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Quantum Affine Symmetry as Generalized Supersymmetry

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The quantum affine $\mathcal{U}_q(\widehat{sl(2)})$ symmetry is studied when q^2 is an even root of unity. The structure of this algebra allows a natural generalization of $N = 2$ supersymmetry algebra. In particular it is found that the momentum operators $P, \overline{P}$, and thus the Hamiltonian, can be written as generalized multi-commutators, and can be viewed as new central elements of the algebra $\mathcal{U}_q(\widehat{sl(2)})$. We show that massive particles in (deformations of) integer spin represention of $sl(2)$ are not allowed in such theories. Generalizations of Witten's index and Bogomolnyi bounds are presented and a preliminary attempt in constructing manifestly $\mathcal{U}_q(\widehat{sl(2)})$ invariant actions as generalized supersymmetric Landau-Ginzburg theories is made.

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1. Introduction

One of the early discoveries in particle physics was the realization of the existence of two different types of particles: Bosons and Fermions. In the algebraic context they are distinguished by the fact that bosonic operators generally satisfy simple commutation relations whereas fermionic operators satisfy simple anti-commutation relations. In quantum field theories which are supersymmetric, there is a strong connection between these two classes of operators. Supersymmetry gives a simple organizing principle for a theory to have both types of these particles in a symmetrical fashion.

In the context of 2 dimensional quantum field theories it is natural to expect that one would encounter more exotic objects than just bosons and fermions. For example, the experience with rational conformal field theories suggests that the braiding properties of non-local operators is a key ingredient in any attempt at understanding the structure of such theories. In these cases the braiding properties are for fields with 'fractional spin'. In 2 dimensions, spin is defined with respect to Lorentz boosts, or equivalently, Euclidean rotations. Fields can have definite spin, however since the Lorentz boost generator L does not commute with the Hamiltonian H, asymptotic particles have no definite spin.

Quantum groups [1][2] can be formulated such that the defining relations are expressed as generalized commutation relations which include phases in the quadratic relations instead of just plus or minus signs. In this way, quantum groups present themselves as natural candidates for generalizations of supersymmetry. Indeed it has been discovered that many integrable 2d QFT's, such as the sine-Gordon theory and its generalizations to the ordinary supersymmetric and fractional supersymmetric sine-Gordon theories, and also the generalization of all of these to other affine Toda theories, enjoy a quantum affine symmetry [3]. Furthermore, at special points the quantum affine symmetry is nothing other than $N = 2$ supersymmetry.

It is natural to ask how much of the structure of supersymmetric theories carries over to the context of quantum affine symmetry. This is an important question to settle in view of the fact that two dimensional quantum field thoeries with $N = 2$ supersymmetry have been shown to have a very rich structure [4][5]. The $N = 2$ superalgebra has important features that have not been explored before in the context of the algebraic properties of quantum affine algebras. These include the existence of chiral rings [6] and of a topological twisting [7]. Our paper is a first step toward understanding these novel properties of quantum affine algebras. In particular we show that many of the properties of the $N = 2$

supersymmetry algebra are present in the quantum affine $\mathcal{U}_q(\widehat{sl(2)})$ algebra with the quantum deformation parameter q^2 satisfying $q^{2p} = 1$, where p is an even integer. In particular we have four (fractional spin) supercharges $Q_\pm, \overline{Q}_\pm$ which are nil-potent $Q_\pm^p = \overline{Q}_\pm^p = 0$, which generalize the supersymmetric ($p = 2$) result. More importantly, introducing the light-cone components P, $\overline{P}$ of the momentum vector operator, where the Hamiltonian $H = P + \overline{P}$, we find that P, $\overline{P}$ can be expressed in terms of the quantum affine generators, again generalizing the $N = 2$ supersymmetric result $P = \{Q_+, Q_-\}$; the new subtlety is that P is no longer quadratic in Q's. Rather P is a polynomial in Q's each term of which has equal number ($p/2$) of Q_+ and Q_-'s. Similar results apply to $\overline{P}$ with the charges $\overline{Q}_\pm$, such that the Hamiltonian is in the universal enveloping algebra of the quantum affine generators. As we will explain, these same elements can also correspond to other local integrals of motion with higher odd integer spin.

The main results of our paper have a purely algebraic characterization. We show that the operators $P, \overline{P}$ are new central elements in the quantum affine $\mathcal{U}_q(\widehat{sl(2)})$ algebra with quantum parameter an even root of unity, which have trivial comultiplication and are 'neutral' (are in the 'purely imaginary direction' in the mathematical description). As we will show, the explicit form of the elements $P, \overline{P}$ is completely characterized by the properties of the algebra $\mathcal{U}_q(\widehat{sl(2)})$.

The organization of the paper is as follows: In section 2 we formulate the problem and provide examples ($p = 2, 4$) of solutions within the purely algebraic framework of the algebra $\mathcal{U}_q(\widehat{sl(2)})$. In section 3 we formulate the problem generally in the context of quantum field theory. The main result of this section is a condition for constructing local conserved quantities out of generalized multi-commutators of the non-local $Q_\pm$'s. The latter condition is expressed in terms of the braiding matrices of the non-local currents. In section 4 we present the solutions to these conditions for some higher examples ($p = 6, 8$). In section 5 we illustrate how these ideas are realized in a concrete example (sine-Gordon (SG) theory at special values of the coupling). In section 6 we describe a first attempt at applying these new ideas. In particular we find the analog of the Bogomolnyi mass bound for kinks, and the absence of particles with integral $sl(2)$ spin. We also discuss some aspects of a superspace Landau-Ginzburg formulation for theories with affine quantum symmetry. In section 7 we present our conclusions and suggest some directions for future investigation. In appendix A, some low dimensional representations of the quantum affine algebra are given. In appendix B, some spectral properties of representations (related to the absence of physically interesting integral spin representations) are proven.

2. Algebraic Formulation

2.1 *The $\mathcal{U}_q(\widehat{sl(2)})$ Quantum Affine Algebra*

In this section we describe how to formulate our results purely algebraically. The $\mathcal{U}_q \equiv \mathcal{U}_q(\widehat{sl(2)})$ loop algebra[1][2] is the universal enveloping algebra generated by $Q_\pm, \overline{Q}_\pm$, and T, satisfying[1]

$$\begin{aligned} &[T, Q_\pm] = \pm 2 Q_\pm, \qquad [T, \overline{Q}_\pm] = \pm 2 \overline{Q}_\pm \\ &Q_\pm \overline{Q}_\pm - q^2 \overline{Q}_\pm Q_\pm = 0 \\ &Q_\pm \overline{Q}_\mp - q^{-2}\, \overline{Q}_\mp Q_\pm = a^2 \frac{(1-q^{\pm 2T})}{(1-q^2)}, \end{aligned} \tag{2.1}$$

where a^2 is some arbitrary constant, which we take to be positive. In a physical realization, a is usually dimensionful. The algebra $\mathcal{U}_q$ is a q-deformation of the affine Lie algebra $\widehat{sl(2)}$, the above generators corresponding to the Cartan-Weyl basis of simple roots. The ordinary $N=2$ supersymmetry algebra corresponds to $q^2 = -1$ (where $a \neq 0$ in the last commutation relation *only* in infinite volume where central terms may exist). Notably absent from the above commutation relations is the commutation properties between Q_+ and Q_- (and the right-moving counterparts). In the quantum affine algebras there is no simple commutation properties for these operators. Instead they are replaced by the deformed Serre relations, which will play an important role in the sequel:

$$\begin{aligned} &Q_\pm^3 Q_\mp - (1+q^2+q^{-2}) Q_\pm^2 Q_\mp Q_\pm + (1+q^2+q^{-2}) Q_\pm Q_\mp Q_\pm^2 - Q_\mp Q_\pm^3 = 0 \\ &\overline{Q}_\pm^3 \overline{Q}_\mp - (1+q^2+q^{-2}) \overline{Q}_\pm^2 \overline{Q}_\mp \overline{Q}_\pm + (1+q^2+q^{-2}) \overline{Q}_\pm \overline{Q}_\mp \overline{Q}_\pm^2 - \overline{Q}_\mp \overline{Q}_\pm^3 = 0. \end{aligned} \tag{2.2}$$

The motivation for imposing these relations is discussed below. For the $N=2$ supersymmetric theories the Serre relations are satisfied in a trivial way as $Q^2 = 0$ for each supercharge. Moreover the anti-commutation of Q_+ and Q_- give P (and their right-moving counterparts). It is precisely the apparent absence of such relations in the general quantum affine algebra that motivated us to find them.

---

[1] The presentation we use for $\mathcal{U}_q$ is not standard, but is more suitable for field theory applications. The more standard presentation of the algebra $\mathcal{U}_q$ is with respect to the basis $\{e_i, f_i, h_i, i=0,1\}$ with generalized Cartan matrix $\{a_{ij}\} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$. Up to constants the isomorphism is $Q_+ = e_1 q^{h_1/2}$, $Q_- = e_0 q^{h_0/2}$, $\overline{Q}_+ = f_0 q^{h_0/2}$, $\overline{Q}_- = f_1 q^{h_1/2}$, $h_1 = T, h_0 = -T + k$, where k is the central extension (which corresponds to the level in the undeformed affine Lie algebra). We have set $k=0$ since this corresponds to the known physical applications.

The algebra $\mathcal{U}_q$ is a Hopf algebra with comultiplication Δ, counit ϵ, and antipode S:

$$\begin{aligned}
\Delta(T) &= T\otimes 1 + 1\otimes T \\
\Delta(Q_\pm) &= Q_\pm \otimes 1 + q^{\pm T}\otimes Q_\pm \\
\Delta(\overline{Q}_\pm) &= \overline{Q}_\pm \otimes 1 + q^{\mp T}\otimes \overline{Q}_\pm \\
\epsilon(Q_\pm) &= \epsilon(\overline{Q}_\pm) = \epsilon(T) = 0 \\
S(T) &= -T, \qquad S(Q_\pm) = -q^{\mp T}\, Q_\pm, \qquad S(\overline{Q}_\pm) = -q^{\pm T}\, \overline{Q}_\pm .
\end{aligned} \tag{2.3}$$

These definitions satisfy the usual properties of a Hopf algebra:

$$\begin{aligned}
\Delta(A)\Delta(B) &= \Delta(AB) \\
S(AB) &= S(B)S(A) \\
\epsilon(AB) &= \epsilon(A)\epsilon(B) \\
m(S\otimes id)\Delta(A) &= m(id\otimes S)\Delta(A) = \epsilon(A) \\
(\epsilon\otimes id)\Delta &= (id\otimes\epsilon)\Delta = id,
\end{aligned} \tag{2.4}$$

where $A, B \in \mathcal{U}_q$, and m is the multiplication map. The first equation defines Δ to be a homomorphism from $\mathcal{U}_q \to \mathcal{U}_q \otimes \mathcal{U}_q$.

In a field theory realization of $\mathcal{U}_q$ as a symmetry algebra, these Hopf algebra properties have a precise meaning[2]. For the discussion in this section we only need to recall that the comultiplication Δ describes how the elements of $\mathcal{U}_q$ act on products of fields at different space-time locations, or how they act on multiparticle states. The antipode S corresponds to a Euclidean rotation by an angle π in the $x-t$ plane, and the counit ϵ is the 1-dimensional vacuum representation of $\mathcal{U}_q$.

Because $\mathcal{U}_q$ is a deformation of an affine Lie algebra, there exists a meaningful derivation d, which in the principal gradation[3] is defined to satisfy

$$[d, T] = 0, \qquad [d, Q_\pm] = Q_\pm, \qquad [d, \overline{Q}_\pm] = -\overline{Q}_\pm . \tag{2.5}$$

[2] See [8][9] for a detailed discussion. Some of this is reviewed in section 3.

[3] This gradation is a twist (inner automorphism) of the more familiar gradation (the homogeneous one) encountered in the affine structure of current algebra. We emphasize that the physical properties (such as Lorentz spin) are not invariant under these changes of gradation, and it is the principal gradation that is relevent in e.g. the SG model. The homogeneous gradation becomes relevent in the twisted version (see sections 6.3, 6.4).

In the field theory, d is proportional to the Lorentz boost generator L. Namely, if $q^2 = \exp(-2\pi i \alpha)$, then $L = s\, d$, where $s = \alpha \ \ \mathrm{Mod}(\mathbb{Z})$[3]. (See section 3.) The equation (2.5) implies that the Lorentz spin of the generators $Q_\pm$ ($\overline{Q}_\pm$) is thus s ($-s$). We take $s = \alpha$.

We introduce the basic two dimensional representation $\widehat{\rho}^{(1/2)}$ of $\mathcal{U}_q$ on the vector space $V \sim \mathbb{C}^2$:

$$\widehat{\rho}^{(1/2)}(T) = \sigma_3, \qquad \widehat{\rho}^{(1/2)}(Q_\pm) = \frac{a}{2}\,\nu\,\sigma_\pm, \qquad \widehat{\rho}^{(1/2)}(\overline{Q}_\pm) = \frac{a}{2}\,\nu^{-1}\,\sigma_\pm, \qquad \nu = e^{s\,\theta}, \tag{2.6}$$

where σ_i are the Pauli spin matrices[4]. In anticipation of the field theory realization, we have set the spectral or 'loop' parameter ν to $\exp(s\,\theta)$. (In e.g. the SG theory V is the vector space of one-soliton states and θ is the rapidity of on-shell particles.) A finite Lorentz boost generated by $\exp(\alpha\, L)$ shifts $\theta \to \theta + \alpha$, thus the above representation satisfies (2.5).

We introduce the following Hermitian conjugation properties:

$$T^\dagger = T, \qquad Q_\pm^\dagger = Q_\mp, \qquad \overline{Q}_\pm^\dagger = \overline{Q}_\mp, \qquad q^\dagger = q^{-1} \tag{2.7}$$

which are compatible with the relations (2.1)(2.2) when q is a phase. In the SG theory these Hermiticity properties can be deduced from the explicit field theory expressions. We remark that this Hermiticity structure is not the conventional one (encountered in e.g. current algebra), and is only possible because we are in the principal gradation.

For an arbitrary Hopf algebra $\mathcal{U}$, one defines adjoint actions as follows. Let

$$\Delta(A) = \sum_i a_i \otimes b_i, \tag{2.8}$$

with $A, a_i, b_i \in \mathcal{U}$, and define

$$\mathrm{ad}\, A(B) = \sum_i a_i\, B\, S(b_i). \tag{2.9}$$

This adjoint action satisfies the following important properties:

$$\begin{aligned} \mathrm{ad}\, A\ \mathrm{ad}\, B(C) &= \mathrm{ad}\ AB\ (C) \\ \mathrm{ad}\, A(BC) &= \sum_i \mathrm{ad}\, a_i(B)\ \mathrm{ad}\, b_i(C). \end{aligned} \tag{2.10}$$

[4] $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\sigma_+ = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$, $\sigma_- = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$.

The first relation in (2.10) is equivalent to saying that viewed as a map $\mathcal{U}\otimes\mathcal{U}\to\mathcal{U}$, ad is a $\mathcal{U}$ homomorphism. Thus the adjoint action generally induces representations of $\mathcal{U}$. For ordinary Lie algebras, $\mathrm{ad}\, A\,(B)=[A,B]$, the first relation in (2.10) implies the Jacobi relations, and the second relation is the statement $[A,BC]=[A,B]C+B[A,C]$.

