Some Calculations of Cobordism Groups and Their Applications in Physics

Some calculations of cobordism groups and their applications in physics

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Abstract

In this paper, we compute several cobordism groups. We use these calculations to classify invertible extended topological field theory with H_n structures and give a classification of Symmetric Protected Topological (SPT) phases with corresponding symmetry groups.

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1. Introduction

Global symmetry plays a crucial role in constraining the fate of macroscopic states or phases of physical systems — its constraint is applicable including but not limited to quantum many body condensed matter, and quantum field theories including gauge theories [27].

In condensed matter physics, one digs into how global symmetry acts on the operators and the states in the local Hilbert space. Symmetric Protected Topological (SPT) phrases are used to describe how phases are different with the presence of symmetry groups [26]. Recent research relates group cohomology to classify Symmetric Protected Topological (SPT) phrases with given symmetry groups in low dimensional cases (lower than 2+1D) [9, 32, 35]. Recently the higher dimension (3+1D) SPT classifications have been more-or-less completed by pioneer works (the bosonic cases in [9, 28], physical intuitive studies of interacting fermionic topological insulators/superconductors (TI/TSC) [29, 30, 22], with the later corrections and refinements from cobordisms [17, 18, 12] or generalized group cohomology [33, 31], see more References therein).

In this paper, we calculate some cobordism groups to classify Symmetric Protected Topological (SPT) phases (invertible topological fields) with certain symmetry groups by the following theorem of Freed and Hopkins [12]: there is a 1:1 correspondence

$$\left\{\begin{array}{l}\text{deformation classes of reflection positive}\\ \text{invertible } n\text{-dimensional extended topological}\\ \text{field theories with symmetry group } H_n\end{array}\right\} \cong [MTH, \Sigma^{n+1} I\mathbb{Z}]_{\text{tors}}.$$

.

First we consider a few cases from the 10 particular global symmetries (see Table 1) that are mostly relevant to the fermionic electrons of condensed matter system in 3+1 dimensional spacetime (3+1D), involving $SU(2)$, $U(1)$, $\mathbb{Z}_2^F$, or $\mathbb{Z}_2^T$ symmetries. If one limits these 10 global symmetries to the quadratic Hamiltonian systems, they correspond to the 10 Cartan symmetry classes, studied since Wigner-Dyson [34, 10, 4]. We focus on the $SU(2)$ and $\mathbb{Z}_2^T$ symmetries. The $SU(2)$ plays the role of the flavor symmetry or the spin-$\frac{1}{2}$'s $SU(2)$ rotational symmetry. The $\mathbb{Z}_2^T$ (or more precisely, $\mathbb{Z}_4^T$) is the time reversal symmetry.

In addition, we also examine global symmetries and topological invariants that are pertinent to quantum chromodynamics (QCD_4) or the cold atom systems with larger flavor/or spin rotational global symmetries: $SU(2) \times SU(2)$ color-flavor symmetry, $SU(3)$ symmetry, and $SU(4)$ symmetry with $\mathbb{Z}_2^T$ time-reversal. See Tabel 2

We also give manifolds generators for each case which are useful to find *topological invariants* and *the partition functions* (or path integrals) in field theoretic form at IR that capture the bulk SPTs (by coupling to *background non-dynamical probed fields*) and also constrain the boundary anomalies (check [14]).

2. Physics

In old days, gapped phased of matter are boring because they all look the same at long distance or time scales. Now it turns out that a variety of topological phases of matter are gapped. To distinguish them one can either consider a nontrivial spatial topology or to look at the edge physics. One key question in physics is to classify gapped local lattice Hamiltonians/phases up to homotopy. Here lattice means that the Hilbert space is a tensor product $\mathcal{H} = \otimes_{v \in V} \mathcal{H}_v$, where V is the set of vertices of a d-dimensional lattice or triangulation and $\mathcal{H}_v$ is finite-dimensional Hilbert space. Local means the Hamiltonian has the form $H = \sum_v H_v$ where H_v acts as identity on all $\mathcal{H}_{v'}$ except for v' in a neighborhood of v. Gapped means the gap between the energies of group states and excited states stays nonzero in the limit of infinite volume.

Using the notion of quantum entanglement, one obtains the following general picture of gapped phases at zero temperature. All gapped zero-temperature phases can be divided into two classes: long-range entangled phases (ie phases with intrinsic topological order) and short-range entangled phases (ie phases with no intrinsic topological order). All short-range entangled phases can be further divided into three classes: symmetry-breaking phases, Symmetry Protected Topological phases, and their mix (symmetry breaking order and SPT order can appear together).

Symmetry-breaking orders are described by group theory. Symmetry Protected Topological phrase (SPT phase) is a special invertible topological field theory. It can be classified by topological field theories shown in Section 3.2. Now let us have a close look into SPT phases in physics.

2.1. **Symmetric Protected Topological phases.** Symmetry Protected Topological phrase (SPT phase) is a kind of phrase in zero-temperature quantum-mechanical states of matter that have a symmetry and a finite energy gap. Before we give a definition of SPT phases, let me introduce what a phase is. Macroscopically, matter has different states, for example, water can be solid, liquid, or gas. They are different states. A phase is an equivalent class of states. A state is equivalent to another one if they an be smoothly connected by changing the Hamiltonian. More precisely, consider two gapped systems with two Hamiltonian operators H, H', corresponding ground energies E, E' and ground states ϕ, ϕ'. We have $H\phi = E\phi$ and $H'\phi' = E'\phi'$. If we can connect H to H' and the corresponding ground energy is a smooth function, we say the two states ϕ and ϕ' are smoothly connected. Water has two phrases: gas and liquid are in the same phase and solid is a different phase.

Two phases may be equivalent if there is no symmetry perturbation but become not equivalent if there is symmetry perturbation for some group G. It behaves as the following Figure 1 [36].

The most well-known example may be the distinction between topological insulators and trivial insulators: in the presence of charge conservation $U(1)$ and time reversal symmetries Z_2^T (more precisely, the symmetry group is $U(1) \rtimes_{Z_2} Z_4^T$), these

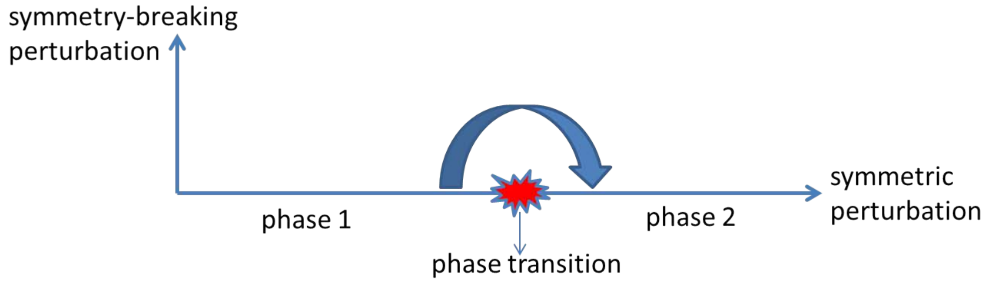

FIGURE 1. [36] The notion of symmetry protected distinction of quantum phases of matter. As long as the relevant symmetries are preserved, the two phases cannot be connected without crossing a phase transition. However, if symmetry-breaking perturbations are allowed, the phase transition can be avoided.

two types of insulators are separated by a phase transition. However, once these symmetries are allowed to be broken, they can be smoothly connected[15, 23, 16].

If phase 2 is the trivial phase, the non equivalent phase 1 is called a symmetry-protected topological (SPT) phases. SPT phases are phases equivalent to the trivial phase in absence of symmetry but not equivalent to the trivial phase with some symmetry group G.

2.2. **Relation to math.** It is believed that SPT phases are special topological field theories. There is a (moduli) space $\mathcal{F}_n$ of invertible reflection positivity extended field theories. It is classified by the homotopy mapping groups $[MTH, \Sigma^{n+1} I\mathbb{Z}]$. Classifications of SPT phases can be considered as maps from a parameter space S to $\mathcal{F}_n$. The idea is that this parameter space S can be converted into a classifying space BG of some group G. Then SPT phases can be classified by the torsion part of $[MTH \wedge S_+, \Sigma^{n+1} I\mathbb{Z}]$. In this idea, the interpretation is that a manifold M with a map to BH_n and a map to S. More detailed discussion is in Section 3.2.

3. Topological field theories

Inspired by Witten, Atiyah[7] gave a axiomatic definition of topological (quantum) field theories (TQFTs) of dimension n. Let me first introduce a category $\mathbf{Cob}_n$.

Definition 3.1. [20] Let n be a positive integer. Define a category $\mathbf{Cob}_n$ as follows:

- An object of $\mathbf{Cob}_n$ is a closed compact $(n-1)$-manifold M
- Given two objects M and N in the category, a morphism from M to N is a bordism from M to N, which is, an n-dimensional manifold B equipped with a diffeomorphism $\partial B \simeq M \coprod N$. We regard two bordisms B and B' as the same morphism in $\mathbf{Cob}_n$ if there is diffeomorphism between B and B' which extends the evident diffeomorphism $\partial B \simeq M \coprod N \simeq \partial B'$ between their boundaries.
- For any object M, the identity map is represented by the product bordism $M \times [0,1]$
- Composition of morphisms is given by gluing bordisms together.

Definition 3.2. Let $\mathcal{C}$ be a a symmetric monoidal category. A topological (quantum) field theories (TQFTs) of dimension n is a symmetric monoidal functor from the category $\mathbf{Cob}_n$ to $\mathcal{C}$.

Remark 3.3. $\mathbf{Cob}_n$ is endowed with a symmetric monoidal category with disjoint union of manifolds.

Remark 3.4. In physics, people usually use the symmetric monoidal category $\mathcal{C}$ as $\mathbf{Vect}_{\mathbb{C}}$. $\mathbf{Vect}_{\mathbb{C}}$ is the category of vector spaces over complex numbers, and is endowed as a symmetric monoidal category with tensor product of vector spaces.

We can extend our definition by requiring that our bordisms are equipped with a tangential structure.

Definition 3.5. Given a fibration $\xi\colon X \to BO(d)$, a X-structure on a d-manifold M is a lift α:

$$\begin{tikzcd} & X \arrow[d, "\xi"] \\ M \arrow[ur, dashed, "\theta"] \arrow[r, "\tau_M"'] & BO(d) \end{tikzcd}$$

where τ classifies the tangent bundle of M. If X is the classifying space BH_d for some group H_d, we say M is a H_d-manifold.

Remark 3.6. If $H_d = SO(d)$, this is the definition of oriented manifolds. If $H_d = \mathrm{Spin}_d$, this is the definition of spin manifolds.

With this structure, we can define a bordism n-category $\mathrm{Bord}_{d-n,d}^{(X,\xi)}$ with (∞, n)-category structure. If $n = 1$ and X is the classifying space BH_d, this is the $(\infty, 1)$-category with objects compact $(d-1)$ H_d-manifold embedded in $\mathbb{R}^\infty$ and morphisms are compact d-cobordisms embedded in $\mathbb{R}^\infty$. We give a definition of this n-category by using n-fold Segal spaces.

3.1. Bordism n-categories.

Definition 3.7. An n-fold Segal space is a functor $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Fun}((\mathbf{\Delta}^{\mathrm{op}})^{\times n-1}, \mathrm{Top}))$ such that

- X_n is is an $(n-1)$-fold Segal space for $n \geq 0$;
- X_0 is essentially constant;
- For each $n > 0$ the Segal map induces a levelwise weak homotopy equivalence

$$s_n : X_n \xrightarrow{\simeq} \underbrace{X_1 \times^h_{X_0} X_1 \times^h_{X_0} \cdots \times^h_{X_0} X_1 \times^h_{X_0} X_1}_{n \text{ factors}}.$$

  Here these homotopy fiber products of $(n-1)$-fold Segal spaces are taken levelwise.

Suppose that the fibration $\xi : X \to BO(d) = Gr_d(\mathbb{R}^\infty)$ factor through $Gr_d(\mathbb{R}^{m+n})$ for some positive integer m and this factorization is GL_{m+n}-equivariant. Given a manifold M, we define the functor

$$\mathrm{Bord}_{d-n;d}^{(X,\xi)}(M) : (\mathbf{\Delta}^{\mathrm{op}})^n \to \mathrm{Top}$$

is defined by assigning to $(m_1, \dots, m_n) \in (\mathbf{\Delta}^{\mathrm{op}})^n$ the space consisting of tuples $((\boldsymbol{t}^i)_{i=1}^n, (W, \theta))$ where $\boldsymbol{t}^i \in \mathbb{R}^{[m_i]} = \{(t_i)_{i=0}^{i=k} \mid t_i \leq t_{i+1}\} \subseteq \mathbb{R}^{k+1}$ for each $1 \leq i \leq n$ and (W, θ) is an embedded submanifold of $M \times \mathbb{R}^n$ with (X, ξ)-structure by a lifting θ. These are required to satisfy the following condition: for all $1 \leq i \leq n$, and $0 \leq j \leq m_i$, W is cylindrical near $\{t_j^i\} \times \mathbb{R}^{\{i+1,\dots,n\}} \subseteq \mathbb{R}^{\{i,i+1,\dots,n\}}$.

Let $M = D^p$, p-dimensional disk in $\mathbb{R}^p$ and $M = D^p \times \mathbb{R}^i$, we have the following weak homotopy equivalence:

Theorem 3.8. [24] *There is a natural levelwise weak homotopy equivalence of* $(n-i)$*-fold simplicial spaces:*

$$B^i \mathrm{Bord}_{d-n;d}^{(X,\xi)}(D^p) \xrightarrow{\simeq} \mathrm{Bord}_{d-n-i;d}^{(X,\xi)}(D^p \times \mathbb{R}^i)$$

where the classifying space functor is applied to the final i*-many simplicial directions* $\{n-i+1, n-i+2, \dots, n\}$.

