orders.

Using the same notation as above, for the phase-space restricted eikonal jet function, we find

$$
T_{\text{eik}}^R(\tau) = \int_{\tau/R}^\infty dx \frac{2}{x + \delta_j} \left[ \frac{1}{x} \right]^\varepsilon = \delta_j^{-\varepsilon} B \left( \frac{R \delta_j}{\tau} ; \varepsilon, 0 \right) = \frac{2}{\varepsilon} - 2 \ln \left( \frac{\tau + \delta_j}{R} \right) + \mathcal{O}(\varepsilon) \quad (2.68)
$$

Note that now there is a UV divergence from the integral up to infinite energy in the $k^-$ direction. This turns into

$$
\left[ J_{\text{eik}}^R(\tau) \right]_{\text{real}} = \frac{\alpha_s C_F}{2\pi} \left( \frac{\mu^2}{Q^2} \right)^\varepsilon
\times \left\{ \delta(\tau) \left[ \frac{2\pi^2}{3} - \frac{2}{\varepsilon} \ln \omega - 2 \ln \omega \ln R + 2 \ln^2 \omega + \mathcal{O}(R) \right] + \left[ \frac{1}{\tau} \left( \frac{2}{\varepsilon} + 2 \ln R - 4 \ln \tau \right) \right]_+ \right\} \quad (2.69)
$$

For the unrestricted eikonal jet function

$$
T_{\text{eik}}^i = \int_0^\infty dx \frac{2}{x + \delta_j} \left[ \frac{1}{x} \right]^\varepsilon = \delta_j^{-\varepsilon} 2\Gamma(\varepsilon) \Gamma(1 - \varepsilon) \quad (2.70)
$$
and

\[
\left[ J_{\text{eik}}^j(\tau) \right]_{\text{real}} = \frac{\alpha_s C_F}{2\pi} \left( \frac{\hat{\mu}^2}{Q^2} \right)^\varepsilon \times \left\{ \delta(\tau) \left( \frac{\pi^2}{3} - \frac{2}{\varepsilon} \ln \omega + \ln \delta_j + \ln^2 \omega \right) + \left[ \frac{1}{\tau} \left( \frac{2}{\varepsilon} - 2 \ln \delta_j - 2 \ln \tau \right) \right]_+ \right\} \tag{2.71}
\]

Finally, the jet functions with real and virtual contributions combined are given in Eqs. (2.28), (2.36), (2.38) and (2.39) for \( j^R, J_{\text{eik}}, j^j \) and \( J_{\text{eik}}^j \) respectively. The structure of IR singularities in virtual and real contributions to the various objects are listed in Table 2.1.

### 2.5.3 Soft functions

Here we focus on the soft function for thrust, defined as

\[
S(\tau) = \sum_{X_s} \int d\Pi_{X_s} \left| \langle X_s | Y_1^\dagger \cdots Y_N | 0 \rangle \right|^2 \delta \left( \tau - \frac{1}{2Q} \Omega_\tau(p_{X_s}) \right) \tag{2.72}
\]

with \( \Omega_\tau(p) \) the measurement function for thrust defined in Eq. (2.22) and \( X_s \) the appropriately phase-space restricted set of states.

The virtual contribution to the soft function comes from the matrix element \( \langle 0 | Y_1^\dagger Y_2 | 0 \rangle \) and is independent of phase-space restrictions. To order \( \alpha_s \),

\[
\langle 0 | Y_1^\dagger Y_2 | 0 \rangle = 1 + g_s^2 \mu^{2\varepsilon} C_F \int \frac{d^d k}{(2\pi)^d} 2 \left( \frac{1}{-k^+ + Q(k^- + Q\omega)} \right) \frac{1}{k^2} \tag{2.73}
\]

so that

\[
[S(\tau)]_{\text{virt.}} = -\delta(\tau) \frac{\alpha_s C_F}{\pi} \frac{1}{\Gamma(1-\varepsilon)} \left( \frac{4\pi \mu^2}{Q^2} \right)^\varepsilon (-\omega)^{-\varepsilon}(\omega)^{-\varepsilon} \left( \frac{\Gamma(\varepsilon)}{\Gamma(1-\varepsilon)} \right)^2 \tag{2.74}
\]

\[
= \delta(\tau) \frac{\alpha_s C_F}{\pi} \left( \frac{\hat{\mu}^2}{Q^2} \right)^\varepsilon \left( -\frac{1}{\varepsilon^2} + \frac{2}{\varepsilon} \ln \omega - \frac{\pi^2}{4} - 2 \ln^2 \omega \right) \tag{2.75}
\]
<table>
<thead>
<tr>
<th>collinear or soft</th>
<th>soft-collinear</th>
<th>UV-IR</th>
<th>UV</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>ln $\omega$</td>
<td>ln $\omega \log R$</td>
<td>ln $\omega \ln \delta_j$</td>
</tr>
<tr>
<td>$\langle p_j \mid \slashed{p} W_j \mid 0 \rangle_{\text{virt.}}$</td>
<td>$\checkmark$</td>
<td>-</td>
<td>$\checkmark$</td>
</tr>
<tr>
<td>$\hat{Z} = \langle 0 \mid Y_j W_j \mid 0 \rangle$</td>
<td>-</td>
<td>-</td>
<td>$\checkmark$</td>
</tr>
<tr>
<td>$[J]_{\text{virt}}$</td>
<td>$\checkmark$</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$[J^R_j]_{\text{real}}$</td>
<td>$\checkmark$</td>
<td>$\checkmark$</td>
<td>-</td>
</tr>
<tr>
<td>$[J^i]_{\text{real}}$</td>
<td>$\checkmark$</td>
<td>-</td>
<td>$\checkmark$</td>
</tr>
<tr>
<td>$J^R_j$</td>
<td>-</td>
<td>$\checkmark$</td>
<td>-</td>
</tr>
<tr>
<td>$J^i$</td>
<td>-</td>
<td>-</td>
<td>$\checkmark$</td>
</tr>
<tr>
<td>$[J^R_{eik}]_{\text{real}}$</td>
<td>-</td>
<td>$\checkmark$</td>
<td>-</td>
</tr>
<tr>
<td>$[J^i_{eik}]_{\text{real}}$</td>
<td>-</td>
<td>-</td>
<td>$\checkmark$</td>
</tr>
<tr>
<td>$J^R_j - J^R_{eik}$</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$J^i - J^i_{eik}$</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 2.1: IR singularities in virtual and real phase-space integrals at order $\alpha_s$. Note that the jet functions and eikonal jet functions are all IR divergent but the IR divergences cancels in their difference. Indeed, the subtracted jet function is infrared safe and has no dependence on $\omega$ or $\delta_j$. Note that log $\omega \log R$ and log $\omega \log \delta_j$ terms describe soft-collinear singularities. After subtracting the eikonal jet function from the inclusive jet function, these singularities drop out.
We see that the virtual contribution has double poles in the UV and IR. In pure dimensional regularization, these double poles would exactly cancel and the virtual contribution exactly vanishes.

Next, we compute the real-emission contributions. First, we consider the inclusive soft function. At order $\alpha_s$, the real emission contribution is given

$$\left[ S(\tau) \right]_{\text{real}} = 2g_s^2 \mu^{2\varepsilon} C_F \int \frac{d^d k}{(2\pi)^d} 2 \left( \frac{1}{k^+ + Q\omega} \frac{1}{k^- + Q\omega} \right) 2\pi \delta(k^+ k^- - k_\perp^2) \theta(k^+) \theta(k^-)$$

$$\times \left[ \theta(k^- - k^+) \delta(\tau - \frac{k^+}{Q}) + \theta(k^+ - k^-) \delta(\tau - \frac{k^-}{Q}) \right]$$

(2.76)

As with the jet function, we write the result as

$$\left[ S \right]_{\text{real}} = \frac{\alpha_s C_F}{2\pi} \frac{1}{\Gamma(1-\varepsilon)} \left( \frac{4\pi\mu^2}{Q^2} \right)^\varepsilon \frac{1}{\tau + \omega} \frac{1}{\tau^\varepsilon} \mathcal{I}_s(\tau)$$

(2.77)

We will use this parameterization for the inclusive and phase-space restricted soft functions.

For the inclusive soft function, we find

$$\mathcal{I}_s(\tau) = \int_\tau^\infty \frac{dx}{x + \omega} \left[ \frac{1}{x} \right]^\varepsilon = \omega^{-\varepsilon} B \left( \frac{\omega}{\tau}; \varepsilon, 0 \right) = \frac{2}{\varepsilon} - 2 \ln (\tau + \omega) + O(\varepsilon)$$

(2.78)

so that, using Eq. (2.61),

$$\left[ S(\tau) \right]_{\text{real}} = \frac{\alpha_s C_F}{\pi} \left( \frac{\mu^2}{Q^2} \right)^\varepsilon \left\{ \delta(\tau) \left( -\frac{2}{\varepsilon} \ln \omega + \frac{\pi^2}{3} + 2 \ln^2 \omega \right) + \left[ \frac{1}{\tau} \left( \frac{2}{\varepsilon} - 4 \ln \tau \right) \right]_+ \right\}$$

(2.79)

adding the virtual contribution, we get to order $\alpha_s$

$$S(\tau) = \delta(\tau) + \frac{\alpha_s C_F}{\pi} \left( \frac{\mu^2}{Q^2} \right)^\varepsilon \left\{ \delta(\tau) \left( -\frac{1}{\varepsilon^2} + \frac{\pi^2}{12} \right) + \left[ \frac{1}{\tau} \left( \frac{2}{\varepsilon} - 4 \ln \tau \right) \right]_+ \right\}$$

(2.80)
which is IR finite (no dependence on $\omega$). The UV poles are removed by renormalization.

The inclusive soft function is IR finite, since all $\omega$ divergences cancel between real and virtual diagrams, as we can see in Eq. (2.79) and Eq. (2.74). As with the jet functions, $\Theta_{\text{Res}}$, represents the phase-space restriction. For the soft function in the ball, the phase space restriction is $\Theta_\Lambda \equiv \theta(\Lambda - k^-)\theta(k^- - k^+) + \theta(\Lambda - k^+)\theta(k^+ - k^-)$; for the soft function outside the cones, $\Theta_R \equiv \theta(R - k^- / k^+)\theta(k^- / k^+ - 1/R)$; finally, for the soft function in the ball and outside the cones, $\Theta_{\Lambda R} \equiv \Theta_\Lambda \Theta_R$.

For the soft function phase-space restricted to radiation outside of cones around the jet axes (but with no cutoff $\Lambda$ on energy), we find

$$
I_s^R(\tau) = \int_\tau^{\tau/R} dx \frac{2}{x + \omega} \left[ \frac{1}{x} \right]^\epsilon = 2 \ln \left( \frac{\tau}{R + \omega} \right) - 2 \ln (\tau + \omega) + \mathcal{O}(\epsilon) \quad (2.81)
$$

Note that this soft function is not UV divergent, since at finite $\tau$ there is an implicit energy cutoff $E < \frac{\tau}{2}(1 + \frac{1}{R})$, as discussed below Eq. (2.32). It leads to

$$
\left[ S^R(\tau) \right]_{\text{real}} = \frac{\alpha_s C_F}{\pi} \left\{ \delta(\tau) \left[ 2 \ln \omega \ln \left( \frac{R}{-1 + R} \right) - \ln^2 \omega - 2 \text{Li}_2 \left( \frac{1 + \omega}{\omega(1 - R)} \right) + 2 \text{Li}_2 \left( \frac{1}{1 - R} \right) \right] + \left[ \frac{2}{\tau} \ln R \right]_+ \right\}
= \frac{\alpha_s C_F}{\pi} \left\{ \delta(\tau) \left( -\frac{\pi^2}{3} + 2 \ln \omega \ln R + \mathcal{O}(R, \omega) \right) + \left[ \frac{2}{\tau} \ln R \right]_+ \right\} \quad (2.82)
$$

Note that the IR-divergent $\ln \omega \ln R$ term does not cancel against the virtual correction. Thus the soft function outside the cones is not infrared safe.

For the soft function phase-space restricted to a ball with energy less than $\Lambda$, but no angular restriction, we find

$$
I_s^\Lambda(\tau) = \int_\tau^{\Lambda/Q} dx \frac{2}{x + \omega} \left[ \frac{1}{x} \right]^\epsilon = 2 \ln \left( \frac{\Lambda}{Q + \omega} \right) - 2 \ln (\tau + \omega) \quad (2.83)
$$
This is also UV finite, as expected. To get the soft function outside the cones with energy less then \( \Lambda \), we can simply combine these two expressions with \( \theta \)-functions:

\[
\mathcal{I}^\Lambda_R(\tau) = \mathcal{I}^\Lambda_R \left( \frac{\tau}{R} - \Lambda \right) + \mathcal{I}^R_s \left( \Lambda - \frac{\tau}{R} \right)
\]

which leads to

\[
\left[ S^\Lambda_R(\tau) \right]_{\text{real}} = \delta(\tau) + C_F \frac{\alpha_s}{\pi} \left( \frac{\mu^2}{Q^2} \right)^\epsilon \left\{ \delta(\tau) \left[ -\frac{\pi^2}{3} + 2 \ln \omega \ln R + \mathcal{O}(R) \right] + \left[ \frac{2}{\tau} \ln R \right]^+ \right\} \theta \left( \Lambda - \frac{\tau}{R} \right) \\
- C_F \frac{\alpha_s}{\pi} \left( \frac{\mu^2}{Q^2} \right)^\epsilon \left\{ \delta(\tau) \left[ \frac{1}{\epsilon^2} + \frac{\pi^2}{4} + 2 \left( \frac{1}{\epsilon} \ln \omega + \ln \omega \ln \frac{\Lambda}{Q} - \frac{1}{2} \ln^2 \omega \right) \right] + \left[ \frac{2}{\tau} \ln \frac{\tau Q^2}{\Lambda} \right]^+ \right\} \theta \left( \frac{\tau}{R} - \Lambda \right)
\]

Combining this with the virtual contribution leads to Eq. (2.29). Note that for \( \tau \) sufficiently small (\( \tau < \Lambda R \)), \( \mathcal{I}^\Lambda_R \) and hence \( S^\Lambda_R \) does not depend on \( \Lambda \). In particular, at leading power in \( \tau \) we have

\[
S^\Lambda_R \simeq S^{(\Lambda=\infty)}R
\]

as in Eq. (2.31).

To repeat, we find that the inclusive soft function is IR finite while \( S^R \) and \( S^\Lambda_R \) are not, due to an incomplete cancellation of IR singularities between virtual and real contributions. Indeed, \( S^R \) contains a \( \ln \omega \ln R \) term, coming from the real-soft region, and a \( \ln^2 \omega \) term, coming from the virtual-soft-collinear region. These singular terms exactly match with those in \( J_{eik^\prime}^R \) according to Eq. (2.82) and Eq. (2.69).
Chapter 3

Factorization for groomed jet substructure beyond the next-to-leading logarithm

3.1 Introduction

The high luminosity proton collisions at the Large Hadron Collider (LHC) enable an unprecedented sensitivity to rare and high scale physics. The cost of such high luminosities is the presence of significant amounts of pile-up radiation present in every event, arising from numerous secondary proton collisions per bunch crossing. Pile-up is truly uncorrelated with the hard scattering and can contaminate any potential measurement. This is particularly important for measurements made on jets, for which pile-up can effect a large systematic bias in observables like the jet mass. In searches for resonances that decay to boosted electroweak objects which have definite masses, pile-up can significantly degrade the ability to separate signal from background. Over the past several years, numerous methods [67–77] have been developed for grooming jets and events for pile-up
mitigation and removal, and are now standard experimental tools at both ATLAS and CMS experiments. Especially in analyses of jets, measurements made at the LHC often involve some form of grooming.

With this motivation, it is imperative to understand these jet grooming techniques from first principles QCD. There have been a few studies of the theoretical aspects of jet groomers [72,74,78,79], with predictions for jet-observable distributions calculated to next-to-leading logarithmic (NLL) accuracy for two widely used jet groomers: the modified mass drop tagger (mMDT) and soft drop. These explicit analytic studies showed that these jet groomers not only have desired experimental properties, but can also dramatically simplify theoretical calculations as compared to their ungroomed counterparts. Non-global logarithms (NGLs) that arise from correlations between in- and out-of-jet scales have proven to be a significant obstruction to resummation of ungroomed jet observables to NLL accuracy and beyond. In particular, it was demonstrated by explicit calculation in Refs. [72,74,78] that mMDT and soft drop groomers eliminate the leading non-global logarithms in jet mass distributions [1]. mMDT and soft drop pave the way for systematically improvable resummed predictions of jet observables.

In this paper, we open the door to systematically improvable jet substructure calculations by presenting an all-orders factorization theorem for the soft-drop [74] groomed observables using soft-collinear effective theory (SCET) [15,29,30,80]. An overview of the method we discuss here and some of our results were presented recently in Ref. [81]. This paper provides a more detailed presentation of those results as well as a derivation of the factorization formula and its remarkable properties.

The soft drop groomer walks through the branching history of a jet, discarding soft branches until a sufficiently hard branching is found. This is enforced by effectively requiring

$$\frac{\min[E_i, E_j]}{E_i + E_j} > z_{cut} \left( \frac{\theta_{ij}}{R} \right) ^{\beta} ,$$

(3.1)
where $E_i$ and $E_j$ are the energies of the particles in that step of the branching, $\theta_{ij}$ is their relative angle, and $R$ is the radius of the jet. $z_{\text{cut}}$ is a parameter that sets the scale of soft, wide angle emissions in the jet; the typical value is $z_{\text{cut}} = 0.1$. $\beta$ is a parameter that controls the aggressiveness of the groomer: $\beta = \infty$ removes the groomer, $\beta = 0$ coincides with mMDT and is simply an energy cut, and $\beta < 0$ removes all soft and collinear singularities. We will consider $\beta \geq 0$. If Eq. (3.1) is not satisfied, the softer of the two branches is removed from the jet, and the grooming procedure continues on the harder branch. When Eq. (3.1) is satisfied, the procedure terminates and the groomed jet is returned. For concreteness, on this groomed jet, we measure the two-point energy correlation functions $e_2^{(\alpha)}$ with angular exponent $\alpha > 0$ [82–84].

In $e^+e^- \rightarrow$ dijets events, the factorization formula we derive in this paper for soft-drop
groomed left and right hemisphere jets is:

\[
\frac{d^2\sigma}{de_{2,L}^{(a)} de_{2,R}^{(a)}} = H(Q^2)S_G(z_{\text{cut}}) \left[ S_C(z_{\text{cut}} e_{2,L}^{(a)}) \otimes J(e_{2,L}^{(a)}) \right] \left[ S_C(z_{\text{cut}} e_{2,R}^{(a)}) \otimes J(e_{2,R}^{(a)}) \right].
\]  

(3.2)

This factorization theorem applies when \( z_{\text{cut}} \ll 1 \) and the left- and right-hemisphere energy correlation functions are asymptotically small: \( e_{2,L}^{(a)}, e_{2,R}^{(a)} \ll z_{\text{cut}} \ll 1 \). We illustrate the physical configuration corresponding to this factorization theorem in Fig. 3.1. In Eq. (3.2), \( H(Q^2) \) is the hard function for \( e^+ e^- \rightarrow q\bar{q} \). \( S_G(z_{\text{cut}}) \) is the global soft function, which is only sensitive to the scale set by \( z_{\text{cut}} \) since all of its emissions fail soft drop. \( S_C(z_{\text{cut}} e_{2,L}^{(a)}) \) is a soft function that is boosted along the direction of the jet in the left hemisphere; its corresponding modes are referred to as collinear-soft [5, 85–89]. Emissions in \( S_C(z_{\text{cut}} e_{2,L}^{(a)}) \) may or may not pass the soft drop requirement and are therefore constrained by both \( z_{\text{cut}} \) and \( e_{2,L}^{(a)} \). Importantly, this collinear-soft mode depends on only a single scale which we generically denote by \( z_{\text{cut}} e_{2,L}^{(a)} \). (For \( \alpha \neq 2 \) or \( \beta > 0 \), the single scale is a different combination of \( z_{\text{cut}} \) and \( e_{2,L}^{(a)} \); we simply call it \( z_{\text{cut}} e_{2,L}^{(a)} \) for notational brevity.) \( J(e_{2,L}^{(a)}) \) is the jet function for the left hemisphere jet, and all emissions in the jet function parametrically pass the soft drop requirement. Thus, the jet function is independent of the scale set by \( z_{\text{cut}} \), and only depends on \( e_{2,L}^{(a)} \). \( \otimes \) denotes convolution in \( e_{2,L}^{(a)} \), and a similar collinear-soft and jet factorization exists for the right hemisphere.

As we will explain in detail, there are several important consequences of this factorization formula. Because the formula depends on the observables \( e_{2,L}^{(a)}, e_{2,R}^{(a)} \) only through collinear objects each of which has a single scale, there are no non-global logarithms. The elimination of the purely soft contribution also makes the shape of soft-drop groomed jet shapes largely independent of what else is going on in the event. For example, the shape of the left hemisphere jet mass is independent of what is present in the right hemisphere. Additionally, the scale associated with the collinear-soft mode is parametrically larger.
than the soft scale associated with ungroomed masses, so non-perturbative corrections such as hadronization are correspondingly smaller.

This factorization theorem allows us to go beyond NLL accuracy to arbitrary accuracy. In this paper, we show that next-to-next-to-leading logarithmic (NNLL) accuracy is readily achievable. We focus on $\alpha = 2$ where the two-point energy correlation function is equal to the squared jet mass (up to a trivial normalization). This lets us extract most of the necessary two-loop anomalous dimensions from the existing literature. For $\beta = 0$, the global soft function $S_G(z_{\text{cut}})$ is closely related to the soft function with an energy veto [89, 90] which is known to two-loop order. There are additional clustering effects from the soft drop algorithm, but these are straightforward to calculate. Interestingly, we find that the clustering effects in the soft drop groomer are intimately related to similar effects observed in jet veto calculations [91–95]. For $\beta = 1$, we compute the two-loop anomalous dimension of $S_G(z_{\text{cut}})$ numerically using the fixed-order code EVENT2 [34].

We thereby achieve full NNLL resummation for the soft-drop groomed jet mass.

It is straightforward to generalize from $e^+e^-$ to $pp$ collisions, since the distribution is determined by collinear physics within the jet, independent of the initial state. The main new ingredient in $pp$ collisions is that jets may be initiated by quarks or gluons. As we will show, soft-drop grooming the jet enables an infrared and collinear safe definition of the jet flavor at leading power in $c_2^{(x)}$ and $z_{\text{cut}}$ by simply summing the flavors of partons in the groomed jet. Using this procedure, we are able to match our NNLL resummed distribution of soft-drop groomed jet mass to fixed order results for $pp \rightarrow Z + j$ events.

1While we will not do it in this paper, one could use the results of Ref. [96] which calculates the anomalous dimension of the soft function for event-wide (recoil-free) angularities [20, 97–99] or energy correlation functions with arbitrary angular exponent. This would enable us to extend our results to the case with $\alpha \neq 2$.

2The jet mass has been calculated at NNLL using other methods [59, 60, 100] as has 2-subjettiness [65]. However, without grooming the jets, there are non-global logarithms which are not resummed (and which may or may not be quantitatively important) and uncontrollable sensitivity to pileup (which is very quantitatively important).
The outline of this paper is as follows. In Sec. 3.2, we review the definition of the soft drop grooming algorithm and the energy correlation functions. In Sec. 3.3, we present the factorization theorem for soft-drop groomed energy correlation functions in $e^+e^-$ → dijets events. In this section, we will also present a detailed power-counting analysis of soft-dropped observables to determine the range of validity of the factorization theorem. Our factorization theorem has many non-trivial consequences, which we review in Sec. 3.4. These include absence of non-global logarithms, process independence, and small hadronization corrections. In Sec. 3.5, we describe and present the ingredients necessary for NNLL resummation. Here, we also describe our method for extracting anomalous dimensions from EVENT2. We then match our NNLL results with fixed-order calculations for $e^+e^-$ collisions in Sec. 3.6 and for $pp \rightarrow Z + j$ events in Sec. 3.7, comparing with Monte Carlo simulations in each case. In Sec. 5.5, we summarize and conclude. The calculational details for NNLL resummation are collected in appendices.

3.2 Observables

In this section, we review the soft drop grooming algorithm and the energy correlation functions. Although previous work has focused on jets produced in $pp$ collisions, we will provide definitions for both lepton and hadron collider environments.

3.2.1 Soft Drop Grooming Algorithm

Given a set of constituents of a jet with radius $R$, the soft drop grooming algorithm [74] proceeds in the following way:

1. Recluster the jet with a sequential $k_T$-type [101–103] jet algorithm. This produces an infrared and collinear (IRC) safe branching history of the jet. The $k_T$ clustering
metric for jets in $e^+e^-$ collisions is

$$d_{ij}^{e^+e^-} = \min \left[ E_i^{2p}, E_j^{2p} \right] (1 - \cos \theta_{ij}), \quad (3.3)$$

where $E_i, E_j$ are the energies of particles $i$ and $j$ and $\theta_{ij}$ is their relative angle. $p$ is a real number that defines the particular jet algorithm. For jets produced in $pp$ collisions, the $k_T$ clustering metric is

$$d_{ij}^{pp} = \min \left[ p_{Ti}^{2p}, p_{Tj}^{2p} \right] R_{ij}^2, \quad (3.4)$$

where $p_{Ti}, p_{Tj}$ are the transverse momenta of particles $i$ and $j$ with respect to the beam and $R_{ij}^2$ is their relative angle in the pseudorapidity-azimuth angle plane.

While the original implementation of soft drop was restricted to reclustering with the Cambridge/Aachen algorithm ($p = 0$) [104–106], we will also briefly consider reclustering with the anti-$k_T$ algorithm ($p = -1$) [107] in Sec. 3.5.2.

2. Sequentially step through the branching history of the reclustered jet. At each branching, check the soft drop criterion. For $e^+e^-$ collisions, we require

$$\frac{\min[E_i, E_j]}{E_i + E_j} > z_{\text{cut}} \left( \sqrt{2 \frac{\sin \frac{\theta_{ij}}{2}}{\sin \frac{R_{ij}}{2}}} \right)^\beta. \quad (3.5)$$

This is known as the **soft drop criterion**. If the branching fails this requirement, then the softer of the two daughter branches is removed from the jet. The soft drop groomer then continues to the next branching in the remaining clustering history.

For $pp$ collisions, the soft drop criterion is

$$\frac{\min[p_{Ti}, p_{Tj}]}{p_{Ti} + p_{Tj}} > z_{\text{cut}} \left( \frac{R_{ij}}{R} \right)^\beta. \quad (3.6)$$
3. The procedure continues until the soft drop criterion is satisfied. At that point, soft drop terminates, and returns the jet groomed of the branches that failed the soft drop criterion.

Once the jet has been groomed, any observable can be measured on its remaining constituents.

### 3.2.2 Energy Correlation Functions

On jets that have been groomed by soft drop, we measure the two-point energy correlation functions \([82–84]\). We do this mainly for concreteness; the general properties of the factorized formula we will present apply for a much broader class of observables. For jets in \(e^+e^-\) collisions, the two-point energy correlation function \(e^{(a)}_2\) is

\[
e^{(a)}_2\big|_{e^+e^-} = \frac{1}{E_J^2} \sum_{i<j} E_i E_j \left( \frac{2p_i \cdot p_j}{E_i E_j} \right)^{\alpha/2},
\]

where \(E_J\) is the sum of the energies of particles in the jet, the sum runs over distinct pairs \(i, j\) of particles in the jet, \(p_i\) is the four-vector momentum of particle \(i\), and the angular exponent \(\alpha\) is required to be greater than 0 for IRC safety. For \(\alpha = 2\) and a jet that has massless constituents, the two-point energy correlation function reduces to the normalized, squared jet mass:

\[
e^{(2)}_2\big|_{e^+e^-} = \frac{m_J^2}{E_J^2}.
\]

The energy correlation functions have the nice property that they are insensitive to recoil effects \([82, 99]\) and do not include explicit axes in their definition.

For jets produced in \(pp\) collisions, the energy correlation functions are appropriately
modified by replacing spherical coordinates with cylindrical coordinates:

\[
\left. e_2^{(a)} \right|_{pp} = \frac{1}{p_{TJ}^2} \sum_{ij \in J} p_{Ti} p_{TJ} R_{ij}^{\alpha},
\]

(3.9)

where \( p_{TJ} \) is the transverse momentum of the jet and \( R_{ij} \) is the separation of particles \( i \) and \( j \) in the pseudorapidity-azimuthal angle plane. For jets at central rapidities and in the limit that all particles in the jet are collinear, Eq. (3.9) reduces to Eq. (3.7). This property in particular will enable us to recycle results calculated in \( e^+ e^- \) collisions to the case of \( pp \) collisions.

### 3.3 Factorization Theorem

In this section, we derive the factorization formula for energy correlation functions measured on soft-drop groomed jets in the region of phase space where \( e_2^{(a)} \ll z_{\text{cut}} \ll 1 \) using SCET [15,29,30,80]. We begin with a power counting analysis based on the scales \( e_2^{(a)} \) and \( z_{\text{cut}} \) relevant to soft-drop groomed jets. This enables us to identify all modes and their momentum scalings that contribute at leading power. Using these scales and the associated modes, we derive the factorization formula. We then show that the jet function in the factorization formula can be re-factorized due to a collinear-soft mode which decouples from the collinear-but-not-soft modes as a result of soft drop.

#### 3.3.1 Power Counting and Modes

For jets on which the soft drop groomer is applied and the energy correlation functions are measured, there are three relevant dimensionless scales: the jet radius \( R \), the soft drop parameter \( z_{\text{cut}} \), and \( e_2^{(a)} \). Typically, jet radii are \( R \sim 1 \). We are interested in the singular region \( e_2^{(a)} \to 0 \) for a fixed value of \( z_{\text{cut}} \). Thus we can assume \( e_2^{(a)} \ll z_{\text{cut}} \). We will also
Figure 3.2: Location of modes appearing in the soft drop factorization theorem in the plane defined by energy fraction $z$ and splitting angle $\theta$ of emissions in the jet. The solid diagonal line separates the regions of phase space where emissions pass and fail soft drop. All emissions along the dashed line that pass soft drop contribute at leading power to the measured value of $e_2^{(a)}$.
assume $z_{\text{cut}} \ll 1$ to refactorize the jet function. The limits $R \ll 1$ or $z_{\text{cut}} \sim 1$ could be considered as well, but are beyond the scope of our analysis.

We will use scaling arguments to identify the regions of phase space that are present at leading power and then take the limit where each region becomes a separate sector, that no longer interacts with the other regions.

For a jet to have $e_2^{(a)} \ll 1$, all particles must be either soft or collinear to the jet axis. In particular, a particle with energy $E = zE_J$ at an angle $\theta$ from the jet axis must satisfy

$$z\theta^a \lesssim e_2^{(a)}.$$  \hfill (3.10)

This is a line in the log$(1/z)$-log$(1/\theta)$ plane, as shown in Fig. 3.2. Anything below the dashed line in this figure is too hard to be consistent with a given value of $e_2^{(a)}$. The soft drop criterion is that

$$z_{\text{cut}} \lesssim z\theta^{-\beta},$$  \hfill (3.11)

This is the region below the solid line in Fig. 3.2.

To find the relevant modes for the factorized expression, we need to identify the distinct characteristic momentum scalings that approach the singular regions of phase space in the limit $e_2^{(a)} \ll z_{\text{cut}} \ll 1$. For a particular scaling, the constraints in Eqs. (3.10) and (3.11) will either remain relevant or decouple. We can characterize the relevant regions by their scalings in light-cone coordinates. Defining $n^\mu$ as the jet direction and $\bar{n}^\mu$ as the direction backwards to the jet, then light-cone coordinates are triplets $p = (p^-, p^+, p_\perp)$ where $p^- = \bar{n} \cdot p$, $p^+ = n \cdot p$ and $p_\perp$ are the components transverse to $n$. On-shell massless particles have $p^+ p^- = p_\perp^2$. The energy fraction is $z = p^0 / Q = \frac{1}{2}(p^+ + p^-) / Q$ and the angle to the jet axis in the collinear limit is $\theta = p_\perp / p^0$.

We start with the soft modes, emitted at large angles $\theta \sim 1$, but still within the jet. If such radiation were to pass soft drop, with energy fraction greater than $z_{\text{cut}}$, it would set
$e_2^{(a)} \gtrsim z_{\text{cut}}$; this contradicts our assumed hierarchy $e_2^{(a)} \ll z_{\text{cut}}$. Therefore, soft wide-angle radiation is removed by soft drop and is not constrained by $e_2^{(a)}$. These modes thus have momenta that scale like

$$p_s \sim z_{\text{cut}} Q(1,1,1).$$

They contribute only to the normalization of the distribution, not to its shape.

Next consider the collinear radiation, emitted at small angles $\theta \ll 1$. All collinear radiation has $p^- \gg p^+$. Then, from Eq. (3.10), we find

$$e_2^{(a)} \sim \frac{(p^+)^{a/2}(p^-)^{1-a/2}}{Q}.$$  

(3.13)

Collinear modes can either have $z \sim 1$ or be parameterically soft $z \ll 1$.

For modes with $z \sim 1$, we have $z \gg z_{\text{cut}}$. Thus $p^- \sim Q$ and $p^+ \sim Q(e_2^{(a)})^{2/a}$ independent of $z_{\text{cut}}$. Their scaling is

$$p_c \sim Q \left(1, (e_2^{(a)})^{2/a}, (e_2^{(a)})^{1/a}\right).$$  

(3.14)

We call these modes **collinear modes**, although strictly they are not-soft collinear modes.

Collinear radiation that can have $z \sim z_{\text{cut}} \ll 1$ we call **collinear-soft**. In this case, $p^- \sim zQ$ and $p^+ \sim \theta^2 zQ$. These modes are simultaneously compatible with Eqs. (3.10) and (3.11). Their scaling is determined by saturating these parametric relationships, which leads to

$$p_{cs} \sim (z_{\text{cut}})^{\frac{a}{\alpha+\beta}} (e_2^{(a)})^{\frac{\beta}{\alpha+\beta}} Q \left(1, \left(\frac{e_2^{(a)}}{z_{\text{cut}}}\right)^{\frac{2}{\alpha+\beta}}, \left(\frac{e_2^{(a)}}{z_{\text{cut}}}\right)^{\frac{1}{\alpha+\beta}}\right).$$  

(3.15)

This is the point in phase space labeled $S_C(z_{\text{cut}} e_2^{(a)})$ in Fig. 3.2.
Figure 3.3: Illustration of the multi-stage matching procedure to derive the soft drop factorization theorem. As discussed in the text, we first match QCD to SCET, then factorize the jet function into collinear and collinear-soft modes. Canonical scales of all modes in the factorization theorem are shown on the right, ordered in virtuality where we assume that $\alpha > 1$ and $\beta \geq 0$.

### 3.3.2 Factorization and refactorization

With the relevant scalings identified, we proceed to derive the factorization formula. For simplicity, we focus on the case of $e^+e^- \rightarrow$ hemisphere jets, with $e_2^{(a)}$ measured on each hemisphere. Jets at hadron colliders can be treated similarly, as we discuss in Sec. 3.4. Fig. 3.3 illustrates the relevant modes and their scales.

We begin with the usual SCET factorization formula, in the absence of soft drop grooming. The hard, collinear and soft modes are separated in the limit of small observables. This leads to [38, 98, 108]

$$
\frac{d^2 \sigma}{d e_{2,L}^{(a)} d e_{2,R}^{(a)}} = H(Q^2) \times S\left(e_{2,L}^{(a)}, e_{2,R}^{(a)}\right) \otimes J(e_{2,L}^{(a)}) \otimes J(e_{2,R}^{(a)})
$$

(3.16)

for the ungroomed hemispheres in $e^+e^- \rightarrow$ dijets events, provided $e_{2,L}^{(a)}, e_{2,R}^{(a)} \ll 1$. Here, $\otimes$ denotes convolution in $e_{2,L}^{(a)}$ or $e_{2,R}^{(a)}$ appropriately. To get to this equation, one can match to full QCD to get the hard function, then decouple the soft and collinear degrees of
freedom to pull the jet and soft functions apart [15, 29, 30, 80]. Alternatively, one can use the method of regions approach [25, 28], or the on-shell phase space approach [8, 9, 109]. Importantly, $e_2^{(a)}$ is insensitive to recoil effects from soft emissions that displace the jet axis from the direction of hard, collinear particles [82, 99], and so the jet and soft functions are completely decoupled.

Next we write down the hard-soft-jet factorization formula in the presence of soft drop grooming, assuming the hierarchy $e_2^{(a)} \ll z_{\text{cut}} \ll 1$. With this assumption, soft radiation emitted at large angles must necessarily fail the soft drop criterion. Thus, all wide angle soft radiation in the jets (in this case, the hemisphere jets) is groomed and cannot contribute to the observable. All that remains of the global soft function is a $z_{\text{cut}}$-dependent normalization factor $S_G(z_{\text{cut}})$. This leads to

$$\frac{d^2\sigma}{de_{2,L}^{(a)}de_{2,R}^{(a)}} = H(Q^2) \times S_G(z_{\text{cut}}) \times J_{ze}(z_{\text{cut}}, e_2^{(a)}) \times J_{ze}(z_{\text{cut}}, e_2^{(a)}) . \quad (3.17)$$

$S_G(z_{\text{cut}})$ gives the cross section for the radiation from a set of Wilson lines that fails the soft drop criterion. An explicit calculation of $S_G$ for hemisphere jets at one-loop is given in Appendix 3.9.5. With the collinear and soft modes decoupled, we can lower the virtuality of the collinear modes without further matching.

The jet function $J_{ze}$ still depends on multiple scales, so to resum all the large logarithms it must be re-factorized. To see that it refactors, note first that in addition to being collinear, radiation in the jet function that is sensitive to the scale set by $z_{\text{cut}}$ must also be soft, by the assumption that $z_{\text{cut}} \ll 1$. Equivalently, emissions with order-1 energy fractions are not constrained by the scale $z_{\text{cut}}$. We can thus factorize the jet function into two pieces depending on their energy fraction:

$$J_{ze}(z_{\text{cut}}, e_2^{(a)}) = J(e_2^{(a)}) \otimes S_C(z_{\text{cut}}e_2^{(a)}) \quad . \quad (3.18)$$
Here, $J(e_2^{(a)})$ is the jet function that only depends on $e_2^{(a)}$ and only receives contributions from emissions with order-1 energy fraction. $S_C(z_{\text{cut}}e_2^{(a)})$ is the soft limit of the unfactorized jet function $J_{ze}(z_{\text{cut}}, e_2^{(a)})$. The scaling of the collinear and collinear-soft modes are given in Eqs. (3.14) and (3.15) as discussed above. Note that, importantly, because the collinear-soft mode arises from refactorization of a jet function, it is a color singlet and only depends on two back-to-back directions. Because the jet function only depends on $e_2^{(a)}$, it is sensitive to a single infrared scale.

The step in Eq. (3.18) is the most unusual and important in the derivation. That the collinear-soft function depends on only a single combination of $z_{\text{cut}}$ and $e_2^{(a)}$ is absolutely critical to being able to resum all the logs of $e_2^{(a)}$. We therefore devote Sec. 3.3.3 to showing explicitly that the collinear-soft function depends on a unique combination of $z_{\text{cut}}$ and $e_2^{(a)}$ as determined by the parametric scaling of the modes of Eq. (3.15), and so is also only sensitive to a single infrared scale.

Inserting Eq. (3.18) into Eq. (3.17) results in the factorization formula for soft drop energy correlation functions:

$$
\frac{d^2\sigma}{de_2^{(a)} de_2^{(a)}} = H(Q^2)S_G(z_{\text{cut}}) \left[ S_C\left(z_{\text{cut}}e_2^{(a)}\right) \otimes J(e_2^{(a)}) \right] \left[ S_C\left(z_{\text{cut}}e_2^{(a)}\right) \otimes J(e_2^{(a)}) \right]
$$

(3.19)

The pieces of the factorization theorem are:

- $H(Q^2)$ is the hard function for production of an $e^+e^- \rightarrow$ dijets event.

- $S_G(z_{\text{cut}})$ is the global soft function. It integrates the radiation coming from Wilson lines in the jet directions that fails the soft drop criterion. Its modes fail soft drop and have momenta that scale as determined in Eq. (3.12).

- $J(e_2^{(a)})$ is the jet function describing the emission of collinear radiation from a jet. Its modes parametrically pass soft drop and have momenta that scale as determined in
Eq. (3.14).

- $S_C \left( z_{\text{cut}} e^{(a)}_2 \right)$ is the collinear-soft function describing the emission of soft radiation boosted along the direction of a jet. Its modes may or may not pass soft drop and have momenta that scale as determined in Eq. (3.15). We denote the single scale that the collinear-soft function depends on as $z_{\text{cut}} e^{(a)}_2$ for brevity; it is shorthand for Eq. (3.36).

We present the operator definitions and explicit one-loop results for all of these functions in the appendices.

The appearance of collinear-soft modes in this factorization theorem has some similarities and differences with respect to the identification of other collinear-soft modes in the literature [5, 85–89]. The original construction of a collinear-soft mode in Ref. [85] followed from boosting two jets in an event far from their center-of-mass frame in an effective theory of collinear dijets called SCET$_+$. The collinear-soft mode in SCET$_+$ is sensitive to three Wilson line directions: the two from the collinear jets and the backward direction from boosting all other jets in the event. This collinear-soft mode was also exploited in Ref. [87] in the resummation of jet observables that are sensitive to multi-prong substructure.

The collinear-soft mode in the factorization theorem presented here, however, is more similar to modes identified from the measurement of multiple observables on a jet, each of which is only sensitive to radiation about a single hard core [5, 86–89]. For example, Ref. [86] presented a factorization theorem for jets on which two angularities [20, 97–99] are measured. At leading power, angularities are only sensitive to the hard jet core, and so the collinear-soft modes only know about two Wilson line directions: the jet axis and the backward direction. More recently, collinear-soft modes of this type have been used to resum NGLs [5, 88] and logarithms of the jet radius [89].
3.3.3 The Single Scale of the Collinear-Soft Function

To demonstrate explicitly that the collinear-soft function only depends on a single scale, we can make the following scaling argument. The collinear-soft function has the following form:

\[
S_C \left( z_{\text{cut}}, \epsilon_2^{(a)} \right) = \sum_n \mu^{2n} \int d\Pi_n |\mathcal{M}_n|^2 \Theta_{\text{SD}} \delta_\epsilon^{(a)}.
\] (3.20)

Here, \( n \) is the number of final state collinear-soft particles, \( d\Pi_n \) is on-shell Lorentz-invariant phase space in \( d = 4 - 2\epsilon \) dimensions:

\[
d\Pi_n = \prod_{i=1}^n \frac{d^4k_i}{(2\pi)^d} 2\pi \delta(k_i^2) \Theta(k_i^0),
\] (3.21)

\( \mu \) is the renormalization scale, and \( \mathcal{M}_n \) is the amplitude for the production of the final state. \( \Theta_{\text{SD}} \) represents the soft drop grooming algorithm, which applies constraints on the final state and \( \delta_\epsilon^{(a)} \) represents the measurement of \( \epsilon_2^{(a)} \) on the final state:

\[
\delta_\epsilon^{(a)} = \delta \left( \epsilon_2^{(a)} - \frac{2^a}{Q} \sum_i (k_i^-)^{1-a/2} (k_i^+)^{a/2} \right),
\] (3.22)

where the sum runs over the set of final state particles \( \{i\} \) that remain in the jet after grooming. To write this expression, we have used the definition of \( \epsilon_2^{(a)} \) from Sec. 3.2.2 and expanded in the collinear-soft limit, as in Eq. (3.13).

Now, we rescale the momenta in light-cone coordinates that appear in the phase space integral in the following way:

\[
k^- \rightarrow (z_{\text{cut}})^{\frac{a}{2+\beta}} (\epsilon_2^{(a)})^{\frac{\beta}{2+\beta}} k^-,
\] (3.23)

\[
k^+ \rightarrow (z_{\text{cut}})^{\frac{a-2}{2+\beta}} (\epsilon_2^{(a)})^{\frac{2+\beta}{2+\beta}} k^+,
\]
At leading power in exactly $d = 4$, the phase space measure $d\Pi_n$ and the squared matrix element $|\mathcal{M}_n|^2$ scale exactly inversely. Therefore, in $d$ dimensions, under this rescaling, we have

$$d\Pi_n |\mathcal{M}_n|^2 \rightarrow \left( z_{\text{cut}} \frac{a-1}{a+p} \left( c_2^{(a)} \right)^{1+\frac{d}{a+p}} \right)^{-2n\epsilon} d\Pi_n |\mathcal{M}_n|^2. \quad (3.24)$$

Next, look at how the measurement functions $\Theta_{\text{SD}}$ and $\delta_{e_2}^{(a)}$ change under the rescaling of Eq. (3.23). First, consider the soft drop groomer $\Theta_{\text{SD}}$. This consists of two parts: one, the reclustering with the Cambridge/Aachen algorithm and the second, the energy requirement on the clustered particles. The clustering metric of the Cambridge/Aachen algorithm is just the pairwise angle

$$d_{ij}^{C/A} = \theta_{ij}^2, \quad (3.25)$$

and a pair $\{i, j\}$ of particles in the jet are clustered if they have the smallest $d_{ij}^{C/A}$. Importantly, the reclustering of the jet with soft drop is completely inclusive: all particles in the jet are clustered with no jet radius parameter. Therefore, for collinear-soft modes, there are only three types of clustering constraints that can be enforced, depending on what $d_{ij}^{C/A}$’s are being compared. If in the clustering history we compare the angles between two collinear-soft particles $i$ and $j$ to the jet axis, this corresponds to the constraint

$$\Theta \left( \frac{k_i^+}{k_i^-} - \frac{k_j^+}{k_j^-} \right). \quad (3.26)$$

This is invariant under the rescalings of Eq. (3.23). If in the clustering history we compare the angle between a collinear-soft particle $i$ to the jet axis and the angle between two
collinear-soft particles $j$ and $k$, we have the constraint

$$\Theta \left( \frac{k_{i}^{+} - k_{j} \cdot k_{k}}{k_{i}^{-} \cdot k_{j} \cdot k_{k}} \right),$$

(3.27)

which is also invariant under the rescalings of Eq. (3.23). Finally, we can compare the angle between a pair of collinear-soft particles $i$ and $j$ to the angle between another pair of collinear-soft functions $k$ and $l$, this leads to

$$\Theta \left( \frac{k_{i} \cdot k_{j} - k_{k} \cdot k_{l}}{k_{i}^{-} \cdot k_{j}^{-} \cdot k_{k}^{-} \cdot k_{l}^{-}} \right).$$

(3.28)

This too is invariant under Eq. (3.23). Therefore, for all possible clustering structures, the Cambridge/Aachen algorithm is invariant under the rescalings of Eq. (3.23).

The soft drop energy requirement on any number of particles that have been reclustered takes the form:

$$\Theta \left( \sum_{i} z_{i} - z_{\text{cut}} \Theta \theta^{\beta} \right),$$

(3.29)

where $z_{i}$ is the energy fraction of particle $i$ and $\theta$ is the angle that the cluster of particles $\{i\}$ makes with the jet axis. In terms of light-cone coordinates, this can be written as:

$$\Theta \left( \sum_{i} z_{i} - z_{\text{cut}} \Theta \theta^{\beta} \right) = \Theta \left( k^{-} - z_{\text{cut}} Q \left( \Theta \theta^{\beta} \right) \right),$$

(3.30)

where

$$k_{\perp} = \left| \sum_{i} k_{\perp i} \right|,$$

(3.31)

and

$$k^{-} = \sum_{i} k_{i}^{-}.$$
Applying the rescalings of Eq. (3.23), this constraint becomes

$$\Theta \left( \sum_i z_i - z_{\text{cut}} \theta^\beta \right) \rightarrow \Theta \left( \sum_i z_i - \theta^\beta \right).$$ \hspace{1cm} (3.33)

Note that the low scale $z_{\text{cut}}$ has been removed from this constraint.

Under the rescaling, the measurement constraint $\delta_{e_2^{(a)}}$ becomes

$$\delta \left( e_2^{(a)} - \frac{2^a}{Q} \sum_i (k_i^-)^{1-a/2}(k_i^+)^{a/2} \right) \rightarrow \frac{1}{e_2^{(a)}} \delta \left( 1 - \frac{2^a}{Q} \sum_i (k_i^-)^{1-a/2}(k_i^+)^{a/2} \right).$$ \hspace{1cm} (3.34)

Therefore, the low scale $e_2^{(a)}$ has been removed from this constraint.

Putting this all together, the collinear-soft function can be rewritten as

$$S_C \left( z_{\text{cut}}, e_2^{(a)} \right) = \sum_n M_n^2 e \left( z_{\text{cut}} \right)^{\frac{a-1}{a+\beta}} \left( e_2^{(a)} \right)^{\frac{1+\beta}{a+\beta}} \frac{1}{e_2^{(a)}} \int d\Pi_n |M_n|^2 \Theta_{SD}^{z_{\text{cut}}=1} \delta_{e_2^{(a)}=1}. \hspace{1cm} (3.35)$$

We have used the notation that $\Theta_{SD}^{z_{\text{cut}}=1}$ is the soft drop grooming algorithm with $z_{\text{cut}} = 1$ and $\delta_{e_2^{(a)}=1}$ is the measurement with $e_2^{(a)} = 1$. All low scales have been explicitly removed from the phase space integral. This function is now seen to be a function only of the single scale

$$^{"} z_{\text{cut}} e_2^{(a)} = (z_{\text{cut}})^{\frac{a-1}{a+\beta}} (e_2^{(a)})^{\frac{1+\beta}{a+\beta}}. \hspace{1cm} (3.36)$$

The quotes just mean that the left-hand side is our abbreviation for the unwieldly quantity on the right-hand side.

This proves that, to all orders, the collinear-soft function has dependence on only a single infrared scale, defined by this combination of $z_{\text{cut}}$ and $e_2^{(a)}$. Notice that the proof relied on the choice of Cambridge/Aachen reclustering in the soft drop grooming algorithm.

This completes the derivation of factorization and re-factorization for soft-drop groomed hemispheres in $e^+e^-$ collisions, on which two-point energy correlation functions have
been measured. All functions in the factorized cross section in Eq. (3.19) are sensitive to a single infrared scale and so all large logarithms can be resummed with the renormalization group.

### 3.4 Consequences of Factorization Theorem

Before we use the factorization theorem of Eq. (3.19) to make predictions for the cross section, we discuss consequences of this formula in some detail. Because the factorization theorem was derived without respect to any fixed order, these results hold to all orders.

Many of these consequences follow from the fact that soft wide angle radiation does not contribute to the shape of the soft-drop groomed $e_{2}^{(a)}$ distribution for $e_{2}^{(a)} \ll z_{\text{cut}} \ll 1$, a property that persists even for jets at hadron colliders. For example, it follows immediately from this fact that the shape of such a distribution is insensitive to contamination from pile-up and underlying event.

In this section, we will furthermore prove that at leading power, there are no NGLs that affect the shape of the soft-drop groomed $e_{2}^{(a)}$ distribution in this regime. This was explicitly shown at $O(a_{s}^2)$ in Refs. [72, 74, 78] and plausibility arguments were presented for all orders, but this is the first proof. The factorization theorem also exhibits sample independence to a large degree, because the shape of the distribution is only sensitive to collinear physics. We will also demonstrate that soft-drop groomed energy correlation functions are less sensitive to hadronization than their ungroomed counterparts.