For the algebra $\mathcal{U}_q$, the adjoint actions take the form

$$
\begin{aligned}
\mathrm{ad}\, T(A) &= [T,A]\ , \qquad\qquad \mathrm{ad}\, T(AB) = \mathrm{ad}\, T(A)B + A\,\mathrm{ad}\, T(B)\\
\mathrm{ad}\, Q_\pm(A) &= Q_\pm A - q^{\pm\,\mathrm{ad}\, T}(A)\, Q_\pm\\
\mathrm{ad}\, \overline{Q}_\pm(A) &= \overline{Q}_\pm A - q^{\mp\,\mathrm{ad}\, T}(A)\, \overline{Q}_\pm\\
\mathrm{ad}\, Q_\pm(AB) &= \mathrm{ad}\, Q_\pm(A)B + q^{\pm\,\mathrm{ad}\, T}(A)\,\mathrm{ad}\, Q_\pm(B)\\
\mathrm{ad}\, \overline{Q}_\pm(AB) &= \mathrm{ad}\, \overline{Q}_\pm(A)B + q^{\mp\,\mathrm{ad}\, T}(A)\,\mathrm{ad}\, Q_\pm(B),
\end{aligned}
\tag{2.11}
$$

where

$$
q^{\pm\,\mathrm{ad}\, T}(A) \equiv q^{\pm T}\, A\, q^{\mp T} = q^{\pm T_A}\, A.
$$

(T_A is defined by the equation $[T,A]=T_A\,A$.) The relations (2.1) and (2.2) may be expressed using these ad 's:

$$
\begin{aligned}
\mathrm{ad}\, T(Q_\pm) &= \pm 2Q_\pm, \qquad \mathrm{ad}\, T(\overline{Q}_\pm) = \pm 2\overline{Q}_\pm\\
\mathrm{ad}\, Q_\pm(\overline{Q}_\pm) &= 0\\
\mathrm{ad}\, Q_\pm(\overline{Q}_\mp) &= \frac{(1-q^{\pm 2T})}{(1-q^2)},
\end{aligned}
\tag{2.12}
$$

$$
\mathrm{ad}^{\,3} Q_\pm(Q_\mp) = \mathrm{ad}^{\,3}\overline{Q}_\pm(\overline{Q}_\mp) = 0. \tag{2.13}
$$

The Serre relations have the following meaning. Let $\mathcal{U}_q^{(0)}$ denote the finite $\mathcal{U}_q(sl(2))$ subalgebra generated by $Q_+,\overline{Q}_-,T$, and let $\mathrm{ad}\,\mathcal{U}_q^{(0)}$ denote $\mathrm{ad}\, A$ for any $A\in\mathcal{U}_q^{(0)}$. Since $\mathrm{ad}\,\mathcal{U}_q^{(0)}$ is a $\mathcal{U}_q^{(0)}$ homomorphism, $\mathrm{ad}\,\mathcal{U}_q^{(0)}(Q_-)$ induces a representation of $\mathcal{U}_q^{(0)}$. The relations $\mathrm{ad}^{\,3} Q_+(Q_-) = \mathrm{ad}\,\overline{Q}_-(Q_-)=0$ mean that $\mathrm{ad}\,\mathcal{U}_q^{(0)}$ on Q_- induces a 3 dimensional representation. As $q\to 1$, the latter representation is of course isomorphic to the adjoint representation of $sl(2)$.

2.2 *Algebraic construction of Local Integrals of Motion*

We will now formulate in purely algebraic terms what one needs in order to construct new local integrals of motion out of Q's. Let us suppose that the elements of $\mathcal{U}_q$ are conserved charges in some Lorentz invariant 2 dimensional physical system. Introduce the

Euclidean space-time translation, or momentum, operators $P, \overline{P}$. They act on fields $\Psi(z, \overline{z})$ as follows:

$$[P, \Psi(z, \overline{z})] = \partial_z \Psi(z, \overline{z}), \qquad \left[\overline{P}, \Psi(z, \overline{z})\right] = \partial_{\overline{z}} \Psi(z, \overline{z}), \tag{2.14}$$

where $z = (t+ix)/2, \overline{z} = (t-ix)/2$. The Hamiltonian of the system is $H = P+\overline{P}$, whereas $P_x = P - \overline{P}$ is the spacial translation operator.

Let us further hypothesize that it is possible to express $P, \overline{P}$ as elements in $\mathcal{U}_q$. Then $P, \overline{P}$ are subject to the following purely algebraic conditions:

(i) P and $\overline{P}$ are self-Hermitian conjugate:

$$P^\dagger = P, \qquad \overline{P}^\dagger = \overline{P}. \tag{2.15}$$

(ii) The operators $P, \overline{P}$ have Lorentz spin ± 1:

$$[L, P] = P, \qquad [L, \overline{P}] = -\overline{P}. \tag{2.16}$$

(iii) $P, \overline{P}$ are central elements of $\mathcal{U}_q$:

$$\mathrm{ad}\, A\,(P) = \mathrm{ad}\, A\,(\overline{P}) = 0 \qquad \forall\ A \in \mathcal{U}_q. \tag{2.17}$$

This is simply the statement that the elements of $\mathcal{U}_q$ are conserved since $\partial_t A = [H, A]$. The centrality of $P, \overline{P}$ implies also that $[P, \overline{P}] = 0$; the latter relation together with (2.16) is the Poincaré algebra in two dimensions.

(iv) $P, \overline{P}$ must have the trivial comultiplication:

$$\Delta(P) = P \otimes 1 + 1 \otimes P, \tag{2.18}$$

and similarly for $\overline{P}$. This is simply the statement that $P, \overline{P}$ are local conserved quantities. This implies for instance that the energy-momentum of a multiparticle state is the sum of the individual energy-momentum. It is also evident from the action of P on a product of two fields:

$$[P, \Psi\, \Psi'] = (\partial_z \Psi)\, \Psi' + \Psi\, (\partial_z \Psi').$$

The above conditions, though easily formulated on physical grounds, are rather non-trivial algebraically. Indeed, for generic q they are impossible to satisfy for numerous reasons. If q is generic, the Lorentz spin $\pm s$ of the charges $Q_\pm, \overline{Q}_\pm$ is irrational, thus one cannot construct operators $P, \overline{P}$ out of them with spin ± 1. Suppose now that q is the following root of unity:

$$q^2 = \exp(-2\pi i/p), \qquad \Rightarrow s = 1/p, \tag{2.19}$$

where p is a positive even integer. One is led to the following minimal ansatz for a solution to the conditions (i)-(iv). Define $\mathcal{U}_q^+ \subset \mathcal{U}_q$ to be generated by $Q_\pm$, and similarly $\mathcal{U}_q^- \subset \mathcal{U}_q$ to be generated by $\overline{Q}_\pm$. We restrict our attention to finding solutions of (i)-(iv) of the following type:

$$P \in \mathcal{U}_q^+, \qquad \overline{P} \in \mathcal{U}_q^-. \tag{2.20}$$

The elements $P, \overline{P}$ we will construct are actually in a subalgebra of $\mathcal{U}_q^\pm$ consisting of the elements obtained from the basic generators via adjoint action. Consider first the element P. The Lorentz spin condition (ii), together with the constraint that P must commute with T, implies that $P[Q_\pm]$ must be a sum of terms each with $p/2$ Q_+'s and $p/2$ Q_-'s. (This is why we are limited to even p.)

Let ω denote the automorphism of $\mathcal{U}_q$:

$$\omega(T) = T, \qquad \omega(Q_\pm) = \overline{Q}_\pm, \qquad \omega(\overline{Q}_\pm) = Q_\pm, \qquad \omega(q) = q^{-1}. \tag{2.21}$$

Then given a solution P to the above conditions, $\overline{P} = \omega(P)$ is a solution.

In the next section we will derive a field theory construction which provides a powerful way of constructing solutions to the conditions (i)-(iv). This field theory construction deals only with elements that may be expressed using multiple adjoint actions. We emphasize however that the above algebraic characterization is overdetermined and if a solution exists one would expect it to completely determine $P, \overline{P}$. Within the present algebraic framework, the existence of solutions to each of the conditions (i)-(iv) is only possible due to the fact that $q^{2p} = 1$. We have verified this by explicit computations for $p = 2, 4, 6$. For the remainder of this section, we describe some general properties of solutions and outline the computations involved in determining them solely algebraically. As we will explain, the conditions (iii) and (iv) are not completely independent. We will present results for $p = 2, 4$ in this section; results for $p = 6, 8$ will be presented in section 4.

It is known from the work of De Concini and Kac that the center of a quantum group is enlarged when q is a root of unity[10]. The paper [10] was mainly concerned with the finite case, however some of its results are valid in the affine case as well. The results in [10] are limited to a subset of the center which does not contain elements of the type $P, \overline{P}$. Firstly, the paper [10] deals with an algebra $\mathcal{U}'_q$ which is slightly different from $\mathcal{U}_q$. In $\mathcal{U}'_q$,

$q^{\pm T} \equiv K_\pm$ are taken as basic generators rather than T itself. In $\mathcal{U}'_q$, as shown in [10], the elements $K^p_\pm, Q^p_\pm, \overline{Q}^p_\pm$ and their generalizations for other positive roots are contained in the center. Thus one sees that $\mathcal{U}_q$ and $\mathcal{U}'_q$ are distinguished by $\mathcal{U}_q$ having a $U(1)$ subalgebra, and $\mathcal{U}'_q$ only a Z_p subalgebra. Strictly speaking, $Q^p_\pm, \overline{Q}^p_\pm$ are not central in $\mathcal{U}_q$, since they do not commute with T. In a physical realization of the symmetry $\mathcal{U}'_q$, since $Q^p_\pm, \overline{Q}^p_\pm$ are central, they must either be zero or proportional to the identity in specific representations. In a theory with $U(1)$ symmetry, since $Q^p_\pm, \overline{Q}^p_\pm$ are not $U(1)$ invariant, they cannot be proportional to the identity. We will explain below that it is algebraically consistent to have

$$Q^p_\pm = \overline{Q}^p_\pm = 0. \tag{2.22}$$

Furthermore we will describe how (2.22) is proven in a quantum field theory realization. We will explain in section 6 how the other possibility corresponding to $\mathcal{U}'_q$ may be realized physically. Thus the primary distinction between our central elements and the ones in [10], is that $P, \overline{P}$ belong in the algebra $\mathcal{U}_q$ and are relevent for theories with $U(1)$ symmetry. $P, \overline{P}$ are also central in $\mathcal{U}'_q$, and are to be associated with imaginary roots.

Some remarks on the comultiplication condition (iv) are in order. More generally, define an element $u \in \mathcal{U}_q$ to be *primitive* if it has the following comultiplication

$$\Delta(u) = u \otimes 1 + \Theta^{(u)} \otimes u, \tag{2.23}$$

for $\Theta^{(u)} \in \mathcal{U}_q$. Primitive elements are important for the following reasons. Consider the Serre relation (2.2), and let u be the LHS of equation (2.2). In order to consistently impose $u = 0$, one must have $\Delta(u) = 0$. The latter is ensured by the fact that u is a primitive element with $\Theta^{(u)} = q^{\pm 2T}$. In fact, the precise form of the LHS of the Serre relation is uniquely fixed by requiring it to be primitive. Primitive elements of $\mathcal{U}_q$ are relatively rare. For example, for a general element a of $\mathcal{U}^+_q$ such as $a = Q^n_+$, one has cross-terms in the comultiplication of the form $\Delta(a) \propto .. + Q^{n_1}_+ \otimes Q^{n_2}_+ + ...$ with $n_1 + n_2 = n$ which cannot be re-expressed in terms of a.

When q is a root of unity, the number of primitive elements increases. It is not difficult to show for example that $Q^p_\pm$ are primitive with $\Theta = q^{\pm pT}$. Again this primitive nature allows one to consistently impose (2.22).

It is very important for us to note that primitive elements can be seen to be automatically central under certain conditions. Suppose that u is primitive, with $\Theta = 1$, and furthermore that $\widehat{\rho}^{(1/2)}(u) \propto 1$, where the constant of proportionality may be zero, i.e.

we suppose that u is central in the particular representation $\widehat{\rho}^{(1/2)}$. Since all the higher dimensional representations of $\mathcal{U}_q$ can be constructed on the space $V^{\otimes N}$ by iterating the comultiplication[5], the primitivity of u ensures that u is central in all of these higher dimensional representations. This implies that u is central in the algebra.

Consider now the centrality condition itself, apart from the primitivity condition. It is sufficient to check centrality with respect to a subset of the generators of $\mathcal{U}_q$. Upon imposing $P = P^\dagger$, one has

$$\operatorname{ad} Q_+(P) = \operatorname{ad} \overline{Q}_-(P) = 0 \quad \Rightarrow \quad \operatorname{ad} Q_-(P) = \operatorname{ad} \overline{Q}_+(P) = 0. \tag{2.24}$$

Thus for Hermitian P, one only has to impose that it is cental with respect to the finite subalgebra $\mathcal{U}_q^{(0)}$ generated by $Q_+, \overline{Q}_-, T$. The only relations available for imposing $\operatorname{ad} Q_+(P) = 0$ are the Serre relations. One can argue that

$$\operatorname{ad} \overline{Q}_-(P) = 0 \quad \Rightarrow \quad \operatorname{ad} Q_+(P) = 0, \tag{2.25}$$

and vice-versa. Recall that $\operatorname{ad} \mathcal{U}_q^{(0)}$ is a $\mathcal{U}_q^{(0)}$ homomorphism. This means that the adjoint action of $\mathcal{U}_q^{(0)}$ on P must define a representation π of $\mathcal{U}_q^{(0)}$. Since $\operatorname{ad} \overline{Q}_-(P) = \operatorname{ad} T(P) = 0$, and $Q_+^p = 0$, π is finite dimensional and must be the trivial one-dimensional representation. Thus $\operatorname{ad} Q_+(P) = 0$. To summarize, it is sufficient to impose $\operatorname{ad} \overline{Q}_-(P) = 0$, the latter being easily computable from (2.11) and (2.12).

For the cases which we have analyzed in detail, P can be completely determined by *either* imposing the primitivity or $\operatorname{ad} \overline{Q}_-(P) = 0$. Again this illustrates how primitivity and centrality are related.

We now outline the details of the computations involved in determining P from the above algebraic structures. One can begin by considering the most general expression for P, which is a sum of $\begin{pmatrix} p \\ p/2 \end{pmatrix}$ terms with arbitrary coefficients, each involving $p/2$ Q_+'s and $p/2$ Q_-'s. For $p \geq 6$ it is important to realize that these terms are not all linearly independent due to the Serre relations. The problem of finding a linearly independent basis has been solved in [11]. The Hermiticity constraint is easily imposed and reduces the number of independent coefficients. A consequence of (2.18) is that $S(P) = -P$, which is also straightforward to impose and further reduces the number of undetermined coefficients. The combined constraints from Hermiticity and the antipode are equivalent to the constraints from CPT invariance.

[5] See the discussion in section 6.