Let $\xi_{m+n} : X_{m+n} \to Gr_d(\mathbb{R}^{m+n})$ be a sequence of GL_{m+n}-equivariant fibrations together with GL_{m+n}-equivariant connecting maps $f_{m+n} : X_{m+n} \to X_{m+n+1}$ making the following diagram commute

$$\begin{tikzcd} X_{m+n} \arrow[r, "f_{m+n}"] \arrow[d, "\xi_{m+n}"'] & X_{m+n+1} \arrow[d, "\xi_{m+n+1}"] \\ Gr_d(\mathbb{R}^{m+n}) \arrow[r] & Gr_d(\mathbb{R}^{m+n+1}) \end{tikzcd}$$

Write $\gamma_d \to Gr_d(\mathbb{R}^{m+n})$ as the canonical d-vector bundle over $Gr_d(\mathbb{R}^{m+n})$ and $\gamma_d^\perp$ as the canonical $(m+n-d)$-vector bundle over $Gr_d(\mathbb{R}^{m+n})$. We have a canonical isomorphism of vector bundles over X_{m+n}

$$f_{m+n}^* \xi_{m+n+1}^* \gamma_d^\perp \cong \xi_{m+n}^* \gamma_d^\perp \oplus \mathbb{R}$$

where $\mathbb{R}$ is trivial bundle of rank one. Hence we have induced maps of Thom spaces:

$$\Sigma Thom(\xi^*_{m+n}\gamma_d^{\perp}) \to Thom(\xi^*_{m+n+1}\gamma_d^{\perp}).$$

Definition 3.9. Let $(\boldsymbol{X}, \boldsymbol{\xi}) = \{(X_{m+n}, \xi_{m+n})\}$ denote a collection of X_{m+n} with connecting maps as above. Then the *Madsen-Tillmann* spectrum is the Thom spectrum $MT\boldsymbol{\xi}$ whose p^{th} space is $Thom(\xi^*_p\gamma_d^{\perp})$ and with the above defined connecting maps.

Now consider the family of n-fold Segal spaces $\mathrm{Bord}_{d-n;d}^{(X_{d+m},\xi_{d+m})}(D^m)$. We have natural connecting maps $\mathrm{Bord}_{d-n;d}^{(X_{d+m},\xi_{d+m})}(D^m) \to \mathrm{Bord}_{d-n;d}^{(X_{d+m+1},\xi_{d+m+1})}(D^m+1)$ as E_p-algebras. Taking colimit, we get a E_∞ n-fold space $\mathrm{Bord}_{d-n;d}^{(\boldsymbol{X},\boldsymbol{\xi})}$. It is a symmetric monoidal (∞, n)-category.

Theorem 3.10. [24] *There is a weak equivalence of E_∞-spaces between the geometric realization $||\mathrm{Bord}_{d-n;d}^{(\boldsymbol{X},\boldsymbol{\xi})}||$ and $\Omega^{\infty-n}MT\boldsymbol{\xi}$, where $MT\boldsymbol{\xi}$ is the Madsen-Tillmann spectrum $MT\boldsymbol{\xi}$.*

The case $n = 1$ is a well-known theorem of Galatius-Madsen-Tillmann-Weiss[13]. The case $d = n$ is established in [20] and [11].

3.2. **Invertible extended field theories.**

Definition 3.11. Let $\mathcal{C}$ be a symmetric monoidal (∞, n)-category. An n-extended d-dimensional topological field theory with a $(\boldsymbol{X}, \boldsymbol{\xi})$-structure is a symmetric monoidal (∞, n) functor $F : \mathrm{Bord}_{d-n;d}^{(\boldsymbol{X},\boldsymbol{\xi})} \to \mathcal{C}$.

If $n = d$, then it is a fully extend d-dimensional topological field theory with a $(\boldsymbol{X}, \boldsymbol{\xi})$-structure.

Definition 3.12. Let $\mathcal{C}$ be a symmetric monoidal (∞, n)-category. A fully extended d-dimensional topological field theory with a $(\boldsymbol{X}, \boldsymbol{\xi})$-structure is a symmetric monoidal (∞, n) functor $F : \mathrm{Bord}_{0;d}^{(\boldsymbol{X},\boldsymbol{\xi})} \to \mathcal{C}$.

Definition 3.13. An n-extended d-dimensional topological field theory F is invertible if there is an extended d-dimensional topological field theory F' such that $F \otimes F' \simeq 1$. Equivalently, it sends every k-morphism to an invertible morphism in $\mathcal{C}$.

In fact, an invertible extended d-dimensional topological field theory F factors through the Picard ∞-category of $\mathcal{C}$. Invertible extended d-dimensional topological field theories are in natural bijection with

$$\pi_0 \mathrm{Map}_{E_\infty}(||\mathrm{Bord}_{d-n;d}^{(\boldsymbol{X},\boldsymbol{\xi})}||, \mathcal{C})$$

A symmetric monoidal (∞, n)-category $\mathcal{C}$ has a higher Picard groupoid quotient $\bar{\mathcal{C}}$, obtained by adjoining inverses of every morphism. Also, a symmetric monoidal (∞, n)-category $\mathcal{C}$ has a maximal subgroupoid $\mathcal{C}^\times$ by removing all noninvertible morphisms. Thus,

Definition 3.14. An fully extended d-dimensional topological field theory $F : \mathrm{Bord}_{0;d}^{(\boldsymbol{X},\boldsymbol{\xi})} \to \mathcal{C}$ is invertible if it factors through

$$\tilde{F} : \overline{\mathrm{Bord}_{0;d}^{(\boldsymbol{X},\boldsymbol{\xi})}} \to \mathcal{C}^\times.$$

Thus, it is equivalent to an infinite map $||\mathrm{Bord}_{d-n;d}^{(\boldsymbol{X},\boldsymbol{\xi})}||, ||\mathcal{C}^\times||$ If we take $\mathcal{C}^\times$ to be the spectrum $I\mathbb{C}$ to be the Brown-Comenetz dual spectrum defined by

$$[X, I\mathbb{C}] = \mathrm{Hom}(\pi_0 X, \mathbb{C})$$

It classifies all isomorphism classes of topological theories. To classify all deformation classes, Freed and Hopkins [12] gives a candidate spectrum $I\mathbb{Z}$.

Theorem 3.15. [12] *There is a 1:1 correspondence*

$$\left\{\begin{array}{l}\textit{deformation classes of reflection positive}\\ \textit{invertible } n\textit{-dimensional extended topological}\\ \textit{field theories with symmetry group } H_n\end{array}\right\} \cong [MTH, \Sigma^{n+1} I\mathbb{Z}]_{tors}.$$

In particular, MTH_n is the Madsen-Tillmann spectrum $MT\boldsymbol{\xi}$ for $\boldsymbol{\xi} : BH \to BO$. $[MTH, \Sigma^{n+1} I\mathbb{Z}]_{\text{tors}}$ stands for the torsion part of homotopy classes of maps from spectrum MTH to the $(n+1)$-th suspension of spectrum $I\mathbb{Z}$. The Anderson dual $I\mathbb{Z}$ is a spectrum that is the fibration of $I\mathbb{C} \to I\mathbb{C}^\times$ where $I\mathbb{C}$ ($I\mathbb{C}^\times$) is the Brown-Comenetz dual spectrum defined by

$$[X, I\mathbb{C}] = \mathrm{Hom}(\pi_0 X, \mathbb{C})$$

$$[X, I\mathbb{C}^\times] = \mathrm{Hom}(\pi_0 X, \mathbb{C}^\times)$$

4. Adams spectral sequence

Adams spectral sequence is a useful tool to compute the stable homotopy groups of spectra. For a prime p and a spectra E of finite type, there is a spectral sequence called Adams spectral sequence converging to $(\pi_{t-s}E)^\wedge_p$ with the following E_2-page

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}_p}(H^*(E, \mathbb{F}_p), \mathbb{F}_p)$$

$\mathcal{A}_p = H\mathbb{F}_p^* H\mathbb{F}_p$ is the mod p Steenrod algebra.

4.1. $\pi_*^S(\mathbb{S})$**-module.** $\pi_{t-s}MTH^\wedge_p$ is a module over the stable homotopy groups of spheres $\pi_*^S(\mathbb{S})$. The E_2-page $E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}_p}(H^*(E, \mathbb{F}_p), \mathbb{F}_p)$ for a spectrum E is also a module over E_2-page $E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}_p}(\mathbb{F}_p, \mathbb{F}_p)$ for the sphere spectra $\mathbb{S}$. There is a few special elements in $E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}_p}(\mathbb{F}_p, \mathbb{F}_p)$ (or $E^{s,t}\infty 2 = \mathrm{Ext}^{s,t}_{\mathcal{A}_p}(\mathbb{F}_p, \mathbb{F}_p)$):

$$h_0 \in E_2^{1,1} = \mathrm{Ext}^{s,t}_{\mathcal{A}_p}(\mathbb{F}_p, \mathbb{F}_p)$$

$$h_1 \in E_2^{1,2} = \mathrm{Ext}^{s,t}_{\mathcal{A}_p}(\mathbb{F}_p, \mathbb{F}_p)$$

(The same notations h_0 and h_1 are used for E_∞-page since both survive to E_∞-page.) h_0 corresponds to the identity map $S^n \to S^n$ and h_1 corresponds to the stable Hopf map $S^{n+3} \to S^{n+2}$.

In Adams spectral sequence diagram, the horizontal axis is degree $t-s$ and the vertical axis is degree s. This is different from Serre spectral sequence diagram. We use a dot to represent that there is a copy of $\mathbb{Z}/p$ in $E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}_p}(H^*(E,\mathbb{F}_p),\mathbb{F}_p)$ (or $E_\infty^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}_p}(H^*(E,\mathbb{F}_p),\mathbb{F}_p)$) and a vertical line connecting two dots if multiple of the generator of the below dot with h_0 is the generator of the above dot. A diagonal line is used to connect two dots if multiple of the generator of one dot with h_1 is the generator of the other dot.

Figure 2 is an example, the E_2-page for $E = ko$, the connective real K-theory, with $p = 2$. $h_0^i \in \mathrm{Ext}^{i,i}_{\mathcal{A}_2}(H^*(ko,\mathbb{F}_2),\mathbb{F}_2)$ is the generator and $h_1^j \in \mathrm{Ext}^{j,2j}_{\mathcal{A}_2}(H^*(ko,\mathbb{F}_2),\mathbb{F}_2)$ is the generator for $1 \leq j \leq 1$. It is also the E_∞-page for ko. If there is a higher differentiable d_r for $r \geq 2$, then the only possible one is $d_r(h_1) = h_0^{r+1}$. Then we would have $0 = d_r(h_0h_1) = h_0 d_r(h_1) + d_r(h_0)h_1 = h_0^{r+2}$. This is not true and so $E_2 = E_\infty$.

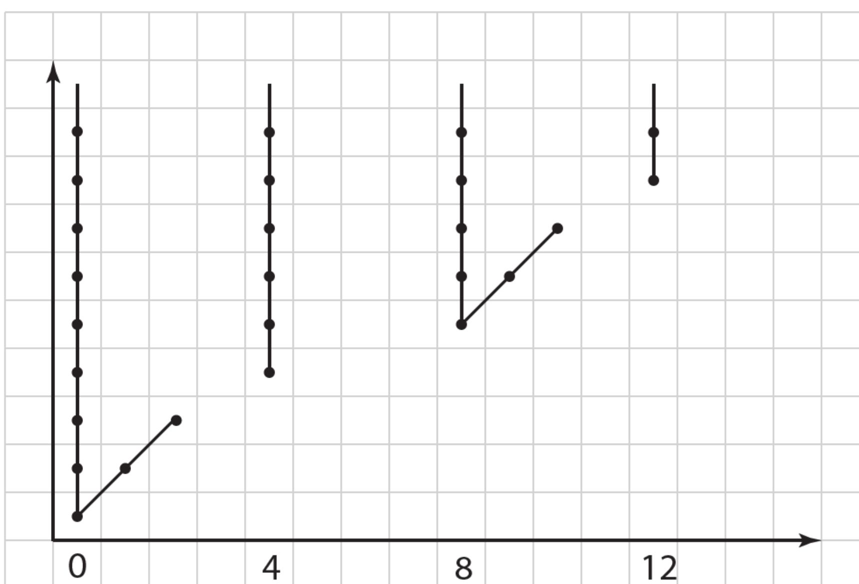

FIGURE 2. The E_2-page for $E = ko$, the connective real K-theory, $p = 2$

4.2. **How to read.** The idea to draw in this way comes from fundamental theorem of finitely generated abelian groups; Every finitely generated abelian group A is isomorphic to a direct sum of p-primary cyclic groups $\mathbb{Z}/p^k$ (for p a prime number and k a natural number) and copies of the infinite cyclic group $\mathbb{Z}$, $A \simeq \mathbb{Z}^n \oplus \bigoplus_i \mathbb{Z}/p_i^{k_i}$.