#### 3.4.1 Absence of Non-Global Logarithms to All Orders

NGLs in cross sections of observables measured on individual hemispheres in $e^{+}e^{-}$ collisions arise from a parametric separation of the scales in the hemispheres. Their leading effects are exclusively non-Abelian and quantify the correlation between the two hemi-
spheres. Clearly, for a correlation to be present, there must be correlated radiation emitted into both hemispheres. If we measure the energy correlation functions \( e_2^{(a)} \) on both hemispheres and demand that \( e_2^{(a)} \ll 1 \), then the radiation in the event must be soft wide-angle or collinear. At leading power, it is not possible to have correlations between different collinear directions (beyond total momentum conservation) as this would violate the collinear factorization of gauge theory amplitudes. Therefore, correlations and NGLs can only arise from soft, wide-angle radiation in the event with these assumptions.

The factorization theorem for ungroomed hemisphere energy correlation functions is [38, 98, 108]

\[
\frac{d^2\sigma}{de_{2,L}^{(a)} de_{2,R}^{(a)}}_{\text{ug}} = H(Q^2) S(e_{2,L}^{(a)}, e_{2,R}^{(a)}) \otimes J(e_{2,L}^{(a)}) \otimes J(e_{2,R}^{(a)}),
\]

where "ug" denotes ungroomed. The cross section explicitly depends on soft wide-angle radiation through the soft function \( S(e_{2,L}^{(a)}, e_{2,R}^{(a)}) \), and so if either \( e_{2,L}^{(a)} \ll e_{2,R}^{(a)} \) or \( e_{2,R}^{(a)} \ll e_{2,L}^{(a)} \), NGLs will be present in this factorization theorem. Because the soft function depends on two scales, all of the singular dependence cannot be determined by renormalization group invariance. More generally, non-global structure present in the soft function has been studied at \( O(a_s^2) \) [110–112] and beyond [4,7] and recently, methods have been developed to control all-orders behavior [5,6,88,113]. However, NGLs represent an obstruction to resummation of the cross section to NLL and beyond.

For groomed hemisphere energy correlation functions, our factorization theorem instead takes the form of Eq. (3.19):

\[
\frac{d^2\sigma}{de_{2,L}^{(a)} de_{2,R}^{(a)}} = H(Q^2) S_G(z_{\text{cut}}) \left[ S_C(z_{\text{cut}} e_{2,L}^{(a)}) \otimes J(e_{2,L}^{(a)}) \right] \left[ S_C(z_{\text{cut}} e_{2,R}^{(a)}) \otimes J(e_{2,R}^{(a)}) \right].
\]

All soft, wide angle radiation throughout the event is described by \( S_G(z_{\text{cut}}) \), which is
sensitive only to the single scale $z_{\text{cut}}$. Therefore, there are no NGLs present in this factorization theorem. Even with a hierarchy between $e_{2,L}^{(a)}$ and $e_{2,R}^{(a)}$, these observables are completely decoupled at leading power in $e_{2}^{(a)}$ and $z_{\text{cut}}$. Additionally, the shape of the distribution is also independent of jet radius effects and the precise way in which the hemispheres are defined.

When we discuss soft-drop groomed jets in $pp$ collisions in Sec. 3.7, we will place no constraint on global soft radiation throughout the event, unlike the case of $e^+e^- \rightarrow$ hemisphere jets. Nevertheless, the shape of the soft-drop groomed $e_{2}^{(a)}$ distribution will still have no NGLs, jet radius effects, etc., due to universality of the collinear limit of QCD amplitudes. The normalization, however, will in general be sensitive to scales both in the jet (set by $z_{\text{cut}}$ and the jet radius $R$) and scales outside of the jet (set by the partonic collision energy). To eliminate these effects in $pp$ collisions, we can normalize the cross section, say, to integrate to unity.

### 3.4.2 Process Independence

Strictly speaking, the factorization theorem of Eq. (3.19) depends on the process. It includes the hard function, which is process dependent, and a soft function, that knows about all hard jet directions. Nevertheless, there is a sense in which the factorization theorem is process independent. Normalizing the cross section completely removes the hard and soft function dependence. Then, by the universal collinear factorization of QCD amplitudes, if we are completely inclusive over the right hemisphere, then the differential cross section of the soft-drop groomed energy correlation function in the left hemisphere is given by

$$\frac{d\sigma}{de_{2,L}^{(a)}} = NS_{C}(z_{\text{cut}}e_{2,L}^{(a)}) \otimes J(e_{2,L}^{(a)}), \quad (3.39)$$
where we assume that $e^{(a)}_{2, L} \ll z_{\text{cut}} \ll 1$ and $N$ is some normalization factor. That is, in the deep infrared where $e^{(a)}_{2, L} \ll z_{\text{cut}} \ll 1$, all radiation in the groomed jet is constrained to be collinear. Therefore, in this limit and for a fixed jet energy, the shape of the distribution for quark jets is independent of the process that created the quark jets, due to the universality of QCD matrix elements in the collinear limit.

This collinear factorization property of soft-drop groomed observables can be exploited for jets in $pp$ collisions. Unlike the dominant case in $e^+e^-$ collisions, jets at a $pp$ collider can be either quark or gluon. Of course, on a jet-by-jet level, we cannot determine whether a jet was initiated by a quark or gluon. However, for a given process, we can determine the relative fraction of quark and gluon jets in the sample. For jets produced at a $pp$ collider, the process independence manifests itself in the cross section as

$$\frac{d\sigma^{pp}}{de^{(a)}_2} = D_q S_{C,q}(z_{\text{cut}}e^{(a)}_2) \otimes J_q(e^{(a)}_2) + D_g S_{C,g}(z_{\text{cut}}e^{(a)}_2) \otimes J_g(e^{(a)}_2),$$

where $D_q$ ($D_g$) is proportional to the fraction of quark (gluon) jets in the sample. The relative fraction of quark and gluon jets can be determined from fixed-order calculations, using a simple algorithm for determining the flavor of a groomed jet. We will describe this in detail in Sec. 3.7 when we match our resummed distribution to fixed order in $pp \rightarrow Z + j$ events.

### 3.4.3 Hadronization Corrections

With a factorization formula, one can estimate the size and importance of non-perturbative corrections to the cross section. We will only consider non-perturbative corrections to the shape, as the normalization can be set by hand. Therefore, non-perturbative corrections can only enter into our factorization theorem via the jet or collinear-soft functions.

From Eqs. (3.14) and (3.15), the scales appearing in the jet and collinear-soft functions
are

$$\mu_j = Q \left( e_2^{(a)} \right)^{1/\alpha}, \quad (3.41)$$

$$\mu_{SC} = Q z_{cut}^{\frac{\alpha-1}{\alpha+\beta}} \left( e_2^{(a)} \right)^{\frac{1+\beta}{\alpha+\beta}}. \quad (3.42)$$

If either of the scales approaches $\Lambda_{QCD}$, then we expect there to be large corrections to the perturbative cross section due to non-perturbative physics. We can estimate when non-perturbative corrections become large by setting these scales to be $\Lambda_{QCD}$. For $\alpha > 1$ and $\beta \geq 0$, the collinear-soft mode has a lower virtuality than the collinear mode, so it will probe the non-perturbative region of phase space first. The value of $e_2^{(a)}$ at which the collinear-soft mode becomes non-perturbative is

$$\mu_{SC} = \Lambda_{QCD} \quad \Rightarrow \quad e_2^{(a)} \Big|_{NP} \simeq \left( \frac{\Lambda_{QCD}}{z_{cut} Q} \right)^{\frac{\alpha-1}{\alpha+\beta}} \cdot \frac{\Lambda_{QCD}}{Q}. \quad (3.43)$$

This estimate can be compared with the Monte Carlo analysis of hadronization corrections to the soft-drop groomed energy correlation functions from Ref. [74]. In particular, the estimate of Eq. (3.43) of when non-perturbative corrections become important for $\alpha = 2$ as a function of $\beta$ agrees exceptionally well with Fig. 10(a) of Ref. [74].

For $\beta < \infty$, the soft drop groomer reduces the effect of non-perturbative corrections with respect to the ungroomed observable. This can be simply seen from Eq. (3.43), in which the prefactor

$$\left( \frac{\Lambda_{QCD}}{z_{cut} Q} \right)^{\frac{\alpha-1}{\alpha+\beta}} \quad (3.44)$$

approaches unity as the grooming is removed ($\beta \to \infty$). This factor is less than 1 for $\beta < \infty$, provided $\alpha > 1$ and $\Lambda_{QCD} < z_{cut} Q$. For high energy jets, this suppression can be substantial. For example, for $\alpha = 2$ (corresponding to jet mass) and $\beta = 0$ (corresponding
to mMDT groomer) non-perturbative effects become important at

$$e^{(2)}_2 \bigg|_{\beta=0}^{NP} \sim \frac{\Lambda_{QCD}^2}{z_{\text{cut}} Q^2}.$$  \hfill (3.45)

This agrees with the estimate of the size of nonperturbative corrections for the mMDT groomer from Ref. [72].

### 3.5 Achieving NNLL Accuracy

In this section, we determine the anomalous dimensions necessary to resum the large logarithms of soft-dropped energy correlation functions through NNLL accuracy. The practical details of how one assembles these ingredients, in the framework of SCET, to construct a resummed cross section are given in App. 3.9.10. We will discuss matching to fixed-order and demonstrate our ability to make phenomenological predictions in subsequent sections.

Resummation in SCET is accomplished with renormalization group evolution. Solving the renormalization group equations to a given logarithmic accuracy requires anomalous dimensions to a particular fixed order. The anomalous dimensions of the functions in the factorization theorem must sum to zero, because the cross section is independent of the renormalization scale.

Recall that, for $e^+e^- \to$ hemisphere jets, the factorization theorem for soft-drop groomed energy correlation functions is

$$\frac{d^2 \sigma}{d e^{(\alpha)}_{2,L} d e^{(\alpha)}_{2,R}} = H(Q^2) S_G(z_{\text{cut}}) \left[ S_C(z_{\text{cut}} e^{(\alpha)}_{2,L}) \otimes J(e^{(\alpha)}_{2,L}) \right] \left[ S_C(z_{\text{cut}} e^{(\alpha)}_{2,R}) \otimes J(e^{(\alpha)}_{2,R}) \right].$$  \hfill (3.46)

Table 3.1 presents the order to which anomalous dimensions and constants of the functions in this factorization theorem must be computed for particular logarithmic accuracy.
Matching LL \( \lambda \) to NLL \( \lambda^2 \) to NNLL \( \lambda^3 \).

Table 3.1: \( \alpha_s \)-order of ingredients needed for resummation to the accuracy given. \( \Gamma_{\text{cusp}} \) is the cusp anomalous dimension, \( \gamma \) is the non-cusp anomalous dimension, and \( \beta \) is the QCD \( \beta \)-function. \( \hat{F}(\partial \omega) \) are the logarithms in the low-scale matrix elements that have been Laplace transformed and \( c_F \) are constants in the low-scale matrix elements. The final column shows the relative order to which the resummed cross section can be matched to fixed-order.

(see, e.g., Ref. [114]). The cusp anomalous dimension \( \Gamma_{\text{cusp}} \) and the QCD \( \beta \)-function are known through three-loop order [115–120] and we present them in App. 3.9.1. The hard function \( H(Q^2) \) for \( e^+e^- \rightarrow q\bar{q} \) is known to high orders and its non-cusp anomalous dimension \( \gamma_H \) is known at three-loop order [121, 122]; we present the relevant pieces in App. 3.9.2. For arbitrary angular exponents \( \alpha \) and \( \beta \), little else in the factorization theorem is known at sufficiently high accuracy to resum to NNLL.

The goal of this section is to fill in the rest of the table, to achieve full NNLL accuracy. We start in Sec. 3.5.1 restricting to \( \alpha = 2 \) (jet mass) and \( \beta = 0 \) (mMDT groomer). For this case, all of the missing ingredients can be determined by recycling results from the literature, up to calculable clustering effects from the soft drop algorithm. In Sec. 3.5.3, we consider \( \alpha = 2 \) and \( \beta \geq 0 \) and demonstrate that one can extract unknown two-loop non-cusp anomalous dimensions with EVENT2. It is possible to extend our analysis to angular exponents for the energy correlation functions beyond \( \alpha = 2 \), but we do not do it in this paper.\(^3\)

\(^3\)The two-loop non-cusp anomalous dimension of the soft function for event-wide (recoil-free) angularities [20,97–99] as a function of the angular exponent has been calculated in Ref. [96]. Recoil-free angularities and two-point energy correlation functions have identical anomalous dimensions [99] and could be used in the same way as the calculation for \( \alpha = 2 \).
3.5.1 NNLL for $\alpha = 2, \beta = 0$

We first consider angular exponents $\alpha = 2$ and $\beta = 0$. In this case, the soft drop requirement enforced at every branching reduces to an energy cut

$$\min[E_i, E_j] > z_{\text{cut}}(E_i + E_j). \quad (3.47)$$

On the soft-drop groomed jets we then measure

$$e_{2}^{(2)} = \frac{m_{g}^{2}}{E_{g}^{2}}, \quad (3.48)$$

where the subscript $g$ denotes that the mass and energy are measured on the groomed jet.

The jet functions in the factorization theorem are independent of the soft drop groomer, so we are able to use results from the literature for these. The inclusive jet function has been calculated to two loops \cite{123–126} and the non-cusp anomalous dimension of the inclusive jet function is known to three loops \cite{127, 128}. We present the relevant expressions in App. 3.9.6.

This leaves the soft function $S_G(z_{\text{cut}})$ and the collinear-soft function $S_C(z_{\text{cut}} e_{2}^{(2)})$ to be determined. Their one-loop expressions are easily calculable, and we present the results in App. 3.9.5 and App. 3.9.9. To determine their two-loop non-cusp anomalous dimensions, we exploit the renormalization group consistency of the factorization theorem. The sum of the anomalous dimensions must vanish at each order:

$$0 = \gamma_H + \gamma_S + 2\gamma_J + 2\gamma_{S_C}, \quad (3.49)$$

where $\gamma_F$ denotes the anomalous dimension of function $F$ in the factorization theorem, and we have used the symmetry of the left and right hemispheres of the event. Therefore,
only one unknown anomalous dimension remains, which we take to be $\gamma_S$.

**Two-Loop Soft Function**

To calculate the two-loop non-cusp anomalous dimension $\gamma_S$ we need to calculate the soft function $S_G(z_{cut})$ with two real emissions. The two-loop expression for the soft function is

$$S_G(z_{cut}) = \int [d^d k_1]_+ [d^d k_2]_+ |\mathcal{M}(k_1, k_2)|^2 \Theta_{SD}.$$  \hspace{1cm} (3.50)

Here, $[d^d k_1]_+$ is the positive-energy on-shell phase space measure in $d = 4 - 2\epsilon$ dimensions:

$$[d^d k_1]_+ = \frac{d^d k_1}{(2\pi)^d} 2\pi \delta(k_1^2) \Theta(k_1^0), \hspace{1cm} (3.51)$$

and $|\mathcal{M}(k_1, k_2)|^2$ is the squared matrix element for two soft emissions from a $q\bar{q}$ dipole. The explicit expression for $|\mathcal{M}(k_1, k_2)|^2$ can be found in Ref. [129]. $\Theta_{SD}$ is the phase space constraint imposed by the soft drop groomer. Recall that, for consistency with the assumed hierarchy $e_2^{(a)} \ll z_{cut}$, soft modes must fail soft drop.

If the particles in the hemispheres are reclustering using the Cambridge/Aachen algorithm, $\Theta_{SD}$ can be written as

$$\Theta_{SD} = \Theta(-\eta_1 \eta_2) \Theta\left(z_{cut} \frac{Q}{2} - k_1^0\right) \Theta\left(z_{cut} \frac{Q}{2} - k_2^0\right)$$

$$+ \Theta(\eta_1 \eta_2) \left[ \Theta(\theta_{1J} - \theta_{12}) \Theta(\theta_{2J} - \theta_{12}) \Theta\left(z_{cut} \frac{Q}{2} - k_1^0 - k_2^0\right) - \Theta(\theta_{1J} - \theta_{12}) \Theta(\theta_{2J} - \theta_{12}) \Theta\left(z_{cut} \frac{Q}{2} - k_1^0\right) \Theta\left(z_{cut} \frac{Q}{2} - k_2^0\right) \right].$$  \hspace{1cm} (3.52)

The first line of Eq. (3.52) corresponds to particles 1 and 2 lying in different hemispheres (opposite rapidity with respect to the $q\bar{q}$ dipole), and so each particle individually must fail soft drop. $Q$ is the center of mass energy and so $Q/2$ is the energy in one hemisphere. The second and third lines correspond to the configuration where both particles lie in
the same hemisphere. $\theta_{12}$ is the angle between the particles and $\theta_{ij}$ (for $i = 1, 2$) is the angle particle $i$ makes with that hemisphere’s axis. If $\theta_{12}$ is less than both $\theta_{1J}$ and $\theta_{2J}$ then, according to the Cambridge/Aachen algorithm, the soft particles are clustered first. Therefore, the sum of the energies of particles 1 and 2 must fail soft drop. If instead one of the particles is closer to the jet axis, then they are clustered separately and must individually fail soft drop.

To proceed, we separate the squared matrix element into Abelian and non-Abelian pieces, according to their color coefficient. At this order, the squared matrix element takes the form

$$|\mathcal{M}(k_1, k_2)|^2 = |\mathcal{M}_{n-A}(k_1, k_2)|^2 + \frac{1}{2!} |\mathcal{M}(k_1)|^2 |\mathcal{M}(k_2)|^2,$$

(3.53)

Here, “n-A” denotes the non-Abelian component of the squared matrix element, which includes the $C_F C_A$ and $C_F n_f T_R$ color channels. The Abelian contribution is just the symmetrized product of the one-loop result, with a color factor of $C_F^2$. We will consider these two pieces separately, starting with the non-Abelian term.

**Non-Abelian Clustering Effects**

Note that except for the effects from Cambridge/Aachen clustering, soft drop is just imposing a soft energy veto on each hemisphere. The two-loop soft function with a soft energy veto was calculated in Ref. [90]. That calculation showed that the two-loop Abelian piece (proportional to $C_F^2$) to the energy vetoed soft function satisfies non-Abelian exponentiation. The two-loop non-cusp anomalous dimension for a hemisphere energy vetoed soft function is then purely non-Abelian and was extracted in Ref. [89]. The non-Abelian
part of the soft function with an energy veto at two-loops is

\[ S_{veto\,n-A,\alpha_s^2} = \int [d^d k_1] + [d^d k_2] + |M_{n-A}(k_1, k_2)|^2 \Theta_{veto}. \]  

(3.54)

The phase space cut \( \Theta_{veto} \) is

\[ \Theta_{veto} = \Theta \left( \Lambda - k_1^0 - k_2^0 \right), \]

where \( \Lambda \) is the veto scale. We can then write the two-loop soft function for soft drop as

\[ S_G(z_{cut})|_{n-A, \alpha_s^2} = S_{veto\,n-A, \alpha_s^2} + \int [d^d k_1] + [d^d k_2] + |M_{n-A}(k_1, k_2)|^2 [\Theta_{SD} - \Theta_{veto}], \]  

(3.55)

where the veto scale is set to \( \Lambda = z_{cut} Q/2 \). The difference between the soft drop and energy veto phase space constraints is purely a clustering effect, given by

\[ \Theta_{SD} - \Theta_{veto} = \left\{ \Theta(\eta_1 \eta_2) \left[ 1 - \Theta(\theta_{1f} - \theta_{12})\Theta(\theta_{2f} - \theta_{12}) \right] + \Theta(-\eta_1 \eta_2) \right\} \]

\[ \times \Theta \left( z_{cut} Q/2 - k_1^0 \right) \Theta \left( z_{cut} Q/2 - k_2^0 \right) \Theta \left( k_1^0 + k_2^0 - z_{cut} Q/2 \right). \]

Eq. (3.55) enables us to calculate much more simply the two-loop non-cusp anomalous dimension of the soft function. The anomalous dimension can then be written as

\[ \gamma_S = \gamma_{veto} + \gamma_{C/A}. \]  

(3.57)

\( \gamma_{veto} \) is the two-loop non-cusp anomalous dimension of \( S_{veto} \) extracted in Ref. [89]:

\[ \gamma_{veto}^2 = \left( \frac{\alpha_s}{4\pi} \right)^2 C_F \left[ \left( \frac{1616}{27} - 56 \zeta_3 \right) C_A - \frac{448}{27} n_f T_R - \frac{2\pi^2}{3} \beta_0 \right], \]  

(3.58)
where \( b_0 \) is the one-loop \( \beta \)-function coefficient:

\[
\beta_0 = \frac{11}{3} C_A - \frac{4}{3} n_f T_R .
\] (3.59)

Then, we only need to determine the contribution to the anomalous dimension from residual Cambridge/Aachen clustering effects, \( \gamma_{C/A} \).

The non-Abelian clustering effects are contained in

\[
S_G(z_{\text{cut}})_{n \cdot A, \alpha_s^2}^{C/A} = \int [d^d k_1] + [d^d k_2] + |\mathcal{M}_{n \cdot A}(k_1, k_2)|^2 [\Theta_{\text{SD}} - \Theta_{\text{veto}}] .
\] (3.60)

The squared non-Abelian matrix element does not have collinear singularities when the angle of the particles from the jet axis is strongly ordered. Therefore, in this integral there is only a collinear divergence when the two emissions become collinear to the jet axis in a non-strongly ordered way. The coefficient of this divergence is proportional to the correction to the two-loop anomalous dimension due to clustering effects in the non-Abelian color channel. The divergence can be extracted with the standard plus-function prescription and the correction to the anomalous dimension can be found. While we were unable to find an analytic expression, its approximate numerical value is

\[
S_G(z_{\text{cut}})_{n \cdot A, \alpha_s^2}^{C/A} = \left( \frac{\alpha_s}{4\pi} \right)^2 C_F \left[ -9.31 C_A - 14.04 n_f T_R \right] \left( \frac{4\mu^2}{Z_{\text{cut}}^2 Q^2} \right)^{2\epsilon} \frac{1}{4\epsilon} + \mathcal{O}(\epsilon^0) .
\] (3.61)

The contribution to the anomalous dimension is then

\[
\gamma_{n \cdot A, \alpha_s^2}^{C/A} = \left( \frac{\alpha_s}{4\pi} \right)^2 C_F \left[ -9.31 C_A - 14.04 n_f T_R \right] .
\] (3.62)

\footnote{This anomalous dimension does not seem to be a linear combination of the usual transcendental numbers appearing in other two-loop anomalous dimensions.}
Abelian Clustering Effects

The Abelian contribution can be calculated similarly. However, unlike the non-Abelian contribution, the exponentiation of the one-loop result will describe at least some of the two-loop Abelian piece. If the square of the one-loop result does not account for all of the two-loop result, then non-Abelian exponentiation breaks down. This does not mean that exponentiation breaks down or that the cross section cannot be resummed, just that the anomalous dimension of the purely Abelian piece will need to be corrected at every logarithmic order. So, for the two-loop non-cusp anomalous dimension, we need to determine the part of the soft function that is not accounted for by non-Abelian exponentiation.

To do this, we start from the full expression for the Abelian term at two-loops:

\[
S_G(z_{\text{cut}}) |_{A,a_s^2} = \frac{1}{2!} \int [d^d k_1] + [d^d k_2] + |\mathcal{M}(k_1)|^2 |\mathcal{M}(k_2)|^2 \Theta_{SD}. \tag{3.63}
\]

We then add and subtract the one-loop phase space constraints:

\[
S_G(z_{\text{cut}}) |_{A,a_s^2} = \frac{1}{2!} \int [d^d k_1] + [d^d k_2] + |\mathcal{M}(k_1)|^2 |\mathcal{M}(k_2)|^2 \Theta \left( z_{\text{cut}} \frac{Q}{2} - k_1^0 \right) \Theta \left( z_{\text{cut}} \frac{Q}{2} - k_2^0 \right)
+ \frac{1}{2!} \int [d^d k_1] + [d^d k_2] + |\mathcal{M}(k_1)|^2 |\mathcal{M}(k_2)|^2 \left[ \Theta_{SD} - \Theta \left( z_{\text{cut}} \frac{Q}{2} - k_1^0 \right) \Theta \left( z_{\text{cut}} \frac{Q}{2} - k_2^0 \right) \right]. \tag{3.64}
\]

The difference between the phase space constraints is a clustering effect, given by

\[
\Theta_{SD} - \Theta \left( z_{\text{cut}} \frac{Q}{2} - k_1^0 \right) \Theta \left( z_{\text{cut}} \frac{Q}{2} - k_2^0 \right) \tag{3.65}
\]

\[
= -\Theta(\eta_1 \eta_2) \Theta(\theta_{1J} - \theta_{12}) \Theta(\theta_{2J} - \theta_{12})
\times \Theta \left( z_{\text{cut}} \frac{Q}{2} - k_1^0 \right) \Theta \left( z_{\text{cut}} \frac{Q}{2} - k_2^0 \right) \Theta \left( k_1^0 + k_2^0 - z_{\text{cut}} \frac{Q}{2} \right).
\]
As with the non-Abelian term, this phase space constraint completely removes all soft divergences and the strongly-ordered collinear limit. The remaining divergence can be isolated by standard plus-function techniques. For the two-loop Abelian Cambridge/Aachen clustering term, we find the numerical result

\[ S_G(z_{\text{cut}}) |_{\text{C/A}} = \left( \frac{\alpha_s}{4\pi} \right)^2 34.01 C_F^2 \left( \frac{4\mu^2}{z_{\text{cut}}^2 Q^2} \right)^{2\epsilon} \frac{1}{4\epsilon} + \mathcal{O}(\epsilon^0), \] (3.66)

for the second integral in Eq. (3.64). The contribution to the anomalous dimension is then

\[ \gamma_{\text{C/A}}^{\alpha_s^2} = \left( \frac{\alpha_s}{4\pi} \right)^2 34.01 C_F^2. \] (3.67)

**Two-Loop Anomalous Dimension and Comparison with EVENT2**

Combining Eqs. (3.58), (3.62) and (3.67), the total two-loop non-cusp anomalous dimension for the soft function is

\[ \gamma_s^{\alpha_s^2} = \left( \frac{\alpha_s}{4\pi} \right)^2 C_F \left[ 34.01 C_F + \left( \frac{1616}{27} - 56\zeta_3 - 9.31 \right) C_A - \left( \frac{448}{27} + 14.04 \right) n_f T_R - \frac{2\pi^2}{3} \beta_0 \right]. \] (3.68)

The two-loop non-cusp anomalous dimension for the collinear-soft function is found by consistency using Eq. (3.49). Note that this anomalous dimension has no log \(z_{\text{cut}}\) terms. Therefore, the anomalous dimensions of no functions in the factorization theorem have log \(z_{\text{cut}}\) dependence. This is a consequence of the fact that each function of our factorization theorem in Eq. (3.19) depends on a single infrared scale, allowing NNLL resummation of all logarithms of \(z_{\text{cut}}\) and \(e_2^{(2)}\) alike. As we discuss in Secs. 3.3.3 and 3.5.2, this result relies on the choice of Cambridge/Aachen reclustering in the soft drop algorithm.

We can verify this result by comparing the resummed distribution, truncated at \(\mathcal{O}(\alpha_s^2)\), with the singular region of the full QCD result, computed to the same fixed order. For the
Figure 3.4: Verification of our factorization theorem at $\mathcal{O}(\alpha_s^2)$ for soft-drop grooming with $z_{\text{cut}} = 0.001$ and $\beta = 0$. Solid curves are numerical results from EVENT2, and dashed curves are $\mathcal{O}(\alpha_s^2)$ terms in our NNLL distribution, plotted in the three color channels $C_F$, $C_F C_A$, and $C_F n_f T_R$. (a) shows a direct comparison and (b) the difference.

full QCD result, we have implemented soft drop into EVENT2 [34], a Monte Carlo code that generates fixed-order results up to $\mathcal{O}(\alpha_s^2)$ in $e^+ e^-$ collisions. Our specific implementation is as follows. We generate $e^+ e^-$ collisions at 1 TeV center of mass energy and identify event hemispheres with the exclusive $k_T$ algorithm [101]. We then reclustering each hemisphere using the Cambridge/Aachen algorithm and apply soft drop with $\beta = 0$. On each of the soft-drop groomed hemispheres, we then measure the energy correlation function $e^{(2)}_2$ and record the larger of the two values, which we denote by $e^{(2)}_{2,H}$ and refer to as the heavy groomed mass. This is simply related to the cross section of our factorization theorem:

$$
\frac{d\sigma}{de^{(2)}_{2,H}} = \int \frac{d\sigma^{(2)}_{e^{(2)}_2}}{de^{(2)}_{2,L} de^{(2)}_{2,R}} \frac{d^2\sigma}{de^{(2)}_{2,L} de^{(2)}_{2,R}} \left[ \Theta \left( e^{(2)}_{2,L} - e^{(2)}_{2,R} \right) \delta \left( e^{(2)}_{2,H} - e^{(2)}_{2,L} \right) + (L \leftrightarrow R) \right].
$$

In Fig. 3.4, we compare EVENT2 results to the prediction of the factorized expression at NNLL expanded to $\mathcal{O}(\alpha_s^2)$. For soft drop with $\beta = 0$, soft logarithms are removed, which
means that at $\mathcal{O}(\alpha_s^2)$, the cross section has the schematic form

$$
e^{(2)}_{2,H} \frac{d\sigma^{\alpha_s^2}}{d\epsilon^{(2)}_{2,H}} \sim \alpha_s^2 C_0 \log \epsilon^{(2)}_{2,H} + \alpha_s^2 C_1,$$

(3.70)

where $C_0$ and $C_1$ are constants. We plot the cross section separated into the three color channels ($C_F^2$, $C_FC_A$, and $C_F n_f T_R$). We set $\zeta_{\text{cut}} = 0.001$ to suppress power corrections of $\zeta_{\text{cut}}$. Excellent agreement between our factorization theorem and EVENT2 is observed at small $\epsilon^{(2)}_{2}$, demonstrating that we have captured all singular terms of the full QCD result in our factorization theorem to $\mathcal{O}(\alpha_s^2)$.

### 3.5.2 Reclustering with anti-$k_T$

It is illuminating to study the clustering effects in the soft function in more detail. In this section, we re-calculate the clustering effects with the anti-$k_T$ algorithm, instead of the standard Cambridge/Aachen algorithm. We find that the clustering effects with the anti-$k_T$ algorithm are intimately related to the corresponding effects calculated in jet veto calculations. This can be understood relatively simply by re-expressing the clustering conditions in a form analogous to the clustering metric of the longitudinally-invariant $k_T$ algorithm.

To calculate the two-loop soft function for soft drop defined with anti-$k_T$ reclustering, we only need to calculate the clustering effects unique to this algorithm. We will denote the phase space constraints for the anti-$k_T$ reclustering as $\Theta_{\text{SD}}^{\alpha_kT}$, but we will not explicitly present them here. The two-loop soft function is

$$S^{\alpha_kT}(\zeta_{\text{cut}}) \bigg|_{\alpha_s^2} = \int [d^d k_1] + [d^d k_2] + |\mathcal{M}(k_1, k_2)|^2 \Theta_{\text{veto}} + \int [d^d k_1] + [d^d k_2] + |\mathcal{M}(k_1, k_2)|^2 \left[ \Theta_{\text{SD}}^{\alpha_kT} - \Theta_{\text{veto}} \right].$$

(3.71)
The relevant phase space constraints can be written as
\[
\Theta_{SD}^{ak_T} - \Theta_{veto} = \Theta(\eta_1\eta_2) \left[ 1 - \Theta \left( \max[k_1^0, k_2^0]\theta_{1J} - \frac{Q}{2}\theta_{12} \right) \Theta \left( \max[k_1^0, k_2^0]\theta_{2J} - \frac{Q}{2}\theta_{12} \right) \right] \\
\times \Theta \left( z_{\text{cut}} \frac{Q}{2} - k_1^0 \right) \Theta \left( z_{\text{cut}} \frac{Q}{2} - k_2^0 \right) \Theta \left( k_1^0 + k_2^0 - z_{\text{cut}} \frac{Q}{2} \right). \tag{3.72}
\]

With this, we can calculate the divergent part of the two-loop soft function from clustering effects and extract the anomalous dimension. As with Cambridge/Aachen, we can write the two-loop non-cusp anomalous dimension as
\[
\gamma_{S}^{ak_T} = \gamma_{veto} + \gamma_{ak_T}, \tag{3.73}
\]
where \(\gamma_{ak_T}\) is the part of the anomalous dimension purely from clustering effects. We find
\[
\gamma_{ak_T} = -8 \left( \frac{\alpha_s}{4\pi} \right)^2 C_F \left\{ \left[ \left( \frac{131}{9} - \frac{4}{3}\pi^2 - \frac{44}{3} \log 2 \right) C_A + \left( -\frac{46}{9} + \frac{16}{3} \log 2 \right) n_f T_R \right] \log z_{\text{cut}} \right. \\
+ \left( -\frac{269}{6} + \frac{7}{2}\zeta_3 + \frac{274}{9} \log 2 + \frac{11\pi^2}{9} + \frac{44}{3} \log^2 2 \right) C_A \\
+ \left( \frac{53}{3} - \frac{4\pi^2}{9} - \frac{116}{9} \log 2 + \frac{16}{3} \log^2 2 \right) n_f T_R \right\}. \tag{3.74}
\]

This anomalous dimension is fascinating. First, note that there is no \(C_F^2\) term, implying that non-Abelian exponentiation holds for anti-\(k_T\) reclustering, in contrast to what we found for the Cambridge/Aachen algorithm. That is, all logarithms at \(O(\alpha_s^2)\) with color factor \(C_F^2\) are accounted for by exponentiating the one-loop result. This is to be expected: Since the anti-\(k_T\) algorithm clusters soft gluons (with energy fractions of order \(z_{\text{cut}}\)) one-by-one with the hard jet core unless two soft gluons have angular separation \(\Delta R \lesssim z_{\text{cut}}\), clustering effects are merely a power correction for Abelian gluons.

Also, unlike the case for Cambridge/Aachen reclustering, there is explicit \(\log z_{\text{cut}}\) dependence in the anomalous dimension of Eq. (3.74). This shows that we do not resum
logarithms of $z_{\text{cut}}$ to full NNLL accuracy when anti-$k_T$ clustering is used in soft drop. The coefficient of the log $z_{\text{cut}}$ term is identical to the coefficient of the logarithm of the jet radius $R$ found from clustering effects in jet veto calculations [91–95]. This connection between soft drop and jet veto calculations can be made clearer by a simple rewriting of the clustering metric.

The $k_T$ class of clustering metrics for $e^+e^-$ collisions can be written as

$$d_{ij} = \min \left[ E_{2p}^i, E_{2p}^j \right] \theta_{ij}^2,$$  \hspace{1cm} (3.75)

for particles $i$ and $j$, with $p$ an integer that defines the jet algorithm. In the soft function, soft particles are either clustered with each other or with the jet axis. For $\beta = 0$, these soft particles have characteristic energy fraction $z_{\text{cut}} \ll 1$. In terms of energy fractions, the clustering metric of two soft particles is

$$d_{ij} = \min \left[ z_{2p}^i, z_{2p}^j \right] Q^{2p} \theta_{ij}^2 \sim z_{\text{cut}}^{2p} Q^{2p} \theta_{ij}^2,$$  \hspace{1cm} (3.76)

and for a soft particle $i$ with the jet axis it is

$$d_{i} = \min [1, z_{\text{cut}}^{2p}] Q^{2p} \theta_{i}^2,$$  \hspace{1cm} (3.77)

where $\theta_{i}$ is the angle between particle $i$ and the jet axis.

Consider $p < 0$. In this case, the two soft gluons are (parametrically) clustered together when

$$z_{\text{cut}}^p \theta_{ij} < \min [\theta_i, \theta_j],$$  \hspace{1cm} (3.78)

or equivalently, when

$$\theta_{ij} < z_{\text{cut}}^{|p|} \min [\theta_i, \theta_j].$$  \hspace{1cm} (3.79)
The effective clustering metric in this case is then

\[ d_{ij}^{\text{eff}} = \min \left( z_i^{2p}, z_j^{2p} \right) \frac{\theta_{ij}^2}{z_{\text{cut}}^{2|p|} \min[\theta_i^2, \theta_j^2]}, \quad d_i^{\text{eff}} = z_i^{2p}. \] (3.80)

With \( p = -1 \), this is the clustering metric for the inclusive anti-\( k_T \) algorithm with effective jet radius \( R = z_{\text{cut}} \ll 1 \). There will now be logarithms of the jet radius that arise. The \( \log z_{\text{cut}} \) term in the anomalous dimension has the identical coefficient as the \( \log R \) term in jet veto calculations because \( z_{\text{cut}} \) and \( R \) act as the angular scale for collinear splittings in the respective soft functions.

In summary, while we could use anti-\( k_T \) to recluster the jet for soft drop grooming, we could not resum all large logarithms to the same precision without a different factorization theorem. Therefore, reclustering in soft drop with the Cambridge/Aachen algorithm is preferred from a theory perspective.

### 3.5.3 NNLL for \( \alpha = 2, \beta \geq 0 \)

For soft drop with angular exponent \( \beta > 0 \), we cannot recycle results from the literature to reach NNLL precision. Instead, a completely new two-loop calculation of either the soft or collinear-soft function is needed. But without such a calculation, we can perform NNLL resummation for particular values of \( \beta > 0 \), using numerical simulations to estimate the ingredients we lack. We will demonstrate this explicitly in the case of \( \beta = 1 \), and the result will allow us to study features of NNLL distributions for energy correlation functions with less aggressive grooming.

The same method we used to validate anomalous dimensions for \( \beta = 0 \) can be used to extract the anomalous dimension for \( \beta > 0 \). This method relies on the fact that all ingredients necessary for NNLL resummation with \( \alpha = 2, \beta > 0 \) are known except the two-loop non-cusp anomalous dimensions of the soft and collinear-soft functions. As
mentioned above, renormalization group invariance determines one of these, say $\gamma_{S_C}^{(1)}$, in terms of the other anomalous dimensions. So only one unknown, $\gamma_S^{(1)}$, remains and we can extract it at fixed order.

To do this for a given $\beta > 0$, we can use EVENT2 to obtain numerical results at $O(\alpha_s^2)$ for the groomed $e_{2,H}^{(2)}$ distribution with several moderately small values of $z_{\text{cut}}$. From each of these distributions, we can subtract the known terms, which we get by expanding the NNLL distribution to fixed order. This leaves a term proportional to the unknown $\gamma_S^{(1)}$, as well as power corrections suppressed by $e_{2,H}^{(2)}$ or $z_{\text{cut}}$. By computing the distribution down to very small $e_{2,H}^{(2)}$, we can ignore the $e_{2,H}^{(2)}$ power corrections. Reducing power corrections from $z_{\text{cut}}$ is limited by the numerical precision of EVENT2 because our factorization theorem only applies for $e_{2,H}^{(2)} \ll z_{\text{cut}}$. Instead, we can fit the $z_{\text{cut}}$ power corrections to linear combinations of $z_{\text{cut}} \log^n(z_{\text{cut}}) \log^m(e_{2,H}^{(2)})$. At $O(\alpha_s^2)$ it is appropriate to use $0 \leq n + m \leq 3$, though in practice we found terms with $m \geq 2$ to be difficult to fit. With the non-negligible power corrections thus removed, we can then extract the remaining anomalous dimension.

While the procedure outlined above is straightforward, an explicit calculation of $\gamma_S^{(1)}$ or $\gamma_{S_C}^{(1)}$ for $\beta > 0$ is of course desirable. On practical time scales, numerical extractions are limited to rough approximations, due to inadequate numerical precision in the deep infrared. Nevertheless, an estimate is sufficient for our purposes here, which are to demonstrate the advantages of resumming jet substructure observables to NNLL, and to examine various levels of grooming. Thus, we will test the above procedure on $\beta = 0$, and learn about the associated uncertainties by comparing with our direct calculation, Eq. (3.68). Then we will move to $\beta = 1$, and extract $\gamma_S^{(1)}$ in that case.

In Fig. ?? we show numerical results at $O(\alpha_s^2)$ from EVENT2 with $\beta = 0$ and $z_{\text{cut}} = 0.1$. Also shown is the NNLL distribution, expanded to fixed order, but without the $\gamma_S^{(1)}$ term. The discrepancy between the curves is thus due to the missing $\gamma_S^{(1)}$ term and $z_{\text{cut}}$ power corrections.
Figure 3.5: Demonstration of non-cusp anomalous dimension extraction in EVENT2. (a) Solid curves are numerical results from EVENT2 at $O(a_s^2)$ with $\beta = 0$ and $z_{\text{cut}} = 0.1$. Dashed curves are $O(a_s^2)$ terms in NNLL distribution, without the term proportional to $\gamma_S^{(1)}$. Discrepancy results from $z_{\text{cut}}$ power corrections in solid curves and missing $\gamma_S^{(1)}$ in dashed curves. Subtracting $z_{\text{cut}} \log^n(z_{\text{cut}}) \log(e^{2H})$ power corrections from dashed curves, we extract the remaining offsets. (b) As $z_{\text{cut}} \to 0$, remaining offsets allow extraction of $\gamma_S^{(1)}$ in rough agreement with Eq. (3.68).

corrections. Using several distributions like this one, with values of $z_{\text{cut}}$ between $10^{-4}$ and $10^{-1}$, we fit the $z_{\text{cut}}$ power corrections. Fig. ?? shows the remaining offsets between our analytical curves and the results of EVENT2, after $z_{\text{cut}} \log^n(z_{\text{cut}}) \log(e^{2H})$ power corrections have been subtracted. On each point in this plot, the error bar represents the standard deviation in EVENT2 output, across $e^{2H}$ bins. The offset that remains as $z_{\text{cut}} \to 0$ is the $\gamma_S^{(1)}$ we would extract using this method. One can see from Fig. ?? that agreement with our analytical calculation, Eq. (3.68), is quite good, with some discrepancy in the $C_F$ channel.

Table 3.2 lists the numerical results of $\gamma_S^{(1)}$ using this method. The uncertainties quoted for the $\beta = 0$ extraction in the table come from the standard deviation in EVENT2 output across $e^{2H}$ bins, which introduces an error in the identification of constant offsets. These should be compared with our direct calculation in the second line of the table.

5In carrying out the procedure just described, we tuned EVENT2 parameters to favor the infrared. In particular, we use of order 1 trillion events, with CUTOFF $= 10^{-15}$ and phase-space sampling exponents NPOW1 = NPOW2 = 5. This procedure corresponded to centuries of CPU time.
<table>
<thead>
<tr>
<th>Soft Drop $\gamma_S^{(1)}$</th>
<th>$C_F$</th>
<th>$C_A$</th>
<th>$n_f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta = 0$ extraction</td>
<td>28 ± 1.5</td>
<td>-40 ± 1</td>
<td>-23 ± 3</td>
</tr>
<tr>
<td>$\beta = 0$ calculation</td>
<td>34.01</td>
<td>-40.90</td>
<td>-21.86</td>
</tr>
<tr>
<td>$\beta = 1$ extraction</td>
<td>6 ± 12</td>
<td>-9.5 ± 2</td>
<td>-8 ± 7</td>
</tr>
</tbody>
</table>

Table 3.2: Extraction of two-loop non-cusp anomalous dimension $\gamma_S^{(1)}$ of wide-angle soft function in the three different color channels. For $\beta = 0$ comparison with our direct calculation is possible. See text for discussion of uncertainties.

The discrepancy in the $C_F$ channel gives us a sense of additional numerical uncertainties, which are significant. Similar disagreements have been encountered before, e.g. in Ref. [42], in the context of $C_F$ channel extractions from EVENT2, and to resolve it might require significantly longer run times.

As stated above, a rough estimate of $\gamma_S^{(1)}$ for $\beta > 0$ is sufficient for our purposes, so we have applied the method described above to the case of $\beta = 1$. See the third line of Table 3.2 for the results of the extraction. Uncertainties quoted in this line of the table have two sources: (i) variance in EVENT2 output, and (ii) additional numerical precision issues, which we took to be the difference (both absolute and relative) between extraction and direct calculation in the $\beta = 0$ test. In each color channel, we took the maximum of these uncertainties and inflated it by a factor of 2.

The estimate in Table 3.2 allows us to study NNLL distributions of $e_2^{(2)}$ groomed with $\beta = 1$. In the resulting distributions, the uncertainties associated with the imperfect extraction are relatively small; e.g. see Fig. ?? below. Still, a direct calculation of either $\gamma_S^{(1)}$ or $\gamma_{SC}^{(1)}$ for $\beta > 0$ would of course be preferred, but we leave this to future work.

### 3.6 Matching NNLL to Fixed Order in $e^+e^- \to$ dijets

Using the results calculated in the previous sections, here we match our resummed differential cross section for soft-drop groomed energy correlation functions to fixed-order
for hemisphere jets produced in $e^+e^-$ collisions. We first match resummed results at NLL and NNLL to $O(\alpha_s)$ and $O(\alpha_s^2)$, respectively, using EVENT2 and demonstrate that theoretical uncertainties are greatly reduced at NNLL. We then compare several Monte Carlo parton shower simulations to our matched NNLL results. We compare both parton and hadron level Monte Carlo to our perturbative analytic results, and include a simple model of hadronization in our calculation. We leave a detailed understanding and justification of incorporating hadronization into the resummed and matched cross section to future work.

### 3.6.1 Matching Resummation to Fixed-Order

With the explicitly calculated and extracted two-loop non-cusp anomalous dimensions of the soft function in the soft drop factorization theorem Eq. (3.19), we are able to resum the differential cross section through NNLL accuracy in the region where $e_2^{(2)} \ll z_{\text{cut}} \ll 1$. Anomalous dimensions of all functions are collected in the appendices and we present the explicit form of the resummed cross section in App. 3.9.10. This resummed cross section is only valid in the region where $e_2^{(2)} \ll z_{\text{cut}} \ll 1$, and will not provide an accurate description of the cross section outside this region. To accurately describe the cross section throughout the full phase space requires matching the resummed result to fixed-order.

While there are many ways to do this at various levels of sophistication, we choose to use simple additive matching. That is, we construct matched distributions according to

$$\frac{d\sigma_{\text{match}}}{de_2^{(2)}} = \frac{d\sigma_{\text{resum}}}{de_2^{(2)}} + \frac{d\sigma_{\text{FO}}}{de_2^{(2)}} - \frac{d\sigma_{\text{resum,FO}}}{de_2^{(2)}}. \tag{3.81}$$

Here, $d\sigma_{\text{resum}}$ is the resummed cross section, calculated to the appropriate logarithmic accuracy. $d\sigma_{\text{FO}}$ is the fixed-order differential cross section calculated to a particular order in $\alpha_s$. $d\sigma_{\text{resum,FO}}$ is the resummed cross section truncated at the same accuracy as the
Figure 3.6: (a) NLL matched distributions for heavy hemisphere $e_2^{(2)}$ in $e^+e^-$ collisions with soft drop grooming $z_{cut} = 0.1$ and $\beta = 0$, $\beta = 1$, and without soft drop. Estimates of theoretical uncertainties are represented by the shaded bands. (b) The corresponding matched distributions at NNLL. For soft drop with $\beta = 1$, the dotted lines represent the extent of the theoretical uncertainties when the variation of the two-loop non-cusp anomalous dimension is included. Note the significant reduction in uncertainties at NNLL.

The logarithmic accuracy of the resummed cross section was defined in Sec. 3.5, and here we specify the fixed orders that we use in the matching procedure. We additively match the analytic NLL distributions to $O(\alpha_s)$ fixed order results, which include one real emission from the $q\bar{q}$ dipole. We match NNLL distributions to $O(\alpha_s^2)$ results, which include up to two real emissions. EVENT2 is able to generate $e^+e^-$ collisions through $O(\alpha_s^2)$, except for the two-loop virtual contribution. The two-loop virtual term only contributes at $e_2^{(2)} = 0$, so our differential distributions are unaffected by this omission.

In Fig. 3.6, we plot the resummed and matched differential cross sections for the larger $e_2^{(2)}$ of the two hemispheres at NLL and NNLL with various levels of soft drop grooming.
Here, we consider dijet production in $e^+e^-$ collisions at 1 TeV center-of-mass energy and identify hemispheres with the exclusive $k_T$ algorithm [101]. The parameters of soft drop are $z_{\text{cut}} = 0.1$ and we show both $\beta = 0$ and $\beta = 1$. We also show the ungroomed heavy hemisphere $e_2^{(2)}$ distribution. In these plots, we include estimates of theoretical uncertainties represented by the lighter bands about the central curve. While more sophisticated methods for estimating uncertainties exist, we simply vary the natural scales that appear in the functions of the factorization theorem up and down by a factor of two. We then take the envelope of these scale variations as an estimate of theoretical uncertainties. This simple prescription is sufficient for our main purpose in showing uncertainty bands: to demonstrate the reduction in theoretical uncertainty in moving from NLL to NNLL.

Included in these uncertainty estimates is a variation in our treatment of the Landau pole of the strong coupling $\alpha_s$. For scales $\mu > 1$ GeV, $\alpha_s$ is evaluated according to its perturbative running. For $\mu < 1$ GeV, we freeze $\alpha_s$ to its value at the scale $\mu = 1$ GeV. This is not intended to be a model for hadronization or non-perturbative physics, but is just intended to maintain finite cross section predictions at small $e_2^{(2)}$ values. To estimate the sensitivity of our results to the scale at which we freeze the coupling, we vary this 1 GeV scale by a factor of two, and include the effect in the uncertainty bands of Fig. 3.6 as well.

Finally, we have shown the uncertainty bands around the $\beta = 1$ curves at NNLL with and without the uncertainty in our estimate of the two-loop non-cusp anomalous dimension of the soft function. One can see from the figure that this imperfect extraction has only a relatively small effect on the overall uncertainty at this order.

Importantly, we allow the normalization of the cross section to change under these scale variations. That is, the curves in Fig. 3.6 are constructed according to Eq. (3.81). The normalization of each distribution displayed is meaningful, since we resum all large logs in both the shape and the normalization. While the central value curves don’t change much in going from NLL to NNLL, the uncertainties are dramatically reduced, and this
Figure 3.7: Comparison between soft-drop groomed $e_2^{(2)}$ distributions with $z_{\text{cut}} = 0.1$ and $\beta = 0$ for NNLL, parton-level, and hadron-level Monte Carlo. All curves integrate to the same value over the range $e_2^{(2)} \in [0.01, 1]$.

is partly due to the increased accuracy in the normalization.

### 3.6.2 Comparison to Monte Carlo

In this section, we compare our NNLL resummed and matched soft-drop groomed $e_2^{(2)}$ distributions to the output of several standard Monte Carlo simulations. We generate $e^+e^- \rightarrow$ dijets events at 1 TeV center-of-mass collision energy with HERWIG++ 2.7.1 [130,131], PYTHIA 8.210 [132,133], and VINCIA 1.2.02 [134–137]. While the HERWIG++ and PYTHIA events are showered from the leading order process $e^+e^- \rightarrow q\bar{q}$, we consider VINCIA with and without fixed-order matching included. The matched VINCIA results
Figure 3.8: Comparison between soft-drop groomed $e_2^{(2)}$ distributions with $z_{cut} = 0.1$ and $\beta = 1$ for NNLL, parton-level, and hadron-level Monte Carlo. All curves integrate to the same value over the range $e_2^{(2)} \in [0.01, 1]$. The uncertainty band for NNLL includes the variation of the two-loop non-cusp anomalous dimension.

are accurate effectively through $O(\alpha_s^2)$. For the most direct comparison of the simulations to our NNLL matched results, we run $\alpha_s$ at two loops in all Monte Carlos (in the CMW scheme [138, 139]) and we fix $\alpha_s(m_Z) = 0.118$, which is the same value used in our analytic calculations. We include Monte Carlo events both at partonic level and after hadronization. These events are then clustered into hemispheres using the exclusive $k_T$ algorithm [101] using FASTJET 3.1.3 [140]. The soft drop grooming and subsequent measurement of the two-point energy correlation functions of these $e^+e^-$ events is implemented in FASTJET with custom code.
We compare the Monte Carlo distributions to our NNLL resummed and matched calculations in Figs. 3.7 and 3.8. In these plots, all distributions integrate to the same value over the range $e_2^{(2)} \in [0.01, 1]$. In Fig. 3.7, we compare the soft-drop groomed $e_2^{(2)}$ distributions with $z_{\text{cut}} = 0.1$, $\beta = 0$. Good agreement between the Monte Carlos and our analytic calculation is observed, with (not surprisingly) the matched Monte Carlo agreeing the best. These distributions also show that parton- and hadron-level Monte Carlos are essentially identical for $e_2^{(2)} \gtrsim 0.001$. In Fig. 3.8, we compare the soft-drop groomed $e_2^{(2)}$ distributions with $z_{\text{cut}} = 0.1$, $\beta = 1$. Again, good agreement between the Monte Carlos and our matched NNLL result is observed, with the parton- and hadron-level Monte Carlos nearly identical for $e_2^{(2)} \gtrsim 0.005$. The uncertainty bands for the analytic curve includes the uncertainty in the two-loop non-cusp anomalous dimension.

