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All Loop $N = 2$ String Amplitudes

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Using the $N = 4$ topological reformulation of $N = 2$ strings, we compute all loop partition function for special compactifications of $N = 2$ strings as a function of target moduli. We also reinterpret $N = 4$ topological amplitudes in terms of slightly modified $N = 2$ topological amplitudes. We present some preliminary evidence for the conjecture that $N = 2$ strings is the large N limit of Holomorphic Yang-Mills in 4 dimensions.

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1. Introduction

One of the simplest types of string theories is $N = 2$ string. It lives in four dimensions, and it has finite number of particles in the spectrum. Moreover it describes self-dual geometries and Yang-Mills fields [1] [2], which are conjectured to describe, through reduction, all 2 and 3 dimensional integrable models. Moreover the 4 dimensional $N = 2$ string itself seems to correspond to an integrable theory, as is evidenced by perturbative vanishing of scattering amplitudes beyond three point functions.

Given all these connections, it seems very important to understand $N = 2$ string amplitudes. In this paper we consider this question and find, rather surprisingly, that one can compute, at least in special cases, the all genus partition function of $N = 2$ strings. This seems to be another evidence for the quantum integrability of self-dual theories. More specifically we consider compactifications of $N = 2$ strings on $T^2 \times R^2$. Using the reformulation of $N = 2$ strings in terms of $N = 4$ topological strings [3], allows one to develop techniques to compute it.

For low genus, this can be done more or less directly, because the structure of the amplitudes are so simple. However for $g \geq 3$ the story gets more complicated. In such cases we have found a modified version of the harmonicity equation of [3] for which the boundary contributions cancel, and are strong enough to yield the genus g partition function up to an overall constant. Specialized to $g = 1, 2$ this result agrees with explicit computations of the amplitudes. This is somewhat analogous to the method used in [4] to compute the topological $N = 2$ string amplitudes, with the replacement of holomorphic anomaly with harmonicity equation.

Another aspect of $N = 2$ string, is the topological interpretation of what it is computing. We show that quite generally $N = 4$ topological strings, are a slightly (but crucially) modified form of $N = 2$ topological string amplitudes. This allows us to give a more clear interpretation of what topological quantities the partition function computes. In particular we see quite explicitly in the cases of genus 1 and 2 in the example of $T^2 \times R^2$ what these topological quantities are, and moreover reproduce in yet another way, the partition function itself by direct topological evaluation.

Given that $N = 2$ string has finite number of particles it is a candidate for a search for a large N limit of a gauge theory. We look for this and find some preliminary evidence that the large N limit of Holomorphic Yang-Mills theory in 4 dimensions (2 complex dimensions)[5][6] is $N = 2$ strings. This theory is a deformation of $N = 2$ topological

Yang-Mills theory, which has been recently solved in the work of Seiberg and Witten [7]. This is another exciting link with $N = 2$ strings.

The organization of this paper is as follows: In section 2 we review relevant aspects of $N = 2$ strings as well as its topological reformulation. We also give a connection between $N = 4$ and $N = 2$ topological amplitudes in this section. In section 3 we show how the modified harmonicity equation manages to avoid boundary contributions (with some of the details postponed to appendix A). In section 4 we consider the target space to be $T^2 \times R^2$ and evaluate the partition function for all g . We do genus 1 and 2 explicitly (with some of the details for the genus 2 case postponed to the appendix B) and then use the harmonicity equation to rederive these results as well as generalize to all g . In section 5, using the topological reinterpretation, we compute the genus 1 and genus 2 contributions topologically and find agreement with the computation of the previous section. Finally in section 6 we present our conclusions and conjectures.

2. Review of $N = 2$ Strings

In this section we briefly review aspects of $N = 2$ strings which are relevant for this paper. $N = 2$ string was first studied in the early days of string theory [8] and its study was resumed with the surge of interest in string theory [9]. It was discovered relatively recently [1],[2] that $N = 2$ string theory has a rich geometric structure related to self-duality phenomena. In particular its critical dimension is four (2 complex dimensions), and the closed string theory describes self-dual gravity, whereas heterotic and open string versions describe self-dual gauge theories in four dimensions coupled to self-dual gravity. Some of these aspects were further studied [10]. More recently it was shown [3] that the loop amplitude computations in $N = 2$ theories can be simplified by proving their equivalence to a new topological string based on the small $N = 4$ superconformal algebra. In this way the ghosts are eliminated and at the same time the matter fields are topologically twisted; this makes computations much easier. The main aim of this paper is to further elaborate on the meaning of the $N = 2$ string amplitudes in light of this development. In this section we will give a brief review of the topological reformulation of [3] referring the interested readers for the detail to that paper. We will mainly concentrate on the closed string case. The generalization to other cases (heterotic and open) are straight forward.

$N = 2$ strings are obtained by gauging the $N = 2$ local supersymmetry on the world-sheet. This consists of the metric $g_{\mu\nu}$, two supersymmetric partners of spin $3/2$, $\psi_{\mu\alpha}^{\pm}$ and

one $U(1)$ gauge field A_μ . In the standard fashion, these give rise to a pair of fermionic ghost (b, c) of spin 2, two pairs of bosonic superghosts (β^\pm, γ^\pm) of spin 3/2 and another pair of fermionic ghost (\tilde{b}, \tilde{c}) of spin 1. The total ghost anomaly is $c = -6$, which is cancelled by a matter with $c = 6$, corresponding to a superconformal theory in 4 dimensions. The vacua of $N = 2$ strings consist of theories in 4d which have Ricci-flat metric [1]. These theories will necessarily have an extended symmetry, by including the spectral flow operators, to the small $N = 4$ superconformal algebra with $c = 6$ ($\hat{c} = 2$).

The $N = 4$ algebra consists of an energy momentum tensor T of spin 2, an $SU(2)$ current algebra of spin 1, whose generators are denoted by J^{++}, J, J^{--} and 4 spin 3/2 supercurrents which form two doublets (G^-, \tilde{G}^+) and (\tilde{G}^-, G^+) under the $SU(2)$ currents. The supercurrents within a doublet have no singularities with each other, while the oppositely charged supercurrents of the different doublets have singular OPE (and in particular give the energy momentum tensor). Moreover G^+ and \tilde{G}^+ have a singular OPE with a total derivative as the residue:

$$G^+(z)\tilde{G}^+(0) \sim \frac{\partial J^{++}(0)}{z}$$

Note in addition that

$$\tilde{G}^+ = G^-(J^{++}) \tag{2.1}$$

which follows from the fact that (G^-, \tilde{G}^+) form an $SU(2)$ doublet. Also note that J^{++} is the left-moving spectral flow operator. This in particular implies that the chiral field V corresponding to the volume form of the superconformal theory can be written as

$$V = J_L^{++} J_R^{++} \tag{2.2}$$

Together with (2.1) this means that

$$G_L^- G_R^- V(z, \bar{z}) = \tilde{G}_L^+ \tilde{G}_R^+(z, \bar{z}) \tag{2.3}$$

It is important to note that the choice of two doublets among the four supersymmetry currents is ambiguous: In particular there is a sphere worth of inequivalent choices given by

$$\begin{aligned} \widehat{\tilde{G}^+}(u) &= u^1 \tilde{G}^+ + u^2 G^+ \\ \widehat{G^-}(u) &= u^1 G^- - u^2 \tilde{G}^- \\ \widehat{\tilde{G}^-}(u) &= u^{2*} \tilde{G}^- - u^{1*} G^- \end{aligned}$$

$$\widehat{G^+}(u) = u^{2*}G^+ + u^{1*}\tilde{G}^+ \quad (2.4)$$

where

$$|u^1|^2 + |u^2|^2 = 1$$

and where the complex conjugate of u_a is $\epsilon^{ab}u_b^*$ (i.e. $\overline{(u^1)} = u^{2*}$ and $\overline{(u^2)} = -u^{1*}$ where $*^2 = -1$). Note that we could do this rotation for left and right $N = 4$ algebras independently, and we will use u_L, u_R to denote the left- and right-moving choices for the rotation.

A theory with $N = 4$ superconformal structure can be deformed, preserving the $N = 4$ structure using chiral field of (left,right) charge (1,1). There are four deformations that can be made out of a given chiral field ϕ^i :

$$S \rightarrow S + \int t_i^{11} G_L^- G_R^- \phi^i - t_i^{21} \tilde{G}_L^- G_R^- \phi^i - t_i^{12} G_L^- \tilde{G}_R^- \phi^i + t_i^{22} \tilde{G}_L^- \tilde{G}_R^- \phi^i$$

Note that for unitary $N = 4$ theories, these deformations are pairwise complex conjugate. In particular there exists a matrix M_i^{j*} so that

$$t_i^{ab} = \epsilon^{ac} \epsilon^{bd} M_i^{j*} t_j^{\bar{c}\bar{d}}$$

with $MM^* = 1$.

The $N = 2$ string amplitudes are computed by integration of the string measure over the $N = 2$ supermoduli. The bosonic piece of this moduli consists of the moduli of genus g Riemann surfaces as well as the g -dimensional moduli of $U(1)$ bundles with a given instanton number n . For a fixed instanton number the dimension of β^\pm zero modes gives the dimension of supermoduli. Since they are charged under the $U(1)$ this dimension will depend on the instanton number. In particular the dimension of these supermoduli is $(2g - 2 - n, 2g - 2 + n)$ for the (β^+, β^-) zero modes. In particular this means that $|n| \leq 2g - 2$ in order to get a non-zero measure. Even though geometrically not obvious, it turns out that we can also assign independent left-moving and right-moving instanton numbers. So at each genus g we have to compute the string amplitudes F_{n_L, n_R}^g with $-2g + 2 \leq n_L, n_R \leq 2g - 2$. It is convenient to collect these amplitudes in terms of a function on u -space. Let

$$\begin{aligned} F^g(u_L, u_R) &= \\ &= \sum_{-2g+2 \leq n_L, n_R \leq 2g-2} \binom{4g-4}{2g-2+n_L} \binom{4g-4}{2g-2+n_R} \cdot F_{n_L, n_R}^g \times \\ &\quad \times (u_L^1)^{2g-2+n_L} (u_R^1)^{2g-2+n_R} (u_L^2)^{2g-2-n_L} (u_R^2)^{2g-2-n_R} \end{aligned}$$

The result of [3] is that F^g can be computed by

$$\begin{aligned}
F^g(u_L, u_R) = & \int_{\mathcal{M}_g} \langle [\prod_{A=1}^{3g-3} (\mu_A, \widehat{G}_L^-(u_L)) (\bar{\mu}_A, \widehat{G}_R^-(u_R))] \int_{\Sigma} J_L J_R \times \\
& \times [\int_{\Sigma} \widehat{G}_L^+(u_L) \widehat{G}_R^+(u_R)]^{g-1} \rangle
\end{aligned} \tag{2.5}$$

where Σ denotes the Riemann surface and \mathcal{M}_g denotes the moduli of genus g surfaces and μ_A denote the Beltrami differentials. In this expression there are no ghosts left over and the $N = 4$ matter field is topologically twisted, i.e. the spin of the fields are shifted by half their charge, so in particular G^+, \tilde{G}^+ have spin 1 and G^-, \tilde{G}^- have spin 2 and J^{++} has spin 0 and J^{--} has spin 2.

Let us give a rough outline of how the above correspondence between $N = 2$ string amplitudes and the $N = 4$ topological amplitude, defined above, arises. The simplest case of constructing this measure corresponds to the $n_L = n_R = 2g - 2$. In this case we have no β^+ zero modes, and $(4g - 4)$ β^- zero modes. If we had instanton number $(g - 1)$, it would have been equivalent to twisting the fields, by the definition of topological twisting (identifying gauge connection with half the spin connection). So for instanton number $(2g - 2)$, we can view the amplitudes as being computed in the topologically twisted version but with an addition of $(g - 1)$ instanton number changing operators inserted. Note that the matter part of the instanton number changing operator is J^{++} . In the topologically twisted measure the (β^-, γ^-) ghost system have the same spin as (b, c) and the (β^+, γ^+) have the same spin as (\tilde{b}, \tilde{c}) , and since they are of the opposite statistics they cancel each other out as far as the non-zero modes are concerned. The zero modes can also be canceled out by a judicious choice of the position of picture changing operators. We have $(4g - 4)$ picture changing operators inserted for integration over the supermoduli which are accompanied from the matter sector with G^- . $(3g - 3)$ of them get folded with the Beltrami differentials in cancelling the zero modes of b . The integration over the $U(1)$ moduli is traded with integration over g operators on Riemann surfaces: $(g - 1)$ of them come from operators where $(g - 1)$ of the instanton changing operators have converted G^- into \tilde{G}^+ and the last one is simply the current J . This would give the correspondence at the highest instanton numbers and the rest are obtained by performing an $SU(2)$ rotation on the $N = 2$ string side and seeing that it corresponds to changing the instanton numbers.