In E_∞-page, the elements in $t-s$ tell a graded associated of the homotopy group $\pi_{t-s}(E)^\wedge_p$ by reading vertically. If there is a vertical segment of k-dots connected to each other, this means that $\pi_{t-s}(E)^\wedge_p$ contains $\mathbb{Z}/p^k$ as a summand. If there is a vertical line of infinitely many dots connected to each other, this means that $\pi_{t-s}(E)^\wedge_p$ contains $\mathbb{Z}_p$ as a summand. For example, in the Adams spectral sequence for sphere spectrum $\mathbb{S}$ at $p=2$, we have:

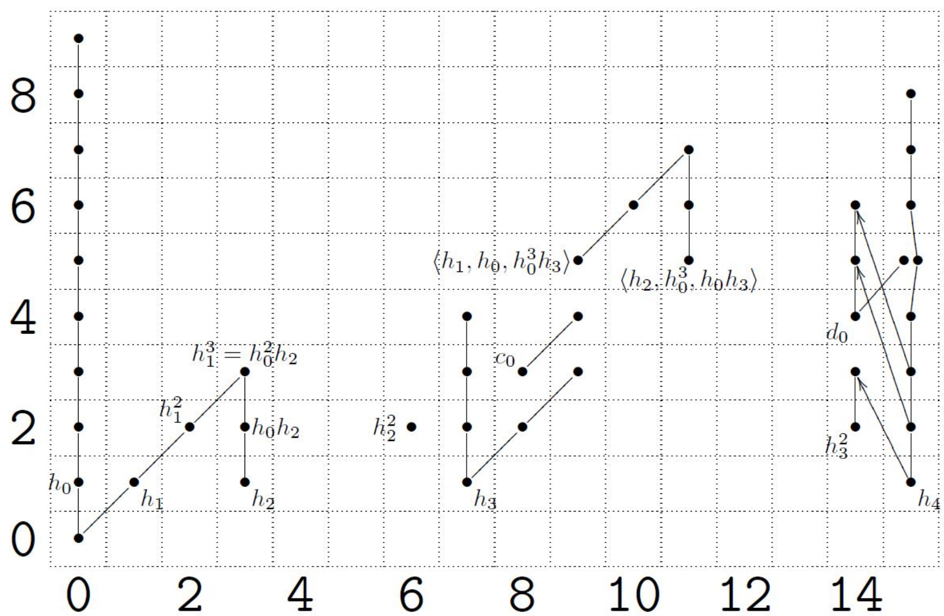

FIGURE 3. [25] The E_∞-page for $\mathbb{S}$, $p=2$

From Figure 3, we see that ($\mathbb{Z}_2$ is 2-adic integer.)

i	0	1	2	3	4	5	6	7	8
$\pi_i^S(\mathbb{S})^\wedge_2$	$\mathbb{Z}_2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/8$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/16$	$\mathbb{Z}/2^{\oplus 2}$

From Figure 2, we see that:

i	0	1	2	3	4	5	6	7	8
$\pi_i(ko)^\wedge_2$	$\mathbb{Z}_2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}_2$	0	0	0	$\mathbb{Z}_2$

From further calculation in odd prime, ko has no odd torsion and thus we see the homotopy groups of ko is

i	0	1	2	3	4	5	6	7	8
$\pi_i(ko)$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$

4.3. **Reduction to $\mathcal{A}(1)$.** In general, the $\mathrm{Ext}_{\mathcal{A}_p}(-,-)$ is hard to compute. It is easier if we can reduce it to compute $\mathrm{Ext}_{\mathcal{A}(1)}(-,-)$, where $\mathcal{A}(1)$ denotes the $\mathbb{F}_2$-algebra generated by Sq^1 and Sq^2. $\mathcal{A}(1)$ is a subalgebra of $\mathcal{A}_2$. In general, given a Hopf algebra A, and B is a Hopf subalgebra of A, a B-module M and a A-module N, there is an isomorphism:

$$\mathrm{Ext}_A(A \otimes_B M, N) \simeq \mathrm{Ext}_B(M, N)$$

In particular, we know $H^*(ko, \mathbb{F}_2) = \mathcal{A}_2 \otimes_{A(1)} \mathbb{F}_2$. Thus,

$$\mathrm{Ext}_{\mathcal{A}_2}(H^*(ko \wedge X), N) \simeq \mathrm{Ext}_{\mathcal{A}(1)}(H^*(X, \mathbb{F}_2), N)$$

The $\mathbb{F}_2$-algebra $A(1)$ is usually depicted at the leftmost side of the Figure 4:

In Figure 4, each node represents one copy of $\mathbb{F}_2$ generated by a chosen homogenous basis, a straight edge represents a Sq^1-action and a curved line represents a Sq^2-action. The middle and the rightmost diagrams are examples of $\mathcal{A}(1)$-modules, the Joker J and the trivial module $\mathbb{F}_2$.

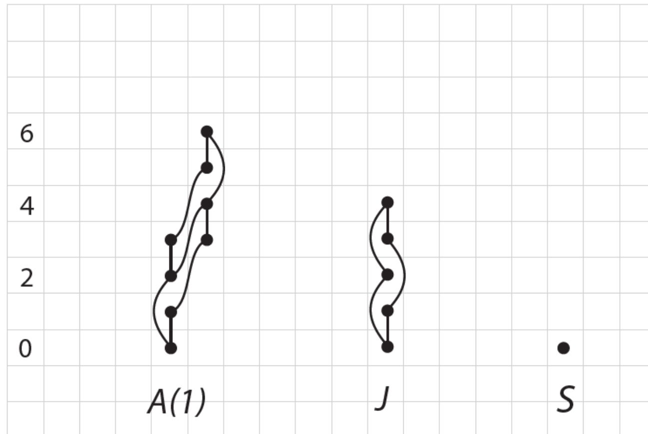

FIGURE 4. Some examples of $\mathcal{A}(1)$-modules

4.4. **Spin bordism** $\pi_* M\mathbf{Spin}\wedge X$**.** In [5], Anderson, Brown and Peterson prove that there is a homotopy equivlence

$$(\pi^J, z_i) : M\mathrm{Spin} \to \bigvee_{\substack{n(J)\mathrm{even},\\ 1\notin J}} ko\langle 4n(J)\rangle \vee \bigvee_{\substack{n(J)\mathrm{odd},\\ 1\notin J}} ko\langle 4n(J)-2\rangle \vee \bigvee_i \Sigma^{|z_i|} H\mathbb{Z}_2.$$

where J is some multiple index, $n(J)$ is the degree of the index and $ko\langle i\rangle$ is $(i-1)$-connected cover of ko. By Bott periodicity, $ko\langle 4n(J)\rangle = \Sigma^{4n(J)} ko$ for $n(J)$ even and $ko\langle 4n(J)-2\rangle = \Sigma^{4n(J)-4} ko\langle 2\rangle$ for $n(J)$ odd. In particular,

$$H^*(ko\langle 4n(J)\rangle, \mathbb{F}_2) = \Sigma^{4n(J)} \mathcal{A}_2 \otimes_{\mathcal{A}(1)} \mathbb{F}_2,$$

$$H^*(ko\langle 4n(J)-2\rangle, \mathbb{F}_2) = \Sigma^{4n(J)-4} \mathcal{A}_2 \otimes_{\mathcal{A}(1)} J,$$

$$H^*(\Sigma^{|z_i|} H\mathbb{Z}_2, \mathbb{F}_2) = \Sigma^{|z_i|} \mathcal{A}_2 \otimes_{\mathcal{A}(1)} \mathcal{A}(1)$$

Note that all the $\mathcal{A}(1)$-modules on the rightmost sides come from Figure 4. In particular, the mod 2 cohomology $H^*(M\mathrm{Spin}, \mathbb{F}_2)$ in the form of

$$\mathcal{A}_2 \otimes_{\mathcal{A}(1)} N$$

for some $\mathcal{A}(1)$-module N. N is determined in their paper [5] and a diagram of N through dimension 28 as Figure 5.

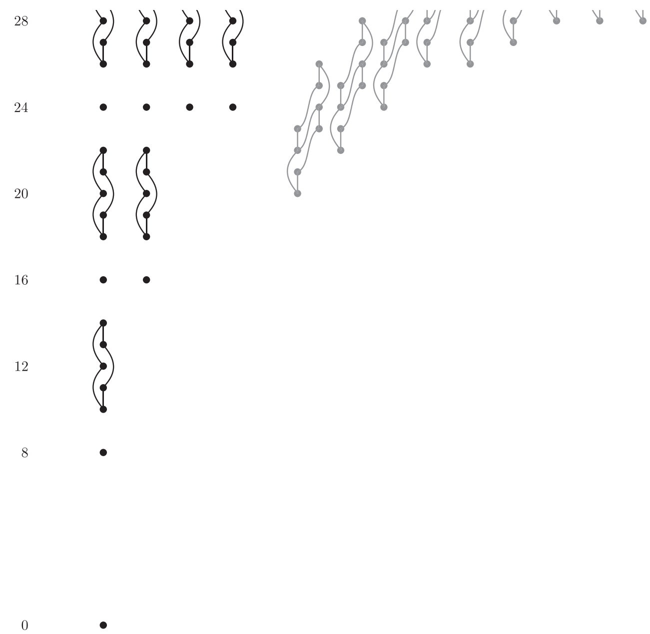

FIGURE 5. N up to dimension 28

Suppose X is a connective spectrum of finite type and write $M_J(X) = \pi_* ko \wedge J \wedge X$. The result of Anderson-Brown-Peterson [5] tells that

$$\pi_* M\mathrm{Spin} \wedge X = ko_*(X) \oplus \Sigma^8 ko_*(X) \oplus \Sigma^{10} M_J(X) \oplus \cdots \oplus \Sigma^{20} H_*(X, \mathbb{F}_2) \oplus \cdots$$

Each term can be computed by spectral sequences as follows;

$$\mathrm{Ext}_{\mathcal{A}(1)}(H^*(X, \mathbb{F}_2), \mathbb{F}_2) \Rightarrow ko_*(X)_2^\wedge$$

$$\mathrm{Ext}_{\mathcal{A}(1)}(J \otimes H^*(X, \mathbb{F}_2), \mathbb{F}_2) \Rightarrow M_J(X)_2^\wedge$$

$$\mathrm{Ext}_{\mathcal{A}(1)}(\mathcal{A}(1) \otimes H^*(X, \mathbb{F}_2), \mathbb{F}_2) \Rightarrow H_*(X)_2^\wedge$$

If M is of the form $M' \oplus F$ with F a free $\mathcal{A}(1)$-module, then

$$\mathrm{Ext}_{\mathcal{A}(1)}(M, \mathbb{F}_2) = \mathrm{Ext}_{\mathcal{A}(1)}(M', \mathbb{F}_2) \oplus \mathrm{Ext}_{\mathcal{A}(1)}(F, \mathbb{F}_2)$$

and the spectral sequence is the sum of two spectral sequences. The latter spectral sequence of F collapses with

$$\mathrm{Ext}^{s,t}_{\mathcal{A}(1)}(F, \mathbb{F}_2) = 0 \quad s > 0$$

$$\mathrm{Ext}^{0,t}_{\mathcal{A}(1)}(F, \mathbb{F}_2) = \mathrm{Hom}_{\mathcal{A}(1)}(F, \mathbb{F}_2)$$

4.5. **Identify $\mathcal{A}(1)$-module structure of $H^*(X;\mathbb{F}_2)$.** To derive the $\mathcal{A}(1)$-module structure, we use a technique introduced by Adams and Margolis [21]. $\mathcal{A}(1)$ contains two of the Milnor operations

$$Q_0 = Sq^1$$

$$Q_1 = [Sq^2, Sq^1]$$

and they generate an exterior algebra $E[Q_0, Q_1] \subset \mathcal{A}(1)$.

Definition 4.1. Let M be an $\mathcal{A}(1)$-module. For $i = 0, 1$ the i^{th} Margolis homology of M is

$$H_*(M; Q_i) = \ker Q_i / \mathrm{image} Q_i.$$

The Margolis homology of a space or spectrum X is the Margolis homology of $H^*(X, \mathbb{F}_2)$ is

$$H_*(X; Q_i) = H_*(H^*(X, \mathbb{F}_2); Q_i)$$

Remark 4.2. The Milnor elements are primitive and the Kunneth isomorphism holds:

$$H_*(M \otimes N; Q_i) = H_*(M; Q_i) \otimes H_*(N; Q_i)$$

The following theorem of Adams and Margolis [1] is useful for identifying $A(1)$-module structure.

Theorem 4.3. [1] *A connected $\mathcal{A}(1)$-module M is free if and only if*

$$H_*(M, Q_i) = 0 \text{ for } i = 0, 1.$$

Given an $\mathcal{A}(1)$-module M, we first find the non-free submodule $N \subset M$ with isomorphic Margolis homology. Thus $M = N \oplus$ free . (See examples, check section 5) To determine the basis of the free part, we now describe a technique.

Let $b(x) = Sq^2Sq^2Sq^2(x)$. If F is a free $\mathcal{A}(1)$-module and $x \in F$, there are elements $a \in \mathcal{A}(1)$ and $y \in F$ with $a \cdot x = b(y) \neq 0$. This is proved by reducing to the case $F = \mathcal{A}(1)$.

Lemma 4.4. *Suppose that F and M are $\mathcal{A}(1)$-module and F is free. A map $F \to M$ is a monomorphism if and only if the induced map $b(F) \to b(M)$ is a monomorphism.*

Proof. The only if part is obvious. For the other direction, suppose that $b(F) \to b(M)$ is monomorphism and $x \in F$. There are $a \in \mathcal{A}(1)$ and $y \in F$ with $a \cdot x = b(y) \neq 0$. Since $b(F) \to b(M)$ is a monomorphism, the image of $b(y) \neq 0$ so is the image of $a \cdot x$ and the image of x. □

Lemma 4.5. *M is an $\mathcal{A}(1)$-module. The following are equivalent:*

(1) If F is a free $\mathcal{A}(1)$-module and $F \subset M$ then $F = 0$.

(2) b(x)=0 for all $x \in M$.

Proof. Suppose that $F \subset$ is a free submodule. If $F \neq 0$ then there is an $x \in F$ such that $b(x) \neq 0$. Conversely if there is $x \in M$ with $b(x) \neq 0$, then the map $\Sigma^{|x|}\mathcal{A}(1) \subset M$ is a nonzero free submodule. □

We call an $\mathcal{A}(1)$-module N has *no free submodule* if it has the above equivalent properties. SInce $\mathcal{A}(1)$ is injective as a module over itself, having a free submodule is equivalent to having a free summand.

Lemma 4.6. *Let M be an $\mathcal{A}(1)$-module, and $N \subset M$ a summand having no free submodules. If F is a free module and $F \to M$ is a monomorphism, then $F \to M/N$ is monomorphism.*

Proof. By Lemma 4.4 it suffices to show that $b(F) \to b(M/N)$ is a monomorphism. Since $b(N) = 0$ and N is a summand, the map $b(M) \to b(M/N)$ is an isomorphism. $\square$

Thus to show a free $\mathcal{A}(1)$-module F is all the free summands of an $\mathcal{A}(1)$-module M, it suffices to give a monomorphism map $F \to M$ and Poincare series of F is equal to Poincare series of M minus Poincare series of no free submodules. For examples, check section 5.