As a more direct comparison of the Monte Carlos, Fig. 3.9 displays the relative difference between each of the hadron-level Monte Carlos and our matched NNLL predictions. Again, soft drop is performed with $z_{\text{cut}} = 0.1$, and both $\beta = 0$ and $\beta = 1$ are shown. All the Monte Carlo curves lie within our shaded band of theoretical uncertainty, but discrep-
ancies between the different simulations are visible.

One striking feature in these plots, especially for $\beta = 0$, is the presence of additional structure in the hadron-level Monte Carlo distributions at small $\epsilon_2^{(2)}$. It is clear that this feature is due to non-perturbative physics, and so is therefore not included in our NNLL calculation. Nevertheless, we can include a simple model of hadronization into our calculation to see if this structure is easily explained.

For additive IRC safe observables, like thrust or jet mass, it can be shown from general principles that hadronization corrections can be incorporated in perturbative distributions by convolution with a model shape function [141, 142]. In general, the energy correlation functions are additive observables, so we should be able to use shape functions to model hadronization corrections. However, once soft drop is applied on the jet, emissions in the jet may or may not contribute to the energy correlation functions, so the observable is no longer strictly additive. We leave a more careful study of whether shape functions can be used to model hadronization effects in groomed observables to future work. Here we convolve our matched results with a simple shape function to see if qualitative agreement with the Monte Carlos can be achieved.

Because shape functions describe non-perturbative physics, they only have support for energies comparable to $\Lambda_{\text{QCD}}$. The shape function we choose is the parametrization suggested by Ref. [143]:

$$F_{\text{shape}}(\epsilon) = \frac{4\epsilon}{\Omega^2}e^{-2\epsilon/\Omega}. \quad (3.82)$$

This is normalized

$$\int_0^\infty d\epsilon F_{\text{shape}}(\epsilon) = 1, \quad (3.83)$$

and has first moment equal to $\Omega$. As discussed in Sec. 3.4.3, of all modes present in our factorization theorem, the collinear-soft mode has the lowest virtuality, so it will have
the largest sensitivity to non-perturbative physics. We thus convolve our perturbative distribution with the shape function, assuming non-perturbative effects are primarily associated with the collinear-soft mode. That is, we include hadronization corrections in the soft drop groomed $\epsilon_2^{(a)}$ distribution according to

$$
\frac{d\sigma_{\text{had}}}{d\epsilon_{\text{cutoff}}^{(a)}} = \int d\epsilon \frac{d\sigma_{\text{pert}}}{d\epsilon_{\text{cutoff}}^{(a)}} \left( \epsilon_{\text{cutoff}}^{(a)} - \left( \frac{\epsilon}{z_{\text{cut}} Q} \right)^{\frac{a-1}{1+b}} \frac{\epsilon}{Q} \right) F_{\text{shape}}(\epsilon),
$$

where the argument of the perturbative distribution is shifted by the virtuality of the collinear-soft mode, Eq. (3.43).

In Fig. 3.10, we compare the matched NNLL distribution of $\epsilon_2^{(2)}$ with and without convolution with the shape function of Eq. (3.82), in which we set $\Omega = 1$ GeV to be comparable to the scale of hadron masses. We show this comparison for soft drop grooming with $\beta = 0$ and $\beta = 1$. The peak at small $\epsilon_2^{(2)}$ for $\beta = 0$ agrees qualitatively with the structure of the hadronized Monte Carlo distributions. Similarly, the shape at small $\epsilon_2^{(2)}$
for $\beta = 1$ agrees with the simulations as well. This suggests that there might exist a shape function for describing hadronization effects in groomed jet observables, though we leave a detailed discussion and justification for such a model to future work.

### 3.7 Matching NNLL to Fixed Order in $pp \rightarrow Z + j$

In this section, we present predictions for soft-drop groomed $e_2^{(\alpha)}$ distributions as measured on the jet in $pp \rightarrow Z + j$ events at the LHC. The definitions of soft drop and energy correlation functions appropriate for jets in $pp$ collisions are given in Sec. 3.2. As with jets from $e^+e^-$ collisions, we match our NNLL resummed distribution to fixed-order results that include relative $O(\alpha_s^2)$ corrections to the Born process.

There are two complications we must deal with. First, at $pp$ collisions, the jets will be both quark and gluon initiated. Second, because we only measure the observable within the jet and do not constrain radiation throughout the rest of the event, the simple hard-soft-jet factorization that we employed for $e^+e^- \rightarrow$ hemisphere jets will not apply here. Nevertheless, while the normalization of the jet-observable distribution will thus be complicated and sensitive to multiple scales, the shape of the distribution will still be controlled exclusively by collinear physics. To address these complications, we first show how soft-drop groomed quark and gluon jets can be unambiguously defined order-by-order in perturbation theory. Then we discuss how the normalization of the distribution can be obtained by matching to full QCD at fixed order. The discussion will focus on the $Z + j$ sample for concreteness, but these ideas apply equally well to any process with hard jets at a hadron collider.
3.7.1 Resummed Cross Section in $pp \rightarrow Z + j$

We define our observable on soft-drop groomed jets in $pp \rightarrow Z + j$ events in the following way. First, we cluster the final state according to a jet algorithm with some jet radius $R \sim 1$. Of the jets with pseudorapidity $|\eta| < \eta_{\text{max}}$, we then identify the jet with the largest transverse momentum $p_{TJ}$ and require that $p_{TJ} > p_{T}^{\min}$. We groom this jet with soft drop and measure $e_{2}^{(a)}$ according to the definitions given in Sec. 3.2 for jets in $pp$ collisions. In this procedure, we remain inclusive over all other hadronic activity in the final state: we only care about the hardest jet.

For this process, the relevant factorization formula is

$$\frac{d\sigma_{\text{resum}}}{de_{2}^{(a)}} = \sum_{k=q,\bar{q},g} D_{k}(p_{T}^{\min}, \eta_{\text{max}}, z_{\text{cut}}, R)S_{C,k}(z_{\text{cut}}e_{2}^{(a)}) \otimes J_{k}(e_{2}^{(a)}). \tag{3.85}$$

Unlike our factorization theorem in $e^{+}e^{-}$ collisions, Eq. (3.85) only resums large logarithms of $e_{2}^{(a)}$ in the limit $e_{2}^{(a)} \ll z_{\text{cut}} \ll 1$. There will be logarithms of $z_{\text{cut}}$ (and other scales in the events) in the $D_{k}$ prefactor that we do not resum. When referring to calculations of this cross section, we will specify the accuracy to which logarithms of $e_{2}^{(a)}$ are resummed (i.e., NLL or NNLL). We now explain the components of this formula in detail.

In Eq. (3.85), $S_{C,k}(z_{\text{cut}}e_{2}^{(a)})$ and $J_{k}(e_{2}^{(a)})$ are the collinear-soft and jet functions for the measurement of soft-drop groomed $e_{2}^{(a)}$ that, by collinear factorization, are identical to the functions defined in $e^{+}e^{-}$ collisions. Unlike in $e^{+}e^{-}$ collisions, however, these functions also have a label $k$ corresponding to the flavor of the jet, and a sum over the possible QCD parton flavors $k$ is included. The symbol $\otimes$ denotes convolution in $e_{2}^{(a)}$ between the collinear-soft and jet functions.

$D_{k}$ is a matching coefficient that can be extracted from fixed-order calculations, and it sets the normalization and relative contributions from the different jet flavors. In addition to the dependence explicitly shown, $D_{k}$ also depends implicitly on parton distributions,
as different initial states produce different flavors of final state jets.

Unlike the case in $e^+e^-$ collisions, where the jet energy was (almost exactly) half the center-of-mass energy, due to the non-trivial parton distributions, the distribution of the jet $p_T$ has a finite width and depends on the cut, $p_T^\text{min}$. For a true precision prediction, we would compute the matching coefficient $D_k$ as a function of $p_{TJ}$ and include an integral in Eq. (3.85) convolving the jet and collinear-soft functions with $D_k(p_{TJ})$. An approach to doing this in a semi-automatic manner was discussed recently in Refs. [144, 145]. But, for simplicity we instead employ the following approximation: we evaluate the jet and collinear-soft functions at $p_{TJ}$, the average $p_{TJ}$.

The average jet transverse momentum $p_{TJ}$ can be estimated by using the fact that the cross section for a jet with transverse momentum $p_{TJ}$ takes the power-law form:

$$\frac{1}{\sigma} \frac{d\sigma}{dp_{TJ}} \simeq \frac{n-1}{p_{TJ}^\text{min}} \left(\frac{p_{TJ}}{p_{TJ}^\text{min}}\right)^n \Theta(p_{TJ} - p_{TJ}^\text{min}).$$

This distribution is normalized and the mean value of $p_{TJ}$ is

$$p_{TJ} = \frac{n-1}{n-2} p_{TJ}^\text{min}.$$  \hspace{1cm} (3.87)

The typical exponent is $n \sim 5$, and we take $n = 5$ in our numerical computations.

The full cross section for soft-dropped $e_2^{(a)}$ (including power corrections) can be expressed as

$$\frac{d\sigma}{de_2^{(a)}} = \sum_{k=q,g} D_k S_{C,k} \otimes J_k + \frac{d\sigma_{\text{pc}}}{de_2^{(a)}}.$$  \hspace{1cm} (3.88)

Here, the right-most term includes all power corrections suppressed by $e_2^{(a)}$ or $z_{\text{cut}}$. The functions $S_{C,k}$ and $J_k$ should be evaluated at $p_{TJ}$ but we have suppressed their arguments for brevity. We will use this form of the cross section to define the matching coefficient $D_k$ at fixed-order. For NNLL resummation, the relative $O(\alpha_s)$ corrections to $D_k$ are required.
First, at leading order in $\alpha_s$, Eq. (3.88) becomes

$$\frac{d\sigma^{(0)}(a)}{de_2^{(a)}} = \sum_{k=q,q',g} D_k^{(0)}(a) \delta(e_2^{(a)}) ,$$

where the superscript $(0)$ denotes the leading order in $\alpha_s$. Here, we have used $J_k^{(0)} = \delta S_{c,k} = \delta(e_2^{(a)})$. Also, since a jet has only one constituent at this order, the distribution has no support away from $e_2^{(a)} = 0$ and there are no partons to soft drop; therefore, there are no $e_2^{(a)}$ or $z_{cut}$ power corrections at this order. Integrating over all $e_2^{(a)}$, we are left with the Born-level cross section for the $k$ flavor channel $a_k^{(0)}$, so that

$$D_k^{(0)} = a_k^{(0)} .$$

At the next-to-leading order in $\alpha_s$, the extraction of $D_k$ requires separating the jets by flavor. Since $D_k$ is defined in each flavor channel, we need to determine the flavor of the hardest jet in each $pp \rightarrow Z + j$ event included in our sample. Ordinarily, any definition of jet flavor based on the constituents of the jet is infrared-unsafe and ill-defined at leading power, because soft wide-angle emissions into a jet can change its flavor.\(^6\) Soft drop eliminates this problem at leading power in $e_2^{(a)}$ and $z_{cut}$ by removing soft wide-angle radiation from the jet. This allows for an infrared and collinear safe definition of jet flavor at leading power in $e_2^{(a)}$ and $z_{cut}$. We define the jet flavor $f_J$ as the flavor sum of the constituents of the groomed jet:

$$f_J = \sum_{i \in I_g} f_i ,$$

where $f_q = 1$, $f_{q'} = -1$ and $f_g = 0$. The subscript on $I_g$ means that one only sums over the jet constituents that remain after grooming with soft drop. If $f_J = \pm 1$, then the jet is quark-type, while if $f_J = 0$, it is gluon-type. With this jet flavor identification, we are able

\(^6\)However, one infrared and collinear safe definition of jet flavor was presented in Ref. [146].
to determine the total fixed-order cross section for each jet flavor channel in \( pp \rightarrow Z + j \).

We will denote the next-to-leading order term in the cross section for a jet of flavor \( k \) as \( \sigma_k^{(1)} \), defined according to the phase space cuts described at the beginning of this section.

Then, at next-to-leading order in the \( k \) flavor channel, Eq. (3.88) becomes

\[
\frac{d\sigma_k^{(1)}}{de_2^{(a)}} = D_k^{(0)} \left[ S_{C,k}^{(1)} + J_k^{(1)} \right] + D_k^{(1)} \delta(e_2^{(a)}) + \frac{d\sigma_{k, \text{pc}}^{(1)}}{de_2^{(a)}}. \tag{3.92}
\]

Here, \( S_{C,k}^{(1)} \) and \( J_k^{(1)} \) are the collinear-soft and jet functions at \( \mathcal{O}(\alpha_s) \). Using \( D_k^{(0)} = \sigma_k^{(0)} \), we can integrate over \( e_2^{(a)} \) to find

\[
D_k^{(1)} = \sigma_k^{(1)} - \sigma_k^{(0)} \int_0^1 de_2^{(a)} \left[ S_{C,k}^{(1)} + J_k^{(1)} \right] - \sigma_{k, \text{pc}}^{(1)}. \tag{3.93}
\]

We computed \( \sigma_k^{(1)} \) using MCFM [147, 148] with settings detailed in the next section. We computed the power corrections according to

\[
\sigma_{k, \text{pc}}^{(1)} \equiv \int de_2^{(a)} \left[ \frac{d\sigma_k^{(1)}}{de_2^{(a)}} - \sigma_k^{(0)} \left( J_k^{(1)} + S_{C,k}^{(1)} \right) \right]. \tag{3.94}
\]

For the first term in the integrand, we use a numerical distribution obtained with MCFM. Since we do not have access to this distribution at arbitrarily small values of \( e_2^{(a)} \), the integral in Eq. (3.94) extends from \( e_2^{(a)} = 10^{-5} \) to 1. This approximation is sufficient for power corrections suppressed by \( e_2^{(a)} \), and the effect of dropping the \( z_{\text{cut}} \delta(e_2^{(a)}) \) term from the integral is negligible in comparison to the scale uncertainties shown in the next section.

This completes our extraction of the matching coefficient \( D_k \) through relative \( \mathcal{O}(\alpha_s) \). With it, the resummed cross section of Eq. (3.85) is complete and ready to be matched to relative \( \mathcal{O}(\alpha_s^2) \) fixed-order results.
3.7.2 Matching Resummation to Fixed-Order

With the resummed differential cross section for soft-drop groomed $e_2^{(a)}$ defined in Eq. (3.85), we next match to fixed order for $pp \rightarrow Z + j$. Our matching procedure will be identical to the procedure we used for $e^+ e^-$ collisions; we add the difference between the exact fixed order and the expansion of the resummed distribution to fixed order:

$$
\frac{d\sigma_{\text{match}}}{d\epsilon_2^{(a)}} = \frac{d\sigma_{\text{resum}}}{d\epsilon_2^{(a)}} + \frac{d\sigma_{\text{FO}}}{d\epsilon_2^{(a)}} - \frac{d\sigma_{\text{resum,FO}}}{d\epsilon_2^{(a)}}. \tag{3.95}
$$

We match the analytic NLL resummed distributions to fixed-order results that include the relative $O(\alpha_s)$ corrections to the Born process for $pp \rightarrow Z + j$. We match NNLL distributions to fixed-order results including relative $O(\alpha_s^2)$ corrections and up to 3 partons in the jet.

We use MCFM v. 6.8 [147, 148] to generate the fixed-order cross sections for soft-drop groomed $e_2^{(a)}$ in $pp \rightarrow Z + j$ events. Currently, MCFM can only generate fixed-order corrections at $O(\alpha_s)$ relative to a Born-level process, and so we will have to use some properties of the observable to be able to calculate to relative $O(\alpha_s^2)$ accuracy. For $e_2^{(a)} > 0$, as we did in $e^+ e^-$ collisions, we can ignore the purely two-loop virtual contribution to $pp \rightarrow Z + j$, as it has no effect on the differential distribution away from $e_2^{(a)} = 0$. MCFM can generate both inclusive $pp \rightarrow Z + j$ and $pp \rightarrow Z + 2j$ processes through relative $O(\alpha_s)$ accuracy. Therefore, we can use $pp \rightarrow Z + 2j$ at relative $O(\alpha_s)$ in MCFM to calculate the relative $O(\alpha_s^2)$ distribution for $pp \rightarrow Z + j$, in the region where $e_2^{(a)} > 0$.

In practice, this procedure requires some care. To define the cross section for $pp \rightarrow Z + 2j$ in MCFM, we must set a minimum $p_T$ for the two jets as identified by MCFM. This is set by the parameter $pT_{\text{jet_min}}$ within MCFM. To compute the fixed-order cross section correctly for $e_2^{(a)}$ as measured on the soft-drop groomed jet in $pp \rightarrow Z + j$ events, $pT_{\text{jet_min}}$ should be set to 0; this would of course produce infinity because $pp \rightarrow Z + 2j$...
lacks the virtual corrections of \( pp \to Z + j \). To regulate this divergence, we set \( \text{pt}_{\text{jet_min}} = 1 \text{ GeV} \) and have verified that for jets with \( p_T J > 500 \text{ GeV} \), this choice has a negligible effect on the differential cross section of \( \epsilon_2^{(a)} \) until deep in the infrared region, well beyond the point where resummation dominates. Additionally, we have verified that the distribution of \( \epsilon_2^{(a)} \) as measured in \( pp \to Z + j \) at relative \( \mathcal{O}(\alpha_s^2) \) is identical to that measured in \( pp \to Z + 2j \) at Born level with \( \text{pt}_{\text{jet_min}} = 1 \text{ GeV} \), up to differences deep in the infrared. Using this procedure, we are therefore able to match to relative \( \mathcal{O}(\alpha_s^2) \) with MCFM.

We generate \( pp \to Z + j \) events through relative \( \mathcal{O}(\alpha_s^2) \) accuracy at the 13 TeV LHC using MSTW 2008 NLO parton distribution functions [149]. We require that the \( p_T \) of the \( Z \) boson is greater than 300 GeV and the absolute value of its pseudorapidity is less than 2.5. Jets are clustered with the anti-\( k_T \) algorithm with radius \( R = 0.8 \). We study the hardest jet in these events that satisfies \( p_T J > 500 \text{ GeV} \) and \( |\eta_J| < 2.5 \). On these identified jets, we then soft-drop groom and measure \( \epsilon_2^{(a)} \) using custom code. This is an exceptionally computationally demanding procedure at relative \( \mathcal{O}(\alpha_s^2) \), due to the complicated phase space of real emissions and the small width of the bins required to calculate the \( \epsilon_2^{(a)} \) distribution. This precision jet substructure study is only possible because of the development of highly efficient methods for generating fixed-order corrections.

In Fig. 3.11 we plot matched distributions for soft-drop \( \epsilon_2^{(2)} \) with \( z_{\text{cut}} = 0.1 \) and both \( \beta = 0 \) and \( \beta = 1 \) at NLL and NNLL. Here, we show both the distributions normalized to the total cross section and normalized over the range \( \epsilon_2^{(2)} \in [0.001, 0.1] \). The shaded bands represent estimates of theoretical uncertainties due to residual infrared scale sensitivity.\(^7\) We show these bands mainly to allow comparison of the uncertainty remaining at different levels of formal precision. For the collinear-soft and jet functions in the resummed cross section, we vary the low scales by a factor of two. To estimate the scale

\(^7\)The relatively large size of the uncertainty bands for \( \epsilon_2^{(2)} \gtrsim 0.1 \) is an artifact of our simplistic additive matching. Additionally, due to the large \( K \) factor, the absolute scale of the matched NNLL distribution in Fig. ?? is roughly twice as large as the matched NLL distribution in Fig. ??.
dependence of the matching coefficient $D_k$ in the resummed cross section is more complicated, and we discuss this in detail in App. 3.9.11. To estimate scale uncertainties in the fixed-order cross section, we vary the factorization and renormalization scales in MCFM by a factor of 2 about 500 GeV $\simeq p_{TJ}$. We then take the envelope of all of these scale variations to produce the shaded bands in Fig. 3.11. For $\beta = 1$ at NNLL, we have also explicitly shown the additional uncertainty due to the two-loop non-cusp anomalous dimension of the collinear-soft function. In going from NLL to NNLL accuracy, the relative size of the scale uncertainty bands decreases by about a factor of 2 or 3 for both choices of normalization of the distributions. However, normalizing the distributions over the range $e_2^{(2)} \in [0.001, 0.1]$ dramatically reduces residual scale uncertainties; at NNLL, these normalized distributions have residual scale uncertainties at the 10% level and smaller.

### 3.7.3 Comparison to Monte Carlo

We now compare our NNLL resummed and matched calculation of soft-drop groomed $e_2^{(2)}$ distributions to Monte Carlo simulations. We generate $pp \rightarrow Z + j$ events at the 13 TeV LHC with HERWIG++ 2.7.1 and PYTHIA 8.210. To improve statistics somewhat, we have turned off $Z/\gamma$ interference in the Monte Carlos. The Z boson is forced to decay to electrons, and we require that the invariant mass of the electrons is within 10 GeV of the mass of the Z boson. We then require that the identified Z boson has $p_{TZ} > 300$ GeV and $|\eta_Z| < 2.5$. Jets are clustered with FASTJET 3.1.3 using the anti-$k_T$ algorithm with radius $R = 0.8$ and we identify the hardest jet in the event with $p_{TJ} > 500$ GeV and $|\eta_j| < 2.5$. We then soft-drop groom this jet and measure $e_2^{(2)}$. Both soft drop and the energy correlation functions are implemented using FASTJET contrib v. 1.019 [140,150].

We have generated two samples from both HERWIG++ and PYTHIA to study the effect of hadronization and underlying event. One sample is purely parton level: both hadronization and underlying event have been turned off and the other sample is the
Monte Carlos run in their default settings, up to the settings of the $Z$ boson mentioned earlier. The distributions of $e_2^{(2)}$ measured on soft-drop groomed jets with $z_{\text{cut}} = 0.1$ and both $\beta = 0, 1$ are illustrated in Fig. 3.12. Here, we compare our matched and normalized NNLL calculation to both the parton-level and hadron-level plus underlying event Monte Carlos. To normalize the Monte Carlo distributions, all curves integrate to the same value on the range $e_2^{(2)} \in [0.001, 0.1]$.

As a more direct comparison of the Monte Carlos, Fig. 3.13 displays the relative difference between each of the hadron-level Monte Carlos and our matched NNLL predictions, with our estimates of theoretical uncertainty shown as shaded bands. Again, soft drop is performed with $z_{\text{cut}} = 0.1$, and both $\beta = 0$ and $\beta = 1$ are shown. Discrepancies between the Monte Carlo results and our predictions are present but not large.

As observed with jets in $e^+e^-$ collisions, there is good agreement between our precision calculation and the Monte Carlos over a wide dynamic range. Importantly, this measurement of the soft-drop groomed $e_2^{(2)}$ is very different from the case in $e^+e^-$. In $e^+e^-$ collisions we calculated the heavy groomed and ungroomed jet masses. By measuring the heavier of the two jet masses, both masses have to be small, and the observable is global. For $pp \rightarrow Z + j$ events, we want to make no restrictions on the out-of-jet radiation. Thus although the soft drop jet mass is still free of non-global contributions, the ungroomed mass will not be. That is, we do not have control over all the large logarithms of ungroomed jet mass in $pp \rightarrow Z + j$ events, and thus cannot predict them using our factorized expression, although other approaches are possible.\(^8\) For this reason, we only show distributions of soft-drop groomed $e_2^{(2)}$ measurements in $pp \rightarrow Z + j$ events.

Fig. 3.12 also illustrates that soft drop grooming eliminates sensitivity to both hadronization and underlying event until deep in the infrared. The parton-level and hadron-level

\(^8\)Calculations of the ungroomed jet mass in $Z + j$ events have been done, with varying approaches to handling the non-global contribution $[59, 60, 100]$. 
distributions for each Monte Carlo agree almost perfectly until below about $e_{2}^{(2)} \lesssim 10^{-3}$. That hadronization effects are small is expected from our $e^{+}e^{-}$ analysis, but this also demonstrates that underlying event effects are negligible. A similar observation was made in Ref. [74], though at a much higher jet $p_T$ ($p_T > 3$ TeV). As in $e^{+}e^{-}$ collisions, we expect that the hadronization effects that are observed in the Monte Carlo can be explained by a shape function, though we leave this to future work.

That the shape of the resummed distribution is both completely determined by collinear dynamics and is insensitive to underlying event suggests that by grooming jets with soft drop, we are able to completely isolate factorization-violating effects into an overall normalization. Therefore, we conjecture that the shape of the leading-power distribution of soft-drop groomed observables as measured in hadron collision events completely factorizes, just like the $p_T$ spectrum in Drell-Yan events [151]. We leave a proof of this conjecture to future work.⁹

3.8 Conclusions

In this paper, we presented the first calculation for an observable measured exclusively on the constituents of a jet to NNLL accuracy and matched to fixed-order results at $O(\alpha_s^2)$ relative to the Born process. The ability to do this calculation required grooming the jet with the soft drop algorithm, which eliminates the complications due to non-global logarithms that afflict ungroomed jet measurements. The soft drop groomer also significantly reduces nonperturbative effects from hadronization and underlying event, rendering the perturbative calculation of energy correlation functions accurate over several decades.

⁹Due to the presence of the complicated object $D_k(p_{T_{\text{min}}}, \eta_{\text{max}}, z_{\text{cut}}, R)$, Eq. (3.85) is not strictly a factorization theorem. It may not be possible to factorize $D_k$ to all orders due to the presence of so-called Glauber modes [151] in the cross section. While it is beyond the scope of this paper, recent work suggests that Glaubers can be included into the cross section directly [152], and our numerical work indicates that the effect may be absorbable into the normalization.
The insensitivity of soft-drop groomed jet observables to underlying event suggests that the normalized cross section fully factorizes in hadronic scattering events.

To complete the resummed calculation to NNLL accuracy required determining the two-loop non-cusp anomalous dimension for the soft function for which all emissions are removed. For $\beta = 0$, we were able to use results from the literature to extract the non-cusp anomalous dimension, up to calculable clustering effects. While not used for results in this paper, the clustering effects when using the anti-$k_T$ algorithm with soft drop are closely related to similar effects found in jet veto calculations. For soft drop angular exponent $\beta > 0$, we demonstrated a numerical procedure for determining the anomalous dimension using EVENT2. This was sufficient to approximate the non-cusp anomalous dimension, but a full calculation of the two-loop soft function for soft drop with $\beta \geq 0$ is desired.

With a complete calculation of the two-loop soft function, including constants, we would be one step closer to resumming to next-to-next-to-next-to-leading logarithmic accuracy ($N^3\text{LL}$). Up to the unknown four-loop cusp anomalous dimension (whose effects have been shown to be small [39, 153, 154]), the only other piece to get to $N^3\text{LL}$ would be the three-loop non-cusp anomalous dimension of the soft-dropped soft function. Without an explicit three-loop calculation, this anomalous dimension could in principle be estimated using a technique similar to what we used at two loops, using a fixed-order code like EERAD3 [155]. If this is possible, then resummation to this accuracy would potentially reduce residual scale uncertainties to the percent-level, assuming a scaling of uncertainties like observed in going from NLL to NNLL.

For our complete predictions, it was vital to match our resummed calculations to high precision fixed-order distributions. Fixed-order calculations have been traditionally used for observables that are inclusive over soft and collinear radiation, like total cross sections or $p_T$ spectra. The generation of fixed-order differential distributions for the plots in this
paper required CPU-centuries, which we attained only by running on thousands of cores. For calculations of more complicated jet observables, precise fixed order computations are likely infeasible with presently available tools. As jet substructure pushes to higher precision, it will be necessary to have fixed-order calculations that more efficiently sample the infrared regions of phase space.

The calculations in this paper represent a new frontier of precision QCD. While jet substructure techniques have been used for some time in experimental analyses at the LHC, they are just now approaching the level of theoretical precision that can be meaningfully compared to data. By soft-drop grooming jets, we greatly reduce the theoretical challenges, enabling the calculation of a wide range of jet substructure observables to full NNLL accuracy.

3.9 Appendix

3.9.1 Three-Loop $\beta$-function and Cusp Anomalous Dimension

The $\beta$-function is defined to be

$$\beta(\alpha_s) = \mu \frac{\partial \alpha_s}{\partial \mu} = -2\alpha_s \sum_{n=0}^{\infty} \beta_n \left( \frac{\alpha_s}{4\pi} \right)^{n+1}. \quad (3.96)$$

For NNLL resummation, we need the $\beta$-function to three-loop order [119, 120]. The first three coefficients are

$$\beta_0 = \frac{11}{3} C_A - \frac{4}{3} T_R n_f, \quad (3.97)$$

$$\beta_1 = \frac{34}{3} C_A^2 - 4 T_R n_f \left( C_F + \frac{5}{3} C_A \right), \quad (3.98)$$

$$\beta_2 = \frac{2857}{54} C_A^3 + T_R n_f \left( 2C_F^2 - \frac{205}{9} C_F C_A - \frac{1415}{27} C_A^2 \right) + T_R^2 n_f^2 \left( \frac{44}{9} C_F + \frac{158}{27} C_A \right). \quad (3.99)$$
For NNLL resummation, we need the cusp anomalous dimension
\[ \Gamma_{\text{cusp}} = \sum_{n=0}^{\infty} \Gamma_n \left( \frac{\alpha_s}{4\pi} \right)^{n+1} \] to three-loop order. The first three coefficients of the cusp anomalous dimension are [115–118]:

\[ \Gamma_0 = 4, \] (3.101)
\[ \Gamma_1 = 4C_A \left( \frac{67}{9} - \frac{\pi^2}{3} \right) - \frac{80}{9} T_{Rnf}, \] (3.102)
\[ \Gamma_2 = 4C_A^2 \left( \frac{245}{6} - \frac{134\pi^2}{27} + \frac{11\pi^4}{45} + \frac{22}{3} \zeta_3 \right) + 32C_A T_{Rnf} \left( \frac{-209}{108} + \frac{5\pi^2}{27} - \frac{7}{3} \zeta_3 \right) \] (3.103)
\[ + 4C_F T_{Rnf} \left( 16\zeta_3 - \frac{55}{3} \right) - \frac{64}{27} T_{Rnf}^2 \]

### 3.9.2 Hard Function

The hard function for dijet production in \( e^+e^- \) collisions is defined by the Wilson coefficient for matching the full QCD current onto the SCET dijet operator. For \( e^+e^- \rightarrow q\bar{q} \) events, the Wilson coefficient \( C(Q^2, \mu) \) is

\[ \langle q\bar{q}|\bar{\psi}\Gamma\psi|0\rangle = C \left( Q^2, \mu \right) \langle q\bar{q}|\chi_nY_n^\dagger\Gamma\chi_n|0\rangle. \] (3.104)

Here, \( \bar{\psi}\Gamma\psi \) is the QCD current for the production of a \( q\bar{q} \) pair from the vacuum. \( \chi_n \) is a quark jet operator collinear quark operator defined in the light-like direction \( \vec{n} \) in SCET. For calculations at leading power, \( \chi_n = W_t^\dagger \psi, \) with \( W_t \) a Wilson line pointing in some direction \( t \) not collinear to \( n \) and \( \psi \) is an ordinary quark field. The soft Wilson lines \( Y_n \) and \( Y_{\bar{n}} \) point in the \( n \) and \( \bar{n} \) directions respectively. \( \Gamma \) represents a generic Dirac matrix. We have ignored contraction with the leptonic tensor for simplicity. The Wilson lines \( Y_n \)
is defined as

\[ Y_n(x^\mu) = P \exp \left( ig \int_0^\infty ds \, n \cdot A(x^\mu + sn^\mu) \right), \quad (3.105) \]

where \( P \) denotes path-ordering. \( Y_n \) and \( W_i \) are defined similarly with \( \bar{n}^\mu \) and \( t^\mu \) replacing \( n^\mu \). In SCET, the gluon fields in the Wilson line are soft gluons for the \( Y \)'s and collinear gluons for the \( W \)'s, but once the sectors are decoupled one can treat any of these gluons simply as a gluon field of full QCD.

The hard function is the square of the Wilson coefficient:

\[ H \left( Q^2, \mu \right) = \left| C \left( Q^2, \mu \right) \right|^2. \quad (3.106) \]

While we do not present its expression here, the hard function for \( e^+e^- \rightarrow gg \) events is defined analogously, by matching the Higgs current \( F_{\mu\nu}F^{\mu\nu} \) onto SCET.

### 3.9.3 \( e^+e^- \rightarrow q\bar{q} \)

The one-loop hard function for the process \( e^+e^- \rightarrow q\bar{q} \) is \([53, 85, 98, 156]\)

\[ H = 1 + \frac{\alpha_s C_F}{2\pi} \left( -L_H^2 - 3L_H - 8 + \frac{7}{6}\pi^2 \right), \quad (3.107) \]

where

\[ L_H = \log \frac{\mu^2}{Q^2}. \quad (3.108) \]

The cusp anomalous dimension of the hard function to all orders is

\[ \Gamma_H = -2C_F\Gamma_{\text{cusp}}, \quad (3.109) \]
where $\Gamma_{\text{cusp}}$ is the cusp anomalous dimension defined in Eq. (3.100). Similar to the cusp anomalous dimension, we define the coefficients of the non-cusp anomalous dimension $\gamma$ via

$$\gamma = \sum_{n=0}^{\infty} \gamma^{(n)} \left( \frac{\alpha_s}{4\pi} \right)^{n+1}.$$  

(3.110)

Through two-loops, the non-cusp anomalous dimension coefficients of the hard function are [121, 122]

$$\gamma^{(0)}_H = -12C_F,$$

$$\gamma^{(1)}_H = \left(-6 + 8\pi^2 - 96\zeta_3\right) C_F^2 + \left(-\frac{1922}{27} - \frac{22}{3} \pi^2 + 104\zeta_3\right) C_F C_A + \left(\frac{520}{27} + \frac{8}{3} \pi^2\right) C_F n_f T_R.$$  

(3.111)

3.9.4 $e^+e^- \rightarrow gg$

In the infinite top quark mass limit or with a finite Yukawa coupling, $e^+e^-$ scattering can produce final state gluon jets. The hard function for such a process can be extracted from $gg \rightarrow H$ calculations. To all orders, the cusp anomalous dimension of the $e^+e^- \rightarrow gg$ hard function is

$$\Gamma_H = -2C_A \Gamma_{\text{cusp}},$$  

(3.112)

where $\Gamma_{\text{cusp}}$ is the cusp anomalous dimension in Eq. (3.100). Through two-loops, the coefficients of the non-cusp anomalous dimension are [157–159]

$$\gamma^{(0)}_H = -4\beta_0,$$

$$\gamma^{(1)}_H = \left(-\frac{236}{9} + 8\zeta_3\right) C_A^2 + \left(-\frac{76}{9} + \frac{2}{3} \pi^2\right) C_A \beta_0 - 4\beta_1.$$  

(3.113)
3.9.5 The Global Soft Function

For arbitrary exponent $\beta$ in the soft-drop groomer, the soft function can be calculated by
requiring that soft gluons in measured jets fail the soft drop criterion. For hemisphere
jets in $e^+e^- \rightarrow q\bar{q}$ events, for example, the soft function is defined by the forward matrix
element of soft Wilson lines:

$$S_{G}(z_{\text{cut}}) = \frac{1}{N_{C}} \text{tr}(0|T\{Y_n Y_{\bar{n}}\}\hat{\Theta}_{SD} T\{Y_n Y_{\bar{n}}\}|0).$$  \hspace{1cm} (3.114)

Here, $n$ and $\bar{n}$ are the light-like directions of the $q\bar{q}$ dipole, $T$ denotes time ordering, and
$\hat{\Theta}_{SD}$ denotes the soft drop groomer operator which requires the final state to fail soft drop.
The action of $\hat{\Theta}_{SD}$ on soft final states cannot be written in a closed form for an arbitrary
final state due to clustering effects, though it can be defined order-by-order. For example,
the matrix element of $\hat{\Theta}_{SD}$ for $\beta = 0$ on a final state with two soft particles was presented
in Sec. 3.5.1.

At one-loop for hemisphere jets in $e^+e^-$ collisions, the soft function $S_{G}$ can be calculated from

$$S_{G} = g^2 \mu^{2\varepsilon} C_i \int \frac{d^4k}{(2\pi)^d} \frac{n \cdot \bar{n}}{n \cdot k - \bar{n}} 2\pi\delta(k^2)\Theta(k^0)\Theta(n \cdot k - n \cdot \bar{n})\Theta\left(z_{\text{cut}} \frac{Q}{2} \left[\frac{n \cdot k}{k^0}\right]^{\beta/2} - k^0\right) + (n \leftrightarrow \bar{n}),$$ \hspace{1cm} (3.115)

where $n$, $\bar{n}$ are back-to-back light-like vectors with $n \cdot \bar{n} = 2$. The requirement $\bar{n} \cdot k > n \cdot k$
restricts the radiation to lie in one hemisphere, while the requirement

$$z_{\text{cut}} \frac{Q}{2} \left[\frac{n \cdot k}{k^0}\right]^{\beta/2} > k^0$$ \hspace{1cm} (3.116)
restricts the soft gluon to fail soft drop. We find

\[ S_G = 1 + \frac{\alpha_s C_i}{\pi} \left[ \frac{1}{2 (1 + \beta)} L_S^2 - \frac{\pi^2}{12} \left( \frac{1}{1 + \beta} + 2 + \beta \right) \right], \quad (3.117) \]

where \( C_i \) is the appropriate color factor (\( C_F \) for \( e^+ e^- \rightarrow q\bar{q} \); \( C_A \) for \( e^+ e^- \rightarrow gg \)) and

\[ L_S = \log \frac{\mu^2}{Q^2 (z_{\text{cut}})^2 4^\beta}. \quad (3.118) \]

To all orders, the cusp anomalous dimension of the hemisphere wide-angle soft function is

\[ \Gamma_S = \frac{2C_i}{1 + \beta} \Gamma_{\text{cusp}}, \quad (3.119) \]

where \( \Gamma_{\text{cusp}} \) is the cusp anomalous dimension from Eq. (3.100). To one-loop order, the non-cusp anomalous dimension is 0:

\[ \gamma^{(0)}_S = 0. \]

For NNLL resummation, we need the non-cusp anomalous dimension to two-loop order. As discussed in Sec. 3.5.1, for soft drop with angular exponent \( \beta = 0 \), this can be extracted from energy veto calculations, up to clustering effects that we calculated. For soft drop with \( \beta = 0 \) and Cambridge/Aachen reclustering, we find the two-loop non-cusp anomalous dimension to be

\[ \gamma^{(1)}_S \bigg|_{\beta=0} = C_i \left[ 34.01 C_F + \left( \frac{1616}{27} - 56\zeta_3 - 9.31 \right) C_A - \left( \frac{448}{27} + 14.04 \right) n_f T_R - \frac{2\pi^2}{3} \beta_0 \right]. \quad (3.120) \]
3.9.6 Jet Functions

Here, we present the quark and gluon jet functions on which the energy correlation function $e_2^{(\alpha)}$ is measured. The quark jet function, for example, is defined by the forward matrix element:

$$J_q(e_2^{(\alpha)}) = \frac{(2\pi)^3}{N_C} \text{tr}(0|\hat{\delta}_n(0)\delta(Q - \vec{n} \cdot \mathcal{P})\delta^{(2)}(\vec{P}_\perp)\delta \left(e_2^{(\alpha)} - \hat{e}_2^{(\alpha)}\right) \hat{\chi}_n(0)|0) \right). \quad (3.121)$$

Here, the jet is collinear to the light-like direction $n$, the operator $\delta(Q - \vec{n} \cdot \mathcal{P})$ restricts the large light-cone component of momentum to be equal to the center-of-mass collision energy $Q$, and $\delta^{(2)}(\vec{P}_\perp)$ restricts the jet function to have zero net momentum transverse to the $n$ direction. The measurement operator is defined by its action on an $n$-particle collinear final state $|X_n\rangle$ as:

$$e_2^{(\alpha)}|X_n\rangle = \frac{2^{3\alpha/2}}{Q^2} \sum_{i<j\in X_n} (\vec{n} \cdot p_i)^{1-\alpha/2}(\vec{n} \cdot p_j)^{1-\alpha/2}(p_i \cdot p_j)^{\alpha/2}|X_n\rangle. \quad (3.122)$$

To write this expression, we have expanded the definition of the energy correlation function from Sec. 3.2.2 to leading power with collinear momenta. The gluon jet function is defined similarly:

$$J_g(e_2^{(\alpha)}) = \frac{(2\pi)^3}{N_C} \text{tr}(0|B_\perp^\mu(0)\delta(Q - \vec{n} \cdot \mathcal{P})\delta^{(2)}(\vec{P}_\perp)\delta \left(e_2^{(\alpha)} - \hat{e}_2^{(\alpha)}\right) B_\perp^\mu(0)|0) \right), \quad (3.123)$$

where $B_\perp^\mu$ is the collinear-gauge invariant operator in SCET that creates physical collinear gluons.

The following expressions will be presented in Laplace space, where renormalization
is multiplicative and the Laplace space conjugate is $\nu$. That is,

$$J(\nu) = \int_0^\infty d\nu_2^{(a)} e^{-\nu_2^{(a)} J(\nu_2^{(a)})}.$$ (3.124)

The one-loop quark and gluon jet functions were first calculated in Ref. [5] for jets on which the two-point energy correlation functions with arbitrary angular exponent are measured.

### 3.9.7 Quark Jets

To one loop, the Laplace-space quark jet function is

$$J_q^{(a)} = 1 + \frac{\alpha_s}{2\pi} C_F \left[ \frac{\alpha}{2(\alpha - 1)} L_C^2 + \frac{3}{2} L_C + \left( \frac{13}{2} - \frac{12}{2\alpha} \right) - \frac{\pi^2}{12} \left( 9 - \frac{3}{\alpha - 1} - \frac{4}{\alpha} \right) \right],$$ (3.125)

where

$$L_C = \log \frac{\mu^2 (\nu e^{\gamma_E})^{2/\alpha}}{E_f^2}.$$ (3.126)

To all orders, the cusp anomalous dimension of the quark jet function is

$$\Gamma_C^{(\alpha)} = \frac{\alpha}{\alpha - 1} C_F \Gamma_{\text{cusp}},$$ (3.127)

where $\Gamma_{\text{cusp}}$ is the cusp anomalous dimension from Eq. (3.100). For all $\alpha$, the one-loop non-cusp anomalous dimension is

$$\gamma_C^{(\alpha)} = 6 C_F.$$ (3.128)

For NNLL resummation, we also need the two-loop non-cusp anomalous dimension. For $\alpha = 2$, corresponding to jet mass or thrust, this is known exactly. In that case, the
non-cusp anomalous dimension is [127]

\[
\gamma_C^{a,(1)}|_{a=2} = C_F \left[ C_F \left( 3 - 4\pi^2 + 48\zeta_3 \right) + C_A \left( \frac{1769}{27} + \frac{22\pi^2}{9} - 80\zeta_3 \right) + T_R n_f \left( -\frac{484}{27} - \frac{8\pi^2}{9} \right) \right].
\]

(3.129)

### 3.9.8 Gluon Jets

To one-loop, the Laplace-space gluon jet function is

\[
J_g(n) = 1 + \frac{\alpha_s}{2\pi} \left[ \frac{\alpha}{2(\alpha - 1)} L_C^2 + \frac{\beta_0}{2} L_C + C_A \left( \frac{67\alpha - 1}{9} - \frac{\pi^2}{3} \frac{2(\alpha - 1)^2 - 1}{\alpha - 1} \right) + n_f T_R \left( \frac{26}{9\alpha} - \frac{23}{9} \right) \right],
\]

(3.130)

where

\[
L_C = \log \frac{\mu^2 (v e^{\gamma_E})^{2/\alpha}}{E_f^2}.
\]

(3.131)

To all orders, the cusp anomalous dimension of the gluon jet function is

\[
\Gamma_C^g = \frac{\alpha}{\alpha - 1} C_A \Gamma_{\text{cusp}},
\]

(3.132)

where \(\Gamma_{\text{cusp}}\) is the cusp anomalous dimension from Eq. (3.100). For all \(\alpha\), the one-loop non-cusp anomalous dimension is

\[
\gamma_C^{g,(0)} = 2\beta_0.
\]

(3.133)

For NNLL resummation, we also need the two-loop non-cusp anomalous dimension. For \(\alpha = 2\), corresponding to jet mass or thrust, this is known exactly. In that case, the
non-cusp anomalous dimension is [54]

$$\gamma_{c}^{(1)} = C_A^2 \left( \frac{2192}{27} - \frac{22\pi^2}{9} - 32\zeta_3 \right) + C_A T_R n_f \left( -\frac{736}{27} + \frac{8\pi^2}{9} \right) - 8 C_F T_R n_f . \quad (3.134)$$

### 3.9.9 Collinear-Soft Function

The final piece in the factorization theorem is the collinear soft function, defined from soft radiation that is collinear to the jet. As it describes soft radiation, the collinear-soft function is defined as a forward matrix element of Wilson lines:

$$S_C(z_{\text{cut}} e_2^{(a)}) = \frac{1}{N_C} \text{tr}(0| T\{Y_n^\dagger W_i\} \delta \left( e_2^{(a)} - (1 - \hat{O}_{SD}) \hat{e}_2^{(a)} \right) \overline{T}\{W_i^\dagger Y_n\}|0) . \quad (3.135)$$

The $Y$ and $W$ Wilson lines are the same as the ones in the soft and jet functions respectively, but depend on collinear-soft fields (which, like any of the others, can be treated as full QCD fields at leading power).

Now, collinear-soft modes only contribute to $e_2^{(a)}$ if emissions pass the soft drop groomer: this is denoted by $1 - \hat{O}_{SD}$ in the measurement function. (Recall that $\hat{O}_{SD}$ removes emissions from the jet according to soft drop.) Again, this operator cannot be written in closed form for an arbitrary final state due to clustering effects, but below, we will calculate it explicitly at one-loop. The $\hat{e}_2^{(a)}$ measurement operator is defined by its action on an $n$-particle collinear-soft final state $|X_{S,n}\rangle$:

$$\hat{e}_2^{(a)} |X_{S,n}\rangle = \frac{2^a}{Q} \sum_{i \in X_{S,n}} (\mathbf{n} \cdot \mathbf{p}_i) 1^{-a/2} (\mathbf{n} \cdot \mathbf{p}_i)^{a/2} |X_{S,n}\rangle . \quad (3.136)$$

This follows from expanding the definition of the energy correlation function from Sec. 3.2.2 to leading power with collinear-soft momenta.
This can be calculated at one-loop accuracy from

\[ S_C = g^2 \mu^{2\epsilon} C_i \int \frac{d^d k}{(2\pi)^d} \frac{n \cdot \bar{n}}{n \cdot k} \frac{2\pi \delta(k^2) \Theta(\bar{n} \cdot k)}{2} \left[ \Theta \left( \frac{z_{\text{cut}}}{\bar{n} \cdot k} \right) - \frac{\bar{n} \cdot k}{Q} \right] \delta \left( e_2^{(a)} \right) \]

\[ + \Theta \left( \frac{\bar{n} \cdot k}{Q} - z_{\text{cut}} \right) \left[ \delta \left( e_2^{(a)} - \frac{2^a}{Q} \right) \left( n \cdot k \right)^{\alpha/2} (\bar{n} \cdot k)^{1-\alpha/2} \right] \]

\[ = g^2 \mu^{2\epsilon} C_i \int \frac{d^d k}{(2\pi)^d} \frac{n \cdot \bar{n}}{n \cdot k} \frac{2\pi \delta(k^2) \Theta(\bar{n} \cdot k)}{2} \Theta \left( \frac{\bar{n} \cdot k}{Q} - z_{\text{cut}} \right) \left[ \delta \left( e_2^{(a)} - \frac{2^a}{Q} \right) \left( n \cdot k \right)^{\alpha/2} (\bar{n} \cdot k)^{1-\alpha/2} \right] - \delta \left( e_2^{(a)} \right) \]

where \( C_i \) is the color factor of the jet. In the second equality, we have rearranged the phase space constraints and explicitly removed scaleless integrals. For this collinear-soft function, at one-loop in Laplace space we find

\[ S_C(v) = 1 + \frac{\alpha_s C_i}{2\pi} \left[ \frac{\alpha + \beta}{2(\alpha - 1)(\beta + 1)} L_{S_C}^2 + \frac{\pi^2 (\alpha + 2 + 3 \beta)}{12 (\alpha + \beta)(\alpha - 1)(\beta + 1)} \right], \quad (3.138) \]

where

\[ L_{S_C} = \log \frac{\mu^2 (\nu e^{\gamma_E})^{2^{\beta+1}/\alpha+\beta}}{E_j^2 (z_{\text{cut}})^{2^{\beta+1}/\alpha+\beta}}. \quad (3.139) \]

To all orders, the cusp anomalous dimension of the collinear-soft function is

\[ \Gamma_{S_C} = -\frac{\alpha + \beta}{(\alpha - 1)(\beta + 1)} \Gamma_{\text{cusp}}, \quad (3.140) \]

where \( \Gamma_{\text{cusp}} \) is the cusp anomalous dimension from Eq. (3.100). To one-loop order, the non-cusp anomalous dimension is 0:

\[ \gamma_{S_C}^{(0)} = 0. \]
For NNLL resummation, we need the non-cusp anomalous dimension to two-loop order. For $\alpha = 2$ and $\beta = 0$, this can be determined by renormalization group consistency of the cross section directly, using either the $e^+e^- \rightarrow q\bar{q}$ or the $e^+e^- \rightarrow gg$ process. For soft drop with Cambridge/Aachen reclustering, the two-loop non-cusp anomalous dimension is

\[
\gamma_{5C}^{(1)} \bigg|_{a=2,\beta=0} = C_i \left[ -17.00 \, C_F + \left( -55.20 + \frac{22\pi^2}{9} + 56\zeta_3 \right) C_A + \left( 23.61 - \frac{8\pi^2}{9} \right) n_f T_R \right].
\] (3.141)

### 3.9.10 Resummation

Because we work in Laplace space, defined according to

\[
F(\nu) = \int_0^{\infty} d\nu e^{(a)}_2 e^{-\nu e^{(a)}_2} F(e^{(a)}_2),
\] (3.142)

the renormalization of all functions in the factorization theorem is multiplicative. For some function $F$ in the factorization theorem, it generically has the renormalization equation

\[
\mu \frac{\partial}{\partial \mu} F(\mu) = \gamma F(\mu),
\] (3.143)

where the anomalous dimension of $F$ is $\gamma$. The anomalous dimension can be written as

\[
\gamma = \Gamma_F(\alpha_s) \log \frac{\mu^2}{\mu_1^2} + \gamma_F(\alpha_s),
\] (3.144)

where $\Gamma_F(\alpha_s)$ is the cusp part of the anomalous dimension, $\mu_1$ is the infrared scale in the logarithm and $\gamma_F(\alpha_s)$ is the non-cusp part of the anomalous dimension. The solution\(^10\) to

---

\(^{10}\) In the plots of resummed distributions in this paper, we have frozen the strong coupling at $\mu_{NP} = 1$ GeV to keep cross sections finite. In the case of frozen $\alpha_s$, the solution to the renormalization group equation for each $F(\mu)$ is quite simple, so we omit the details of the prescription below $\mu_{NP}$ here.
the renormalization group equation can be written more conveniently as an integral with respect to $a_s$, by using the definition of the $\beta$-function as

$$\frac{d\mu}{\mu} = \frac{d a_s}{\beta(a_s)}. \quad (3.145)$$

Then, the solution to Eq. (3.144) can be expressed as

$$F(\mu) = F(\mu_0) \exp \left[ 2 \int_{a_s(\mu_0)}^{a_s(\mu)} \frac{da}{\beta(a)} \Gamma_F(a) \int_{a_s(\mu_0)}^{a} \frac{da'}{\beta(a')} + \int_{a_s(\mu_0)}^{a_s(\mu)} \frac{da}{\beta(a)} \gamma_F(a) \right. \quad (3.146)$$

$$+ \left. \log \frac{\mu_0^\prime 2}{\mu_1^\prime} \int_{a_s(\mu_0)}^{a_s(\mu)} \frac{da}{\beta(a)} \Gamma_F(a) \right],$$

where $\mu_0$ is a reference scale.