From (2.3) and (2.4) one can compute $\Delta(P)$. Setting to zero all of the terms not of the form (2.18) yields an overdetermined system of linear equations for the coefficients. For $p \geq 6$, in doing this computation one must bear in mind that some of the unwanted terms in $\Delta(P)$ may be automatically zero due to the Serre relations, and this generally leads to weaker conditions on the coefficients. One can systematically take this into account by re-expressing all terms in $\Delta(P)$ in terms of linearly independent basis elements. For $p = 2, 4, 6$, due to the fact that $q^{2p} = 1$, one obtains a unique solution up to an overall arbitrary constant (which we call c_p below). For $p = 2, 4$ they are the following:

$$\begin{aligned} P &= c_2 \left(Q_+Q_- + Q_-Q_+\right) = c_2 \,\mathrm{ad}\, Q_+(Q_-) \qquad\qquad (p=2) \\ P &= c_4 \left((Q_+Q_-)^2 - Q_+Q_-^2Q_+ - Q_+^2Q_-^2 - Q_-Q_+^2Q_- + (Q_-Q_+)^2 - Q_-^2Q_+^2\right) \\ &= \frac{c_4}{2}\left(\mathrm{ad}\,(Q_+Q_-Q_+)(Q_-) - \mathrm{ad}\,(Q_+^2Q_-)(Q_-) - \mathrm{ad}\,(Q_-Q_+^2)(Q_-)\right), \quad (p=4) \end{aligned} \tag{2.26}$$

For $p = 4$ the primitivity of P entails the cancelation of 70 unwanted cross-terms!

The centrality condition can be imposed as follows. Imagine again starting with a completely general expression for P as a sum of linearly independent terms with arbitary coefficients. From (2.11) and (2.12) one can easily compute $\mathrm{ad}\,\overline{Q}_-(P)$, and setting the result to zero leads to linear algebraic equations for the coefficients of the independent terms in P. Again, for $p \geq 6$, one must first re-express the result of the computation of $\mathrm{ad}\,\overline{Q}_-(P)$ in a linearly independent basis before setting it to zero, otherwise the resulting equations are too strong. Let us illustrate the centrality properties of P for $p = 2, 4$. The case of $p = 2$ is in a certain sense degenerate, due to the fact that one cannot prove that $Q_\pm^2$ commute with $Q_\pm$ from the Serre relations. However since $Q_\pm^2$ are primitive, it is consistent to set them to zero. The Serre relations degenerate in this case since they are now automatically satisfied. One has that $[Q_\pm, P] = 0$ due to (2.22), and one can verify easily that $\mathrm{ad}\,\overline{Q}_\pm(P) = 0$, using the fact that $q^2 = -1$.

The first non-degenerate case is $p = 4$. One can easily check that $\widehat{\rho}^{(1/2)}(P) \propto e^\theta$ for the expressions in (2.26). Thus the above arguments indicate that since P is primitive, it must be central. Indeed, (2.26) is the unique solution to $\mathrm{ad}\,\overline{Q}_-(P) = 0$, as verified by explicit computation. The simplest way to show explicitly that $\mathrm{ad}\,Q_+(P) = 0$ is to use the expression for P in terms of ad's. Let $S_\pm$ denote the LHS of the first Serre relation in (2.2). From the basic properties (2.10) of the adjoint action, one has

$$\mathrm{ad}\,A \;\, \mathrm{ad}\,S_\pm \;\, \mathrm{ad}\,B\,(C) = 0 \qquad \forall\, A, B, C \in \mathcal{U}_q. \tag{2.27}$$

Using this, one finds

$$
\begin{aligned}
\operatorname{ad} Q_+(P) \;\propto\; & \operatorname{ad}\left(Q_+^2 Q_- Q_+ - Q_+^3 Q_- - Q_+ Q_- Q_+^2\right)(Q_-) \\
= & - \operatorname{ad} Q_- \operatorname{ad}^3 Q_+ (Q_-) = 0,
\end{aligned}
\tag{2.28}
$$

where we have used (2.13).

As we alluded to above, the main difficulty encountered for $p \geq 6$ is that due to the Serre relations, there are linear relations among elements of $\mathcal{U}_q^+$, and one is forced to work with a linearly independent basis. In dealing with this, we did not utilize the basis described in [11], but rather adopted a simpler scheme which will be described when we present the solutions in section 4. For $p \geq 8$, the above computations are intractible, and we rely instead on the field theory construction of the next section to generate solutions. As we will argue, the field theory formulation provides an efficient means of generating primitive elements. Furthermore, the field theory construction apparently automatically incorporates the linear dependencies of elements of $\mathcal{U}_q^+$ due to the Serre relations.

Many physical systems which are known to have the $\mathcal{U}_q$ algebra as a symmetry (e.g. SG theory) are integrable, and thus have an infinite number of commuting integrals of motion $I_n, \overline{I}_n$ of integer Lorentz spin $\pm n$. If these higher integrals of motion are also contained in $\mathcal{U}_q$, then they must also satisfy the above conditions with (2.16) replaced by

$$
[L, I_n] = n I_n, \qquad [L, \overline{I}_n] = -n \overline{I}_n. \tag{2.29}
$$

The question arises as to whether we expect to be able to construct infinitely many commuting integrals of motion purely from the $\mathcal{U}_q$ algebra. In view of the fact that ordinary $N = 2$ supersymmetry algebra is a special case of $\mathcal{U}_q$ one would expect that the answer to the above question be no, since in the ordinary $N = 2$ theory the generic theory is *not* integrable, and so we are not able to construct infinitely many integrals of motion. We expect the same is true for higher p, though the above argument is not conclusive in these cases since, as explained above, the $p = 2$ point is degenerate[6].

However if the exact form of q is different from (2.19), the conserved charge that we construct satisfies all the requisite properties except that its spin, though integral, is

6 In the case of $p = 4$ we searched unsuccessfully for a spin 3 integral of motion as a sum of terms with 6 Q_+'s and 6 Q_-'s using the field theory construction of the next section.

different from that of P and cannot be identified with it. In particular consider instead of (2.19)

$$q^2 = \exp\left(-2\pi i \frac{p'}{p}\right), \qquad \Rightarrow s = p'/p \tag{2.30}$$

where p', p are relatively prime integers. Since all one needs in the above construction is that $q^p = -1$, then P is still a solution to (i),(iii),(iv), except that now it has Lorentz spin p' (which is an odd integer), and can be associated with the higher spin integral of motion $I_{p'}$, in addition to the usual energy and momentum. This shows that for example in the case of the SG model, where q is a fixed function of coupling, one can construct integrals of motion of any odd integer spin, but one must vary the coupling in order to do so. See section 5. The existence of this additional higher integral of motion in a dense subset of the coupling constant space strongly suggests that the theory in integrable, but does not constitute a proof that this is indeed the case. Isolated theories that are $\mathcal{U}_q$ invariant with q given in (2.19) are not necessarily integrable. Henceforth, unless otherwise stated, we are dealing with $p' = 1$.

3. Field Theory Formulation

The $\mathcal{U}_q$ symmetry of a 2d quantum field theory implies the existence of conserved currents $J^\mu(x)$, satisfying $\partial_\mu J^\mu = 0$, for each basic generator Q of $\mathcal{U}_q$, such that $Q = \frac{1}{2\pi i}\int_{-\infty}^{\infty} dx\ J^t(x)$ is a conserved charge. The conserved charge T corresponds to a $U(1)$ symmetry of the theory, and is generated by a local Lorentz spin 1 current. Let $J^\mu_\pm(x), \overline{J}^\mu_\pm(x)$; $\mu = z, \overline{z}$ denote the conserved currents for the charges $Q_\pm, \overline{Q}_\pm$ respectively. The specific construction of these currents is of course model-dependent. However these currents are characterized by some fundamental model-independent features from which the Hopf algebra properties of $\mathcal{U}_q$ follow. We first review these facts, following [3]. For simplicity, we will focus on the currents $J^\mu_\pm$, however all of the results in this section are valid with $J_\pm \to \overline{J}_\pm, Q_\pm \to \overline{Q}_\pm, q \to q^{-1}, s \to -s$.

Let $\Psi(x)$ be a general field of the quantum field theory, and let the braiding of the currents with these fields take the following form

$$J^\mu_\pm(x,t)\,\Psi(y,t) \;=\; q^{\pm T_\Psi}\,\Psi(y,t)\,J^\mu_\pm(x,t), \qquad x < y, \tag{3.1}$$

where T_Ψ is the $U(1)$ charge of Ψ. These braiding relations define a time-ordering prescription $\widehat{T}$:

$$\begin{aligned} \widehat{T}\left(J^\mu_\pm(x,t+\epsilon)\,\Psi(y)\right) &= J^\mu_\pm(x,t+\epsilon)\,\Psi(y) \qquad\qquad \epsilon>0 \\ &= q^{\pm T_\Psi}\,\Psi(y)\,J^\mu_\pm(x,t+\epsilon) \qquad \epsilon<0. \end{aligned} \tag{3.2}$$

These braiding relations arise due to the fact that the currents $J^\mu_\pm$ are generally non-local. Define the adjoint action of the charges $Q_\pm$ on the field as follows:

$$\mathrm{ad}\,Q_\pm\left(\Psi(y)\right) = \frac{1}{2\pi i}\widehat{T}\left(\oint_{\mathcal{C}(y)} dx^\nu \varepsilon_{\nu\mu}\, J^\mu_\pm(x)\,\Psi(y)\right) \tag{3.3}$$

where $\mathcal{C}(y)$ is a contour beginning and ending at $x=-\infty$ and surrounding the point y. The precise shape of the contour is irrelevant due to the conservation of the currents. Break up the contour $\mathcal{C}(y)$ into two pieces $\mathcal{C}_1(y)+\mathcal{C}_2(y)$ where $\mathcal{C}_1$ is above y, in the time-ordered sense, and extends from $-\infty$ to $+\infty$, whereas $\mathcal{C}_2$ is below y and goes from $+\infty$ to $-\infty$. The contour $\mathcal{C}_1$ contributes $Q_\pm\Psi(y)$ to (3.3), whereas $\mathcal{C}_2$ contributes $-q^{\pm T_\Psi}\Psi(y)Q_\pm$. Thus $\mathrm{ad}\,Q_\pm$ is a braided commutator:

$$\mathrm{ad}\,Q_\pm\left(\Psi(y)\right) = Q_\pm\,\Psi(y) - q^{\pm T_\Psi}\,\Psi(y)\,Q_\pm. \tag{3.4}$$

If one takes Ψ to be a current itself, and integrates over y in (3.4), then one sees that the adjoint action we have defined in the quantum field theory is equivalent to the adjoint action (2.11) in the algebra $\mathcal{U}_q$. The comultiplication (2.3) is also a straightforward consequence of the braiding properties of the currents. One has

$$\mathrm{ad}\,Q_\pm\left(\Psi(x)\Psi'(y)\right) = \mathrm{ad}\,Q_\pm\left(\Psi(x)\right)\,\Psi'(y) + q^{\pm T_\Psi}\,\Psi(x)\,\mathrm{ad}\,Q_\pm\left(\Psi'(y)\right). \tag{3.5}$$

This is shown by defining the LHS as in (3.3) with the contour $\mathcal{C}$ surrounding both x and y, and decomposing $\mathcal{C}$ into two separate contours, one surrounding x the other y, and again taking into account the time-ordering. The equation (3.5) translates directly into (2.3) if one interprets the two spaces in the tensor product as the space of fields at the points x and y respectively. From the usual correspondence between fields and the Hilbert space, one sees that the comultiplication in (2.3) is valid on the multiparticle Hilbert space as well. The other Hopf algebra properties also have a clear field theoretic meaning[9][8]. The antipode S corresponds to a Euclidean rotation by an angle π in the $x-t$ plane, and the counit ϵ is the 1-dimensional vacuum representation of $\mathcal{U}_q$.

Let L be the generator of Lorentz boosts, and let the spin s of the charges be defined by $[L, Q_\pm] = s\, Q_\pm$. For the currents $J_\pm$, the braiding relations (3.1) read

$$J^\mu_\epsilon(x)\, J^\nu_{\epsilon'}(y) = q^{\epsilon\epsilon' 2}\; J^\nu_{\epsilon'}(y)\, J^\mu_\epsilon(x) \qquad x < y, \tag{3.6}$$

$(\epsilon, \epsilon' = \pm)$. If $q^2 = \exp(-2\pi i\alpha)$, then from the braiding relations (3.6) with $\epsilon = \epsilon'$, one deduces that $s = \alpha \text{ Mod}\mathbb{Z}$.

We remark that it follows from the above formulation that for local conserved charges and currents with trivial braiding, Δ is the trivial one, and the adjoint action is the usual commutator.

In order to make contact with the previous section, one needs to consider multiple adjoint actions of charges on fields. Consider the expression

$$\text{ad}\, Q_{a_1}\, \text{ad}\, Q_{a_2} \cdots \text{ad}\, Q_{a_{n-1}} \left(J^\mu_{a_n}(y) \right), \tag{3.7}$$

where $a_i \in \{\pm\}$. In general one cannot shrink the contours in this expression due to presence of multiple cuts. However under certain special circumstances the contours in (3.7) may be shrunk arbitrarily close to y, and the operator product expansion used, to yield a new meaningful operator. In this way one can obtain a local operator, such as the energy-momentum tensor. These contour shrinkage conditions were studied in [12], based on ideas introduced in [13]. In [12], the contour shrinking conditions were derived in the special case where $a_1 = a_2 = \ldots = a_{n-1}$. We now generalize this result.

To simplify the discussion we introduce a vector space notation. Let $W \sim \mathbb{C}^2$ denote the 2-dimensional vector space corresponding to the $\pm$ indices of the charges $Q_\pm$. For reasons that will become clear, we express the braiding relations (3.6) as follows

$$J_1(x)\, J_2(y) \;=\; \widehat{R}_{12}\, J_1(y) J_2(x) \qquad x < y, \tag{3.8}$$

where $\widehat{R}_{12}$ is a 4×4 matrix acting on $W \otimes W$, and the subscripts label individual W components of $W \otimes W$. Specifically,

$$\widehat{R}_{12} = R_{12}\, \sigma_{12}, \tag{3.9}$$

where $R_{12} = \text{diag}(q^2, q^{-2}, q^{-2}, q^2)$, and σ_{12} is the operator that permutes two spaces:

$$\sigma_{12} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.10}$$

The expression (3.7) may then be viewed as a specific vector in the 2^n dimensional vector space $W^{\otimes n}$.

Consider now the expression

$$\mathrm{ad}\, Q_1 \,\mathrm{ad}\, Q_2 \cdots \mathrm{ad}\, Q_{n-1}\left(J_n^\mu(y)\right) = \frac{1}{(2\pi i)^{n-1}} \widehat{T}\left(\prod_{i=1}^{n-1} \oint_{\mathcal{C}_i} dx_i J_i^*(x_i)\right) J_n^\mu(y) \tag{3.11}$$

where $J^{*\mu} \equiv \epsilon^\mu_\nu J^\nu$. The above expression is viewed as a general vector $v \in W^{\otimes n}$, where the subscipts label one of the components W of the n-fold tensor product. All the contours $\mathcal{C}_i$ begin and end at $x_\infty \equiv -\infty$, and surround y as shown in figure 1.

