Multidimensional treatment of photon emission from accretion discs around black holes

Mon. Not. R. Astron. Soc. **381,** 1267–1274 (2007) doi:10.1111/j.1365-2966.2007.12319.x

Multidimensional treatment of photon emission from accretion discs around black holes

Kazunori Kohri,[1,2★] Ken Ohsuga[3] and Ramesh Narayan[1]

[1]*Harvard–Smithsonian Center for Astrophysics, MS-51, 60 Garden Street, Cambridge, MA 02138, USA*
[2]*Physics Department, Lancaster University, Lancaster LA1 4YB*
[3]*Department of Physics, Rikkyo University, Toshimaku, Tokyo 171-8501, Japan*

Accepted 2007 August 3. Received 2007 August 3; in original form 2007 February 1

ABSTRACT

We consider photon emission from a supercritical accretion disc in which photons in the inner regions are trapped and advected in towards the centre. Such discs are believed to be present in failed supernovae and gamma-ray bursts, and possibly also in ultraluminous X-ray sources. We show that the luminosity from a supercritical accretion disc is proportional to the logarithm of the mass accretion rate when the vertical profile of the matter density is exponential. We obtain analytical results also for other density profiles, but the dependence of the luminosity on the accretion rate is no longer logarithmic.

Key words: accretion, accretion discs – black hole physics – hydrodynamics – radiative transfer.

1 INTRODUCTION

There is wide consensus that active galactic nuclei (AGN), gamma-ray bursts (GRBs), X-ray binaries, etc. are powered by accretion flows on to compact relativistic objects, most often a black hole (BH). A well-known model of such accretion flows is the standard disc model proposed by Shakura & Sunyaev (1973), which has been used to explain a variety of observations of AGN and X-ray binaries.

Recently, it has been recognized that observations of many objects cannot be explained by the standard disc model. For instance, this model cannot produce the very high temperature ($T > 10^{10}$ K) and broad-band spectrum (extending from 10^9 to $>10^{18}$ Hz) observed in the Galactic Centre source Sgr A* and in X-ray binaries in the hard state. This has led to the idea of an advection-dominated accretion flow (Narayan & Yi 1995; Abramowicz et al. 1995,[1]) which is also called a radiatively inefficient accretion flow (RIAF). The RIAF model explains the low-luminosity, high-temperature and optically thin emission of these systems.[2]

The standard disc cannot also be applied to very high-luminosity accretion discs in which the mass accretion rate $\dot{M}$ exceeds the critical mass accretion rate $\dot{M}_{\rm crit}$ ($\equiv L_{\rm E}/c^2$, where $L_{\rm E}$ is the Eddington luminosity). In this regime, the disc becomes optically very thick and as a result photons are partially trapped inside the accreting gas. This is the regime of interest to us.

Photon trapping is important at any radius where the accretion time-scale is longer than the photon diffusion time-scale from the disc mid-plane to the surface (Begelman 1978). The photons produced in this region of the disc are advected into the central BH and are unable to escape from the flow. Begelman considered only spherically symmetric accretion, but it is recognized that photon trapping is important even for systems with a disc geometry.

Photon trapping has been included approximately in the so-called 'slim disc model' proposed by Abramowicz, Igumenshchev & Lasota (1998), and in recent calculations by Szuszkiewicz et al. (1996) and Watarai et al. (2000), as well as in numerical simulations by Honma, Matsumoto & Kato (1991), Szuszkiewicz & Miller (1997) and Watarai & Mineshige (2003). However, these studies do not model the effect fully since they consider vertically integrated quantities in the disc and reduce the problem to a 1D model. Full 2D radiation-hydrodynamical (2D-RHD) simulations were done for the first time by one of the current authors in Ohsuga et al. (2005).[3]

A realistic model of supercritical accretion must include the following two effects. (i) Even at disc radii inside the trapping radius, photons that are emitted near the surface of the disc can escape, though most of the photons emitted deeper inside are advected into the BH. (ii) The vertical profiles of various quantities in the disc such as density, optical depth, etc. play a critical role in determining what fraction of the photons escape, and thereby influence the total luminosity of the disc.

The first effect was pointed out and studied by Ohsuga et al. (2002) using simple analytical models and numerical simulations. In this paper, we include the second effect and study the combined

★E-mail: k.kohri@lancaster.ac.uk

[1] We note that Ichimaru (1977) proposed similar ideas 20 yr earlier.

[2] For recent developments on the RIAF model of Sgr. A*, see Manmoto, Kusunowse & Mineshige (1997), Yuan, Cui & Narayan (2005) and references therein.

[3] Eggum, Coroniti & Katz (1988) performed the first 2D numerical simulations of supercritical accretion discs, and their work was later improved by Okuda (2002). However, the computation time in these studies was short and the numerical models did not reach steady state. Ohsuga et al. (2005) were the first to successfully compute models approaching steady state.

influence of both effects for a broad class of accretion-disc models. We develop an analytical model and show that it agrees well with the numerical results of Ohsuga et al. (2005).

The results obtained here may change the current understanding of models such as the slim disc and convection-dominated accretion flow model (CDAF, Stone, Pringle & Begelman 1999; Igumenshchev, Abramowicz & Narayan 2000; Narayan, Igumenshchev & Abramowicz 2000; Quataert & Gruzinov 2000), as well as the neutrino-dominated accretion flow (NDAF) model (Popham, Woosley & Fryer 1999; Narayan, Piran & Kumar 2001; Di Matteo, Perna & Narayan 2002; Kohri & Mineshige 2002; Kohri, Narayan & Piran 2005; Gu, Liu & Lu 2006) which is believed to play an important role in the central engine of GRBs (Narayan, Paczynski & Piran 1992).[4]

In the following we use R to represent the cylindrical radius of a point with respect to the central mass and z to represent its vertical distance from the equatorial plane of the disc. We also assume that the temperature T in the disc is lower than the electron mass ($kT < m_e c^2$), so that the electrons are non-relativistic.