The exponentiated kernels can be explicitly evaluated to any logarithmic accuracy given the anomalous dimensions. The cusp-part of the anomalous dimension, $\Gamma_F(a_s)$, is proportional to the cusp anomalous dimension, $\Gamma_F(a_s) = d_F \Gamma_{cusp}$, where $d_F$ includes an appropriate color factor. The cusp anomalous dimension has an expansion in $a_s$ given by Eq. (3.100). The non-cusp anomalous dimension has a similar expansion defined in Eq. (3.110). For resummation to NNLL accuracy, we need the $\gamma_0$ and $\gamma_1$ coefficients, corresponding to computing the anomalous dimensions of the functions in the factorization theorem to two-loops.

With these expansions, we are able to explicitly evaluate the exponentiated kernel to NNLL accuracy. We have:

$$K_F(\mu_0, \mu_0) \equiv 2 \int_{a_s(\mu_0)}^{a_s(\mu)} \frac{da}{\beta(a)} \Gamma_F(a) \int_{a_s(\mu_0)}^{a} \frac{da'}{\beta(a')} + \int_{a_s(\mu_0)}^{a_s(\mu)} \frac{da}{\beta(a)} \gamma_F(a) \quad (3.147)$$

$$= C_i \frac{\Gamma_0}{2\beta_0^2} \left\{ \frac{4\pi}{a_s(\mu_0)} \left( \log r + \frac{1}{r} - 1 \right) + \left( \frac{\Gamma_1}{\Gamma_0} - \frac{\beta_1}{\beta_0} \right) (r - 1 - \log r) - \frac{\beta_1}{2\beta_0} \log^2 r \right. \right.$$

$$\left. + \frac{a_s(\mu_0)}{4\pi} \left[ \left( \frac{\Gamma_1}{\Gamma_0} \frac{\beta_1}{\beta_0} \right) (r - 1 - r \log r) - \left( \frac{\beta_1^2}{\beta_0^2} - \frac{\beta_2}{\beta_0} \right) \log r \right] \right. \right.$$

$$\left. \right. \left. + \frac{a_s(\mu_0)}{4\pi} \left[ \left( \frac{\Gamma_1}{\Gamma_0} \frac{\beta_1}{\beta_0} \right) (r - 1 - r \log r) - \left( \frac{\beta_1^2}{\beta_0^2} - \frac{\beta_2}{\beta_0} \right) \log r \right] \right. \right.$$

132
\[
\left. \begin{array}{c}
\left\{ \frac{\Gamma_2}{\Gamma_0} - \frac{\Gamma_1 \beta_1}{\Gamma_0 \beta_0} + \frac{\beta_1^2}{\beta_0^2} - \frac{\beta_2}{\beta_0} \right\} r^2 - 1 \frac{\Gamma_2}{\Gamma_0} - \frac{\Gamma_1 \beta_1}{\Gamma_0 \beta_0} (1 - r) + \left( \frac{\Gamma_2}{\Gamma_0} - \frac{\Gamma_1 \beta_1}{\Gamma_0 \beta_0} \right) r^2 - 1 \frac{\Gamma_2}{\Gamma_0} - \frac{\Gamma_1 \beta_1}{\Gamma_0 \beta_0} (1 - r) \right)
\end{array} \right\}
- \frac{\gamma_0}{2\beta_0} \log r - \frac{\gamma_0 \alpha_s(\mu_0)}{4\pi} \left( \frac{\gamma_1}{\gamma_0} - \frac{\beta_1}{\beta_0} \right) (r - 1),
\]

where

\[
r = \frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}.
\]

The other exponentiated factor is

\[
\omega_F(\mu, \mu_0) \equiv \int_{\alpha_s(\mu)}^{\alpha_s(\mu_0)} \frac{d\alpha}{\beta(\alpha)} \Gamma_F(\alpha)
\]

\[
= -C_i \frac{\Gamma_0}{2\beta_0} \left\{ \log r + \frac{\alpha_s(\mu_0)}{4\pi} \left( \frac{\Gamma_1}{\Gamma_0} - \frac{\beta_1}{\beta_0} \right) (r - 1) + \frac{1}{2} \frac{\alpha_s^2(\mu_0)}{(4\pi)^2} \left( \frac{\beta_1^2}{\beta_0^2} - \frac{\beta_2}{\beta_0} + \frac{\Gamma_2}{\Gamma_0} \right) (r^2 - 1) \right\}.
\]

Then, we can write the solution in Laplace space to the renormalization group equation in Eq. (3.146) as

\[
F(\mu) = e^{K_F(\mu, \mu_0)} F(\mu_0) \left( \frac{\mu_0^2}{\mu_1^2} \right)^{\omega_F(\mu, \mu_0)}.
\]

Because the hard function and the wide-angle soft function are independent of the observable \( e^{(a)} \), their renormalization group equations are identical in real space and Laplace conjugate space. For the jet functions and the collinear-soft function, the inverse Laplace transform is non-trivial.

For any of the jet functions appearing in the factorization theorem, the Laplace space solution can be written as

\[
J(\nu, \mu) = e^{K_J(\nu, \mu_0)} J(\nu, \mu_0) \left[ \frac{\mu_0^2}{E_j^2} (\nu e^{\gamma_E})^2 / \pi \right]^{\omega_J(\nu, \mu_0)}.
\]

Note that the logarithms that appear in the low-scale jet function \( J(\nu, \mu_0) \) have the same
argument as the factor that is raised to the $\omega_j$ power. Therefore, using the relationship (noted by Ref. [128])

$$\frac{\partial^n}{\partial q^{\frac{n}{3}}} v^q = v^q \log^n v,$$

we can re-write the jet function as

$$J(v, \mu) = e^{K_j(\mu, \mu_0)} J(L \to \partial_{\omega_j}) \left[ \frac{\mu_0^2}{E_j^2} (v e^{\gamma E})^{2/\alpha} \right]^{\omega_j(\mu, \mu_0)}.$$

(3.152)

Here $J(L \to \partial_{\omega_j})$ means that the logarithms in the low-scale jet function $J(v, \mu_0)$ are replaced by derivatives with respect to the exponentiated factor $\omega_j(\mu, \mu_0)$. The exact same replacement can be made for the collinear-soft function. In that case, we have

$$S_C(v, z_{cut}, \mu) = e^{K_{SC}(\mu, \mu_0)} S_C(L \to \partial_{\omega_{SC}}) \left[ \frac{\mu_0^2}{E_j^2} (v e^{\gamma E})^{2/\alpha} \right]^{\omega_{SC}(\mu, \mu_0)}.$$

(3.153)

This re-writing of the jet and collinear-soft functions allows for very straightforward inverse Laplace transformation. In Laplace space, the total differential cross section for left and right hemisphere jets in $e^+e^-$ collisions is

$$\sigma(v) = \exp \left[ K_H(\mu, \mu_H) + K_S(\mu, \mu_S) + K_{SC}(\mu, \mu_{SC}^{(L)}) + K_{SC}(\mu, \mu_{SC}^{(R)}) + K_j(\mu, \mu_j^{(L)}) + K_j(\mu, \mu_j^{(R)}) \right] \times H(Q, \mu_H) S(z_{cut}, \mu_S) S_C(L \to \partial_{\omega_{SC}}) S_C(L \to \partial_{\omega_{SC}}) J(L \to \partial_{\omega_j}^{(L)}) J(L \to \partial_{\omega_j}^{(R)}) \times \left( \frac{\mu_0^2}{Q^2} \right) \omega_{H}(\mu, \mu_H) \left( \frac{\mu_0^2}{4 \Delta^2 Q^2} \right) \omega_{S}(\mu, \mu_S) \left[ \frac{(\mu_{SC}^{(L)})^2 (v e^{\gamma E})^{2/\alpha}}{E_j^2 (z_{cut})^{2/\alpha}} \right] \omega_{SC}(\mu, \mu_{SC}^{(L)}) \times \left( \frac{\mu_{SC}^{(R)} (v e^{\gamma E})^{2/\alpha}}{E_j^2 (z_{cut})^{2/\alpha}} \right) \omega_{SC}(\mu, \mu_{SC}^{(R)}) \times \left( \frac{\mu_j^{(L)} (v e^{\gamma E})^{2/\alpha}}{E_j^2} \right) \omega_{j}(\mu, \mu_j^{(L)}) \times \left( \frac{\mu_j^{(R)} (v e^{\gamma E})^{2/\alpha}}{E_j^2} \right) \omega_{j}(\mu, \mu_j^{(R)}).$$

(3.154)
Note that the inverse Laplace transform commutes with the derivatives, and we have
\[
L^{-1} [\mu^q] = \frac{(e^{(a)}_2)^{-q-1}}{\Gamma(-q)} .
\] (3.155)

Therefore, the differential cross section in real space can be written as:
\[
e_{2L}e_{2R} \frac{d^2 \sigma}{d e_{2L} d e_{2R}}
= \exp \left[ K_H(\mu, \mu_H) + K_S(\mu, \mu_S) + K_{SC}(\mu, \mu_{SC}^{(L)}) + K_{SC}(\mu, \mu_{SC}^{(R)}) + K_J(\mu, \mu_J^{(L)}) + K_J(\mu, \mu_J^{(R)}) \right]
\times H(Q, \mu_H) S(z_{cut}, \mu_S) S_C(L \to \partial_{\omega_S}^{(L)}) S_C(L \to \partial_{\omega_S}^{(R)}) I(L \to \partial_{\omega_J}^{(L)}) I(L \to \partial_{\omega_J}^{(R)})
\times \left( \frac{\mu_H^2}{Q^2} \right)^{\omega_{H}(\mu, \mu_H)} \left( \frac{\mu_S^2}{4 \beta^2 z_{cut} Q^2} \right)^{\omega_{S}(\mu, \mu_S)} \left[ \left( \frac{e_{2L}^{(a)} e^{-\gamma_E}}{E_J^2 (z_{cut})^{a + p}} \right)^{-2} \right] \omega_{SC}(\mu, \mu_{SC}^{(L)}) \omega_{SC}(\mu, \mu_{SC}^{(R)})
\times \left( \frac{\mu_J^{(L)}}{E_J^2} \right)^{2/\alpha \gamma_{J}^{(L)}} \left( \frac{\mu_J^{(R)}}{E_J^2} \right)^{2/\alpha \gamma_{J}^{(R)}} \omega_{J}(\mu, \mu_J^{(L)}) \omega_{J}(\mu, \mu_J^{(R)})
\times \Gamma \left( -\frac{2(\beta + 1)}{\alpha + \beta} - \frac{2}{\alpha} \omega_{SC}(\mu, \mu_{SC}^{(L)}) - \frac{2}{\alpha} \omega_{SC}(\mu, \mu_{SC}^{(R)}) \right) .
\] (3.156)

### 3.9.11 Renormalization Group Evolution of $D_k$

In this appendix we discuss in detail the renormalization group evolution of the jet flavor coefficient $D_k$ and explain the procedure we used to estimate the scale uncertainty introduced by neglecting higher-order terms.

The cross section for soft-drop groomed jets in $pp \to Z + j$ events factorizes in the limit $e_{2L}^{(a)} \ll z_{cut} \ll 1$, where
\[
\frac{d \sigma_{\text{resum}}}{d e_{2L}^{(a)}} = \sum_{k=0,1,2,3} D_k(p_T^{\min}, \eta_{\max}, z_{cut}, R) S_{C,k}(z_{cut} e_{2L}^{(a)}) \otimes J_k(e_{2L}^{(a)}) .
\] (3.157)
The fact that $D_k$ depends on multiple scales prohibits its resummation to all orders. Nevertheless, its renormalization scale dependence is completely determined by renormalization group invariance of the cross section. We can improve our prediction by solving the following renormalization group equation, which holds at leading power in $z_{\text{cut}}$:

$$
\frac{\partial \log D_k}{\partial \log \mu} = - \frac{\partial \log (J_k \otimes S_{C,k})}{\partial \log \mu} = \Gamma_{D_k}(\alpha_s) \log \left( \frac{\mu^2}{Q^2} \right) + \gamma_{D_k}(\alpha_s, z_{\text{cut}}),
$$

(3.158)

where $Q = 2\vec{p}_{TJ}$. The anomalous dimensions $\Gamma_{D_k}$ and $\gamma_{D_k}$ are

$$
\Gamma_{D_k} = -\frac{\beta}{1 + \beta} C_k \Gamma_{\text{cusp}},
$$

(3.159)

$$
\gamma_{D_k} = -\left( \gamma_{J_k} + \gamma_{S_{C,k}} \right) - \frac{C_k}{1 + \beta} \Gamma_{\text{cusp}} \log z_{\text{cut}}^2.
$$

(3.160)

Here, $C_k$ is the color Casimir for the jet of flavor $k$. The anomalous dimension has $\log z_{\text{cut}}$ dependence, which means $Q$ is not a natural scale of $D_k$ where all logarithms are minimized.

Nevertheless, we can still formally evolve $D_k$ from a scale $\mu_0 \sim Q$ to a renormalization scale $\mu$ common to the jet and collinear-soft function. Solving the renormalization group evolution Eq. (3.158), the improved $D_k$ takes the form

$$
D_k(\mu, \mu_f) \equiv D_k(\alpha_s, \mu_0, \mu_f) \left( \frac{\mu_0^2}{Q^2} \right)^{\omega_{D_k}(\mu, \mu_0)} e^{K_{D_k}(\mu, \mu_0)}. \tag{3.161}
$$

Here, $\mu_f$ represents the factorization scale; i.e., the scale at which the parton distribution functions in $D_k$ are defined. The $\omega_{D_k}$ and $K_{D_k}$ functions are defined in App. 3.9.10. To estimate uncertainties from higher-order corrections due to residual scale dependence in
\( D_k \), we will vary both \( \mu_0 \) and \( \mu_f \) over the values

\[
\begin{align*}
\mu_0 &= \left\{ \frac{Q}{2}, Q, 2Q \right\}, \\
\mu_f &= \left\{ \frac{Q}{2}, Q, 2Q \right\}.
\end{align*}
\]  

(3.162)  

(3.163)

For evaluating \( D_k \) at fixed-order, we keep the full leading and next-to-leading terms as well as singular terms at the next-to-next-to-leading order in the following expansion of the solution to the renormalization group equation, Eq. (3.161). Expanding \( D_k \) in powers of \( \alpha_s \) as

\[
D_k(\alpha_s, \mu, \mu_f) = \sum_{n=0} \left( \frac{\alpha_s(\mu)}{4\pi} \right)^n D_k^{(n)}(\mu, \mu_f),
\]  

(3.164)

we have the solutions:

\[
D_k^{(0)} = c_D^{(0)},
\]  

(3.165)

\[
D_k^{(1)} = \Gamma_D^{(0)} c_D^{(0)} \log^2 \frac{\mu}{Q} + \left( \gamma_D^{(0)} c_D^{(0)} + 2\beta_0 c_D^{(0)} \right) \log \frac{\mu}{Q} + c_D^{(1)},
\]  

(3.166)

\[
D_k^{(2)} = \frac{1}{2} \left( \Gamma_D^{(0)} \right)^2 c_D^{(0)} \log^4 \frac{\mu}{\mu_0} + \left( \gamma_D^{(0)} \Gamma_D^{(0)} + \frac{8}{3} \beta_0 \Gamma_D^{(0)} \right) c_D^{(0)} \log^3 \frac{\mu}{Q}
\]  

(3.167)

\[
+ \left( \Gamma_D^{(1)} + \frac{1}{2} \left( \gamma_D^{(0)} \right)^2 + 3\beta_0 \gamma_D^{(0)} \right) c_D^{(0)} \log^2 \frac{\mu}{Q}
\]  

(3.168)

\[
+ \left( \gamma_D^{(1)} + 2\beta_1 \right) c_D^{(0)} \log \frac{\mu}{Q} + \left( \gamma_D^{(0)} + 4\beta_0 \right) c_D^{(1)} \log \frac{\mu}{Q} + c_D^{(2)}.
\]  

(3.169)

The non-singular terms \( c_D^{(n)} \) are defined such that at \( \mu = Q \),

\[
D_k^{(n)}(Q, \mu_f) = c_D^{(n)}(Q, \mu_f).
\]  

(3.170)

Therefore one can extract the value of \( c_D^{(0)}(Q, \mu_f) \) and \( c_D^{(1)}(Q, \mu_f) \) from MCFM and then extrapolate \( D_k \) to arbitrary value of \( \mu_0 \). Given the current level of precision of MCFM, this procedure can be done through the next-to-leading order. At \( \mathcal{O}(\alpha_s^3) \), the \( c_D^{(2)} \) term cannot
be determined without the next-to-next-to-leading $pp \rightarrow Z + j$ cross section. Note that the size of $c^{(2)}_{D_k}$ is no greater than $\mathcal{O}(\alpha_s^3 \log^4 z_{\text{cut}})$. Thus we can estimate that the size of uncertainty introduced by the unknown higher-loop non-singular term $c^{(2)}_{D_k}$ is roughly a factor of $1 \pm \alpha_s^2 \log^4 z_{\text{cut}} e^{\alpha_s L n + 1 + \cdots}$, which is beyond NNLL accuracy.
Figure 3.11: NLL matched (left) and NNLL matched (right) distributions for hardest jet \( e_2^{(2)} \) in \( pp \rightarrow Z + j \) events with soft drop grooming \( z_{\text{cut}} = 0.1 \) and \( \beta = 0 \) and \( \beta = 1 \). Estimates of theoretical uncertainties are represented by the shaded bands. For soft drop with \( \beta = 1 \), the dotted lines represent the extent of the theoretical uncertainties when the variation of the two-loop non-cusp anomalous dimension is included. The distributions in the two upper figures are normalized to the total cross section (in femtobarns), while in the bottom figures, the distributions integrate to the same value over the range \( e_2^{(2)} \in [0.001, 0.1] \). Note the reduction in uncertainties as one moves from NLL to NNLL, and also as one considers the normalized distribution.
Figure 3.12: Comparison between soft-drop groomed $e_2^{(2)}$ distributions with $z_{cut} = 0.1$ and $\beta = 0$ (top) and $\beta = 1$ (bottom) for matched and normalized NNLL, parton-level, and hadron-level Monte Carlo. All curves integrate to the same value over the range $e_2^{(2)} \in [0.001, 0.1]$. The uncertainty band for soft drop with $\beta = 1$ at NNLL includes the variation of the two-loop non-cusp anomalous dimension.
Figure 3.13: Direct comparison of hadron-level output from HERWIG++ and PYTHIA already shown in Fig. 3.12. Soft drop is performed with $z_{\text{cut}} = 0.1$ and both $\beta = 0$ (left) and $\beta = 1$ (right). Curves are displayed as relative differences between Monte Carlo output and our matched NNLL predictions, with theoretical uncertainties shown as a shaded band.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure3.13}
\caption{Direct comparison of hadron-level output from HERWIG++ and PYTHIA already shown in Fig. 3.12. Soft drop is performed with $z_{\text{cut}} = 0.1$ and both $\beta = 0$ (left) and $\beta = 1$ (right). Curves are displayed as relative differences between Monte Carlo output and our matched NNLL predictions, with theoretical uncertainties shown as a shaded band.}
\end{figure}
Chapter 4

Collinear factorization violation and effective field theory

4.1 Introduction

That scattering cross sections factorize is critical to the predictive power of quantum field theory at particle colliders. At proton colliders, such as the Large Hadron Collider currently running at CERN, factorization allows a cross section to be computed by a convolution of separate components: a hard part, sensitive only to the net momenta going into particular directions, a soft part, describing the distribution of radiation of parametrically lower energy than going into the hard collision, a collinear part, describing the branching of partons into sets of partons going in nearly the same direction, parton distribution functions, describing the probability of finding a parton with a given fraction of the proton’s energy, and fragmentation functions, describing the probability of a quark or gluon to transition into a given color-neutral baryon or meson. Then each of these components can be computed or measured separately.

While the above qualitative description of factorization certainly holds to some extent,
it is sometimes violated. The violation might contribute only a phase to the amplitude and therefore cancel at cross-section level, or it might contribute to the magnitude of the amplitude and have observable consequences. An example such consequence is the appearance of super-leading logarithms [160–162]. These are contributions to a cross section in a region with no hard particles that scale like $a^n \ln^m x$ with $m > n$ for some observable $x$. The prediction from factorization is that only soft radiation should be relevant in the region (since it has no hard particles) and soft contributions scale at most like $a^n \ln^n x$.

In this paper, we progress towards understanding when factorization fails by connecting different approaches to factorization-violation.

To begin, it is important to be precise about what is meant by factorization. There are certain situations where factorization has been proven to hold. For example, amplitudes of quarks and gluons in quantum chromodynamics (QCD) are known to factorize when all the particles are in the final state. Suppose we have a state of going particles $\langle X \rangle = \langle X_s \rangle \langle X_1 \rangle \cdots \langle X_N \rangle$ comprising quarks or gluons all of which are either soft, in $\langle X_s \rangle$, or collinear to one of $N$ sectors, in $\langle X_1 \rangle$ through $\langle X_N \rangle$. By soft, we mean the energies of these particles are all small compared to a scale $Q$ associated with the energy of the particles: $E_{\text{soft}} \leq \lambda_s Q$ with $\lambda_s \ll 1$. By collinear to a direction $n_j^\mu$ we mean that the projection on $n_j^\mu$ of any momentum $p^\mu$ in the $\langle X_j \rangle$ sector is small, $p \cdot n_j \leq \lambda_j Q$ with $\lambda_j \ll 1$. In such a situation, the amplitude for producing $\langle X \rangle$ in full QCD is equal at leading power to the amplitude for producing $\langle X \rangle$ through a product of matrix elements [163, 164]

$$\langle X | \phi^* \cdots \phi | 0 \rangle \cong \mathcal{C}(S_{ij}) \frac{\langle X_1 | \phi^* W_1 | 0 \rangle}{\langle 0 | Y_1^\dagger W_1 | 0 \rangle} \cdots \frac{\langle X_N | W_N^\dagger \phi | 0 \rangle}{\langle 0 | W_N^\dagger Y_N | 0 \rangle} \langle X_s | Y_1^\dagger \cdots Y_N | 0 \rangle \quad (4.1)$$

The objects $W_j$ and $Y_j$ are collinear and soft Wilson lines (path ordered exponentials of gluon fields). The difference between $W_j$ and $Y_j$ is that the $W_j$ represent the radiation from all the non-collinear sectors and hence point away from the $j$ direction while the
$Y_j$ represent sources for eikonal radiation and point along the $j$ direction. The function $\mathcal{C}(S_{ij})$ is an infrared-finite Wilson coefficient depending only on the net momenta going into each collinear sector.

The symbol $\cong$ means that the two sides of Eq. (4.1) are equal at leading power. In particular, the infrared divergences (soft and collinear) agree on both sides and the finite parts agree to leading order in $\lambda_s$ to leading order in each of the $\lambda_j$. The amplitude level equivalence implies that and if one computes some infrared safe observable, such as the sum of the masses $\tau = \sum m_i^2$ of all the jets in an event, then all of the singular parts of the observable distribution will also agree (any term which is singular as $\tau \to 0$).

The operator $\mathcal{O} = \phi^* \cdots \phi$ is chosen to be a local operator made from scalar fields, for simplicity, but a similar formula holds for gauge-invariant operators made from quarks or gluons, or for scattering amplitudes. A more general version of this formula, including spin and color indices, and the full factorization proof can be found in [164].

In Eq. (4.1) all the fields are those of full QCD: ordinary quarks and gluons. This formula is closely related to factorization in Soft-Collinear Effective Theory (SCET) [26, 27, 30, 80, 165]. In SCET, only leading power interactions among fields are kept in the SCET Lagrangian, making the Feynman rules more complicated than those in QCD. Also in SCET, the soft-collinear overlap is removed through a diagram-by-diagram zero-bin subtraction procedure [21], rather than through operator-level subtractions, as in Eq. (4.1). The two formulations are equivalent, and also equivalent to factorization formulas in traditional QCD [114].

In the derivation of Eq. (4.1) in [164], which is similar to traditional factorization proofs [14, 151, 166, 167], it is essential that the eikonal approximation can be used. The eikonal approximation amounts to equating the limit where momenta are soft ($k^\mu \to 0$) with the limit where the energies of all the collinear particles are large ($Q \to \infty$). The $Q \to \infty$ limit allows collinear particles to be replaced by classical sources (Wilson lines).
so that the soft radiation is insensitive to the structure of the collinear sector. The subtlety is that there are regions of virtual momenta phase-space in which all of the components are soft, $k^\mu \ll Q$, but $Q$ is still relevant. For example, $k^\mu \to 0$ holding $Q = \vec{k}^2 / k_0$ fixed is not possible after the eikonal limit ($Q \to \infty$) has been taken. It is phase space regions associated with limits like this that are dangerous for factorization, as we will review.

In Section 4.3 we show that the only time the Glauber scaling subtlety might spoil amplitude-level factorization is when there are initial states collinear to final states. This is of course a well-known result, but it is helpful to revisit the derivation to set the stage for investigations into factorization-violation. If we explicitly avoid such configurations, the factorization formula in Eq. (4.1) has a natural generalization. Let $\langle X \rangle = \langle X_1, \cdots, X_N \rangle$ be the collinear sectors of final state and $|Z\rangle = |Z_{N+1} \cdots Z_M\rangle$ be the collinear sectors of the initial state and let $\langle X_s \rangle$ and $|Z_s\rangle$ be the soft particles in the initial and final states respectively.\footnote{Soft particles in the initial state are not particularly interesting physically, but since factorization holds if they are included, we allow for $|Z_s\rangle$ to be nonzero for completeness} We assume no two sectors are collinear. Note that this is not a strong requirement – for any set of external momenta, one can always define the threshold $\lambda_j$ for collinearity to be so small that each momentum is in its own sector (the only time this cannot be done is if two momenta are exactly proportional). Then Eq. (4.1) generalizes to\footnote{For physical, positive-energy momenta, incoming Wilson lines are denoted with a bar, like $\bar{Y}_j$ or $\bar{W}_j$ (see [163]) and have a different $i\epsilon$ prescription than outgoing, un-barred, Wilson lines. In this paper, our convention is that all momenta are outgoing and so distinguishing incoming and outgoing Wilson lines is unnecessary.}

$$\langle X | \phi^* \cdots \phi | Y \rangle \cong C(S_{ij}) \frac{\langle X_1 | \phi^* W_1 | 0 \rangle}{\langle 0 | Y_1^* W_1 | 0 \rangle} \cdots \frac{\langle 0 | W_N^* \phi | Z_M \rangle}{\langle 0 | W_N^* Y_N | 0 \rangle} \langle X_s | Y_1^* \cdots Y_N | Z_s \rangle$$

This is a non-trivial generalization of Eq. (4.1). It requires the Glauber gluon contributions to be identical on both sides, even when there are initial states involved.

The reason the factorization formula in Eq. (5.2) holds is because for hard scattering with no initial state collinear to any final state, the Glauber contribution is always con-
tained in the eikonal contribution. This containment holds in QCD when contours can be
deformed out of the Glauber region into the eikonal region (where the eikonal approxi-
mation can be used). That is, it holds when there is no pinch in the Glauber region. We
review these observations in Section 4.3. That the eikonal limit contains the Glauber con-
tribution is often assumed to study Glauber effects through matrix elements of Wilson
lines (see e.g. [168]), so it is important to understand when it holds.

In the context of Soft-Collinear Effective Theory, being able to deform contours from
the Glauber region into the eikonal region is closely related to the “soft-Glauber corre-
spodence” or the “Cheshire Glauber” discussed in [152]. To be clear, the soft-Glauber correspondence is a statement about when Glauber operators in SCET reproduce the Glauber limit of soft graphs in SCET. One expects that the soft-Glauber correspondence holds if there is no pinch in the Glauber region, however this has not been shown. Indeed, it is not even a precise statement, since the pinches are properties of graphs in QCD and the soft-Glauber correspondence refers to SCET. One goal of our analysis is to clarify the relationship between Glauber operators in SCET and pinches in QCD.

A corollary of Eq. (5.2) is that infrared divergences of purely virtual graphs in full QCD, including all Glauber contributions, are exactly reproduced by the factorized ex-
pression. Factorization violation can only show up in graphs with emissions, that is, in relating an amplitude to one with an additional collinear or soft particle. In particular, when there are collinear emissions, it is known that factorization does not hold. To be precise, amplitude-level splittings are often described through an operator $S_p$ which acts on an $n$-body matrix element $|\mathcal{M}\rangle$ turning it into a matrix element with $n + 1$ partons $|\mathcal{M}\rangle$:

$$|\mathcal{M}\rangle = S_p \cdot |\mathcal{M}\rangle$$

(4.3)

In a $1 \rightarrow 2$ splitting a parton with momentum $P^\mu$ splits into two partons with momenta
and $p_2^\mu$ with $P^\mu \equiv p_1^\mu + p_2^\mu$. Factorization implies that this splitting function $S_p$ should depend only on the momenta and colors of particles collinear to the $P^\mu$ direction. This requirement, called *strict factorization*, was shown by Catani, de Florian and Rodrigo to be violated in certain situations [169]. In particular, when $p_1^\mu$ is incoming, $p_2^\mu$ is outgoing, there is another incoming colored particle with some momentum $p_3^\mu$ not collinear to $P^\mu$, and another outgoing colored particle with momentum $p_j^\mu$ not collinear to $p_2^\mu$, then $S_p$ can depend on the color of the $p_3^\mu$ and $p_j^\mu$ partons.

We review this calculation of strict factorization violation in Section 4.5. The approach of [169], and also [170], is to calculate $S_p$ by taking the limit of a matrix element with $n + 1$ directions as it reduces to a matrix element with $n$ directions. Then $S_p$ can be deduced from known results about IR singularities of $n + 1$ parton matrix elements. In other words, one calculates $S_p$ by turning $|M|$ into $|\bar{M}|$.

The necessity of using $n + 1$-body matrix elements to calculate $S_p$ is a little counterintuitive. Since the splitting originates from $|\bar{M}|$ it seems one should not need information about a general $n + 1$ body matrix element $|M|$ to deduce it. Indeed, for timelike splittings (as in $e^+ e^- \rightarrow$ hadrons), one can start from a factorized expression, like in Eq. (4.1), and calculate $S_p$ from within the collinear sector of $|\bar{M}|$. This calculation is done at leading order explicitly in [163]. In this paper we show that one can still calculate $S_p$ from $|\bar{M}|$ when strict factorization is violated, through the inclusion of Glauber operators in the effective theory.

It is natural to propose including Glauber modes [171–174] into Soft-Collinear Effective Theory (SCET). However, since Glauber gluons have transverse components much larger than their energies, they cannot be represented by on-shell fields in a Lagrangian, like soft and collinear modes are. Recently, a framework to incorporate Glauber contributions into SCET was proposed by Rothstein and Stewart [152]. The Glauber gluons are introduced not through new on-shell fields, but as potential interactions among pre-
existing collinear and soft fields. We briefly review this approach in Section 4.4.4. One of
the main new results of this paper is the direct calculation of 1-loop and (partial) 2-loop
factorization violating contributions to $S_p$ in Sections 4.6 and 4.7 with the SCET Glauber
formalism. The calculations are highly non-trivial, depending critically on the effective
field theory interactions and the rapidity regulator. They therefore provide a satisfying
cross check on both SCET and the factorization-violating splitting amplitudes in [169,170].

Stepping back from the technical calculations, we make some general observations
about properties that the Glauber operator contributions in SCET must have. For ex-
ample, they must not spoil factorization when factorization holds (as for all outgoing
particles). This forces the Glauber contributions to be non-analytic functions of exter-
nal momentum. It is impressive that this required non-analyticity is exactly produced
through the Glauber operators with the non-analytic rapidity regulator. We summarize
some of the features of the SCET calculations that allow this to work in Section 4.8.

In this paper, we use the convention that all momenta are outgoing, so that incoming
momenta $p^\mu$ have negative energy, $p^0 < 0$. With this convention $p_1 \cdot p_2 < 0$ for a spacelike
splitting (one incoming and one outgoing) and $p_1 \cdot p_2 > 0$ for a timelike splitting (both
incoming or both outgoing). It also means energy fractions $z = \frac{E_2}{E_1 + E_2}$ will be negative for
spacelike splittings and positive for timelike splittings. In Section 4.2, we review the var-
ious modes appearing in hard-soft-collinear factorization of scattering amplitudes, and
their scaling. In Section 4.3, we show that for large-angle hard scattering, the effects of
Glauber exchange is contained in the eikonal limit. Therefore amplitude-level factoriza-
tion formula is not modified. In Section 4.4, we discuss several approaches to isolating the
Glauber contribution. In Section 4.5, we summarize known results about factorization-
violation for timelike splitting. We review the connection between imaginary terms in
1-loop graphs and factorization violation, and how the factorization-violating splitting
amplitude is derived from the infrared structure of $n + 1$-parton amplitudes in QCD. In
Section 4.6, we compute the 1-loop Glauber contributions to timelike splitting amplitudes that are not contained in the soft contributions using SCET. Both the IR-divergent and the IR-finite contributions to the 1-loop factorization-violating effects are reproduced. In Section 4.7, we compute the contributions to two-loop factorization violation in timelike splittings involving double Glauber exchange. These contributions exactly reproduce the real part of the 2-loop splitting amplitude from [169]. We summarize some rather remarkable non-analytic properties of the Glauber contributions in Section 4.8 and summarize our results in Section 4.9.

4.2 Elements of factorization and Glauber scaling

Glauber gluons play a central role in understanding violations of factorization, so we devote this section to explaining their origin and relevance. Our goal here is to clarify the concepts of scaling, the relationship between soft and Glauber regions of momentum space, and the reasons that Glauber gluons can spoil factorization.

Broadly speaking, the goal of factorization is to write some scattering amplitude \( M \), which is a function of lightlike external momenta \( p_1 \cdots p_n \) as

\[
| M(p_1, \cdots, p_n) \rangle \approx | M_{\text{factorized}}(p_1, \cdots, p_n) \rangle
\]

(4.4)

where \( | M_{\text{factorized}} \rangle \) is simpler in some way. Here the symbol \( \approx \) implies that the two are not exactly equal, but equal up to parametrically small power corrections in some function of the momenta (e.g. in \( \lambda = p_1 \cdot p_2 / Q^2 \) with \( Q \) the center of mass energy).

A necessary condition for Eq. (4.4) to hold is that the infrared divergences agree on both sides. Since factorization involves writing an amplitude as products of simpler amplitudes each of which contain some subset of the infrared divergences, a first step to
understanding factorization is to classify infrared divergences.

Classifying the infrared divergences amounts the following consideration. Take a given Feynman diagram written as an integral over various loop momenta $k^\mu_i$. We associate given values $k^\mu_{i,0}$ of these momenta to an infrared divergence if the integral is infinite when integrated in an infinitesimal compact volume around $k^\mu_{i,0}$. The singularity requires a pole in the integration region, so at least one of the propagators must go on-shell within this volume. More precisely, the pole must be on the integration contour at $\epsilon = 0$ ($\epsilon$ here refers to the $i\epsilon$ in the Feynman propagator). However, at small finite $\epsilon$, the poles are necessarily off the contour of integration, so the singularity only occurs if there are two coalescing poles on different sides of the contour, or a pole at the end-point of the contour. This condition is often described as saying that the integration contour cannot be deformed away from the singular region, so that the singularity becomes pinched on the integration contour as $\epsilon \to 0$. This necessary condition is encoded in the Landau equations [175]. For a theory with massless particles with external momenta $p^\mu_j$, the Landau equations imply that the possible values for the infrared singular regions of integration are either soft, $k^\mu_{i,0} = 0$ exactly, or collinear $k^\mu_{i,0} = z p^\mu_j$ for some $z$ and some $p^\mu_j$. Thus, we say that the pinch surface for a theory with massless particles comprises the soft region ($k_i = 0$) and regions collinear to the direction of each external momentum.

The Landau equations are derived using only the denominators in a Feynman diagram. It can certainly happen that the numerator structure makes the diagram infrared-finite even though the contour is pinched according to the denominators. In addition, whether a diagram is divergent or not can depend on gauge. So the Landau conditions give a necessary but not sufficient condition for a singularity.

Although knowing the pinch surface, that is, the exactly singular region of loop momenta is a good start, this surface does not immediately tell us anything about factorization. It does however refine the problem: to match the infrared divergences of a given
amplitude, it is enough to match the integral in all the regions around the pinch surface. Thus it inspires us to look for a factorized expression by Taylor expanding the integrand around the pinch surface.

To expand around the $k_i^\mu = 0$ part of the pinch surface, we can equivalently expand around the limit where all of the external momenta become infinitely energetic ($|p_j^0| \to \infty$). Generically, this lets us replace propagator involving a loop momentum $k^\mu$ and an external lightlike momentum $p^\mu$ as

$$\frac{1}{(p + k)^2 + i\epsilon} \to \frac{1}{2p \cdot k + i\epsilon}$$

(4.5)

This replacement is known as taking the **eikonal limit.** Treating the momentum $p^\mu$ as infinite allows us to ignore the recoil of $p^\mu$ when $k^\mu$ is emitted, so that the $p^\mu$ is essentially a classical background source. This approximation is at the heart of all factorization proofs.

The subtlety where Glauber scaling comes in is that in taking the eikonal limit, in deriving Eq. (4.5), it is not enough to have $k^\mu \ll Q$ for all components of $k^\mu$, where $Q = p^0$ is the external particle’s energy. Rather, we must also have $k^2 \ll p \cdot k$. To appreciate the difference between these two limits, rather than taking the limit, as in Eq. (4.5), let us write the propagator as its eikonal version plus a remainder

$$\frac{1}{(p + k)^2 + i\epsilon} = \frac{1}{2p \cdot k + i\epsilon} - \frac{k^2}{((p + k)^2 + i\epsilon)(2p \cdot k + i\epsilon)}$$

(4.6)

This exact relation lets us write a diagram in the full theory as the sum of diagrams, each of which represents an explicit integral with the original propagator replaced by one of these two terms. The first term generates the soft part $\langle X_s | Y_1^+ \cdots Y_N | 0 \rangle$ of factorized expressions like Eq. (4.1), and the second term generates collinear parts. Factorization holds only if the collinear parts do not have infrared divergences associated with soft singularities.
Scaling arguments are powerful tools for determining whether soft singularities are present. The replacement in Eq. (4.5) amounts to taking $k^2 / k \cdot p \rightarrow 0$. One can apply this limit by rescaling all the loop momenta as $k_i^\mu \rightarrow \kappa^2 k_i^\mu$ and keeping the leading terms as $\kappa \rightarrow 0$. This guarantees that the remaining integral, that is the difference between the original integral and the one with this replacement, must scale like $\kappa^n$ with $n > 0$.

Let us call this type of scaling, where all the components of all the loop momenta scale the same way, eikonal scaling. We could take $k^\mu \rightarrow \kappa^2 k^\mu$, which we call ultrasoft scaling or $k^\mu \rightarrow \kappa k^\mu$, which we call soft scaling. Soft and ultrasoft scaling are both examples of eikonal scaling and equivalent for determining whether an integral is superficially divergent. To see this, consider changing variables from $k^\mu$ to $\{\kappa, \Omega\}$, with $\kappa$ the radial variable and $\Omega$ representing generically some angular variables. Then a diagram which scales like $\kappa^n$ transforms to a $\int d\Omega \int \kappa^{n-1} d\kappa$ integral which is divergent near $\kappa = 0$ if and only if $n \leq 0$. Using ultrasoft rather than soft scaling would have the diagram scale like $\kappa^{2n}$ which is still divergent under the same conditions.

Next, we must ask whether a diagram could still be divergent when integrated around $k^\mu = 0$ pinch surface even if it is power-counting finite under eikonal scaling. After transforming $k^\mu$ to $\{\kappa, \Omega\}$, such a divergence could come from the angular integrals over the $\Omega$’s. A sufficient condition to guarantee infrared finiteness is if the integral scales like $\kappa^n$ with $n > 0$ under any possible scalings of the different components of $k^\mu$ [164].

To examine this possibility, let us go to lightcone coordinates. We can decompose an arbitrary momentum $k^\mu$ with respect to two given lightlike momenta $p^\mu$ and $q^\mu$ as

$$k^\mu = \frac{1}{Q} k^- p^\mu + \frac{1}{Q} k^+ q^\mu + k^\mu_\perp$$

(4.7)

where

$$k^- = \frac{2}{Q} k \cdot q, \quad k^+ = \frac{2}{Q} k \cdot p$$

(4.8)
with $Q^2 = 2p \cdot q$. We also often use the 2-vector perpendicular component $\vec{k}_\bot$ with $\vec{k}_\bot^2 = -(k^\mu)^2$. Then Eq. (5.11) becomes (ignoring the $i\epsilon$ prescription temporarily)

$$\frac{1}{(p+k)^2} = \frac{1}{Qk^+} \left[ 1 - \frac{k^+k^- - \vec{k}_\bot^2}{Qk^+ + k^+k^- - \vec{k}_\bot^2} \right] (4.9)$$

Under ultrasoft (eikonal) scaling,

$$(k^+, k^-, \vec{k}_\bot) \rightarrow (\kappa^2 k^+, \kappa^2 k^-, \kappa^2 \vec{k}_\bot) (4.10)$$

and the second term on the right in Eq. (4.9) is suppressed by a factor of $\kappa^2$ with respect to the first term. Since diagrams are at most logarithmically divergent, the strongest a divergence can be is $\kappa^0$, and therefore integrals involving the second term are finite under eikonal scaling [164].

What other scalings can we consider? We need to send $k^\mu \rightarrow 0$, so let us normalize $k^+ \rightarrow \kappa^2 k^+$. Then we can generally write $k^- \rightarrow \kappa^{2a} k^-$ and $\vec{k}_\bot \rightarrow \kappa^b \vec{k}_\bot$ with $a > 0$ and $b > 0$. The second term then scales like

$$\frac{k^+k^- - \vec{k}_\bot^2}{Qk^+ + k^+k^- - \vec{k}_\bot^2} \rightarrow \frac{\kappa^{2a}k^+k^- - \kappa^{2b}\vec{k}_\bot^2}{\kappa^2 Qk^+ + \kappa^{2a}k^+k^- - \kappa^{2b}\vec{k}_\bot^2} (4.11)$$

Now, if $b > 1$, then the $\vec{k}_\bot^2$ terms in the denominator can be neglected and the integral scales like $\kappa^{\min(2a,2b-2)}$. However, since $a > 0$ (so that we approach the soft pinch), this term is power-counting finite. Thus we must have $b \leq 1$. For $b \leq 1$, the $\vec{k}_\bot^2$ term dominates both numerator and denominator and this term scales like $\kappa^0$. The scaling is independent of $a$ and $b$ (for $b \leq 1$) so we can take $a = 2$ and $b = 1$ to represent this case. Thus the only possible approach to $k^\mu = 0$ which is not automatically infrared finite is

$$(k^+, k^-, \vec{k}_\bot) \rightarrow (\kappa^2 k^+, \kappa^2 k^-, \kappa \vec{k}_\bot) (4.12)$$
This is known as **Glauber scaling**. It is the only possible scaling towards the $k^\mu = 0$ pinch under which the substitution in Eq. (5.11) might not leave an infrared-finite remainder. Gluons with momenta that have Glauber scaling are often called **Glauber gluons**. These gluons are spacelike and purely virtual: as $\kappa \to 0$ they approach the soft singularity from a direction in which $k^2 < 0$, in contrast with soft or collinear gluons, which can have $k^2 = 0$ for finite $\kappa$.

As an aside, note that we are taking $a > 0$ and $b > 0$ so that we zoom in on the soft region of the pinch surface. If we take $a = 0$, then the numerator and denominator of Eq. (4.11) both scale the same way and there is no additional suppression. However, if $a = 0$ then $k^\mu$ remains finite as $\kappa \to 0$. In fact, it approaches the direction $p^\mu$ of the line that we are expanding around (as in Eq. (4.9)). Thus this is collinear scaling. We can represent this scaling with $b = 1$ so that under **collinear scaling**

$$(k^+, k^-, k_\perp) \rightarrow (\kappa^2 k^+, k^-, \kappa k_\perp)$$

That the expansion in Eq. (4.9) does not improve the convergence under collinear scaling is neither surprising nor a problem. That collinear singularities are completely reproduced in the factorized expression was shown in [164].

### 4.3 Glauber containment in hard scattering

In this section, we will show that, for scattering amplitudes, singularities associated with Glauber scaling are already contained in the soft factor when no incoming momentum is collinear to an outgoing momentum. Because of the simplicity of the pinch surface with massless external particles, all of the relevant issues already appear at 1 loop in a vertex correction graph. Thus we begin with the 1-loop example, then work out the general
4.3.1 1-loop example

The example we will study in great detail is the Sudakov form factor in scalar QED:

\[ I_{\text{full}} = \int \frac{d^4 k}{(2\pi)^4} \frac{(2p_1 - k)\mu}{(p_1 - k)^2 + i\epsilon} \frac{\Pi_{\mu\nu}(k)}{k^2 + i\epsilon} \frac{(2p_2 + k)^\nu}{(p_2 + k)^2 + i\epsilon} \]  

(4.14)

Here, \( i\Pi_{\mu\nu} \) is the numerator of the photon propagator. In Feynman gauge, \( i\Pi_{\mu\nu} = -i\gamma_{\mu\nu} \). While Feynman gauge can be very efficient for calculations, physical gauges are often better for understanding factorization. In physical gauges, such as lightcone gauge, the numerator \( \Pi_{\mu\nu} \) represents a sum over physical polarizations and ghosts decouple. In lightcone gauge

\[ \Pi^\mu_{\nu}(k) = -\delta_{\mu\nu} + \frac{r^\mu k^\nu + r^\nu k^\mu}{r \cdot k} \]  

(4.15)

with \( r^\mu \) a reference lightlike 4-vector. Note that \( r_{\mu} \Pi^{\mu\nu} = 0 \) and \( k_{\mu} \Pi^{\mu\nu} = \frac{k^2}{r \cdot k} r^\nu \).

According to the Landau equations, which ignore the numerator structure, a necessary condition for a singularity is that all of the propagators go on-shell at once. This can happen when \( k^\mu = z p_1^\mu \) for some \( z \), \( k^\mu = z p_2^\mu \) for some \( z \) or when \( k^\mu = 0 \). Under ultrasoft scaling, the integration measure scales like \( \kappa^8 \) and the denominator factors scale like \( \kappa^2, \kappa^4 \) and \( \kappa^2 \) respectively. The numerator scales like \( \kappa^0 \) in either Feynman gauge or lightcone gauge, thus this integral is soft-sensitive. Under collinear scaling, the measure scales like \( \kappa^4 \) and the denominator factors scale like \( \kappa^2, \kappa^2 \) and \( \kappa^0 \) respectively. Thus, in Feynman gauge where \( \Pi^\mu_{\nu} \sim \kappa^0 \), the graph is collinear-sensitive. However in lightcone gauge, since \( k_{\mu} \Pi^{\mu\nu} \sim \frac{k^2}{r \cdot k} r^\nu \sim \kappa^1 \) and \( k^\mu \propto p_1^\mu \) on the collinear pinch surface, there is extra...
suppression. Thus this graph is actually collinear-finite in physical gauges. (In general, graphs which involve lines connecting different collinear sectors are collinear finite in physical gauges according to Lemma 3 of [164].) Thus for this graph, integrations around the collinear regions of the pinch surface, \( k^\mu = z p^\mu_1 \) and \( k^\mu = z p^\mu_2 \) for \( z \neq 0 \) are finite.

To study the singularity structure of this diagram, it is useful to go to lightcone coordinates with respect to \( p^\mu_1 \) and \( p^\mu_2 \), with \( k^+ = 2 k \cdot p_2 / Q \), \( k^- = 2 k \cdot p_1 / Q \), and \( Q^2 = 2 p_1 \cdot p_2 \):

\[
I_{\text{full}} = i g^2 s \int \frac{dk^+ dk^- d^2k_\perp}{2(2\pi)^4} \frac{(2p_1 - k)\mu}{[Q + k^+ k^- - \vec{k}_\perp^2 + i\epsilon]} \frac{\Pi_{\mu\nu}(k)}{[k^+ k^- - \vec{k}_\perp^2 + i\epsilon]} \frac{(2p_2 + k)^\nu}{[Q k^+ + k^+ k^- - \vec{k}_\perp^2 + i\epsilon]} \tag{4.16}
\]

The denominator has zeros in the complex plane at

\[
k^- = -\frac{\vec{k}_\perp^2}{Q - k^+} + i\epsilon \frac{1}{Q - k^+}, \quad k^- = \frac{\vec{k}_\perp^2}{k^+} - i\epsilon \frac{1}{k^+}, \quad k^- = \frac{\vec{k}_\perp^2 - Qk^+}{k^+} - i\epsilon \frac{1}{k^+} \tag{4.17}
\]

These are on the same side of the real axis unless \( 0 < k^+ < Q \). Thus the integral vanishes outside of this range and we can restrict \( 0 \leq k^+ \leq Q \). This configuration is shown on the left side of Fig. 4.1. Looking at the poles in the \( k^+ \) plane shows that we must also have \( -Q \leq k^- \leq 0 \).

Since we are only interested here in the soft pinch surface, we can restrict the integration region so that all components of \( k^\mu \) have magnitude less than \( Q \). We also take \( \Pi^{\mu\nu} = g^{\mu\nu} \), since the other terms in lightcone gauge do not affect the power counting around the soft pinch surface. Then,

\[
I_{\text{full}} \approx i 2Q^2 g^2 \int_{-\kappa Q}^{\kappa Q} \frac{dk^+ dk^- d^2k_\perp}{2(2\pi)^4} \frac{1}{[Qk^- - \vec{k}_\perp^2 + i\epsilon]} \frac{1}{[k^+ k^- - \vec{k}_\perp^2 + i\epsilon]} \frac{1}{[Qk^+ - \vec{k}_\perp^2 + i\epsilon]} \tag{4.18}
\]

Here, \( \approx \) means we are restricting the integral to the soft pinch surface, with \( \kappa \ll 1 \). This
has poles in the complex $k^-$ plane at

$$k^- = -\frac{k^2}{Q} + i\varepsilon, \quad k^- = \frac{k^2}{k^+} - i\varepsilon$$  \hspace{1cm} (4.19)

The third pole from the original integral in Eq. (4.17) has moved off to $k^- = -\infty - i\varepsilon$.