5. Computation

This part aims to fill calculations of bordism groups in more detail[1]. We have the following 1:1 correspondence

$$\left\{\begin{array}{l}\text{deformation classes of reflection positive}\\ \text{invertible } n\text{-dimensional extended topological}\\ \text{field theories with symmetry group } H_n\end{array}\right\} \cong [MTH, \Sigma^{n+1} I\mathbb{Z}]_{\text{tors}}.$$

From the fibration of $I\mathbb{Z} \to I\mathbb{C} \to I\mathbb{C}^\times$, there is an exact sequence

$$0 \longrightarrow \operatorname{Ext}^1(\pi_n MTH, \mathbb{Z}) \longrightarrow [MTH, \Sigma^{n+1} I\mathbb{Z}] \longrightarrow \operatorname{Hom}(\pi_{n+1} MTH, \mathbb{Z}) \longrightarrow 0$$

The torsion part $[MTH, \Sigma^{n+1} I\mathbb{Z}]_{\text{tors}}$ is

$$\operatorname{Ext}^1((\pi_n MTH)_{\text{tors}}, \mathbb{Z}) = \operatorname{Hom}((\pi_n MTH)_{\text{tors}}, U(1))$$

In this section we compute homotopy groups $\pi_* MTH$ for groups $H = \mathrm{Pin}^+ \times_{\{\pm 1\}} SU(2)$, $H = \mathrm{Pin}^- \times_{\{\pm 1\}} SU(2)$, $H = \mathrm{Pin}^+ \times_{\{\pm 1\}} SO(4)$, $H = \mathrm{Pin}^+ \times SU(3)$, and $H = \mathrm{Pin}^+ \times_{\{\pm 1\}} SU(4)$. In the following note, BG stands for the classifying space associated to a group G.

[1]Here we adopt the notations widely used in mathematics community. We write $\mathbb{Z}_n$ (or $\mathbb{Z}/n$ or $\mathbb{Z}/(n\mathbb{Z})$) for the finite group of order n. We write $\{\pm 1\}$ for a $\mathbb{Z}_2$ finite group.

We can think of $\pi_k MTH$ as bordism group of k-manifolds with H-principal structure on stable tangent bundles. In particular, MTH is the colimit of $\Sigma^n MTH_n$, where $\Sigma^n MTH_n = \text{Thom}(BH_n; \mathbb{R}^n - V_n)$, where V_n is the induced vector bundle (of dimension n) by the map $BH_n \to BO_n$. In the cases we are interested in, $BH_n \to BO_n$ is the projection

$$H_n \xrightarrow{pr_1} \text{Pin}_n^{\pm}/\{\pm 1\} = O(n)$$

In another way, we can think of $MTH = \text{Thom}(BH, -V)$, where V is the induced virtual bundle (of dimension 0) by the map $BH \to BO$. In the case we are interested in, $BH \to BO$ is the projection

$$H \xrightarrow{pr_1} \text{Pin}^{\pm}/\{\pm 1\} = O$$

Note: "T" in MTH denotes that the H structures are on tangent bundles instead of normal bundles. In the following sections, w_i denotes the ith Stiefel-Whitney class. $H^*(-)$ stands for mod 2 cohomology $H^*(-; \mathbb{F}_2)$.

5.1. **From condensed matter to math.** We list the corresponding tables of SPT symmetry groups and the stable group H for computing bordism groups in Table 1 and Table 2.

Cartan	Condensed Matter Symmetry (for fermionic electrons)	Full Symmetry: Minkowski vs. Euclidean	Cobordism Ω^4; Classification (3+1d)
CII	fTI ($T^2 = C^2 = (-1)^F$, $C \in \mathbb{Z}_2^C$): $U(1)^{\mathrm{c}} \rtimes [\mathbb{Z}_2^T \times \mathbb{Z}_2^C]$ $[U(1)^{\mathrm{c}} \rtimes \mathbb{Z}_2^C] \times \mathbb{Z}_2^{CT}$	$\frac{[U(1)^{\mathrm{c}} \rtimes \mathbb{Z}_4^C]}{\mathbb{Z}_2} \times \mathbb{Z}_2^{CT}$, $SU(2)^{\mathrm{c}} \times \mathbb{Z}_2^T$ vs. $\frac{[U(1)^{\mathrm{c}} \rtimes \mathbb{Z}_4^C] \times \mathbb{Z}_4^{CT}}{(\mathbb{Z}_2)^2}$ or $\frac{SU(2) \times \mathbb{Z}_4^T}{\mathbb{Z}_2}$	$\mathrm{Pin}^- \times_{\mathbb{Z}_2^F} SU(2)$; $(\nu_{\mathrm{CII}}, \alpha, \beta) \in \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
C	fTSC: $SU(2) \supset \mathbb{Z}_2^F$	$SU(2)$	$\mathrm{Spin} \times_{\mathbb{Z}_2^F} SU(2)$; No class
CI	fTSC ($T^2 = C^2 = (-1)^F$, $C \in \mathbb{Z}_{2,y}^{\mathrm{spin}}$): $SU(2)^{\mathrm{spin}} \times \mathbb{Z}_2^T$, $[U(1)_z^{\mathrm{spin}} \rtimes \mathbb{Z}_{2,y}^{\mathrm{spin}}] \times \mathbb{Z}_2^T$, $U(1)_z^{\mathrm{spin}} \rtimes [\mathbb{Z}_{2,y}^{\mathrm{spin}} \times \mathbb{Z}_2^{CT}]$	$\frac{SU(2)^{\mathrm{spin}} \times \mathbb{Z}_4^T}{\mathbb{Z}_2}$ vs. $SU(2) \times \mathbb{Z}_2^T$, $\frac{[U(1)_z^{\mathrm{spin}} \rtimes \mathbb{Z}_{4,y}^{\mathrm{spin}}] \times \mathbb{Z}_4^T}{(\mathbb{Z}_2)^2}$ vs. $\frac{[U(1)_z^{\mathrm{spin}} \rtimes \mathbb{Z}_{4,y}^{\mathrm{spin}}] \times \mathbb{Z}_2^T}{\mathbb{Z}_2}$, $\frac{U(1)_z^{\mathrm{spin}} \rtimes [\mathbb{Z}_{4,y}^{\mathrm{spin}} \times \mathbb{Z}_2^{CT}]}{\mathbb{Z}_2}$ vs. $\frac{U(1)_z^{\mathrm{spin}} \rtimes [\mathbb{Z}_{4,y}^{\mathrm{spin}} \times \mathbb{Z}_4^{CT}]}{(\mathbb{Z}_2)^2}$	$\mathrm{Pin}^+ \times_{\mathbb{Z}_2^F} SU(2)$; $(\nu_{\mathrm{CI}}, \alpha) \in \mathbb{Z}_4 \times \mathbb{Z}_2$
AI	fTSC ($T^2 = +1$):$U(1)^{\mathrm{c}} \rtimes \mathbb{Z}_2^T$	$U(1) \rtimes \mathbb{Z}_2^T$ vs. $\frac{U(1) \rtimes \mathbb{Z}_4^T}{\mathbb{Z}_2}$	$\mathrm{Pin}^- \ltimes_{\mathbb{Z}_2^F} U(1)$; $\alpha \in \mathbb{Z}_2$
BDI	fTSC ($T^2 = +1$): $\mathbb{Z}_2^T \times \mathbb{Z}_2^F$	$\mathbb{Z}_2^T \times \mathbb{Z}_2^F$ vs. $\mathbb{Z}_4^T$	Pin^-; No class
D	only $\mathbb{Z}_2^F$	$\mathbb{Z}_2^F$	Spin; No class
DIII	fTSC ($T^2 = (-1)^F$)	$\mathbb{Z}_4^T$ vs. $\mathbb{Z}_2^T \times \mathbb{Z}_2^F$	Pin^+; $\nu_{\mathrm{DIII}} \in \mathbb{Z}_{16}$
AII	fTI ($T^2 = (-1)^F$): $U(1)^{\mathrm{c}} \rtimes \mathbb{Z}_2^T$	$\frac{U(1) \rtimes \mathbb{Z}_4^T}{\mathbb{Z}_2}$ vs. $U(1) \rtimes \mathbb{Z}_2^T$	$\mathrm{Pin}^+ \ltimes_{\mathbb{Z}_2^F} U(1)$; $(\nu_{\mathrm{AII}}, \alpha, \beta) \in \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
A	$U(1) \supset \mathbb{Z}_2^F$	$U(1)$	Spin^c; No class
AIII	fTSC ($T^2 = (-1)^F$): $U(1)_z^{\mathrm{spin}} \times \mathbb{Z}_2^T$	$\frac{U(1)_z^{\mathrm{spin}} \times \mathbb{Z}_4^T}{\mathbb{Z}_2}$ vs. $U(1) \times \mathbb{Z}_2^T$	Pin^c; $(\nu_{\mathrm{AIII}}, \alpha) \in \mathbb{Z}_8 \times \mathbb{Z}_2$

TABLE 1. We list down symmetry groups related to 10 Cartan symmetry classes that contain $U(1)$, time reversal T, and/or charge/spin conjugation C symmetries. The second column shows symmetry notation in condensed matter. fTI/fTSC means fermionic Topological Insulator/Superconductor. The $U(1)^{\mathrm{c}}$ means the electromagnetic $U(1)^{\mathrm{charge}}$ symmetry. The $SU(2)^{\mathrm{c}}$ means the approximate charge symmetry, but there is no obvious $SU(2)$-charge symmetry from the electronic condensed matter. The $U(1)^{\mathrm{spin}}$ means the spin or orbital like $U(1)$ symmetry.

Particle Physics / QCD (or Cold Atom) Realization	Full Sym Minkowski vs. Euclidean	Cobordism Ω^4; Classification (3+1d)
$SU(2)_{\text{color}} \times SU(2)_{\text{flavor}}$, $T^2 = (-1)^F$	$\frac{(SU(2))^2 \times \mathbb{Z}_4^T}{(\mathbb{Z}_2)^2}$ vs. $\frac{(SU(2))^2}{\mathbb{Z}_2} \times \mathbb{Z}_2^T$	$\mathrm{Pin}^+ \times_{\mathbb{Z}_2^F} SO(4)$; $(\nu, \alpha, \beta, \gamma) \in \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
$SU(2)$, $T^2 = (-1)^F$	$\frac{SU(2) \times \mathbb{Z}_4^T}{\mathbb{Z}_2}$ vs. $SU(2) \times \mathbb{Z}_2^T$	$\mathrm{Pin}^+ \times_{\mathbb{Z}_2^F} SU(2)$; $(\nu_{\text{CI}}, \alpha) \in \mathbb{Z}_4 \times \mathbb{Z}_2$
$SU(3)$, $T^2 = (-1)^F$	$SU(3) \times \mathbb{Z}_4^T$ vs. $SU(3) \times \mathbb{Z}_2^F \times \mathbb{Z}_2^T$	$\mathrm{Pin}^+ \times SU(3)$; $(\nu, \alpha) \in \mathbb{Z}_{16} \times \mathbb{Z}_2$
$SU(4)$, $T^2 = (-1)^F$	$\frac{SU(4) \times \mathbb{Z}_4^T}{\mathbb{Z}_2}$ vs. $SU(4) \times \mathbb{Z}_2^T$	$\mathrm{Pin}^+ \times_{\mathbb{Z}_2^F} SU(4)$; $(\alpha, \beta, \gamma) \in \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.
$SU(2n+1)$, $T^2 = (-1)^F$	$SU(2n+1) \times \mathbb{Z}_4^T$ vs. $SU(2n+1) \times \mathbb{Z}_2^F \times \mathbb{Z}_2^T$	$\mathrm{Pin}^+ \times SU(2n+1)$; $(\nu, \alpha) \in \mathbb{Z}_{16} \times \mathbb{Z}_2$

TABLE 2. Time Reversal and $SU(N)$ Symmetry-Protected Topological Invariants:

5.2. $H = \mathbf{Pin}^+ \times_{\{\pm 1\}} SU(2)$. In this section, we first prove that

$$MTH \simeq M\mathrm{Spin} \wedge \Sigma^{-3} MO(3)$$

Then describe the $\mathcal{A}(1)$-module structure of $H^*(\Sigma^{-3}MO(3))$ and compute the homotopy groups by Adams spectral sequence.

5.2.1. *Understanding BH.* Recall that Pin^+ is an extensions of O by $\mathbb{Z}/2$. In particular, the classifying space $B\mathrm{Pin}^+$ is classified by the following fibration

$$\begin{array}{ccc} B\mathrm{Pin}^+ & & \\ \downarrow & & \\ BO & \xrightarrow{w_2} & K(\mathbb{Z}/2, 2) \end{array}$$

where $K(\mathbb{Z}/2, 2)$ is the Eilenberg-MacLane space.