Topological Meaning of $N = 2$ String Amplitudes

Given the fact that the physical $N = 2$ string amplitudes have been reformulated in terms of topologically twisted $N = 4$ theories, it is natural to ask if there is any topological meaning to the latter. Recall that if we have any $N = 2$ superconformal theory we can consider the twisted version and couple it to topological gravity, which has critical dimension 3. The geometrically interesting examples of such theories are sigma models on Calabi-Yau manifolds and depending on how the left- and right-moving degrees of freedom are twisted we get a topological theory which counts holomorphic maps (A-model) or quantizes the variations of complex structure on the Calabi-Yau (the Kodaira-Spencer theory [4] obtained from B-model). If the complex dimension of Calabi-Yau is not equal to three the topological string amplitude vanishes because the $(3g - 3)$ negative charges of the G^- insertions is not balanced by the $d(g - 1)$ charge violation of the $U(1)$ of the $N = 2$ algebra if $d \neq 3$. Only in the case of complex dimension 2 one can still try to get a non-vanishing amplitude by inserting $(g - 1)$ chiral operators to the action of the form $G_L^- G_R^- V$ where V is the unique chiral field with charge two² and it corresponds to the volume form of the complex 2d manifold. Note using (2.3) that these $(g - 1)$ insertions are the same as the $(g - 1)$ insertions of $\tilde{G}_L^+ \tilde{G}_R^+$. In other words it gives exactly the same result as the partition function for the highest instanton number of the $N = 2$ string (2.5) with the exception of the insertion of $\int J_L J_R$. It was argued in [4] that this $N = 2$ topological amplitude vanishes even with this charge insertion. In fact it was directly argued in [3] that this follows rather simply from the underlying $N = 4$ algebra. So the $N = 2$ string amplitude manages to be non-trivial precisely because of the extra insertion of $\int J_L J_R$. Therefore there must be a simple topological meaning for the highest instanton amplitude of the $N = 2$ string.

For concreteness let us consider the A-model version which is set up to count the holomorphic maps from Riemann surfaces to Calabi-Yau manifolds. In the limit that $\bar{t}_i \rightarrow \infty$ one can show that the measure is concentrated near the holomorphic maps [4]. In this case we are considering holomorphic maps which map the Riemann surface with $(g - 1)$ points on the Riemann surface mapped to specific $(g - 1)$ points on the target which is dual to the volume form. Actually to go to the Poincare dual of the volume form one has to use G^+ trivial operators to deform the field, but that may change the amplitude in this case because we have J insertion which does not commute with G^+ . So we have

² In dimension bigger than 3 we need a negative charged chiral field which does not exist, and in dimension 1, there is no chiral field with charge bigger than one.

to use the precise representative given by $G_L^- G_R^- V$. Each time we choose a cohomology representative in target of degree d (corresponding to d -forms), it gives rise to a $d - 2$ form on moduli space (which translating degree to charge, in the operator language means that the charge is decreased by two units because of the insertion of $\oint G_L^- \oint G_R^-$)³. In our case each volume form will give a $(1, 1)$ form on the moduli space of holomorphic maps which we denote by k . So consider the moduli space \mathcal{M}^g of holomorphic maps from genus g to the 2 complex manifold. The formal complex dimension of \mathcal{M} is $(g - 1)$, however it typically has a dimension bigger than $(g - 1)$. In such cases the topological amplitude computation is done by considering the bundle \mathcal{V} on \mathcal{M} whose fibers are the anti-ghost zero modes which is $H^1(N)$ where N is the pull back of the normal bundle piece of the tangent bundle on the manifold restricted to the holomorphic image of the Riemann surface. Let n be the dimension of \mathcal{V} . Then the complex dimension of \mathcal{M} is $(g - 1 + n)$. Therefore if it were not for the $\int J_L J_R$ insertion, the usual arguments of topological strings, in the simple cases, would lead to the computation of

$$\int_{\mathcal{M}} k^{g-1} c_n(\mathcal{V}) \tag{2.6}$$

where c_n denotes the n -th chern class of \mathcal{V} . However as mentioned before this amplitude vanishes. The effect of the $\int J_L J_R$ insertion, will correspond on the moduli of holomorphic maps to a $(1,1)$ form which we denote by \mathcal{J} . This has the effect of absorbing one of the fermion zero modes which was responsible for the vanishing of the amplitude. Thus the characteristic class that we will end up with from \mathcal{V} will be of dimension $(n - 1)$. The precise form of it may depend on the case under consideration. Therefore using the same reasoning as for topological theories we see that the top instanton number amplitude for $N = 2$ strings in the $\bar{t} \rightarrow \infty$ computes

$$F_{2g-2, 2g-2}^g |_{\bar{t} \rightarrow \infty} = \int_{\mathcal{M}^g} k^{g-1} \wedge c_{n-1}(\mathcal{V}) \wedge \mathcal{J} \tag{2.7}$$

Later in this paper we will see how this works in detail in the case of the four manifold $T^2 \times R^2$ for $g = 1, 2$. It happens that for some topological strings the formula (2.6) is modified. An example of this is discussed in [11]. In such cases some of the insertions of operators corresponding to fields (the analog of k in the above) will be replaced by

³ The form on the moduli space can be described by considering the canonical map from the total space of the Riemann surface and the moduli space of holomorphic maps to the target manifold, and using the pull-back of the d -form and integrating it over the Riemann surface.

modifying the bundle V . It turns out that this does happen for us for $g \geq 3$ for the example of $T^2 \times R^2$. For $g \geq 3$ the above formula in this case gets replaced by

$$F_{2g-2,2g-2}^g|_{\bar{t} \rightarrow \infty} = \int_{\mathcal{M}^g} k \wedge c_{n+g-3}(\tilde{\mathcal{V}}) \wedge \mathcal{J} \quad (2.8)$$

for some $\tilde{\mathcal{V}}$. Unfortunately there is no general prescription for computing this that we are aware of, and it very much depends on the models. We have not computed $\tilde{\mathcal{V}}$ for $T^2 \times R^2$, which is relevant for $g > 2$ amplitudes.

3. Harmonicity Equation

In this section, we will prove that the g -loop amplitude $F^g(u_L, u_R)$ solves the equations

$$\epsilon^{ab} u_R^c \frac{\partial}{\partial u_L^a} D_{tbc} F^g(u_L, u_R) = 0 \quad (3.1)$$

$$\epsilon^{ab} u_L^c \frac{\partial}{\partial u_R^a} D_{tcb} F^g(u_L, u_R) = 0. \quad (3.2)$$

In the paper [3], Berkovits and one of the authors have pointed out that the stronger version of these equations

$$\begin{aligned} \epsilon^{ab} \frac{\partial}{\partial u_L^a} D_{tbc} F^g(u_L, u_R) &= 0 \\ \epsilon^{ab} \frac{\partial}{\partial u_R^a} D_{tcb} F^g(u_L, u_R) &= 0 \end{aligned} \quad (3.3)$$

would hold if the contributions from the boundary of the moduli space \mathcal{M}_g and contact terms in operator products were absent. The purpose of this section is to examine these contributions carefully. As we shall see, there are in fact contact terms which spoil (3.3). This is also in accord with the fact that if there were no corrections to (3.3) it would lead to puzzling conclusions [12]. Fortunately these contact terms are cancelled out in (3.1) and (3.2). In the following, we will refer to these as *harmonicity equations*.

In the next section, we will examine the case when the target space is $M = T^2 \times R^2$. For $g = 2$, we can compute F^g directly and check explicitly that the F^g directly and check explicitly that the harmonicity equations (3.1) and (3.2) are satisfied. Furthermore the harmonicity equations will make it possible to determine F^g for all $g \geq 3$ up to a constant factor independent of target space moduli at each g .

Now let us prove the harmonicity equations. The steps in the proof are parallel to those used in [3] except we will have to be very careful with many boundary contributions

that arise. The covariant derivative D_{tab} is defined so that its action on F^g generates an insertion of a marginal operator corresponding to the target space moduli t^{ab} ;

$$\begin{aligned} w_L^a w_R^b D_{tab} F^g(u_L, u_R) &= \\ &= \int_{\mathcal{M}_g} \langle [\prod_{A=1}^{3g-3} (\mu_A, \widehat{G}_L^-(u_L)) (\bar{\mu}_A, \widehat{G}_R^-(u_R))] \int_{\Sigma} J_L J_R \times \\ &\quad \times \left[\int_{\Sigma} \widehat{G}_L^+(u_L) \widehat{G}_R^+(u_R) \right]^{g-1} \int_{\Sigma} \{ \widehat{Q}_L^+(w_L), [\widehat{Q}_R^+(u_R), \bar{\phi}] \} \rangle \end{aligned}$$

where $\bar{\phi}$ is an anti-chiral primary field coupled to the moduli t^{ab} . Therefore the left-hand side of (3.1) can be written as

$$\begin{aligned} \epsilon^{ab} u_R^c \frac{\partial}{\partial u_L^a} D_{t^{bc}} F^g &= \\ &= \epsilon^{ab} \frac{\partial^2}{\partial u_L^a \partial w_L^b} [w_L^b u_R^c D_{t^{bc}} F^g] = \\ &= \epsilon^{ab} \frac{\partial^2}{\partial u_L^a \partial w_L^b} \int_{\mathcal{M}_g} \langle [\prod_{A=1}^{3g-3} (\mu_A, \widehat{G}_L^-(u_L)) (\bar{\mu}_A, \widehat{G}_R^-(u_R))] \times \\ &\quad \times \int_{\Sigma} J_L J_R \left[\int_{\Sigma} \widehat{G}_L^+(u_L) \widehat{G}_R^+(u_R) \right]^{g-1} \times \\ &\quad \times \int_{\Sigma} \{ \widehat{Q}_L^+(w_L), [\widehat{Q}_R^+(u_R), \bar{\phi}] \} \rangle. \end{aligned} \tag{3.4}$$

Note that the marginal operator inserted here is $\{ \widehat{Q}_L^+(w_L), [\widehat{Q}_R^+(u_R), \bar{\phi}] \}$ with u_R in the right-mover while $w_L \neq u_L$ in the left-mover. Since

$$\begin{aligned} \epsilon^{ab} \frac{\partial^2}{\partial u_L^a \partial w_L^b} \widehat{G}_L^+(u_L) \widehat{Q}_L^+(w_L) &= \tilde{G}_L^+ Q_L^+ - G_L^+ \tilde{Q}_L^+ \\ &= (\epsilon_{ab} u_L^a w_L^b)^{-1} \left(\widehat{G}_L^+(u_L) \widehat{Q}_L^+(w_L) - \widehat{G}_L^+(w_L) \widehat{Q}_L^+(u_L) \right) \end{aligned}$$

and similarly

$$\begin{aligned} \epsilon^{ab} \frac{\partial^2}{\partial u_L^a \partial w_L^b} \widehat{G}_L^-(u_L) \widehat{Q}_L^+(w_L) &= G_L^- Q_L^+ + \tilde{G}_L^- \tilde{Q}_L^+ \\ &= (\epsilon_{ab} u_L^a w_L^b)^{-1} \left(\widehat{G}_L^-(u_L) \widehat{Q}_L^+(w_L) - \widehat{G}_L^-(w_L) \widehat{Q}_L^+(u_L) \right), \end{aligned}$$

the differential operator $\epsilon^{ab} \partial_{u_L^a} \partial_{w_L^b}$ exchanges u_L and w_L . In the following, we will show that if we pick any of $(4g-4)$ u_L 's in the correlation function

$$\begin{aligned} f^g(u_L, u_R; w_L) &= \int_{\mathcal{M}_g} \langle [\prod_{A=1}^{3g-3} (\mu_A, \widehat{G}_L^-(u_L)) (\bar{\mu}_A, \widehat{G}_R^-(u_R))] \int_{\Sigma} J_L J_R \times \\ &\quad \times \left[\int_{\Sigma} \widehat{G}_L^+(u_L) \widehat{G}_R^+(u_R) \right]^{g-1} \int_{\Sigma} \{ \widehat{Q}_L^+(w_L), [\widehat{Q}_R^+(u_R), \bar{\phi}] \} \rangle, \end{aligned}$$

and exchange it with w_L , f^g remains invariant. This implies that f^g obeys

$$\epsilon^{ab} \frac{\partial^2}{\partial u_L^a \partial w_L^b} f^g(u_L, u_R; w_L) = 0,$$

which is equivalent to the harmonicity equation (3.1) by (3.4).