What we we would like is to drop the $\vec{k}^2$ terms compared to $Qk^+$ and $Qk^-$. This can only be justified if it is parametrically true everywhere in the integration region. It cannot be justified in the Glauber region, that is, in regions of $k^\mu$ where $k^- \lesssim k^2 / Q$. But let us look at the complex $k^-$ plane in more detail at fixed $k_\perp$ and $k^+$, both of which are soft ($< \kappa Q$). The dangerous Glauber region is shown as the hatched area that nearly touches the pole at $k^- = -\frac{k^2}{Q} + i\varepsilon$. The integration contour (the real $k^-$ line) goes right through this region. To avoid this region, we note that since $k^- < \kappa Q$, the pole at $k^- = \frac{k^2}{k^+} - i\varepsilon$ is parametrically far away from the Glauber region. Thus we can deform the contour downward into the complex plane avoiding the Glauber region explicitly. For example, we could take move onto the arc with $|k^-| = \frac{k^2}{Q\kappa}$. This arc avoids the Glauber region without crossing the non-Glauber pole. Once the contour is out of the Glauber region,
we can use eikonal scaling $k_\perp^2 \ll k^- Q$ to simplify the integrand. Then we can deform the contour back. Note that we can do this deformation for any $k^+$ and $\vec{k}_\perp$. We can then do the same manipulation for the $k^+$ integral to justify dropping $\vec{k}_\perp^2 \ll k^+ Q$. The result is

$$I_{\text{soft}} = -2iQ^2 g_s^2 \int_{-\kappa Q}^{\kappa Q} \frac{dk^- dk^+ d^2 k_\perp}{2(2\pi)^4} \frac{1}{[\kappa^2 k^- + i\epsilon]} \frac{1}{[k^- k^+ - \vec{k}_\perp^2 + i\epsilon]} \frac{1}{[Qk^+ + i\epsilon]}$$

(4.20)

which is the same integral we would get from taking the eikonal limit of $I_{\text{full}}$. Thus all of the soft singularities of $I_{\text{full}}$, including those from the Glauber region (the hatched region), are contained in $I_{\text{soft}}$.

One can also confirm directly that the Glauber region is contained in the soft integral through direct calculation. In pure dimensional regularization, $I_{\text{soft}}$ is scaleless and vanishes. The UV and IR divergences can be separated using a photon mass and a rapidity regulator, as in Eq. (10.3) of [152]. There it is shown that the imaginary part of the soft amplitude agrees with the Glauber contribution. In pure dimensional regularization, the imaginary part of the soft (or Glauber) contribution is in Eq. (4.37) below.

An implication of the contour deformation argument is that all the integrals coming from the use of the second (not eikonal) term in the replacement in Eq. (5.11) (diagrams with all-blue lines, in the language of [164]) are completely IR finite. For example, performing this replacement on the $p_2$ line results in

$$I_{\text{remain}} = 2iQ^2 g_s^2 \int_{-\kappa Q}^{\kappa Q} \frac{d^4 k}{(2\pi)^4} \frac{1}{[\kappa^2 k^- - \vec{k}_\perp^2 + i\epsilon]} [k^- k^+ - \vec{k}_\perp^2 + i\epsilon] Qk^+ + i\epsilon$$

(4.21)

That this integral is IR finite is easy to see – the $k^2$ pole is canceled by the expansion and the remaining poles in the $k^+$ plane (or $k^-$ plane) are on the same side of the real axis and so the integral vanishes. A more general argument is that once the contour is deformed out of the Glauber region, the new term suppresses the IR divergent part of the amplitude.
in the entire integration region. Thus the amplitude is power-counting finite (scaling like $\kappa^n$ with $n > 0$), as it would be under eikonal scaling without the contour deformation.

### 4.3.2 Spacelike example

Next, let us look at a diagram with a particle in the initial state and one in the final state:

$$I_{\text{full}}^{\text{SL}} = \frac{p_1 - k}{p_1} \frac{p_2 + k}{p_2} = ig^2 \int \frac{d^4k}{(2\pi)^4} \frac{(2p_1 - k)^\mu}{(p_1 - k)^2 + i\varepsilon} \frac{\Pi_{\mu\nu}(k)}{k^2 + i\varepsilon} \frac{(2p_2 + k)^\nu}{[(p_2 + k)^2 + i\varepsilon]}$$

(4.22)

Recall our convention that the incoming momentum, $p_1^\mu$, is treated as outgoing with negative energy. Let us assume that $p_1^\mu$ and $p_2^\mu$ are not proportional to each other. Then we define $Q^2 = -2p_1 \cdot p_2 > 0$. As $Q$ is the only scale in the problem, we zoom in on the soft singularity again by applying $k^\mu \ll Q$. We can go to lightcone coordinates in the $p_1^\mu, p_2^\mu$ frame, as before. The integral then becomes

$$I_{\text{full}}^{\text{SL}} \approx 2iQ^2 g^2 \int_{-\kappa Q}^{\kappa Q} \frac{dk^- dk^+ d^2k_\perp}{2(2\pi)^4} \frac{1}{[Qk^- - k_\perp^2 + i\varepsilon]} \frac{1}{[k^- k^+ - k_\perp^2 + i\varepsilon]} \frac{1}{[Qk^+ - k_\perp^2 + i\varepsilon]}$$

(4.23)

Now the poles are at

$$k^- = \frac{k_\perp^2}{Q} - i\varepsilon, \quad k^- = \frac{k_\perp^2}{k^+} - \operatorname{sign}(k^+) i\varepsilon$$

(4.24)

As in the spacelike case, one pole is in the Glauber region, but the other is not. Thus we can deform the contour away from the Glauber region and justify the eikonal expansion here as well. As it happens (see Section 4.4 below), the Glauber contribution to this integral exactly vanishes. But the point is that the Glauber contribution is contained in the eikonal limit for either timelike or spacelike kinematics, whether or not it happens to

159
vanish.

4.3.3 General argument

Now let us generalize the argument from the previous section to the $n$-loop case with arbitrary final and initial states. All we will assume is that none of the final states are collinear to any of the initial states. The proof of factorization in [164] did not use any information about initial or final states or about integration contours. What was shown is that all the terms which are power-counting divergent under eikonal scaling and collinear scaling in any of the directions in the full theory are reproduced in the factorized expression.

What we need to show is that integrals involving the second term when the substitution in Eq. (5.11) is used, do not have any soft singularities despite their being power-counting divergent under Glauber scaling. Since the factorized expression in Eq. (4.1) reproduces the self-energy graphs in the full theory exactly, the only graphs we need to be concerned about are the ones which would ordinarily be part of the hard vertex. That is, those connecting to more than one collinear direction. We need to show that the eikonal expansion can be justified for such diagrams.

In order for there to be a singularity in a diagram, according to the Landau criteria, all the propagators have to be either exactly zero or proportional to an external momentum. So let us put all the loop momenta except for one exactly on the singular surface. This leaves a single loop integration variable we call $k^\mu$. We can trace this momentum along
the diagram. For example,

\[
\int \frac{d^4k}{(2\pi)^4} \frac{N}{(k^2 + i\epsilon)(k - p_2)^2 + i\epsilon} \frac{1}{(k - p_2)^2 + i\epsilon} \frac{1}{(k + p_1)^2 + i\epsilon}
\]

(4.25)

where \(N\) is some numerator structure. This is almost identical to the Sudakov form factor diagram we studied in the previous subsection. The only difference is that now there are multiple poles at each singular point. Although the other loop integrations that we are neglecting make the actual diagram much less singular than this (no diagram can scale to a negative power of \(\kappa\) under any soft scaling), we do not need to make use of the extra cancellations. The only thing to observe is that the poles are all in the same parts of the \(k^\mu\) phase space as in the simple vertex correction. Thus the contour deformation works in exactly the same way and the eikonal expansion can be justified.

What happens if the loop momentum connects to an incoming and an outgoing direction? As long as \(p_1^\mu\) and \(p_2^\mu\) are not proportional to each other, one can still justify the eikonal expansion, as in Section 4.3.2.

The only time a problem can arise is if an incoming momentum and an outgoing momentum are collinear. For example, consider a configuration like this

\[
\int_{\kappa Q} d^4k \frac{1}{(p_2 + k)^2 + i\epsilon} \frac{1}{((p_1 + p_2 + k)^2 + i\epsilon)} \frac{1}{[k^2 + i\epsilon]} \dots
\]

(4.26)
If $p_1^\mu$ and $p_2^\mu$ are collinear then $2 p_1 \cdot k = -Q_1 k^-$ and $2 p_2 \cdot k = Q_2 k^-$ for two energies $Q_1$ and $Q_2$ and the same lightcone component $k^-$ is in both products. Moreover $Q_1 \geq Q_2$ by momentum conservation. Then

$$I \approx \int_{-\kappa Q}^{\kappa Q} \frac{d^4 k}{(2\pi)^4} \frac{1}{[Q_2 k^- - \vec{k}_\perp^2 + i\epsilon]} \frac{1}{[-(Q_1 - Q_2) k^- - \vec{k}_\perp^2 + i\epsilon]} \frac{1}{[k^- k^+ - \vec{k}_\perp^2 + i\epsilon]} \cdots \quad (4.27)$$

This type of diagram has poles at

$$k^- = \frac{\vec{k}_\perp^2}{Q_2} - i\epsilon, \quad k^- = -\frac{\vec{k}_\perp^2}{Q_1 - Q_2} + i\epsilon, \quad k^- = \frac{\vec{k}_\perp^2}{k^+} - i\epsilon, \quad (4.28)$$

The first two poles, coming from the two collinear propagators are on opposite sides of the real axis and not parametrically separated so one cannot deform the contour out of the Glauber region. Therefore one cannot justify dropping $\vec{k}_\perp^2 \ll Q_i k^-$ in the integrand since such a modification may miss infrared divergences.

From this example, we see that for the Glauber region to deserve special concern, we need to have the momentum $k^\mu$ flowing in opposite directions through two lines that are collinear to each other. Thus a graph like Eq. (4.22) is not problematic, even if $p_1^\mu$ and $p_2^\mu$ are proportional. Graphs in which the two collinear lines that the loop momenta runs through are in the same final state sector are also not problematic – the collinear sector in the factorized expression is the same as in full QCD so all of the singularities are necessarily reproduced in this region.

So far, we have considered only virtual graphs. Equivalently, we assumed that each sector has one particle, with no two momenta proportional to each other. We can weaken this requirement and allow for multiple particles in each sector. For particles with momenta $p^\mu$ and $q^\mu$ to be in the same collinear sector, we require $p \cdot q < \lambda^2 Q^2$ with $Q$ a hard scale (e.g. the center of mass energy). The parameter $\lambda$ is presumed to be small.
and the factorization formula is supposed to hold to leading power in \( \lambda \). Soft external particles can have energies up to \( \lambda^2 Q \). The effect of having multiple collinear particles in a sector or soft particles is to make some lines in diagrams like Eq. (4.25) off-shell, so that \( p^2 = \lambda^2 Q^2 \). For \( \lambda = 0 \), these lines would be massless and the graph would be infrared divergent, however, at finite \( \lambda \), the IR divergence is regulated. That is, a graph which would be logarithmically divergent may now be finite proportional to \( \ln \lambda \). In this way, the factorization formula which reproduces the infrared singularities at \( \lambda = 0 \) also reproduces the amplitude to leading power in \( \lambda \) when there are soft particles or multiple collinear particles.

The key point is that adding soft or collinear particles does not invalidate the argument that justifies the eikonal approximation. On the pinch surface, which characterizes both the IR divergences and the leading power behavior at small \( \lambda \), the contours can always be deformed out of the Glauber region. When a graph only involves particles in a single collinear sector with only outgoing (or only incoming) particles, there can be Glauber singularities, however these graphs are identical in the factorized and full theory amplitudes, and so factorization still holds. Thus factorization holds for arbitrary soft and collinear sectors, as long as no final state particles are collinear to any initial state particles.

### 4.3.4 Summary

In this section, we analyzed when the eikonal approximation can be trusted to reproduce all the soft singularities of an amplitude in full QCD. This is important because using the eikonal approximation is crucial to proving factorization formulas like Eq. (4.1). We found that the eikonal approximation can be used in any situation in which no incoming and outgoing particles are collinear. Thus Eq. (4.1) can be generalized. Take any initial state \( |Z\rangle = |Z_1\rangle \cdots |Z_M\rangle |Z_s\rangle \), with the momenta in \( |Z_j\rangle \) all collinear to a direction \( n_j \) and
all the momenta in $|Z_s\rangle$ soft, and any final state $\langle X | = \langle X_1 | \cdots \langle X_N | \langle X_s |$, with similar definitions, and assuming no initial state direction is collinear to any final state direction, then to leading power in $\lambda$

$$\langle X | \phi^* \cdots \phi | Z \rangle \equiv_{\text{IR}} C(S_{ij}) \frac{\langle X_1 | \phi^* W_1 | 0 \rangle}{\langle 0 | Y_1^\dagger W_1 | 0 \rangle} \cdots \frac{\langle 0 | W_N^\dagger \phi | Z_N \rangle}{\langle 0 | W_N^\dagger Y_N | 0 \rangle} \langle X_s | Y_1^\dagger \cdots Y_N | Z_s \rangle \quad (4.29)$$

Since $\lambda$ is arbitrary, a corollary is that for all virtual diagrams for hard scattering in which no final state and initial state momenta are proportional, the complete IR divergences of the full graphs in QCD, including any coming from the Glauber region, are exactly reproduced by the factorized expressions. Another corollary is that all possible violations of factorization are associated with situations where initial states and final state momenta are collinear.

### 4.4 Isolating the Glauber contribution

We have seen that most of the time, singularities associated with Glauber scaling are automatically contained in the expansion around zero momentum, using homogeneous, eikonal scaling. Factorization violation is associated with situations where this containment fails, so that the eikonal limit does not reproduce all of the soft singularities. To clarify the role that Glauber gluons play in amplitude-level factorization, it may be helpful to isolate their contributions. In this section, we explore some approaches to identifying the Glauber contribution and we explore the connection between the Glauber limit and the imaginary part of amplitudes.

In this section we will mostly be concerned, once again, with the 1-loop vertex correc-
tion diagram in scalar QED, Eq. (4.14):

$$I_{\text{full}} = \begin{array}{c}
\begin{array}{c}
p_1 \\
\uparrow k \\
p_2
\end{array}
\end{array} \approx -i2Q^2g^2_s \int \frac{d^4k}{(2\pi)^4} \frac{1}{[-Qk^- - \vec{k}_\perp^2 + i\epsilon]} \frac{1}{[k^-k^+ - \vec{k}_\perp^2 + i\epsilon]} \frac{1}{[Qk^+ - \vec{k}_\perp^2 + i\epsilon]}$$

(4.30)

We have ignored the numerator structure because it is irrelevant to our discussion. The following analysis is very similar for QCD.

In QED, the amplitude in $4-2\epsilon$ dimensions is

$$I_{\text{full}} = \frac{\alpha_s}{2\pi} \left( \frac{\mu^2}{-2p_1 \cdot p_2 - i\epsilon} \right)^\epsilon \left( \frac{1}{\epsilon^2} + \text{finite} \right)$$

(4.31)

This is an analytic function of $p_1 \cdot p_2$ with a branch cut when $p_1 \cdot p_2 > 0$. Expanding around $\epsilon = 0$ gives

$$I_{\text{full}} = \frac{\alpha_s}{2\pi} \left( \frac{1}{\epsilon^2} - \frac{1}{\epsilon} \ln \frac{-2p_1 \cdot p_2 - i\epsilon}{\mu^2} + \text{finite} \right)$$

(4.32)

This has an imaginary part if and only if $p_1 \cdot p_2 > 0$. The Glauber contribution produces only the imaginary part of this result, as we will now see in a number of different ways.

### 4.4.1 Method of regions

According to the method of regions, we can isolate the Glauber contribution by assuming $k^\pm \sim \kappa^2$, $\vec{k}_\perp \sim \kappa$ and expanding the integrand to leading order in $\kappa$. This gives

$$I_{\text{Glauber}} = -i2Q^2g^2_s \int \frac{d^4k}{(2\pi)^4} \frac{1}{[-Qk^- - \vec{k}_\perp^2 + i\epsilon]} \frac{1}{[\vec{k}_\perp^2 + i\epsilon]} \frac{1}{[Qk^+ - \vec{k}_\perp^2 + i\epsilon]}$$

(4.33)
In this form, one cannot integrate over $k^-$ and $k^+$ using Cauchy’s theorem since the integrand does not die off fast enough as $k^\pm \to \infty$. Changing to $k^- = k_0 - k_z$ and $k^+ = k_0 + k_z$ gives

$$I_{\text{Glauber}} = -i2Q^2g_s^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{[Qk_0 + Qk_z - \vec{k}_\perp^2 + i\epsilon][\vec{k}_\perp^2 + i\epsilon]} \frac{1}{[Qk_0 + Qk_z - \vec{k}_\perp^2 + i\epsilon]}$$

Ordinarily, the $k^2$ term is quadratic in $k_0$ and $k_z$ so using $k^\pm$ is simpler, but in the Glauber limit the transverse components dominate so there is no advantage.

For timelike separation (both outgoing) as written, the poles in the $k_0$ plane are at $k_0 = \mp \vec{k}_\perp^2 / Q \pm k_z \pm i\epsilon$. Closing the $k_0$ contour downwards to pick up the $k_0 = \vec{k}_\perp^2 / Q - k_z - i\epsilon$ pole results in an integral which is divergent at large $|k_z|$. This divergence can be regulated by a rapidity regulator. Following Ref. [152], we use the $\eta$ regulator of Ref. [176].

Adding also a small mass $m$ to regulate the infrared singularity and working in $d = 4 - 2\epsilon$ dimensions to regulate the UV divergence gives

$$I_{\text{Glauber}} = -2Qg_s^2\mu^{4-d} \int \frac{d^{d-2}k_\perp dk_z}{(2\pi)^{d-1}} v^{2\eta} \frac{1}{[2k_z]^2} \frac{1}{[2Qk_z - 2\vec{k}_\perp^2 + i\epsilon][\vec{k}_\perp^2 - m^2 + i\epsilon]}$$

$$= -\frac{\alpha_s}{2\pi} (i\pi) \left( \frac{1}{\epsilon_{\text{UV}}} + \ln \frac{\mu^2}{m^2} \right)$$

Note that the result is independent of $\eta$ (see also Appendix B.2 of [152]). Essentially, adding the $|2k_z|^{2\eta}$ term allows us to do the $k_z$ integral. The imaginary part of the full soft graph with these regulatoris is identical [152] confirming that the Glauber is contained in the soft.

If we also use dimensional regularization to regulate IR, the integral would be scaleless and vanish. We then deduce that in pure dimensional regularization

$$I_{\text{Glauber}} = \frac{\alpha_s}{2\pi} (-i\pi) \left( \frac{1}{\epsilon_{\text{UV}}} - \frac{1}{\epsilon_{\text{IR}}} \right)$$
Thus the Glauber contribution is IR divergent and purely imaginary.

Note that if $p_2^\mu$ were incoming, then $Qk^+$ term would flip sign and the poles in the $k_0$ plane would be at $k_0 = -\bar{\vec{k}}^2 / Q \pm k_z + i\epsilon$. Since both poles are above the integration contour, we can close the contour downwards giving zero. In other words, the Glauber limit with spacelike separation gives zero for this diagram.

Thus we find that the Glauber limit gives zero in the spacelike case ($p_1 \cdot p_2 < 0$) and a purely imaginary number in the timelike case ($p_1 \cdot p_2 > 0$). Since the amplitude is zero in a compact region of $p_1 \cdot p_2$, it cannot be an analytic function of $p_1 \cdot p_2$ without vanishing completely. The non-analyticity comes from the rapidity regulator, since $|2k_z|^2 \eta$ is a non-analytic function. This non-analyticity is therefore unavoidable if the Glauber contribution is to give only the imaginary part of the amplitude.

### 4.4.2 Cut-based approach

Yet another way to study Glauber gluon is through discontinuity of scattering amplitudes. This is natural from the viewpoint that Glauber gluon is associated with the imaginary part of scattering amplitudes at lowest order.

Using Cutkosky’s cutting rule, the s-channel discontinuity of Eq. (4.30) can be written as

$$
\text{Disc}_s I_{\text{full}} = i(-2\pi i)^2 g_s^2 \int \frac{d^4k}{2(2\pi)^4} \delta((p_1 - k)^2)\theta((p_1 - k)^0)\delta((p_2 + k)^2)(\theta(p_2 + k)^0) \\
\times \frac{(2p_1 - k)^\mu \Pi_{\mu\nu}(k)(2p_2 + k)^\nu}{[k^- k^+ - \bar{\vec{k}}^2 + i\epsilon]} \tag{4.38}
$$

where the only possible cut is through the top and bottom line in the vertex diagram. We shall see that at this order, the Glauber contribution is given by the forward limit of the discontinuity, $\bar{\vec{k}} \ll Q$. The on-shell conditions of $p_1 - k$ and $p_2 + k$ automatically enforce
the Glauber scaling, $k^- \sim k^+ \sim k^2_\perp / Q$. In the forward limit, the two-body phase space measure becomes

$$
\int \frac{d^2 k_\perp}{(2\pi)^2} \delta((p_1 - k)^2)\theta((p_1 - k)^0)\delta((p_2 + k)^2)\theta(p_2 + k)^0)
$$

and the matrix element to the right of the cut becomes forward scattering amplitude

$$
\frac{(2p_1 - k)\Pi_{\mu\nu}(k)(2p_2 + k)^\nu}{[k - k_\perp + i\epsilon]} \rightarrow \frac{2Q^2}{k_\perp^2 - i\epsilon}
$$

The one-loop Glauber contribution is thus given by

$$
I_{\text{Glauber}} \equiv \lim_{k_\perp / Q \rightarrow 0} \text{Disc}_s I_{\text{full}} = -\frac{i\alpha_s}{2\pi^2} \int \frac{d^2 k_\perp}{k_\perp^2}
$$

in agreement with the other definition of Glauber contribution (Eq. (4.36) after the $k_z$ integral is done). It is interesting to note that defining the Glauber contribution in this way avoids the use of the rapidity regulator in intermediate steps of the calculation. This is comforting, as the result should be regulator independent.

Isolating the Glauber contribution in momentum space at 1-loop level with this approach was introduced in [115], and a similar cutting prescription in position space was explored more in [168]. The idea is similar to the s-channel unitarity approach for extracting the Reggie trajectory of scattering amplitudes [177]. However, it should be noted that only at 1-loop level can one identify the leading term in the $\epsilon$ expansion of the Glauber contribution with a discontinuity. At higher loop, there is no direct relation anymore. For example, at 2 loops, double Glauber exchange contributes $(i\pi)^2 = -\pi^2$, which is real and has no discontinuity.
4.4.3 Position space

One can interpret Eq. (4.41) as describing a potential \( \tilde{V}(k) \sim \frac{g_s^2}{k_\perp^2} \) between the two outgoing particles. Fourier transforming, the potential \( V(x) \sim g_s^2 \ln |x_\perp| \) depends only on the transverse separation \( x_\perp \) between the particles. Like the Coulomb potential, \( V(r) \sim \frac{g_s^2}{r} \), the Glauber potential is time-independent, but unlike the Coulomb potential, the Glauber potential additionally does not depend on the longitudinal separation \( x_L \).

Since the Glauber contribution is contained in the soft contribution for this graph, we can simplify the calculation from one in full QCD by taking the energies of \( p_{m_1}^\mu \) and \( p_{m_2}^\mu \) to infinity. Following [168], to regulate some of the divergence, we also take \( p_{m_1}^\mu \) and \( p_{m_2}^\mu \) timelike. Thus we write \( p_{m_1}^\mu = Q_1 n_1^\mu \) and \( p_{m_2}^\mu = Q_2 n_2^\mu \) with \( n_1^2 = n_2^2 = 1 \). The integral then becomes

\[
I_{\text{soft}} = g_s^2 \int \frac{d^4k}{(2\pi)^4} \frac{n_{12}}{(-n_1 \cdot k + i\epsilon)(n_2 \cdot k + i\epsilon)(k^2 + i\epsilon)}
\]

\[
= g_s^2 \int_0^\infty ds_1 \int_0^\infty ds_2 \frac{n_{12}}{k^2 + i\epsilon} e^{-i s_1 (n_1 \cdot k) + i s_2 (n_2 \cdot k)}
\]

\[
= g_s^2 \int_0^\infty ds_1 \int_0^\infty ds_2 \frac{n_{12}}{(s_1 n_q - s_2 n_2^\mu)^2 - i\epsilon}
\]

where \( n_{12} = n_1 \cdot n_2 \). Schwinger parameters \( s_1 \) and \( s_2 \) have been introduced on the second line. They represent the proper time that the particles have travelled from the hard vertex. The result of course matches the matrix element of timelike Wilson lines, which we could have written down directly.

A nice observation from [168] is that, in this form, we can see that an imaginary part can only come from times \( s_1 \) and \( s_2 \) for which particles along the \( n_1^\mu \) and \( n_2^\mu \) direction are lightlike separated. That is, the Glauber contribution is associated with spacetime points under which particles moving in the two directions can causally influence each other. As an analogy, think of passengers on trains going in two different directions shining
a light at each other. The light from one train can be seen on the other train only at
the appropriate spacetime point. In contrast, if one particle is incoming and the other
outgoing, as in Eq. (4.22), it is impossible for the two to be lightlike separated – one cannot
see light from a train which arrived in the station before your train left (except at the origin
of time). Thus, there is no imaginary part in that situation, and no Glauber contribution.
Taking the limit where the trajectories become lightlike in the timelike separation (both
outgoing) case, the support of the imaginary part lies on the lightcone, \( s_1 = 0 \) or \( s_2 = 0 \).
For spacelike separation, the lightlike limit is zero.

The amplitude in Eq. (4.44) is the soft amplitude, from reducing the full diagram in
QCD in the eikonal limit. As shown in the previous section, it contains the complete
Glauber contribution, which is now identified as the imaginary part of the diagram.

Defining the cusp angle \( \gamma \) through \( \cosh \gamma = -n_{12} \) and changing variables
\( s_1 = s e^\tau \) and \( s_2 = s \), the integral becomes [178]:

\[
I_{\text{soft}} = \frac{\alpha_s}{4\pi} \int_0^{\infty} ds \int_{-\infty}^{\infty} d\tau \frac{\cosh \gamma}{\cosh \tau + \cosh \gamma} = \frac{\alpha_s}{8\pi} \int_0^{\infty} ds \frac{\gamma \cosh \gamma}{s}
\]

(4.45)

This is again UV and IR divergent. In the spacelike separation case (one incoming and
one outgoing) \( n_{12} < 0 \) and \( \gamma > 0 \) is real. Then this integral is real. The timelike case (both
outgoing) corresponds to \( n_{12} > 0 \) whereby \( \cosh \gamma < 0 \) and \( \gamma \) is complex. Thus only in the
case of timelike separation does the amplitude have an imaginary part.

The fact that the Glauber contribution is purely imaginary at 1-loop implies that it will
necessarily cancel in cross section calculations at next-to-leading order. More generally,
the 1-loop Glauber contribution exponentiates into a phase which cancels in cross sections
to all orders. This exponentiation comes about in the same way that the exponentiation
of the Coulomb phase comes about. One way to see it is by computing the energy of a
moving charge in the potential of another moving charge, either classically or in quan-
tum mechanics [179] or by mapping to AdS [178]. These considerations also lead to the shockwave picture of Glauber exchange in forward scattering [152]. As exponentiation of the Glauber phase corresponds very closely to Abelian exponentiation, it naturally is also limited in the non-Abelian theory. For example, irreducible 2-loop contributions or beyond with both Glauber and soft/collinear loops present may not exponentiate.

4.4.4 Effective field theory Glauber operator

We have seen that taking the Glauber limit of the integrand, according to the method of regions, gives a result which is purely imaginary and non-vanishing only in the time-like case. The effective field theory (EFT) approach tries to write down a Lagrangian whose Feynman rules generate the integral so that one no longer has to take limits of integrands. This Lagrangian can be derived by matching (writing down all possible operators consistent with symmetries and working out their coefficients to agree with QCD) or by performing a multipole expansion in the classical theory, keeping the leading interactions according to some specified scalings. In general, there is not a 1-to-1 correspondence between diagrams in the effective theory and diagrams in QCD, even after those diagrams are expanded according to some scaling. For example, EFT operators often include Wilson lines for gauge invariance. These Wilson lines represent the leading power contribution of many diagrams in QCD.

The effective field theory that isolates the infrared singular regions of QCD is called Soft-Collinear Effective Theory [26, 27, 30, 80, 165] (see [28] for a review). If a process involves hard directions $p_1^\mu$ and $p_2^\mu$, then in SCET collinear fields denoted by $\xi_j$ are introduced in the $p_1^\mu$ and $p_2^\mu$ direction. These fields have labels fixing the large and perpendicular components of their momenta, with the dynamics determined by fluctuations around these parametrically large components. The extension of SCET to include Glauber contributions was recently achieved in [152]. The prescription is to add Glauber operators to
the Lagrangian for each pair of directions in the theory. For example, in QED we would add

\[ \mathcal{O}_{\text{QED}}^{G_{12}} = 8\pi\alpha \left[ \frac{x_1}{2} \alpha_1 \frac{1}{p_2} \right] \left[ \frac{x_2}{2} \alpha_2 \right] \]  

(4.46)

Here \( P_\perp \) is an operator which picks out the \( \perp \) component of the collinear fields it acts. In QCD, the operator is significantly more complicated [152]

\[ \mathcal{O}_{\text{QCD}}^{G_{12}} = 8\pi\alpha_s \left[ \frac{x_1}{2} W_1^A W_1^{A_1} \right] \frac{1}{p_2} \]

\[ \times \left[ P_{\perp}^{\mu} \gamma_1^\mu \gamma_2^\mu P_{\perp} - g P_{\perp}^{\mu} B_{1S \perp}^{\mu} \gamma_1^\mu \gamma_2^\mu - g B_{1S \perp}^{\mu} \gamma_1^\mu B_{2S \perp}^{\mu} - \frac{i g}{2} n_1^{\mu} n_2^{\mu} \gamma_1^\mu \gamma_2^\mu \right]^{AB} \]

(4.47)

The additional terms involve collinear Wilson lines \( W_j \), soft Wilson lines \( \gamma_j \) in the adjoint representation, soft gluon fields \( B_{\perp,\mu} \), and soft gluon field strengths \( \vec{G}^S_{\mu\nu} \). For definitions of all of these objects, see [152]. The collinear Wilson lines are necessary to ensure collinear gauge-invariance. The terms on the second line all involve soft fields and are separately soft-gauge-invariant. The variety of terms in this expression is determined by matching amplitudes in SCET and in full QCD. Although one might imagine their coefficients get corrected order-by-order in perturbation theory through this matching, it seems that in fact they do not; remarkably this operator appears to not receive any corrections and is not renormalized.

In the Abelian limit, \( \vec{G}^S_{\mu\nu} = B^{\mu} = 0 \) since these are both proportional to \( f^{abc} \) and the adjoint Wilson lines are trivial: \( \gamma_1 = \gamma_2 = 1 \). With these substitutions, Eq. (4.47) reduces to Eq. (4.46).

At leading order in \( g_s \), the QCD operator reduces to the QED one up to the group theory factors. The leading order interaction of this operator is a 4-point interaction connecting lines in the \( n_1^{\mu} \) direction with lines in the \( n_2^{\mu} \) directions. The Feynman rule pro-
duces a factor of $\frac{i k^2}{k^2}$ with $k^\mu$ the momentum transferred between the two lines. Operator insertions are drawn either with a red oval, indicating its pointlike nature, or with a red dotted line, indicating its origin in gluon exchange. For example,

$$I_{O_{G12}} = \frac{1}{p_1} \frac{1}{p_2} \uparrow k \quad \approx \quad -i2Q^2g_s^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{[-Qk^- - \vec{k}_\perp + ie]} \frac{1}{[-\vec{k}_\perp + ie]} \frac{1}{[Qk^+ - \vec{k}_\perp + ie]}$$

(4.48)

This is identical to the Glauber limit of the original diagram, in Eq. (4.30).

Just adding this operator to the SCET Lagrangian with no other modification will lead to overcounting. As we already observed, in most situations the Glauber contribution is contained in the soft diagrams. The resolution proposed in [152] is to subtract off the overlap, diagram by diagram. The viewpoint of [152] is that the Glauber mode is a separate mode from the soft. Thus the true soft contribution, $S$, meaning soft without Glauber, should be defined as the naive soft contribution $\tilde{S}$ (including Glaubers) with its Glauber limit $S^{(G)}$ subtracted: $S = \tilde{S} - S^{(G)}$. This subtraction is done using the zero-bin subtraction method [21] – subtract from the soft diagram its limit where only the terms to leading power in Glauber scaling are kept. For those cases where the soft-Glauber contribution is identical to the Glauber contribution, which includes active-active and active-spectator interactions [152], the result is the same as using the naive soft graph only.

### 4.5 Factorization-violation in collinear splittings

As observed in Section 4.3.3, the only time hard-soft-collinear factorization can break down is in situations where an incoming particle is collinear to an outgoing one, the space-like collinear limit. In fact, it is known that in the space-like limit, amplitude-level
factorization is violated [169]. We now proceed to reproduce and discuss this important result.

Collinear factorization implies that the matrix element $|M\rangle$ for an amplitude with $m$ particles collinear to a particular direction is related to the amplitude $|\overline{M}\rangle$ with only one particle collinear to that direction. For simplicity, we consider here only the case with $m = 1$, describing $1 \rightarrow 2$ collinear splittings. Let us say there are $n$ total particles in $|M\rangle$, of which only $p_2^\mu$ is collinear to $p_1^\mu$. In this case, the splitting function $S_\mu$ is defined as

$$|M(p_1, \cdots, p_n, \rangle \approx S_\mu(p_1, p_2; p_3, \cdots, p_n) \cdot |\overline{M}(P, p_3, \cdots, p_n)\rangle \quad (4.49)$$

Here, $P^\mu \equiv p_1^\mu + p_2^\mu$, meaning $P^\mu$ is an onshell momentum ($P^2 = 0$) which is equal to the sum of the two momenta that split, up to power corrections in $\lambda^2 = 2p_1 \cdot p_2 / Q^2$, with $Q$ the center-of-mass energy or some other hard scale. The object $S_\mu$ is an (amplitude-level) splitting function, or splitting amplitude. The matrix elements should be thought of as vectors in color space and $S_\mu$ as an operator acting on these vectors.

Eq.(5.3), which has the splitting function depending on all the momenta, is called generalized collinear factorization [169]. Even generalized factorization is non-trivial. The non-trivial part is that the splitting function is universal, independent of the short-distance physics encoded in the matrix element $|\overline{M}\rangle$.

Generalized factorization is not terribly useful for computing cross sections. For example, since generalized splitting amplitudes depend on all the hard directions and colors in the processes, they do not allow us to use the semi-classical parton-shower simulation method to generate jet substructure to all orders. The parton shower can be justified when the $S_\mu$ depends only on the momenta $p_1^\mu$ and $p_2^\mu$ in the relevant collinear sector, $S_\mu = S_\mu(p_1, p_2)$, and only on the color $T_1$ in that sector. When $S_\mu$ has this special form, we say, following [169] that strict collinear factorization holds.
Using results from the previous sections, we first confirm that strict factorization holds when there are zero or one colored particles in the initial state. We then discuss the case with two or more initial state particles where strict factorization may fail. We review the calculation of the 1-loop factorization-violating effect in the IR-divergent part of $Sp$ from [169] and summarize other results from QCD. In the next sections we will reproduce these results from SCET with Glauber operators.

### 4.5.1 Strict factorization

Let us start with the situation where $p_1^\mu$ is the momentum of an outgoing quark, $p_2^\mu$ is the momentum of an outgoing gluon and none of the other $n-2$ momenta are collinear to $p_1^\mu$ and $p_2^\mu$. We also take our matrix elements to be of operators with $n-1$ fields, e.g. $O = \bar{\psi}_1 \cdots \psi_{n-1}$. In this situation, the hard-soft-collinear factorization formula in Eq. (5.2) holds for $|\overline{\mathcal{M}}\rangle$ and for $|\mathcal{M}\rangle$. We can write

$$|\overline{\mathcal{M}}\rangle \cong \frac{\langle P | \bar{\psi} W_1 | 0 \rangle}{\text{tr} \langle 0 | Y_1 W_1 | 0 \rangle} \cdot |\mathcal{M}_{\text{rest}}\rangle, \quad |\mathcal{M}\rangle \cong \frac{\langle p_1, p_2 | \bar{\psi} W_1 | 0 \rangle}{\text{tr} \langle 0 | Y_1 W_1 | 0 \rangle} \cdot |\mathcal{M}_{\text{rest}}\rangle \quad (4.50)$$

Here, spin and color indices are suppressed (see Section 12 of [164] for more details) and tr indicates a color trace. $|\mathcal{M}_{\text{rest}}\rangle$ represents the product of the Wilson coefficient with the matrix element of soft Wilson lines and the collinear matrix elements involving momenta in other directions. Critically, the form of $|\mathcal{M}_{\text{rest}}\rangle$ is identical for both factorization formulas. Thus the splitting function is the ratio of the two:

$$Sp = \frac{\langle p_1, p_2 | \bar{\psi} W_1 | 0 \rangle}{\langle P | \bar{\psi} W_1 | 0 \rangle} \quad (4.51)$$
To be explicit, at tree-level, the splitting function for a right-handed quark and a negative helicity gluon can be derived in this way [164]

\[ S^{0}_{R-}(p_1, p_2) = g_s \frac{\sqrt{2} z}{[p_2 p_1] \sqrt{1 - z}} T_1 \]  

(4.52)

Which agrees with the well-known tree-level QCD splitting amplitudes [180]. Derivations for other amplitude-level splitting functions in this way can be found in [164].

Note that in this case (both \( p^\mu_1 \) and \( p^\mu_2 \) outgoing), the splitting function depends only on these two momenta and also only on the net color in the 1 direction. This is what is referred to as strict factorization.

4.5.2 Strict factorization violation from \( i\pi/\epsilon \) terms

Next, let us consider a general process where \( p^\mu_1 \) and \( p^\mu_2 \) are to become collinear and the other momenta point in generic directions. We do not yet specify which particles are incoming or outgoing and want to see what sufficient conditions are for strict factorization to be violated.

Since \( p^\mu_1 \) or \( p^\mu_2 \) may be incoming, we cannot assume the hard-soft-collinear factorization formula in Eq. (5.2) is correct at leading power in \( \lambda^2 = p_1 \cdot p_2 / Q^2 \). However, as long as \( p^\mu_1 \) and \( p^\mu_2 \) are not pointing in exactly the same direction (so \( \lambda \neq 0 \)), then the factorization formula guarantees that all of the infrared divergences of the full theory are reproduced in the factorized expression. That is, the Glauber contribution is contained in the soft except at the exceptional point in phase space where \( p^\mu_1 \propto p^\mu_2 \). This is very powerful, as we can then determine the IR divergences of \( |M\rangle \) and \( |\bar{M}\rangle \) separately and then explore the limit where \( p^\mu_1 \) becomes collinear to \( p^\mu_2 \) [169].

Consider the matrix element \( |\bar{M}\rangle \) of a hard operator with \( n - 1 \) fields. \( |\bar{M}\rangle \) is the matrix element before the emission, where the parton in the 1 direction has momentum
\[ P^\mu \simeq p_1^\mu + p_2^\mu. \]

The color associated with \( P^\mu \) we write as \((12)\). This means that the color operator acts as the sum of the color operators for \( p_1^\mu \) and \( p_2^\mu \):

\[ \mathbf{T}^{(12)} \cdot \mathbf{X} = (\mathbf{T}^1 + \mathbf{T}^2) \cdot \mathbf{X}. \]

Now, as long as \( p_1^\mu \) and \( p_2^\mu \) are not proportional, nothing stops us from treating them as separate hard directions. Then \( |\mathcal{M}| \) is also given by a the matrix element of a local operator whose infrared divergences are reproduced by the factorized expression

\[ |\mathcal{M}| = \left( \begin{array}{c} 1 \\ 3 \\ \vdots \\ n-1 \end{array} \right) \]

(4.53)

For strict factorization to hold, these two expressions should be related by a splitting function \( \text{Sp} \) that depends only on the \((12)\) system, independent of the rest of the process.

Let us now look at the 1-loop corrections to \( |\mathcal{M}| \) and \( |\mathcal{M'}| \). In particular, we are interested in the Glauber contribution since that is where factorization violation might come from. The Glauber contribution is contained in the soft contribution and at 1-loop is purely imaginary, as shown in Section 4.4. Indeed, at 1-loop the Glauber contribution is particular simple: it gives a factor of \( \frac{\alpha_s}{2\pi} \left( \frac{i\pi}{\epsilon_{\text{IR}}} - \frac{i\pi}{\epsilon_{\text{UV}}} \right) \), as in Eq. (4.37) if the two lines the loop connects are both outgoing or both incoming. Here we keeps the IR poles only as the UV poles will be renormalized away. In QCD, one gets this 1-loop contribution multiplied by
a group theory factor of $T_i \cdot T_j$. For example, a gluon connecting lines 1 and 3 gives

$$|\mathcal{M}^{G1}\rangle = \sum_{i<j} T_i \cdot T_j \frac{\alpha_s i \pi}{2 \pi \epsilon} \cdot |\mathcal{M}^0\rangle$$  \hspace{1cm} (4.55)$$

where $G1$ stands for the 1-loop Glauber contribution to $|\mathcal{M}\rangle$ and $|\mathcal{M}^0\rangle$ is $|\mathcal{M}\rangle$ to lowest order in $g_s$. We have neglected the UV pole in Eq. (4.55). Although we draw the contribution as a dotted red line, as in the SCET approach, all we are using here is that the Glauber contribution is identified with the imaginary part at 1-loop which follows from any method of computation. The 1-loop Glauber corrections to $|\mathcal{M}\rangle$ is given by the same formula, but summed over the $n - 1$ partons.

Let us first consider the case where all the particles are outgoing or all incoming. To simplify the expression we can use color conservation, $\sum_j T_j = 0$ and $T_i \cdot T_i = C_i$ where $C_i$ is the group Casimir (a number not an operator). We then find

$$|\mathcal{M}^{G1}\rangle = \frac{\alpha_s i \pi}{2 \pi \epsilon} \frac{1}{2} \sum_{i=1}^n T_i \cdot (-T_i) \cdot |\mathcal{M}^0\rangle = -\frac{\alpha_s i \pi}{4 \pi \epsilon} \sum_{i=1}^n C_i \cdot |\mathcal{M}^0\rangle$$  \hspace{1cm} (all outgoing) (4.56)

For $|\overline{\mathcal{M}}\rangle$ we have

$$|\overline{\mathcal{M}}^{G1}\rangle = \frac{\alpha_s i \pi}{4 \pi \epsilon} \left[ -T_{(12)} \cdot T_{(12)} - \sum_{i=3}^n T_i \cdot T_i \right] \cdot |\overline{\mathcal{M}}^0\rangle = -\frac{\alpha_s i \pi}{4 \pi \epsilon} \left[ C_{(12)} + \sum_{i=3}^n C_i \right] \cdot |\overline{\mathcal{M}}^0\rangle$$  \hspace{1cm} (4.57)

The splitting function has to reproduce the ratio of these two. Thus

$$i \text{Im} \mathbf{S}^1 = \frac{\alpha_s i \pi}{4 \pi \epsilon} \left[ C_{(12)} - C_1 - C_2 \right] \cdot \mathbf{S}^0 + \text{finite terms}$$  \hspace{1cm} (all outgoing) (4.58)$$

This color factor only depends on the colors of the particles in the 12 sector, independent
of the rest of the event, consistent with strict collinear factorization.

Next, suppose that $p_1^\mu$ is in the initial state but all other particles are in the final state. Then there is no Glauber contribution from anything connecting to $p_1^\mu$ nor from anything connecting to $P^\mu = p_{(12)}^\mu$ in the case of $|\mathcal{M}\rangle$. With this arrangement,

$$|\mathcal{M}^{G1}\rangle = \frac{\alpha_s}{4\pi} \frac{i\pi}{e} \sum_{i=2}^{n} T_i \cdot (-T_i - T_{1}) \cdot |\mathcal{M}^{0}\rangle = -\frac{\alpha_s}{4\pi} \frac{i\pi}{e} \left[ -C_1 + \sum_{i=2}^{n} C_i \right] \cdot |\mathcal{M}^{0}\rangle$$

(4.59)

where $\sum_{i=2}^{n} T_i = -T_1$ has been used twice. Similarly,

$$|\mathcal{M}^{G1}\rangle = \frac{\alpha_s}{4\pi} \frac{i\pi}{e} \sum_{i=3}^{n} T_i \cdot (-T_i - T_{(12)}) \cdot |\mathcal{M}^{0}\rangle = -\frac{\alpha_s}{4\pi} \frac{i\pi}{e} \left[ -C_{(12)} + \sum_{i=3}^{n} C_i \right] \cdot |\mathcal{M}^{0}\rangle$$

(4.60)

Thus, with one incoming colored particle

$$i\text{Im} \mathbf{S}^1 = \frac{\alpha_s}{4\pi} \frac{i\pi}{e} \left[ -C_{(12)} + C_1 - C_2 \right] \cdot \mathbf{S}^0 + \text{finite terms}$$

(one incoming)

(4.61)

Although this splitting function is different from the all outgoing case, Eq. (4.58), both depend only on the colors of the particles in the 12 sector. Thus with one incoming particle, we cannot conclude that strict collinear factorization is violated.

Now suppose $p_1^\mu$ and another particle, $p_3^\mu$ are both in the initial state, with $p_3^\mu$ still generic (not collinear to any other direction). For $\mathcal{M}$, which has $p_2^\mu$ outgoing, we get

$$|\mathcal{M}^{G1}\rangle = \frac{\alpha_s}{2\pi} \frac{i\pi}{e} \left[ T_1 \cdot T_3 + \frac{1}{2} \sum_{i=2,4,\ldots} T_i \cdot (-T_i - T_1 - T_3) \right] \cdot |\mathcal{M}^{0}\rangle$$

(4.62)

$$= \frac{\alpha_s}{2\pi} \frac{i\pi}{e} \left[ 2T_1 \cdot T_3 + \frac{1}{2} C_1 + \frac{1}{2} C_3 - \frac{1}{2} C_2 - \frac{1}{2} \sum_{i=4}^{n} C_i \right] \cdot |\mathcal{M}^{0}\rangle$$

(1 and 3 incoming)

(4.63)

The matrix element $|\mathcal{M}|$, correspondingly has (12) and 3 incoming. Its 1-loop Glauber
contribution is

$$|\overline{\mathcal{M}}^{G_1}\rangle = \frac{\alpha_s}{2\pi} \frac{i\pi}{\epsilon} \left[ 2T_{(12)} \cdot T_3 + \frac{1}{2} C_{(12)} + \frac{1}{2} C_3 - \frac{1}{2} \sum_{i=4}^{n} C_i \right] \cdot |\overline{\mathcal{M}}^0\rangle$$  \hspace{1cm} (4.64)

So we find

$$Sp^{1,\text{non-fact}} = i\text{Im} \; Sp^1 = \frac{\alpha_s}{4\pi} \frac{i\pi}{\epsilon} \left[ -C_{(12)} + C_1 - C_2 - 4T_2 \cdot T_3 \right] \cdot Sp^0 + \text{finite terms} \quad (1 \text{ and } 3 \text{ incoming})$$  \hspace{1cm} (4.65)

where $Sp^{\text{non-fact}}$ denotes strict-factorization-violating contributions. This form indicates a violation of strict collinear factorization: the splitting function depends on the color of particles in the matrix element other than those involved in the splitting ($T_3$ in this case).

The result in Eq. (4.65) was first derived in Ref. [169] by examining IR singularities of full QCD amplitudes [181–183] in different kinematical regions. The authors of Ref. [169] also derived the 1-loop finite part and the 2-loop IR singular part of factorization violating splitting amplitudes, as we discuss next.

### 4.5.3 Strict-factorization violation from full QCD

Next, we summarize some known additional results about factorization-violation from full QCD, including the 1-loop finite parts and the 2-loop divergent parts of $Sp^{\text{non-fact}}$.

The IR divergent part of a 1-loop amplitude is defined relative to the tree-level amplitude as

$$|\mathcal{M}^1\rangle = \Gamma^1(\epsilon) |\mathcal{M}^0\rangle + |\mathcal{M}^{1,\text{fin.}}\rangle$$  \hspace{1cm} (4.66)

where $|\mathcal{M}^{1,\text{fin.}}\rangle$ is a finite, analytic function of the momenta. The general expression for $\Gamma^1(\epsilon)$ follows from Eq. (4.32) with the appropriate sum over pairs of external legs to which the virtual gluon can attach and appropriate color factors. The $1/\epsilon^2$ poles are color diagonal. Using color conservation to simplify the result, an amplitude in QCD with $n$ external
partons with colors $T_i$ has IR divergences given by \[181\]

\[
I^1(\epsilon) = \frac{\alpha_s}{2\pi} \left[ -\sum_{i=1}^{n} \left( \frac{C_i}{\epsilon^2} + \frac{\gamma_i}{\epsilon} \right) - \frac{1}{\epsilon} \sum_{i \neq j} T_i \cdot T_j \ln \frac{-s_{ij} - i\epsilon}{\mu^2} \right] \tag{4.67}
\]

Here $\gamma_i$ is the regular (non-cusp) anomalous dimension: $\gamma_q = 3C_F/2$ for quarks and $\gamma_g = \beta_0 = \frac{11}{6}C_A - \frac{2}{3}T_F n_f$ for gluons. Although anomalous dimensions are usually associated with UV divergences, they appear in this expression because they can be extracted using properties of scaleless integrals in dimensional regularization, in which the UV and IR divergences cancel \[182\].

We are interested here in the order-by-order expansion of the splitting amplitudes. We write

\[
|\mathcal{M}^0 + \mathcal{M}^1 + \cdots \rangle \approx (\mathbf{S}^0 + \mathbf{S}^1 + \cdots) |\overline{\mathcal{M}}^0 + \overline{\mathcal{M}}^1 + \cdots \rangle \tag{4.68}
\]

So that $|\mathcal{M}^0\rangle = \mathbf{S}^0 |\overline{\mathcal{M}}^0\rangle$ at tree-level, $|\mathcal{M}^1\rangle = \mathbf{S}^0 |\overline{\mathcal{M}}^1\rangle + \mathbf{S}^1 |\overline{\mathcal{M}}^0\rangle$ at 1-loop, and so on. It can be helpful to separate out the divergent parts of the splitting function too. We define,

\[
\mathbf{S}^1 = I_C^1 \cdot \mathbf{S}^0 + \mathbf{S}^{1,\text{fin}}. \tag{4.69}
\]

where $\mathbf{S}^{1,\text{fin}}$ is an IR-finite analytic function of momenta. All the IR divergences are absorbed in $I_C^1$. It is not hard to show that \[169, 170\]

\[
I_C^1 = I^1 - \bar{I}^1 \tag{4.70}
\]

where $\bar{I}^1$ is the divergent part of $|\overline{\mathcal{M}}^1\rangle$, as in Eq. (4.66).