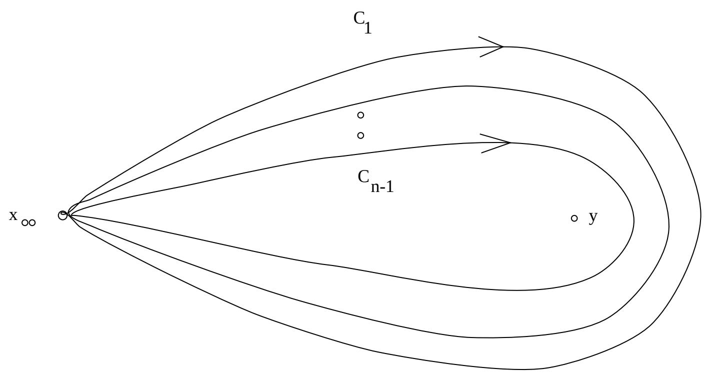

Figure 1. The contours representing the expression (3.11). The time and space axes run vertically and horizontally respectively.

Now consider deforming the point $x_\infty \to x_\infty + \epsilon$, where ϵ is some infinitestimal vector. In general, the point x_∞ cannot be moved due to the presence of multiple cuts. To derive the conditions underwhich x_∞ can be moved, we compute the derivative with respect to ϵ of the expression in (3.11). When this is zero, then the contours may be shrunk to a region arbitrarily close to y. The derivative with respect to ϵ of the expression (3.11) yields a sum of $2(n-1)$ terms, which arise from the two boundaries of each integral $\oint dx_i$. Let $J(x_\infty^+)$ and $J(x_\infty^-)$ respectively denote the currents at the upper and lower boundaries of each contour integral. As space-time points, $x_\infty^+ = x_\infty^-$, however the two fields $J(x_\infty^+)$ and $J(x_\infty^-)$ are essentially different due to the non-locality of the currents. For concreteness consider first the contributions to ∂_ϵ coming from $\oint dx_1$:

$$\left(J_1^*(x_\infty^+) - J_1^*(x_\infty^-)\right) \left[\prod_{i=2}^{n-1} \oint dx_i \; J_i^*(x_i)\right] J_n^*(y). \tag{3.12}$$

In order to understand this expression, it is useful to represent the non-locality of the currents with a string which runs from $-\infty$ to the location of the current and represents the cut associated to the current; this string is analagous to the disorder line defining disorder fields in 2 dimensions, and is manifest in the example of the SG theory. The fields $J_1^*(x_\infty^\pm)$ in the expression (3.12) are represented graphically in figure 2.

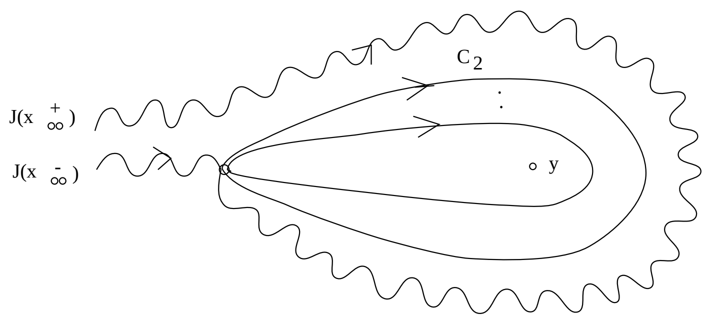

Figure 2. Graphical representation of equation (3.12), where wavy lines denote the strings.

One sees that the two terms in (3.12) are equivalent up to braiding factors that arise when $J_1^*(x_\infty^+)$ completely encircles the fields $J_i^*(x_i), i = 2,..,n$. Thus eq. (3.12) can be rewritten as

$$\begin{aligned}&\left(\widehat{R}_{12}\widehat{R}_{23}\cdots\widehat{R}_{n-2,n-1}\widehat{R}^2_{n-1,n}\widehat{R}_{n-2,n-1}\cdots\widehat{R}_{23}\widehat{R}_{12}-1\right)\cdot\\ &\qquad J_1^*(x_\infty)\left(\prod_{i=2}^{n-1}\oint dx_i J_i^*(x_i)\right)\; J_n^*(y).\end{aligned}\tag{3.13}$$

All of the other terms in ∂_ϵ of (3.11) are of the form of the term in (3.13) up to braiding factors. Consider for example the 2 terms arising from the boundaries of $\oint dx_2$:

$$\oint dx_1 J_1^*(x_1)\left[J_2(x_\infty^+)-J_2(x_\infty^-)\right]\left(\prod_{i=3}^{n-1}\oint dx_i J_i^*(x_i)\right)\; J_n^\mu(y).\tag{3.14}$$

As above, the current $J_2(x_\infty^+)$ can be encircled back around $J_i, i = 3,..,n$ yielding the factor

$$\widehat{R}_{23}\widehat{R}_{34}\cdots\widehat{R}_{n-2,n-1}\widehat{R}^2_{n-1,n}\widehat{R}_{n-2,n-1}\cdots\widehat{R}_{34}\widehat{R}_{23}$$

so that the two terms in (3.14) are of the same type. Then to obtain a term of the type in (3.13) the current $J_1(x_1)$ can be braided through J_2 and the change of integration variables $x_1 \to x_2$ performed. One obtains for (3.14)

$$\begin{aligned}\widehat{R}_{12}\left(\widehat{R}_{23}\widehat{R}_{34}\cdots\widehat{R}_{n-2,n-1}\widehat{R}^2_{n-1,n}\widehat{R}_{n-2,n-1}\cdots\widehat{R}_{34}\widehat{R}_{23}-1\right)\cdot \\ J^*_1(x_\infty)\left(\prod_{i=2}^{n-1}\oint dx_i J^*_i(x_i)\right)\, J^*_n(y).\end{aligned} \tag{3.15}$$

Putting all of this together, one finds

$$-\frac{\partial}{\partial\epsilon}\ \mathrm{ad}\, Q_1\, \mathrm{ad}\, Q_2\cdots\, \mathrm{ad}\, Q_{n-1}\left(J^\mu_n(y)\right) = M\ \ J_1(x^-_\infty)\, \mathrm{ad}\, Q_2\cdots\, \mathrm{ad}\, Q_{n-1}\left(J^\mu_n(y)\right), \tag{3.16}$$

where M is a matrix acting on $W^{\otimes n}\to W^{\otimes n}$:

$$\begin{aligned}M &\equiv \mathcal{P} - B\widetilde{\mathcal{P}} \\ \mathcal{P} &= 1+\widehat{R}_{12}+\widehat{R}_{12}\widehat{R}_{23}+\ldots+\widehat{R}_{12}\widehat{R}_{23}\cdots\widehat{R}_{n-2,n-1} \\ \widetilde{\mathcal{P}} &= 1+\widehat{R}_{n-2,n-1}+\widehat{R}_{n-2,n-1}\widehat{R}_{n-3,n-2}+\ldots+\widehat{R}_{n-2,n-1}\cdots\widehat{R}_{12} \\ B &= \widehat{R}_{12}\widehat{R}_{23}\cdots\widehat{R}_{n-2,n-1}\widehat{R}^2_{n-1,n}.\end{aligned} \tag{3.17}$$

The conditions underwhich the contours may be shrunk are thereby reduced to the following problem in linear algebra. Denote by $v(a_1,a_2,\ldots,a_n), a_i=\pm$ the obvious basis vectors of $W^{\otimes n}$. For any vector v define the field $J_v(y)$ such that for the basis vectors one has

$$J_{v(a_1..a_n)} \equiv\ \mathrm{ad}\, Q_{a_1}\cdots\, \mathrm{ad}\, Q_{a_{n-1}}\left(J^\mu_{a_n}(y)\right). \tag{3.18}$$

Similarly define $Q_v\in\mathcal{U}_q$ by

$$Q_{v(a_1\ldots,a_n)} \equiv\ \mathrm{ad}\, Q_{a_1}\cdots\, \mathrm{ad}\, Q_{a_{n-1}}\left(Q_{a_n}\right). \tag{3.19}$$

One has $Q_v = \frac{1}{2\pi i}\int_{-\infty}^{\infty} dy J_v(y)$. Given a null vector $v^{(0)}$ of M, $M\, v^{(0)}=0$, then equation (3.16) shows that the contours may be shrunk in the expression $J_{v^{(0)}}$. The resulting current in conserved, as is the charge $Q_{v^{(0)}}$.

The above contour shrinking condition is actually too strong, due to the existence of the Serre relations which imply that some of the terms on the RHS of (3.16) may automatically be zero. We now describe the weaker form. For $v\in W^{\otimes n}$ define $\widehat{v}\in W^{\otimes(n-1)}$ as the evident projection of v onto its last $n-1$ components $W_2\otimes\cdots\otimes W_n$. Let $Q_{\widehat{v}}\in\mathcal{U}_q$

be defined as in (3.19). For some $\widehat{v}$, $Q_{\widehat{v}}$ will equal zero due to the Serre relations. Define $W_0^{(n)} \subset W^{\otimes n}$ as

$$W_0^{(n)} \equiv \{v \in W^{\otimes n} : Q_{\widehat{v}} = 0\}. \tag{3.20}$$

We state our final result. The contours can be shrunk in the expression $J_{v^{(0)}}$ if

$$M\, v^{(0)} = 0 \qquad \mathrm{Mod}\, W_0. \tag{3.21}$$

The contour shrinking condition (3.21) has a definite algebraic significance. Consider the expression

$$\mathrm{ad}\, Q_{v^{(0)}} \left(\Psi(y)\, \Psi'(y')\right) = \frac{1}{2\pi i} \int dx^\mu \epsilon_{\mu\nu} J^\nu_{v^{(0)}}(x)\, \Psi(y)\Psi'(y'), \tag{3.22}$$

where as usual one integrates the current $J_{v^{(0)}}(x)$ along a contour surrounding y and y'. Since the multiple contours in the definition of $J_{v^{(0)}}(x)$ can be shrunk arbitarily close to x, one can safely decompose the x-contour in (3.22) into two pieces, one surrounding y the other y' to obtain

$$\mathrm{ad}\, Q_{v^{(0)}} \left(\Psi(y)\, \Psi'(y')\right) = \mathrm{ad}\, Q_{v^{(0)}} \left(\Psi(y)\right)\, \Psi'(y') + \mathrm{ad}\, \Theta \left(\Psi(y)\right) \mathrm{ad}\, Q_{v^{(0)}} \left(\Psi'(y')\right), \tag{3.23}$$

for some $\Theta \in \mathcal{U}_q$. If the contours could not be shrunk in the definition of $J_{v^{(0)}}$, one would obtain additional terms in (3.23) of the type $\mathrm{ad}\, A \left(\Psi(y)\right) \mathrm{ad}\, B \left(\Psi'(y')\right)$, where $A, B \in \mathcal{U}_q$ are of lower degree than $Q_{v^{(0)}}$. Thus the contour shrinking condition ensures $Q_{v^{(0)}}$ is a primitive element.

We remark that for local currents, where $R_{12} = 1$, *any* vector $v \in W^{\otimes n}$ satisfies (3.21). Algebraically this corresponds to the fact that in this case any element of the form $[Q_{a_1}, [Q_{a_2}, \cdots [Q_{a_{n-1}}, Q_{a_n}] \cdot \cdot]]$ is primitive.

We first illustrate the above construction with some simple examples. Consider first the special case where all of the charges $Q_{a_i}, i = 1, .., n-1$ are the same, so that we consider solutions of (3.21) of the form $v^{(0)} = v(a, a, ..., a, a_n), a = +$ or $-$. Note that for $v^{(0)}$ of this type, $\mathcal{P} v^{(0)} = \widetilde{\mathcal{P}} v^{(0)} \propto v^{(0)}$, where the constant of proportionality is a sum of phases. Therefore, in this case the contour shrinking condition is reduced to

$$(1 - B) v^{(0)} = 0. \tag{3.24}$$

The simplest solutions to (3.24) for generic q are the vectors $v^{(0)} = v(+,+,+,-)$ and $v(-,-,-,+)$. These solutions allow one to prove the Serre relations, as was done in

[12]. Namely, the above arguments show that $J_{v(\pm\pm\pm\mp)} = \mathrm{ad}^{\,3} Q_\pm\,(J_\mp)$ are well defined operators in the quantum field theory. By using (model-dependent) scaling arguments, one shows that $\mathrm{ad}^{\,3} Q_\pm(J_\mp) = 0$. For generic q, there exist other solutions of (3.21) for $n > 4$. However they do not contain anything new, since for these other solutions $v^{(0)}$, $Q_{v^{(0)}} = 0$ as a consequence of the Serre relations. For example $\mathrm{ad}^{\,m} Q_\pm(Q_\mp) = 0$, $m \geq 3$.

When q is a root of unity, there are new solutions to the contour shrinking conditions. As before let $q^{2p} = 1$, such that the charges $Q_\pm$ have Lorentz spin $1/p$. The simplest example is the following. Consider the vector $v^{(0)} = v(+,+,+,..,+,+) \in W^{\otimes p}$ with all $+$'s. One finds that $Bv^{(0)} = q^{2p} v^{(0)}$. Thus the contours can be shrunk in the expression $\mathrm{ad}^{\,p-1} Q_+\,(J_+(y))$ and similarly for the expression $\mathrm{ad}^{\,p-1} Q_-\,(J_-(y))$. This implies that we have charged conserved integrals of motion $Q_\pm^p$. Since in the generic theory one does not expect to have any *extra* integrals of motion other than the energy and momentum, which are neutral, these operators better vanish identically. In particular, in section 5, within the context of the SG model, we will use scaling arguments to show that

$$\mathrm{ad}^{\,p-1} Q_\pm\,(J_\pm) = 0 \quad \Rightarrow Q_\pm^p = 0. \tag{3.25}$$

We now apply the above construction to the momentum operators P. Since the adjoint action involves mixed Q_+, Q_-'s, there is no simplification of (3.21). First consider the case of $p = 4$. Here $W_0^{(4)} = \emptyset$. One finds a unique solution to (3.21) (up to an overall arbitrary constant):

$$v^{(0)} = v(+-+-) - v(++--) - v(-++-) \qquad (p=4). \tag{3.26}$$

The conserved current $J_{v^{(0)}}$ has Lorentz spin 2. In the specific case of the SG theory, $J_{v^{(0)}}$ may be identified with the component of the energy-momentum tensor $\mathcal{T}(y)$:

$$\mathrm{ad}\,\left(Q_+Q_-Q_+ - Q_+^2 Q_- - Q_- Q_+^2\right)\left(J_-^\mu(y)\right) \propto \mathcal{T}_z^\mu(y). \tag{3.27}$$

Integration of this equation yields (2.26).

Solutions to the contour shrinking conditions for $p = 6, 8$ will be presented in the next section. We finish this section with some remarks. In general, the solutions to (3.21) are not unique. For the cases we have examined in detail, this multiplicity of solutions corresponds to the same element P as a consequence of the Serre relations. Given an element of $\mathcal{U}_q^+$ such as P, generally there is no unique expression for it in terms of ad's due to the identity (2.27). We assert without proof the following: Given a solution $v^{(0)}$ to (3.21) and its associated $Q_{v^{(0)}}$, then any other $\widetilde{v}^{(0)}$ such that due to the Serre relations

$Q_{\widetilde{v^{(0)}}} = Q_{v^{(0)}}$ is also a solution. For example, at $p = 6$ we obtained a 4-parameter family of solutions to (3.21), and all are equivalent due to the Serre relations (see section 4). Thus the field theory approach to the construction of P automatically resolves some of the complications which arise due to the linear dependencies of basis elements in the algebraic approach.