Since

$$\begin{aligned} \widehat{G}_L^-(u_L) &= [\widehat{\tilde{Q}}_L^+(u_L), J_L^-] \\ \widehat{G}_L^+(u_L) &= -[\widehat{\tilde{Q}}_L^+(u_L), J_L], \end{aligned} \tag{3.5}$$

exchanging u_L and w_L is same as exchanging locations of $\widehat{\tilde{Q}}_L^+(u_L)$ and $\widehat{\tilde{Q}}_L^+(w_L)$ in $f^g(u_L, u_R; w_L)$. We can exchange their locations just like two automobile drivers would do when they try to pass each other on a narrow country road. We can first move $\widehat{\tilde{Q}}_L^+(w_L)$ off from $[\widehat{\tilde{Q}}_R^+(u_R), \bar{\phi}]$, park it in a “turn-out” at $\int_{\Sigma} J_L J_R$. We then move $\widehat{\tilde{Q}}_L^+(u_L)$ off from J_L^- or J_L in (3.5), let it pass $\widehat{\tilde{Q}}_L^+(w_L)$, and stop it at $[\widehat{\tilde{Q}}_R^+(u_R), \bar{\phi}]$. Finally we move $\widehat{\tilde{Q}}_L^+(w_L)$ out from the turn-out and stop it at J_L^- or J_L . In this way, we can exchange locations of $\widehat{\tilde{Q}}_L^+(w_L)$ and $\widehat{\tilde{Q}}_L^+(u_L)$.

This is not the complete story since we have neglected the anti-commutators of $\widehat{\tilde{Q}}_L^+$ with \widehat{G}_L^- and \widehat{G}_L^+ ;

$$\begin{aligned} \{\widehat{\tilde{Q}}_L^+(w_L), \widehat{G}_L^-(u_L)\} &= 2(\epsilon_{ab} u_L^a w_L^b) T_L \\ \{\widehat{\tilde{Q}}_L^+(w_L), \widehat{G}_L^+(u_L)\} &= \frac{1}{2}(\epsilon_{ab} u_L^a w_L^b) \partial J_L^{++}, \end{aligned}$$

which appear when we move $\widehat{\tilde{Q}}_L^+(u_L)$ and $\widehat{\tilde{Q}}_L^+(w_L)$ back and forth in f^g . Therefore what we have shown so far is that (3.4) is a linear combination of the following four types of terms;

$$\begin{aligned} &\int_{\mathcal{M}_g} \langle (\mu_{A'}, T_L) \prod_{A \neq A'} (\mu_A, \widehat{G}_L^-(u_L)) \prod_{A=1}^{3g-3} (\bar{\mu}_A, \widehat{G}_R^-(u_R)) \rangle \times \\ &\quad \times \int_{\Sigma} J_L J_R \left[\int_{\Sigma} \widehat{G}_L^+(u_L) \widehat{G}_R^+(u_R) \right]^{g-1} \int_{\Sigma} [\widehat{\tilde{Q}}_R^+(u_R), \bar{\phi}], \end{aligned} \tag{3.6}$$

$$\begin{aligned} &\int_{\mathcal{M}_g} \langle (\mu_{A'}, T_L) (\mu_{A''}, J_L^-) \prod_{A \neq A', A''} (\mu_A, \widehat{G}_L^-(u_L)) \prod_{A=1}^{3g-3} (\bar{\mu}_A, \widehat{G}_R^-(u_R)) \rangle \times \\ &\quad \times \int_{\Sigma} J_L J_R \left[\int_{\Sigma} \widehat{G}_L^+(u_L) \widehat{G}_R^+(u_R) \right]^{g-1} \int_{\Sigma} \{ \widehat{\tilde{Q}}_L^+(u_L), [\widehat{\tilde{Q}}_R^+(u_R), \bar{\phi}] \}, \end{aligned} \tag{3.7}$$

$$\int_{\mathcal{M}_g} \langle [\prod_{A=1}^{3g-3} (\mu_A, \widehat{G}_L^-(u_L)) (\bar{\mu}_A, \widehat{G}_R^-(u_R))] \int_{\Sigma} J_L J_R \times \int_{\Sigma} \partial J_L^{++} \widehat{G}_R^+(u_R) \left[\int_{\Sigma} \widehat{G}_L^+(u_L) \widehat{G}_R^+(u_R) \right]^{g-2} \int_{\Sigma} [\widehat{Q}_R^+(u_R), \bar{\phi}] \rangle. \quad (3.8)$$

$$\int_{\mathcal{M}_g} \langle (\mu_{A'}, J_L^-) \prod_{A \neq A'} (\mu_A, \widehat{G}_L^-(u_L)) \prod_{A=1}^{3g-3} (\bar{\mu}_A, \widehat{G}_R^-(u_R)) \times \int_{\Sigma} \partial J_L^{++} J_R \left[\int_{\Sigma} \widehat{G}_L^+(u_L) \widehat{G}_R^+(u_R) \right]^{g-1} \int_{\Sigma} [\widehat{Q}_R^+(u_R), \bar{\phi}] \rangle, \quad (3.9)$$

where $A', A'' = 1, \dots, 3g-3$ ($A' \neq A''$). We did not write terms which are related to one of these four types by contour deformation of the operators. To prove the harmonicity equation (3.1), we want to show that these four terms vanish.

The first two, (3.6) and (3.7), contain $(\mu_{A'}, T_L)$ thus are total derivative in the moduli space \mathcal{M}_g of smooth Riemann surfaces. Then these integrals reduce to integrals on the boundaries of \mathcal{M}_g . In Appendix A, it is shown that there is no boundary contribution to these integrals and therefore we can ignore (3.6) and (3.7).

The third (3.8) and fourth (3.9) will also be zero if we can integrate away the total derivatives $\partial J_L^{++} \widehat{G}_R^+(u_R)$ and $\partial J_L^{++} J_R$. This will be possible if there is no singularity in the domains of these integrals. Let us first consider the third term (3.8). By the Cauchy theorem, the surface integral of $\partial J_L^{++} \widehat{G}_R^+(u_R)$ becomes anti-holomorphic contour integrals around other operators in the correlation function. Although there are operators in (3.8) which have operator product singularities with $J_L^{++} \widehat{G}_R^+(u_R)$, none of these singularities survive after the contour integrals since they all have wrong powers in the holomorphic and anti-holomorphic coordinates. Thus (3.8) vanishes by integration-by-parts.

Let us examine the last piece (3.9). Again $J_L^{++} J_R$ has singularities with other operators in (3.9), but the only ones that survive the contour integrals are those at $\widehat{G}_R^-(u_R)$, which gives

$$\int_{\mathcal{M}_g} \langle (\mu_{A'}, J_L^-) (\mu_{A'''}, J_L^{++} \widehat{G}_R^-(u_R)) \prod_{A \neq A'} (\mu_A, \widehat{G}_L^-(u_L)) \prod_{A \neq A'''} (\bar{\mu}_A, \widehat{G}_R^-(u_R)) \times \left[\int_{\Sigma} \widehat{G}_L^+(u_L) \widehat{G}_R^+(u_R) \right]^{g-1} \int_{\Sigma} [\widehat{Q}_R^+(u_R), \bar{\phi}] \rangle,$$

where $A''' = 1, \dots, 3g-3$. However it is easy to see that this in fact is also zero. To show this, we can just move $\widehat{Q}_R^+(u_R)$ off from $\bar{\phi}$. Since everything in the above (anti-) commutes

with $\widehat{\widetilde{Q}}_R^+(u_R)$, its contour drops off from the Riemann surface. Therefore the last term (3.9) also vanishes by integration-by-parts.

We have shown that (3.6), (3.7), (3.8) and (3.9) are all zero. Thus the harmonicity equation (3.1) is proven.

We should point out that in this proof it is crucial that $\widehat{\widetilde{Q}}_R^+(u_R)$ in the marginal operator has the same u_R as the rest of the operators \widehat{G}_R^- and \widetilde{G}_R^+ in F^g . Otherwise the operators $J_L^{++}\widehat{\widetilde{G}}_R^+$ would have a singularity with $[\widehat{\widetilde{Q}}_R^+(u_R), \bar{\phi}]$ with a non-zero pole residue, and the last part of the proof would not go through. This is where the stronger version of the harmonicity equation (3.3) breaks down.

4. $N = 2$ String Amplitudes on $T^2 \times R^2$ to All Order in Perturbation

In this section, we will examine the $N = 2$ string amplitudes on $T^2 \times R^2$ in detail. We consider the A -model only. The corresponding amplitudes in the B -model are obtained by simply replacing the Kähler moduli σ by the complex moduli ρ .

At genus one, the string amplitude has been computed in our previous paper [1] as

$$F^1 = -\log \left(\sqrt{\text{Im}\sigma \text{Im}\rho} |\eta(\sigma)|^2 |\eta(\rho)|^2 \right),$$

where σ and ρ are Kähler and complex moduli of T^2 respectively. At genus two, we will carry out explicit computation below and derive⁴

$$F^2(u_L, u_R) = \sum_{(n,m) \neq (0,0)} \left(\frac{u_L^1 u_R^1}{n + m\sigma} + \frac{u_L^2 u_R^2}{n + m\bar{\sigma}} \right)^4.$$

We will show that $F_{2,2}^2$ has a nice topological interpretation as counting of number of holomorphic maps from genus two surfaces to T^2 .

We will also verify that these expressions for F^1 and F^2 are consistent with the harmonicity equations, (3.1) and (3.2). We will then apply these equations to $g \geq 3$ amplitudes. It turns out that the harmonicity equations determine F^g up to an overall constant at each genus as

$$F^g(u_L, u_R) = (\text{const}) \times \sum_{(n,m) \neq (0,0)} |n + m\sigma|^{2g-4} \left(\frac{u_L^1 u_R^1}{n + m\sigma} + \frac{u_L^2 u_R^2}{n + m\bar{\sigma}} \right)^{4g-4}.$$

Topological interpretation of $g \geq 3$ amplitudes will be discussed in section 5.

⁴ This is up to an overall normalization. To obtain the topological normalization discussed in section 5.1 we need to multiply the above result by $\frac{1}{4(2\pi)^4}$.

4.1. Genus One

At genus one, the $N = 2$ string amplitude F^1 is given by

$$F^1 = \frac{1}{4} \int \frac{d\tau}{(\text{Im}\tau)^2} \left\langle \left(\int J_L J_R \right)^2 \right\rangle$$

where τ is the modulus of the worldsheet torus. This is defined in such a way that the derivative $D_{t^{ab}}$ with respect to the target space moduli t^{ab} gives

$$u_L^a u_R^b D_{t^{ab}} F^1 = \frac{1}{2} \int \frac{d\tau}{(\text{Im}\tau)^2} \left\langle \int J_L J_R \int \widehat{G}_L^-(u_L) \widehat{G}_R^-(u_R) \phi \right\rangle$$

which is natural generalization of (2.5) ($1/2$ is due to the Z_2 symmetry of the torus).

When the target space is $T^2 \times R^2$, this expression reduces to

$$F^1 = \frac{1}{2} \sum_{n,m,r,s \in Z} \int \frac{d^2\tau}{(\text{Im}\tau)^2} \exp(-S)$$

where

$$S = \frac{1}{\text{Im}\tau \text{Im}\rho} \left(t p_L \bar{p}_R + \bar{t} \bar{p}_L p_R \right)$$

p_L and p_R are string momenta on T^2 given by

$$\begin{aligned} p_L &= (n + \rho s) - (m + \rho r) \bar{\tau} \\ \bar{p}_L &= (n + \bar{\rho} s) - (m + \bar{\rho} r) \bar{\tau} \\ p_R &= (n + \rho s) - (m + \rho r) \tau \\ \bar{p}_R &= (n + \bar{\rho} s) - (m + \bar{\rho} r) \tau \end{aligned}$$

with ρ being the complex moduli of T^2 , and t is the Kähler modulus whose real and imaginary parts are the volume of T^2 and the theta parameter of the sigma-model. In [1] (See also [4] and [13]) , this integral is evaluated with the result

$$F^1 = -\log \left(\sqrt{\text{Im}\sigma \text{Im}\rho} |\eta(\sigma)|^2 |\eta(\rho)|^2 \right) \quad (4.1)$$

where $\sigma = (8\pi i)^{-1} t$.

It is easy to show that this expression is consistent with the harmonicity equation. In fact, in this case, the stronger version of the harmonicity equation (3.3) also holds. On F^1 , the stronger version takes the form

$$\left(\frac{D}{Dt^{1a}} \frac{D}{Dt^{2b'}} - \frac{D}{Dt^{2a}} \frac{D}{Dt^{1b'}} \right) F^1 = 0 \quad (4.2)$$

where $a, b = 1, 2$. Let us first consider the case when $(a, b) = (1, 2)$ and choose t^{11} to be the Kähler modulus t and $t^{22'}$ to be its complex conjugate \bar{t} . In this case $D_{t^{21}}$ and $D_{t^{12'}}$ insert operators $\partial X^2 \bar{\partial} X^1$ and $\partial X^2 \bar{\partial} X^{\bar{1}}$ on the worldsheet where X^1 and X^2 are coordinates on T^2 and R^2 parts of the target space respectively. Since there is no winding mode in R^2 direction, ∂X^2 and $\bar{\partial} X^2$ are contracted according to the Wick rule, and its residue is proportional to $\bar{\partial} X^1 \bar{\partial} X^{\bar{1}}$. This is equal (up to a factor $(t + \bar{t})^{-1}$) to the T^2 part of the energy-momentum tensor T_R , which we can convert into a derivative $\partial_{\bar{\tau}}$ with respect to the worldsheet modulus. Thus we find

$$D_{t^{21}} D_{t^{12'}} F^1 = \frac{1}{\pi i (t + \bar{t})} \sum_{n, m, r, s} \int d^2 \tau \frac{\partial}{\partial \bar{\tau}} \left(\frac{1}{\text{Im} \tau} \exp(-S) \right).$$

Since this integral is total derivative in $\bar{\tau}$, it will receive contribution only from the boundary of the moduli space at $\text{Im} \tau \rightarrow \infty$. There the sum over n, m, r, s becomes an integral and we obtain

$$\sum_{n, m, r, s} \exp(-S) \rightarrow \frac{\pi \text{Im} \tau}{t + \bar{t}}.$$

The contribution from the boundary of the moduli space then gives

$$D_{t^{21}} D_{t^{12'}} F^1 = \frac{1}{2(t + \bar{t})^2}.$$

By combining this with the harmonicity equation (4.2),

$$D_t D_{\bar{t}} F^1 = \frac{1}{2(t + \bar{t})^2}$$

and this is consistent with the expression (4.1) of F^1 .