2 STANDARD 1D APPROXIMATIONS

In this section we introduce the standard 1D treatment of accretion discs, following the approach described in Ohsuga et al. (2002). In this approach, we integrate along the z-axis and omit the z dependences of various astrophysical quantities. For instance, we write the optical depth from the mid-plane of the disc ($z = 0$) to the surface ($z = z_{\max}$) as

$$\tau = \int_0^{z_{\max}} \mathrm{d}z\, \sigma_{\gamma e} n_e \tag{1}$$

$$\simeq \sigma_T n_e H, \tag{2}$$

where $\sigma_{\gamma e}$ is the scattering cross-section of a photon off an electron. Although $\sigma_{\gamma e}$ depends in general on the photon energy E_γ, we limit ourselves to Thomson scattering, for which the cross-section is independent of energy. This is reasonable for the low-energy photons $E_\gamma \ll \mathcal{O}$ (MeV) of interest to us. H is the disc half-thickness and n_e is the electron number density in the disc mid-plane.

The diffusion time-scale for a photon to escape from the disc mid-plane is given by

$$t_{\rm diff} \approx N_{\rm scatt} \frac{\lambda}{c}, \tag{3}$$

where $N_{\rm scatt}$ is the number of scatterings, $\lambda = 1/(\sigma_T n_e)$ is the mean free path and c is the speed of light. Since $N_{\rm scatt}$ is given by

$$N_{\rm scatt} \approx 3\tau^2 \tag{4}$$

(assuming a random walk in three dimensions, see Appendix B), we write

$$t_{\rm diff} \approx 3\tau H/c. \tag{5}$$

The accretion time-scale is

$$t_{\rm acc} = -\frac{R}{v_r}, \tag{6}$$

where the radial velocity v_r is related to the mass accretion rate $\dot{M}$ and the density ρ by

$$v_r = -\frac{\dot{M}}{4\pi R \int_0^H \rho\, \mathrm{d}z} \tag{7}$$

$$= -\frac{\dot{M}}{4\pi R \rho H}. \tag{8}$$

Thus

$$t_{\rm acc} \approx \frac{2\tau}{\dot{m}} \frac{R^2}{c\, R_g}, \tag{9}$$

where $R_g = 2GM/c^2$ is the Schwarzschild radius, and we have set $\rho \approx m_p n_e$ and introduced a dimensionless mass accretion rate $\dot{m}$,

$$\dot{m} = \frac{\dot{M}}{\dot{M}_{\rm crit}}. \tag{10}$$

The critical mass accretion rate $\dot{M}_{\rm crit}$ is defined in terms of the Eddington luminosity L_E as follows: $\dot{M}_{\rm crit} = L_E/c^2$, where $L_E = 4\pi c G m_p M/\sigma_T \simeq 1.3 \times 10^{38}\,{\rm erg\,s^{-1}}\ (M/{\rm M_\odot})$ for a BH of mass M.

Photons can escape from the disc only if the condition $t_{\rm diff} < t_{\rm acc}$ is satisfied. Therefore, the disc radiates freely only from radii larger than a certain limit,

$$R > R_{\rm trap}, \tag{11}$$

where the trapping radius $R_{\rm trap}$ is given by

$$R_{\rm trap} = \frac{3}{2} h_{\rm in} \dot{m} R_g \tag{12}$$

and

$$h_{\rm in} \equiv \left.\frac{H}{R}\right|_{R \leqslant R_{\rm trap}}, \tag{13}$$

assuming that, when $R < R_{\rm trap}$ and radiation is trapped, the disc is geometrically thick with $h_{\rm in} \sim 1$ and that h here is independent of R. As we show in Appendix A, h is indeed constant ($\approx$0.5) for $R \leqslant R_{\rm trap}$ and decreases $\propto 1/R$ for $R_{\rm trap} < R$. This validates equation (12). Note that $R_{\rm trap}$ is proportional to $\dot{m}$.

The energy released in the disc per unit time by viscous dissipation is given by

$$L = 2\int_{R_{\rm in}}^{R_{\rm out}} 2\pi R\, Q_{\rm vis}(R)\, \mathrm{d}R, \tag{14}$$

where the viscous heating rate is

$$Q_{\rm vis}(R) = \frac{3}{8\pi} \frac{GM}{R^3} \dot{M} \left[1 - \left(\frac{R_{\rm in}}{R}\right)^{1/2}\right]. \tag{15}$$

Here $R_{\rm in}$ and $R_{\rm out}$ are the radii of the inner and outer edge of the disc. However, not all the released energy is radiated because of photon trapping. We discuss two cases below, depending on whether $R_{\rm trap}$ is larger or smaller than $R_{\rm in}$. For readers' convenience we introduce a dimensionless inner radius normalized by the Schwarzschild radius.

$$r_{\rm in} \equiv \frac{R_{\rm in}}{R_g}. \tag{16}$$

[4] The relativistic magnetohydrodynamics jet model is another attractive model of energetic accretion sources (e.g. McKinney 2005, 2006, and references therein). This model is less likely to be affected by our work.

2.1 $R_{\rm trap} < R_{\rm in}$

When $R_{\rm trap} < R_{\rm in}$, there is no trapped region and we estimate the total luminosity to be

$$L_{\rm 1D,tot,\mathit{R}_{trap}<\mathit{R}_{in}} = 2\int_{R_{\rm in}}^{\infty} 2\pi R Q_{\rm vis}(R)\,{\rm d}R = \frac{1}{6h_{\rm in}} L_{\rm E}\frac{R_{\rm trap}}{R_{\rm in}} = \frac{L_{\rm E}}{4r_{\rm in}}\dot{m}, \quad (17)$$

which is linearly proportional to $\dot{m}$. This is a well-known result for the standard thin accretion disc model in Newtonian gravity.