4. Explicit Solutions for $p = 6, 8$

As stated previously, for $p \geq 6$, one must work in a linearly independent basis of $\mathcal{U}_q$ in order to construct $P, \overline{P}$ from the algebraic construction of section 2. The scheme we adopted for finding a linearly independent basis is the following. Consider a general expression in $\mathcal{U}_q^+$. The Serre relations (2.2) can be used to reexpress any terms involving 3 consecutive Q_+'s or Q_-'s as a sum of terms which do not. In this way, we constucted a basis in $\mathcal{U}_q^+$ with the Q_-'s to the left of the expression as much as possible, by repeatedly using the identities:

$$
\begin{aligned}
A\, Q_+^3 Q_-\; B &= A\left((1+q^2+q^{-2})Q_+^2 Q_- Q_+ - (1+q^2+q^{-2})Q_+ Q_- Q_+^2 + Q_- Q_+^3\right) B \\
A\, Q_+ Q_-^3\; B &= A\left((1+q^2+q^{-2})Q_- Q_+ Q_-^2 - (1+q^2+q^{-2})Q_-^2 Q_+ Q_- + Q_-^3 Q_+\right) B.
\end{aligned} \tag{4.1}
$$

The same scheme can be used to remove linear dependencies amoung elements of $\mathcal{U}_q$ that are expressed in terms of ad's. One has $\mathrm{ad}\, L\,(C) = \mathrm{ad}\, R\,(C)$ where L, R are the left and right hand sides of eq. (4.1).

Though the above scheme has served for our purposes, it has the drawback of not completely removing linear dependencies. For example, one can re-express $Q_+^3 Q_-^3$ by either moving the Q_+'s to the right using the first relation in (4.1), or alternatively move the Q_-'s to the left using the second relation. The resulting expressions are not identical. Setting them equal, one obtains the identity:

$$
\begin{aligned}
&Q_+ Q_-^2 Q_+ Q_- Q_+ - Q_- Q_+ Q_-^2 Q_+^2 - Q_- Q_+ Q_- Q_+^2 Q_- - Q_+ Q_- Q_+ Q_-^2 Q_+ \\
&\quad + Q_+^2 Q_-^2 Q_+ Q_- - Q_+ Q_- Q_+^2 Q_-^2 + Q_- Q_+^2 Q_- Q_+ Q_- + Q_-^2 Q_+^2 Q_- Q_+ = 0.
\end{aligned} \tag{4.2}
$$

We now outline the analysis of the contour shrinking conditions required for the construction of P at $p = 6$. The action of the matrix M in (3.21) on basis vectors $v(a_1, .., a_n)$ leaves the index a_n unchanged, therefore it is sufficient to study solutions on the vector space spanned by basis vectors with $a_n = -$. The most general vector $v^{(0)}$ of this type with $n=6$ consists of a sum of 10 terms. The $\mathrm{Mod}\, W_0$ condition may be imposed as follows. Due to the Serre relations,

$$Q_{\widehat{v}(--+++-)} = 0, \qquad Q_{\widehat{v}(-++++--)} = 2\, Q_{\widehat{v}(-++-+-)} - 2\, Q_{\widehat{v}(-+-++-)}.$$

Thus one should make the replacements $v(--+++-) = 0$ and $v(-+++--) = 2v(-++-+-) - 2v(-+-++-)$ after computing $M\, v^{(0)}$ and before setting the result to zero. In this way one obtains 8 linear equations for the 10 unknown coefficients in the vector $v^{(0)}$. After a long but straightforward computation, one finds a 4-parameter family of solutions:

$$\begin{aligned} v^{(0)} &= 2a\, v(+-+-+-) + 2b\, v(+++---) + 2c\, v(-+++--) \\ &\quad + 2d\, v(--+++-) + (a-4b)\, v(++-+--) \\ &\quad + (2a-4b-4c)\, v(-++-+-) + (4b-2a)\, v(+-++--) \\ &\quad + (4b+4c-4a)\, v(-+-++-) - a\, v(++--+-). \end{aligned} \tag{4.3}$$

Remarkably, for arbitrary constants a, b, c, d, $Q_{v^{(0)}}$ represents a unique element of $\mathcal{U}_q$ up to an overall constant. This is because the terms in $Q_{v^{(0)}}$ proportional to b, c, or d are all zero due to the Serre relations. This example illustrates how the contour shrinking conditions automatically incorporate Serre's consequences. Indeed, one may find solutions to $M\, v^{(0)} = 0$ ignoring the $\mathrm{Mod}\, W_0$ condition altogether. The result of this computation is a two parameter family of solutions, which are again equivalent up to Serre relations, and are the same as (4.3). Setting $b = c = d = 0$, and identifying $Q_{v^{(0)}}$ with P, one obtains the basic expression

$$\begin{aligned} P &= \frac{c_6}{6}\ \mathrm{ad}\,(2Q_+Q_-Q_+Q_-Q_+ + Q_+^2Q_-Q_+Q_- + 2Q_-Q_+^2Q_-Q_+ \\ &\qquad - 2Q_+Q_-Q_+^2Q_- - 4Q_-Q_+Q_-Q_+^2 - Q_+^2Q_-^2Q_+)\ \ (Q_-)\,. \end{aligned} \tag{4.4}$$

Writing out the ad's in (4.4) and reexpressing the result in a linearly independent basis using (4.1)(4.2), one obtains

$$\begin{aligned} P &= \frac{c_6}{2}\,(2(Q_+Q_-)^3 + 2(Q_-Q_+)^3 + Q_+^2Q_-Q_+Q_-^2 - 2Q_-Q_+^2Q_-Q_+Q_- \\ &\qquad + 2Q_+Q_-Q_+^2Q_-^2 - 5Q_+^2Q_-^2Q_+Q_- - 9Q_-^2Q_+Q_-Q_+^2 - 7Q_+Q_-^2Q_+Q_-Q_+ \\ &\qquad + 7Q_-Q_+Q_-^2Q_+^2 + 5Q_+Q_-Q_+Q_-^2Q_+ + 5Q_-^3Q_+^3). \end{aligned} \tag{4.5}$$

It is simple to check that $\widehat{\rho}^{(1/2)}(P) \propto e^{\theta}$, as only the first two terms in (4.5) contribute. We have checked explicitly that P is hermitian, primitive, and central, in the way described in section 2. One must use the Serre relations to verify each of these properties.

For $p = 8$ we found the following solution to the contour shrinking condition:

$$P \quad \propto \quad \mathrm{ad}\, Y\,(Q_-) \qquad\qquad (p=8), \tag{4.6}$$

where

$$\begin{aligned} Y = {} & 2Q_-Q_+^2Q_-^2Q_+^2 - (6+4\sqrt{2})Q_-Q_+^2(Q_-Q_+)^2 + (2+2\sqrt{2})Q_-Q_+^2Q_-Q_+^2Q_- \\ & - 2Q_+Q_-Q_+Q_-^2Q_+^2 + (6+4\sqrt{2})Q_+Q_-Q_+Q_-Q_+Q_-Q_+ \\ & + (4+4\sqrt{2})Q_+Q_-Q_+^2Q_-^2Q_+ - 2\sqrt{2}Q_+^2Q_-^2Q_+Q_-Q_+ + 2Q_+^2Q_-^2Q_+^2Q_- \\ & - (10+4\sqrt{2})Q_+^2Q_-Q_+Q_-^2Q_+ - 4Q_+^2Q_-Q_+^2Q_-^2 + (7\sqrt{2}-2)Q_+^3Q_-^3Q_+ \\ & + (\sqrt{2}-2)Q_+^3Q_-^2Q_+Q_- + (6+\sqrt{2})Q_+^3Q_-Q_+Q_-^2 - 3\sqrt{2}Q_+^4Q_-^3 \\ & - (2+2\sqrt{2})Q_+Q_-Q_+Q_-Q_+^2Q_- \end{aligned} \tag{4.7}$$

Due to the voluminousness of the computations involved, we have not checked that the above P satisfies the requisite algebraic properties described in section 2. However we did verify that P is proportional to the identity in the 2-dimensional representation.

5. Generalized Supersymmetries in the Sine-Gordon Theory

In this section we describe how the above structures are realized in a specific model, namely the SG theory. The following discussion generalizes easily to the fractional supersymmetric SG theories[3]. The SG theory is defined by the action

$$S = \frac{1}{4\pi}\int d^2z\,(\partial_z\Phi\partial_{\overline{z}}\Phi + 4\lambda : \cos(\beta\Phi) :)\,. \tag{5.1}$$

With the above normalization of the kinetic term[7], one has

$$< \Phi(z,\overline{z})\Phi(0) >= -\log(z\overline{z}).$$

The coupling constant has scaling dimension $2-\beta^2$. For $0 \le \beta^2 \le 2$, the theory may be viewed as a relevant perturbation of the $c=1$ conformal field theory of a single real scalar field, where the conformal field theory is recovered in the ultraviolet (u.v.) limit $\lambda \to 0$.

[7] With our normalization of the kinetic term, the free fermion point occurs at $\beta = 1$.

The SG theory is known to possess a $\mathcal{U}_q$ symmetry, with

$$q = \exp\left(-2\pi i/\beta^2\right)$$

which we now review[3]. The generator T is the usual topological charge

$$T = \frac{\beta}{2\pi}\int_{-\infty}^{+\infty} dx\,\partial_x\Phi. \tag{5.2}$$

The conserved currents $J^\mu_\pm, \overline{J}^\mu_\pm$ for the charges $Q_\pm, \overline{Q}_\pm$ are non-local and take the following form

$$\begin{aligned}
J_{\pm,z}(z,\overline{z}) &= \exp\left(\pm\frac{2i}{\beta}\phi\right), \qquad J_{\pm,\overline{z}}(z,\overline{z}) = \lambda\gamma\exp\left(\pm i(\frac{2}{\beta}-\beta)\phi \mp i\beta\overline{\phi}\right)\\
\overline{J}_{\pm,\overline{z}}(z,\overline{z}) &= \exp\left(\mp\frac{2i}{\beta}\overline{\phi}\right), \qquad \overline{J}_{\pm,z}(z,\overline{z}) = \lambda\gamma\exp\left(\mp i(\frac{2}{\beta}-\beta)\overline{\phi} \pm i\beta\phi\right),
\end{aligned} \tag{5.3}$$

where $\gamma \equiv \beta^2/(2-\beta^2)$, and $\phi, \overline{\phi}$ are the quasichiral[8] components of the scalar field $\Phi = \phi + \overline{\phi}$,

$$\begin{aligned}
\phi(x,t) &= \frac{1}{2}\left(\Phi(x,t) + \int_{-\infty}^{x} dy\partial_t\Phi(y,t)\right)\\
\overline{\phi}(x,t) &= \frac{1}{2}\left(\Phi(x,t) - \int_{-\infty}^{x} dy\partial_t\Phi(y,t)\right).
\end{aligned} \tag{5.4}$$

The conservation of these currents is established to all orders in λ using conformal perturbation theory of the kind developed in [14], and is primarily a consequence of the relations

$$\begin{aligned}
\mathrm{res}_{z=w}\quad J_{+,z}(z)\ \exp(-i\beta\phi(w)) &= \gamma\partial_z\chi^*(z)\\
\mathrm{res}_{z=w}\quad J_{-,z}(z)\ \exp(+i\beta\phi(w)) &= \gamma\partial_z\chi(z),
\end{aligned} \tag{5.5}$$

where

$$\chi = \exp\left(i(\beta - 2/\beta)\phi\right), \qquad \chi^* = \exp\left(-i(\beta-2/\beta)\phi\right),$$

and similar relations for $\overline{J}_\pm$. From the conformal scaling dimensions of the currents, one finds that the charges $Q_\pm$ and $\overline{Q}_\pm$ have Lorentz spin $s = 2/\beta^2 - 1$ and $1 - 2/\beta^2$ respectively.

The last relation in (2.1) is a consequence of the following relations

$$\begin{aligned}
Q_+\overline{Q}_- - q^{-2}\overline{Q}_-Q_+ &= \frac{\lambda}{2\pi i}\gamma^2\int dx\ \partial_x X^*\\
\overline{Q}_+Q_- - q^2 Q_-\overline{Q}_+ &= -\frac{\lambda}{2\pi i}\gamma^2\int dx\ \partial_x X,
\end{aligned} \tag{5.6}$$

[8] In this context ‘chiral’ refers to the distinction between left and right movers in the conformal limit.

where X is the local spinless field

$$X(z,\overline{z}) = \chi\overline{\chi} = \exp\left(i(\beta - \frac{2}{\beta})\Phi\right), \tag{5.7}$$

and X^* its complex conjugate. We will use this result in the next section.

The spectrum of the theory contains soliton doublets of topological charge $T = \pm 1$. Denote by $|\theta, \pm >$ the 1-soliton states of rapidity θ, where as usual

$$P = m\, e^{\theta}, \qquad \overline{P} = m\, e^{-\theta}. \tag{5.8}$$

The representation $\widehat{\rho}^{(1/2)}$ of $\mathcal{U}_q$ on the one-soliton states is as in (2.6). The complete spectrum for all β and the S-matrices are known[15].

At the special values of the coupling

$$\beta = \sqrt{\frac{2p}{p+1}}, \tag{5.9}$$

one has $q = -\exp(-i\pi/p)$, and the charges $Q_\pm, \overline{Q}_\pm$ have Lorentz spin $\pm 1/p$. All of the general arguments of the previous sections apply. To complete the identification, we provide the necessary scaling arguments.

Let us assume that all fields in the SG theory have a smooth u.v. limit where they can be identified with fields in the $c = 1$ conformal field theory, as is normally done in conformal perturbation theory[14]. Consider first the relation $Q_+^p = 0$. The results of the previous section show that $\mathrm{ad}^{\,p-1} Q_+(J_+)$ is a well-defined field of scaling dimension 2. In the u.v. limit as $\lambda \to 0$, this field must be in the chiral conformal family of the field $\exp(\frac{2pi}{\beta}\phi)$. All fields in this family have dimension greater than or equal to $p(p+1)$, thus to zero-th order in λ, $\mathrm{ad}^{\,p-1} Q_+(J_+) = 0$. From the form of the currents J_+ (5.3), one infers that higher order contributions in λ to $\mathrm{ad}^{\,p-1} Q_+(J_+)$ must be in the conformal family of the fields $\exp\left(\frac{2pi}{\beta} + in\beta(\phi + \overline{\phi})\right)$ for any integer n. Since the dimensions of these fields are greater than or equal to $p(p+1)$, and since λ has positive scaling dimension, one concludes that $\mathrm{ad}^{\,p-1} Q_+(J_+) = 0$ to all orders in λ. Similar arguments apply to the other relations in (3.25).

Next consider the relations involving P. Let $v^{(0)}$ be a solution to the contour shrinking conditions relevent toward the identification of P. The field $J_{v^{(0)}}(y)$ is a dimension 2 field with zero topological charge. In the u.v. limit the only such field is the component of the energy-momentum tensor $\mathcal{T}_{zz}(z)$. Thus to zero-th order in λ, $J_{v^{(0)}} \propto \mathcal{T}_{zz}$. Any higher order contributions must come from the families $[\exp(in\beta(\phi+\overline{\phi}))]$ for n an integer, with dimensions greater than or equal to $\frac{2n^2p}{p+1}+k$, for k equal to some positive integer. In order for such a contribution to arise at order λ^m, by dimensional analysis one must have

$$\frac{2m}{p+1}+\frac{2n^2p}{p+1}+k=2.$$

The only possibility for even p is $k=0, m=n=1$. This corresponds to a first order correction in λ, which is nothing other than the order λ correction to $\mathcal{T}$. Thus to all orders in λ one has

$$J^{\mu}_{v^{(0)}}(z,\overline{z}) \;\propto\; \mathcal{T}^{\mu}_{z}(z,\overline{z}).$$

The constants c_p in e.g. (2.26), (4.5), can in principle be computed exactly in the u.v. limit.