By considering other cases such as $(a, b) = (1, 1)$ and $t^{11} = t, t^{12'} = \rho$, we can derive identities such as

$$D_t D_{\rho} F^1 = 0,$$

which are also consistent with the expression (4.1). In fact, the harmonicity equation (4.2) together with the modular invariance in ρ and the duality in t uniquely determine F^1 to be of the form (4.1).

4.2. Genus Two; Evaluation I

Genus two computation of the amplitude is much easier than $g > 2$, because all the fermionic fields in the definition of the partition function are absorbed by the fermion zero modes (this is mirrored, as we will explain later, in the simplicity in its topological reinterpretation). We leave this aspect of the genus 2 amplitude computation to Appendix B, where it is shown that the genus 2 amplitude F^2 on T^4 is given by

$$F^2 = \sum_{P_L, P_R} \int \left(\frac{\det g}{\det \text{Im}\Omega} \right)^2 \langle [\det(\widehat{P}_L + \widehat{r}_L) \det(\widehat{P}_R + \widehat{r}_R)]^2 \rangle \times \exp[-S(P_L, P_R)] \frac{d^3\Omega d^3\bar{\Omega}}{[\det \text{Im}\Omega]^3}. \quad (4.3)$$

where \widehat{P}_L and \widehat{P}_R are given by

$$\begin{aligned} \widehat{P}_{La}^i &= u_L^1 P_{La}^i + u_L^2 \epsilon^{ij} g_{j\bar{k}} P_{La}^{\bar{k}} \\ \widehat{P}_{Ra}^{\bar{i}} &= u_R^1 P_{Ra}^{\bar{i}} + u_R^2 \epsilon^{\bar{i}\bar{j}} g_{\bar{j}k} P_{Ra}^k \end{aligned}$$

with P_L and P_R being parametrized by a set of integers n, m, r, s as

$$\begin{aligned} P_{La}^i &= (n_a^i + \rho_j^i s_a^j) - (m^{ib} + \rho_j^i r^{jb}) \bar{\Omega}_{ba} \\ P_{La}^{\bar{i}} &= (n_a^{\bar{i}} + \bar{\rho}_j^{\bar{i}} s_a^j) - (m^{ib} + \bar{\rho}_j^{\bar{i}} r^{jb}) \bar{\Omega}_{ba} \\ P_{Ra}^i &= (n_a^i + \rho_j^i s_a^j) - (m^{ib} + \rho_j^i r^{jb}) \Omega_{ba} \\ P_{Ra}^{\bar{i}} &= (n_a^{\bar{i}} + \bar{\rho}_j^{\bar{i}} s_a^j) - (m^{ib} + \bar{\rho}_j^{\bar{i}} r^{jb}) \Omega_{ba}, \end{aligned}$$

\widehat{r}_L and \widehat{r}_R are quantum variables obeying

$$\begin{aligned} \langle \widehat{r}_{La}^i \widehat{r}_{Lb}^j \rangle &= \langle \widehat{r}_{Ra}^{\bar{i}} \widehat{r}_{Rb}^{\bar{j}} \rangle = 0 \\ \langle \widehat{r}_{La}^i \widehat{r}_{Rb}^{\bar{j}} \rangle &= -g^{i\bar{j}} (u_L^1 u_R^1 + u_L^2 u_R^2) (\text{Im}\Omega)_{ab}, \end{aligned} \quad (4.4)$$

the action S is given by

$$S(P_L, P_R) = \left(t_{i\bar{j}} P_{La}^i P_{Rb}^{\bar{j}} + \bar{t}_{i\bar{j}} P_{La}^{\bar{j}} P_{Rb}^i \right) (\text{Im}\Omega^{-1})^{ab},$$

with $t_{i\bar{j}} = g_{i\bar{j}} + i\theta^\alpha k_{i\bar{j}}^\alpha$ and Ω_{ab} is the period matrix of the genus 2 surface.

It is straightforward to perform the Wick contraction of $\langle [\det(\widehat{P}_L + \widehat{r}_L) \det(\widehat{P}_R + \widehat{r}_R)]^2 \rangle$ using (4.4) as

$$\begin{aligned}
& \left(\frac{\det g}{\det \text{Im}\Omega} \right)^2 \langle [\det(\widehat{P}_L + \widehat{r}_L) \det(\widehat{P}_R + \widehat{r}_R)]^2 \rangle = \\
& = \left(\frac{\det g \det \widehat{P}_L \det \widehat{P}_R}{\det \text{Im}\Omega} \right)^2 - \\
& - 4 \frac{\det g \det \widehat{P}_L \det \widehat{P}_R}{\det \text{Im}\Omega} (\widehat{P}_L, \widehat{P}_R) (u_L^1 u_R^1 + u_L^2 u_R^2) + \\
& + \left(16 \frac{\det g \det \widehat{P}_L \det \widehat{P}_R}{\det \text{Im}\Omega} + 2(\widehat{P}_L, \widehat{P}_R) \right) (u_L^1 u_R^1 + u_L^2 u_R^2)^2 - \\
& - 12(\widehat{P}_L, \widehat{P}_R) (u_L^1 u_R^1 + u_L^2 u_R^2)^3 + 12(u_L^1 u_R^1 + u_L^2 u_R^2)^4
\end{aligned} \tag{4.5}$$

where

$$(\widehat{P}_L, \widehat{P}_R) = g_{i\bar{j}} \widehat{P}_{La}^i (\text{Im}\Omega^{-1})^{ab} \widehat{P}_{Rb}^{\bar{j}}.$$

To compute F^2 on $T^2 \times R^2$, we set

$$(g_{i\bar{j}}) = \begin{pmatrix} r_1/\text{Im}\rho & 0 \\ 0 & r_2 \end{pmatrix}$$

and send $r_2 \rightarrow \infty$ while keeping r_1 finite. In order for the action S to remain finite, we must impose the momenta in the r_2 -direction to vanish, $P_{La}^2 = P_{Ra}^2 = 0$. The action then becomes

$$S = t(p_L, \bar{p}_R) + \bar{t}(\bar{p}_L, p_R), \tag{4.6}$$

where

$$(p_L, \bar{p}_R) = \frac{p_{La} (\text{Im}\Omega^{-1})^{ab} \bar{p}_{Rb}}{\text{Im}\rho},$$

p_L and p_R are string momenta on T^2 given by

$$\begin{aligned}
p_{La} &= P_{La}^1 = (n_a + \rho s_a) - (m^b + \rho r^b) \bar{\Omega}_{ab} \\
\bar{p}_{La} &= P_{La}^{\bar{1}} = (n_a + \bar{\rho} s_a) - (m^b + \bar{\rho} r^b) \bar{\Omega}_{ab} \\
p_{Ra} &= P_{Ra}^1 = (n_a + \rho s_a) - (m^b + \rho r^b) \Omega_{ab} \\
\bar{p}_{Ra} &= P_{Ra}^{\bar{1}} = (n_a + \bar{\rho} s_a) - (m^b + \bar{\rho} r^b) \Omega_{ab}
\end{aligned}$$

with ρ being the complex moduli of T^2 , and t is the Kähler moduli of T^2 , whose real and imaginary parts are r_1 and θ respectively. In this limit, we can make the following substitutions in (4.5):

$$\begin{aligned} \frac{\det g \det \widehat{P}_L \det \widehat{P}_R}{\det \text{Im}\Omega} &= (t + \bar{t})^2 u_L^1 u_R^1 u_L^2 u_R^2 \left[(p_L, \bar{p}_R)(\bar{p}_L, p_R) - (p_L, p_R)(\bar{p}_L, \bar{p}_R) \right] \\ (\widehat{P}_L, \widehat{P}_R) &= (t + \bar{t}) \left[u_L^1 u_R^1 (p_L, \bar{p}_R) + u_L^2 u_R^2 (\bar{p}_L, p_R) \right]. \end{aligned} \quad (4.7)$$

We can now expand $F^2(u_L, u_R)$ in powers in u_L and u_R and extract $F_{n,m}^2$. Since F^2 depends on u_L^i and u_R^i only through the combinations $u_L^1 u_R^1$ and $u_L^2 u_R^2$ as one can see from (4.7) and (4.5), the off-diagonal terms $F_{n,m}^2$ ($n \neq m$) all vanish for $T^2 \times R^2$ (This is not the case for a generic T^4). Since the unitarity of the sigma-model implies $\overline{F_{n,m}^g} = F_{-m,-m}^g$, we only need to compute $F_{2,2}^2$, $F_{1,1}^2$ and $F_{0,0}^2$. Let us examine them one by one. In the following, we drop the superscript 2 from $F_{n,n}^2$ to simplify expressions.

(1) $F_{2,2}$:

From (4.5) and (4.7), it is easy to read off the following expression for $F_{2,2}$.

$$F_{2,2} = 2 \sum_{p_L, p_R} \int \left((t + \bar{t})^2 (p_L, \bar{p}_R)^2 - 6(t + \bar{t})(p_L, \bar{p}_R) + 6 \right) \exp(-S) \frac{d^3\Omega d^3\bar{\Omega}}{[\det \text{Im}\Omega]^3}.$$

Since S for T^2 is given by (4.6), we can also write it as

$$F_{2,2} = 2 \left((t + \bar{t})^2 \frac{\partial^2}{\partial t^2} + 6(t + \bar{t}) \frac{\partial}{\partial t} + 6 \right) \mathcal{Z}, \quad (4.8)$$

with

$$\mathcal{Z}(t, \bar{t}) = \sum_{p_L, p_R} \int \exp(-S) \frac{d^3\Omega d^3\bar{\Omega}}{[\det \text{Im}\Omega]^3} \quad (4.9)$$

By doing the Poisson resummation in p_L and p_R , one can show that the combination $(t + \bar{t})^2 \mathcal{Z}(t, \bar{t})$ is invariant under the T -duality transformation. Thus (4.8) can also be written as

$$\begin{aligned} F_{2,2} &= \frac{2}{(t + \bar{t})^2} \frac{\partial}{\partial t} \left[(t + \bar{t})^2 \frac{\partial}{\partial t} \left((t + \bar{t})^2 \mathcal{Z} \right) \right] \\ &= 2D_t^2 \left[(t + \bar{t})^2 \mathcal{Z}(t, \bar{t}) \right] \end{aligned}$$

where D_t is the duality covariant derivative. This means in particular that the weight of $F_{2,2}$ is such that $F_{2,2}(t, \bar{t})(dt)^2$ is invariant under the duality transformation.