2.2 $R_{\rm in} < R_{\rm trap}$

When $R_{\rm in} < R_{\rm trap}$, only the region of the disc outside $R_{\rm trap}$ radiates freely and the luminosity from this region of the disc is

$$L_{{\rm 1D},R_{\rm trap}<R} = 2\int_{R_{\rm trap}}^{\infty} 2\pi R Q_{\rm vis}(R)\,{\rm d}R = \frac{1}{2h_{\rm in}}L_{\rm E}\left[1-\frac{2}{3}\left(\frac{R_{\rm trap}}{R_{\rm in}}\right)^{-1/2}\right] = \frac{1}{2h_{\rm in}}L_{\rm E}\left[1-\sqrt{\frac{8r_{\rm in}}{27h_{\rm in}}}\,\dot{m}^{-1/2}\right]. \quad (18)$$

Within the 1D approximation being discussed in this section, it is unclear how much luminosity is emitted from the region of the disc $R_{\rm in} < R < R_{\rm trap}$. One extreme assumption is that there is no luminosity at all from this region, as in Ohsuga et al. (2002). Alternatively, one might assume that the disc flux here is limited to the local Eddington flux $F \approx F_{\rm E}(R) = L_{\rm E}/(4\pi R^2) = GMm_{\rm p}c/(R^2 \sigma_{\rm T})$. The luminosity from $R_{\rm in} < R < R_{\rm trap}$ of the disc is then given by

$$L_{{\rm 1D},R<R_{\rm trap}} = 2\int_{R_{\rm in}}^{R_{\rm trap}} 2\pi R F_{\rm Edd}(R)\,{\rm d}R = L_{\rm E}\ln\left(\frac{R_{\rm trap}}{R_{\rm in}}\right) = L_{\rm E}\ln\left(\frac{3h_{\rm in}}{2r_{\rm in}}\dot{m}\right). \quad (19)$$

If we include the term given in equation (19), keeping in mind that it is uncertain, then the 'total luminosity' becomes

$$L_{\rm 1D,tot,R_{in}<R_{trap}} = L_{{\rm 1D},R_{\rm trap}<R} + L_{{\rm 1D},R<R_{\rm trap}} = \frac{1}{2h_{\rm in}}L_{\rm E}\left[1+2h_{\rm in}\ln\left(\frac{R_{\rm trap}}{R_{\rm in}}\right)-\frac{2}{3}\left(\frac{R_{\rm trap}}{R_{\rm in}}\right)^{-1/2}\right] = \frac{1}{2h_{\rm in}}L_{\rm E}\left[1+2h_{\rm in}\ln\left(\frac{3h_{\rm in}}{2r_{\rm in}}\dot{m}\right)-\sqrt{\frac{8r_{\rm in}}{27h_{\rm in}}}\,\dot{m}^{-1/2}\right]. \quad (20)$$

Note that the luminosity is reduced relative to the result in equation (17) for the untrapped case, and increases only logarithmically with $\dot{m}$. This explains why discs with supercritical accretion ($\dot{m} \gg 1$, 'slim' discs) are radiatively inefficient.

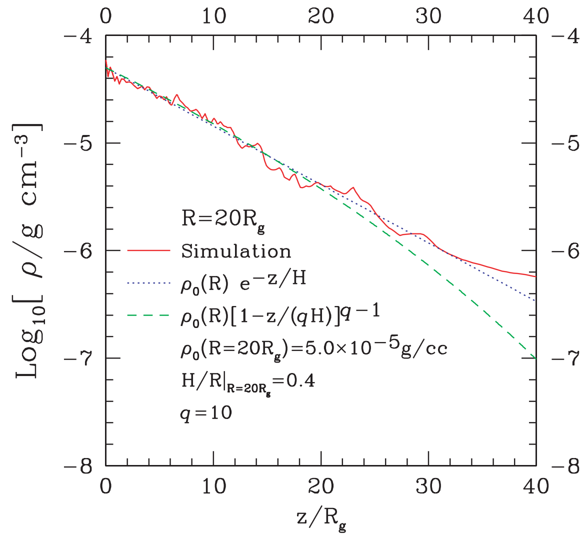

Figure 1. Variation of the matter density ρ with height z at a radius $R = 20R_{\rm g}$. The solid line represents the result from the numerical simulation of Ohsuga et al. (2005). The data are fitted well with either an exponential (dotted line) or a power-law (dashed) profile, with $H/R = 0.4$ and (for the power-law profile) $q = 10$. The numerical results in Ohsuga et al. (2005) can be fitted with the same value of $H/R = 0.4$ for all radii up to $\sim 50R_{\rm g}$.

3 2D TREATMENT

So far we have discussed a 1D approximation and assumed somewhat arbitrarily that the region $R < R_{\rm trap}$ either does not radiate at all or radiates at the local Eddington rate. In this section we carry out a more careful 2D analysis. We focus on the region $R_{\rm in} \leqslant R < R_{\rm trap}$ discussed in Section 2.2, but now we carefully consider the 2D distribution of quantities.

In the following, we consider most of the quantities in the disc to depend on both R and z (the vertical height). However, we assume that the thin-disc approximation is approximately valid in the sense that we omit the z dependence when considering hydrostatic balance in the z direction ($c_{\rm S} = \Omega_{\rm K}H$, with the sound speed $c_{\rm S}$ and the Keplerian angular velocity $\Omega_{\rm K} = \sqrt{GM/R^3}$) and angular momentum conservation ($\nu\int\rho{\rm d}z = \dot{M}(1-\sqrt{R_{\rm in}/R})/(3\pi)$, with the kinetic viscosity ν). Then the z-dependent optical depth at a given R is given by

$$\tau(z) = \int_z^{z_{\rm max}} {\rm d}z\,\sigma_{\gamma e}n_{\rm e}(z) \quad (21)$$

$$\simeq \sigma_{\rm T}\int_z^{z_{\rm max}} {\rm d}z\,n_{\rm e}(z), \quad (22)$$

where $n_{\rm e}(z)$ is the electron number density at height z, and $z_{\rm max}$ corresponds to the upper surface of the disc. We see that the expression for τ given in equation (1) in the 1D approximation corresponds to $\tau(0)$ in equation (21).