Consider now the SG theory at the coupling

$$\beta=\sqrt{\frac{2p}{p+p'}}\ , \qquad \Rightarrow q^2=\exp\left(-2\pi i p'/p\right)\ , \quad s=p'/p\ , \tag{5.10}$$

where p' is a positive, odd integer, relatively prime with p. By the reasoning in sections 2,3, the elements $P,\overline{P}$ constructed at $p'=1$ are still well defined integrals of motion, but now are viewed as higher spin integrals of motion with spin p'.

6. Applications

In this section we consider some applications of the results obtained in the previous sections. In section 6.1 we review the rerpesentation theory of quantum affine algebras. In section 6.2, using the existence of P and $\overline{P}$ constructed out of the Q's we show that particles exist only in deformations of the half-integral spin representations of $sl(2)$. Some of the technical details of this are presented in appendix B. In section 6.3 we discuss some generalizations of Witten's index and the recently discovered index [5] in the context of affine quantum algebras. We also discuss the notion of 'topological twisting' for these theories. In section 6.4 we make a first attempt in formulating a Landau-Ginzburg analog for such theories. This subsection, though far from complete, will discuss some aspects which need to be better understood in order to derive powerful results analogous to differential equations satisfied by the new index [4].

6.1 *Representation Theory*

Recall the representation theory of classical (undeformed) loop algebras $\widehat{g}$. To any finite dimensional representation ρ_g of g, one can associate a finite dimensional representation $\widehat{\rho}_g \in \rho_g(\nu, \nu^{-1})$ of $\widehat{g}$, where ν is the loop parameter. Furthermore, every finite dimensional representation of $\widehat{g}$ is isomorphic to tensor products of such representations. These tensor product representations are irreducible if and only if the loop parameters ν_i are distinct. As was shown in [16], when q is not a root of unity, this structure persists for affine quantum algebras as well. Let us describe more precisely these finite dimensional representations at generic q.

Let e, f, h satisfy the relations of the finite $\mathcal{U}_q(sl(2))$ algebra in the standard presentation:

$$[h, e] = 2\, e, \qquad [h, f] = -2\, f, \qquad [e, f] = \frac{q^h - q^{-h}}{q - q^{-1}}. \tag{6.1}$$

Given any representation of this algebra, a representation of affine $\mathcal{U}_q$ can be constructed from the isomorphism:

$$Q_+ = a\, \nu\, e\, q^{h/2}, \qquad Q_- = a\, \nu\, q^{-h/2} f, \qquad \overline{Q}_- = a\, \nu^{-1}\, f\, q^{h/2}, \qquad \overline{Q}_+ = a\, \nu^{-1}\, q^{-h/2} e. \tag{6.2}$$

The $\mathcal{U}_q$ relations (2.1)(2.2) are easily verified using (6.1). The relations (2.1) are straightforward; for the Serre relations, one can show they are equivalent to

$$-[f, e^3] + (1 + q^2 + q^{-2}) e\, [f, e]\, e = 0, \tag{6.3}$$

which again easily follows from (6.1). Note that for representations that satisfy $f = e^\dagger$, which is compatible with (6.1), the representations of $\mathcal{U}_q$ constructed in this fashion have the hermiticity properties (2.7). All finite dimensional representations depending on a single free loop parameter can be constructed in this way. A few low dimensional examples are given in appendix A. Other finite dimensional representations correspond to tensor products of the above ones, and depend on more than one loop variable. These tensor product representations are generally irreducible, unless certain relations are satisfied among the various ν_i. (See [16].)

When q is a root of unity, the representation theory is significantly modified. We make the reasonable assumption that the above structure of finite dimensional representations of $\mathcal{U}_q$ is still valid, except that one must replace the representations of the finite $\mathcal{U}_q(sl(2))$ with the appropriate ones at a root of 1. The representation theory of the finite $\mathcal{U}_q(sl(2))$ in this situation is well-studied[17][18][19][11]. One finds that due to the fact that e^p, f^p

are central, finite dimensional representations are at most p-dimensional. The type A representations $\rho^{(j)}$ are characterized by $e^p = f^p = 0$, and are deformations of the classical spin j representations of dimension $2j+1$, where $j = 0, 1/2, .., (p-1)/2$. Within this class, one distinguishes between Type II, with $j \leq p/2 - 1$, and Type I, with $j = (p-1)/2$ plus certain 'mixed' representations with q-dimension $[n] = (q^n - q^{-n})/(q - q^{-1})$ equal to zero. Type B, the so-called periodic representations, are p dimensional, and have no classical analogue. They are characterized by $e^p = f^p = 1$, which is also consistent with centrality of these elements[9].

In view of the relations (2.22), it is only the Type A representations that are relevent for the considerations of this paper. However, one can apply the basic constructions of our framework to hypothetical theories with Type B periodic representations. In the periodic representations one would have $Q^p_\pm, \overline{Q}^p_\pm \neq 0$. The results of section 3 show that the contours can be shrunk in $\mathrm{ad}\, Q^{p-1}_\pm (Q_\pm)$. In any theory with genuine $U(1)$ symmetry generated by T, one must have (2.22). However in other models with no $U(1)$ symmetry but rather a Z_p symmetry (which occurs in the algebra $\mathcal{U}'_q$), the central elements $Q^p_\pm, \overline{Q}^p_\pm$ are neutral, and given the Lorentz spin of these operators, the following identification is consistent:

$$Q^p_\pm = P, \qquad \overline{Q}^p_\pm = \overline{P}, \tag{6.4}$$

where here $P, \overline{P}$ have Lorentz spin p'. Note that here p can be even or odd. This is reminiscent of the structure that was found in certain perturbations of Z_p invariant conformal field theories[20]. However the latter models do not have the full quantum affine structure (they only have two charges rather than four), but rather a $\mathcal{U}_q(sl(2))$ restriction of $\mathcal{U}_q$ that is rather a generalization of $N = 1$ supersymmetry; moreover these models have no apparent Type B periodic representations. In [21] it is proposed that the relation (6.4) is relevent to the chiral Potts model of classical statistical mechanics at the Z_p point, however it is unclear if this structure persists in the continuum quantum field theory. Thus we know of no quantum field theory which is a precise realization of this possibility.

6.2 *Spectral Properties*

We now use the above results to derive some general properties of quantum field theories with the symmetry $\mathcal{U}_q$. We assume we are given a theory with $\mathcal{U}_q$ symmetry, such that

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[9] There is also a Z_2 doubling of representations [16] which is not relevent for what we are considering here.

at certain points $q^2 = \exp(-2\pi i/p)$, and $P, \overline{P}$ are given by constructions of the previous sections. Let us further suppose that the $\mathcal{U}_q$ symmetry is realized on asymptotic particle states. The basic finite dimensional representations $\widehat{\rho}^{(j)}$ constructed above which depend on a single loop parameter can be interpreted as single particle representations, where, as discussed above for $\widehat{\rho}^{(1/2)}$, the loop parameter is $\nu = e^{s\,\theta}$, where θ is the rapidity of the particle, and s is the Lorentz spin of the charges $Q_\pm$. The tensor product representations are of course understood as multiparticle states.

$\mathcal{U}_q$ invariant S-matrices for the integrable theories can be constructed as follows[10]. Let $S^{j,j'}(\theta_1 - \theta_2)$ denote the 2-particle to 2-particle S-matrix for the scattering of spin j with spin j' particles. Since the $\mathcal{U}_q$ symmetry acts on 2-particle states via the comultiplication Δ, the invariance of the S-matrix is the statement

$$S^{j,j'}(\theta_1 - \theta_2)\, \widehat{\rho}^{(j)} \otimes \widehat{\rho}^{(j')}\, (\Delta(A)) \;=\; \widehat{\rho}^{(j)} \otimes \widehat{\rho}^{(j')}\, (\Delta'(A))\; S^{j,j'}(\theta_1 - \theta_2) \qquad \forall\, A \in \mathcal{U}_q, \tag{6.5}$$

where Δ' is the permuted comultiplication, i.e. $\Delta' = \sigma\,\Delta$, where here σ is the permutation operator: $\sigma(u \otimes v) = v \otimes u$. These equations were studied in [2], where it was shown that solutions are unique up to an overall scalar factor and automatically satisfy the Yang-Baxter equation. Solutions can also be constructed from the fusion procedure[22]. Thus the S-matrices take the form $S^{j,j'}(\theta) = s_0(\theta)\, R^{j,j'}(\theta)$, where $R^{j,j'}$ are the R-matrices constructed in [2], and $s_0(\theta)$ is an overall scalar factor which makes $S^{j,j'}$ crossing symmetric and unitary. The minimal solution for $j = j' = 1/2$ is the known SG S-matrix[11]. For $j = j' = 1$, the same S-matrix was constucted independently in [23] by imposing the Yang-Baxter equation directly. The advantage of constructing the S-matrices from (6.5) is that it fixes the dependence of the S-matrices on the physically relevent parameters, which are encoded in the value of q.

One can evaluate $P, \overline{P}$ for each of the finite dimensional representations of $\mathcal{U}_q$. One may do this by using explicitly the formulas for $P, \overline{P}$ and the explicit representations $\widehat{\rho}^{(j)}$. However the result can be determined purely from the abstract algebraic properties of $P, \overline{P}$ without knowing explicit formulas for $P, \overline{P}$ and $\widehat{\rho}^{(j)}$. We henceforth assume simply that $P, \overline{P}$ exist for any p. We normalize the constants c_p in e.g. the formulas (2.26), (4.5), by rescaling the Q's if necessary (which may amount to a redefinition of a in (2.1)) so that

$$P = \epsilon \Big(\, (Q_+ Q_-)^{p/2} + (Q_- Q_+)^{p/2} + ...\Big)$$

[10] Subsets of these S-matrices may be valid for the non-integrable cases as well.

[11] See [3] for details of this correspondence.

where $\epsilon = \pm 1$. It is not a priori clear which choice of sign is realized physically (except in the $p = 2$ case where positivity of P implies that $\epsilon = +1$). We find that $P = \epsilon\, a^p e^\theta$ in the representation $\widehat{\rho}^{(1/2)}$. Then one can show that for any p, in the representations $\widehat{\rho}^{(j)}$ one has

$$\begin{array}{lll} P = \epsilon\, a^p\, e^\theta\; (-1)^{j-1/2}, \qquad \overline{P} = \epsilon\, a^p\, e^{-\theta}\; (-1)^{j-1/2}, & & \text{if} \quad j = 1/2, 3/2, .., (p-1)/2 \\ \qquad\qquad P = \overline{P} = 0 & & \text{if} \quad j = 0, 1, .., (p-2)/2. \end{array} \tag{6.6}$$

The proof of these results relies on certain elementary aspects of the fusion procedure and is given in appendix B. The basic idea behind the proof is that due to its trivial comultiplication P acts very simply on tensor product representations, thus one can deduce its value in any representation obtained by decomposing tensor products. One may verify the above results explicitly for the representations listed in appendix A using the explicit form of $P, \overline{P}$ given in the previous sections for $p = 2, 4, 6$.

The result (6.6) has interesting implications, which we now discuss. The formula (6.6) implies that integer spin j (including $j = 0$ singlet) representations of massive particles are excluded, since for any a these representations have zero energy. If anything, they can only correspond to degenerate vacuum representations. If this happens this means that affine quantum symmetry is spontaneously broken as the affine quantum charges do not annihilate the vacuum. One also sees that in a given model, with fixed ϵ, the energy alternates in sign for the half-integer spin representations, thus they are not all allowed if one requires positivity of the energy. In a given theory, one can have at most either the spin j representations with $2j = 1$ mod 4, or $2j = 3$ mod 4 depending on whether $\epsilon = +1$ or -1. The mass m of the particles in these representations is given by

$$m^2 = P\overline{P} = a^{2p}, \tag{6.7}$$

and is independent of the spin j.

One may ask whether all of the latter half-integral spin representations are allowed in a single theory. Consider an integrable theory where the S-matrix bootstrap axioms hold. If a theory contains spin j' particles in addition to spin j, and if the S-matrix $S^{j,j}$ contains poles for intermediate spin j' particles, then the S-matrix $S^{j,j'}$ can be reconstructed from $S^{j,j}$ using the closure axioms of the bootstrap. The fusion procedure, which describes how tensor products of $\mathcal{U}_q$ representations decompose, can be understood as this closure property of the S-matrix. For half-integral j, since $\widehat{\rho}^{(j)} \otimes \widehat{\rho}^{(j)}$ can only be decomposed into integral spin j' representations, this implies it is impossible to find poles corresponding to other half-integral spin j' particles in S^{jj}.[12] Thus it appears likely that one can only have one-particle states in a single representation $\widehat{\rho}^{(j)}$ of $\mathcal{U}_q$, and j must be half integral.

Let us compare the above selection rules with the spectrum of the known models with $\mathcal{U}_q$ symmetry. It can be inferred from the papers [3] that all of the known models can be described universally as anisotropic perturbations of the level k $su(2)$ Wess-Zumino-Witten (WZW) models with the action

$$S = S^{(k)}_{WZW} \; + \; \lambda \int d^2x \left(J^+_\mu J^-_\mu + J^-_\mu J^+_\mu + g\, J^3_\mu J^3_\mu \right), \tag{6.8}$$

where $J^a_\mu(x)$ are the local level k $su(2)$ currents of the WZW theory, and g is the anisotropy parameter. When $g = 1$, the diagonal $su(2)$ is unbroken. When $g \neq 1$, this theory has a $\mathcal{U}_q$ symmetry for any k, where q is related to g in a precise way that can be derived by bosonizing the WZW theory with a Z_k parafermion and a free boson and relating the model to the k-th fractional supersymmetric SG model[13]. The SG model occurs at $k = 1$. All of these models have points in the coupling space where q is as given in (2.19). The higher k theories would be likely candidates for realizations of the higher spin representations since they contain chiral primary fields of spin $j \leq k/2$. Certainly, the above results imply there cannot be any integer spin j representations in the spectrum. In the case of the SG theory this confirms the well-known result that for values of the coupling (5.9) there are no singlet bound states (breathers). Actually, it was found that for all k the particles in the spectrum still transform in the spin $j = 1/2$ representation, the S-matrix being the tensor product of the SG S-matrix and an RSOS S-matrix.

The field theories, if they exist, which correspond to the scattering given by $S^{j,j}$ for $j > 1/2$ have not yet been found. At least for integer spin j our results rule out the existence of such theories with $q^2 = \exp(-2\pi i/p)$ and $P, \overline{P}$ given by the above constructions. However it is possible that some theories do exist which exhibit the scattering S^{jj}, $j \in \mathbb{Z}$, if the expressions we have constructed for $P, \overline{P}$ cannot be identified the momentum operators but rather are identically zero. This could be the case for instance if the theory is such

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[12] The $N = 2$ case is exceptional in this regard, as any shift $T \to T + \alpha$, where α is an arbitrary constant, still forms a doublet representation of the $N = 2$ superalgebra with a different value of a.