(2) $F_{1,1}$:

By extracting a coefficient of $(u_L^1 u_R^1)^3 u_L^2 u_R^2$ from (4.5) using (4.7) for the limit $T^4 \rightarrow T^2 \times R^2$, we obtain

$$\begin{aligned}
F_{1,1} = & \frac{1}{4} \int \left(-(t + \bar{t})^3 (p_L, \bar{p}_R) \left[(p_L, \bar{p}_R)(\bar{p}_L, p_R) - (p_L, p_R)(\bar{p}_L, \bar{p}_R) \right] + \right. \\
& + (t + \bar{t})^2 \left[(p_L, \bar{p}_R)^2 + 5(p_L, \bar{p}_R)(\bar{p}_L, p_R) - 4(p_L, p_R)(\bar{p}_L, \bar{p}_R) \right] - \\
& \left. - (t + \bar{t}) \left[9(p_L, \bar{p}_R) + 3(\bar{p}_L, p_R) \right] + 12 \right) \exp(-S) \frac{d^3 \Omega d^3 \bar{\Omega}}{[\det \text{Im} \Omega]^3}
\end{aligned} \tag{4.10}$$

This expression is simplified significantly by using the following formula for the variation of the action (4.6) with respect to the worldsheet moduli Ω_{ab} .

$$\frac{\partial S}{\partial \Omega_{ab}} = \frac{i}{2} (t + \bar{t}) \left((p_L \frac{1}{\text{Im} \Omega})_a (\bar{p}_L \frac{1}{\text{Im} \Omega})_b + (p_L \frac{1}{\text{Im} \Omega})_b (\bar{p}_L \frac{1}{\text{Im} \Omega})_a \right) \tag{4.11}$$

This formula can be derived either by computing the derivative of S directly or by noting that $(\partial S / \partial \Omega_{ab}) \omega_a \omega_b$ is proportional to an expectation value of the energy-momentum tensor $T = g_{i\bar{j}} \partial X^i \partial X^{\bar{j}}$. By using this formula repeatedly, we are going to reduce $F_{1,1}$ given by (4.10) to

$$\begin{aligned}
F_{1,1} = & \frac{3}{2} \int \left(-(t + \bar{t})(p_L, \bar{p}_R) + 2 \right) \exp(-S) \frac{d^3 \Omega d^3 \bar{\Omega}}{[\det \text{Im} \Omega]^3} \\
= & \frac{3}{2} \left((t + \bar{t}) \frac{\partial}{\partial t} + 2 \right) \int \exp(-S) \frac{d^3 \Omega d^3 \bar{\Omega}}{[\det \text{Im} \Omega]^3} \\
= & \frac{3}{2} \left((t + \bar{t}) \frac{\partial}{\partial t} + 2 \right) \mathcal{Z} = (t + \bar{t})^{-1} \partial_t \left[(t + \bar{t})^2 \mathcal{Z} \right]
\end{aligned} \tag{4.12}$$

In particular, this shows that $F_{1,1}(t, \bar{t})(\sqrt{dt})^3 \sqrt{d\bar{t}}$ is invariant under the duality transformation.

Now let us prove (4.12). We first note that the first term in the integrand of (4.10) can be rearranged as

$$\begin{aligned}
& (p_L, \bar{p}_R) \left[(p_L, \bar{p}_R)(\bar{p}_L, p_R) - (p_L, p_R)(\bar{p}_L, \bar{p}_R) \right] = \\
& = (p_L, \bar{p}_R) \left[(p_L, \bar{p}_R)(\bar{p}_L, p_R) + (p_L, p_R)(\bar{p}_L, \bar{p}_R) \right] - \\
& \quad - 2(p_L, p_R)(\bar{p}_L, \bar{p}_R)(p_L, \bar{p}_R) \\
& = \frac{2}{i(t + \bar{t})} \left(\frac{\partial S}{\partial \Omega_{ab}} p_{Ra} \bar{p}_{Rb} \right) (p_L, \bar{p}_R) - \frac{2}{i(t + \bar{t})} \left(\frac{\partial S}{\partial \Omega_{ab}} \bar{p}_{Ra} \bar{p}_{Rb} \right) (p_L, p_R).
\end{aligned}$$

We can then perform the integration-by-parts on \mathcal{M}_2 as

$$\begin{aligned}
& \int (p_L, \bar{p}_R) \left[(p_L, \bar{p}_R)(\bar{p}_L, p_R) - (p_L, p_R)(\bar{p}_L, \bar{p}_R) \right] \exp(-S) \frac{d^3\Omega d^3\bar{\Omega}}{[\det \text{Im}\Omega]^3} = \\
&= \frac{2}{i(t+\bar{t})} \int \frac{\partial}{\partial \Omega_{ab}} \left(\frac{p_{Ra}\bar{p}_{Rb}(p_L, \bar{p}_R) - \bar{p}_{Ra}\bar{p}_{Rb}(p_L, p_R)}{[\det \text{Im}\Omega]^3} \right) \exp(-S) d^3\Omega d^3\bar{\Omega} \\
&= \frac{1}{(t+\bar{t})} \int \left((p_L, \bar{p}_R)^2 + 4(p_L, \bar{p}_R)(\bar{p}_L, p_R) - 5(p_L, p_R)(\bar{p}_L, \bar{p}_R) \right) \times \\
&\quad \times \exp(-S) \frac{d^3\Omega d^3\bar{\Omega}}{[\det \text{Im}\Omega]^3}
\end{aligned}$$

it is easy to show that there is no contribution from the boundaries of \mathcal{M}_2 .

Substituting this into (4.10), we obtain

$$\begin{aligned}
F_{1,1} = \frac{1}{4} \int & \left((t+\bar{t})^2 \left[(p_L, \bar{p}_R)(\bar{p}_L, p_R) + (p_L, p_R)(\bar{p}_L, \bar{p}_R) \right] - \right. \\
& \left. - 9(t+\bar{t})(p_L, \bar{p}_R) - 3(\bar{p}_L, p_R) + 12 \right) \exp(-S) \frac{d^3\Omega d^3\bar{\Omega}}{[\det \text{Im}\Omega]^3}
\end{aligned} \tag{4.13}$$

We then note

$$(p_L, \bar{p}_R)(\bar{p}_L, p_R) + (p_L, p_R)(\bar{p}_L, \bar{p}_R) = \frac{2}{i(t+\bar{t})} \left(\frac{\partial S}{\partial \Omega_{ab}} p_{Ra}\bar{p}_{Rb} \right),$$

and do the integration-by-parts again.

$$\begin{aligned}
& \int \left((p_L, \bar{p}_R)(\bar{p}_L, p_R) + (p_L, p_R)(\bar{p}_L, \bar{p}_R) \right) \exp(-S) \frac{d^3\Omega d^3\bar{\Omega}}{[\det \text{Im}\Omega]^3} = \\
&= \frac{2}{i(t+\bar{t})} \int \frac{\partial}{\partial \Omega_{ab}} \left(\frac{1}{[\det \text{Im}\Omega]^3} p_{Ra}\bar{p}_{Rb} \right) \exp(-S) d^3\Omega d^3\bar{\Omega} \\
&= \frac{3}{(t+\bar{t})} \int \left((p_L, \bar{p}_R) + (\bar{p}_L, p_R) \right) \frac{d^3\Omega d^3\bar{\Omega}}{[\det \text{Im}\Omega]^3}.
\end{aligned}$$

Substituting this into (4.13), we recover (4.12) and this is what we wanted to show.

(3) $F_{0,0}$:

By extracting a coefficient of $(u_L^1 u_R^1 u_L^2 u_R^2)^2$ from (4.5) using (4.7) for the limit $T^4 \rightarrow T^2 \times R^2$, we obtain

$$\begin{aligned}
F_{0,0} = \frac{1}{36} \int & \left((t+\bar{t})^4 \left[(p_L, \bar{p}_R)(\bar{p}_L, p_R) - (p_L, p_R)(\bar{p}_L, \bar{p}_R) \right]^2 + \right. \\
& - 4(t+\bar{t})^3 \left[(p_L, \bar{p}_R) + (\bar{p}_L, p_R) \right] \left[(p_L, \bar{p}_R)(\bar{p}_L, p_R) - (p_L, p_R)(\bar{p}_L, \bar{p}_R) \right] + \\
& + (t+\bar{t})^2 \left[2(p_L, \bar{p}_R)^2 + 2(\bar{p}_L, p_R)^2 + 40(p_L, \bar{p}_R)(\bar{p}_L, p_R) - 32(p_L, p_R)(\bar{p}_L, \bar{p}_R) \right] \\
& \left. - 36(t+\bar{t}) \left[(p_L, \bar{p}_R) + (\bar{p}_L, p_R) \right] + 72 \right) \exp(-S) \frac{d^3\Omega d^3\bar{\Omega}}{[\det \text{Im}\Omega]^3}
\end{aligned} \tag{4.14}$$

As in the case of $F_{1,1}$, we can use the formula (4.11) to reduce this to

$$\begin{aligned}
F_{0,0} &= \int \left((t + \bar{t})^2 (p_L, \bar{p}_R)(\bar{p}_L, p_R) - \right. \\
&\quad \left. - 2(t + \bar{t}) \left[(p_L, \bar{p}_R) + (\bar{p}_L, p_R) \right] + 2 \right) \exp(-S) \frac{d^3\Omega d^3\bar{\Omega}}{[\det \text{Im}\Omega]^3} \\
&= \left((t + \bar{t})^2 \frac{\partial^2}{\partial t \partial \bar{t}} + 2(t + \bar{t}) \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \bar{t}} \right) + 2 \right) \int \exp(-S) \frac{d^3\Omega d^3\bar{\Omega}}{[\det \text{Im}\Omega]^3} \\
&= \left((t + \bar{t})^2 \frac{\partial^2}{\partial t \partial \bar{t}} + 2(t + \bar{t}) \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \bar{t}} \right) + 2 \right) \mathcal{Z} = \partial_t \partial_{\bar{t}} [(t + \bar{t})^2 \mathcal{Z}].
\end{aligned} \tag{4.15}$$

Thus in particular $F_{0,0}(t, \bar{t}) dt d\bar{t}$ is invariant under the duality transformation.

Let us prove (4.15). We note that the first term in the integrand of (4.14) can be written as

$$\begin{aligned}
&\left[(p_L, \bar{p}_R)(\bar{p}_L, p_R) - (p_L, p_R)(\bar{p}_L, \bar{p}_R) \right]^2 = \\
&= \frac{2}{i(t + \bar{t})} \left(\frac{\partial S}{\partial \Omega_{ab}} p_{Ra} \bar{p}_{Rb} \right) \left((p_L, \bar{p}_R)(\bar{p}_L, p_R) + (p_L, p_R)(\bar{p}_L, \bar{p}_R) \right) - \\
&\quad - \frac{4}{i(t + \bar{t})} \left(\frac{\partial S}{\partial \Omega_{ab}} p_{Ra} p_{Rb} \right) (p_L, \bar{p}_R)(\bar{p}_L, \bar{p}_R).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\int \left((p_L, \bar{p}_R)(\bar{p}_L, p_R) - (p_L, p_R)(\bar{p}_L, \bar{p}_R) \right)^2 \exp(-S) \frac{d^3\Omega d^3\bar{\Omega}}{[\det \text{Im}\Omega]^3} = \\
&= \frac{2}{i(t + \bar{t})} \int \frac{\partial}{\partial \Omega_{ab}} \left[\left(p_{Ra} \bar{p}_{Rb} \{ (p_L, \bar{p}_R)(\bar{p}_L, p_R) + (p_L, p_R)(\bar{p}_L, \bar{p}_R) \} - \right. \right. \\
&\quad \left. \left. - 2p_{Ra} p_{Rb} (p_L, \bar{p}_R)(\bar{p}_L, \bar{p}_R) \right) [\det \text{Im}\Omega]^{-3} \right] \exp(-S) d^3\Omega d^3\bar{\Omega} \\
&= \frac{1}{(t + \bar{t})} \int \left(4(p_L, \bar{p}_R)^2 (\bar{p}_L, p_R) + 2(\bar{p}_L, p_R)(p_L, p_R)(\bar{p}_L, \bar{p}_R) - \right. \\
&\quad \left. - 6(p_L, \bar{p}_R)(p_L, p_R)(\bar{p}_L, \bar{p}_R) \right) \exp(-S) \frac{d^3\Omega d^3\bar{\Omega}}{[\det \text{Im}\Omega]^3}.
\end{aligned}$$

By substituting this into (4.14), we obtain

$$\begin{aligned}
F_{0,0} &= \frac{1}{36} \int \left((t + \bar{t})^3 \left[6(p_L, p_R)(\bar{p}_L, \bar{p}_R)(\bar{p}_L, p_R) - \right. \right. \\
&\quad \left. - 2(p_L, p_R)(\bar{p}_L, \bar{p}_R)(p_L, \bar{p}_R) - 4(p_L, \bar{p}_R)(\bar{p}_L, p_R)^2 \right] + \\
&\quad + (t + \bar{t})^2 \left[2(p_L, \bar{p}_R)^2 + 2(\bar{p}_L, p_R)^2 + \right. \\
&\quad \left. + 40(p_L, \bar{p}_R)(\bar{p}_L, p_R) - 32(p_L, p_R)(\bar{p}_L, \bar{p}_R) \right] - \\
&\quad \left. - 36(t + \bar{t}) \{ (p_L, \bar{p}_R) + (\bar{p}_L, p_R) \} + 72 \right) \exp(-S) \frac{d^3\Omega d^3\bar{\Omega}}{[\det \text{Im}\Omega]^3}.
\end{aligned} \tag{4.16}$$

Next we note that the first term in the integrand of (4.16) can be written as

$$\begin{aligned}
& 6(p_L, p_R)(\bar{p}_L, \bar{p}_R)(\bar{p}_L, p_R) - 2(p_L, p_R)(\bar{p}_L, \bar{p}_R)(p_L, \bar{p}_R) - \\
& \quad - 4(p_L, \bar{p}_R)(\bar{p}_L, p_R)^2 = \\
& = \frac{2}{i(t + \bar{t})} \frac{\partial S}{\partial \Omega_{ab}} \left(-4p_{Ra}\bar{p}_{Rb}(\bar{p}_L, p_R) + 5p_{Ra}p_{Rb}(\bar{p}_L, \bar{p}_R) - \right. \\
& \quad \left. - \bar{p}_{Ra}\bar{p}_{Rb}(p_L, p_R) \right)
\end{aligned}$$