Note that $SU(2) = \mathrm{Spin}(3) = S^3$ so it has the following fibration

$$\begin{array}{ccc} BSU(2) & & \\ \downarrow & & \\ BSO(3) & \xrightarrow{w_2} & K(\mathbb{Z}/2, 2) \end{array}$$

We have a commutative diagram that each square is a homotopy pullback square

$$\begin{tikzcd}
BH \arrow[r] \arrow[d] & BSO(3) \arrow[d, "w_2'"] \\
BO \arrow[r, "w_2"'] & K(\mathbb{Z}/2,2)
\end{tikzcd}$$

There is a homotopy equivalent $f : BO \times BSO(3) \xrightarrow{\sim} BO \times BSO(3)$ by $(V,W) \to (V-W+3,W)$. Note that $f^*(w_2) = w_2(V-W) = w_2(V)+w_1(V)w_1(W)+w_2(W) = w_2 + w_2'$ since W is oriented. Then we have the following homotopy pullback

$$\begin{tikzcd}
BH \arrow[r, "\sim"] \arrow[d] & B\mathrm{Pin}^+ \times BSO(3) \arrow[d] & \\
BO \times BSO(3) \arrow[r, "f"] \arrow[d, "pr_1{,}V"] \arrow[rr, bend right, "w_2+w_2'"'] & BO \times BSO(3) \arrow[r, "w_2+0"] \arrow[dl, "V+W-3"] & K(\mathbb{Z}/2,2) \\
BO & &
\end{tikzcd}$$

This implies that

$$BH \sim B\mathrm{Pin}^+ \times BSO(3) \tag{5.1}$$

We note that there is a pullback diagram

$$\begin{tikzcd}
B\mathrm{Spin} \times BO(1) \arrow[r, "\sim", "V+L-1"'] \arrow[d] & B\mathrm{Pin}^+ \arrow[d] & \\
BSO \times BO(1) \arrow[r, "V+L-1"] \arrow[rr, bend right, "w_2"'] & BO \arrow[r, "w_2"] & K(\mathbb{Z}/2,2)
\end{tikzcd}$$

and we have $BO(3) \sim BSO(3) \times BO(1)$ by $V \mapsto (V \otimes \mathrm{Det}\, V, \mathrm{Det}\, V)$. Thus, we have the following homotopy pullback square

$$\begin{tikzcd}
BH \arrow[r, "\sim"] \arrow[d] & B\mathrm{Pin}^+ \times BSO(3) \arrow[r] \arrow[d] & B\mathrm{Spin} \times BO_3 \arrow[d] & \\
BO \times BSO(3) \arrow[r, "(V+W-3{,}W)"'] \arrow[d, "(V{,}W)\mapsto V"] & BO \times BSO(3) \arrow[r, "(V-\mathrm{Det}V+1{,}W\otimes\mathrm{Det}V)"'] & BSO \times BO(3) \arrow[r, "w_2"] \arrow[dll, bend left, "(E{,}F)\mapsto E+\mathrm{Det}F-F\otimes\mathrm{Det}F+2"] & K(\mathbb{Z}/2,2) \\
BO & & &
\end{tikzcd}$$

5.2.2. *Understanding $B\hat{H}$.*

Write $P = K(\mathbb{Z}/2,1) \times K(\mathbb{Z}/2,2)$ with the group structre

$$(x_1, x_2) * (y_1, y_2) = (x_1 + y_1, x_2 + y_2 + x_1 y_1)$$

in which $x_i,\ y_i \in H^i(-)$ With this choice the map $BO \xrightarrow{(w_1,w_2)} P$ is a group homomorphism.

Define $B\hat{H} \to BO$ by the homotopy pullback square

$$\begin{tikzcd}
B\hat{H} \arrow[r] \arrow[d] & BO(3) \arrow[d, "{(w_1,w_2)}"] \\
BO \arrow[r, "{(w_1,w_2+w_1^2)}"'] & P
\end{tikzcd}$$

Then we have a homotopy square involving $B\hat{H} \to BO$ like below

$$\begin{tikzcd}
B\hat{H} \arrow[r] \arrow[d] & B\mathrm{Spin} \times BO(3) \arrow[d] & \\
BO \times BO(3) \arrow[r, "{id-(V_3-s)}"'] \arrow[d, "{(V,W)\mapsto V}"'] & BO \times BO(3) \arrow[r, "{(w_1,w_2)}"'] \arrow[dl, "{(W,V_3)\mapsto -W-(V_3-3)}"] & P \\
BO & &
\end{tikzcd}$$

Thus $B\hat{H} \to BO$ can be identified with the map

$$B\mathrm{Spin} \times BO(3) \to BO$$

$$(W, V_3) \mapsto -W - (V_3 - 3)$$

This leads to the following equivalence

$$MT\hat{H} \sim M\mathrm{Spin} \wedge \Sigma^{-3} MO(3) \tag{5.2}$$

5.2.3. *Identification of $B\hat{H} \to BO$ with $BH \to BO$.*

The homotopy fiber of $B\hat{H} \to BO$ being the same as the homotopy fiber of $BO(3) \to P$ is $B\mathrm{Spin}_3$. We can identify $B\hat{H}$ with BH. We know that BH sit in a homotopy pullback

$$\begin{tikzcd} BH \arrow[r] \arrow[d] & BSO_3 \arrow[d, "w_2'"] \\ BO \arrow[r, "w_2"] & K(\mathbb{Z}/2,2) \end{tikzcd}$$

and $B\hat{H}$ sit in the pullback

$$\begin{tikzcd} B\hat{H} \arrow[r] \arrow[d, "(w_1,w_2)"'] & BO(3) \arrow[d, "(w_1,w_2)"] \\ BO(n) \arrow[r, "(w_1,w_2+w_1^2)"'] & P \end{tikzcd}$$

To identify them, expand the first homotopy pullback to the diagram

$$\begin{tikzcd} B\hat{H} \arrow[r] \arrow[d] & BO(3) \arrow[r, "V_3\otimes \mathrm{Det}V_3"] \arrow[d, "(w_1,w_2)"] & BSO(3) \arrow[d, "w_2'"] \\ BO \arrow[r, "(w_1,w_2+w_1^2)"'] \arrow[rr, bend right, "w_2"'] & P \arrow[r, "w_1^2+w_2"'] & K(\mathbb{Z}/2,2) \end{tikzcd}$$

Thus, we can identify $B\hat{H} \to BO$ with $BH \to BO$. With these identification, we have

$$MTH \sim M\mathrm{Spin} \wedge \Sigma^{-3}MO(3) \tag{5.3}$$

These are useful for computing homotopy groups of MTH.

Remark 5.1. From the following diagram,

$$\begin{tikzcd} BH \arrow[r] \arrow[d] & BSO(3) \arrow[d, "w_2'"] \\ BO \arrow[r, "w_2"] & K(\mathbb{Z}/2,2) \end{tikzcd}$$

we can think of the nth homotopy group $\pi_n MTH$ as the bordism group of n-manifolds with a $SO(3)$-bundle $V_{SO(3)}$ such that the 2nd Stiefel-Whitney classes of tangent bundle TM and of $V_{SO(3)}$ agrees, $w_2(TM) = w_2(V_{SO(3)})$. If we use the other model $B\hat{H} \simeq B\mathrm{Spin} \times BO(3) \to BO$ by $(W, V_3) \mapsto -W - (V_3 - 3)$, then V_3 can be identified by $V_{SO(3)} \otimes (TM - n)$.

5.2.4. *Identify $\mathcal{A}(1)$-module structure of $H^*(\Sigma^{-3}MO(3))$.*

$$H^*(\Sigma^{-3}MO(3)) = \mathbb{F}_2[w_1, w_2, w_3]U$$

U stands for Thom class of the universal 3-bundle E_3 over $BO(3)$ and w_i is the ith Stiefel-Whitney class of E_3 over $BO(3)$.

It would be helpful to use the equivalence $BO(1) \times BSO(3) \to BO(3)$ classifying the tensor product of the defining vector bundles. Write

$$w_i \in H^i(BO(3))$$

$$v_i \in H^i(BSO(3))$$

$$v_1 \in H^1(BO(1))$$

for the corresponding Stiefel-Whitney classes, so under the equivalence we have

$$w_1 = v_1$$

$$w_2 = v_2 + v_1^2$$

$$w_3 = v_3 + v_2 v_1 + v_1^3$$

Note for $H^*(MO(1)) = \mathbb{F}_2[v_1]U_1$ and Wu formula,

$$Q_0 U_1 = v_1 U_1,\ Q_0 v_1 = v_1^2,$$

$$Q_1 U_1 = v_1^3 U_1,\ Q_1 v_1 = 0,$$

We can easily get that

$$H^*(MO(1); Q_0) = 0$$

$$H^*(MO(1); Q_1) = \mathbb{F}_2\{v_1\}U_1$$

For $H^*(MSO(3)) = \mathbb{F}_2[v_2, v_3]U_3$ and Wu formula,

$$Q_0U_3 = 0,\ Q_0v_2 = v_3,\ Q_0v_3 = 0.$$

$$Q_1U_3 = v_3U_3,\ Q_1v_2 = v_2v_3,\ Q_0v_3 = v_3^2$$

We can easily get that

$$H^*(MSO(3); Q_0) = \mathbb{F}_2[v_2^2]U_3$$

For Q_1 homology, $H^*(MSO(3))$, as a module over the exterior algebra $E[Q_1]$, is a sum of vector spaces of basis

$$\{v_2^j, v_2^jv_3, w_2^j, v_2^jv_3^2, v_2^j, v_2^jv_3^3, \cdots\}U_3$$

Using this, we can see that

$$H^*(MSO(3); Q_1) = \mathbb{F}_2\{v_2^{2j+1}\}U_3$$

$H^*(MO(3)) = H^*(MSO(3) \otimes H^*(MO(1))$ and so (write $U = U_1U_3$)

$$H^*(MO(3); Q_0) = 0$$

$$H^*(MO(3); Q_1) = \mathbb{F}_2\{v_1v_2^{2j+1}\}U$$

Let M and N be the $\mathcal{A}(1)$-modules depicted in Figure 6.

Then the map

$$(M \oplus N) \otimes \mathbb{F}_2[v_2^4] \to H^*(MO(3))$$

is a monomorphism and induces an isomorphism of Margolis homology groups. It follows by Lemma 4.4 that

$$H^*(\Sigma^{-3}MO(3)) = (M \oplus N) \otimes \mathbb{F}_2[v_2^4] \oplus \text{free}$$

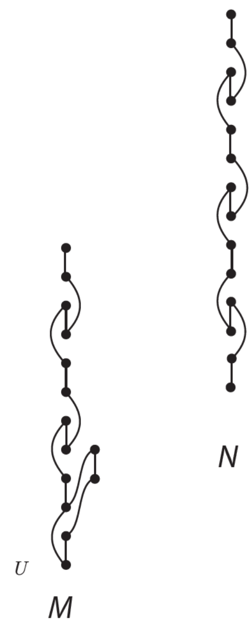

FIGURE 6. Non-free summand of $H^*(MO(3))$

The Poincare series for the indecomposables of the free modules is the quotient of

$$\frac{1}{(1-t)(1-t^2)(1-t^3)} - \frac{(1-t)^{-1} + t^3 + t^4 + t^6(1-t)^{-1}}{1-t^8}$$

by the Poincare series $(1+t)(1+t^2)(1+t^3)$ of $\mathcal{A}(1)$. It turns out to be

$$\frac{t^2}{(1-t^4)(1-t^8)} + \frac{t^4 + t^5 + t^6 + t^9 + t^{10} + t^{11} + t^{12} + t^{15}}{(1-t^4)(1-t^8)(1-t^{12})}$$

The free modules correspond to

$$\mathcal{A}(1)[w_1^4, w_2^4]\{w_1^2\}U \oplus \mathcal{A}(1)[w_1^4, w_2^4, w_3^4]\{w_2^2, w_2w_3, w_3^2, w_2^3w_3, w_1^2w_2^3w_3, w_1^2w_2^2w_3^2, w_2^3w_3^3\}U$$

By Lemma 4.6, it suffices to show that $Sq^2Sq^2Sq^2$ of all basis in

$$\mathcal{A}(1)[w_1^4, w_2^4]\{w_2\}U \oplus \mathcal{A}(1)[w_1^4, w_2^4, w_3^4]\{w_2^2, w_2w_3, w_3^2, w_2^3w_3, w_1^2w_2^3w_3, w_1^2w_2^2w_3^2, w_2^3w_3^3\}U$$

are

$$\{(w_1^4w_2^3 + w_1^3w_2^2w_3 + w_1^2w_2w_3 + w_1w_3^3), (w_1^4w_2^2w_3 + w_1^2w_3^3),$$

$$(w_1^4w_3^2 + w_1^3w_3^3), (w_1^2w_2^2w_3^3 + w_3^5), (w_1^2w_2w_3^4 + w_1w_3^5), (w_1^6w_2^4w_3 + w_1^2w_3^5),$$

$$(w_1^6w_2^3w_3^2 + w_1^5w_2^2w_3^3 + w_1^4w_2w_3^4 + w_1^3w_3^3), (w_1^4w_2^4w_3^3 + w_3^7)\}w_1^{4k}w_2^{4i}w_3^{4j}U$$

linearly independent, which we can check.