Plugging in Eq. (4.67) gives

\[
I_C^1 = \frac{\alpha_s}{2\pi} \left[ \frac{C_{(12)} - C_1 - C_2}{\epsilon^2} - \frac{\gamma_{(12)} - \gamma_1 - \gamma_2}{\epsilon} - \frac{2}{\epsilon} T_1 \cdot T_2 \ln \frac{-s_{12} - i\epsilon}{\mu^2} \right]
\]
\[- \frac{2}{\epsilon} \sum_{j=3}^{n} \left( T_1 \cdot T_j \ln \frac{-s_{1j} - i\epsilon}{\mu^2} + T_2 \cdot T_j \ln \frac{-s_{2j} - i\epsilon}{\mu^2} - T_{(12)} \cdot T_j \ln \frac{-s_{(12)j} - i\epsilon}{\mu^2} \right) \]  

(4.71)

The factorization violation in $S p^1$ is contained in the imaginary part of $I^1_C$. Explicitly, when there are two incoming momenta,

$$i \text{Im} I^1_C = \frac{\alpha_s}{4\pi} \frac{i\pi}{\epsilon} \left[ -C_{(12)} + C_1 - C_2 - 4T_2 \cdot T_3 \right] \quad \text{(1 and 3 incoming)} \quad (4.72)$$

in agreement with Eq. (4.65).

The same approach can be used to extract the finite parts of the 1-loop splitting functions. The expression for $I^1_C$ from [169], including terms that are IR-finite is

$$I^1_C = c \Gamma \left( \frac{-s_{12} - i\epsilon}{\mu^2} \right)^{-\epsilon} \frac{\alpha_s}{2\pi} \left\{ \frac{C_{(12)} - C_1 - C_2}{\epsilon^2} + \frac{\gamma_{(12)} - \gamma_1 - \gamma_2 + \beta_0}{\epsilon} \right. \right.
\left. + \left. \frac{2}{\epsilon} \left[ \sum_{j=3}^{n} T_1 \cdot T_j f(\epsilon, 1 - z) + \sum_{j=3}^{n} T_2 \cdot T_j f(\epsilon, z - i\epsilon s_{(12)}) \right] \right\} \right\} \quad (4.73)$$

where $c \Gamma \equiv \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{(4\pi)^{2-\epsilon}\Gamma(1-2\epsilon)}$. The function $f(\epsilon, z)$ is a hypergeometric function, defined by

$$f(\epsilon, 1/x) \equiv \frac{1}{\epsilon} \left[ 2F_1(1, -\epsilon; 1 - \epsilon; 1 - x) - 1 \right] = \ln x - \epsilon \text{Li}_2(1 - x) + O(\epsilon^2) \quad (4.74)$$

The convention taken is that $1 - z > 0$. This can be assumed for timelike or spacelike splittings without loss of generality since $z + (1 - z) = 1$ (i.e. $z$ can still be positive or negative). Thus the $f(\epsilon, 1 - z)$ factor in Eq. (4.73) is real and unambiguous. The function $f(\epsilon, z)$ has a cut for negative real $z$, i.e. for spacelike splittings. Writing $f(\epsilon, z - i\epsilon p_j \cdot p_2)$ in Eq. (4.73) makes the result well-defined. In particular, the sign of the imaginary part of the function is determined by the sign of $p_j \cdot p_2$. This dependence of the analytic continuation on the momentum $p_j$ obstructs the reduction of Eq. (4.73) to a form that obeys strict factorization.
To see the factorization violating part of the $\epsilon^0$ term, we can use

$$\text{Im } \text{Li}_2(1 - \frac{1}{z \pm i\epsilon}) = \pm \pi \ln(1 - \frac{1}{z}) \quad (4.75)$$

Then the factorization violating contribution is seen to be

$$\mathbf{S}^{1,\text{non-fact}} = i \text{Im } \mathbf{S}^1$$

$$= c\Gamma \frac{\alpha_s}{4\pi} (i\pi) \left( \frac{1}{\epsilon} + \ln \frac{z - 1}{z} + \ln \frac{\mu^2}{s_{12}} \right) [-4T_2 \cdot T_3] \mathbf{S}^0 + \cdots \quad (1 \text{ and } 3 \text{ incoming}) \quad (4.76)$$

where the $\cdots$ include terms that do not violate strict factorization, such as the $C_1$ and $C_2$ terms in Eq. (4.72).

The factorization-violating IR-divergent part of the 2-loop splitting function is also presented in [169]. The 2-loop splitting amplitude can be written as

$$\mathbf{S}^2 = \left[ D_C^2(\epsilon) + \frac{1}{2}(I_C^1)^2 \right] \cdot \mathbf{S}^0 + \frac{1}{\epsilon} \text{ terms} + \text{finite} \quad (4.77)$$

where

$$D_C^2(\epsilon) = \left( \frac{\alpha_s}{2\pi} \right)^2 \left( \frac{-s_{12} - i\epsilon}{\mu^2} \right)^{-2\epsilon} \pi f_{abc} \sum_{i=1,2} \sum_{j,k=3}^{n} T_i^a T_j^b T_k^c \Theta(-z_i) \text{sign}(s_{ij}) \Theta(-s_{jk})$$

$$\times \ln \left( \frac{s_{jk}s_{12}z_1z_2}{s_{jks_{12}}} - i\epsilon \right) \left[ -\frac{1}{2\epsilon^2} + \frac{1}{\epsilon} \ln \left( \frac{-s_{ij}}{z_i} \right) \right] \quad (4.78)$$

This contribution is non-vanishing only if the amplitude involves both incoming and outgoing colored partons. The most important part of this 2-loop splitting function is the real component, since it can contribute to the cross section. The real part is contained in the
anti-Hermitian combination of $D_C^2(\epsilon)$ is

$$
\frac{1}{2} \left( D_C^2(\epsilon) - \tilde{D}_C^{2,4}(\epsilon) \right) = -i \frac{\alpha^2_s}{4} \frac{(-s_{12} - i\epsilon)}{\mu^2} - 2\epsilon f_{abc} \sum_{i=1,2} \sum_{j,k=3}^{n} T_i^a T_j^b T_k^c \Theta(-z_i) \text{sign}(s_{ij}) \Theta(-s_{jk})
$$

\times \left[ -\frac{1}{2\epsilon^2} + \frac{1}{\epsilon} \ln \left( \frac{-z_i}{1-z_i} \right) \right] \quad (4.79)

In a representation where the $T_j^a$ are purely imaginary, this contribution is purely real.

### 4.6 Factorization violation from SCET

We have seen that splitting functions violate strict factorization starting at 1-loop. The condition for strict-factorization violation is that there be more than one colored particle in both the initial and final state. In such a situation the amplitude for producing a final-state particle collinear to one of of the initial-state particles (a spacelike splitting) depends on the colors and momenta of particles not collinear to it. The factorization-violating contribution was derived in [169, 170] by taking limits of the full $n + 1$-body matrix elements in QCD. We reviewed the procedure for the 1-loop IR divergent part and discussed the extension to also include the finite part and to 2-loops.

In SCET the IR divergences of both $|\mathcal{M}|$ and $|\overline{\mathcal{M}}|$ agree with full QCD, so the derivation of Eq. (4.71), which encodes 1-loop factorization violation, could be done in SCET by taking limits of $n + 1$ amplitudes, as in QCD. Note that all the IR divergences of QCD are reproduced in SCET without any special consideration of the Glauber contribution (i.e. because Glauber modes are contained in soft modes for hard scattering, as discussed in Section 4.3). However, SCET should also be able to produce the splitting functions using an effective field theory constructed using only the $n$ collinear directions of $|\mathcal{M}|$, i.e. not by taking the limit of an effective field theory with $n + 1$ collinear directions. In that case, we do need the Glauber operators. We we would like to know is whether the general-
ized splitting function can be derived without knowing features of the full \( n + 1 \) body amplitudes by the addition of Glauber operators to SCET.

### 4.6.1 Tree-level splitting amplitudes in SCET

To begin, let’s discuss how a splitting amplitude would be calculated in SCET with soft and collinear modes, but no Glauber operators. SCET without Glaubers can produce the strict-factorization-preserving splitting amplitude \( S^{\text{fact}} \), but not the factorization-violating parts in \( S^{\text{non-fact}} \). Recall our notation that \( P^\mu \cong p_1^\mu + p_2^\mu \) is mother parton momentum. We take \( P^\mu \) and the daughter momentum \( p_1^\mu \) to be incoming and \( p_2^\mu \) to be outgoing. The strictly-factorizing splitting amplitude is then as in Eq. (5.5):

\[
S^{\text{fact}} = \frac{\langle p_2 | W_1^\dagger \bar{\psi} | p_1 \rangle}{\langle 0 | W_1^\dagger \bar{\psi} | P \rangle} \tag{4.80}
\]

Here \( \bar{\psi} \) is an ordinary Dirac fermion and \( W_1 \) is a collinear Wilson line pointing in a direction \( t^\mu \) not collinear to \( P^\mu \). In the traditional formulation of SCET \( \bar{\psi} \) carries a label specifying the large and perpendicular components of its momentum, and the interactions of \( \psi \) with collinear gluons are power expanded. However it is simpler to use the full QCD Feynman rules as we do here.

The tree-level splitting amplitudes are easily computed from Eq. (4.80) (see [163]):

\[
S^{0} = \frac{\langle p_2 | W_1^\dagger \psi | p_1 \rangle^{\text{tree}}}{\langle 0 | W_1^\dagger \psi | P \rangle^{\text{tree}}} = p_1 \quad + \quad p_2 \quad - \quad \tag{4.81}
\]

The first diagram has the gluon coming off the fermion from a Lagrangian interaction, the second diagram has the gluon coming out of the Wilson line. These graphs evaluate
to
\[
\langle p_2 | W_1^+ \psi | p_1 \rangle_{\text{tree}} = g_s T_1 \left[ \frac{t \cdot \epsilon}{t \cdot p_2} - \frac{g (\psi_1 + \psi_2)}{(p_1 + p_2)^2} \right] v(p_1) \tag{4.82}
\]

The dependence on the Wilson line direction \( t^\mu \) drops out for physical polarizations. The result is [163]

\[
\begin{align*}
\mathbf{S}^0_{R-} &= g_s \frac{\sqrt{2}}{|p_2 p_1|} \frac{z}{\sqrt{1 - z}} T_1, \\
\mathbf{S}^0_{R+} &= g_s \frac{\sqrt{2}}{|p_1 p_2|} \frac{1}{\sqrt{1 - z}} T_1, \tag{4.83}
\end{align*}
\]

\( \mathbf{S}^0_{L\pm} \) are related to \( \mathbf{S}^0_{R\mp} \) by parity conjugation. Here \( R \) and \( L \) refer to the spin of the fermion (right or left) and \( \pm \) refer to the helicities of the emitted gluon. These tree-level splitting functions hold for any kinematical configuration, as strict-factorization holds at tree level.

In Eq. (4.82), we used QCD Feynman rules, as is appropriate for evaluating the splitting function defined in Eq. (4.80). This is based on the reformulation of SCET in [163] and [164]. In traditional SCET, the splitting function is computed from \( \langle p_2 | W_1^+ \bar{\zeta}_n | p_1 \rangle \) with \( \bar{\zeta}_n \) a collinear quark, i.e. one whose interactions are truncated to leading power. The denominator in Eq. (4.80) is replaced by the diagram-level zero-bin subtraction procedure outlined in [21]. The SCET Feynman rules [29] then give

\[
\begin{align*}
\langle p_2 | W_1^+ \bar{\zeta}_n | p_1 \rangle_{\text{tree}} &= -g_s T_1 \left[ \frac{n \cdot (p_1 + p_2)}{2 (p_1 + p_2)^2} \left( n \cdot \epsilon + \frac{\psi_2, \perp}{n \cdot (p_1 + p_2)} \right) \frac{\epsilon}{2} + \frac{n \cdot \epsilon}{\bar{n} \cdot p_2} \right] v_n(p_1) \tag{4.84}
\end{align*}
\]

where \( p_1^\mu \propto n^\mu \) has been used and the Wilson line direction \( t^\mu \) is set to \( \bar{n}^\mu \). Simplifying this expression for the various helicity/spin combinations gives the same splitting functions as in Eq. (4.83).

To compute the 1-loop corrections to \( \mathbf{S}^0 \text{fact} \) we need to evaluate Eq. (4.80) to next order. non-collinear sectors encapsulated by the Wilson lines. Evaluating the relevant graphs should produce a result equivalent to known results about 1-loop splitting amplitudes
from full QCD. This calculation has not been done in SCET, to our knowledge, and would certainly be an interesting check on the formalism that we leave to future work.

### 4.6.2 Factorization violating contributions

To compute the contributions to the generalized splitting function that violate strict factorization, we obviously cannot start from the factorized expression Eq. (5.2) which leads to Eq. (4.80). The advantages of writing a factorized expression as in Eq. (5.2) include first, that it involves only QCD fields and the familiar QCD Feynman rules, and second, that the soft-collinear overlap is removed through an operator matrix element. The inclusion of Glauber effects has so far only been formulated in the traditional presentation of SCET [152], with collinear and soft fields and their associated SCET Feynman rules and with the overlap removed by a diagram-by-diagram zero-bin subtraction procedure. In this approach, one writes matrix elements in the effective theory as one big operator product

\[ |\mathcal{M}\rangle \cong C(S_{ij}) \langle p_2; \cdots X_N; X_S | \phi^* W_1 Y_1^\dagger \cdots W_N^\dagger Y_N \phi | p_1; X_3 \rangle \]  

(4.85)

Similarly,

\[ |\overline{\mathcal{M}}\rangle \cong C(S_{ij}) \langle \cdots X_N; X_S | \phi^* W_1 Y_1^\dagger \cdots W_N^\dagger Y_N \phi | P; X_3 \rangle \]  

(4.86)

The generalized splitting function can be computed as the ratio of these matrix elements using SCET Feynman rules and appropriate zero-bin subtractions.

As indicated in Section 4.4.4, Glauber effects are included in SCET through the addition of potential operators that couple pairwise all possible fields in all possible collinear sectors. These operators are schematically of the form \( \mathcal{O}_G \sim \bar{\psi}_i \psi_j \frac{1}{p_2^+} \bar{\psi}_j \psi_i \), as in Eq. (4.46), with a plethora of terms in QCD, as in Eq. (4.47). There are different operators coupling quarks to quarks, gluons to gluons and quarks to gluons. See [152] for all the details.

Most of the time, the effects of these Glauber operators are identical to the effects of
the Glauber limit of the soft diagrams (connecting the soft Wilson lines $Y_j$ in Eq. (4.85)).

Thus one must either subtract the overlap, as is done in [152], or more simply compute
the full soft graphs without the soft-Glauber subtraction, and not bother including the
Glauber operator contribution when it is not needed. Thus for example, the following
graphs contribute to $|\mathcal{M}|$:

$$I_{Sa} = \quad , \quad I_{Ga} =$$

Here the horizontal gluon with a line through it is a collinear emission, the vertical gluon
is soft, and the dots are Glauber. Since the Glauber limit of $I_{Sa}$ gives exactly $I_{Ga}$, we can
simply compute $I_{Sa}$ without zero-bin subtracting and not include $I_{Ga}$. Moreover, $I_{Sa}$ gives
the same result as the analogous contribution to $|\mathcal{M}|$ (the graph is the same without
the emitted gluon), thus we can ignore both of these graphs when computing the 1-loop
splitting amplitude.

The Glauber graphs that cannot be ignored are those for which there is not a corre-
sponding soft graph. In SCET, the interactions of soft gluons with collinear fields are com-
pletely removed from the Lagrangian; they only come from the Wilson lines in Eqs. (4.85)
and (4.86). These can be drawn coming out of the blob, since that represents the operator
containing the Wilson lines, or slightly shifted away from the blob, as in the diagrams in
Eq. (4.87). Thus graphs like

\[ I_{Sb} = \text{[diagram]} , \quad I_{Gb} = \text{[diagram]} \] (4.88)

give identical Glauber contributions to \(|\mathcal{M}\rangle\). We can therefore ignore both due to the soft-Glauber zero-bin subtractions. Note that while the soft graph factorizes into the product of a soft matrix element and a collinear emission, the Glauber graph does not factorize. Thus we cannot claim that \(I_{Gb}\) is identical to the contribution from the analogous graph contributing to \(|\overline{\mathcal{M}}\rangle\). Instead, we need to look at the graphs contributing to \(|\overline{\mathcal{M}}\rangle\):

\[ \bar{I}_{Sb} = \text{[diagram]} , \quad \bar{I}_{Gb} = \text{[diagram]} \] (4.89)

These have identical Glauber contributions and can be dropped by the zero-bin subtraction.

The Glauber graphs that have no corresponding soft graph are:

\[ \bar{I}_{Gc} = \text{[diagram]} , \quad \bar{I}_{Gd} = \text{[diagram]} , \quad \bar{I}_{Ge} = \text{[diagram]} \] (4.90)
The upper Glauber vertex in $I_{Gc}^j$ comes from the expansion of the collinear Wilson line in the quark-quark Glauber operator connecting the $1$ and $j$ directions. One must consider these three graphs for any direction $j = 3 \cdots N$.

To evaluate $I_{Gc}^j$, we first note that the collinear Wilson line direction $t_1^\mu$ can be anything not collinear to the $p_1^\mu$ direction. We can therefore choose a basis of polarization vectors $\epsilon_{\pm}(p_2)$ for the outgoing collinear gluon with momentum $p_2^\mu$ so that $t_1 \cdot \epsilon_{\pm} = 0$. Doing so makes graph $I_{Gc}^j = 0$ for any $j$.

Next, we turn to $I_{Gd}^j$. In position space, this graph describes a Glauber exchange that takes place earlier in time than the collinear emission. That is, the emission interrupts the Glauber loop. In such a situation, a general argument as given in [152] that the graph must vanish. We can also see it directly from the integral itself. Working in lightcone coordinates, $p_1^\mu = -\frac{1}{2}Qn^\mu$ and $p_2^\mu = \frac{1}{2}p_2^+n^\mu + \frac{1}{2}p_2^-\bar{n}^\mu + p_{2,\perp}^\mu$. For $j \geq 3$, we have $p_j^\mu = \frac{1}{2}Q_j n_j^\mu$ (for outgoing $p_j^\mu$) or $p_j^\mu = -\frac{1}{2}Q_j n_j^\mu$ (for incoming $p_j^\mu$). And we decompose the Glauber momentum $k^\mu$ with respect to $n^\mu$ and $n_j^\mu$ so that for each diagram $k^- = n \cdot k$, $k^+ = n_j \cdot k$. This gives

$$I_{Gd}^j = \frac{S_s^2}{(2\pi)^4} \frac{d^4 k}{2k_\perp^2} \frac{1}{2} \frac{n \cdot n_j}{\mp Q_j k^+ - k_\perp^2 + i\epsilon} \times 1 \frac{Qk^- - k_\perp^2 + i\epsilon}{1} \frac{Qk^- - (k_\perp + p_{2,\perp})^2 - Qp_2^+ + i\epsilon}{Qk^- - (k_\perp + p_{2,\perp})^2 - Qp_2^+ + i\epsilon} \times \cdots$$

(4.91)

with the $\cdots$ at the end representing the numerator and spin structure that are irrelevant here. Here the $\mp$ sign denotes either outgoing ($-$) or incoming ($+$) $p_j$. We note that only two propagators depend on $k^-$, and both of the corresponding poles in the complex $k^-$ plane are below the real $k^-$ axis. Thus we can close the $k^-$ contour downward and the
Note that if there were no emission, the graph would reduce to the Glauber vertex correction in Eq. (4.30) which does not vanish. The extra emission adds a propagator that causes the integral to be convergent at $|k^-| = \infty$ allowing us to evaluate it using Cauchy’s theorem. Adding more propagators interrupting the Glauber loop will only add more poles on the same side of the real $k^+$ axis. This is the momentum-space version of the argument in [152] that Glauber exchanges cannot be interrupted.

Finally, we turn to graph $I_{Ge}^j$. This one will not vanish, so we have to work out the full numerator structure. It is

$$I_{Ge}^j = \frac{1}{2} \delta_s^3(T_2 \cdot T_j)T_1 \mathcal{M}^0 \int \frac{d^d k}{(2\pi)^d} \frac{1}{2k^2 |\eta|} \frac{n \cdot n_j}{2} \frac{1}{k^+ - \delta_j + i\epsilon} \frac{1}{k^- - \delta_2 + i\epsilon} \frac{1}{-k^- + \delta_1 + i\epsilon} \epsilon_{\mu}(p_2)$$

(4.92)

with the upper (lower) signs on the second line corresponding to $p_j^\mu$ outgoing (incoming). Here, the $|2k^2|^{-\eta}$ factor comes from the rapidity regulator. The denominator factors are

$$\delta_j = \frac{\vec{k}^2}{Q_j}, \quad \delta_2 = \frac{\left(\vec{p}_2^\perp + \vec{k}^\perp\right)^2}{p_2^+} - p_2^-, \quad \delta_1 = -\frac{\left(\vec{p}_2^\perp + \vec{k}^\perp\right)^2}{Q - p_2^+} - p_2^-.$$  

(4.93)

Recall that $p_1 = -\frac{1}{2} Q n^\mu$, which explains how $\delta_2$ becomes $\delta_1$ under $p_2^\mu \rightarrow p_2^\mu + p_1^\mu$. The numerator factor is

$$N_{\mu}(p_1, p_2, k) = \frac{\not{q}}{2} \left[ -\frac{2(p_2^\perp + k^\perp)^\mu}{p_2^+} + \frac{(p_2^\perp + k^\perp)\gamma_\perp^\mu}{-Q + p_2^+} \right] \not{q} = \mathcal{O}(p_1)$$

(4.94)

This argument only works if the integral is convergent, which requires the rapidity regulator. See the longer discussion in Section 4.8.
where \( \gamma_{\perp}^\mu \) are the perpendicular components of \( \gamma^\mu \), projected out as with a 4-vector:
\[
\gamma_{\perp}^\mu = \gamma^\mu - \frac{1}{2} \not{\mathbf{u}} n^\mu - \frac{1}{2} \not{\mathbf{v}} \bar{n}^\mu.
\]
In the numerator expression, the \( \not{\mathbf{u}} \) and \( \not{\mathbf{v}} \) factors at the beginning and the end project onto the collinear spinors. The part in bracket comes from expanding the QCD vertex at leading power, according to the SCET Feynman rules, using the vertex coming from the quark-gluon Glauber operator, and simplifying. That this numerator factor depends only on \( k_{\perp}^\mu \), not on \( k^+ \) or \( k^- \) greatly simplifies the calculation. This simplification comes from keeping only the leading-power interactions, as in SCET.

To evaluate this graph we first write \( k^\pm = k^0 \pm k^z \) and perform the \( k^0 \) integration by contours. The poles at \( k^0 = k^z + \delta_2 - i\epsilon \) and \( k^0 = k^z + \delta_1 + i\epsilon \) pinch the contour in the Glauber region [184]. If we take \( p_j^\mu \) to be outgoing (upper signs in Eq. (4.92)), then the pole from the first propagator is at \( k^0 = -k^z - \delta_j + i\epsilon \). We then close the contour downwards setting \( k^0 = k^z + \delta_2 \) so that \( k^- = \delta_2 \) and \( k^+ = 2k^z + \delta_2 \). This gives

\[
I_{Ge}^j = -2g_s^2 (T_2 \cdot T_j) T_1 \mathcal{M}^0 \int \frac{d^{d-2} k_{\perp}}{(2\pi)^{d-1}} \frac{N^\mu}{k_{\perp}^2} \frac{1}{\delta_1 - \delta_2} \epsilon^\mu(p_2) \int dk^z \frac{1}{|2k^z|^\eta} \frac{1}{-2k^z - \delta_2 - \delta_j + i\epsilon} \tag{4.95}
\]

The \( k^z \) integral is straightforward to evaluate (see Eq. (B.4) of [152]):

\[
\int_{-\infty}^{\infty} \frac{dk^z}{2\pi} \frac{1}{|2k^z|^\eta} \frac{1}{2k^z + 2\Delta + i\epsilon} = \frac{1}{4\pi} \left[ (-2\pi i) \csc(2\pi \eta) \sin(\pi \eta)(-i\Delta)^{-2\eta} \right] \tag{4.96}
\]
\[
= \frac{1}{4\pi} (-i\pi) + \mathcal{O}(\eta) \tag{4.97}
\]

Note that the \( k^z \) integral cares only about the discontinuity of \( [-2k^z - \delta_2 - \delta_j + i\epsilon]^{-1} \), which is independent of the value of \( \delta_2 \) and \( \delta_j \). Even though the result is independent of \( \eta \) as \( \eta \to 0 \), one still needs the rapidity regulator to make the integral well-defined. Indeed, the rapidity regulator imparts critical non-analyticity allowing the graph to vanish for \( p_2^\mu \) incoming but not for \( p_2^\mu \) outgoing.
If the non-collinear leg $p^\mu_j$ that the Glauber gluon connects to is incoming (+ sign in Eq. (4.92)), then the pole for from the first propagator is at $k^0 = -k^z + \delta_j - i\epsilon$. We then close the contour upwards setting $k^0 = k^z + \delta_j$ so that $k^- = \delta_1$ and $k^+ = 2k^z + \delta_1$. This gives

$$I^3_{Ge} = 2g_s^3(T_2 \cdot T_j) T_1 \cdot \overline{M}^0 \frac{d^{d-2}k_\perp}{(2\pi)^{d-2} k_\perp^2} \frac{1}{\delta_1 - \delta_2} \epsilon_\mu(p_2) \int \frac{dk^z}{2\pi} \frac{1}{2k^z|\epsilon|} \frac{1}{2k^z + \delta_1 - \delta_j + i\epsilon}$$

(4.98)

Compared to Eq. (4.95), we have a relative minus sign from the integrand. The result is the same as Eq. (4.95) with a $-\epsilon$ out front. That is,

$$I^j_{Ge} = \pm 2g_s^3(T_2 \cdot T_j) T_1 \cdot \overline{M}^0 \frac{d^{d-2}k_\perp}{(2\pi)^{d-2} k_\perp^2} \frac{1}{p_2^+} \frac{\gamma_{\perp \mu} \gamma_{\perp} \epsilon_\perp}{Q - p_2^+} \frac{i\epsilon}{2} u(p_1)$$

$$\times \frac{1}{4\pi i\epsilon} \frac{1}{p_2^+(Q - p_2^+)} \int \frac{d^{d-2}k_\perp}{(2\pi)^{d-2} (p_{2,\perp} + k_\perp)^2 k_\perp^2}$$

(4.99)

Here the $\mp$ sign denotes either outgoing ($-$) or incoming ($+$) $p_j$. The $k_\perp$ integral is regulated in $d - 2 = 2 - 2\epsilon$ dimensions:

$$\mu^{d-4} \int \frac{d^{d-2}k_\perp}{(2\pi)^{d-2} k_\perp^2} \frac{p_2^\mu + k_\perp^\mu}{(p_{2,\perp} + k_\perp)^2} = \frac{1}{4\pi} \frac{p_2^\mu}{p_{2,\perp}^2} \left( \frac{4\pi \mu^2}{p_{2,\perp}^2} \right) \frac{\Gamma(-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)}$$

$$= \frac{1}{4\pi} \frac{p_2^\mu}{p_{2,\perp}^2} \left( \frac{4\pi e^{-\gamma_E} \mu^2}{p_{2,\perp}^2} \right) \left( -\frac{1}{\epsilon} + O(\epsilon) \right)$$

(4.100)

Putting things together, this diagram is

$$I^j_{Ge} = \pm \frac{\alpha_s}{2\pi} \left( \frac{4\pi e^{-\gamma_E} \mu^2}{p_{2,\perp}^2} \right) \left( \frac{i\pi}{\epsilon} + O(\epsilon) \right) (T_2 \cdot T_j) T_1 \cdot \overline{M}^0$$

$$\times g_s \frac{p_2^+(Q - p_2^+)}{Q p_{2,\perp}^2} \frac{i\epsilon}{2} \left[ -2 \frac{p_{2,\perp} \cdot \epsilon_\perp}{p_2^+} + \frac{p_{2,\perp} \cdot \gamma_{\perp}}{Q + p_2^+} \right] \frac{i\epsilon}{2} u(p_1)$$

(4.101)
where the \(+\)(\(-\)) sign corresponds to \(p_j\) outgoing (incoming). Using the on-shell condition: 
\[ p_{2,\perp} \cdot \epsilon_\perp = -\frac{1}{2} p_2^+ n \cdot \epsilon - \frac{1}{2} p_2^- \bar{n} \cdot \epsilon = -\frac{1}{2} p_2^+ n \cdot \epsilon, \]
we recognize the spin structure in Eq. (4.101) to be the same as the tree-level splitting amplitude, as in Eq. (4.84). Thus,

\[
I_j^{1_{\text{Ge}}} = \pm \frac{\alpha_s}{2\pi} \left( \frac{4\pi e^{-\gamma_E} \mu^2}{p_{2,\perp}^2} \right)^\epsilon \left( \frac{i\pi}{\epsilon} + \mathcal{O}(\epsilon) \right) (T_2 \cdot T_j) \mathbf{S}^0 \cdot \mathcal{M}^0
\]

(4.102)

\[
= \pm \frac{\alpha_s}{2\pi} (4\pi e^{-\gamma_E})^\epsilon (i\pi) \left( \frac{1}{\epsilon} + \ln \frac{\mu^2}{p_{2,\perp}^2} + \mathcal{O}(\epsilon) \right) (T_2 \cdot T_j) \mathbf{S}^0 \cdot \mathcal{M}^0
\]

(4.103)

Recall that
\[
\frac{z}{1-z} = \frac{p_2^+}{p_1^+}
\]

(4.104)

we have
\[
p_{2,\perp}^2 = (-2 p_1 \cdot p_2) \frac{z}{(z-1)},
\]
and

\[
I_j^{1_{\text{Ge}}} = \pm \frac{\alpha_s}{2\pi} (4\pi e^{-\gamma_E})^\epsilon (i\pi) \left( \frac{1}{\epsilon} + \ln \frac{\mu^2}{-2 p_1 \cdot p_2} + \ln \frac{z-1}{z} + \mathcal{O}(\epsilon) \right) (T_2 \cdot T_j) \mathbf{S}^0 \cdot \mathcal{M}^0
\]

(4.105)

Summing over all \(j\), and let \(p_3\) be an incoming parton, we then get

\[
\mathbf{S}^1_{\text{non-fact}} = \frac{\alpha_s}{2\pi} (4\pi e^{-\gamma_E})^\epsilon (i\pi) \left( \frac{1}{\epsilon} + \ln \frac{\mu^2}{-2 p_1 \cdot p_2} + \ln \frac{z-1}{z} \right) \left( -T_2 \cdot T_3 + \sum_{j>3} T_2 \cdot T_j \right) \mathbf{S}^0
\]

(4.106)

\[
= i\alpha_s (4\pi e^{-\gamma_E})^\epsilon \left( \frac{1}{\epsilon} + \ln \frac{\mu^2}{-2 p_1 \cdot p_2} + \ln \frac{z-1}{z} \right) (-T_2 \cdot T_3) \mathbf{S}^0 + \cdots
\]

(4.107)

where the \(\cdots\) respect strict factorization. This reproduces both the singular and the finite parts of the 1-loop factorization-violating splitting function, as in Eq. (4.76).
4.7 Two-loop factorization-violation from SCET

In SCET, the 2-loop splitting function comes from expanding the ratio of $|\mathcal{M}|$ to $|\mathcal{M}|$ as in Eqs. (4.85) and (4.86) to 2-loop order. Physical effects of factorization violation must occur at the cross-section level, thus the most important effect we are looking for is a real contribution to $S_{P}^{2,\text{non-fact}}$. We will therefore focus on isolating this real part, which should match Eq. (4.79). Factorization violating effects at 2-loops will necessarily involve insertions of the Glauber operator, and at 2-loops there can be 1 or 2 exchanged Glauber gluons.

Not all 2-loop diagrams involving Glauber gluons can contribute to factorization-violation. For example, none of the diagrams in Fig. 4.2 are relevant. Fig. 4.2(a) and Fig. 4.2(b) describe Glauber exchange right next to the hard interaction. For these graphs, as at 1-loop, the Glauber is contained in the soft contribution and does not generate factorization violation. Diagrams like Fig. 4.2(c) with a disconnected soft loop cancel with the product of a 1-loop Glauber exchange and a 1-loop soft contribution. Fig. 4.2(d) is an example of diagram with collinear loops in the $n_{j}$ sector, with $j \neq 1$. Since the flow of Glauber momentum follows the direction of energy flow in the $n_{j}$—collinear propagators, the $n_{j}$—component of Glauber momentum is not pinched. Thus the Glauber contribution is contained in the soft contribution, or, more physically, the Glauber contribution acts coherently on the on the $n_{j}$—collinear fields. The sum of such diagrams will cancel with the product of the 1-loop Glauber contribution and the 1-loop collinear contribution. It is not hard to see that that loop corrections due to interactions between fields in the non-collinear sectors do not contribute to the factorization-violating part of the splitting amplitude.

Two-loop diagrams that can contribute to factorization violation must involve color exchange between the daughter gluon and hard non-collinear partons. They can be cate-
Figure 4.2: Example Glauber diagrams that can be ignored for the 2-loop splitting amplitude.

Figure 4.3: Example Glauber-soft and Glauber-collinear mixing diagrams that may break strict collinear factorization.

Corporized as double Glauber exchange, Glauber-soft mixing and Glauber-collinear mixing diagrams. The splitting amplitude is thus given by

$$\mathbf{S} p_{2,\text{non-fac.}} \cdot |\mathcal{M}^0_i| = |\mathcal{M}^2_{\text{double Glauber}}\rangle - \mathbf{S} p_{1,\text{non-fac.}} \cdot |\mathcal{M}^1_{\text{Glauber loops}}\rangle + |\mathcal{M}^2_{\text{non-fac. Glauber-soft}}\rangle + |\mathcal{M}^2_{\text{non-fac. Glauber-coll.}}\rangle - \mathbf{S} p_{1,\text{non-fac.}} \cdot |\mathcal{M}^1_{n_1-\text{coll. loops}}\rangle$$  (4.108)

The first term on the right-hand-side of Eq. (4.108) corresponds to double Glauber diagrams, with examples given in Fig. 4.4 and Fig. 4.5. Since each Glauber sub-loop produces a factor of $(i\pi)$, these terms are purely real. Similar to the 1-loop Glauber diagrams, we expect that double Glauber diagrams to be rapidity-finite and have no logarithmic dependence on the momenta of non-collinear partons. The second term in Eq. (4.108) takes away the 1-loop factorization-breaking effect. The second line of Eq. (4.108) comes from
Glauber-soft and Glauber-collinear diagrams that violates factorization. Representative diagrams for each set are shown in Fig. 4.3. These diagrams have highly non-trivial kinematic dependence on all external partons and involve two-loop multi-leg loop integrals with rapidity divergences. A full explicit calculation of these diagram is beyond the scope of our paper. In the following section we will carry out the calculation of double Glauber diagrams and show that the first line of Eq. (4.108) reproduces the leading pole of the real part of $Sp^{2,\text{non-fac.}}$.

### 4.7.1 Double-Glauber diagrams

In this section, we give explicit results for double-Glauber exchange diagrams that can violate strict factorization. To evaluate the diagrams, we use the rapidity-regularization scheme given in [152], which adds a convergence factor of $\frac{1}{|k_{\perp}|}$ to the integrand for both Glauber momenta. We refer to $p_1$ and $p_2$ as collinear partons and all other partons as non-collinear. We discuss and summarize the calculation here, leaving the details of some representative calculations to Appendix 4.10.1.

All of the relevant double Glauber diagrams have at least one Glauber attached to the $p_2$ gluon. Diagrams where neither Glauber attaches to $p_2$ either vanish or are contained in $|\mathcal{M}^1_{\text{Glauber loops}}$. None of the relevant diagrams have Glauber gluons attached to $p_1$; when the Glauber gluon attaches to $p_1$, the Glauber loop is interrupted by the real emission and the diagram will vanish (just as $J_{\text{Gd}}^j = 0$ in Section 4.6.2). The relevant diagrams can be divided into two categories, those involving two hard-collinear directions (Fig. 4.4), and those involving three hard-collinear directions (Fig. 4.5).

We will start with diagrams in Fig. 4.4 involving two hard-collinear directions. These can either have two Glauber vertices on $p_2$ and two on a non-collinear parton $p_j$ or they can have one Glauber vertex on $p_2$, one on the internal collinear line labeled as $p_{(12)}$, and the other two on $p_j$. 197
Figure 4.4: Double Glauber exchange diagrams that involve two collinear sectors.

Fig. 4.4(a) and Fig. 4.4(b) describe Glauber exchange with outgoing non-collinear partons. Fig. 4.4(a) has two parallel Glauber rungs which can be ordered in time. We find

\[
\frac{1}{2!} = \left( T^b_2 T^c_2 \right) \left( T^b_j T^c_j \right) S p^0 \mathcal{M}^0
\]

\[
\times \left( \frac{\alpha_s}{2\pi} \right)^2 (i\pi)^2 \left( \frac{4\pi^2 \mu^2}{p_{2,\perp}^2} \right)^{2\epsilon} \left[ \Gamma(-\epsilon) \right]^2 \frac{\Gamma(1-\epsilon) \Gamma(1+2\epsilon)}{\Gamma(1-3\epsilon)}
\]

\[ 4.109 \]

The $1/2!$ is a symmetry factor coming from time-ordering of the two Glaubers. Fig. 4.4(b) has two crossed Glauber rungs such that the ordering of the Glauber vertices in light-cone time are the opposite on each line. The integral vanishes since all poles lie on the same side of $k^0$—complex contour:

\[ 4.110 \]
The sum of these two diagrams gives

\[ \begin{array}{c}
\text{double pole} \\
= \frac{1}{2!} (T_2 \cdot T_j)^2 \mathcal{S} p^0 \mathcal{M}^0 \left( \frac{\alpha_s}{2\pi} \right)^2 (i\pi)^2 \frac{1}{e^2} \end{array} \tag{4.111} \]

Note that this is exactly half of the 1-loop Glauber exchange diagram squared (see Eq. (4.103)). This contribution must be summed over all outgoing legs \( j \).

Fig. 4.4(c) and Fig. 4.4(d) describe Glauber exchanges between \( p_2 \) and the other incoming parton \( p_3 \):

\[
\begin{array}{c}
\text{double pole} \\
= \frac{1}{2!} (T_2 \cdot T_3)^2 \mathcal{S} p^0 \mathcal{M}^0 \\
\times \frac{\alpha_s}{2\pi} (i\pi)^2 \left( \frac{4\pi \mu^2}{p_{2\perp}^2} \right)^2 \left[ \Gamma(\epsilon) \right]^2 \frac{\Gamma(1-\epsilon) \Gamma(1+2\epsilon)}{\Gamma(1-3\epsilon)} \end{array} \tag{4.112} \]

Compared with Fig. 4.4(a), the order of color generators switches on the non-collinear leg. Similar to Fig. 4.4(b), the cross Glauber graph vanishes:

\[ = 0 \tag{4.113} \]

Now we write \( (T_2^b T_2^b) (T_3^b T_3^b) \) as \( (T_2 \cdot T_3)^2 + C_A T_2 \cdot T_3 \), and focus on the leading pole. The sum of these two diagrams is

\[ \begin{array}{c}
\text{double pole} \\
= \frac{1}{2!} \left[ (T_2 \cdot T_3)^2 + C_A T_2 \cdot T_3 \right] \mathcal{S} p^0 \mathcal{M}^0 \left( \frac{\alpha_s}{2\pi} \right)^2 (i\pi)^2 \frac{1}{e^2} \end{array} \tag{4.114} \]
Note that this result is similar to half the square of the leading pole from the corresponding 1-loop diagram; it contains, however, an additional $C_A$ term that comes from the switching the order of color generators as compared to Fig. 4.4(a).

So far we considered diagrams with two Glauber vertices on $p_2$. Now we move on to diagrams with one Glauber vertex on $p_2$ and the other on the internal collinear line, namely $p_{(12)}$. The diagram where a Glauber completes a loop connecting $p_{(12)}$ to a non-collinear parton $p_j$ looks like the vertex diagram $I_{Glauber}$ discussed in Section 4.4. It vanishes if $p_j$ is outgoing such that $p_{(12)} \cdot p_j < 0$. When $p_j$ is incoming, we can write down two such diagrams Fig. 4.4(e) and Fig. 4.4(f); both are non-vanishing and they only differ in color structure:

$$ \left( T_2^b \right) \left( T_3^b T_3^c \right) S^0 \left( - T_{(12)}^c \right) \overline{M}^0 $$

$$ \times \frac{1}{2!} \left( \frac{\alpha_s}{2\pi} \right)^2 \left( i \pi \right)^2 \left( \frac{4\pi \mu^2}{p_{2,\perp}^2} \right)^{2\epsilon} ( -\epsilon ) \Gamma(1 - \epsilon) \Gamma(1 + \epsilon) \left( \frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon} \right) \frac{\Gamma(1 - 2\epsilon)}{\Gamma(1 - 2\epsilon)} $$

$$ \left( T_2^b \right) \left( T_3^b T_3^c \right) S^0 \left( - T_{(12)}^c \right) \overline{M}^0 $$

$$ \times \frac{1}{2!} \left( \frac{\alpha_s}{2\pi} \right)^2 \left( i \pi \right)^2 \left( \frac{4\pi \mu^2}{p_{2,\perp}^2} \right)^{2\epsilon} ( -\epsilon ) \Gamma(1 - \epsilon) \Gamma(1 + \epsilon) \left( \frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon} \right) \frac{\Gamma(1 - 2\epsilon)}{\Gamma(1 - 2\epsilon)} $$

where $1/\epsilon$ is associated with the IR divergence. Using color conservation relation: $T_{(12)} = -T_3 - \cdots - T_m$, the sum of these two diagrams can be written as

$$ \left( - T_3 \cdot T_2 \right) S^0 \left( T_{(12)} \cdot T_3 \right) \overline{M}^0 $$

200
The first line in Eq. (4.117) will be removed by the corresponding term in $\text{Sp}^{(1)\text{non-fac.}}\mathcal{M}_\text{Glauber loops}$ in Eq. (4.108).

Now we turn to the diagrams with three different collinear directions, as shown in Fig. 4.5. Here we omitted diagrams where four Glauber vertices are on different legs, since those diagrams trivially factorize into the product of two one-loop results, which are contained in $\text{Sp}^{1,\text{non-fac.}}[\mathcal{M}^1]_{\text{Glauber loop}}$. Therefore we focus on diagrams where two Glauber vertices are on the same leg. The first line of Fig. 4.5 are diagrams with two Glauber vertices on $p_2$. Diagrams in the second line have only one Glauber vertex on $p_2$.

First consider diagrams in the first line with two Glaubers exchanged between $p_2$ and two non-collinear partons $p_i$ and $p_j$ with $i \neq j$ and $(i, j = 3, \cdots, m)$. One immediate concern is that with 3 directions one cannot choose lightcone coordinates aligned with all of them. Fortunately, this is not a problem for Glauber graphs – at leading power in the Glauber expansion the $k_\perp$ components dominate over the projection of $k^\mu$ on any
lightcone direction. Thus we can choose one lightcone direction $n_1^\mu$ aligned with the collinear $p_1^\mu$ direction and the other $n_2^\mu$ aligned with any other direction and the calculation is basically the same as if there are only two directions involved. More details of the decomposition are given in Appendix 4.10.1. The result is:

\[ -(T_2 \cdot T_j)(T_2 \cdot T_3) \, Sp^0 \, \mathcal{M}^0 \]

\[ \times \left( \frac{\alpha_s}{2\pi} \right)^2 \, (i\pi)^2 \, \left( \frac{4\pi\mu^2}{\bar{p}_{2,\perp}^2} \right)^{2\epsilon} \, [\Gamma(-\epsilon)]^2 \frac{\Gamma(1-\epsilon)\Gamma(1+2\epsilon)}{\Gamma(1-3\epsilon)} \]

In Fig. 4.5(b) the Glauber vertices on the outgoing gluon line are switched. The $k^-$ integral then vanishes since all three poles are on the same side of the contour:

\[ = 0 \] (4.118)

The sum of the two diagram is

\[ \left( -\frac{1}{2}(T_2 \cdot T_j)(T_2 \cdot T_3) - \frac{i}{2} f_{abc} T_2^a T_3^b T_j^c \right) \, Sp^0 \, \mathcal{M}^0 \left( \frac{\alpha_s}{2\pi} \right)^2 \, (i\pi)^2 \, \frac{1}{\epsilon^2} \] (4.119)

We have broken up the color factor into terms that are symmetric and antisymmetric under $3 \leftrightarrow j$. The first term can be identified with the cross terms coming from the exponentiation of the sum of $I^3_{Ge}$ and $I^j_{Ge}$ computed in Section 4.6.2.

Graphs with two outgoing legs as in Fig. 4.5(c) are similar with an extra factor of $1/2$
from the time-ordering:

\[ \frac{1}{2!} \left( (T_2 \cdot T_j) (T_2 \cdot T_k) \right) \mathcal{S} \mathcal{P}^0 \mathcal{M}^0 \]

\[ \times \left( \frac{\alpha_s}{2\pi} \right)^2 (i\pi)^2 \left( \frac{4\pi\mu^2}{p_{2,\perp}^2} \right)^{2\varepsilon} \left[ \Gamma(-\varepsilon) \right]^2 \frac{\Gamma(1-\varepsilon) \Gamma(1+2\varepsilon)}{\Gamma(1-3\varepsilon)} \]

(4.120)

Fig. 4.5(d) can be obtained from Fig. 4.5(c) by switching \( j \) and \( k \):

\[ \frac{1}{2!} \left( (T_2 \cdot T_j) + (T_2 \cdot T_k) \right) \mathcal{S} \mathcal{P}^0 \mathcal{M}^0 \times \left( \frac{\alpha_s}{2\pi} \right)^2 (i\pi)^2 \frac{1}{\varepsilon^2} \]

(4.121)

These diagrams produce the cross term from the exponentiation of \( I_{Ge}^j \) and \( I_{Ge}^k \).

The remaining diagrams on the second line of Fig. 4.5 can be computed in the same way. We find for the sum:

\[ \frac{1}{2!} \left( (T_2 \cdot T_j) + (T_2 \cdot T_k) + (j \leftrightarrow k) \right) \mathcal{S} \mathcal{P}^0 \mathcal{M}^0 \]

\[ \times \left( \frac{\alpha_s}{2\pi} \right)^2 (i\pi)^2 \left( \frac{4\pi\mu^2}{p_{2,\perp}^2} \right)^{2\varepsilon} \left[ \Gamma(-\varepsilon) \right]^2 \frac{\Gamma(1-\varepsilon) \Gamma(1+2\varepsilon)}{\Gamma(1-3\varepsilon)} \left( \frac{1}{\varepsilon_{UV}} - \frac{1}{\varepsilon} \right) \]

\[ \text{double pole} \]

\[ \left( (T_2 \cdot T_j) + (T_2 \cdot T_k) \right) \mathcal{S} \mathcal{P}^0 (T_j \cdot T_k) \mathcal{M}^0 \left( \frac{\alpha_s}{2\pi} \right)^2 (i\pi)^2 \frac{1}{\varepsilon^2} \]

(4.122)

The sum of these last four diagrams will cancel the corresponding terms in the contribution \( \mathcal{S} \mathcal{P}^{1,\text{non-fac.}} \mathcal{M}_{\text{Glauber loops}}^1 \) to Eq. (4.108).
Let us summarize and put together the results for double Glauber diagrams in Fig. 4.4 and Fig. 4.5. As expected, these diagrams have no explicit dependence on the momenta of non-collinear partons. They are only sensitive to the physical scale associated with the splitting. All the non-vanishing double Glauber graphs have a ladder-type topology, where two vertical rungs represent Glauber interactions ordered in time.

Fig. 4.4 + Fig. 4.5 = \( \mathcal{S}^{(1),\text{non-fac}} \mathcal{M}_{\text{Glauber loops}}^{1} \)

\[
= \left\{ \frac{1}{2} \left( - \mathbf{T}_2 \cdot \mathbf{T}_3 + \sum_{j=4}^{m} \mathbf{T}_2 \cdot \mathbf{T}_j \right)^2 + \sum_{j=4}^{m} i f_{abc} \mathbf{T}_2^a \mathbf{T}_3^b \mathbf{T}_j^c \right\} \mathcal{S}^0 \mathcal{M}^0
\]

\[
\times \left( \frac{\alpha_s}{2\pi} \right)^2 (i\pi)^2 \left( \frac{4\pi \mu^2}{p_{T_{2,\perp}}^2} \right)^2 e \left( \frac{1}{\epsilon^2} - \frac{\pi^2}{6} + \mathcal{O}(\epsilon) \right)
\]

\[
\text{double pole } \left\{ \frac{1}{2} \left[ \frac{\alpha_s}{2\pi} \frac{i\pi}{\epsilon} \left( - \mathbf{T}_2 \cdot \mathbf{T}_3 + \sum_{j=4}^{m} \mathbf{T}_2 \cdot \mathbf{T}_j \right) \right]^2 - \frac{\alpha_s^2}{4\epsilon^2} \sum_{j=4}^{m} i f_{abc} \mathbf{T}_2^a \mathbf{T}_3^b \mathbf{T}_j^c \right\} \mathcal{S}^0 \mathcal{M}^0
\]

The first term in Eq. (4.124) comes directly from exponentiating the one-loop Glauber phase given by Eq. (4.106). The second term with a purely non-abelian color structure \( i f_{abc} \mathbf{T}_2^a \mathbf{T}_3^b \mathbf{T}_j^c \) corresponds to the anti-hermitian part of \( \Delta_C^2(\epsilon) \) shown in Eq. (4.79). Thus we find that the real part of the \( 1/\epsilon^2 \) IR poles in \( \mathcal{S}^{\text{non-fact}} \) at 2-loops from [169] are exactly reproduced by SCET. The subleading terms and the imaginary part of \( \mathcal{S}^{\text{non-fact}} \) at 2-loops involve graphs other than the double Glauber ones. We leave the complete computation of \( \mathcal{S}^{\text{non-fact}} \) at 2-loops to future work.

### 4.8 Analytic properties of Glauber gluons in SCET

We have shown that SCET with the addition of Glauber operators as proposed in [152] reproduces known results about factorization violating contributions to splitting ampli-
tudes from QCD. For the effective theory to be consistent it is critical that the Glauber operators do not destroy factorization in situations where it supposed to hold.

SCET graphs involving Glauber gluons have unusual properties compared to graphs in QCD. For example, the form-factor graph vanishes if \( p_1 \cdot p_2 < 0 \) but is non-zero if \( p_1 \cdot p_2 > 0 \):

\[
\begin{align*}
p_1 & = \frac{a_s i \pi}{2\pi \epsilon_{\text{IR}}} , \\
p_2 & = 0
\end{align*}
\] (4.125)

This implies in particular that the Glauber graph is not an analytic function of the external momenta. The non-analyticity comes about through the non-analytic rapidity regulator. This regulator is an essential part of the definition of SCET with Glaubers.

Glauber gluons have an intimate connection to soft gluons. For example, the amplitudes in Eq. (4.125) are exactly the imaginary part of the corresponding soft graphs. More generally, the Glauber region corresponds to a particular approach to the soft singularity: \( k^\mu \to 0 \) with \( k^\pm \lesssim \frac{k^2}{Q} \) for a hard scale \( Q \). Thus, Glauber gluons can be understood by studying the region around the soft pinch surface, as we did in Section 4.3. In that section, we showed that when factorization holds, Glauber gluons can be safely ignored.

More precisely, we showed that when there is no pinch in the Glauber region the eikonal approximation can be justified to reproduce the complete soft singularity. While these observations about pinched contours are useful for studying factorization in QCD, they do not immediately translate to observations about graphs involving Glauber operator insertions in SCET.

In SCET, when the Glauber operator contribution is entirely contained in the soft contribution, the soft-Glauber correspondence is said to hold [152]. When the soft-Glauber correspondence holds, the Glauber contributions can be completely ignored due to the
zero-bin subtraction. More precisely, in [152], Rothstein and Stewart called a soft graph in SCET without its Glauber subtraction a "naive" soft graph denoted by \( S \). Then the "pure" soft contribution \( S \) is the naive soft contribution with its Glauber limit subtracted off: \( S = \tilde{S} - S^{(G)} \). The pure soft graphs have nice properties, such as that they are independent of the direction of the soft Wilson lines; all of the unusual properties of the Glauber-gluon graphs, such as the non-analytic behavior of Eq. (4.125) and the necessity of a rapidity regulator are eliminated by this subtraction. When the soft-Glauber correspondence holds, \( S^{(G)} = G \).