We now introduce a detailed model for the vertical electron density profile. As seen in Fig. 1, we can fit the density profile found in the numerical simulation in Ohsuga et al. (2005) quite well with the following power-law form,

$$n_{\rm e}(z) = n_{\rm e}(0)\left(1-\frac{z}{qH}\right)^{q-1} \quad (q \geqslant 1,\ \text{and } 0 \leqslant z \leqslant qH), \quad (23)$$

where $n_e(0)$ and the power-law index q may depend on the radius R. In the limit as $q \to \infty$, this model takes the form of an exponential[5]

$$n_e(z) = n_e(0) \exp\left(-\frac{z}{H}\right) \qquad (0 \leqslant z \leqslant \infty). \tag{24}$$

The optical depth from z to the surface is

$$\tau(z) = \tau(0) \left(1 - \frac{z}{qH}\right)^q, \tag{25}$$

except for $q \to \infty$ when it takes an exponential form.

Because we adopt the numerical results corresponding to $\dot{m} \gtrsim 100$ in Ohsuga et al. (2005), the radii R of the order of tens of R_g that we consider are smaller than $R_{trap} \sim 10^2 R_g$ and so we are in the radiation-trapped regime. Note that Fig. 1 gives $H/R \approx 0.4$ for the numerical simulation of Ohsuga et al. (2005), which is approximately consistent with the approximation $h_{in} = 0.5$ that we made in the previous section and in Appendix A.

The photon diffusion time-scale from height z to the surface of the disc at z_{max} is given by

$$t_{diff}(z) \approx N_{scatt}(z, z_{max})\, \lambda(z)/c, \tag{26}$$

where $N_{scatt}(z, z_{max})$ is the number of scatterings in the 3D random walk,

$$N_{scatt}(z, z_{max}) = 3\ [\tau(z)]^2, \tag{27}$$

and $\lambda(z)$ is the mean free path which is estimated using the properties of the disc at the starting position z of the photon

$$\lambda(z) = \frac{1}{\sigma_T\, n_e(z)}. \tag{28}$$

The detailed derivation of the above relation is given in Appendix B. By using equations (27) and (28), we obtain an expression for the z-dependent diffusion time-scale,

$$\begin{aligned} t_{diff}(z) &= 3\tau(z)^2 \lambda(z)/c \\ &= 3\tau(0) \frac{H}{c} \left[\frac{\tau(z)}{\tau(0)}\right]^{(q+1)/q}. \end{aligned} \tag{29}$$

By requiring the diffusion time-scale $t_{diff}(z)$ to be shorter than the accretion time-scale t_{acc} given in equation (9), we find that photons can escape only from the region where the following condition is satisfied,

$$\frac{\tau(z)}{\tau(0)} < \left(\frac{R}{R_{trap}}\right)^{q/(q+1)}. \tag{30}$$

The photon-trapping radius R_{trap} is determined by equation (12) and is proportional $\dot{m}$. A schematic picture indicating the region of the disc from which radiation can escape is shown in Fig. 2. We see that trapping is not determined by simply a critical radius R_{trap}, but is described in terms of a 2D surface $z_{trap}(R)$, where

$$\frac{\tau(z_{trap})}{\tau(0)} \equiv \left(\frac{R}{R_{trap}}\right)^{q/(q+1)}. \tag{31}$$

[5] If on the other hand we have a constant z-independent density, as considered by Ohsuga et al. (2002), it would correspond to $q = 1$. The corresponding results can be recovered by setting $q = 1$ in all the expressions in the current paper.

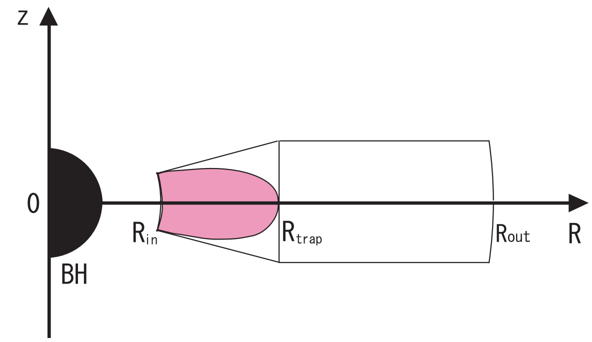

Figure 2. Schematic representation of a supercritical accretion disc around a BH. The shaded region shows the photon-trapping region from which no photons can escape. We see that, even for $R < R_{trap}$, there exist regions near the surface of the disc from which photons can diffuse out (white region).

Note that z_{trap} is a function of R in this case given by

$$z_{trap} = qH \left[1 - \left(\frac{R}{R_{trap}}\right)^{1/(q+1)}\right], \tag{32}$$

and its second-order derivative by R is always negative $d^2 z_{trap}/dR^2 < 0$ for $R < R_{trap}$ because

$$\frac{dz_{trap}}{dR} = q h_{in} \left[1 - \frac{q+2}{q+1} \left(\frac{R}{R_{trap}}\right)^{1/(q+1)}\right] \tag{33}$$

and

$$\frac{d^2 z_{trap}}{dR^2} = -\frac{q(q+2)}{q+1} h_{in} \frac{1}{R_{trap}} \left(\frac{R}{R_{trap}}\right)^{-q/(q+1)} \tag{34}$$

for $R < R_{trap}$. The luminosity from the region $R_{in} \leqslant R \leqslant R_{trap}$ is then given by

$$L_{2D, R<R_{trap}} = 2 \int_{R_{in}}^{R_{trap}} 2\pi R Q_{vis}(R) \frac{\tau(z_{trap})}{\tau(0)}\, dR. \tag{35}$$

Here we assume that all the energy released at $z \geqslant z_{trap}$ is radiated, while the energy released at $z < z_{trap}$ is completely trapped. Also, because Q_{vis} and $\tau(z)$ are both proportional to $\int \rho dz$, we express the vertical distribution of viscous dissipation directly by means of the factor $\tau(z_{trap})/\tau(0)$.