5.2.5. *Computation of* $\mathrm{Ext}_{\mathcal{A}(1)}(-, \mathbb{F}_2)$. Now we can compute

$$\mathrm{Ext}_{\mathcal{A}(1)}(M, \mathbb{F}_2) \Rightarrow (\pi_* ko \wedge M)_2^\wedge$$

$$\mathrm{Ext}_{\mathcal{A}(1)}(N, \mathbb{F}_2) \Rightarrow (\pi_* ko \wedge N)_2^\wedge$$

The Ext term are depicted as Figure 7 and 8

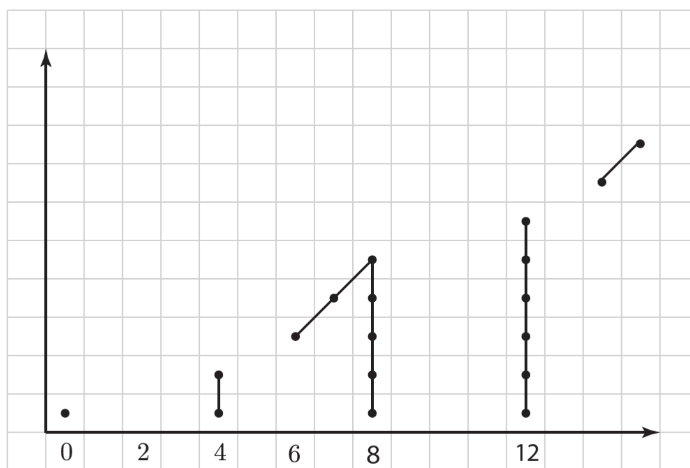

FIGURE 7. Ext of M

Both spectral sequences collapse. we have

i	0	1	2	3	4	5	6	7	8	9	10	11	12
$\pi_i ko \wedge M$	$\mathbb{Z}/2$	0	0	0	$\mathbb{Z}/4$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/32$	0	0	0	$\mathbb{Z}/64$

i	0	1	2	3	4	5	6	7	8	9	10	11	12
$\pi_{i+6} ko \wedge N$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/8$	0	0	0	$\mathbb{Z}/16$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/128$	0

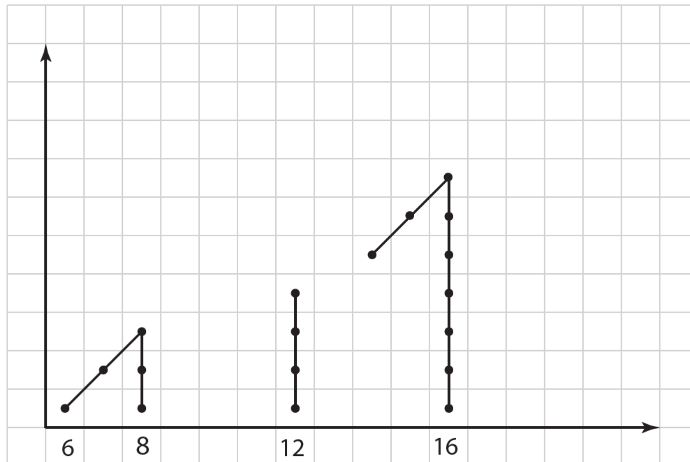

FIGURE 8. Ext of N

Recall that

$$\pi_* M\mathrm{Spin} \wedge X = ko_*(X) \oplus \Sigma^8 ko_*(X) \oplus \Sigma^{10} M_J(X) \oplus \cdots \oplus \Sigma^{20} H_*(X, \mathbb{F}_2) \oplus \cdots$$

we have

Theorem 5.2. *The bordism groups of MTH are*

i	0	1	2	3	4	5	6	7	8	9
$\pi_i MTH$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/4 \oplus \mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2^{\oplus 4}$	$\mathbb{Z}/2^{\oplus 2}$	$\mathbb{Z}/32 \oplus \mathbb{Z}/2^{\oplus 3}$	$\mathbb{Z}/2^{\oplus 2}$

5.2.6. *Manifold generators of the 4^{th} homotopy group.*

Theorem 5.3. $(\mathbb{CP}^2, L_{\mathbb{C}}+1)$ *and* $(\mathbb{RP}^4, 3)$ *generate* $\pi_4 MTH$*, where* $L_{\mathbb{C}}$ *is tautological complex line bundle over* $\mathbb{CP}^2$*.*

Proof. First, check that $(\mathbb{CP}^2, L_{\mathbb{C}}+1)$ and $(\mathbb{RP}^4, 3)$ are elements in $\pi_4 MTH$.

$$\begin{tikzcd} \mathbb{CP}^2 \arrow[r, "L_{\mathbb{C}}+1"] \arrow[d, "T\mathbb{CP}^2"'] & BSO(3) \arrow[d, "w_2'"] \\ BO \arrow[r, "w_2"] & K(\mathbb{Z}/2, 2) \end{tikzcd}$$

with $w_2(T\mathbb{CP}^2) = w_2(L_{\mathbb{C}} + 1)$

$$\begin{tikzcd} \mathbb{RP}^4 \arrow[r, "3"] \arrow[d, "T\mathbb{RP}^4"'] & BSO(3) \arrow[d, "w_2'"] \\ BO \arrow[r, "w_2"] & K(\mathbb{Z}/2, 2) \end{tikzcd}$$

with $w_2(T\mathbb{RP}^4) = w_2(3)$

From the above spectral sequence, we have a map

$$\pi_4 MTH = \mathbb{Z}/4 \oplus \mathbb{Z}/2 \to \mathbb{Z}/2 \oplus \mathbb{Z}/2$$

$$(M, V_{SO(3)}) \mapsto (\int_M w_1 w_3(V_{SO(3)} \otimes (TM - 4)), \int_M w_2^2(V_{SO(3)} \otimes (TM - 4)))$$

In particular,

$$w_1(V_{SO(3)} \otimes (TM - 4)) = w_1(TM)$$

$$w_2(V_{SO(3)} \otimes (TM - 4)) = w_1^2(TM) + w_2(TM)$$

$$w_3(V_{SO(3)} \otimes (TM - 4)) = w_1^3(TM) + w_1(TM)w_2(TM) + w_3(V_{SO(3)})$$

$(\mathbb{R}P^4, 3)$ is sent to (1,1) and $(\mathbb{C}P^2, L + 1)$ is sent to $(0, 1)$. So they generates. If the invariants are chosen to be $w_1^4(TM)$ and $w_2^2(TM)$, it gives the same results. □

5.3. $H = \mathbf{Pin}^- \times_{\{\pm 1\}} SU(2)$**.** In the following sections, we first prove that

$$MTH \simeq M\mathrm{Spin} \wedge \Sigma^3 MTO(3)$$

Then describe the $\mathcal{A}(1)$-module structure of $H^*(\Sigma^3 MTO(3))$ and compute the homotopy groups by Adams spectral sequence.

5.3.1. *Understanding BH.* Recall that Pin^- is an extensions of O by $\mathbb{Z}/2$ with the following fibration

$$\begin{tikzcd} B\mathrm{Pin}^- \arrow[d] & \\ BO \arrow[r, "w_1^2 + w_2"] & K(\mathbb{Z}/2, 2) \end{tikzcd}$$

Thus, the case of $H = \mathrm{Pin}^- \times_{\{\pm 1\}} SU(2)$ is analogous to case of $H = \mathrm{Pin}^+ \times_{\{\pm 1\}} SU(2)$ by just exchanging w_2 and $w_1^2 + w_2$.

We have a commutative diagram that each square is a homotopy pullback square

$$\begin{tikzcd}
BH \arrow[r, "\sim"] \arrow[d] & B\mathrm{Pin}^- \times BSO(3) \arrow[d] & \\
BO \times BSO(3) \arrow[r, "f", "\sim"'] \arrow[d, "pr_1=V"] \arrow[rr, bend right, "w_1^2+w_2+w_2'"'] & BO \times BSO(3) \arrow[r, "(w_1^2+w_2+0)"] \arrow[dl, "V+W-3"] & K(\mathbb{Z}/2,2) \\
BO & &
\end{tikzcd}$$

This implies that

$$BH \sim B\mathrm{Pin}^- \times BSO(3) \tag{5.4}$$

We have the following homotopy pullback square

$$\begin{tikzcd}
BH \arrow[r, "\sim"] \arrow[d] & BPin^- \times BSO_3 \arrow[r] \arrow[d] & BSpin \times BO_3 \arrow[d] & \\
BO \times BSO(3) \arrow[r, "(V-W+3,W)"'] \arrow[d, "(V,W)\mapsto V"] & BO \times BSO(3) \arrow[r, "(V+\mathrm{Det}V-1,W\otimes \mathrm{Det}V)"'] & BSO \times BO(3) \arrow[r, "(w_1,w_2)"] \arrow[dll, bend left, "(E,F)\mapsto E-\mathrm{Det}F+F\otimes \mathrm{Det}F-2"] & K(\mathbb{Z}/2,2) \\
BO & & &
\end{tikzcd}$$

5.3.2. *Understanding $B\hat{H}$.* Write $P = K(\mathbb{Z}/2,1) \times K(\mathbb{Z}/2,2)$ with the group structre

$$(x_1, x_2) * (y_1, y_2) = (x_1 + y_1, x_2 + y_2 + x_1 y_1)$$

in which x_i, $y_i \in H^i(-)$ With this choice the map $BO \xrightarrow{(w_1,w_2)} P$ is a group homomorphism.

Then define $B\hat{H} \to BO$ to be the composition of $B\hat{H} \to BO$ of case $H = \mathrm{Pin}^+ \times_{\{\pm 1\}} SU(2)$ with $BO \xrightarrow{-id} BO$, so we have the following homotopy pullback square

$$\begin{tikzcd}
B\hat{H} \arrow[r] \arrow[d] & BO(3) \arrow[d, "{(w_1,w_2)}"] \\
BO \arrow[r, "{(w_1,w_2)}"'] & P
\end{tikzcd}$$

Then we have a homotopy square involving $B\hat{H} \to BO$ like below

$$\begin{tikzcd}
B\hat{H} \arrow[r] \arrow[d] & B\mathrm{Spin} \times BO(3) \arrow[d, "{(-id,id)}"] & \\
BO \times BO_3 \arrow[r, "id-(V_3-s)"'] \arrow[d, "{(V,W)\mapsto V}"'] & BO \times BO(3) \arrow[r, "{(w_1,w_2)}"'] \arrow[dl, "{(W,V_3)\mapsto W+(V_3-3)}"] & P \\
BO & &
\end{tikzcd}$$

Thus $B\hat{H} \to BO$ can be identified with the map

$$B\mathrm{Spin} \times BO(3) \to BO$$

$$(W, V_3) \mapsto -W + (V_3 - 3)$$

This leads to the following equivalence

$$MT\hat{H} \sim M\mathrm{Spin} \wedge \Sigma^3 MTO(3) \tag{5.5}$$

5.3.3. *Identification of $B\hat{H} \to BO$ with $BH \to BO$.*

The homotopy fiber of $B\hat{H} \to BO$ being the same as the homotopy fiber of $BO(3) \to P$ is $B\mathrm{Spin}_3$. We can identify $B\hat{H}$ with BH. We know that BH sit in a homotopy pullback

$$\begin{tikzcd}
BH \arrow[r] \arrow[d] & BSO(3) \arrow[d, "w_2'"] \\
BO \arrow[r, "w_1^2+w_2"] & K(\mathbb{Z}/2,2)
\end{tikzcd}$$

and $B\hat{H}$ sit in the pullback

$$\begin{tikzcd} B\hat{H} \arrow[r] \arrow[d] & BO(3) \arrow[d, "{(w_1,w_2)}"] \\ BO \arrow[r, "{(w_1,w_2)}"'] & P \end{tikzcd}$$

To identify them, expand the first homotopy pullback to the diagram

$$\begin{tikzcd} B\hat{H} \arrow[r] \arrow[d] & BO(3) \arrow[r, "V_3\otimes \mathrm{Det}V_3"] \arrow[d, "{(w_1,w_2)}"] & BSO(3) \arrow[d, "w_2'"] \\ BO \arrow[r, "{(w_1,w_2)}"'] \arrow[rr, bend right, "w_1^2+w_2"'] & P \arrow[r, "w_1^2+w_2"'] & K(\mathbb{Z}/2,2) \end{tikzcd}$$

Thus, we can identify $B\hat{H} \to BO$ with $BH \to BO$. With these identification, we have

$$MTH \sim M\mathrm{Spin} \wedge \Sigma^3 MTO(3) \tag{5.6}$$

Remark 5.4. From the following diagram,

$$\begin{tikzcd} BH \arrow[r] \arrow[d] & BSO(3) \arrow[d, "w_2'"] \\ BO \arrow[r] & K(\mathbb{Z}/2,2) \end{tikzcd}$$

we can think of $\pi_n MTH$ as the bordism group of n-manifolds with a $SO(3)$-bundle $V_{SO(3)}$ such that $w_1^2 + w_2(TM) = w_2(V_{SO(3)})$. If we use the other model $B\hat{H} \simeq B\mathrm{Spin} \times BO(3) \to BO$ by $(W, V_3) \mapsto -W + (V_3 - 3)$, then V_3 can be identified by $V_{SO(3)} \otimes (TM - n)$.

5.3.4. *Identify $\mathcal{A}(1)$-module structure of $H^*(\Sigma^3 MTO(3))$.*

$$H^*(\Sigma^3 MTO(3)) = \mathbb{F}_2[w_1, w_2, w_3]U$$

where U stands for Thom class of $-E_3$ over $BO(3)$ and w_i is the ith Stiefel-Whitney class of E_3 over $BO(3)$.