What we would like to be true is that, in momenta configurations for which there is no Glauber pinch in QCD, then the soft-Glauber correspondence holds. This is not easy to show, since there is not a 1-to-1 correspondence between the Glauber limit of graphs in QCD (via the method of regions) and graphs in SCET with Glauber operator insertions. Thus, the check that Glauber operators do not destroy factorization in SCET is non-trivial. Here, we provide a general argument for why it should be true in general.

To study factorization violating effects of Glauber gluons, it is not particularly useful to subtract off the Glauber limit of each soft graph and add it back in through a pure Glauber contribution. The graphs in Eq. (4.125) are irrelevant to factorization violation since there is no pinch in the Glauber region for either kinematic configuration. Instead, we want to start with factorization in SCET without Glaubers, and look at what new effects adding Glaubers will have. That is, we would like to consider \( G_{\text{non-fact}} = G - G_{\text{fact}} \) where \( G \) refers to any graph with Glaubers and \( G_{\text{fact}} \) are the Glauber graphs that double count contributions from the factorized expression in SCET without Glaubers.

A critical property that \( G_{\text{non-fact}} \) must have is that it does not spoil factorization in situations where factorization is supposed to hold. For example, consider the canonical Glauber pinch graph as in Eq. (4.25). We suppose there are two collinear momenta \( p_1 \parallel p_2 \) and want to look at how the contribution changes when \( p_2^{\parallel} \) goes from incoming to
outgoing:

\[
\begin{align*}
\text{left: } & \quad p_1 - p_2 = 0, \\
\text{right: } & \quad p_1 - p_2 \neq 0 
\end{align*}
\] (4.126)

In the configuration on the left, \( p_1 \cdot p_2 > 0 \), there is no Glauber pinch, and factorization should hold. For this graph, there is no corresponding soft graph for the Glauber to be contained in.\(^4\) Thus the graph on the left must vanish or else factorization \textit{would} be violated. The right graph, with \( p_1 \cdot p_2 < 0 \), must reproduce known factorization-violating results from QCD (as we have shown it does). Note that the corresponding graphs in QCD are analytic functions of momenta and generically do not vanish for either sign of \( p_1 \cdot p_2 \).

The remarkable non-analytic property of the diagrams in Eq. (4.126) as a function of \( p_2^\mu \) is achieved through a conspiracy of the power expansion in SCET and the rapidity regulator. The power expansion sequesters all of the \( k^+ \) and \( k^- \) dependence:

\[
\int \frac{d^4k}{(2\pi)^4} \frac{f(k, p_i)}{k^- - \delta_1 - i\epsilon} \frac{1}{k^- - \delta_2 + i\epsilon} \frac{1}{k^+ - \delta_3 + i\epsilon} \sim (4.127)
\]

As indicated in the figure, the power expansion lets us drop certain components of the

\(^4\) Soft graphs are only sensitive to the net momenta in the collinear sector. The soft graph with a gluon exchanged from the \( p_1 \) sector has a Glauber limit given by the graph where the Glauber connects between the two legs closest to the hard vertex, as in Eq. (4.88).
loop momentum $k^\mu$ on each leg that it flows through. The propagator for the red, dotted Glauber leg, with momentum $k^\mu$, only depends on the largest component, which is $k_\perp$ according to Glauber scaling. This propagator has been absorbed into the $f(k_\perp, p_i)$ function in the numerator. Similarly, the $k^+$ component is dropped when $k^\mu$ is added to $p_1^\mu$ or $p_2^\mu$ and the $k^-$ component is dropped when $k^\mu$ is added to whatever momentum $q^\mu$ flows into the hard vertex from the rest of the diagram. Note that this momentum $q^\mu$ must be lightlike or there is no pinch and the entire diagram is not infrared sensitive and can be dropped at leading power. The leading power expansion also forces both the locations of the poles $\delta_i$ and the numerator to depend only on the largest components of $k^\mu$, namely $k_\perp$. That is, $\delta_1, \delta_2, \delta_j$ and $f$ depend only on $k_\perp$ and components of the external momenta $p_i$, but not on $k^+$ or $k^-$. An explicit example is given in Eq. (4.92).

In the case where $p_2^\mu$ is incoming the diagram has the same form but the $\delta_2$ pole crosses the real axis. That is, we make the replacement $k^- - \delta_2 + i\epsilon \rightarrow k^- - \delta'_2 - i\epsilon$ in Eq. (4.127). This flip removes the pinch from the Glauber region. In fact, it naively seems that since both the poles in the $k^+$ plane are on the same side of the axis, then Eq. (4.127) vanishes, as we expect for $p_2^\mu$ incoming. Unfortunately, things are not that simple: if the integral vanishes for $p_2^\mu$ incoming and is an analytic function of momenta, then it must also vanish for $p_2^\mu$ outgoing. In fact, we cannot conclude that it vanishes simply because the $k^-$ integral seems to give zero. The problem is that the power expansion has made the $k^-$ integral infinite. Although $k^-$ has nothing to do with the pinch in $k^+$ in the Glauber region, we need to regulate the whole integral to make the calculation well-defined.

After adding a factor of $|k_z|^{-\eta}$ from the rapidity regulator it is natural to change variables from $(k^-, k^+)$ to $(k^-, k_z)$. Doing the $k^-$ integral in Eq. (4.127) then gives

$$\int \frac{dk_z d^2k_\perp}{(2\pi)^3} \frac{1}{|k_z|^{\eta}} \frac{f(k_\perp, p_i)}{2k_z + \delta_1(k_\perp) - \delta'_1(k_\perp) - i\epsilon} \frac{1}{\delta_1(k_\perp) - \delta_2(k_\perp)} \neq 0$$

(4.128)
For the spacelike splitting case, the $k^-$ poles are on the same side of the real axis and the integral gives zero.

This 1-loop example is all that is required to show that Glauber operator contributions do not destroy factorization for timelike splittings in general. In the $n$-loop case, the infrared sensitive region has all the loop momenta near the pinch surface. Thus we can focus on a single loop, over a momentum $k^\mu$, with the other momenta placed on the pinch surface ($k_i^\mu = 0$ or $k_i^\mu$ proportional to some external momentum). For timelike splittings, the energies and large light-cone components of all the momenta in each collinear sector have the same sign. This places all the poles in poles in $k^-$ on the same side of the real axis. Therefore, once the integral is regulated with the rapidity regulator, the integral over $k^-$ will give zero just as in the 1-loop case.

In this way, the Glauber contribution with the rapidity regulator remarkably produces diagrams with the right properties: they vanish when factorization holds but contribute when factorization is violated. If a QCD diagram does not have a pinch in the Glauber region, then the corresponding diagram with Glauber gluon exchange in SCET with a rapidity regulator will vanish. This is a non-trivial consistency check on the SCET-Glauber formulation, requiring both the power expansion and the rapidity regulator. It is only possible because Glauber contributions in SCET are non-analytic functions of external momenta.

The above arguments suggest that there may be a way to identify the contribution from operators in SCET with properties of amplitudes computed in QCD. Since the Glauber contributions are non-analytic and vanish when $p_2^\mu \to -p_2^\mu$, we might identify the Glauber contribution as $G = \mathcal{M}(p_2, p_j) - \mathcal{M}(-p_2, p_j)$. At 1-loop order, this is equivalent to half the discontinuity across the cut on the real $p_2 \cdot p_j$ axis. Beyond 1-loop, taking the discontinuity across the cut can only reproduce the imaginary part of the amplitude, not terms like $(i\pi)^2$ coming from double-Glauber exchange; flipping the sign of $p_2$ could get all
of the multi-Glauber effects correct. It would certainly be interesting to investigate the connection between Glauber contributions in SCET and analyticity of QCD amplitudes in greater detail.

4.9 Summary and conclusions

In this paper we have studied factorization-violation in collinear splittings from the effective field theory point of view. The first few sections of the paper discussed situations where factorization holds. In particular, the importance of Glauber scaling was reviewed. We discussed how factorization requires application of the eikonal approximation to preserve all of the singularities in a small ball around the soft pinch surface. This requirement fails when there is a pinch in the Glauber region. Understanding the interplay between factorization, the eikonal approximation, and Glauber pinches allowed us to extend the precise amplitude-level formulation of factorization developed in [109, 163, 164] to situations where there are colored particles in the initial state. In particular, as long as no incoming direction and outgoing direction are collinear, strict factorization holds. This result, although implicit in much of the early literature on factorization, has never been stated explicitly or proven to our knowledge, so we include it here for completeness. Understanding where and why factorization does hold is a firm starting point for an analysis of factorization violation.

Regarding factorization violation, it had been shown from full QCD that in spacelike splittings (as in initial state radiation) the splitting amplitude is different from timelike splittings (as in final state radiation) [169, 170]. In particular, for spacelike splittings, strict factorization is violated, in that the collinear splitting amplitude depends on the colors and kinematics of non-collinear partons. These results were derived in QCD by looking at the IR divergences of amplitudes with $n + m$ well-separated partons and taking the limit
where \( m \) of the partons become collinear. We showed that these results can be reproduced using SCET with the inclusion of Glauber operators as proposed in [152]. In particular, we confirmed the divergent and finite factorization-violating terms at 1-loop and the leading real divergent part at 2-loops, for \( m = 1 \). These calculations are non-trivial in SCET and require careful use of the rapidity regulator and the power expansion.

In the SCET approach, the splitting is computed from emissions off an amplitude with \( n \) collinear sectors, rather than by taking limits of \( m + n \) parton amplitudes. This conceptual difference might be advantageous in studying physical implications of factorization-violation, for example, by sequestering factorization-violating effects to certain operator matrix elements. However, this is not yet possible as it is not clear how the Glauber contributions in SCET can be disentangled from the factorization-preserving soft and collinear contributions.

In [152] it was shown that much of the time the contribution from Glauber operators is identical to the Glauber limit of the soft contribution. This equivalence, \( G = S^{(G)} \) was called the “soft-Glauber correspondence” in [152]. It is important to understand when the soft-Glauber correspondence holds, as the soft-Glauber overlap (as well as the collinear-Glauber overlap) must be zero-bin subtracted to avoid overcounting. Unfortunately, it seems very hard to establish the soft-Glauber correspondence to all orders. Some examples and suggestive general arguments were given in [152]. In this paper, we connected the soft-Glauber correspondence (a feature of SCET) to situations in which integration contours can be deformed out of the Glauber region into the eikonal region in full QCD. When this deformation is possible, as in situations where no incoming parton is collinear to an outgoing parton, the soft-Glauber correspondence must hold.

One intriguing feature of the SCET-Glauber contributions is that they produce necessarily non-analytic functions of external momentum. Non-analyticity is critical for the Glauber contributions both to vanish when a momentum is outgoing (\( E > 0 \)) and to not
vanish when a momentum is incoming \((E < 0)\). In QCD, amplitudes are analytic functions of momenta (up to poles and branch cuts) but in SCET they are not. An example of how this works is the 1-loop Sudakov form factor, where QCD gives a \(\frac{1}{\varepsilon} \ln(-p_1 \cdot p_2 - i\varepsilon)\) term, which is analytic, while the Glauber contribution gives just the discontinuity of this result, \(-\frac{i\pi}{\varepsilon} \theta(p_1 \cdot p_2)\), which is non-analytic. For this form factor, the Glauber contribution is not factorization-violating, as the soft-Glauber correspondence holds, but the same non-analyticity is critical in factorization-violating cases. Indeed, the 1-loop 1-emission Glauber graphs are also non-analytic functions of a momentum \(p_2\), as they must vanish when \(p_2\) is outgoing (so as not to spoil factorization when it holds) and reproduce factorization-violating results from QCD when \(p_2\) is incoming. The SCET formalism achieves this through a combination of the power expansion, which sequesters all the dependence on certain momentum components into certain parts of the Feynman diagrams so that Glauber graphs can exactly vanish when factorization holds, and the rapidity regulator, which is non-analytic.

These observations, summarized in Section 4.8 are suggestive that the factorization-violating Glauber contributions may be identified with a sort of generalized discontinuity of the QCD amplitude: they reproduce the difference between \(|M(p_2, p_j)|\) and \(|M(-p_2, p_j)|\). Another corollary of these observations is that the rapidity divergences in the Glauber graphs must be regulated with a non-analytic regulator. The non-analyticity is helpful, in that it allows for the Glauber graphs to isolate the factorization-violating effects, but it also makes computing Glauber contributions beyond 1-loop order more challenging than for graphs where dimensional regularization can be used.
4.10 Appendix

4.10.1 Double Glauber integrals

In the Appendix we give more details of some representative 2-loop double-Glauber-exchange diagrams.

First consider Fig. 4.4(c) with two parallel Glauber rungs between \( p_2 \) and \( p_3 \):

\[
\text{Fig. 4.4(c)} = \quad (4.129)
\]

\[
= 4g_5^2 (T_2^b T_2^c) (T_3^c T_3^b) T_1 \mathcal{M}^0
\]

\[
\times \int \frac{d^d \ell}{(2 \pi)^d} \frac{d^d k}{(2 \pi)^d} \frac{1}{k^- - \delta'_2 + i \epsilon (\ell^+ - k^+)} \frac{1}{\delta'_2 + i \epsilon \ell^+ - \delta_3 + i \epsilon} \frac{1}{-\ell^- - \delta_1 + i \epsilon}
\]

\[
= 4g_5^2 (T_2^b T_2^c) (T_3^c T_3^b) T_1 \mathcal{M}^0
\]

\[
\times \int \frac{d^{d-2} \ell}{(2 \pi)^{d-2}} \frac{d^{d-2} k}{(2 \pi)^{d-2}} N^\mu (p_1, p_2, \ell_\perp) \varepsilon_\mu (p_2) \frac{1}{(\ell_\perp - k_\perp)^2} \frac{1}{\delta'_2 - \delta_1}
\]

\[
\times \int \frac{d \ell^z}{2 \pi} \frac{d k^z}{2 \pi} |2(\ell_z - k_z)|^{-\eta} |2k_z|^{-\eta} \frac{1}{2(\ell^z - k^z) - \delta_1 - \delta'_2 - \delta'_3 + i \epsilon} \frac{1}{2\ell^z - \delta_1 - \delta_3 + i \epsilon}
\]

(4.131)

where

\[
\delta_1 = - \frac{(\vec{p}_{2,\perp} + \vec{\ell}_{\perp})^2}{Q - p_{2,\perp}^+} - p_{2,\perp}^-,
\]

\[
\delta_2 = \frac{(\vec{p}_{2,\perp} + \vec{\ell}_{\perp})^2}{p_{2,\perp}^+} - p_{2,\perp}^-,
\]

\[
\delta'_2 = - \frac{(\vec{p}_{2,\perp} + \vec{k}_{\perp})^2}{Q - p_{2,\perp}^+} - p_{2,\perp}^-,
\]

\[
\delta'_3 = - \frac{(\vec{p}_{2,\perp} + \vec{k}_{\perp})^2}{Q - p_{2,\perp}^+} - p_{2,\perp}^-.
\]
\[ \delta_3 = \frac{\ell_1^2}{Q_3}, \quad \delta'_3 = \frac{(\ell_1 - \bar{k}_1)^2}{Q_3}, \quad N^\mu(p_1, p_2, l) = \frac{\eta}{2} \left[ -\frac{2(p_{2+} + \ell_\perp)^\mu}{p_2^+} + \frac{(p_{2+} + l_\perp + \gamma^\mu_\perp)}{-Q + p_2^+} \right] \frac{d}{2} v_n(p_1) \]

(4.132)

The \( \ell^2 \) and \( k^2 \) integrals can be conveniently carried out in position space. After Fourier transforming the light-cone propagators, the integral becomes integrals of light-cone coordinates, transforming the light-cone propagators, the integral becomes integrals of light-cone coordinates, and

\[
\text{Fig. 4.4(c) } = 4s^5 (T_2^b T_3^c) (T_3^c T_3^b) T_1 \overline{\mathcal{M}}^0 \left[ -\frac{2\varepsilon_{\perp\mu}}{p_2^+} - \frac{\gamma_{\perp\mu} \ell_\perp}{Q - p_2^+} \right] v_n(p_1) \quad (4.133)
\]

\[
\times \frac{p_{2+}^2 (Q - p_2^+)}{Q} \int \frac{d^{d-2} \ell_\perp}{(2\pi)^{d-2}} \frac{d^{d-2} k_\perp}{(2\pi)^{d-2}} \frac{p_{2+}^\mu + \ell_\perp^\mu}{(p_{2+} + \ell_\perp)^2 (\ell_\perp - \bar{k}_1)^2 \bar{k}_1^2} \times \frac{1}{4 \times 2!} (1 + \mathcal{O}(\eta)) \]

where \( \kappa_\eta = 2^{-\eta} \Gamma(1 - \eta) \sin(\pi\eta/2)/(\pi\eta/2) = 1 + \mathcal{O}(\eta) \). The effective diagram with the \( \eta \)–regulator preserves the physical property of Glauber interactions: they are instantaneous Coulomb interactions that are ordered in time. The \( \theta \)-functions in Eq. (4.133) guarantee that the Glauber exchanges take place at light-cone time \(-x < \bar{y} < 0\), both earlier than the hard interaction. Time ordering between the two Glaubers produces a \( 1/2! \) symmetry factor.

The \( \ell_\perp, \bar{k}_1 \) integral contains an \( 1/\varepsilon^2 \) divergence,

\[
\mu^{2(4-d)} \int \frac{d^{d-2} \ell_\perp}{(2\pi)^{d-2}} \frac{d^{d-2} k_\perp}{(2\pi)^{d-2}} \frac{p_{2+}^\mu + \ell_\perp^\mu}{(p_{2+} + \ell_\perp)^2 (\ell_\perp - \bar{k}_1)^2 \bar{k}_1^2} \frac{1}{(2\pi)^{d-2}} \frac{1}{(2\pi)^{d-2}}
\]

214
\[
\begin{align*}
&= \left( \frac{1}{4\pi} \right)^2 \frac{p_{2,\perp}^\mu}{\vec{p}_{2,\perp}^2} \left( \frac{4\pi \mu^2}{\vec{p}_{2,\perp}^2} \right)^{2e} \left[ \Gamma(-e) \right]^2 \frac{\Gamma(1-e)\Gamma(1+2e)}{\Gamma(1-3e)} \\
&= \left( \frac{1}{4\pi} \right)^2 \frac{p_{2,\perp}^\mu}{\vec{p}_{2,\perp}^2} \left( \frac{4\pi e^{-\gamma_e} \mu^2}{\vec{p}_{2,\perp}^2} \right)^{2e} \left( \frac{1}{e^2} - \frac{\pi^2}{6} + O(e) \right)
\end{align*}
\] (4.135)

As expected, the two-loop Glauber diagram has the same spin structure as the tree-level diagram, and thus proportional to the \(Sp^0\),

\[
\text{Fig. 4.4(c)} = \frac{1}{2!} \left( \frac{ig_s^2}{2} \right)^2 g_s \left( T^b_2 T^c_3 \right) \left( T^b_3 T^c_1 \right) \frac{\overrightarrow{M}^0}{\overrightarrow{P}^2} \left[ -2e_{\perp,\mu} - \frac{\gamma_{\perp,\mu} \ell_{\perp}}{Q - p_2^+} \right] v_{n}(p_1) \]

\[
\times \frac{p^+_2 (Q - p^+_2)}{Q} \frac{p_{2,\perp}^\mu}{\vec{p}_{2,\perp}^2} \left( \frac{1}{4\pi} \right)^2 \left( \frac{4\pi \mu^2}{\vec{p}_{2,\perp}^2} \right)^{2e} \left[ \Gamma(-e) \right]^2 \frac{\Gamma(1-e)\Gamma(1+2e)}{\Gamma(1-3e)}
\]

\[
= \left( T^b_2 T^c_3 \right) \left( T^c_3 T^b_1 \right) \frac{\overrightarrow{M}^0}{\overrightarrow{P}^2} \left[ -2e_{\perp,\mu} - \frac{\gamma_{\perp,\mu} \ell_{\perp}}{Q - p_2^+} \right] v_{n}(p_1)
\] (4.136)

Next we consider diagrams with two Glaubers gluons exchanged between \(p_2^\mu\) and two non-collinear partons \(p_i^\mu\) and \(p_j^\mu\), \((i, j = 3, \cdots, m)\). In these diagrams we insert Glauber operators \(O_{n_1} \frac{1}{P_{\perp}^\perp} \partial_{n_1}\) and \(O_{n_1} \frac{1}{P_{\perp}^\perp} \partial_{n_1}\), where \(n_i^\mu\) and \(n_j^\mu\) are not generically related to \(n_1^\mu\).

Strictly speaking, the transverse label momentum \(P_{\perp}^\perp\) should be replaced by \(P_{1j}^\perp\) to extract the components of a 4-vector in the plane transverse to \(n_1^\mu\) and \(n_j^\mu\) in Minkowski space. Take 1\rangle\langle j\rangle and 1\rangle\langle j\rangle to be an orthogonal basis in the transverse plane with respect to \(n_1^\mu\) and \(n_j^\mu\). Bases with different choice of light-cone directions can be related through the following equations

\[
\begin{align*}
1\rangle j &= 1\rangle i \frac{[ji]}{[ji]} + 1\rangle i \frac{[ij]}{[ij]} \\
\langle j\rangle 1 &= 1\langle i \frac{\langle ij \rangle}{\langle ij \rangle} + 1\langle i \frac{\langle ji \rangle}{\langle ji \rangle}
\end{align*}
\] (4.137)
Projecting a 4-vector $p^\mu$ onto transverse direction $1)[j$ and $j)[1$, we have

$$p_{1j}^{\perp,\mu} = \frac{n_1 \cdot p}{n_1 \cdot n_j} (n_j)_{1i}^{\perp,\mu} + p_{1i}^{\perp,\mu}$$

(4.138)

If $p^\mu$ has collinear or Glauber scaling, $(n_1 \cdot p) \sim \lambda^2$ while $p_{1i}^{\perp,\mu} \sim \lambda$, then at leading power we can drop $(n_1 \cdot p)$ with respect to $p_\perp$. Thus, Thus,

$$p_{1i}^{\perp,\mu} \approx p_{1j}^{\perp,\mu}, \quad (p^\mu \text{ Glauber or collinear to } n_1^{\mu})$$

(4.139)

The operator $P_\perp^2$ sandwiched between $O_{n_i}$ and $O_{n_j}$ will pull out the virtuality of the exchanged Glauber gluon, which takes the same form no matter what light-cone coordinates we choose.

We will show the explicit calculation of Fig. 4.5(a) and Fig. 4.5(c) in the following. In Fig. 4.5(a), one Glauber connecting parton $p_2$ and an incoming parton $p_3$ has virtual momentum $\ell - k$, the other Glauber with virtual momentum $k$ connects $p_2$ and outgoing parton $p_j$. As we argued above, at leading power both Glaubers can be treated in the same way as those exchanged between back-to-back jets. The integrand is the following, where the propagators still depend linearly on the light-cone components of loop momenta,

$$\text{Fig. 4.5(a) = } \int \frac{d\ell}{(2\pi)^d} \frac{dk}{(2\pi)^d} \frac{1}{1 - \frac{n_1 \cdot (\ell - k) - n_2 \cdot (\ell - k)^{-\eta} n_1 \cdot k - n_j \cdot k^{-\eta} N\mu(p_1, p_2, \ell_\perp) \epsilon\mu(p_2)}{N\mu(p_1, p_2, \ell_\perp) \epsilon\mu(p_2)} \frac{1}{(\ell_\perp - k_\perp)^2} \frac{1}{k^{2\perp}_\perp}$$

$$\times \frac{1}{n_1 \cdot k - \delta_2^\prime + i\epsilon} \frac{1}{n_3 \cdot (\ell - k) - \delta_3^\prime + i\epsilon} \frac{1}{n_j \cdot k - \delta_1 + i\epsilon} \frac{1}{n_1 \cdot \ell - \delta_2^\prime + i\epsilon} \frac{1}{n_1 \cdot \ell - \delta_1 + i\epsilon}$$

(4.141)
Here, $\delta_1, \delta_2, \delta'_2, \delta'_3$ are the same as those defined in Eq. (4.132). We also define $\delta_j = \bar{k}_j^2 / Q_j$ and $\delta'_j = (\bar{\ell}_j - \bar{k}_j)^2 / Q_j$, with $p_j^\mu \equiv \frac{1}{2} Q_j n_j^\mu$, for any outgoing non-collinear parton $j$. In Eq. (4.141), $\kappa_{ij} \equiv (n_i \cdot n_j) / 2$, which is equal to 1 only for back-to-back directions. These factors are inserted at the operator level to guarantee the RPI-III invariance of the SCET Lagrangian [152]. Let us choose integration variables to be

$$\ell^- = n_1 \cdot \ell, \quad k^- = n_1 \cdot k, \quad k_1^\perp = \frac{n_3 \cdot (\ell - k) - n_1 \cdot (\ell - k)}{2}, \quad k_2^\perp = \frac{n_j \cdot k - n_1 \cdot k}{2} \quad (4.142)$$

so that

$$\int d^d \ell \, d^dk \to \frac{1}{\kappa_{13} \kappa_{1j}} \int d\ell^- \, dk^- \, dk_1^\perp \, dk_2^\perp \, d^{d-2} \ell_\perp \, d^{d-2} k_\perp \quad (4.143)$$

After change of integration variables, the $\kappa_{ij}$ terms drops out and the integrand looks independent of the direction of $p_j$. After integrating over $\ell^-$ and $k^-$, each collinear propagator depends linearly on $k_1^\perp, k_2^\perp$. Hence we can easily transform into position space.

$$\text{Fig. 4.5(a)} = 4 \tilde{g}_s^5 \left( T_j^b T_z^c \right) \left( - T_3^z \right) T_j^b T_1 \overline{\mathcal{M}}^0$$

$$\times \int \frac{d^{d-2} \ell_\perp \, d^{d-2} k_\perp}{(2\pi)^{d-2}} \frac{N^\mu(p_1, p_2, \ell_\perp) \epsilon_\mu(p_2)}{(\ell_\perp - k_\perp)^2} \frac{1}{k_1^\perp} \frac{1}{k_2^\perp} \frac{1}{\delta_2 - \delta_1} \quad (4.144)$$

$$\times \int \frac{dk_1^\perp \, dk_2^\perp}{2\pi} \left| 2k_1^\perp \right|^{-\eta} \left| 2k_2^\perp \right|^{-\eta} \frac{1}{2 \delta_1 - \delta_2 - \delta_3 + i \epsilon} \frac{1}{-2 \delta_2 - \delta_j + i \epsilon} \quad (4.145)$$

$$= 4 \tilde{g}_s^5 \left( T_j^b T_z^c \right) \left( - T_3^z \right) T_j^b T_1 \overline{\mathcal{M}}^0$$

$$\times \frac{p_2^+ (p_1^+ - p_2^+)}{p_1^+} \int \frac{d^{d-2} \ell_\perp \, d^{d-2} k_\perp}{(2\pi)^{d-2}} \frac{N^\mu(p_1, p_2, \ell_\perp) \epsilon_\mu(p_2)}{(\ell_\perp - k_\perp)^2} \frac{1}{k_1^\perp} \frac{1}{k_2^\perp} \frac{1}{\delta_2 - \delta_1} \quad (4.144)$$

$$\times \frac{1}{4} \left( \kappa_{ij} \frac{\eta}{2} \right)^2 \int dx dy \theta(x) \theta(y) \frac{1}{|x|^{1+\eta}} \frac{1}{|y|^{1+\eta}} e^{-ix(\delta'_3 + \delta'_2 + \delta_1)/2 - iy(\delta'_3 + \delta'_2)/2} \quad (4.145)$$

$$= (i)^2 \tilde{g}_s^5 \left( T_j^b T_z^c \right) \left( - T_3^z \right) T_j^b T_1 \overline{\mathcal{M}}^0$$

217
\[ \times \frac{p_2^+(Q - p_2^+)}{Q} \int \frac{d^{d-2} \ell_\perp}{(2\pi)^{d-2}} \frac{d^{d-2} k_\perp}{(2\pi)^{d-2}} \frac{N^\mu(p_1, p_2, \ell_\perp)}{(p_{2,\perp}^2 + \ell_\perp^2)^2} \frac{1}{(\ell_\perp - k_\perp)^2} \frac{1}{k_\perp^2} \left( \frac{1}{2} \right)^2 (1 + O(\eta)) \]

(4.146)

The position-space picture describes Glauber exchange before and after the hard interaction, taking place at light-cone time \(-x\) and \(-y\) with \(-x < 0 < y\). The integration region is the quarter \((x, y)\)-plane. At \(O(\eta^0)\), the \(dx\) and \(dy\) integrals are symmetric, the result being twice of the parallel box diagrams in Fig. 4.4.

Fig. 4.5(a) = \(-\langle T_{2} \cdot T_{3} \rangle \langle T_{2} \cdot T_{j} \rangle S \mathbf{p}^0 \overline{\mathcal{M}}^0 \)

\[ \times \left( \frac{\alpha_s}{2\pi} \right)^2 \left( \frac{i\pi}{\mu^2} \right)^2 \left[ \Gamma(1 - \epsilon) \Gamma(1 + 2\epsilon) \right] \frac{[\Gamma(-\epsilon)]^2}{\Gamma(1 - 3\epsilon)} \]

(4.147)

Fig. 4.5(c) has two Glaubers connecting \(p_2\) with two outgoing partons \(p_j\) and \(p_k\), with virtual momenta \(k\) and \(\ell - k\), respectively. Choose integration variables to be

\[ \ell^- = n_1 \cdot \ell, \quad k^- = n_1 \cdot k, \quad k_1^z = \frac{n_j \cdot (\ell - k) - n_1 \cdot (\ell - k)}{2}, \quad k_2^z = \frac{n_k \cdot k - n_1 \cdot k}{2} \]

(4.148)

then

Fig. 4.5(c) = \[ \times \left( \frac{\alpha_s}{2\pi} \right)^2 \left( \frac{i\pi}{\mu^2} \right)^2 \left[ \Gamma(1 - \epsilon) \Gamma(1 + 2\epsilon) \right] \frac{[\Gamma(-\epsilon)]^2}{\Gamma(1 - 3\epsilon)} \]

(4.149)

\[ = 4g_s^5 \langle T_{2}^b T_{2}^c \rangle T_j^b T_k^c T_1^a \overline{\mathcal{M}}^0 \]

\[ \times \int \frac{d^{d-2} \ell_\perp}{(2\pi)^{d-2}} \frac{d^{d-2} k_\perp}{(2\pi)^{d-2}} \frac{d\ell^-}{2\pi} \frac{dk^-}{2\pi} \frac{dk_1^z}{2\pi} \frac{dk_2^z}{2\pi} \left( \frac{1}{(\ell_\perp - k_\perp)^2} k_\perp^2 \right) \]

(4.150)
$$= 4g_s^5 (T^b_2 T^c_2) T^b_j T^c_k T_1 \overline{\mathcal{M}}^0$$

$$\times \frac{p^+_2 (Q - p^+_2)}{Q} \int \frac{d^{d-2} \ell}{(2\pi)^d-2} \frac{d^{-2} k}{(2\pi)^d-2} N^\mu(p_1, p_2, \ell) \varepsilon_\mu(p_2) \frac{1}{(\ell - k)^2} \frac{1}{k^2} \frac{1}{(\ell + \bar{p}_{2,\perp})^2}$$

$$\times \frac{1}{4} \left( \frac{\kappa \eta}{2} \right)^2 \int dxdy \, \theta(x) \theta(y - x) \frac{1}{|x|^{1+\eta}} \frac{1}{|y|^{1+\eta}} e^{-ix(\delta_j^\ell + \delta_2^\ell) / 2 - i(\eta - x)(\delta_j + \delta_2) / 2}$$

$$= (i)^2 g_s^5 (T^b_2 T^c_2) T^b_j T^c_k T_1 \overline{\mathcal{M}}^0$$

$$\times \frac{p^+_2 (Q - p^+_2)}{Q} \int \frac{d^{d-2} \ell}{(2\pi)^d-2} \frac{d^{-2} k}{(2\pi)^d-2} N^\mu(p_1, p_2, \ell) \frac{1}{(\bar{p}_{2,\perp} + \ell)^2} \frac{1}{(\ell - k)^2} \frac{1}{k^2} \times \frac{1}{2!} \left( \frac{1}{2} \right)^2 (1 + \mathcal{O}(\eta))$$

(4.151)

Here the $\theta$-functions ensure that both Glaubers are produced after the hard interaction with a particular ordering. The time-ordering between the two Glaubers gives a $\frac{1}{2!}$ symmetry factor.

Fig. 4.5(c) = \frac{1}{2!}(T_2 \cdot T_j)(T_2 \cdot T_k) \text{Sp}^0 \overline{\mathcal{M}}^0$

$$\times \left( \frac{\alpha_s}{2\pi} \right)^2 (i\pi)^2 \left( \frac{4\pi \mu^2}{p^2_{2,\perp}} \right)^{2\varepsilon} [\Gamma(-\varepsilon)]^2 \frac{\Gamma(1 - \varepsilon) \Gamma(1 + 2\varepsilon)}{\Gamma(1 - 3\varepsilon)}$$

(4.152)
Chapter 5

Factorization Violation and Scale Invariance

5.1 Introduction

Factorization is essential to the predictive power of perturbative QCD at hadron colliders. The essential point of factorization is that it lets us separate the dynamics of the proton from the dynamics of the scattering that produces hard radiation. Unfortunately, factorization is known not to hold universally. There is both theoretical evidence for non-factorization through effects like super-leading logarithms and experimental evidence, as calculations performed assuming factorization can have significant deviations from experiment. In order to continue to push the effective precision of predictions for the Large Hadron Collider and other machines, it will be essential to understand more about how and when factorization violation occurs. The hope is that with better understanding we might be able to either choose observables for which factorization violation is minimal or find some universality in factorization-violating effects.

Factorization has been shown to hold rigorously in perturbation theory in certain cir-
cumstances. For example, it holds at the amplitude level for processes with only final-state radiation. It also holds at the amplitude level for processes with only one colored particle in the initial state (like deep-inelastic scattering at the parton level). In addition, it holds for any process with a fixed number of external particles. These results were reviewed and clarified recently in [109, 163, 164]. Factorization is known to be violated both at the amplitude and cross section level for processes with two colored initial state particles and at least one final-state particle collinear to an initial-state one [169, 170, 185]. In such situations, while the amplitude can still be written as a splitting amplitude times the amplitude with the collinear pair replaced by their mother particle, the splitting amplitude necessarily depends on the quantum number of non-collinear particles. It’s therefore called generalized splitting amplitude in Ref. [169].

Even if factorization is violated at the amplitude level, it may still cancel when amplitudes are squared and integrated over phase space. The celebrated example of this phenomenon is the Collins-Soper-Sterman (CSS) proof of factorization for Drell-Yan [184]. There, when all relevant processes describing the hadro-production of a lepton pair are included, the cross section still can be written as the convolution of parton-distribution functions and a hard scattering kernel.

In recent years, there has been renewed interest in extending the CSS argument to other processes, and understanding its failure. Partly this has been motivated by the advent of jet substructure techniques [68, 186–189], where predictions of observables like jet mass or beam thrust are apparently more sensitive to factorization-violating effects than traditional kinematic observables, like the jet $p_T$ spectrum. Partly, it has been driven by theory developments that give new handles on factorization-violating effects. In particular, Soft-Collinear Effective Theory (SCET) [15, 27, 29, 30, 80, 165] has allowed for higher precision jet substructure calculations. Recently, Rothstein and Stewart [152] have explained how to account for Glauber effects within the SCET framework (see also [171–174]).

221
In this current paper, we attempt to shed some light on when factorization occurs by combining the SCET Glauber picture with observations by CSS and others. In particular, it has been argued in Ref. [152, 190] that factorization can be violated by diagrams with pure Glauber gluon exchange, but not involving any real gluon radiation. We revisit these arguments and show that in fact factorization is not violated by the pure Glauber graphs. Our calculation amounts to showing that after summing over all possible cuts, contribution from pure Glauber graphs vanish,

$$\sum_{\text{cuts}} \delta(X - f_X(p^\mu_3, p^\mu_4)) = 0$$

(5.1)

Here, the vertical dashed lines denote possible cuts and $p^\mu_3$ and $p^\mu_4$ are the momenta of the two outgoing spectators generated by the cuts.

The cancellation we find implies that the differential cross-section for any single-scale observable, like hadronic transverse energy, or beam thrust, gets no contribution from pure-Glauber graphs. The leading factorization-violating effects therefore must involve diagrams with the Lipatov vertex [191], that is, diagrams where soft gluons are exchanged between Glauber rungs. Our argument does not apply to doubly-differential observables, like those studied in [192].

We begin in Section 5.2 with a review of some results about factorization at the amplitude and cross section level. This section can be skipped by an informed reader, but may be useful to a reader who finds the literature on factorization marginally impenetrable. Section 5.3 presents our main new result, that the contribution of pure Glauber ladder graphs to cross section exactly vanishes for any single-scale observable. To not disrupt the logic, we include in this section only summaries of the calculations, with de-
tails relegated to Appendices ?? through ??.

While elements of our calculation are similar to the summation of Glauber ladder graphs into a phase in position space for forward scattering calculations, we work instead in momentum space where a choice of variables respecting scale invariance can be made. Section 5.4 discusses graphs beyond the Glauber ladders, necessarily involving soft gluons and the Lipatov vertex. For these graphs, scale-invariance is violated by quantum effects and so factorization can be violated. A brief summary and conclusions are in Section 5.5.

5.2 Factorization and Factorization Violation

We begin with a review of some known results about factorization. There are no new results in this section. The goal of the section is merely to clarify what we mean by factorization, and what is known, using relatively clear and precise language.

Almost all the literature about factorization refers to statements about soft and collinear divergences in perturbative QCD. A clean way to describe perturbative factorization was developed in [109, 163, 164, 185]. Consider an initial state $|Z\rangle$ and a final state $\langle X|$ each of which is made up of some quarks and gluons with various momenta. We can group those momenta into sectors either collinear to a set of directions $n_j^\mu$ or soft. Each sector has an associated scale. For example, the scale $\lambda_j$ associated with the $n_j$-collinear sector may be defined so that each momentum $p_j$ in the $n_j$ collinear sector has $n_j \cdot p_j \leq \lambda_j p_j^0$. For the soft sector a parameter $\lambda_s$ can be defined so that the energy of each soft momentum $k^\mu$ has $k^0 \leq \lambda_s Q$ where $Q$ is a hard scale, such as the center-of-mass energy, or the energy of some jet. Thus we can write $\langle X| = \langle X_s| \langle X_1| \cdots \langle X_N|$ and $|Z\rangle = |Z_s\rangle |Z_{N+1} \cdots Z_M\rangle$.

For any such decomposition in which no initial-state direction is collinear to a final-state
direction, amplitudes factorize:

$$\langle X | \phi^* \cdots \phi | Z \rangle \simeq C(S_{ij}) \frac{\langle X_1 | \phi^* W_1 | 0 \rangle}{\langle 0 | Y_1^+ W_1 | 0 \rangle} \cdots \frac{\langle 0 | W_M^+ \phi | Z_M \rangle}{\langle 0 | W_M^+ Y_M | 0 \rangle} \langle X_s | Y_1^+ \cdots Y_M | Z_s \rangle$$  \hspace{1cm} (5.2)

In this relation \(\simeq\) means the two sides agree at leading power in all of the \(\lambda_j\) and \(\lambda_s\), including all infrared divergences (soft and collinear divergences). Another way to say this is that the Wilson coefficient \(C(S_{ij})\), defined as the ratio of the left-side amplitude to the factorized amplitude on the right, is finite as all the \(\lambda\) go to zero. Here, the matrix elements are written as amplitudes of operators in scalar QED, for simplicity; the same factorization formula holds in QCD but the color and spin notation is more cumbersome. The \(W_j\) and \(Y_j\) are Wilson lines. All operators in this factorization formula are written in terms of the fields of an ordinary quantum field theory (scalar QED or QCD) – no effective field theory interactions are required. This factorization is closely related to factorization in SCET where there are soft fields and collinear fields for each direction with leading-power interactions.

One important implication of factorization is that it relates amplitudes with different external states. For example, the Wilson coefficient \(C(S_{ij})\) depends only on hard scales \(S_{ij} = (P_i^\mu + P_j^\mu)^2\), with \(P_j^\mu\) are the net momenta in the \(n_j\)-collinear sector. Thus the factorization formula relates processes with different distributions of soft and collinear particles.

An application of factorization is to prove the universality of collinear splittings. Treating amplitudes as vectors in color space, a splitting amplitude is defined as the ratio between an amplitude \(|\overline{M}\rangle\) with \(n - m\) particles to an amplitude \(|M\rangle\) with \(n\) particles. For \(m = 1\) the relationship can be written as

$$| M(p_1, \cdots, p_n) \rangle \simeq Sp(p_1, p_2; p_3, \cdots, p_n) \cdot | \overline{M}(P, p_3, \cdots, p_n) \rangle$$  \hspace{1cm} (5.3)
Here \( M \) and \( \overline{M} \) are amplitudes like in Eq. (5.2). The momentum \( P^\mu \) on the right is a single particle momentum that splits into \( p_1^\mu \) and \( p_2^\mu \): \( P^\mu \approx p_1^\mu + p_2^\mu \). Factorization in Eq. (5.2) implies that

\[
|\overline{M}\rangle \approx \frac{\langle P| \overline{\psi} W_1 |0\rangle}{\text{tr} \langle 0| Y_1^\dagger W_1 |0\rangle} \cdot |\mathcal{M}_{\text{rest}}\rangle, \quad |M\rangle \approx \frac{\langle p_1, p_2| \overline{\psi} W_1 |0\rangle}{\text{tr} \langle 0| Y_1^\dagger W_1 |0\rangle} \cdot |\mathcal{M}_{\text{rest}}\rangle
\]

and therefore

\[
\mathbf{S} p = \frac{\langle p_1, p_2| \overline{\psi} W_1 |0\rangle}{\langle P| \overline{\psi} W_1 |0\rangle}
\]

The key point is that the splitting amplitude is \textit{universal} – it only depends on the the fields in the direction collinear to the splitting.

Now consider a situation where some outgoing particles are collinear to some incoming ones, such as in forward scattering. Then factorization at the amplitude level does not hold and Eq. (5.2) is invalid. Instead, we can write something similar by combining all the operators into one:

\[
\langle X| \phi^* \cdots \phi |Z\rangle \approx C(S_{ij}) \langle X| \phi^* W_1 Y_1^\dagger \cdots Y_N W_N^\dagger \phi |Z\rangle
\]

To evaluate the right-hand side one needs to use the SCET Lagrangian which has collinear fields, soft fields, and Glauber interactions

\[
\mathcal{L}_{\text{SCET}} = \sum_j \mathcal{L}_j + \mathcal{L}_s + \sum_{ij} \mathcal{L}_{ij}^G
\]

The labels \( j \) and \( s \) have become quantum numbers labelling the sectors. Thus interactions between different collinear sectors are forbidden by \( j \) and \( s \) superselection rules (the terms in each \( \mathcal{L}_j \) or \( \mathcal{L}_s \) only involve fields with the same quantum numbers). The denominator factors on the right-hand-side of Eq. (5.2) are replaced by a diagram-level zero-bin sub-
traction procedure to remove the soft/collinear overlap. With this understanding, when $L_{ij}^G$ can be ignored then Eq. (5.2) and Eq. (5.6) are equivalent.

The terms $L_{ij}^G$ in the SCET Lagrangian are the “Glauber interactions”, introduced in [152] (see also [173, 174]). They contain interactions among collinear particles in the $i$ and $j$ sectors, as well between these collinear directions and soft particles. The Glauber interactions are non-local, involving explicit factors of $\frac{1}{\tilde{k}_\perp^2}$ where $\tilde{k}_\perp$ is the transverse-momentum transfer. The diagrams involving Glauber exchange to lowest order in $g_s$ are the same as those arising by taking the Glauber-scaling limit of diagrams in full QCD. For higher-order diagrams there is not a simple method-of-regions correspondence with QCD. Instead the all-orders form of these terms is fixed by symmetry arguments (reparameterization invariance) and direct matching calculations. In addition to adding these new interactions, SCET with $L_{ij}^G$ requires a new zero-bin subtraction: the Glauber limit of soft and collinear graphs must be subtracted diagram-by-diagram so as not to double count. Finally, there is an implicit non-analytic rapidity regulator required to define the theory. Some understanding of why the regulator must be non-analytic was discussed in [185], but understanding the regulator in more detail (e.g. what properties it must have, what is regulator-independent) is an open area of research.

As the Glauber interactions involve different collinear sectors, they violate factorization. It is nevertheless a non-trivial check that the Glauber interactions can reproduce known factorization-violating effects in full QCD. One such check was performed in [185] where the generalized splitting function (i.e. one that depends on multiple directions) was calculated using the SCET formalism, finding a result in agreement with full QCD. Other checks, such as reproducing the Glauber phase in forward scattering or the BFKL equation, were discussed in [152].

For factorization to hold in hadronic collisions, we would like to be able to separate the dynamics of the proton from the hard scattering. Let’s start with a situation where
factorization does hold, deep-inelastic scattering (DIS). In pictures, factorization implies

\[
\frac{\psi_1 \psi_2}{\lambda_1 \lambda_2} = \frac{\psi_1}{\lambda_1} \times \frac{\psi_2}{\lambda_2}
\]

(5.8)

The first diagram on the right-hand side further factorizes, into soft and collinear (not shown). The \( n_1 \) and \( n_2 \) labels indicate to which directions the various lines are collinear. The red shallow curled lines are meant to indicate soft gluons interacting between the two collinear sectors. In words, the dynamics of the initial state, collinear to the \( n_1^\mu \) direction, factorizes from the hard scattering. In equations

\[
\langle X_1; X_2; X_3 | \bar{\psi} \gamma^\mu \psi | \gamma, p_1 \rangle \cong \langle X_2, X_3 | \bar{\psi} \gamma^\mu \psi | \gamma, p_1' \rangle \times \mathbf{Sp}(p_1 \rightarrow p_1', X_1)
\]

(5.9)

Where \( p_1'^\mu = \xi p_1^\mu \) is the momentum participating in the hard partonic scattering.

Does this imply that factorization holds in terms of parton distribution functions for the proton? Not necessarily. Indeed, factorization of DIS at the amplitude level is neither a necessary condition (factorization could hold at the cross section level) or a sufficient condition (we are just working in perturbation theory here, so we cannot say anything about non-perturbative physics). Nevertheless it is suggestive of factorization and a good start.

For Drell-Yan, the analogous factorization does not hold

\[
\langle X_1; X_2; X_3 | \bar{\psi} \gamma^\mu \psi | \gamma, p_1 \rangle \not\cong \langle X_2, X_3 | \bar{\psi} \gamma^\mu \psi | \gamma, p_1' \rangle \times \mathbf{Sp}(p_1 \rightarrow p_1', X_1)
\]

(5.10)
The problem originates with the soft sector, but affects purely collinear emissions too through virtual effects.

There is a quick way to see what the problem is with soft radiation. Soft radiation factorizes when it is sensitive to only the net color charge going in a particular direction. This is similar to Gauss’s law in electromagnetism – at large distances only the net charge in a region matters to the leading approximation. But what is the net charge when there are incoming and outgoing particles in the same sector? Not so obvious. For DIS, we can use a trick and move all the color of the \( n_1 \)-collinear sector to a Wilson line in the scattering operator. That is, by color conservation, we know the net color that the outgoing \( n_2 \) jet sees is negative of the net \( n_1 \) color. For Drell-Yan, this trick does not work. Indeed, the reason the minimal number of colored particles needed for factorization breaking is four is that with three or fewer, color conservation can be used to ensure that the scattering is only sensitive to the net color. This argument can be found in [169] and is explained in depth in [185].

The obstruction to factorization can be traced to the invalidity of the eikonal approximation in describing soft radiation. Soft radiation refers to regions of real or virtual phase space in which all the components of a gluon’s momentum are small compared to the energy scale \( Q \) of the particles emitting the soft radiation, \(|k^\mu| \ll Q\). For two momenta \( p^\mu \) and \( q^\mu \) we can always write

\[
\frac{1}{(p+k)^2 + i\epsilon} = \frac{1}{2p \cdot k + i\epsilon} - \frac{k^2}{((p+k)^2 + i\epsilon)(2p \cdot k + i\epsilon)} \tag{5.11}
\]

The eikonal or Grammer-Yennie approximation amounts to dropping the second term on the right with respect to the first. When the eikonal approximation can be used, soft-collinear factorization holds [163, 164]. For \( p^\mu \) hard \((p^0 \sim Q)\) and \( k^\mu \) soft \((k^0 \ll Q)\), it seems like the second term can always be dropped since \( \frac{k^2}{p \cdot k} \sim \frac{k^0}{Q} \ll 1 \). Unfortunately, it
is not enough for $k^0 \ll Q$. The problem is that it is possible for $k^\mu \ll Q$ with $\frac{k^2}{p \cdot k} \sim 1$. For example, $k^\mu \to 0$ holding $\frac{k^2}{\epsilon_0 Q}$ fixed. In light-cone coordinate the scaling of such mode can be written as $k \sim Q(\lambda^2, \lambda^2, \lambda)$. This kind of scaling toward $k^\mu = 0$ is known as Glauber scaling and can foil factorization.

An intuitive way to understand the obstruction from Glauber scaling is to contrast the soft limit $k^\mu \to 0$ with the limit $Q \to \infty$. If $Q$ decouples, so we can take $Q \to \infty$ to describe soft radiation, then the sources emitting soft radiation can be treated as scale-invariant Wilson lines $Y_n$. Soft-collinear factorization is based on being able to use this scale-invariant limit. Glauber scaling involves $Q$ in an essential way, so it obstructs scale-invariance.

To prove that the eikonal limit can be used, one must show that the region of soft momenta described by Glauber scaling is contained in scaleless integrations over soft momenta. In CSS language, this happens when there is no pinch in the Glauber region. In the language of SCET, it is when contributions form the Glauber Lagrangian ($L^G_{ij}$ in Eq.(5.7)) are not exactly canceled by the Glauber-soft and Glauber-collinear zero-bin subtractions [152]. The relation between these two ideas was further explored in [185].

So factorization does not hold in Drell-Yan at the amplitude level, as in Eq. (5.10), because the Glauber region is not contained in the soft region. Graphs for Drell-Yan have a pinch in the Glauber region. What CSS showed was that despite the non-factorization at the amplitude level, factorization still holds for Drell-Yan as long as the observable is inclusive over all QCD radiation. In pictures

$$
\sum_X \int d\Pi_X \left| \ldots \right|^2 \cong \sum_X \int d\Pi_X \left| \ldots \right|^2 \times \int |S_p|^2 \times \int |S_p|^2
$$

(5.12)

To prove this, CSS showed that for soft gluons with momenta $k^\mu_i$ that interact with
collinear momenta $p^\mu_j$, the transverse components of the soft momenta can be neglected. That is, we can replace $k^\mu = (k^+, k^-, \vec{k}_\perp)$ by $k^\mu = (k^-, 0, 0)$ where $k^-$ is the component backwards to the jet direction (i.e. $k \cdot p \simeq k^+ p^-$). Once we know the result is unaffected by dropping transverse components then there is only one way to scale $k^\mu \to 0$; there is no Glauber region and the eikonal approximation $k^2 \to 0$ is always justified around $k^2 = 0$. In more detail, CSS’s argument used old-fashioned perturbation theory. They showed that the same cross section results with and without neglecting $k_\perp$ and $k_+$ by summing over all possible cuts in spectator-spectator graphs

\[ \text{as well as in active-spectator graphs. In addition, their argument exploited observations about the polarizations of the gluons coupling to the collinear sector. While there is no doubt that the CSS proof is correct, it gives little guidance as to what we might learn about situations where factorization is violated.} \]

\[5.2.1 \quad \text{Factorization violation}\]

Factorization holds for the Drell-Yan process where only the lepton momenta are measured. A typical Drell-Yan observable is

- **Lepton transverse momentum** $q_T$: defined in the Drell-Yan process as the transverse momentum of the lepton pair $q_T = |\vec{p}_{1,\perp} + \vec{p}_{2,\perp}|$ where $p_{1,\perp}^\mu$ and $p_{2,\perp}^\mu$ are the lepton momenta.
By momentum conservation, $q_T$ is also equal to the net transverse momentum of all the hadronic final state particles.