Evaluating the above integral, we obtain the luminosity of the disc from the region $R < R_{trap}$:

$$\begin{aligned} &L_{2D, R<R_{trap}} \\ &= 2 \int_{R_{in}}^{R_{trap}} 2\pi R Q_{vis}(R) \left(\frac{R}{R_{trap}}\right)^{q/(q+1)} dR \\ &= \frac{1}{2h_{in}} L_E \left(\frac{R_{trap}}{R_{in}}\right)^{1/(q+1)} \\ &\quad \times \int_1^{R_{trap}/R_{in}} dx \frac{1}{x^{(q+2)/(q+1)}} \left(1 - \frac{1}{x^{1/2}}\right) \\ &= \frac{1}{2h_{in}} L_E \\ &\quad \times \left[\frac{2(q+1)}{q+3} \left(\frac{R_{trap}}{R_{in}}\right)^{-1/2} + \frac{(q+1)^2}{q+3} \left(\frac{R_{trap}}{R_{in}}\right)^{1/(q+1)} - (q+1)\right] \end{aligned}$$

$$= \frac{1}{2h_{\rm in}} L_{\rm E} \times \left[\frac{2(q+1)}{q+3} \left(\frac{3h_{\rm in}\dot{m}}{2r_{\rm in}} \right)^{-1/2} + \frac{(q+1)^2}{q+3} \left(\frac{3h_{\rm in}\dot{m}}{2r_{\rm in}} \right)^{1/(q+1)} - (q+1) \right]. \quad (36)$$

The dominant contribution comes from the second term, which is proportional to $\dot{m}^{1/(q+1)}$ for $\dot{m} \gg 1$. The luminosity from the region $R_{\rm trap} < R$ is, of course, the same as in the 1D case (equation 18), i.e. $L_{{\rm 2D},R_{\rm trap}<R} = L_{{\rm 1D},R_{\rm trap}<R}$, since there is no photon trapping. Therefore, we have the following expression for the total radiated luminosity from the entire disc,

$$L_{\rm 2D,tot}(q) = L_{{\rm 2D},R<R_{\rm trap}} + L_{{\rm 2D},R_{\rm trap}<R}$$
$$= \frac{1}{2h_{\rm in}} L_{\rm E} \times \left[\frac{4q}{3(q+3)} \left(\frac{R_{\rm trap}}{R_{\rm in}} \right)^{-1/2} + \frac{(q+1)^2}{q+3} \left(\frac{R_{\rm trap}}{R_{\rm in}} \right)^{1/(q+1)} - q \right]$$
$$= \frac{1}{2h_{\rm in}} L_{\rm E} \times \left[\frac{4q}{3(q+3)} \left(\frac{3h_{\rm in}\dot{m}}{2r_{\rm in}} \right)^{-1/2} + \frac{(q+1)^2}{q+3} \left(\frac{3h_{\rm in}\dot{m}}{2r_{\rm in}} \right)^{1/(q+1)} - q \right]. \quad (37)$$

In a similar fashion, we obtain the following result for the case of an exponential density profile ($q \to \infty$),

$$L_{\rm 2D,tot}(q=\infty) = \frac{1}{2h_{\rm in}} L_{\rm E} \left[\frac{4}{3} \left(\frac{R_{\rm trap}}{R_{\rm in}} \right)^{-1/2} + \ln \left(\frac{R_{\rm trap}}{R_{\rm in}} \right) - 1 \right]$$
$$= \frac{1}{2h_{\rm in}} L_{\rm E} \left[\sqrt{\frac{32r_{\rm in}}{27h_{\rm in}}}\, \dot{m}^{-1/2} + \ln \left(\frac{3h_{\rm in}}{2r_{\rm in}} \dot{m} \right) - 1 \right]. \quad (38)$$

To verify that this result is consistent with the limit $q \to \infty$ of equation (36), we make use of the following useful relation:

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(\frac{1}{2\epsilon + 1} x^{\epsilon} - 1 \right) = \ln x - 2. \quad (39)$$

This shows that the dominant terms in equation (36) do not diverge as $q \to \infty$ but tend to a finite value that is proportional to $\ln(\dot{m})$ for $\dot{m} \gg 1$.

4 COMPARISON WITH NUMERICAL SIMULATION

Here we compare the analytical formulae derived above for the luminosity L of the disc with the results of multidimensional numerical simulations reported by Ohsuga et al. (2005).

In Fig. 3 we plot the disc luminosity in Eddington units as a function of the dimensionless mass accretion rate $\dot{m}$. Analytical formulae are denoted by solid lines and correspond, in descending order, to the following cases: no photon-trapping (equation 17), power-law density profile with $q = 1$ and 10 (equation 37), exponential density profile (equation 38, i.e. $q \to \infty$), and the standard 1D model which ignores photons emitted from $R_{\rm in} < R < R_{\rm trap}$ (equation 19). Triangles, circles and squares connected by long-dashed lines indicate the numerical results from fig. 7 of Ohsuga et al. (2005) corresponding to metallicity $Z = 0$, 1 and 10 $Z_\odot$, respectively.[6] The dotted line represents the results of the hydrodynamical simulations of the slim disc model carried out by Watarai et al. (2000), which did not explicitly include the effects discussed here. In these calculations, we adopted $R_{\rm in} = 3R_{\rm g}$ and $h = 0.4$.

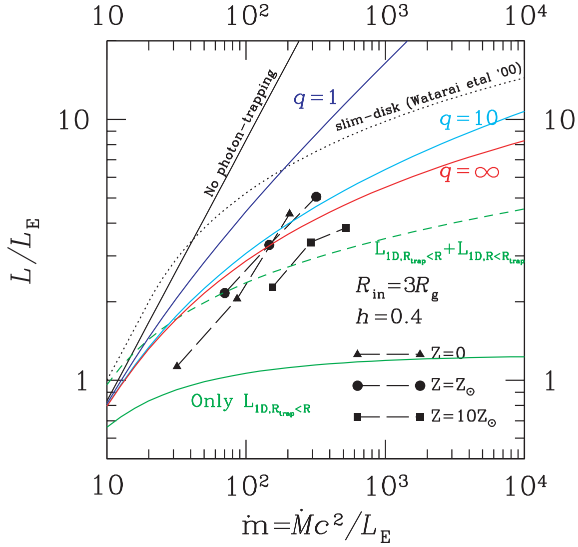

Figure 3. Variation of the disc luminosity as a function of the mass accretion rate. Analytical formulae are denoted by solid lines. From above, the lines correspond to the case with photon trapping ignored, power-law density profiles with $q = 1$ (constant density), $q = 10$ and $q \to \infty$ (exponential profile), and a standard 1D model in which photons are assumed to be fully trapped for $R_{\rm in} < R < R_{\rm trap}$. The triangles, circles and squares connected by long-dashed lines indicate the numerical results shown in fig. 7 in Ohsuga et al. (2005), for metallicity $Z = 0$, 1 and 10 $Z_\odot$, respectively. The dotted line represents the case of the hydrodynamical slim disc simulations of Watarai et al. (2000), without explicitly including 2D effects. The short-dashed line shows the profile of luminosity in an approximate analytical 1D model in which we assume that the disc emits at the local Eddington rate at radii $R_{\rm in} < R < R_{\rm trap}$. All results correspond to $R_{\rm in} = 3R_{\rm g}$ and $h = 0.4$.