The Margolis homology is the same as that of $\Sigma^{-3} MO(3)$ since the ratio of the two

Thom class is $\Sigma^{-6}w_3^2$ which is annihilated by the Milnor operations. The non-free modules for $H^*(\Sigma^3 MTO(3))$ are depicted in Figure **??**:

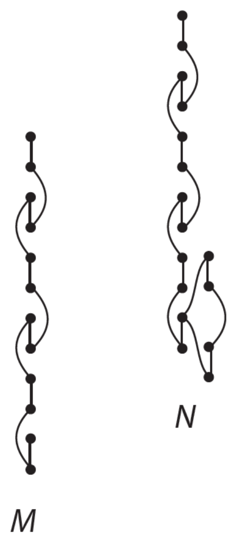

FIGURE 9. non-free modules M and N

The Poincare series for the free modules as the quotients of

$$\frac{1}{(1-t)(1-t^2)(1-t^3)} - \frac{t^2(1-t)^{-1} + t^6(1-t)^{-1} + t^5 + t^6 + t^8 + t^9}{1-t^8}$$

by the Poincare series $(1+t)(1+t^2)(1+t^3)$ of $\mathcal{A}(1)$. It turns out to be

$$\frac{t^7}{(1-t^4)(1-t^8)} + \frac{1+t^4+t^6+t^9+t^{10}+t^{11}+t^{15}+t^{17}}{(1-t^4)(1-t^8)(1-t^{12})}$$

The free modules correspond to the direct sum of

$$\mathcal{A}(1)[w_1^4, w_2^4]\{w_1^2 w_2 w_3\}U$$

and

$$\mathcal{A}(1)[w_1^4, w_2^4, w_3^4]\{1, w_2^2, w_1^2 w_2^2, w_2^3 w_3, w_2^2 w_3^2, w_2 w_3^3, w_2^3 w_3^3, w_1^2 w_2^3 w_3^3\}U$$

5.3.5. *Computation of* $\mathrm{Ext}_{\mathcal{A}(1)}(-, \mathbb{F}_2)$. Now we can compute

$$\mathrm{Ext}_{\mathcal{A}(1)}(M, \mathbb{F}_2) \Rightarrow (\pi_* ko \wedge M)_2^\wedge$$

$$\mathrm{Ext}_{\mathcal{A}(1)}(N, \mathbb{F}_2) \Rightarrow (\pi_* ko \wedge N)_2^\wedge$$

The Ext term are depicted as Figure 10 and 11

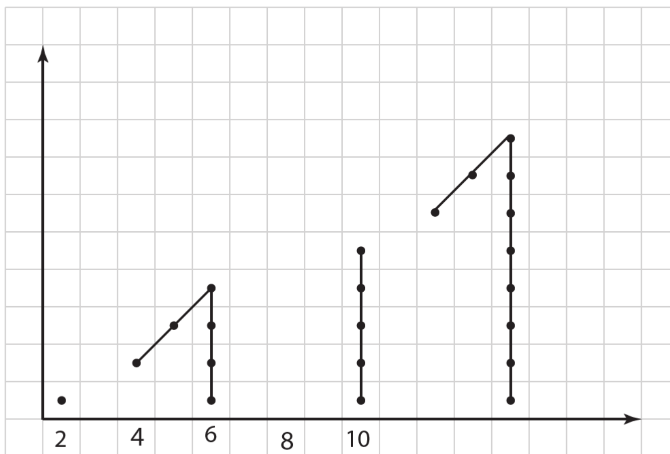

FIGURE 10. Ext of M

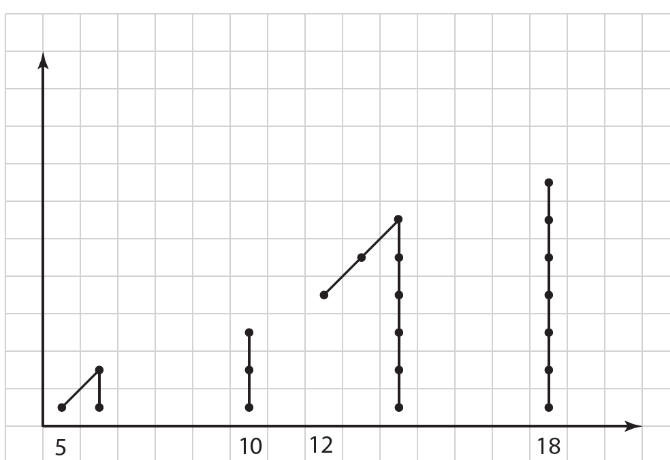

FIGURE 11. Ext of N

Both spectral sequences collapse. we have

i	0	1	2	3	4	5	6	7	8	9	10	11	12
$\pi_{i+2} ko \wedge M$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/16$	0	0	0	$\mathbb{Z}/32$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/256$

i	0	1	2	3	4	5	6	7	8	9	10	11	12
$\pi_{i+5} ko \wedge N$	$\mathbb{Z}/2$	$\mathbb{Z}/4$	0	0	0	$\mathbb{Z}/8$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/64$		

Recall that

$$\pi_* M\mathrm{Spin} \wedge X = ko_*(X) \oplus \Sigma^8 ko_*(X) \oplus \Sigma^{10} M_J(X) \oplus \cdots \oplus \Sigma^{20} H_*(X, \mathbb{F}_2) \oplus \cdots$$

we have

Theorem 5.5. *The bordism groups of MTH are*

i	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*
$\pi_i MTH$	$\mathbb{Z}/2$	*0*	$\mathbb{Z}/2$	*0*	$\mathbb{Z}/2^{\oplus 3}$	$\mathbb{Z}/2^{\oplus 2}$	$\mathbb{Z}/16 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2^{\oplus 2}$	$\mathbb{Z}/2^{\oplus 2}$

5.3.6. *Manifold generators of the 4^{th} homotopy group.*

Theorem 5.6. *The generators of $\pi_4 MTH = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$ are (S^4, H), $(\mathbb{CP}^2, L_{\mathbb{C}} + 1)$, and $(\mathbb{RP}^4, 2L_{\mathbb{R}} + 1)$, where H is the induced oriented 3-dimensional vector bundle from Hopf bundle $S^7 \to S^4$, $L_{\mathbb{C}}$ is the tautological complex line bundle over $\mathbb{CP}^2$ and $L_{\mathbb{R}}$ is the tautological real line bundle over $\mathbb{RP}^4$.*

Proof. First, check that (S^4, H), $(\mathbb{CP}^2, L_{\mathbb{C}} + 1)$, and $(\mathbb{RP}^4, 2L_{\mathbb{R}} + 1)$ are elements in $\pi_4 MTH$.

$$\begin{tikzcd} \mathbb{CP}^2 \arrow[r, "L_{\mathbb{C}}+1"] \arrow[d, "T\mathbb{CP}^2"'] & BSO(3) \arrow[d, "w_2'"] \\ BO \arrow[r, "w_1^2 + w_2"] & K(\mathbb{Z}/2, 2) \end{tikzcd}$$

with $w_1^2 + w_2(T\mathbb{CP}^2) = w_2(L_{\mathbb{C}} + 1)$

$$\begin{array}{ccc} \mathbb{RP}^4 & \xrightarrow{2L_{\mathbb{R}}+1} & BSO(3) \\ {\scriptstyle T\mathbb{RP}^4}\downarrow & & \downarrow{\scriptstyle w_2'} \\ BO & \xrightarrow{w_1^2+w_2} & K(\mathbb{Z}/2,2) \end{array}$$

with $w_1^2 + w_2(T\mathbb{RP}^4) = w_2(2L_{\mathbb{R}} + 1)$.

$$\begin{array}{ccc} S^4 & \xrightarrow{H} & BSO(3) \\ {\scriptstyle TS^4}\downarrow & & \downarrow{\scriptstyle w_2'} \\ BO & \xrightarrow{w_1^2+w_2} & K(\mathbb{Z}/2,2) \end{array}$$

with $w_1^2 + w_2(TS^4) = w_2(H)$.

From the spectral sequence in the previous section, we have a map

$$\pi_4 MTH \to \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$$

$$(M, V_{SO(3)}) \mapsto (\text{mod 2 index of Dirac operator}, \int_M w_1^4(TM), \int_M w_2^2(TM))$$

We can see (S^4, H), $(\mathbb{CP}^2, L+1)$, and $(\mathbb{RP}^4, 2L_{\mathbb{R}} + 1)$ are the generators. □

Remark 5.7. There is a map $MTH \to MTO$ if we forget the H-structure on stable tangent bundles. We know the latter one is isomorphic to $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ generated by $\mathbb{CP}^2$ and $\mathbb{RP}^4$. The kernel of this map is generated by (S^4, H) where H is the induced $SO(3)$ bundle from Hopf bundle $S^7 \to S^4$.

5.4. $H = \mathbf{Pin}^+ \times_{\{\pm 1\}} SO(4)$**.** In this section, we first prove that

$$MTH \simeq M\mathrm{Spin} \wedge \Sigma^{-3} MO(3) \wedge \Sigma^{-3} MSO(3)$$

Then describe the $\mathcal{A}(1)$-module structure of $H^*(\Sigma^{-3}MO(3)) \otimes H^*(\Sigma^{-3}MSO(3))$ and and compute the homotopy groups by Adams spectral sequence.

5.4.1. *Understanding BH and $B\hat{H}$.* There is a homotopy pullback square:

$$\begin{tikzcd} BH^+ \arrow[r] \arrow[d] & BSO(3)\times BSO(3) \arrow[d, "w_2'+w_2''"] \\ BO \arrow[r, "w_2"'] & K(\mathbb{Z}/2,2) \end{tikzcd}$$

Similar to computation of $MT\mathrm{Pin}^+ \times_{\pm 1} SU(2)$, if we define a new space $B\hat{H}$ to sit in the following homotopy pullback

$$\begin{tikzcd} B\hat{H} \arrow[r] \arrow[d] & BO(3)\times BSO(3) \arrow[d, "(w_1',w_2')+(w_1'',w_2'')"] \\ BO \arrow[r, "w_1^2+w_2"'] & P \end{tikzcd}$$

Remark 5.8. From the following diagram,

$$\begin{tikzcd} BH \arrow[r] \arrow[d] & BSO(3)\times BSO(3) \arrow[d, "w_2'+w_2''"] \\ BO \arrow[r, "w_2"'] & K(\mathbb{Z}/2,2) \end{tikzcd}$$

we can think of $\pi_n MTH$ as the bordism group of n-manifolds with two oriented 3-dimensional vector bundle V_1 and V_2 such that then second Stiefel-Whitney class $w_2(TM)$ of tangent bundle TM agrees with the sum of second Stiefel-Whitney classes $w_2(V_1)+w_2(V_2)$ of V_1 and V_2. If we use the other model $B\hat{H} \simeq B\mathrm{Spin}\times BO(3)\times BSO(3) \to BO$ by $(W, V_3, V_2) \mapsto -W-(V_3+V_2-6)$, then V_3 can be identified by $V_1 \otimes (TM-n)$.

5.4.2. *Identify $\mathcal{A}(1)$-module structure of $H^*(\Sigma^{-6}MO(3)\wedge MSO(3))$.*

$$H^*(\Sigma^{-6}MO(3)\wedge MSO(3)) = H^*(\Sigma^{-3}MO(3))\otimes H^*(\Sigma^{-3}MSO(3))$$

Write as w_i, v_i' section 5.2.4 for the corresponding Stiefel-Whitney classes for $MO(3)$ and $v_i \in H^i(MSO(3))$ for the corresponding Stiefel-Whitney classes. Thus by Kunneth formula,

$$H^*(MO(3)\wedge MSO(3); Q_0) = 0$$

$$H^*(MO(3)\wedge MSO(3); Q_1) = \mathbb{F}_2\{v_1 v_2^{2j+1} {v'}_2^{2i+1}\}U$$

We know from previous section,

$$H^*(MSO(3); Q_0) = \mathbb{F}_2[w_2^2]U_3$$

$$H^*(MSO(3); Q_1) = \mathbb{F}_2\{v_2^{2j+1}\}U_3$$

Let P and Q be the $\mathcal{A}(1)$-modules depicted in Figure 12.

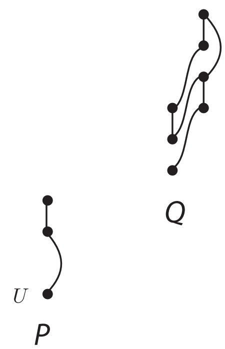

FIGURE 12. Non-free summand of $H^*(MSO(3))$

Then the map

$$(P \oplus Q) \otimes \mathbb{F}_2[v_2^4] \to H^*(MSO(3))$$

is a monomorphism and induces an isomorphism of Margolis homology groups. It follows by Lemma 4.4 that

$$H^*(\Sigma^{-3}MSO(3)) = (P \oplus Q) \otimes \mathbb{F}_2[v_2^4] \oplus \text{free}$$

The Poincare series for the indecomposables of the free modules is the quotient of

$$\frac{1}{(1-t^2)(1-t^3)} - \frac{1+t^2+t^3+t^4(1+t+2t^2+t^3+t^4+t^5)}{1-t^8}$$

by the Poincare series $(1+t)(1+t^2)(1+t^3)$ of $\mathcal{A}(1)$. It turns out to be

$$\frac{t^9}{(1-t^6)(1-t^9)}$$

The summand of free modules is

$$\mathcal{A}(1)[v_3^2, v_2^4] \cdot v_2^3 v_3 U_3$$

We can check they are linear independent because $Sq^2Sq^2Sq^2$ of basis are $\{v_3^{2i}v_2^{4j}v_3^6\}$ which are linear independent. Now let M, N as in section 5.2.4 and focus on $P \otimes M$, $P \otimes N$, $Q \otimes M$ and $Q \otimes N$.

Let M_i, $1 \leq i \leq 4$ be the $\mathcal{A}(1)$-modules depicted in Figure 13.

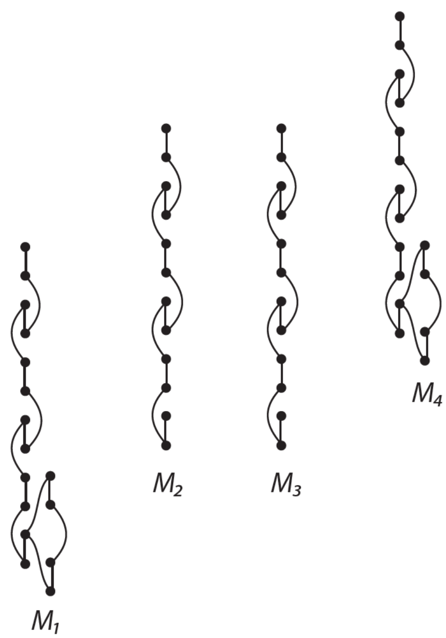

FIGURE 13. Non-free summand of $H^*(MSO_3 \wedge MO_3)$

From the same technique, we can see that

$$P \otimes M = M_1 \oplus \mathcal{A}(1)[w_1^4]\{w_1^3 w_3\}U_1U_3 \oplus \mathcal{A}(1)\{1, w_2\}U_1U_3$$

$$P \otimes N = M_2 \oplus \mathcal{A}(1)[w_1^4]\{w_1w_3v_2^2, w_1^2w_3v_2^2, w_1^3w_3v_2^2\}U_1U_3 \oplus \mathcal{A}(1)\{v_2^2, v_2v_3, w_1v_2v_3, w_2v_2^2, v_2^2v_3\}U_1U_3$$

$$Q \otimes M = M_3 \oplus \mathcal{A}(1)[w_1^4]\{w_1w_2w_3 + w_2^3 + w_3^2\}U_1U_3$$

$$Q \otimes N = M_4 \oplus \mathcal{A}(1)[w_1^4]\{(w_1w_2w_3 + w_2^3 + w_3^2)v_2^2, (w_1w_2w_3 + w_2^3 + w_3^2)v_3^2, (w_1^2w_3^2 + w_2w_3^2)v_2^2\}U_1U_3$$

Then the direct sum of

$$(\oplus_i M_i) \otimes \mathbb{F}_2[v_2^4, {v'}_2^4] \to H^*(\Sigma^{-6} MO(3) \wedge MSO(3))$$

is a monomorphism and induces an isomorphism of Margolis homology groups. The free summand is direct sum of

$$\mathbb{F}_2[v_2^4, {v'}_2^4] \otimes \text{above free summand}$$

and

$$\mathbb{F}_2[v_2, v_3] \otimes \text{ free summand of } H^*(MO_3)$$

and

$$(M + N) \otimes \text{ free summand of } H^*(MSO_3)$$

5.4.3. *Computation.* $B\hat{H} \to BO$ is identified with $BH \to BO$. We can see that the spectrum MTH is homotopy equivalent to the spectrum $M\mathrm{Spin} \wedge MO(3) \wedge MSO(3)$.