Therefore

$$\begin{aligned}
& \int \left(6(p_L, p_R)(\bar{p}_L, \bar{p}_R)(\bar{p}_L, p_R) - 2(p_L, p_R)(\bar{p}_L, \bar{p}_R)(p_L, \bar{p}_R) - \right. \\
& \quad \left. - 4(p_L, \bar{p}_R)(\bar{p}_L, p_R)^2 \right) \exp(-S) \frac{d^3\Omega d^3\bar{\Omega}}{[\det \text{Im}\Omega]^3} = \\
& = \frac{2}{i(t + \bar{t})} \int \frac{\partial}{\partial \Omega_{ab}} \left(\left[-4p_{Ra}\bar{p}_{Rb}(\bar{p}_L, p_R) + 5p_{Ra}p_{Rb}(\bar{p}_L, \bar{p}_R) - \right. \right. \\
& \quad \left. \left. - \bar{p}_{Ra}\bar{p}_{Rb}(p_L, p_R) \right] [\det \text{Im}\Omega]^{-3} \right) \exp(-S) d^3\Omega d^3\bar{\Omega} \\
& = \frac{2}{(t + \bar{t})} \int \left(-8(p_L, \bar{p}_R)(\bar{p}_L, p_R) + 10(p_L, p_R)(\bar{p}_L, \bar{p}_R) - \right. \\
& \quad \left. - (\bar{p}_L, p_R)^2 - (p_L, \bar{p}_R)^2 \right) \exp(-S) \frac{d^3\Omega d^3\bar{\Omega}}{[\det \text{Im}\Omega]^3}.
\end{aligned}$$

Substituting this into (4.16), we obtain

$$\begin{aligned}
F_{0,0} & = \frac{1}{36} \int \left((t + \bar{t})^2 \left[24(p_L, \bar{p}_R)(\bar{p}_L, p_R) - 12(p_L, p_R)(\bar{p}_L, \bar{p}_R) \right] - \right. \\
& \quad \left. - 36(t + \bar{t}) \left[(p_L, \bar{p}_R) + (\bar{p}_L, p_R) \right] + 72 \right) \exp(-S) \frac{d^3\Omega d^3\bar{\Omega}}{[\det \text{Im}\Omega]^3} \tag{4.17}
\end{aligned}$$

Finally we note that the first term in the integrand of (4.17) can be written as

$$\begin{aligned}
& 24(p_L, \bar{p}_R)(\bar{p}_L, p_R) - 12(p_L, p_R)(\bar{p}_L, \bar{p}_R) \\
& = 36(p_L, \bar{p}_R)(\bar{p}_L, p_R) - \frac{24}{i(t + \bar{t})} \frac{\partial S}{\partial \Omega_{ab}} p_{Ra}\bar{p}_{Rb}.
\end{aligned}$$

Therefore

$$\begin{aligned}
& \int (t + \bar{t})^2 \left(\{ 24(p_L, \bar{p}_R)(\bar{p}_L, p_R) - 12(p_L, p_R)(\bar{p}_L, \bar{p}_R) \} \exp(-S) \frac{d^3\Omega d^3\bar{\Omega}}{[\det \text{Im}\Omega]^3} = \right. \\
& = 36 \int \left((t + \bar{t})^2 (p_L, \bar{p}_R)(\bar{p}_L, p_R) - (t + \bar{t}) \{ (p_L, \bar{p}_R) + (\bar{p}_L, p_R) \} \right) \times \\
& \quad \times \exp(-S) \frac{d^3\Omega d^3\bar{\Omega}}{[\det \text{Im}\Omega]^3}.
\end{aligned}$$

Substituting this into (4.17), we find that $F_{0,0}$ is expressed as (4.15), and this is what we wanted to show.

4.3. Genus Two; Evaluation II

We have shown that, for $T^2 \times R^2$, the $N = 2$ string amplitudes at genus 2 are given by

$$\begin{aligned} F_{2,2} &= 2(t + \bar{t})^2 D_t^2 \mathcal{Z} \\ F_{1,1} &= \frac{3}{2}(t + \bar{t}) D_t \mathcal{Z} \\ F_{0,0} &= (t + \bar{t})^2 D_t D_{\bar{t}} \mathcal{Z} \end{aligned} \tag{4.18}$$

where

$$\mathcal{Z} = \sum_{p_L, p_R} \int_{\mathcal{M}_2} \exp(-S(p_L, p_R)) \frac{d^3 \Omega d^3 \bar{\Omega}}{[\det \operatorname{Im} \Omega]^3}.$$

We shall see that these expressions are consistent with the harmonicity equation.

To understand $F_{n,n}$ better, we shall first prove the following two key properties of \mathcal{Z} ;

(1) \mathcal{Z} is a sum of two terms, one depends only on the Kähler moduli $\sigma = (8\pi i)^{-1} t$ and $\bar{\sigma}$ and another depends only on the complex moduli ρ and $\bar{\rho}$ up to a factor $(\operatorname{Im} \sigma)^{-2}$.

$$\mathcal{Z} = f(\sigma, \bar{\sigma}) + (\operatorname{Im} \sigma)^{-2} \tilde{f}(\rho, \bar{\rho}) \tag{4.19}$$

(2) f and \tilde{f} are eigen-functions of Laplacians on the Kähler and the complex moduli spaces respectively.

$$\begin{aligned} \partial_t \partial_{\bar{t}} [(t + \bar{t})^2 f] &= 2f \\ 4(\operatorname{Im} \rho)^2 \partial_\rho \partial_{\bar{\rho}} \tilde{f} &= 2\tilde{f} \end{aligned} \tag{4.20}$$

These properties, combined with the large t behavior of \mathcal{Z} ,

$$\mathcal{Z} \rightarrow \int \frac{d^3 \Omega d^3 \bar{\Omega}}{[\det \operatorname{Im} \Omega]^3} \sim \int_{\mathcal{M}_2} (c_1)^3 \quad (t, \bar{t} \rightarrow \infty)$$

where c_1 is the first Chern class of the Hodge bundle over the moduli space \mathcal{M}_2 and the mirror symmetry $\sigma \leftrightarrow \rho$, completely determines $f(\sigma, \bar{\sigma})$ and $\tilde{f}(\rho, \bar{\rho})$ as

$$\begin{aligned} f(\sigma, \bar{\sigma}) &= \sum_{n,m} \frac{1}{(n + m\sigma)^2 (n + m\bar{\sigma})^2} \\ \tilde{f}(\rho, \bar{\rho}) &= \sum_{n,m} \frac{(\operatorname{Im} \rho)^2}{(n + m\rho)(n + m\bar{\rho})}. \end{aligned}$$

By substituting this into (4.18) and (4.19), we obtain the following expression for $F_{n,n}$.

$$\begin{aligned}
F_{0,0} &= 2f = \sum_{n,m} \frac{2}{(n+m\sigma)^2(n+m\bar{\sigma})^2} \\
F_{1,1} &= \frac{3}{2}(t+\bar{t})D_t f = \sum_{n,m} \frac{3}{(n+m\sigma)^3(n+m\bar{\sigma})} \\
F_{2,2} &= 2(t+\bar{t})^2 D_t^2 f = \sum_{n,m} \frac{12}{(n+m\sigma)^4}
\end{aligned} \tag{4.21}$$

In particular, $F_{2,2}$ is holomorphic in σ

$$\partial_{\bar{\sigma}} F_{2,2} = 0$$

and is given by the Eisenstein series of degree 4. These expressions for $F_{n,n}$ are combined nicely as

$$\begin{aligned}
F(u_L, u_R) &= \sum_{n=-2}^2 \binom{4}{2+n}^2 F_{n,n} (u_L^1 u_R^1)^{2+n} (u_L^2 u_R^2)^{2-n} \\
&= 12 \sum_{(n,m) \neq (0,0)} \left(\frac{u_L^1 u_R^1}{n+m\sigma} + \frac{u_L^2 u_R^2}{n+m\bar{\sigma}} \right)^4.
\end{aligned}$$

Now let us prove (4.19) and (4.20). We will use

$$\begin{aligned}
\partial_{\rho}(p_L, \bar{p}_R) &= \partial_{\rho}(\bar{p}_L, p_R) = \frac{i}{2\text{Im}\rho}(\bar{p}_L, \bar{p}_R) \\
\partial_{\rho} \left[\frac{1}{\text{Im}\rho}(p_L, p_R) \right] &= \frac{i}{2(\text{Im}\rho)^2} \left((p_L, \bar{p}_R) + (\bar{p}_L, p_R) \right) \\
\partial_{\rho} \left[\text{Im}\rho(\bar{p}_L, \bar{p}_R) \right] &= 0
\end{aligned}$$

and

$$\partial_{\rho} S = \frac{i(t+\bar{t})}{2\text{Im}\rho}(\bar{p}_L, \bar{p}_R),$$

which follows from the definition of (p_L, \bar{p}_R) etc. Therefore

$$\begin{aligned}
\partial_{\rho} D_t \mathcal{Z} &= \partial_{\rho} \int \left(- (p_L, \bar{p}_R) + \frac{2}{(t+\bar{t})} \right) \exp(-S) \frac{d^3\Omega d^3\bar{\Omega}}{[\det \text{Im}\Omega]^3} \\
&= \int \left(\frac{i(t+\bar{t})}{2\text{Im}\rho}(\bar{p}_L, \bar{p}_R)(p_L, \bar{p}_R) - \frac{3i}{2\text{Im}\rho}(\bar{p}_L, \bar{p}_R) \right) \exp(-S) \frac{d^3\Omega d^3\bar{\Omega}}{[\det \text{Im}\Omega]^3}.
\end{aligned}$$

We then note

$$i(t+\bar{t})(\bar{p}_L, \bar{p}_R)(p_L, \bar{p}_R) = \frac{1}{\text{Im}\rho} \bar{p}_{La} \bar{p}_{Rb} \frac{\partial S}{\partial \Omega_{ab}}.$$

We can then perform integration-by-parts to obtain

$$\begin{aligned}
& \int \frac{i(t + \bar{t})}{2\text{Im}\rho} (\bar{p}_L, \bar{p}_R)(p_L, p_R) \exp(-S) \frac{d^3\Omega d^3\bar{\Omega}}{[\det \text{Im}\Omega]^3} = \\
& = \int \frac{-1}{2(\text{Im}\rho)^2} \frac{\partial}{\partial \Omega_{ab}} \left(\frac{\bar{p}_{La} \bar{p}_{Rb}}{[\det \text{Im}\Omega]^3} \right) \exp(-S) d^3\Omega d^3\bar{\Omega} \\
& = -\frac{3i}{2\text{Im}\rho} \int (\bar{p}_L, \bar{p}_R) \exp(-S) \frac{d^3\Omega d^3\bar{\Omega}}{[\det \text{Im}\Omega]^3}.
\end{aligned}$$

Thus we found

$$\partial_\rho D_\sigma \mathcal{Z} = (\text{Im}\sigma)^{-2} \partial_\rho \partial_\sigma \left[(\text{Im}\sigma)^2 \mathcal{Z} \right] = 0.$$

Similarly we can prove

$$\partial_\rho \partial_\sigma \left[(\text{Im}\sigma)^2 \mathcal{Z} \right] = \partial_{\bar{\rho}} \partial_\sigma \left[(\text{Im}\sigma)^2 \mathcal{Z} \right] = \partial_{\bar{\rho}} \partial_\sigma \left[(\text{Im}\sigma)^2 \mathcal{Z} \right] = 0.$$

Therefore \mathcal{Z} is a sum of $f(t, \bar{t})$ and $(t + \bar{t})^{-2} \tilde{f}(\rho, \bar{\rho})$ as in (4.19).