From Fig. 3 we see that the luminosity is larger than with the standard analytical 1D model if we ignore radiation from the photon-trapped region. The increase is the result of the multidimensional nature of photon diffusion, which allows radiation to partially escape even from radii $R < R_{\rm trap}$.

For reference, we have also plotted by a short-dashed line the approximate formula (20) for the disc luminosity in the 1D model, which includes the photon luminosity emitted from $R_{\rm in} < R < R_{\rm trap}$ by assuming a simple local Eddington flux. Because of the logarithmic functional form of this approximation, this model resembles the exponential density model in the 2D treatment. Correspondingly, we conclude that this model provides a reasonable 1D approximation.

[6] The existence of metallicity has two conflicting effects which influence the disc structure: (i) larger radiation coefficient induced by the sizable amount of metals helps to cool the gas more, which means that the gas can be easily accreted and (ii) the larger absorption coefficient by such a metallicity pushes the gas outside, which means that it is difficult to be accreted. However, there is an uncertainty of metallicity in the system. Therefore, Ohsuga et al. (2005) studied three cases of the metallicity in a reasonable range, i.e. $Z/Z_\odot = 0$, 1 and 10. The variation of the luminosity that comes from this uncertainty should be treated as an error in the numerical simulation. For details see Ohsuga et al. (2005).

Ohsuga et al. (2005) pointed out that photons tend to have momenta pointed inwards because of advection by the accretion flow, which causes the radiation to spend a longer time in the disc before escaping relative to the simple diffusion estimate we have used in our model. However, we believe this introduces only a minor correction to our results because we are focusing mainly on photons for which $t_{\rm diff} < t_{\rm acc}$.

In addition, Ohsuga et al. (2002) obtained outflows driven by the super-Eddington luminosity of their models. Although our analytical model does not include outflows, we believe our discussion is still applicable in a time-averaged mean sense.

5 DISCUSSION AND CONCLUSION

In this paper, we have studied photon emission from a supercritical accretion disc. In the usual 1D treatment of discs, one integrates all quantities along the vertical (z) direction. This is reasonable for a thin accretion disc when the accretion rate is well below the Eddington rate. However, we show that, when the accretion rate is large, it is quite important to consider the z dependence of fluid quantities and to detreat the vertical diffusion of photons carefully. If we omit the z dependence of the diffusion process in the simplest version of the 1D model, we tend to significantly underestimate the luminosity. On the other hand, if we omit the z dependences of the matter density and assume uniform density, we overestimate the luminosity. Only when we consider the z dependence of the matter density and adopt the appropriate density profile such as a power-law or exponential form as a function of z, as seen in 2D-RHD simulations, are we able to reproduce the luminosity found in the numerical simulations.

Using the analytical approach developed here, we should be able to model how the peak energy of the photon spectrum decreases when the photons produced near the BH are trapped, as pointed out by Ohsuga, Mineshige & Watarai (2003). This will be discussed in a forthcoming paper (Kohri & Ohsuga, in preparation).

When we simultaneously include radiative, advective, convective and neutrino cooling processes in an analytical model of the accretion disc, the effects discussed here will play a crucial role. In NDAFs, for instance, the neutrino luminosity and annihilation rate could be modified substantially relative to the predictions of 1D models. The dominant cooling process could be changed, and the entire picture of the accretion might be dramatically modified, e.g. for the produced energy through the $\nu\,\bar{\nu}$ annihilation (Di Matteo et al. 2002; Chen & Beloborodov 2007; Gu et al. 2006), r-process nucleosynthesis (Surman, McLaughlin & Hix 2006) and so on. We plan to discuss these effects in forthcoming papers.

Multidimensional hydrodynamic simulations of accretion discs including all of these cooling processes have not been done so far (except for the 2D numerical simulations of neutrino-cooled discs by Lee & Ramirez-Ruiz 2006). To clarify the role of multidimensionality on CDAFs, NDAFs, etc., it would be useful to compare the analytical estimates obtained in this paper with full multidimensional numerical simulations. It is hoped that these simulations will be done in the near future.

ACKNOWLEDGMENTS

KK would like to thank A. E. Broderick, J. C. McKinney and S. Mineshige for valuable discussions at an early stage of this work. This work was supported in part by NASA grant NNG04GL38G, PPARC grant, PP/D000394/1, EU grant MRTN-CT-2006-035863, the European Union through the Marie Curie Research and Training Network 'UniverseNet' (MRTN-CT-2006-035863) and Research Grant from Japan Society for the Promotion of Science (17740111).