For other observables, results about factorization are murkier. Two other variables we will discuss are

- **Hadronic transverse energy $E_T$.** Assuming all measured particles are massless, transverse energy is the scalar sum of the particles’ transverse momenta $E_T = \sum_j |\vec{p}_{jT}|$.

- **Beam thrust.** This is a hadronic event shape observable. For a process involving vector boson production ($pp \rightarrow V + X$), beam thrust defined in hadronic center-of-mass frame is $\tau_B = \frac{1}{Q} \sum_j |\vec{p}_{jT}| \exp(-|Y_j| - Y_V)$ where $Q$ and $Y_V$ are the vector-boson mass and rapidity and $\vec{p}_{jT}$ and $Y_j$ the transverse momentum and rapidity of the other final state particles [193, 194].

Some of these observables have been considered for some time and various observations have been made about them. $q_T$ was shown to factorize by CSS. For the other observables no rigorous results are known. The general lore is that factorization violation shows up in event generator simulations as sensitivity of an observable to the underlying event. [152, 190, 192, 195–197].

Consider for example beam thrust. In the original papers [193, 194, 198], beam thrust was thought to factorize. A rough argument along the lines of CSS about why the Glauber contributions should cancel was presented in [193]. Unfortunately, beam thrust is extremely sensitive to models of the underlying event; turning underlying event on or off in simulation has an order-one effect on the beam thrust distribution, completely destroying predictivity of the theoretical calculation [199]. Beam thrust was revisited by Gaunt [190] who argued that the pure Glauber contribution is factorization-violating. Similar conclusions were reached in [152].
Let us briefly review Gaunt’s argument. He first considered summing cuts of a diagram with a single Glauber exchange. There are two possible cuts.

![Diagram of two cuts](image)

The two cuts contribute to the cross section with opposite signs and the sum of them vanishes, independent of the observable. Next, he looked at diagrams with two Glauber gluons exchanged. There are three cuts through such graphs:

![Diagram of three cuts](image)

Gaunt argued that for the observables $E_T$ and beam thrust, the Glauber effects do not cancel when summing over these cuts and there is a factorization-violating effect. Gaunt showed that for one Glauber exchange diagram, one can achieve the cancellation by relabel the transverse component of integrated momentum. However, for two Glauber exchange, it was argued that no change of variables exist that can achieve the cancellation.

A similar argument to Gaunt’s can be found in Stewart and Rothstein’s treatise on Glauber gluons in SCET. These authors argued that when all the ladder graphs are summed, the result is a Glauber phase $\exp(i\phi(\vec{b}_\perp))$ with $\vec{b}_\perp$ the impact parameter conjugate to the relative transverse momentum of the two spectators $\Delta \vec{p}_\perp$. When one calculates the cross section there is an integral over this $\vec{b}_\perp$ as well as the $\vec{b}_\perp'$ for the complex-conjugate amplitude. The argument is that these phases do not cancel unless the observable is indepen-
dent of $\Delta \vec{p}_\perp$. This means that factorization is violated by the Glauber ladder diagrams for any observable other than $q_T$.

While both Gaunt and Rothstein/Stewart provide strong arguments, these arguments rely on assuming no unusual cancellations can happen. Their choices of variables are certainly suggestive that it would take a miracle for cancellation to happen. But miracles do happen when there are symmetries. In this paper, we show that a choice of variables that respects the scale invariance of the problem makes a general cancelation manifest.

Our main result is that the sum of all the ladder graphs is zero when integrated over the kinematic variables other than a single infrared safe observable:

$$
\int dz d\bar{z} \sum_{\text{cuts}} = 0 \quad (5.16)
$$

The variables $z$ and $\bar{z}$ that respect the symmetry of the scattering will be discussed in the next section.

The leading factorization violating effect comes from graphs with soft gluons exchanged between the Glauber rungs:

These graphs involve the Lipatov vertex – the 3-gluon vertex connected 2 Glauber gluons to a soft gluons [191]. The Lipatov vertex is currently known to one loop in QCD [200,201]. The Lipatov vertex is embedded in the Glauber operator in SCET in Eq. (5.7).
5.3 Glauber ladder graphs

In this section, we prove that summing over all Glauber ladder graphs, factorization is preserved. We start with the 2-loop result which demonstrates all the essential features of our argument. We then discuss the all-orders result.

5.3.1 2-loop Glauber cancellation

Consider the amplitude for Drell-Yan production from quark initial states with two outgoing gluons:

\[ M_{\text{h}}q(p_1) + \bar{q}(p_2) \rightarrow g(p_3) + g(p_4) + V \]  

(5.18)

At up to 2-loop order order, the diagrams that can produce factorization violating effects are

\[ M_0 = \frac{1}{2} \frac{3}{4} a b, \quad M_{G1} = \frac{1}{2} \frac{3}{4} k, \quad M_{G2} = \frac{1}{2} \frac{3}{4} k \]  

(5.19)

In these diagrams, the red dotted exchanges are Glauber gluons. The Feynman rules for these gluons can be found in [29] using SCET, or more simply by power expanding the amplitude in QCD using Glauber scaling: \( |\vec{k}_\perp| \gg k^+, k^- \). Glauber scaling makes the propagators for a momentum \( k^\mu \) depend only on \( \vec{k}_\perp \), i.e. \( k^2 = k^+ k^- - \vec{k}_\perp^2 \simeq -\vec{k}_\perp^2 \). Since the propagators only involve transverse momentum, they are more like contact interactions than propagators but we draw them as extended lines to exhibit their origin from diagrams in QCD.
The leading order matrix element is

$$\mathcal{M}_0 = \frac{1}{s_{13} s_{24}} N_{\mu \nu} p_{3,\perp}^\mu p_{4,\perp}^\nu$$  \hspace{1cm} (5.20)$$

We assume the outgoing gluons are collinear to the incoming quarks, so the tree-level matrix element is given at leading power by the product of splitting amplitudes:

$$\mathcal{M}_0 \approx \mathbf{S} p^0(1_q, 3_g) \mathbf{S} p^0(2_q, 4_g) \bar{\sigma}_n \Gamma u_n$$  \hspace{1cm} (5.21)$$

This leads to a simplified form for the matrix $N_{\mu \nu}$ in Eq. (5.20):

$$N_{\mu \nu} = \bar{\sigma}_n(p_2) \frac{d}{2} \int \left( \frac{2 \varepsilon_{4,\nu} + \ell_{4,\perp} \gamma_{\perp,\nu}}{n \cdot (p_2 - p_4)} \right) \Gamma \left( \frac{2 \varepsilon_{3,\mu}}{\bar{n} \cdot p_3} + \frac{\gamma_{\perp,\mu} \ell_{3,\perp}}{\bar{n} \cdot (p_1 - p_3)} \right) \frac{d}{2} u_n(p_1)$$  \hspace{1cm} (5.22)$$

By inserting the Glauber potential operators and integrating out the light-cone components of the Glauber momenta, the one-loop and two-loop Glauber ladder graphs reduce to the following integrals in the $d = 2 - 2\varepsilon$ transverse plane,

$$\mathcal{M}_{G1} = \left( -\frac{ig_s^2}{2} \right) (T_3 \cdot T_4) \frac{1}{s_{13} s_{24}} N_{\mu \nu}$$

$$\times \bar{p}_{3,\perp}^2 \bar{p}_{4,\perp}^2 \int \frac{d^{d-2} \ell}{(2\pi)^{d-2}} \frac{1}{(\bar{p}_{3,\perp} + \ell_{\perp})^2} \frac{1}{(\bar{p}_{4,\perp} - \ell_{\perp})^2} \frac{1}{\ell_{\perp}^2} (p_{3,\perp}^\mu + \ell_{\perp}^\mu) (p_{4,\perp}^\nu - \ell_{\perp}^\nu)$$  \hspace{1cm} (5.23)$$

where $s_{ij} = (p_i + p_j)^2$ and

$$\mathcal{M}_{G2} = \left( -\frac{g_s^4}{8} \right) (T_3 \cdot T_4)^2 \frac{1}{s_{13} s_{24}} N_{\mu \nu}$$

$$\times \bar{p}_{3,\perp}^2 \bar{p}_{4,\perp}^2 \int \frac{d^{d-2} \ell}{(2\pi)^{d-2}} \frac{d^{d-2} k}{(2\pi)^{d-2}} \frac{1}{(\bar{p}_{3,\perp} + \ell_{\perp})^2} \frac{1}{(\bar{p}_{4,\perp} - k_{\perp})^2} \frac{1}{(k_{\perp} - \ell_{\perp})^2} \frac{1}{k_{\perp}^2} (p_{3,\perp}^\mu + \ell_{\perp}^\mu) (p_{4,\perp}^\nu - \ell_{\perp}^\nu)$$  \hspace{1cm} (5.24)$$
where \( s_{ij} = (p_i + p_j)^2 \), and we define the momentum of all particle as incoming. Outgoing momentum is obtained by crossing. We have computed the amplitudes \( M_0, M_{G1} \) and \( M_{G2} \) in \( d \) dimensions as a series in \( \epsilon = \frac{4-d}{2} \), using some master integrals from [202]. Details are given in the appendices and will be summarized here.

In computing the matrix-element squared, we find that up to 2-loop order the IR divergences exactly cancel. This cancellation is non-trivial, but also expected from general results. Of the 8 possible helicity combinations, only 2 are independent. If both pairs of collinear quark and gluon have the same (or opposite) helicity, such that \( h_3 = \pm h_1, h_4 = \pm h_2 \), then we find (see Appendix ??)

\[
|M|^2_{++} \equiv |M_0 + M_{G1} + M_{G2} + \cdots |^2_{h_3 = \pm h_1, h_4 = \pm h_2}
= |M_0(++,+,\pm,\pm)|^2 \left[ 1 + \frac{c_{2G}}{c_0} \frac{\alpha_s^2}{8} (u + v + 1) \ln^2 \frac{u}{v} + \mathcal{O}(\alpha_s^3) \right]
\] (5.25)

If one pair of quark and gluon have the same helicity, the other pair have the opposite helicity, such that \( h_3 = \pm h_1, h_4 = \mp h_2 \), then

\[
|M|^2_{+-} \equiv |M_0 + M_{G1} + M_{G2} + \cdots |^2_{h_3 = \pm h_1, h_4 = \mp h_2}
= |M_0(+-,+,+,+)|^2 \left[ 1 + \frac{c_{2G}}{c_0} \frac{\alpha_s^2}{4} \ln u \ln v + \mathcal{O}(\alpha_s^3) \right]
\] (5.26)

where the color sum is implicit in \( |M|^2 \). The 2-loop and tree-level color factors are \( c_{2G} = \frac{1}{8} C_A^2 (C_A^2 + 2) C_F, c_0 = C_A C_F^2 \), and

\[
|M_0(h_1, h_2, h_3, h_4)|^2 = c_0 \frac{1}{q_T^4 uv} \frac{s_{23} s_{14}}{s_{12}^2} \left| \text{Split}_{-h_1 h_3} \left( \frac{1}{z_{13}} \right) \right|^2 \left| \text{Split}_{-h_2 h_4} \left( \frac{1}{z_{24}} \right) \right|^2,
\] (5.27)

with

\[
z_{13} \equiv \frac{\bar{n} \cdot (p_1 + p_3)}{\bar{n} \cdot p_1} < 1, \quad \text{and} \quad z_{24} \equiv \frac{n \cdot (p_2 + p_4)}{n \cdot p_2} < 1.
\] (5.28)
\[ |\text{Split}^{++}(z_{ij})|^2 = 8s^2 \frac{1}{1-z_{ij}}, \quad |\text{Split}^{+-}(z_{ij})|^2 = 8s^2 \frac{z_{ij}^2}{1-z_{ij}}, \]  

These results are expressed in terms of the dimensionful variable \( q_T = |\vec{q}_\perp| \) and two dimensionless variables \( u \) and \( v \). These are defined as

\[ \vec{q}_\perp \equiv \vec{p}_{3,\perp} + \vec{p}_{4,\perp}, \quad u \equiv \frac{\vec{p}_{3,\perp}^2}{q_T^2}, \quad v \equiv \frac{\vec{p}_{4,\perp}^2}{q_T^2}. \]  

At fixed \( q_T \), the collinear limits are \( u \to 0, v \to 1 \) or \( v \to 0, u \to 1 \).

To proceed, let’s consider how to integrate over \( d^2 p_{3,\perp} \), or equivalently \( u \) and \( v \). Since \( \vec{q}_\perp \equiv \vec{p}_{3,\perp} + \vec{p}_{4,\perp} \), the three vectors form a triangle in the transverse plane. We can rotate this triangle so that \( \vec{q}_\perp \) is conveniently oriented along the real axis, and rescale out \( q_T \), leading to a simpler triangle defined by one point \( z \) in the complex plane.

The relation between \( u, v \) and \( z \) is then

\[ u = z \bar{z}, \quad v = (1 - z)(1 - \bar{z}) \]  

Using \( z \) and \( \bar{z} \) facilitates integrating over \( d^2 p_{3,\perp} \). Explicitly,

\[ d^2 p_{3,\perp} = dp_{3,x} dp_{3,y} = q_T^2 d \text{Re}(z) d \text{Im}(z) = q_T^2 d^2 z \]  

The phase space integrals then become regular two dimensional integral over conformal
coordinates.

\[ q_T^2 \int \frac{d^2p_{3\perp}}{(2\pi)^2} |\mathcal{M}|^2_{l_1l_2} \equiv c_2G \frac{s^{23}s^{14}}{s^{12}} \left|\text{Split}(z_{13}^{-1})\right|^2 \left|\text{Split}(z_{24}^{-1})\right|^2 \left(\frac{\alpha_s}{4\pi}\right)^2 I_{l_1l_2}^{\text{reg}} \]  (5.33)

Explicitly, for \(|\mathcal{M}|^2_{++}\), the relevant integral is (see Appendix ??)

\[ I_{\text{reg}}^{++} \equiv \int d^2z \, \frac{1}{uv} \frac{u + v - 1}{2} \ln^2 \left(\frac{u}{v}\right) = 4\pi\zeta_3 \]  (5.34)

For \(|\mathcal{M}|^2_{+-}\), the relevant integral is

\[ I_{\text{reg}}^{+-} \equiv \int d^2z \, \frac{1}{uv} \ln u \ln v = 4\pi\zeta_3 \]  (5.35)

Intriguingly both integrals give the same result. This result is non-zero, and not power suppressed as \(q_T \to 0\) so it would seem to indicate factorization violation.

If this were the end of the story, we would have found a non-zero factorization-violating contribution to any observable, even \(q_T\) for which factorization is proven to hold. In fact, there is another piece contributing to the cross section. The term of order \(\epsilon^1\) in \(|\mathcal{M}|^2\) has the form

\[ |\mathcal{M}|^2 = c_2G \frac{\alpha_s^2}{c_0} \frac{2\epsilon}{\hat{q}_T^2} e^{2\epsilon\gamma_E c_T^2} \left\{ |\mathcal{M}_0|^2 (uv)^{-2\epsilon} [6\zeta_3 + \mathcal{O}(\epsilon)] + \text{integrable} \right\} + \cdots \]  (5.36)

where \(\hat{\mu}^2 \equiv 4\pi\mu^2\), and \(c_T = \frac{\Gamma(1-\epsilon)^2\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)}\). Due to the factor \(\frac{1}{uv}\) in \(|\mathcal{M}_0|^2\) in Eq. (5.27), the first term in braces is singular when integrated over 4-dimensional phase space. Thus in \(d\) dimensions it will give a \(\epsilon^{-1}\) factor which cancels the \(\epsilon^1\) prefactor giving an \(\mathcal{O}(\epsilon^0)\) contribution. The remaining terms in the braces are integrable over phase space thus do not contribute as \(\epsilon \to 0\). The singular term is independent of helicity, so we leave the helicity labels implicit.
Performing the integral over the singular piece in \( d \) dimensions, we find

\[
\mu^2 q_T^2 \int \frac{d^{d-2} p_{3,\perp}}{(2\pi)^{d-2}} |\mathcal{M}|^2 = c_2 G^{S_{23} S_{14}} s^2_{12} |\text{Split}(z^{-1}_{13})|^2 |\text{Split}(z^{-1}_{24})|^2 \\
\times \left( \frac{\alpha_s}{4\pi} \right)^2 \epsilon \left\{ 6 \zeta_3 I_{\text{sing}}(\epsilon) + \mathcal{O}(\epsilon^0) \right\} \tag{5.37}
\]

where

\[
I_{\text{sing}}(\epsilon) \equiv \pi e^{2\epsilon} \gamma e^2 q_T^2 (\mu^2)^{1+2\epsilon} p_{3,\perp} \int \frac{d^{2-2\epsilon} p_{3,\perp}}{\pi^{1-\epsilon}} \frac{1}{[\tilde{q}_T^2 p_{3,\perp}^2]^{1+2\epsilon}} \Gamma(1+4\epsilon) \Gamma(-6\epsilon) \Gamma^2(1+2\epsilon) \tag{5.38}
\]

\[
= \pi e^{2\epsilon} \gamma e^2 q_T^2 \left( \frac{\mu}{q_T} \right)^{3\epsilon} \frac{\Gamma^2(-3\epsilon) \Gamma(1+4\epsilon)}{\Gamma(-6\epsilon) \Gamma^2(1+2\epsilon)} \tag{5.39}
\]

\[
= -\frac{2\pi}{3\epsilon} + \mathcal{O}(\epsilon^0) \tag{5.40}
\]

When added to the contributions from \( \mathcal{O}(\epsilon^0) \) in \(|\mathcal{M}|^2\), Eqs. (5.33) and (5.34)-(5.35), we see that the net contribution is zero:

\[
q_T^2 \left. \frac{d\sigma}{dq_T^2} \right|_{2\text{-loop \text{Glauber}}} = c_0 \mu^2 e^2 q_T^2 \int \frac{d^{d-2} p_{3,\perp}}{(2\pi)^{d-2}} \left( |\mathcal{M}|^2_{1,2} + |\mathcal{M}|^2_{1,2} \right) = 0 + \mathcal{O}(\epsilon) \tag{5.41}
\]

Thus at 2 loops, the Glauber ladder graphs do not generate a factorization-violating effect for \( q_T \).

What changes if we use a different observable, like \( E_T \) or beam thrust? We know by dimensional analysis that \( q_T^{4+4\epsilon} |\mathcal{M}|^2_{2\text{-loop}} \) is dimensionless, depending only on \((z, z)\), thus we define

\[
|\tilde{\mathcal{M}}|^2_{2\text{-loop}}(z, \bar{z}) \equiv q_T^4 \left( \frac{q_T}{\mu} \right)^{4\epsilon} |\mathcal{M}|^2_{2\text{-loop}} \tag{5.42}
\]

\(^1\)Although we extend the integration region outside of the small-\( p_T \) region to establish the cancellation, this is exactly what is required by the effective field theory. The region where \( p_T \) is not small has no Glauber pinch and is correctly described by a factorized expression.
and a rescaled transverse momentum \( \vec{p}_{3,\perp}^\mu \) so that

\[
|\vec{p}_{3,\perp}|^2 = |z|^2, \quad |1 - \vec{p}_{3,\perp}|^2 = |1 - z|^2
\]  

(5.43)

Working with \( q_T \) and \( \vec{p}_{3,\perp}^\mu \) as the independent variables, Eq. (5.41) can be written as the sum of the following two integrals canceling each other at \( \mathcal{O}(\epsilon^0) \),

\[
\frac{1}{2} \frac{d\sigma}{d\ln q_T} \bigg|_{\text{2-loop Glauber}} = \sigma_0 \int \frac{d^2z}{(2\pi)^2} |\tilde{M}|^2_{j_1,j_2}(z,\bar{z}) + \sigma_0 \int \frac{d^{d-2}\vec{p}_{3,\perp}^\mu}{(2\pi)^{d-2}} \left( \frac{\mu}{q_T} \right)^{6\epsilon} |\tilde{M}|^2_{\epsilon}(z,\bar{z}) + \mathcal{O}(\epsilon)
\]  

(5.44)

Then we can change variables from \( q_T \) to \( E_T \) easily. For \( 2 \rightarrow 2 \) scattering

\[
E_T = |\vec{p}_{3,\perp}| + |\vec{p}_{4,\perp}| = (\sqrt{u} + \sqrt{v})q_T = (|z| + |1 - z|)q_T
\]  

(5.45)

Thus, noting that the Jacobian \( \frac{\partial \ln q_T}{\partial \ln E_T} = 1 \),

\[
\frac{1}{2} \frac{d\sigma}{d\ln E_T} \bigg|_{\text{2-loop Glauber}} = \sigma_0 \int \frac{d^2z}{(2\pi)^2} |\tilde{M}|^2_{j_1,j_2}(z,\bar{z}) + \sigma_0 \left( \frac{\mu}{E_T} \right)^{6\epsilon} \int \frac{d^{d-2}\vec{p}_{3,\perp}^\mu}{(2\pi)^{d-2}} (|z| + |1 - z|)^{6\epsilon} |\tilde{M}|^2_{\epsilon}(z,\bar{z}) + \mathcal{O}(\epsilon)
\]  

(5.46)

The integration over \( z \) and \( \bar{z} \) in the first integral is unaffected by the change of variables. The second integral is changed by a factor of \( (|z| + |1 - z|)^{6\epsilon} \). However, the behavior of the integrand in the singular regions \( z \rightarrow 0 \) and \( z \rightarrow 1 \) is unchanged. This new factor only contributes at order \( \mathcal{O}(\epsilon) \). Therefore the cancellation between the two integrals still holds at \( \mathcal{O}(\epsilon^0) \). The key point is that the inclusive integration over \( z \) and \( \bar{z} \) is observable independent. A more detailed discussion of the change of variables is given in Appendix ??.
For another example, consider beam thrust. For 2 → 2 scattering,

\[
\ln \tau_B = 2 \ln q_T + \ln \left( \frac{|z|^2}{(z_{13}^{-1} - 1)q^+} + \frac{|1 - z|^2}{(z_{24}^{-1} - 1)q^-} \right)
\]

(5.47)

where \( q^\mu = (q^+, q^-, q_T) \) is the total momentum of the lepton pair. Changing from \( q_T \) to \( \tau_B \) results in an expression similar to Eq. (5.46). The integration around the singular region again only contributes new terms that start at \( \mathcal{O}(\epsilon) \), and thus there is no factorization-violating effect from the 2-loop Glauber ladder graphs.

The same argument holds for any infrared-safe single transverse observable \( X \). Any such observable must be expressible as

\[
\ln X = a \ln q_T + g(z, \bar{z})
\]

(5.48)

for some \( a > 0 \) and some function \( g(z, \bar{z}) \) that is regular as \( z \to 0 \) and \( z \to 1 \). If \( g(z, \bar{z}) \) were not regular in the collinear limits, or if \( a \leq 0 \), then the observable cannot be infrared safe. Thus the cancellation holds for any infrared safe observable \( X \).

### 5.3.2 All-orders Glauber cancellation

We showed that at 2-loop order, the Glauber ladder diagrams alone do not violate factorization for any single scale observable. Now we will show that the cancellation we found persists to all orders.

By direct calculation, we find that to all orders in perturbation theory, the sum of Glauber ladder diagrams has the form

\[
\mathcal{M}_G(\{h_i\}) = e^{i\Phi_G(\bar{a}_s(T_3 \cdot T_4))} \left[ 1 + f_{\text{reg}}^{\{h_i\}}(z, \bar{z}, \frac{\alpha_s}{2}(T_3 \cdot T_4)) + \epsilon f_{\text{sing}}^{\{h_i\}}(\bar{a}_s(T_3 \cdot T_4)) + \cdots \right] \mathcal{M}^0(\{h_i\})
\]

(5.49)
where the \( \cdots \) are terms that do not contribute to the cross section. The combination

\[
\hat{\alpha}_s(\mu) \equiv \frac{\alpha_s(\mu)}{2} c_\Gamma e^{\gamma_E} (q_T^2 u v)^{-\varepsilon}
\]  

(5.50)

with \( c_\Gamma = \frac{\Gamma(1-\varepsilon)^2 \Gamma(1+\varepsilon)}{\Gamma(1-2\varepsilon)} \) appears naturally and the phase is

\[
\mathcal{E}_g(\alpha) \equiv \alpha \left( \frac{1}{\varepsilon} + 2 \gamma_E \right) - i \ln \frac{\Gamma(1 + i\alpha)}{\Gamma(1 - i\alpha)}
\]

(5.51)

A key property of the expression in Eq. (5.49) is that all of the infrared divergences are contained in the phase. This is shown in Appendix ???. Another property is that once the phase is factored out, the order \( \varepsilon^0 \) term denoted \( f_{\text{reg}} \) is integrable over the collinear regions. This decomposition and the calculation of \( \mathcal{E}_g(\hat{\alpha}_s) \) are given in Appendix ???. The expression for \( f_{\text{reg}} \) at 2-loops, which appeared in Eqs. (5.25) and (5.26), is computed in Appendix ???.

The leading non-integrable piece we call \( f_{\text{sing}} \). It can be extracted from a general formula we derive for the singular part of \( \mathcal{M}_G \). At order \( \alpha_s^n \), this singular part is

\[
\mathcal{M}^{n}_{G,\text{sing}} = \frac{1}{n!} \left( \frac{-i\alpha_s}{2} \right)^n (T_3 \cdot T_4)^n e^{n\varepsilon \gamma_E} (q_T^2)^{-n\varepsilon} (uv)^{-n\varepsilon} \left[ \Gamma(-\varepsilon) \right]^n \Gamma(1 - \varepsilon) \Gamma(1 + n\varepsilon) \Gamma(1 - (n + 1)\varepsilon) \mathcal{M}_0
\]

(5.52)

Expanding \( e^{-i\phi_g(\hat{\alpha}_s)} \mathcal{M}^{n}_{G,\text{sing}} \) gives

\[
e f_{\text{sing}}(\alpha; \varepsilon) = (3\zeta_3 \varepsilon + \frac{9}{2} \zeta_4 \varepsilon^2 + \cdots) \alpha^2 + (\zeta_4 \varepsilon - 16 \zeta_5 \varepsilon^2 + \cdots) i\alpha^3 + (-5 \zeta_5 \varepsilon + \cdots) \alpha^4 + \mathcal{O}(\alpha^5)
\]

(5.53)

The first term in this expansion was used in Eq. (5.36).
Since all of the IR divergences cancel in $|\mathcal{M}|^2$, the cross section has the form

$$\left.\frac{d\sigma}{d\ln q_T}\right|_{\text{Glauber ladders}} = \int dz d\bar{z} F_{\text{reg}}(z, \bar{z}) + e \frac{1}{q_T^2} \left( \frac{\mu}{q_T} \right)^{2(n+1)e} \int d^{d-2} p_{3,\perp} F_{\text{sing}}(z, \bar{z}) = 0$$

(5.54)

In Section 5.3.1 we showed that at two loops both of these terms are separately non-zero and only their sum vanishes. For observable $q_T$, this must also be true to all orders as a result of CSS proof of factorization for $q_T$.

Now say we change variables from $\ln q_T$ to $\ln E_T$, using Eq. (5.45). The first integral in Eq. (5.54) is unaffected since the integrand depends only on the conformal coordinates. Changing $q_T$ into $E_T$ also has no effect on the singular behavior of the second integral, since $E_T \to q_T$ in the collinear limits. Thus both integrals give the same result at $O(e^0)$, and there is no factorization-violating effect from Glauber ladders for $E_T$, to all orders. The same argument holds for any infrared-safe single observable $X$.

### 5.4 Factorization-violating effects

Does $q_T$ factorization imply factorization for other observables to all orders? The answer is, not surprisingly, no. The scaling arguments we have been using only go so far. They do however give some indications of how factorization violation can show up.

Non-ladder potentially factorization-violating contributions to the cross section involve soft emission from the Glauber legs. For example, virtual or real contributions could look like

$$\mathcal{M}_V = \quad \mathcal{M}_R = \quad (5.55)$$

243
In this section we will focus on the potentially factorization-violating contributions from graphs with soft gluon exchanges between the Glauber ladders. Similar graph has also been briefly discussed in Ref. [190].

For graphs like these, there is neither proof nor expectation that the IR divergences should resum into a phase. Instead, we expect there to be IR divergences in $|\mathcal{M}|^2$ that cancel only when the phase space integral is done. Let us denote the cross section from all graphs with Glaubers as $\sigma_G$. Quite generally, we can write

$$\frac{d\sigma_G}{d\ln q_T} = \int dudv \ldots dw f(q_T, u, v, w, \ldots)$$  \hspace{1cm} (5.56)

with $q_T$ the only dimensionful variable and the $u, v, w, \ldots$ variables dimensionless. For an IR or rapidity divergent contribution, another scale $\mu$ or $\nu$ can appear. This scale can only appear logarithmically, so the $q_T$ dependence must be logarithmic and we can write a series expansion

$$\frac{d\sigma_G}{d\log q_T} = \int dudv \ldots dw \left[ f_0(u, v, \ldots, w) + \ln \frac{q_T}{\mu} f_1(u, v, w, \ldots) + \ln^2 \frac{q_T}{\mu} f_2(u, v, \ldots, w) + \ldots \right]$$  \hspace{1cm} (5.57)

Some of these integrals may be IR divergent. We know however, that the final result must be IR finite, so the IR divergences must cancel among the various contributions. Moreover, we know by $q_T$-factorization that the sum of all of these contributions is exactly zero. Since each term multiplies a different power of $\ln q_T$, each one separately must integrate to zero.

Now consider the cross section for $E_T$ rather than $q_T$. For $2 \to 2$ scattering, $E_T = q_T(\sqrt{u} + \sqrt{v})$. So $\ln q_T = \ln E_T + \ln(\sqrt{u} + \sqrt{v})$. For graphs with more final-state gluons, we would have a more complicated function but still a linear relation between $\ln q_T$ and
\ln E_T. So let us write

\[ \ln q_T = \ln E_T + g(u, v, \ldots w) \tag{5.58} \]

Other single scale variables, like beam thrust, will have a similarly linear relation with a different function \( g \). Then

\[
\frac{d\sigma}{d\log E_T} = \int dudv \ldots dw \left[ f_0(u, v, \ldots, w) \ln \frac{E_T}{\mu} f_1(u, v, w, \ldots) \\
+ g(u, v, \ldots, w)f_1(u, v, \ldots, w) + 2\ln \frac{E_T}{\mu} g(u, v, \ldots, w)f_2(u, v, \ldots, w) + \cdots \right] \tag{5.59}
\]

The terms on the first line vanish by \( q_T \) factorization, but the terms on the second may not.

### 5.4.1 Observable dependence

Let us now put a little more detail into Eq. (5.59). We discuss two types of observables for which factorization may not hold

1. Non-global observables that do not include the collinear region, so that the real soft gluon phase space is not fully integrated over.

2. Global observables that are rapidity-independent in the collinear region.

The first category includes non-global observables only sensitive to soft emissions within some region of the detector, but inclusive over the collinear particles along the beam directions. An example of such an observable is the mass of the hardest jet. The second category includes global hadronic event shapes, like \( E_T \) or transverse thrust.

To understand the factorization breaking effects due to the these two types of measurements, let us look at diagrams with one soft gluon as an example. The leading order
real-emission diagram has a single soft gluon coming off of a Glauber line:

\[
M_R^1 = \quad (5.60)
\]

This diagram contributes to the cross section at order \( g_s^6 |\mathcal{M}_0|^2 \). Virtual diagrams that contribute at this same order are the square of 1-loop graphs with a Glauber gluon and a single soft loop:

\[
M_V^1 = \quad (5.61)
\]

There are also 2-loop graphs with two Glauber gluons relevant at this order:

\[
M_V^2 = \quad (5.62)
\]

These contribute at order \( g_s^6 |\mathcal{M}_0|^2 \) as well as through interference with the tree-level graphs \( \mathcal{M}_0 \). Note that there are no graphs where a soft gluon connects to a collinear line – collinear fields only interact with ultrasoft gluons at leading power.

First, let’s consider observables in class 1, that are inclusive over the beam. Let’s call the observable \( X \) and the measurement function on the emitted radiation \( f_X(k) \). For these observables we know that if we were inclusive over everything, all the Glauber graphs would exactly cancel, like for \( q_T \). So the uncanceled part is only due to the real emission in the measured region of phase space. Thus we only need to consider Lipatov diagrams, like \( M_R^1 \) with real soft emissions. The leading factorization violating effect can then be
computed by
\[
\frac{d\hat{\sigma}_G}{dX} = \int d^2q_T d^2p_{3,\perp} \int_\Omega d^4k 2\pi\delta(k^2) |\mathcal{M}_k^2|^2 \delta(f_X(k) - X) \tag{5.63}
\]

where \(\Omega\) is the area of phase space of the observable. In the limit that the angular region \(A = \Delta y \Delta \phi\) in phase space is small, the real emission amplitude is going to be independent of rapidity \(y\) and azimuthal angle \(\phi\). Then we expect the factorization violating effect to be proportional to the area and integrand to only depend on the transverse momentum of the emitted gluon. Thus,

\[
\frac{d\hat{\sigma}_G}{dX} \sim A \int d^2q_T d^2p_{3,\perp} d^2k_\perp \gamma_{\text{Lip}}(q_T, p_{3,\perp}, k_\perp) \delta(f_X(k_\perp) - X) \tag{5.64}
\]

where \(\gamma_{\text{Lip}}\) is given by

\[
\gamma_{\text{Lip}}(q_T, p_{3,\perp}, k_\perp) = N_{\mu\nu} N_{\rho\sigma} \int \frac{d^{d-2}\ell_{1,\perp} d^{d-2}\ell_{2,\perp}}{(2\pi)^{d-2}} \frac{(p_{3,\perp}^\mu + \ell_{1,\perp}^\mu)}{(p_{3,\perp} + \ell_{1,\perp})^2} \frac{(q_T^\nu - p_{3,\perp}^\nu - \ell_{1,\perp}^\nu)}{(q_T - p_{3,\perp} - \ell_{1,\perp})^2} \frac{(p_{3,\perp}^\rho + \ell_{2,\perp}^\rho)}{(p_{3,\perp} + \ell_{2,\perp})^2} \frac{(q_T^\rho - p_{3,\perp}^\rho - \ell_{2,\perp}^\rho)}{(q_T - p_{3,\perp} - \ell_{2,\perp})^2} \times \frac{1}{\ell_{1,\perp}^2 \ell_{2,\perp}^2} \left[ \frac{2\ell_{1,\perp}^2}{(\ell_{1,\perp} - \ell_{2,\perp})^2 (\ell_{2,\perp} + \ell_\perp)^2} + \frac{2\ell_{2,\perp}^2}{(\ell_{1,\perp} + \ell_{2,\perp})^2 (\ell_{1,\perp} - \ell_\perp)^2} \right] \tag{5.65}
\]

Holding \(k_\perp\) fixed, the phase-space integral \(\int d^2q_T d^2p_{3,\perp} \gamma_{\text{Lip}}(q_T, p_{3,\perp}, k_\perp)\) is IR finite. The \(k_\perp \to 0\) limit is not allowed by the measurement function. Therefore Eq. (5.64) will give us a positive finite number proportional to the area \(A\) and to \(\alpha_s^4\).

Now let’s proceed with the second type of observable, which measures particles emitted in all rapidity regimes. In particular, we will assume that the observable is independent of the rapidity of soft or collinear particles but only sensitive to their transverse momenta. The measurement function acting on two-body and three-body final state can
be written as

\[ f_X(\vec{p}_{3,\perp}, \vec{q}_{\perp} - \vec{p}_{3,\perp}) = |q_T|^a f_X(z, \bar{z}) \]  
(5.66)

\[ f_X(\vec{p}_{3,\perp}, \vec{k}_{\perp}, \vec{q}_{T} - \vec{p}_{3,\perp} - \vec{k}_{\perp}) = |q_T|^a f_X(z, \bar{z}, w, \bar{w}) \]  
(5.67)

where \(z, w\) are complex dimensionless variables defined by

\[ zz \equiv \frac{p_{3,\perp}^2}{q_T^2}, \quad w\bar{w} \equiv \frac{k_{\perp}^2}{q_T^2}, \quad (1 - z - w)(1 - \bar{z} - \bar{w}) \equiv \frac{p_{4,\perp}^2}{q_T^2} \]  
(5.68)

By infrared safety, \(f_X(z, \bar{z})\) and \(f_X(z, \bar{z}, w, \bar{w})\) have the following properties,

\[ \lim_{z \to 0} f(z, \bar{z}) = 1, \quad \lim_{w \to 0} f(z, \bar{z}, w, \bar{w}) = f(z, \bar{z}), \]  
(5.69)

Unlike the non-global observables, here all the diagrams contribute to \(\frac{d\sigma}{dX}\). Since the diagrams have entangled virtual and real IR divergences in different regimes, it is hard to see directly whether \(\frac{d\sigma}{dX}\) is non-vanishing. The way we will deal with it is to add and subtract a counterterm, which is similar to the idea of subtraction. We do this by defining a new measurement function \(f'\) that is not sensitive to soft radiation collinear to the spectators. Then

\[ f'_X(\vec{p}_{3,\perp}, \vec{k}_{\perp}, \vec{q}_{T} - \vec{p}_{3,\perp} - \vec{k}_{\perp}) = f_X(\vec{p}_{3,\perp}, \vec{q}_{T} - \vec{p}_{3,\perp}) \]  
(5.70)

For 2-body final states \(f'_X = f_X\).

Now we can write down the cross section as the sum of two terms

\[ \frac{d\sigma}{d\ln X} = \left( \frac{d\sigma}{d\ln X} - \frac{d\sigma'}{d\ln X} \right) + \frac{d\sigma'}{d\ln X} \]  
(5.71)
By $\sigma'$ we mean the cross section computed with $f'_X$ instead of $f_X$. The first term only acts non-trivially on wide-angle soft real emission diagrams, where the divergence as $k_{\perp}$ goes to zero are cured by the measurement function. To compute the second term, we can first integrate inclusively over soft momentum $k$, after which soft real and virtual divergences cancel. Then the integrand contains only finite integrable functions in two-body phase space. In the following we will show in detail that either term in Eq. (5.71) could be non-vanishing.

Let’s start with the first term. To better understand the behavior of this integral, especially how it depends on the measurement function, we need to study the scaling behavior of the squared amplitude. Since the measurement is independent of the soft gluon rapidity, integrating out its light-cone component give us a rapidity divergence. Regulating this divergence with a scale $\nu$ gives a rapidity log term and breaks scale invariance of the integral in the transverse plane. Thus, we we can no longer prove the cancellation using scale invariance as we did for Glauber ladder diagrams., even for a single-scale observable. In order to show the failure of cancellation, we only need to keep track of the rapidity logs. Expanding the matrix element, we see

$$|\mathcal{M}|^2_{\text{Lip}}(q_T, p_{3,\perp}, k_{\perp}) = \int \frac{dk^0 dk^z}{(2\pi)^2} |\mathcal{M}|^2_{\text{R}} \text{regulated} = \left(\frac{2}{\eta} + \ln \frac{\nu^2}{k_{\perp}^2}\right) \gamma_{\text{Lip}}(q_T, p_{3,\perp}, k_{\perp}) + \eta\text{-finite}. \quad (5.72)$$

Note that the $k_z$ integral in Eq. (5.72) leads to rapidity divergence. We have used the rapidity regulator of Ref. [49, 176] to regularize it, which leads to $1/\eta$ divergence in Eq. (5.72).

The rapidity scale $\nu$ in Eq. (5.72) is similar to the $\mu$ in dimensional regularization. This determines the scaling behavior of the squared amplitude

$$|\mathcal{M}|^2_{\text{Lip}}(q_T, p_{3,\perp}, k_{\perp}) = \frac{1}{q_T^8} \left(-\ln q_T^2 \tilde{\gamma}_{\text{Lip}}(z, \bar{z}, w, \bar{w}) + O(\epsilon) + |\tilde{\mathcal{M}}|^2_{\text{Lip}}(z, \bar{z}, w, \bar{w})\right), \quad (5.73)$$

249
where we use $\tilde{\gamma}_{\text{Lip}}$ and $\tilde{\mathcal{M}}_{\text{Lip}}$ to denote the dimensionless version of $\gamma_{\text{Lip}}$ and $\mathcal{M}_{\text{Lip}}$ by dividing with appropriate power of $q_T$. The second term in Eq. (5.73) respects scale invariance and therefore insensitive to the difference in measurement functions. The first term in Eq. (5.73) is of the form anticipated in Eq. (5.57). It will give us a non-vanishing integral

$$
\frac{d\sigma'}{d\ln X} - \frac{d\sigma}{d\ln X} = \int \frac{d^2q_T d^2p_{3,\perp} d^2k_{\perp}}{(2\pi)^6} |\mathcal{M}|^2_{\text{Lip}}(q_T, p_{3,\perp}, k_{\perp}) \left[ \delta[f_X(q_T, p_{3,\perp}, k_{\perp}) - X] - \delta[f'_X(q_T, p_{3,\perp}) - X] \right]
$$

$$
(5.74)
$$

$$
\mathcal{O}(\epsilon) \int d^2z d^2w \gamma_{\text{Lip}}(z, \bar{z}, w, \bar{w}) \int d\ln q_T (-\ln q_T^2) \left[ \delta(f_X - X) - \delta(f'_X - X) \right]
$$

$$
= \int d^2z d^2w \gamma_{\text{Lip}}(z, \bar{z}, w, \bar{w}) \frac{2}{a} \ln \frac{f_X(z, \bar{z}, w, \bar{w})}{f_X(z, \bar{z})}
$$

$$
(5.75)
$$

Now let’s move onto the second term in Eq. (5.71). In order to compute $\frac{d\sigma'}{dX}$, let us take the Lipatov diagram and fully integrate out the real soft momentum $k$, then add it to the virtual diagrams. Doing so allow us to write down a squared amplitude $|\mathcal{M}_V|^2_{\text{inc}}$, which corresponds to the sum of all four-loop cut diagrams with fixed $q_T$ and $p_{3,\perp}$,

$$
|\mathcal{M}_V|^2_{\text{inc}}(q_T, p_{3,\perp}) \equiv \mathcal{M}_V^2(\mathcal{M}^0)^* + \mathcal{M}^0(\mathcal{M}_V^2)^*
$$

$$
+ \mathcal{M}_V^1(\mathcal{M}_V^1)^* + \int \frac{d^d k}{(2\pi)^d} 2\pi \delta(k^2) \mathcal{M}_R^1(\mathcal{M}_R^1)^*
$$

$$
(5.76)
$$

Then the cross section becomes a two-body phase-space integral over $|\mathcal{M}_V|^2_{\text{inc}}$

$$
\frac{d\sigma'}{d\ln X} = \int \frac{d^2q_T d^2p_{3,\perp}}{(2\pi)^4} |\mathcal{M}_V|^2_{\text{inc}}(q_T, p_{3,\perp}) \delta[f'(q_T, p_{3,\perp}) - X]
$$

$$
(5.77)
$$
Focusing on the divergent terms, we are able to determine the scaling behaviour of $|M_V|_{inc}^2$, which takes the following form

$$
|M_V|_{inc}^2(q_T, p_{3,\perp}) = \frac{1}{q_T^4} \left[ \left( \frac{1}{2} \ln^2 q_T^2 - \ln \frac{v^2}{\ln q_T^2} \right) \Gamma_{inc}(z, \bar{z}) - \ln q_T^2 \gamma_{inc}(z, \bar{z}) + O(\epsilon) + |\tilde{M}_V|_{inc}^2(z, \bar{z}) \right]
$$

(5.78)

where $\Gamma_{inc}$ and $\gamma_{inc}$ are determined by integrals in transverse dimensions over the soft and glauber momenta. Again, the cross section with measurement $f'_X$ can be written as its difference with measurement $q_T$,

$$
\frac{d\sigma}{d\ln X} - \frac{1}{a} \frac{d\sigma}{d\ln q_T} = \frac{2}{a^2} \int d^2z \Gamma_{inc}(z, \bar{z}) \ln^2 f(z, \bar{z})
+ \frac{2}{a} \int d^2z \left[ \ln \frac{v^2}{X^2} \Gamma_{inc}(z, \bar{z}) + \gamma_{inc}(z, \bar{z}) \right] \ln f(z, \bar{z})
$$

(5.79)

Putting the pieces together

$$
\frac{d\sigma}{d\ln X} \sim \frac{2}{a} \ln \frac{v^2}{X^2} \int d^2z \Gamma_{inc}(z, \bar{z}) \ln f(z, \bar{z}) + \frac{2}{a^2} \int d^2z \Gamma_{inc}(z, \bar{z}) \ln^2 f(z, \bar{z})
+ \frac{2}{a} \int d^2z \left[ \gamma_{inc}(z, \bar{z}) \ln f(z, \bar{z}) + \int d^2w \tilde{\gamma}_{Lip}(z, \bar{z}, w, \bar{w}) \ln \frac{f(z, \bar{z}, w, \bar{w})}{f(z, \bar{z})} \right]
$$

(5.80)

Generically we expect the integrals above not to vanish. The $v$ dependence here should cancel with contributions from Glauber diagrams with one additional collinear gluon, and $v^2$ should be replaced by a hard scale related to $q^+ q^-$. It would certainly be interesting to see whether the explicit forms of these integrals can help determine some observables for which the integrals do vanish, and factorization violation is then postponed to higher order.

So far our discussion is restricted to diagrams with soft emissions from the Glauber line. At leading order, real-emission diagrams can also have a single soft gluon coming
of an active collinear parton. Example tree-level and 1-loop diagrams are

\[ \mathcal{M}_{Ra}^0 = \quad , \quad \mathcal{M}_{Ra}^1 = \] (5.81)

A factorization-violating effect can come from interference of \( \mathcal{M}_{Ra}^0 \) with two-loop real-emission diagrams \( \mathcal{M}_{R}^2 \):

\[ \mathcal{M}_{R}^2 = \quad + \quad \] (5.82)

and from interference between \( \mathcal{M}_{Ra}^1 \) and the one-loop diagram \( \mathcal{M}_{R}^1 \) in Eq. (5.60). These interference terms contribute to the cross section at order \( g_s^6 |\mathcal{M}_{0}|^2 \). Both \( \mathcal{M}_{Ra}^1(q, p_3, k) \) and \( \mathcal{M}_{R}^2(q, p_3, k) \) contain IR divergences from the Glauber loop labeled in the diagram. With the presence of Lipatov vertex, the color generators of the two Glauber operator insertions in \( \mathcal{M}_{R}^2 \) do not commute, therefore IR divergences do not cancel at squared amplitude level. The interference term \( \mathcal{M}_{Ra}^1(\mathcal{M}_{R}^1)^* + \mathcal{M}_{Ra}^0(\mathcal{M}_{R}^2)^* \) could therefore contain \( q_T \)-dependent IR poles, generating factorization-violating effects.

The same issue appears in \( \mathcal{M}_{R} \) at higher loop order, for example, though diagrams

\[ \mathcal{M}_{R}^n = \quad , \quad \mathcal{M}_{Ra}^n = \] (5.83)

Since color generators to the left of the Lipatov vertex do not commute to the right, IR divergences in real-emission diagrams do not cancel in the squared amplitudes \( |\mathcal{M}_{R}|^2 \).
or \((M_R M_{R-1}^{<})_{N\text{-Glauber}}\) for \(N > 3\). Understanding the structure of the IR divergences of the active/spectator soft-emission diagrams terms could be important to understanding factorization-violation in more detail and should be an interesting area for future research.

5.5 Summary and Conclusion

The main result of this paper is that for any single scale observable, factorization is not violated due to processes involving only Glauber exchange between spectator quarks. Previous studies, working in position space \([152]\) or using more standard kinematic variables \([\_]\), suggested that the sum of all of these graphs would not vanish for a generic observable without some magical cancellation. In our proof, we show that the magical cancellation does happen, and is clearest using a natural set of conformal coordinates \(z\) and \(\bar{z}\). With these variables, integral over phase space for spectator emission is completely inclusive over \(z\) for all single-scale observables.

To check our result, we work out the complete contribution for the Glauber graphs to next-to-leading order. To see the cancellation explicitly, we compute the terms up to order \(\epsilon^0\) at 2 loops and the terms up to order \(\epsilon\) at 1-loop. These order \(\epsilon\) terms are needed because a \(\frac{1}{\epsilon}\) phase space divergence appears when integrating over the spectator transverse momentum. We find a non-trivial cancellation between the \(O(\epsilon^0)\) IR-finite 2-loop contribution and the \(\frac{1}{\epsilon} \times \epsilon\) 1-loop contribution, confirming our general result.

To see the cancellation to all orders, we show that all of the IR divergences in the ladder diagrams exponentiate into a phase. When the phase is removed, the \(O(\epsilon^0)\) terms are integrable over phase space while the \(O(\epsilon)\) terms are not. Thus at each order in perturbation theory, a non-trivial cancellation between finite contributions and phase-space singular contributions will occur. While we have focus on Drell-Yan process in this paper, the method we use is general and can also be applied to more complicated processes, like
jet production at hadron collider. For example, it has been shown that dijet production at hadron collider violate factorization, if the produced dijet is back-to-back and has small total transverse momentum [203]. It would be interesting to see whether or how Glauber gluon violate factorization from explicit calculation along the line of this work. Another interesting process to consider using our formalism is $t\bar{t}$ production at small transverse momentum [204–206], where it has been shown that final state interaction of top quark and spectator leads to factorization violation [207].

Our result implies that the leading order factorization-violating effects must involve soft radiation, through the Lipatov vertex associated with the Glauber lines. The graphs involving such soft radiation can have IR divergences that do not exponentiate and/or rapidity divergences. Such divergences, when regulated, generate a scale in the amplitude which prohibits the application of our scale-invariance argument for observable independence. We briefly study the forms that these higher order terms can have. Diagrammatically these contributions are similar to the factorization violation in Regge factorization [152, 208]. It will be interesting to evaluate these diagrams explicitly, and see if there is any universality in the perturbative factorization violating terms. With the explicit expression for the factorization-violating contribution, it will also be interesting to search for hadronic observable that can delay or avoid factorization violation.
Bibliography


256


[37]


[42] Y.-T. Chien and M. D. Schwartz, “Resummation of heavy jet mass and comparison


Determination of $\alpha_s$ from the C-parameter Distribution,” arXiv:1501.04111
[hep-ph].


[48] S. Catani, G. Turnock, and B. Webber, “Jet broadening measures in $e^+e^-$


[hep-ph].

[53] A. V. Manohar, “Deep inelastic scattering as $x \to 1$ using soft collinear effective

[54] T. Becher and M. D. Schwartz, “Direct photon production with effective field


[56] R. J. Gonsalves, N. Kidonakis, and A. Sabio Vera, “$W$ production at large
transverse momentum at the large hadron collider,” *Phys.Rev.Lett.* **95** (2005)


263


[165]


[187]

[188]


inclusive cross sections for hadronic collisions,” Phys. Rev. D86 (2012) 114038,

factorization beyond next-to-leading logarithmic accuracy,” JHEP 02 (2015) 029,