To prove (4.20), we first compute

$$\begin{aligned}
& \left((t + \bar{t})^2 D_t D_{\bar{t}} + 4(\text{Im}\rho)^2 \partial_\rho \partial_{\bar{\rho}} \right) \mathcal{Z} = \\
& = \int \left((t + \bar{t})^2 [(p_L, \bar{p}_R)(\bar{p}_L, p_R) + (p_L, p_R)(\bar{p}_L, \bar{p}_R)] - \right. \\
& \quad \left. - 3(t + \bar{t}) [(p_L, \bar{p}_R) + (\bar{p}_L, p_R)] + 2 \right) \exp(-S) \frac{d^3\Omega d^3\bar{\Omega}}{[\det \text{Im}\Omega]^3}.
\end{aligned}$$

By using

$$(t + \bar{t})^2 [(p_L, \bar{p}_R)(\bar{p}_L, p_R) + (p_L, p_R)(\bar{p}_L, \bar{p}_R)] = -2i(t + \bar{t}) p_{Ra} \bar{p}_{Rb} \frac{\partial S}{\partial \Omega_{ab}},$$

one can show

$$\begin{aligned}
& \int \left((t + \bar{t})^2 [(p_L, \bar{p}_R)(\bar{p}_L, p_R) + (p_L, p_R)(\bar{p}_L, \bar{p}_R)] - \right. \\
& \quad \left. - 3(t + \bar{t}) [(p_L, \bar{p}_R) + (\bar{p}_L, p_R)] \right) \exp(-S) \frac{d^3\Omega d^3\bar{\Omega}}{[\det \text{Im}\Omega]^3} = 0
\end{aligned}$$

by integration-by-parts. Thus \mathcal{Z} is an eigen-function of a Laplacian

$$(t + \bar{t})^2 D_t D_{\bar{t}} + 4(\text{Im}\rho)^2 \partial_\rho \partial_{\bar{\rho}} = 4 \left((\text{Im}\sigma)^2 D_\sigma D_{\bar{\sigma}} + (\text{Im}\rho)^2 \partial_\rho \partial_{\bar{\rho}} \right)$$

as

$$\left((\text{Im}\sigma)^2 D_\sigma D_{\bar{\sigma}} + (\text{Im}\rho)^2 \partial_\rho \partial_{\bar{\rho}} \right) \mathcal{Z} = 2\mathcal{Z}.$$

Since \mathcal{Z} is a sum of $f(\sigma, \bar{\sigma})$ and $(\text{Im}\sigma)^{-2} \tilde{f}(\rho, \bar{\rho})$ as in (4.19), this means that f and \tilde{f} are also eigen-functions of $(\text{Im}\sigma)^2 D_\sigma D_{\bar{\sigma}}$ and $(\text{Im}\rho)^2 \partial_\rho \partial_{\bar{\rho}}$ as in (4.20), and this is what we wanted to show.

4.4. Topological Interpretation at $g = 2$

We have found that $F_{2,2}$ is holomorphic in t and is given by the Eisenstein series of degree 4. The holomorphicity of $F_{2,2}$ implies [14],[4] that $F_{2,2}$ should “count” the number of holomorphic maps from genus-2 Riemann surfaces to T^2 .

Since $F_{2,2}$ is independent of \bar{t} , let us regard t and \bar{t} to be independent and take $\bar{t} \rightarrow \infty$ limit in (4.8) while keeping t to be finite. This limit imposes constraint on the period matrix Ω_{ab} as

$$\Omega_{ab}(m^b + \rho r^b) = (n_a + \rho s_a). \quad (4.22)$$

In this case, the map $X : \Sigma \rightarrow T^2$ characterized by the string momenta p_L, p_R become a holomorphic map. There are 2 equations for 3 independent components of Ω_{ab} constraints. Thus a solution to the constraint should be parametrized by one complex parameter. It is easy to write down the most general solution. Since Ω_{ab} is symmetric, we can parametrize it by 3 complex parameters u, v, w as

$$\begin{aligned} \Omega_{ab} = & u(Im\Omega\bar{\alpha})_a(Im\Omega\bar{\alpha})_b + \\ & + v[\epsilon_{ac}\alpha^c(Im\Omega\bar{\alpha})_b + (Im\Omega\bar{\alpha})_a\epsilon_{bc}\alpha^c] + w\epsilon_{ac}\alpha^c\epsilon_{bd}\alpha^d, \end{aligned}$$

where $\alpha^a = m^a + \rho r^a$. For fixed u, v, w , this is a non-linear equation since $Im\Omega$ in the right hand side also depends on u, v, w . This however will not cause complication later since the values of u and v are fixed by the constraints and the dependence on w turns out to be simple as we shall see. In this parametrization, the solutions to the constraints correspond to

$$\begin{aligned} u = u_0 &= (\alpha Im\Omega\bar{\alpha})^{-2}(n_a + \rho s_a)\alpha^a \\ v = v_0 &= (\alpha Im\Omega\bar{\alpha})^{-2}(n_a + \rho s_a)\epsilon^{ab}(Im\Omega\bar{p})_b \end{aligned}$$

and w is arbitrary. The term of the action which blows up in the $\bar{t} \rightarrow \infty$ limit is now of the form

$$\bar{t}(\bar{p}_L, p_R) = \frac{\bar{t}}{Im\rho}(\alpha Im\Omega\bar{\alpha})^3(|u - u_0|^2 + (\det Im\Omega)^{-1}|v - v_0|^2).$$

The exponentiated action becomes in this limit

$$\begin{aligned} \exp(-S) &\sim \\ &\sim \left(\frac{Im\rho}{\bar{t}}\right)^2 [\alpha Im\Omega\bar{\alpha}]^{-6} [\det Im\Omega] \delta^{(2)}(u - u_0) \delta^{(2)}(v - v_0) \exp(-t(p_L, \bar{p}_R)). \end{aligned}$$

Since $\Omega_{ab}\alpha^b = (n_a + \rho s_a)$, it follows

$$\begin{aligned} (p_L, \bar{p}_R) &= 4 \frac{\alpha^a \text{Im} \Omega_{ab} \bar{\alpha}^b}{\text{Im} \rho} \\ &= 4 \frac{\text{Im}(\alpha^a \Omega_{ab} \bar{\alpha}^b)}{\text{Im} \rho} \\ &= 4(s_a m^a - r_a n^a), \end{aligned}$$

namely (p_L, \bar{p}_R) is a degree of the holomorphic map from Σ to T^2 . It is convenient to change the integration variables from Ω_{ab} to u, v, w . The Jacobian is easily computed as

$$d^3 \Omega d^3 \bar{\Omega} = (\alpha \text{Im} \Omega \bar{\alpha})^6 d^2 u d^2 v d^2 w$$

Thus $(p \text{Im} \Omega \bar{p})^{-6}$ from the exponentiated action cancels with the Jacobian.

To compute $F_{2,2}$, we need to apply D_t^2 on \mathcal{Z} as in (4.8). In the $\bar{t} \rightarrow \infty$ limit, D_t^2 acting on $\exp(-S)$ reduces to ∂_t^2 , and the integrand for $F_{2,2}$ becomes

$$\left(\frac{p \text{Im} \Omega(w) \bar{p}}{\det \text{Im} \Omega(w)} \right)^2 \exp[2\pi i \sigma (s_a m^a - r_a n^a)] d^2 w$$

where $\sigma = (8\pi i)^{-1} t$ and

$$\Omega_{ab}(w) = \Omega_{ab}^0 + w \epsilon_{ac} \alpha^c \epsilon_{bd} \alpha^d,$$

with Ω_{ab}^0 being a special solution to the constraint (4.22).

Since

$$\frac{\partial}{\partial w} \Omega_{ab}(w) = \epsilon_{ac} \alpha^c \epsilon_{bd} \alpha^d,$$

we can write

$$\begin{aligned} \left(\frac{p \text{Im} \Omega(w) \bar{p}}{\det \text{Im} \Omega(w)} \right)^2 &= (\text{Im} \Omega^{-1})^{ab} \partial_w \Omega_{bc}(w) (\text{Im} \Omega^{-1})^{cd} \partial_{\bar{w}} \bar{\Omega}_{da}(\bar{w}) \\ &= \partial_w \partial_{\bar{w}} \text{trace} \log \text{Im} \Omega \\ &= \partial_w \partial_{\bar{w}} \log \det \text{Im} \Omega \end{aligned}$$

Thus we can interpret that $F_{2,2}$ computes the first Chern class of the Hodge bundle over the one dimensional moduli space of holomorphic maps from Σ to T^2 . In section 5, we will further elaborate on this point and show that we can reproduce the Eisenstein series of degree 4 from this topological point of view.

4.5. Harmonicity Equation on $T^2 \times R^2$

Now that we have the explicit expression (4.21) for $F_{n,n}$, we would like to check whether the harmonicity equations (3.1) and (3.2) are consistent with it.

Let us first write down the harmonicity equation (3.1) on $T^2 \times R^2$ for general value of g . In terms of the components, the equation is

$$D_{t^{22}}F_{n,m}^g - D_{t^{12}}F_{n-1,m}^g + \frac{2g-2+m}{2g-2-m+1} \left(D_{t^{21}}F_{n,m-1}^g - D_{t^{11}}F_{n-1,m-1}^g \right) = 0. \quad (4.23)$$

Suppose t^{22} couples to the marginal operator $\partial_z X^{\bar{1}} \partial_{\bar{z}} X^1$ where X^1 is the coordinate in the T^2 direction, namely $t^{22} = \bar{t}$ in the notation in this section. In this case, t^{12} , t^{21} and t^{11} couple to $\partial_z X^2 \partial_{\bar{z}} X^1$, $\partial_z X^{\bar{1}} \partial_{\bar{z}} X^2$ and $\partial_z X^2 \partial_{\bar{z}} X^2$ respectively. In this case, it is easy to see that the only nontrivial case in (4.23) is when $n = m$, otherwise each term in the equation vanishes identically. Since X^2 is in the R^2 direction, $X^2(z, \bar{z})$ is a single valued function on the Riemann surface Σ . It is then straightforward to compute insertions of these operators in F^g and obtain

$$\begin{aligned} (t + \bar{t})D_{t^{12}}F_{n-1,n}^g &= (2g - 2 + n)F_{n-1,n-1}^g \\ (t + \bar{t})D_{t^{21}}F_{n,n-1}^g &= (2g - 2 + n)F_{n-1,n-1}^g \\ (t + \bar{t})D_{t^{11}}F_{n,n}^g &= (g + n)F_{n,n}^g \end{aligned}$$

We can derive these formula by writing, for example, $\partial_z X^2 \partial_1 X^1 = \partial_z (X^2 \partial_1 X^1)$ and by doing integration-by-parts. By substituting them into (4.23), we obtain

$$(t + \bar{t})D_{\bar{t}}F_{n,n}^g = \frac{2g-2+n}{2g-2-n+1} (g-n)F_{n-1,n-1}^g \quad (4.24)$$

when $t^{22} = \bar{t}$. Similarly when $t^{11} = t$, (4.23) becomes

$$(t + \bar{t})D_t F_{n,n}^g = \frac{2g-2-n}{2g-2+n+1} (g+n)F_{n+1,n+1}^g \quad (4.25)$$

By combining these two equations, we also find that $F_{n,n}^g$ is an eigen-function of the Laplace operator

$$(t + \bar{t})^2 D_t D_{\bar{t}} F_{n,n}^g = (g-n)(g+n-1)F_{n,n}^g. \quad (4.26)$$

When $g = 2$, the holomorphic anomaly equations, (4.24) and (4.25), gives

$$\begin{aligned} D_{\bar{t}}F_{1,1} &= \frac{3}{2}F_{0,0} \\ D_tF_{1,1} &= \frac{3}{4}F_{2,2} \\ D_tF_{0,0} &= \frac{4}{3}F_{1,1}. \end{aligned}$$

It is straightforward to check that, combined with the Laplace equation (4.26), they are consistent with the explicit expressions (4.21) for $F_{n,n}^2$. Now we can apply the harmonicity equations, (4.24) and (4.25), to compute F^g for all g .

4.6. $g \geq 3$

We have verified that the harmonicity equations (4.24) and (4.25) are consistent with the explicit computation at genus 2. Let us now use the harmonicity equations to determine F^g for all $g \geq 3$. The two equations imply

$$\begin{aligned} (t + \bar{t})^2 D_t D_{\bar{t}} F_{n,n}^g &= (g - n)(g + n - 1) F_{n,n}^g \\ (t + \bar{t})^2 D_{\bar{t}} D_t F_{n,n}^g &= (g + n)(g - n + 1) F_{n,n}^g, \end{aligned}$$

and therefore

$$[D_t, D_{\bar{t}}] F_{n,n}^g = -2g(t + \bar{t})^{-2} F_{n,n}^g.$$

Combined with the hermiticity condition $\overline{F_{n,n}^g} = F_{-n,-n}^g$, we find $F_{n,n}^g (\sqrt{d\bar{t}})^{(g+n)}$ $(\sqrt{d\bar{t}})^{(g-n)}$ is invariant under the duality transformation.

Using the fact that

$$\tilde{F}_{0,0}^g = F_{0,0}^g (t + \bar{t})^g$$

is a modular function of weight zero, and that it is an eigenstate of Laplacian (4.26) and that as $t \rightarrow \infty$ it can at most have power law singularity in $t + \bar{t}$ (as it is becoming equivalent to R^4) allows us to solve for it (up to an overall constant). In particular we learn from (4.26) that for large t

$$\tilde{F}_{0,0}^g \sim (t + \bar{t})^g$$

Now using the modular invariance we can get the rest by acting with $SL(2, Z)$ (note that $SL(2, Z)$ transformations commute with the Laplace operator and so will give you another

function with the same eigenvalue for Laplace operator)⁵. We thus learn, in this way, that (4.26) has a solution for $F_{0,0}^g$ as

$$F_{0,0}^g = (\text{const}) \times \sum_{(n,m) \neq (0,0)} \frac{1}{|n + m\sigma|^{2g}}.$$

That this solution is unique follows from the fact that if we had another solution, by subtracting the two solutions we get a function which vanishes at infinity—this means that it is the eigenstate of Laplacian with a positive eigenvalue, which is the wrong sign. Thus we have a unique solution. Note that the constant appearing in front of our solution cannot depend on the complex structure of T^2 because we can always take $t \rightarrow \infty$ in which case the answer will be the partition function on R^4 which clearly is independent of which complex structure we chose for T^2 before blowing it up.