[13] When $g = 1$, the free boson is compactified at a radius $r = 1/\sqrt{2k}$, and deforming g away from 1 simply changes r. The resulting value of q is $q = -\exp(-i\pi(1/(2k^2r^2) - 1/k))$ [3].

that the Lorentz spin of the charges $Q_\pm$ is not $1/p$. As an illustration of this statement consider again the SG theory. When $\beta^2 < 1$, singlet breather states are known to appear in the spectrum. From eq. (5.10) one sees that this requires $p < p'$. Thus, precisely in the region of the coupling where singlets appear, the elements $P, \overline{P}$ we have constructed have Lorentz spin $p' > 1$, and are not the true momentum operators; integer $sl(2)$ spin states are now allowed and indeed breathers found in the spectrum.

The relation (6.7) shows that the basic one-particle states in each representation satisfy the Bogomolnyi bound just as in the $p = 2$ case. In specific models one can relate a and thus the mass to certain topological characteristics of the theory. For example in the SG model, from (5.6) we have

$$\frac{\lambda^2 \gamma^4}{4\pi^2} \, |\Delta X|^2 = a^4 \Big| \frac{1 - q^{2T}}{1 - q^2} \Big|^2, \tag{6.9}$$

where $\Delta X = X(\infty) - X(-\infty)$. In the spin $j = 1/2$ representation, the right hand side of (6.9) is just a^4, thus one obtains

$$m^2 = \left(\frac{\lambda \gamma^2}{2\pi} \, |\Delta X| \right)^p \qquad (j = 1/2). \tag{6.10}$$

(A choice of mass units is implied in the above to avoid an overall constant ambiguity).

6.3 *Generalizations of Supersymmetric Indices*

An important tool in the study of supersymmetric theories is Witten's index $Tr(-1)^F$, where F is fermion number. Let $\mathcal{H}$ denote the Hilbert space and consider

$$I = Tr_{\mathcal{H}} \, (-1)^{T/2} \, e^{-\widetilde{\beta} H}, \tag{6.11}$$

where H is the Hamiltonian and $\widetilde{\beta}$ is a constant (inverse temperature). At the supersymmetric point ($p = 2$), fermion number F is defined to be ± 1 for the charges $Q_\pm$, thus $F = T/2$, and (6.11) is the Witten index. For higher p, the index I shares many of the desirable properties at the $p = 2$ point. First it is easy to see that all of the allowed half-integer spin j one-particle representations of $\mathcal{U}_q$ contribute zero to I. Note that the odd-dimensional integer spin j representations, if they exist in the theory, are not projected out by $(-1)^{(T/2)}$, but since they have $H = 0$, they also do not introduce any $\widetilde{\beta}$ dependency! So it is precisely the miracle of having no (massive) odd dimensional representation in the theory which makes this index independent of $\widetilde{\beta}$ for one particle states.

The above results indicate that it is possible to show that I is independent of $\widetilde{\beta}$ for one particle states by using the algebraic properties described in the previous sections. Namely,

$$\partial_{\widetilde{\beta}} I = -Tr_{\mathcal{H}}(-1)^{T/2}(P+\overline{P})\, e^{-\widetilde{\beta}H} = 0. \tag{6.12}$$

To prove this result using the $\mathcal{U}_q$ algebraic properties, one can insert the expressions for $P, \overline{P}$ in terms of $Q_\pm, \overline{Q}_\pm$. Since $Q_\pm, \overline{Q}_\pm$ commute with H, they can be cycled in the trace:

$$Tr(-1)^{T/2} A\, Q_\pm\, e^{-\widetilde{\beta}H} = -Tr(-1)^{T/2} Q_\pm\, A\, e^{-\beta H}, \qquad A \in \mathcal{U}_q. \tag{6.13}$$

For $p=2,4$, using the expressions (2.26) one sees that the terms in P and $\overline{P}$ cancel in pairs in the expression (6.12). For higher p we have not proven (6.12) in this algebraic fashion, but the arguments of the last paragraph indicate it must be possible.

Now we have to consider whether this index is independent of $\widetilde{\beta}$ also for multi-particle states. One would *naively* expect that any tensor products of such representations contribute zero. This expectation is based on assuming that the degeneracies of multiparticle states are the same in a given representation of affine quantum group. However, it is well known that such expectations are generally invalid and that the degeneracy of states in a given multiplet will not generally be the same. This was indeed the reason that the new index defined in [5] is $\widetilde{\beta}$ dependent, in a non-trivial way. So a priori it could be that the I defined above depends on $\widetilde{\beta}$ from multi-particle contributions. In the context of Sine Gordon model this has been recently checked [24] and it has been found that I *does* pick up a two particle contribution[14]. In fact this result is not surprising, because in the conformal limit we do not have any natural index, and all twisting (changing of the boundary conditions) of the $c=1$ model give rise to partition functions which depend on $\widetilde{\beta}$ (or the moduli of torus). It would be very interesting to find a simple differential equation characterizing the $\widetilde{\beta}$ dependence of this object.

We can also go on and define another 'index' in analogy with [5]. The object to consider would be

$$Q = Tr(-1)^{T/2} T\ e^{-\widetilde{\beta}H}$$

It is easy to see that just as in [5] this index receives contributions from one particle states and again *naively* receives no contribution from multi-particle states. However there is an anomaly and one can compute for example the two particle contribution to Q which is

[14] This is different from the $p=2$ case where Witten's index is independent of $\widetilde{\beta}$.

non-vanishing. It would be very interesting to develop the machinary just as in [4] and derive differential equations which characterize Q for all even p.

The first step toward a deeper understanding of the higher p algebras is to find the analog of 'topological twisting' which exists in the $p = 2$ case. In the $p = 2$ case one considers coupling the theory to a background $U(1)$ gauge field, which is coupled to the $T/2$ current, and is set equal to $\frac{1}{2}$ of the spin connection [7][25]. In this way the spin of fields change according to

$$s \to s - \frac{\widetilde{q}}{2}$$

where $\widetilde{q}$ is the charge $T/2$ of the state. In this way Q_+ and $\overline{Q}_-$ become scalars and Q_- and $\overline{Q}_+$ become spin ± 1. Moreover the theory has an energy momentum tensor which can be written as (anti)commutators of $Q_+ + \overline{Q}_-$ with some operator. In the conformal case this means that the twisted theory has $c = 0$.

The natural generalization of this for higher p is to set the gauge field which couples to $T/2$ to be $\frac{1}{p}$ of the spin connection. In this way the spins shift according to

$$s \to s - \frac{\widetilde{q}}{p}. \tag{6.14}$$

Alternatively this corresponds to shifting $L \to L - T/2p$. In this way Q_+ and $\overline{Q}_-$ become scalars and Q_- and $\overline{Q}_+$ have spin $\pm 2/p$. Note that Q_+ and $\overline{Q}_-$, which generate the quantum group algebra $\mathcal{U}_q^{(0)}$ as a subalgebra of $\mathcal{U}_q$ is the true symmetry in the twisted version[15]. This suggests a major difference with the $p = 2$ case, in that both Q_+ and Q_- cannot have integral spins in any given sector (as they have spins which are different mod integers in the twisted version). Put differently, if we quantize on a circle the operators Q_+ and Q_- are not defined on all states in a Hilbert space. In particular if Q_+ is defined on the subsector $\mathcal{H}_+$ and acts by $\mathcal{H}_+ \to \mathcal{H}_-$ then Q_- is defined as the adjoint operator acting as $\mathcal{H}_- \to \mathcal{H}_+$. This structure is reminiscent of Felder's construction of the minimal models from $c = 1$ theory[26], and shows that there exists a massive analog of Felder's cohomology, based on (2.22). Indeed the above topological twisting applied to the $c = 1$ model is nothing but changing the energy momentum tensor by background charge, which shifts the central charge to $c = 1 - \frac{6}{p(p+1)}$. Note that unlike the $p = 2$ case the twisted theory has $c \neq 0$. This is mirrored by the fact that the P that we obtained cannot be

[15] In the mathematical terminology this twist corresponds to going from the principal to the homogeneous gradation.

written solely as a (generalized) Q_+ commutator. We also need Q_- commutators. At any rate in the conformal case the 'topological states' are to be identified with the non-trivial elements of Felder cohomology which is simply the Hilbert space of minimal models. So we see a major difference with the $p=2$ case, in that there are apparently infinitely many topological states, as opposed to the $p=2$ case where one typically obtains a finite number of states.

There exists another meaningful trace which is motivated by the fact that $\mathcal{U}_q^{(0)}$ is a symmetry of the twisted theory. To introduce this let us first note the following: The adjoint actions used above are left actions. Using the same notation as in (2.9) define a right action as

$$\mathrm{ad}^R A(B) = \sum_i S'(b_i)\, B\, a_i,$$

where S' is the skew antipode defined to satisfy $m(S' \otimes 1)\Delta' = \epsilon$. This satisfies

$$\mathrm{ad}^R\, A\, \mathrm{ad}^R\, B(C) = \mathrm{ad}^R\, BA(C).$$

For $\mathcal{U}_q$ these take the form

$$\mathrm{ad}^R\, Q_\pm(B) = -q^{\mp T_B}\ \mathrm{ad}\, Q_\pm(B) \qquad \mathrm{ad}^R\, \overline{Q}_\pm(B) = -q^{\pm T_B}\ \mathrm{ad}\, \overline{Q}_\pm(B).$$

Now let us consider the following 'index'

$$I' = Tr_{\mathcal{H}}\, q^{-T}\, e^{-\widetilde{\beta} H}.$$

This trace satisfies

$$Trq^{-T}\, A\, \mathrm{ad}\, Y(B) = Trq^{-T}\ \mathrm{ad}\,^R Y(A) B, \qquad \forall\ Y \in \mathcal{U}_q^{(0)}.$$

Since this trace is only invariant under the $\mathcal{U}_q^{(0)}$ subalgebra, I' is not independent of $\widetilde{\beta}$. However since the symmetry of the twisted theory is nothing but $\mathcal{U}_q^{(0)}$, this is presumably an important object to consider for the topologically twisted theory. The fact that it is not independent of $\widetilde{\beta}$ may be related to the fact that there are infinitely many topological states and they appear at all energy levels, just as in the minimal model case. An interesting property of I' is that it projects out all of the Type AI representations[16].

[16] One can show that

$$Tre^{2\pi i L} A\ \mathrm{ad}\, Y(B) = Tre^{2\pi i L} \mathrm{ad}^R\, Y(A)\, B, \qquad \forall\ Y \in \mathcal{U}_q,$$

6.4 *Generalized Superspace Landau-Ginzburg Reformulation*

In the $p = 2$ case, i.e. the standard $N = 2$ supersymmetric theories, a deeper understanding of the structure of the theory in which the supersymmetry is manifest comes from a superspace formulation. In addition to the spatial coordinates one introduces anti-commuting odd coordinates $\theta^{\pm}$ (and the right-moving counterparts). The fact that they anti-commute is modeled after $Q_{\pm}^2 = 0$ and $\{Q_+, Q_-\} = 0$ (on the $P = 0$ subspace). In the affine quantum case with $p > 2$ in order to understand the appearance of $\mathcal{U}_q$ symmetry in a natural way one would still have to introduce two extra coordinates $\theta_{\pm}$ (and the right-moving counterparts). However now the commutation properties of them are a little more complicated. Modeled after $Q_{\pm}^p = 0$ one should impose $\theta_{\pm}^p = 0$. The complication arises when we consider assigning commutation properties *between* θ_+ and θ_-. In this case the natural relation we can impose is modeled after the Serre relation (2.2):

$$\theta_{\pm}^3\theta_{\mp} - (1 + q^2 + q^{-2})\theta_{\pm}^2\theta_{\mp}\theta_{\pm} + (1 + q^2 + q^{-2})\theta_{\pm}\theta_{\mp}\theta_{\pm}^2 - \theta_{\mp}\theta_{\pm}^3 = 0.$$

To this we should also add the constraint $P = 0$ in determining the commutation properties of θ's (just as in the $p = 2$ case). Apart from these, there is no other natural commutation relations among these θ's. From this it seems clear that the superspace formulation is going to be much more involved than in the $p = 2$ case, as it seems that the fields do not have a finite expansion when expanded in superspace. In other words, we will end up with infinitely many *auxiliary fields*. It would be very challenging and interesting to unravel the structure of the corresponding field theory; this project is beyond the scope of the present paper.

At this point one has the option of postponing a discussion of a Landau-Ginzburg theory in a superspace until such a formulation has been studied in more detail. However, assuming that the superspace can be made sense out of, and that the basic structure is not too drastically different from the $p = 2$ case, we can make some natural extrapolations to higher p and guess at least some aspects of such LG theories. Furthermore, one may

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where L is the Lorentz boost operator. The virtue of the trace $Tre^{2\pi iL}$ is these invariance properties under adjoint action for the full quantum affine algebra $\mathcal{U}_q$. However, one cannot use this trace to define an index since the operator $e^{2\pi iL}$ is not well-defined on eigenstates of H, due to the fact that L does not commute with H. This was to be expected as there is no twisting which makes all Q's have integral spins at the same time. However this trace is well-defined on the space corresponding to the action of fields at a single space-time point on the vacuum.

develop many aspects of the theory without reference to superspace, i.e. in 'components'. This is what we will presently do in the context of Sine-Gordon model. The following results are not completely rigorous, but are noteworthy enough for us to present them at this stage.

First let us recall some basic facts about the $p = 2$ case [27]. In this case one considers 'chiral'[17] fields X (which commute with Q_+) and form two types of terms in the action: The D–term which involves integration over all super-coordinates and involves a function of both X and its complex conjugate $K(X, X^*)$. Then there is also an F–term, which involves integration over half the superspace of a function $W(X)$ holomorphic in X (and its conjugate). W is known as the superpotential. The action takes the form

$$S = \int d^2 z d^4\theta \; K(X, X^*) + \Big(\int d^2 z d\theta_- d\overline{\theta}_+ \; W(X) + h.c.\Big)$$

which gives the equation of motion

$$D^- \overline{D}^+ (\frac{\partial K}{\partial X}) + \frac{\partial W}{\partial X} = 0 \tag{6.15}$$

A basic fact is that under topological twisting X has 0 left (and right) dimension which naturally leads to the concept of chiral rings.

Now we try to mimic this structure for the $p > 2$ case of Sine-Gordon theory. From the relations (5.6), one sees that the $\mathcal{U}_q$ algebra closes on topological charges for the fields X, X^*. Since in the topological setting $\mathcal{U}_q^{(0)}$ is a good symmetry, based on analogy with $p = 2$ one would naturally take X as the analog of (anti-)chiral field. This identification is strengthened by the fact that in the topologically (or anti-topologically) twisted version X^* (or X) will have zero dimension, as follows from (6.14). Thus we take X as the basic LG field.