Now we can compute

$$\mathrm{Ext}_{\mathcal{A}(1)}(M_1, \mathbb{F}_2) \Rightarrow (\pi_* ko \wedge M_1)_2^\wedge$$

$$\mathrm{Ext}_{\mathcal{A}(1)}(M_2, \mathbb{F}_2) \Rightarrow (\pi_* ko \wedge M_2)_2^\wedge$$

$$\mathrm{Ext}_{\mathcal{A}(1)}(M_3, \mathbb{F}_2) \Rightarrow (\pi_* ko \wedge M_3)_2^\wedge$$

$$\mathrm{Ext}_{\mathcal{A}(1)}(M_4, \mathbb{F}_2) \Rightarrow (\pi_* ko \wedge M_4)_2^\wedge$$

From section 5.2.5 and 5.3.5, we known

i	0	1	2	3	4	5	6	7	8	9	10	11	12
$\pi_{i+3} ko \wedge M_1$	$\mathbb{Z}/2$	$\mathbb{Z}/4$	0	0	0	$\mathbb{Z}/8$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/64$		

i	0	1	2	3	4	5	6	7	8	9	10	11	12
$\pi_{i+8} ko \wedge M_2$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/16$	0	0	0	$\mathbb{Z}/32$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/256$

i	0	1	2	3	4	5	6	7	8	9	10	11	12
$\pi_{i+8} ko \wedge M_3$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/16$	0	0	0	$\mathbb{Z}/32$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/256$

i	0	1	2	3	4	5	6	7	8	9	10	11	12
$\pi_{i+11} ko \wedge M_4$	$\mathbb{Z}/2$	$\mathbb{Z}/4$	0	0	0	$\mathbb{Z}/8$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/64$		

Recall that

$$\pi_* M\mathrm{Spin} \wedge X = ko_*(X) \oplus \Sigma^8 ko_*(X) \oplus \Sigma^{10} M_J(X) \oplus \cdots \oplus \Sigma^{20} H_*(X, \mathbb{F}_2) \oplus \cdots$$

we have

Theorem 5.9. *The bordism groups of MTH are*

i	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*
$\pi_i MTH$	$\mathbb{Z}/2$	*0*	$\mathbb{Z}/2^{\oplus 2}$	$\mathbb{Z}/2$	$\mathbb{Z}/4 \oplus \mathbb{Z}/2^{\oplus 3}$	$\mathbb{Z}/2^{\oplus 3}$	$\mathbb{Z}/2^{\oplus 8}$	$\mathbb{Z}/2^{\oplus 4}$	$\mathbb{Z}/8 \oplus \mathbb{Z}/2^{\oplus 12}$	$\mathbb{Z}/2^{\oplus 8}$

5.4.4. *Manifold generators of the 4th homotopy groups.*

Theorem 5.10. *The generators of $\pi_4 MTH = \mathbb{Z}/4 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$ are $(\mathbb{RP}^4, 3, 3)$, $(\mathbb{CP}^2, L_{\mathbb{C}}+1, 3)$, $(\mathbb{RP}^4, 2L_{\mathbb{R}}+1, 2L_{\mathbb{R}}+1)$, $(\mathbb{CP}^2, 3, L_{\mathbb{C}}+1)$. $L_{\mathbb{R}}$ ($L_{\mathbb{C}}$) is the tautological (complex) line bundle over $\mathbb{RP}^4$ ($\mathbb{CP}^2$).*

Proof. First check that they are elements in $\pi_4 MTH$

$$\begin{tikzcd} \mathbb{RP}^4 \arrow[r, "{(3,3)}"] \arrow[d, "TM"'] & BSO(3) \times BSO(3) \arrow[d, "w_2' + w_2''"] \\ BO \arrow[r, "w_2"'] & K(\mathbb{Z}/2, 2) \end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{CP}^2 \arrow[r, "(L_{\mathbb{C}}+1{,}3)"] \arrow[d, "TM"'] & BSO(3)\times BSO(3) \arrow[d, "w_2'+w_2''"] \\
BO \arrow[r, "w_2"'] & K(\mathbb{Z}/2,2)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{RP}^4 \arrow[r, "(2L_{\mathbb{R}}+1{,}2L_{\mathbb{R}}+1)"] \arrow[d, "TM"'] & BSO(3)\times BSO(3) \arrow[d, "w_2'+w_2''"] \\
BO \arrow[r, "w_2"'] & K(\mathbb{Z}/2,2)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{CP}^2 \arrow[r, "(3{,}L_{\mathbb{C}}+1)"] \arrow[d, "TM"'] & BSO(3)\times BSO(3) \arrow[d, "w_2'+w_2''"] \\
BO \arrow[r, "w_2"'] & K(\mathbb{Z}/2,2)
\end{tikzcd}$$

The corresponding invariants mapping to $\mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2$ are $w_1^4(TM)+w_1^2(TM)w2(V_1)$, $w_1^4(TM)+w_2^2(V_1)$, $w_1^2(TM)w_2(V_2)$ and $w_2^2(V_2)$ respectively. We can check they generates $\mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2$, and thus generates $\mathbb{Z}/4\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2$. $\square$

5.5. $H = \mathbf{Pin}^+\times SU(3)$**.** In this section, we first prove that

$$MTH \simeq MT\mathrm{Pin}^+\wedge BSU(3)_+$$

Then describe the $\mathcal{A}(1)$-module structure of $H^{*-1}(MTO(1))\otimes H^*(BSU(3))$ and and compute the homotopy groups by Adams spectral sequence.

5.5.1. *Identify $\mathcal{A}(1)$-module structure.* The integral cohomology $H^*(BSU(3);\mathbb{Z}) = \mathbb{Z}[c_2,c_4]$, where c_2, c_3 are Chern classes. The mod 2 cohomology $H^*(BSU(3)) = \mathbb{F}_2[w_4,w_6]$, where $w_{2i}=c_i$ (mod 2). Thus, we have

$$H_*(BSU(3);Q_0)=\mathbb{F}_2[w_4,w_6]$$

$$H_*(BSU(3);Q_1)=\mathbb{F}_2[w_4,w_6]$$

Therefore,

$$H_*(MTO(1) \wedge BSU(3); Q_0) = 0$$

$$H_*(BSU(3) \wedge BSU(3); Q_1) = \mathbb{F}_2[w_4, w_6]w_1U$$

Let M_1, M_2 and N depicted as Figure 14

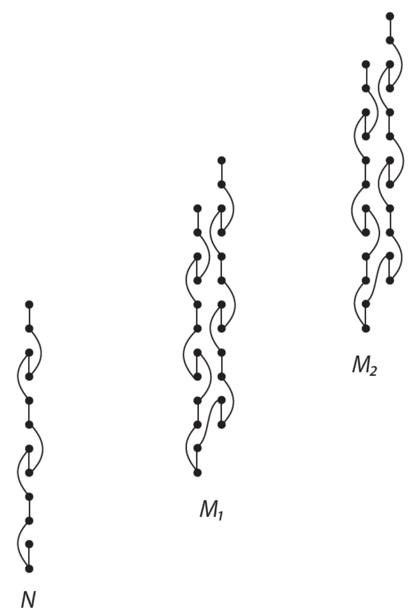

FIGURE 14. M_1, M_2 and N

The map

$$(M_1 \oplus M_2) \otimes \mathbb{F}_2[w_4^2, w_6^2] \bigoplus N \otimes \mathbb{F}_2[w_4^2] \to H_*(MTO(1) \wedge BSU(3))$$

is an isomorphism on Margolis homology and thus is a monomorphism. And Poincare series of the free part is

$$\frac{1}{(1-t)(1-t^4)(1-t^6)} - \frac{t^4(1-t)^{-1} + t^6(1-t)^{-1} + t^{10}(1-t)^{-1} + t^{12}(1-t)^{-1}}{(1-t^8)(1-t^{12})} - \frac{1}{(1-t)(1-t^8)}$$

is 0, which is trivial.

5.5.2. *Computation.* Now we can compute

$$\mathrm{Ext}_{\mathcal{A}(1)}(M_1, \mathbb{F}_2) \Rightarrow (\pi_* ko \wedge M_1)_2^\wedge$$

$$\mathrm{Ext}_{\mathcal{A}(1)}(M_2, \mathbb{F}_2) \Rightarrow (\pi_* ko \wedge M_2)_2^\wedge$$

$$\mathrm{Ext}_{\mathcal{A}(1)}(N, \mathbb{F}_2) \Rightarrow (\pi_* ko \wedge N)_2^\wedge$$

The Ext term are depicted as Figure 15 and 16

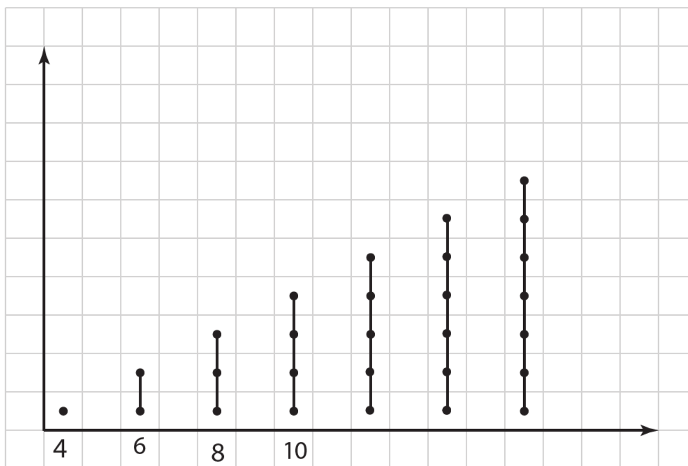

FIGURE 15. Ext of M

Note the Atiyan-Hirzebruch spectral sequence of $M\mathrm{Pin}_*^+(BSU(3))$ collapses and Adams spectral sequence here collapses too.

i	0	1	2	3	4	5	6	7	8	9	10	11	12
$\pi_{i+4} ko \wedge M_1$	$\mathbb{Z}/2$	0	$\mathbb{Z}/4$	0	$\mathbb{Z}/8$	0	$\mathbb{Z}/16$	0	$\mathbb{Z}/32$	0	$\mathbb{Z}/64$	0	$\mathbb{Z}/128$

i	0	1	2	3	4	5	6	7	8	9	10	11	12
$\pi_{i+10} ko \wedge M_2$	$\mathbb{Z}/2$	0	$\mathbb{Z}/4$	0	$\mathbb{Z}/8$	0	$\mathbb{Z}/16$	0	$\mathbb{Z}/32$	0	$\mathbb{Z}/64$	0	$\mathbb{Z}/128$

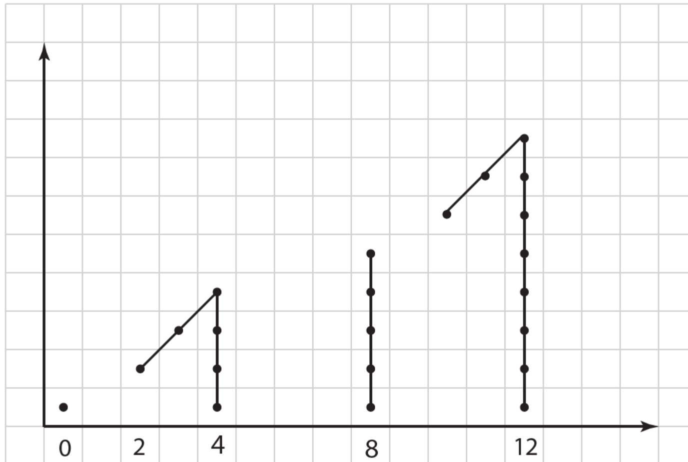

FIGURE 16. Ext of N

i	0	1	2	3	4	5	6	7	8	9	10	11	12
$\pi_i ko \wedge N$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/16$	0	0	0	$\mathbb{Z}/32$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/256$

Recall that

$$\pi_* M\mathrm{Spin} \wedge X = ko_*(X) \oplus \Sigma^8 ko_*(X) \oplus \Sigma^{10} M_J(X) \oplus \cdots \oplus \Sigma^{20} H_*(X, \mathbb{F}_2) \oplus \cdots$$

we have

Theorem 5.11. *The bordism groups of MTH are*

i	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*
$\pi_i MTH$	$\mathbb{Z}/2$	*0*	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/16 \oplus \mathbb{Z}/2$	*0*	$\mathbb{Z}/4$	0	$\mathbb{Z}/32 \oplus \mathbb{Z}/8 \oplus \mathbb{Z}/2$	*0*

5.5.3. *Manifold generators of the 4th homotopy groups.*

Theorem 5.12. *The generators of $\pi_4 MTH = \mathbb{Z}_{16} \oplus \mathbb{Z}_2$ are $(\mathbb{RP}^4, \mathbb{RP}^4 \times SU(3))$ and (S^4, H) where H is the Hopf fibration $S^7 \to S^4$ considered as a $SU(2)$ bundle by $SU(2) \to SU(3)$.*

Proof. If we think of MTH as a Pin^+ 4-manifold with a $SU(3)$- bundles W, the corresponding invariants are eta invariant and $c_2 \pmod 2$ of W. They generate the bordism groups of MTH.

$\square$

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