REFERENCES

Abramowicz M. A., Chen X., Kato S., Lasota J.-P., Regev O., 1995, ApJ, 438, L37
Abramowicz M. A., Igumenshchev I. V., Lasota J.-P., 1998, MNRAS, 293, 443
Begelman M. C., 1978, MNRAS, 184, 53
Chen W.-X., Beloborodov A. M., 2007, ApJ, 657, 383–399
Di Matteo T., Perna R., Narayan R., 2002, ApJ, 579 706
Eggum G. E., Coroniti F. V., Katz J. I., 1988, ApJ, 330 142
Gu W.-M., Liu T., Lu J.-F., 2006, ApJ, 643, L87
Honma F., Matsumoto R., Kato S., 1991, PASJ, 43, 147
Ichimaru S., 1977, ApJ, 214, 840
Igumenshchev I. V., Abramowicz M. A., Narayan R., 2000, ApJ, 537, L271
Kohri K., Mineshige S., 2002, ApJ, 577, 311
Kohri K., Narayan R., Piran T., 2005, ApJ, 629, 341
Lee W. H., Ramirez-Ruiz E., 2006, ApJ, 641, L961
Manmoto T., Kusunowse M., Mineshige S., 1997, ApJ, 489, 791
McKinney J. C., 2005, ApJ, 630, L5–L8
McKinney J. C., 2006, MNRAS, 368, 1561
Narayan R., Paczynski B., Piran T., 1992, ApJ, 395, L83
Narayan R., Igumenshchev I. V., Abramowicz M. A., 2000, ApJ, 539, 798
Narayan R., Piran T., Kumar P., 2001, ApJ, 557, 949
Narayan R., Yi I., 1995a, ApJ, 444, 231–243
Narayan R., Yi I., 1995b, ApJ, 452, 710
Ohsuga K., Mineshige S., Mori M., Umemura M., 2002, ApJ, 574, 315
Ohsuga K., Mineshige S., Watarai K., 2003, ApJ, 596, 429
Ohsuga K., Mori T., Nakamoto M., Mineshige S. M., 2005, ApJ, 628, 368
Okuda T., 2002, PASJ, 54, 243
Popham R., Woosley S. E., Fryer C., 1999, ApJ, 518, 356
Quataert E., Gruzinov A., 2000, ApJ, 539, 809
Shakura N. I., Sunyaeu R. A., 1973, A&A, 24, 337–355
Stone J. M., Pringle J. E., 2000, MNRAS, 322, 461
Stone J. M., Pringle J. E., Begelman M. C., 1999, MNRAS, 310, 1002–1016
Surman R., McLaughlin G. C., Hix W. R., 2006, ApJ, 643, 1057
Szuszkiewicz E., Miller J. C., 1997, MNRAS, 287, 165
Szuszkiewicz E., Malkan M. A., Abramowicz M. A., 1996, ApJ, 458, 474
Watarai K., Mineshige S., 2003, ApJ, 596, 421
Watarai K., Fukue J., Takeuchi M., Mineshige S., 2000, PASJ, 52, 133
Yuan F., Cui W., Narayan R., 2005, ApJ, 620, 905

APPENDIX A: ANALYTICAL ESTIMATES OF DISC HALF-THICKNESS H

In this section, we simply try to analytically estimate the R dependence on disc half-thickness H. The force balance along z-axis is generally expressed by

$$\frac{\sigma_T}{m_{\rm p}c}F = \frac{GM}{R^2}\frac{H}{R}, \tag{A1}$$

with a photon flux of F.

If $R_{\rm trap} < R$, then it would be reasonable to assume that the flux is equal to the viscous heating rate, $F \approx Q_{\rm vis}$. Then, we obtain

$$\left.\frac{H}{R}\right|_{R_{\rm trap}<R} \approx \frac{3}{4}\frac{R_{\rm g}}{R}\dot{m}. \tag{A2}$$

Therefore H/R is approximately proportional to $\propto 1/R$ for $R_{\rm trap} < R$.

On the other hand, if $R < R_{\rm trap}$, we may assume that the flux would be approximately the order of the Eddington flux $F \approx (1/2)$ $F_{\rm E} = (1/2)\, L_{\rm E}/(4\pi R^2)$. The factor of 1/2 in front of $F_{\rm E}$ is attached as a matter of convenience for the consistency. That comes from the viewpoint of continuities of astrophysical quantities although physics does not change at all by this artificial factor. From equation (A1) we find that H/R is constant for $R < R_{\rm trap}$,

$$\left.\frac{H}{R}\right|_{R<R_{\rm trap}} \approx \frac{1}{2}. \tag{A3}$$

Then we see that $h = H/R$ is a smooth function of R,

$$h = \begin{cases} 1/2 & (R \leqslant R_{\rm trap}), \\ \\ \frac{1}{2} R_{\rm trap}/R & (R_{\rm trap} < R), \end{cases} \tag{A4}$$

if we take $R_{\rm trap} = 3/2 h_{\rm in} \dot{m} R_{\rm g}$ with $h_{\rm in} = 1/2$.

APPENDIX B: DERIVATION OF THE *Z*-DEPENDENT/INDEPENDENT DIFFUSION TIME-SCALE

Here we discuss the diffusion time-scale along with z-axis in 3D random walk processes when the scattering length depends on the position, i.e. in the case that we consider the z-dependent number density of electron, the cross-section and so on in the electron–photon scattering processes.

In general, we can write the summed distance measured along the 3D random walk process from the stating point $z = z_0$ as

$$\mathrm{d}(z_0, z_{\max}) = \sum_{i_z=1}^{N_{\rm scatt}(z_0,z_{\max})} \int_{z_{i_z-1}}^{z_{i_z}} \mathrm{d}z, \tag{B1}$$

where $N_{\rm scatt}(z_0, z_{\max})$ is the number of the scattering from $z = z_0$ to a maximum of z ($\equiv z_{\max}$). The integral interval $[z_{i_z-1},\ z_{i_z}]$ is determined by solving the following differential equation for the number of the scattering,

$$\frac{\mathrm{d}N}{\mathrm{d}z} = \sigma_{\gamma e} n_{\rm e}(z). \tag{B2}$$

When we solve it, we may assume that the interval between the scatterings is determined by the approximate relation,

$$\left| \int_{z_{i_z-1}}^{z_{i_z}} \sigma_{\gamma e} n_{\rm e}(z)\, \mathrm{d}z \right| = \frac{1}{3}, \tag{B3}$$

where the meaning of dividing by 3 comes from the contribution only to z-axis in the 3D random walk. Of course the right-hand side would not have to be one-third if only it were the order of $\mathcal{O}(1)$. For simplicity, here we have just taken it one- third. In addition, generally z_{i_z} must not be larger than z_{i_z-1}.