We first rewrite the SG potential $\cos(\beta\Phi)$ in terms of the fields X, X^* and the $\mathcal{U}_q$ charges. Let $2 : \cos(\beta\Phi) := V + V^*$, where $V =: \exp(i\beta\Phi) :$. At the values of the coupling (5.9), $X = \exp(-i\sqrt{\frac{2}{p(p+1)}}\Phi)$. The fields V, V^* have (untwisted) scaling dimension $2p/(p+1)$ and X, X^* have dimension $2/(p(p+1))$. The coupling λ has dimension $2/(p+1)$. One has the basic formulas

$$\mathrm{ad}\, Q_- \overline{Q}_+ (V) = \gamma^2 \, \partial_z \partial_{\overline{z}} X, \qquad \mathrm{ad}\, Q_+ \overline{Q}_- (V^*) = \gamma^2 \, \partial_z \partial_{\overline{z}} X^* \tag{6.16a}$$

$$\mathrm{ad}\, Q_+ \overline{Q}_- (V) = \mathrm{ad}\, Q_- \overline{Q}_+ (V^*) = 0. \tag{6.16b}$$

[17] This notion of chirality is not to be confused with the usage of the term in previous sections of this paper. See footnote 8.

The above equations are proven by breaking up X and V into their quasi-chiral components and using conformal operator product expansions. In fact, the chiral part of (6.16) is what ensures that the $\mathcal{U}_q$ charges are conserved (see (5.5)), so these relations are intimately tied to the existence of the $\mathcal{U}_q$ symmetry itself.

The expressions for $P, \overline{P}$ can now be used to express V and V^* as an action of elements of $\mathcal{U}_q$ on the fields X, X^*. Define $\widetilde{Q}_\pm, \widetilde{\overline{Q}}_\pm \in \mathcal{U}_q(\widehat{sl(2)})$ by the formulas

$$P = Q_+ \widetilde{Q}_- + Q_- \widetilde{Q}_+, \qquad \overline{P} = \overline{Q}_+ \widetilde{\overline{Q}}_- + \overline{Q}_- \widetilde{\overline{Q}}_+. \tag{6.17}$$

The $\widetilde{Q}_\pm, \widetilde{\overline{Q}}_\pm$ can be read off from the explicit expressions for $P, \overline{P}$. For example for $p = 4$, $\widetilde{Q}_- = c_4(Q_- Q_+ Q_- - Q_-^2 Q_+ - Q_+ Q_-^2)$. Let us further suppose that the analog of 'chirality' conditions hold:

$$\mathrm{ad}\, \widetilde{Q}_-(X) = \mathrm{ad}\, \widetilde{\overline{Q}}_+(X) = \widetilde{Q}_+(X^*) = \widetilde{\overline{Q}}_-(X^*) = 0. \tag{6.18}$$

The relations (6.18) can be established purely algebraically. Consider the first relation at $p = 4$. From (5.6) one has

$$\mathrm{ad}\, \widetilde{Q}_-(\Delta(X)) \propto \mathrm{ad}\, \widetilde{Q}_- \,\mathrm{ad}\, Q_- \left(\overline{Q}_+\right).$$

From the Serre relations one finds $\widetilde{Q}_- Q_- = -Q_-^3 Q_+$, thus

$$\mathrm{ad}\, \widetilde{Q}_-(\Delta(X)) \propto \mathrm{ad}\, Q_-^3 \,\mathrm{ad}\, Q_+ \left(\overline{Q}_+\right) = 0,$$

where we have used (2.12). We have also verified directly in the field theory using contour shrinking conditions and scaling arguments that the chirality conditions (6.18) are valid in some simple cases. For $p = 2$ this is easily proven. The contour shrinking conditions which are generally needed to prove (6.18) are of precisely the same kind as in section 3, due to the following braiding relations

$$J_\pm(x)\, \chi(y) \;=\; q^{\mp 2} \chi(y)\, J_\pm(x), \qquad J_\pm(x)\, \chi^*(y) \;=\; q^{\pm 2} \chi^*(y)\, J_\pm(x) \qquad x < y. \tag{6.19}$$

This means that χ, χ^* behave just like J_- and J_+ respectively in the braiding relations. Thus to prove for example that $\mathrm{ad}\, \widetilde{Q}_-(X) = 0$ when $p = 4$, one must show that

$$M\ \left(v(-+--) - v(--+-) - v(+---)\right) = 0.$$

This is easily verified; in fact $\mathrm{ad}\,\widetilde{Q}_-(Q_-) = 0$ is just a different expression for the Serre relation, which is only possible since $q^4 = -1$. Assuming (6.18), (6.16) and (6.17) imply

$$V = \gamma^2 \ \mathrm{ad}\,\widetilde{Q}_+\widetilde{\overline{Q}}_- \,(X)\,, \qquad V^* = \gamma^2 \ \mathrm{ad}\,\widetilde{Q}_-\widetilde{\overline{Q}}_+(X^*). \tag{6.20}$$

Define powers $X^n(0)$ of the field X by the (regularized) operator product expansions: $X^2(0) = \lim_{\epsilon\to 0} X(\epsilon)X(0)$, $X^3(0) = \lim_{\epsilon\to 0} X(\epsilon)X^2(0)$, etc. The operator product expansion implies

$$X^p(0) = \lim_{\epsilon\to 0} \epsilon^{\frac{2(p-1)}{(p+1)}} \ V^*(0). \tag{6.21}$$

Let us *define* the integrals $\int d\theta_\pm$, $\int d\overline{\theta}_\pm$ by the formulas

$$\int d\theta_\pm\, W \equiv \ \mathrm{ad}\,\widetilde{Q}_\mp(W), \qquad \int d\overline{\theta}_\pm W \equiv \ \mathrm{ad}\,\widetilde{\overline{Q}}_\mp(W). \tag{6.22}$$

Now we propose, at the conformal point, the action of the form

$$\int d^2z d^4\theta \ K(X,X^*) + \int d^2z d\theta_- d\overline{\theta}_+ \ W(X) + h.c.$$

where to leading order we take $K = XX^*$ and $W = X^{p+1}/(p+1)$. Indeed the equations of motion (6.15), with this choice of K and W becomes

$$\mathrm{ad}\,\widetilde{Q}_-\widetilde{\overline{Q}}_+(X^*) + X^p = 0$$

which (up to normalization questions) follows from (6.21) and (6.20)! Next we add perturbation $V + V^*$ which takes us aways from the conformal point. In view of (6.20) we can represent this perturbation by the addition of X to the superpotential, and so we end up with the action

$$S = \int d^2z d^4\theta\, XX^* + \left(\int d^2z d^2\theta \left(\frac{1}{p+1}X^{p+1} - \widehat{\lambda}X\right) + h.c.\right) \tag{6.23}$$

(where $\widehat{\lambda}$ is a rescaled λ). In the $p = 2$ case the superpotential would suggest a 'chiral ring' of the form $X^p = \widehat{\lambda}$. Is this the case for us?[18] Consider $< X^p(0) >$ in conformal perturbation theory. One has

$$< X^p(0) >=< X^p(0) >_{\lambda=0} + \frac{\lambda}{2\pi i}\int d^2w < V(w,\overline{w})X^p(0) > + \cdots. \tag{6.24}$$

[18] The definition of such a ring for general p is bound to be more subtle than in the $N = 2$ case, and is beyond the scope of the present paper.

Clearly $< X^p(0) >= 0$ in the conformal field theory at $\lambda = 0$. The only correction is to order λ and it is finite. Evaluating this using conformal operator products, and regularizing the integral over d^2w, one finds $< X^p(0) >= -\frac{1}{2}\left(\frac{p+1}{p-1}\right)\lambda$. This is a further confirmation of the existence of the above LG formulation of Sine-Gordon theory.

The action (6.23) is manifestly $\mathcal{U}_q$ invariant. To see this, consider e.g. a variation of F–term generated by Q_-. One has $\delta F = \int d^2z \, \mathrm{ad}\, Q_- \widetilde{Q}_+ \widetilde{\overline{Q}}_-(W) = 0$, since by (6.17) the integrand is a total derivative. (The variation of the h.c. term is zero by (6.16b).) A similar argument applies to the D–term.

The above reformulation of the SG theory has the following interpretation. The original $\cos(\beta\Phi)$ potential has an infinite number of minima occuring at $\Phi = 2\pi n/\beta$. The SG solitons are kinks that interpolate between two neighboring pairs of vacua. In terms of the field X, from (5.7) one finds that the minima in X occur at $X = (q^2)^n$. But these are precisely the solutions of the equations of motion $X^p = \widehat{\lambda}$ which follow from the action (6.23), for $n = 0, 1, \cdots, p-1$. Thus one may consider the theory (6.23) as equivalent to a version of the SG theory where one limits the number of minima of the $\cos(\beta\Phi)$ potential to p in number, i.e. one identifies $\Phi \equiv \Phi + \frac{2\pi}{\beta}(p-1)$. This type of SG theory was recently discussed in [28].

7. Conclusions

Based on similarities between $N = 2$ supersymmetric algebras and a special point of the $\mathcal{U}_q$ quantum affine $sl(2)$ algebra, we considered finding the structures present in the $N = 2$ theories in the quantum affine algebras. This led us to discovering new central elements in these algebras when q^2 is an even root of unity which we identify with $P, \overline{P}$. The existence of such elements required solutions to overdetermined systems of equations. We found these explicitly for a few non-trivial cases, and indicated how in principle one could find solutions for all cases. Although we did not prove the existence of solutions for all cases, the very fact that in the first few non-trivial cases that we checked there were unique solutions, suggests strongly that this is going to be the case in general. It would be very interesting to prove this in full generality and find a general expression for $P, \overline{P}$ in terms of Q's.

Assuming the existence of $P, \overline{P}$ in general, we showed that representations of integral $sl(2)$ spin cannot be realized physically, as they will have $P = \overline{P} = 0$. This was also explicity checked in the cases where we had expressions for P and $\overline{P}$. It would be interesting to find examples of theories with affine quantum $sl(2)$ symmetry where particles in higher half-integral spin (other than spin 1/2) representations appear. It would also be interesting to see what physical implications the absence of integral spin representations has.

Motivated by similarities with the $p = 2$ case we considered topological twisting for $p > 2$. In the conformal limit with $c = 1$ this is precisely Felder's construction of minimal models. However one obtains infinitely many 'topological states' corresponding to the Felder cohomology. It would be interesting to understand how one could use this structure more effectively, as one does with ordinary topological theories with $p = 2$. This may lead to a different point of view and perhaps a deeper understanding of the restricted sine-Gordon theories, which describe integrable perturbations of the $c < 1$ minimal models[29][12].

Related to the above issue, we considered the question of superspace for higher p and found that it is bound to be rather subtle. In particular superfields will contain infinitely many auxiliary fields. This would be very exciting and challenging to develop in a field theory set up. Assuming this can be done, we proposed a Landau-Ginzburg like theory for Sine-Gordon theory at even roots of unity. It would be of great interest to understand the structure of such Landau-Ginzburg theories. This may open the door to obtaining a deeper understanding to a class of 2d quantum field theories with quantum affine symmetry. It should also be interesting to generalize the above results from $sl(2)$ to other groups.

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Appendix A. Examples of low-dimensional representations

Here we provide the spin $j = 1, 3/2, 2, 5/2$ representations of $\mathcal{U}_q$, in the following presentation.

$$\begin{aligned} &T = \mathrm{diag}(2j, 2j-1, \ldots, -2j) \\ &Q_+ = a\,\nu\,E, \qquad \overline{Q}_- = a\,\nu^{-1}\,F, \qquad Q_- = a\,\nu\,E^\dagger, \qquad \overline{Q}_+ = a\,\nu^{-1}\,F^\dagger. \end{aligned} \tag{A.1}$$

with $F = E^t$. In all cases, E is an upper off-diagonal matrix. Let $E_{rs},\ r,s = 1,..,2j+1$ denote the matrix entries of E. Below we list the non-zero entries $\{E_{12}, E_{23}, ..., E_{2j,2j+1}\}$.

$$
\begin{aligned}
j=1: &\quad \{\sqrt{1+q^2}, \sqrt{1+q^{-2}}\} \\
j=3/2: &\quad \{\sqrt{1+q^2+q^4}, \sqrt{2+q^{-2}+q^2}, \sqrt{1+q^{-4}+q^{-2}}\} \\
j=2: &\quad \{\sqrt{1+q^2+q^4+q^6}, \sqrt{2+q^{-2}+2q^2+q^4}, \\
&\quad\ \ \sqrt{2+q^{-4}+2q^{-2}+q^2}, \sqrt{1+q^{-6}+q^{-4}+q^{-2}}, \} \\
j=5/2: &\quad \{\sqrt{1+q^2+q^4+q^6+q^8}, \left(q^{-1}+q^1\right)\sqrt{1+q^4}, 1+q^{-2}+q^2, \\
&\quad\ \ \left(q^{-3}+q^{-1}\right)\sqrt{1+q^4}, \sqrt{1+q^{-8}+q^{-6}+q^{-4}+q^{-2}}\}
\end{aligned}
\tag{A.2}
$$

Appendix B. Proof of Spectral Properties

Here we prove the result (6.6). The aspects of the fusion procedure we will need are the following. Consider the tensor product of two representations of $\mathcal{U}_q$, $\widehat{\rho}^{(j)}(\nu_1) \otimes \widehat{\rho}^{(j')}(\nu_2)$. If ν_1 and ν_2 are related in a precise way, then this representation is reducible[16]. Let $|j,m>, m \in \{-j, -j+1, ...j\}$ denote the vectors in the representation $\widehat{\rho}^{(j)}$ and consider e.g. the conditions for $\widehat{\rho}^{(1/2)}(\nu_1) \otimes \widehat{\rho}^{(j)}(\nu_2)$ to be reducible and contain a spin $j+1/2$ representation. This requires that

$$
Q_- \left(|1/2, 1/2> \otimes |j,j>\right) = K_{j+1/2} \overline{Q}_- \left(|1/2, 1/2> \otimes |j,j>\right), \tag{B.1}
$$

where $K_{j+1/2}$ is some constant of proportionality. Using $\Delta(Q_-, \overline{Q}_-)$ one can compute the action of $Q_-, \overline{Q}_-$ on the tensor product, and one finds

$$
\nu_1^2 = \nu_2^2 q^{-2} = K_{j+1/2}.
$$

Now consider more specifically the case of $\widehat{\rho}^{(1/2)} \otimes \widehat{\rho}^{(1/2)}$. On this tensor product space, $P = \epsilon\, a^p (\nu_1^p + \nu_2^p) = 0$ since $q^p = -1$. This shows that $P = 0$ on $\widehat{\rho}^{(1)}$. Since the repeated tensor product of $\widehat{\rho}^{(1)}$ with itself yields higher dimensional integer representations, this shows that $P = 0$ on all integer spin representations.

Next consider reducing $\widehat{\rho}^{(1/2)}(\nu_1) \otimes \widehat{\rho}^{(j-1/2)}(\nu_2)$ for j half integer to obtain the representation $\widehat{\rho}^{(j)}$. One has that $P = \epsilon\, a^p \nu_1^p$, where $\nu_1^2 = K_j$. The constant K_j can be computed from the definition (B.1) and the defining relations of $\mathcal{U}_q$. We describe representations using the notation of appendix A. In the representation $\widehat{\rho}^{(j)}(\nu)$ one has that $\widehat{\rho}^{(j)}(\overline{Q}_-)/\nu$ is a $2j+1$ dimensional matrix with only upper off diagonal entries F_{rs}, (with $r = s+1$), and $\widehat{\rho}^{(j)}(Q_-)\nu^{-1} = F^*$. One has then $K_j = F_{21}/F_{21}^*$. From the relation (2.1), one finds that $F_{21} = \left((1-q^{4j})/(1-q^2)\right)^{1/2}$. Thus $K_j = q^{2j-1}$, and this proves (6.6).

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