As we will see later, the diffusion tends to proceed outward with increasing diffusion length or mean free path. Then the time spent in inner regions tends to be much longer than that in outer ones. That means that phenomena of the diffusion which started from $z = z_0$ are mainly determined by the local physics at around $z = z_0$. Then equation (B3) might be rewritten as a definition of the z-dependent mean free path $\lambda(z)$ through

$$\int_{z_{i_z-1}}^{z_{i_z}} \sigma_{\gamma e} n_{\rm e}(z)\, \mathrm{d}z \approx \sigma_{\rm T} n_{\rm e}(z_{i_z-1}) \int_{z_{i_z-1}}^{z_{i_z}} \mathrm{d}z, \tag{B4}$$

by

$$\lambda(z_{i_z-1}) \equiv \left| \int_{z_{i_z-1}}^{z_{i_z}} \mathrm{d}z \right| \approx \frac{1}{3} \frac{1}{\sigma_{\rm T}\ n_{\rm e}(z_{i_z-1})}, \tag{B5}$$

where we assumed that $\sigma_{\gamma e} = \sigma_{\rm T}$. Although there would be a lot of ways to define the z-dependent mean free path, the definition in equation (B5) would be relatively natural in the current context of the accretion discs because the phenomena are mainly locally determined.

Then d (z_0, $z_{\max}$) in equation (B1) is rewritten as

$$\mathrm{d}(z_0, z_{\max}) = \sum_{i_z=1}^{N_{\rm scatt}(z_0,z_{\max})} \ell(z_{i_z-1}), \tag{B6}$$

with

$$\ell(z_{i_z}) = \begin{cases} \lambda(z_{i_z}) & (z_{i_z+1} \geqslant z_{i_z}), \\ -\lambda(z_{i_z}) & (z_{i_z+1} < z_{i_z}). \end{cases} \tag{B7}$$

Next let us consider the averaged value of d (z) after sufficiently a lot of tries.

$$\langle \mathrm{d}(z_0, z_{\max}) \rangle = \sum_{i_z=1}^{N_{\rm scatt}(z_0,z_{\max})} \langle \ell(z_{i_z-1}) \rangle, \tag{B8}$$

where $\langle X \rangle$ means the average of X after such trials. Here we can assume $\langle \ell(z_{i_z-1}) \rangle \approx 0$. That is because we have assumed that the local physics at around $z = z_{i_z-1}$ determines the phenomena, and surely then this approximation would not be so bad. Then we see that the averaged value of d (z_0, $z_{\max}$) vanishes,

$$\langle \mathrm{d}(z_0, z_{\max}) \rangle \approx 0. \tag{B9}$$

On the other hand, however, the averaged value of the square of d(z) must be finite.

$$\begin{aligned} &\langle \mathrm{d}(z_0, z_{\max})^2 \rangle \\ &= \sum_{i_z=1}^{N_{\rm scatt}(z_0,z_{\max})} \sum_{j_z=1}^{N_{\rm scatt}(z_0,z_{\max})} \langle \ell(z_{i_z-1}) \ell(z_{j_z-1}) \rangle \\ &= \sum_{i_z=j_z=1}^{N_{\rm scatt}(z_0,z_{\max})} \langle \ell(z_{i_z-1})^2 \rangle + \sum_{i_z \neq j_z} \langle \ell(z_{i_z-1}) \ell(z_{j_z-1}) \rangle \\ &\approx \sum_{i_z=j_z=1}^{N_{\rm scatt}(z_0,z_{\max})} \langle \lambda(z_{i_z-1})^2 \rangle, \end{aligned} \tag{B10}$$

where we approximated

$$\begin{aligned} &\sum_{i_z \neq j_z} \langle \ell(z_{i_z-1}) \ell(z_{j_z-1}) \rangle \\ &\approx \sum_{i_z \neq j_z} \ell(z_{i_z-1}) \langle \ell(z_{j_z-1}) \rangle \left(\approx \sum_{i_z \neq j_z} \langle \ell(z_{i_z-1}) \rangle \ell(z_{j_z-1}) \right) \\ &\approx 0, \end{aligned} \tag{B11}$$

because the step i_z does not correlate with that of j_z among the independent trials for $\langle \ell(z_{i_z-1}) \rangle = 0$. Now we can approximate $\langle \lambda(z_{i_z-1})^2 \rangle \approx \lambda(z_0)^2$ because the scatterings at the outer regions do not frequently occur and hardly contribute to the summation at all.

Then from (B10), we find that the number of the scattering of the photon diffused from z to $z_{\max}$, i.e. $\langle \mathrm{d}^2(z, z_{\max}) \rangle = (\int_z^{z_{\max}} \mathrm{d}z)^2$, is represented by[7]

$$\begin{aligned} N_{\rm scatt}(z, z_{\max}) &\approx 3 \left[\sigma_{\rm T} n_{\rm e}(z)\right]^2 \left(\int_z^{z_{\max}} \mathrm{d}z \right)^2 \\ &\approx 3 \left[\sigma_{\rm T} \int_z^{z_{\max}} \mathrm{d}z\ n_{\rm e}(z) \right]^2 \\ &= 3\tau(z)^2, \end{aligned} \tag{B12}$$

[7] Note that we have removed the subscript '0' in z.

where we have used the same logic in equation (B4) to transform the first line to the second.

Here we get the expression of the z-dependent diffusion time-scale,

$$t_{\rm diff}(z) \equiv N_{\rm scatt}(z, z_{\rm max})\lambda(z)/c$$
$$\approx 3\tau(z)^2\lambda(z)/c. \quad \text{(B13)}$$

Of course, if we omit the z dependence, we immediately get the z-independent diffusion time-scale shown in equations (3) and (5).

When we consider the power-law density profiles $\propto (1 - z/q\,H)^{q-1}$ (or the exponential one as their large-q limit), surely $t_{\rm diff}(z)$ decreases rapidly as $\propto (1 - z/q\,H)^{q+1}$ as a function of z. This means that the time spent in the inner regions is much longer than that in the outer ones, which validates our assumption in the current context in the accretion discs.

This paper has been typeset from a TEX/LATEX file prepared by the author.