We can now use (4.24) and (4.25) to compute the rest of $F_{n,n}^g$ to obtain

$$F^g(u_L, u_R) = (\text{const}) \times \sum_{(n,m) \neq (0,0)} |n + m\sigma|^{2g-4} \left(\frac{u_L^1 u_R^1}{n + m\sigma} + \frac{u_L^2 u_R^2}{n + m\bar{\sigma}} \right)^{4g-4}.$$

5. Topological Interpretation of $N = 2$ String Amplitudes on $T^2 \times R^2$

Having seen that the genus 1 and genus 2 computation of $N = 2$ string amplitudes on $T^2 \times R^2$ have a topological interpretation we now ask the same question in all genera. In the general case we use the result discussed in section 2, and in particular apply equation (2.7) to our special case where target space is $T^2 \times R^2$.

Let us recall equation (2.7):

$$F_{2g-2, 2g-2}^g \Big|_{\bar{t} \rightarrow \infty} = \int_{\mathcal{M}^g} k^{g-1} \wedge c_{n-1}(\mathcal{V}) \wedge \mathcal{J}$$

This equation shows that the top instanton number amplitude, in the limit $\bar{t} \rightarrow \infty$ can be reinterpreted topologically by doing a topological computation on the moduli space of holomorphic maps. Let us see how this works. In the case of genus one the above computation is exactly the same as counting the holomorphic maps from torus to torus, because the \mathcal{J} insertion precisely absorbs the zero mode in the direction of R^2 and so we are back to counting holomorphic maps from genus one to genus one, which was done in [4].

⁵ We are thankful to A. Lesniewski for discussion on this point.

For genus g , the moduli of holomorphic maps \mathcal{M} has dimension $(2g - 2 + 1)$ for degree bigger than zero. This corresponds to double covering of the torus by the Riemann surface having $(2g - 2)$ branch points and $(+1)$ comes from choice of the R^2 coordinate of the holomorphic map. Note that all holomorphic maps to $T^2 \times R^2$ will lead to constant maps as far as the R^2 factor is concerned. Thus pulling back the volume form V and integrating over the Riemann surface will lead us to the statement that k is precisely the $(1, 1)$ form on \mathcal{M} in the direction of changing the R^2 image. The bundle \mathcal{V} in our case is the same as holomorphic one forms, simply because the normal bundle is simply the R^2 direction (i.e. the fermion zero modes in the R^2 direction). In other words \mathcal{V} is simply the Hodge bundle \mathcal{H} on the moduli of genus g surfaces, restricted in our case to the moduli of Riemann surfaces which holomorphically cover a fixed torus. The top chern class is g , but we are instructed to take the $(top - 1)$ class, which is $c_{g-1}(\mathcal{H})$. Note that the dimension of \mathcal{M} agrees with $(g - 1) + (g - 1) + 1$ as expected from (2.7). Let us first consider the case of $g = 2$. In this case we are instructed to compute $\int k \wedge c_1 \wedge \mathcal{J}$ over the moduli space of holomorphic maps which is of dimension 3; 2 coming from the choices of two branch points and 1 from the image of the map on R^2 . As discussed above the k integrates over the R^2 part and gives the volume in the R^2 direction. Moreover \mathcal{J} gives the volume form over the torus, i.e. absorbs the zero mode corresponding to shift of the origins of the map on the torus direction. Note that if we did not have \mathcal{J} and if we have c_2 instead of c_1 the computation would have been the standard $N = 2$ topological computation which would have vanished because of the flatness of the torus. This agrees with the general argument that the J insertion is crucial for a non-vanishing answer. We are thus left with $\int c_1$ over the moduli of holomorphic maps from genus 2 to torus, up to a shift in the origin of the torus. This is precisely the object we encountered in explicit computation in section 3.

5.1. Genus 2 Topological Computation

We have seen that the $\bar{t} \rightarrow \infty$ of genus 2 computation of the top component amplitude is the same as integration of the first Chern class c_1 of the Hodge bundle over the one dimensional space of moduli of holomorphic maps. Moreover using other argument we have shown that the top component is proportional to E_4 . We will now prove that the answer being proportional to E_4 could have also been derived using the direct topological computation.

To this end we have to use the fact that c_1 for genus 2 can be written as

$$c_1 = 2\pi i \partial \bar{\partial} \log \det \text{Im} \Omega$$

and try to use integration by parts to integrate over moduli of holomorphic curves. However in order to do this we cannot directly use the above expression because $\det \operatorname{Im}\Omega$ is not a modular invariant object. Instead we write it as⁶.

$$c_1 = \frac{1}{2\pi i} \partial \bar{\partial} \log \left[\det \operatorname{Im}\Omega \left(\prod_{\text{even } \theta \text{ functions}} \vartheta^{\bar{\vartheta}} \right)^{1/5} \right]$$

which is modular invariant. Note that product of even θ functions has no zeroes in the interior of the moduli space for $g = 2$ (a fact that fails to be true for higher genera). Since we have a total derivative we can integrate by parts and we thus come to the point on the moduli of holomorphic maps which corresponds either to a handle degeneration or to splitting to two genus 1 curves. The product of even theta functions in the handle degeneration case has a zero of the order $z^{1/2}$ and in the case of splitting a zero of the order z . So in order to compute $\int_{\mathcal{M}} c_1$ we simply have to count how many holomorphic curves exist which go from a handle degenerated genus 2 to torus and multiply it by $1/10$ and add to it the number of holomorphic curves which exist when we have the splitting case and multiply it by $1/5$. This is described mathematically by the statement that

$$c_1 = \frac{1}{10} (2\delta_1 + \delta_0) \tag{5.1}$$

where δ_1 denotes the first chern class of a bundle whose divisor is the boundary of moduli space corresponding to genus 2 splitting to two genus 1 curves and δ_0 denotes the the corresponding one where the divisor is the boundary of moduli space where the genus 2 curve has a handle degeneration. Note that we have chosen coordinates on the moduli space such that a symmetry factor of $1/2$ in the δ_0 and δ_1 degenerations are included.

Using (5.1) we are in a position to compute the genus 2 topological amplitude in terms of genus 1 amplitude⁷. First note that a genus 2 covering of a torus will lead to two branch points. The degenerate genus 2 curves can occur only when the two branch points collide. Not every colliding branch points give rise to degenerate Riemann surfaces, as some of them simply convert 2 branch points of order 2 to a single one of order three. Those would not contribute to our amplitude. To count the degenerations of the other type, note that if you remove the degenerate preimage we end up in the handle degeneration case to a holomorphic map from torus to torus where we have marked two of the covering sheets

⁶ Which is the same trick that give the 2 loop bosonic string amplitude [15]

⁷ We are grateful to R. Dijkgraaf for explaining this to us.

(the ones which get glued over the handle degeneration) and in the splitting case to two genus one curves connected by a tube. In the handle degeneration case if the remaining genus 1 to torus map is of degree n , we have $n(n-1)/2$ ways to choose the sheets, and so putting all the contributions of these together, and denoting the genus 1 answer by F_1 (the topological part of it which is $dF_1/dt = \eta'/2\pi i\eta$) we see that the handle degeneration gives (noting that $1/2$ is already counted in the definition of δ_0)

$$\frac{1}{10} \cdot \left[\frac{d^2 F_1}{dt^2} - \left(\frac{dF_1}{dt} + \frac{1}{24} \right) \right]$$

(note that each d/dt gives a factor of n —note that since we are in the topological limit of $\bar{t} \rightarrow \infty$ we do not have covariantization of d/dt). We have added $+1/24$ to dF_1/dt to eliminate the degree zero part of the map which we take into account separately below. Similarly when we get the splitting case we get two maps from two different genus 1 curves to our torus. We simply have to choose a sheet from each one to identify with the other. If one of them is a covering of order n and the other of order m , we get nm ways of doing this. We also have to divide by a symmetry factor of $1/2$ because of the Z_2 symmetry of exchanging the two genus 1 curves. We thus get a contribution from the splitting case (noting that the symmetry factor $1/2$ is already included in the definition of δ_1)

$$\frac{1}{5} \cdot \left(\frac{dF_1}{dt} + \frac{1}{24} \right)^2$$

In addition to these two contributions we have bubbling type contributions, which correspond to degenerate maps from a genus 2 to the torus, where the genus 2 curve is itself a torus glued to another torus, where one torus gets mapped to a constant, and the other gets holomorphically mapped to the torus. The c_1 of this family will simply be the c_1 of the genus 1 curves times the one point function of the genus 1 answer. Since c_1 on the genus 1 moduli space gives $1/12$, the bubbling contribution is given by

$$\frac{1}{12} \left(\frac{dF_1}{dt} + \frac{1}{24} \right)$$

There is also going to be an overall constant contribution coming from genus 2 curves which map to a constant. Using the topological formula (2.7), and the fact that in this case $\int k \wedge \mathcal{J}$ absorb the volume integral over $T^2 \times R^2$, this should be $c_3(\mathcal{H} \oplus \mathcal{H}) = 2c_1(\mathcal{H})c_2(\mathcal{H})$ and using the fact that $2c_2 = (c_1)^2$ it is given by $(c_1)^3$. Integrated over moduli of genus 2 curves, this gives $\frac{1}{2880}$. Putting all these three contributions together we find

$$\frac{1}{10} \cdot \left[\frac{d^2 F_1}{dt^2} - \left(\frac{dF_1}{dt} + \frac{1}{24} \right) \right] + \frac{1}{5} \cdot \left(\frac{dF_1}{dt} + \frac{1}{24} \right)^2 + \frac{1}{12} \left(\frac{dF_1}{dt} + \frac{1}{24} \right) + \frac{1}{2880}$$

It is quite miraculous that all the terms which are not second order in derivatives of t disappear as they should in order to end up with a function of a definite modular weight. Moreover E_4 which was shown to be proportional to the genus 2 answer is proportional to $\frac{d^2 F_1}{dt^2} + 2\left[\frac{dF_1}{dt}\right]^2$ as expected. We thus learn that

$$F_{2,2}^2 = \frac{1}{2880} E_4 = \frac{1}{10} \left(\frac{d^2 F_1}{dt^2} + 2 \left(\frac{dF_1}{dt} \right)^2 \right)$$

5.2. $g \geq 3$

If we consider $g \geq 3$ the above topological computation formally vanishes, because we get a higher power of k . Since all of them are in the direction of R^2 , and there is only one such direction on the moduli space and if the topological amplitude were given by the above formula, we would get zero. In fact this is precisely an example of the type mentioned at the end of section 2, where the extra insertions go to modifying the bundle \mathcal{V} . This is clear from the explicit attempt in computation of the amplitude for $g \geq 3$ because then we can no longer replace the fermion fields by the zero mode wave functions, as is possible for genus 2—some of the fermions are contracted, giving us Greens functions, which are reinterpreted as curvature of a bundle, as in [11]. In such a case presumably methods similar to those of [11] should be applicable to determine the new bundle $\tilde{\mathcal{V}}$ which we expect to be of rank $2g - 2$, and for which the amplitude can be written as

$$F_{2g-2, 2g-2}^g \Big|_{\bar{t} \rightarrow \infty} = \int k \wedge c_{2g-3}(\tilde{\mathcal{V}}) \wedge \mathcal{J}$$

We have not determined this bundle.

6. Speculations and Conjectures

From the discussions in this paper it is clear that the $N = 2$ string theory on $R^2 \times T^2$ has a lot of resemblance to the large N description of 2d Yang-Mills as a string theory [16]. Even though the precise topological computation we ended up with was not exactly the one appearing in the large N limit of 2d Yang-Mills on a torus it is very close to it, in that the primary objects in both cases is the moduli of holomorphic maps from Riemann surfaces to a torus. It is thus natural to ask if there is any gauge theory which would give us, as a large N expansion, the $N = 2$ string.

In order to narrow down the search we should recall the natural setting in which the $N = 2$ string theory is defined. First of all the target space dimension of $N = 2$

string is four thus we are looking for a $4d$ gauge theory. Another fact motivated from the connection between large N limit of $2d$ Yang-Mills theory and string theory is that the latter has no propagating degrees of freedom and it has only global topological degrees of freedom (which was the reason for its exact solvability [17]). Another fact is that we are looking for a theory which makes sense only in two *complex* dimensions, as that is the natural setting for $N = 2$ strings. We only know of one class of gauge theories which satisfies all these requirements: It is known as the *holomorphic Yang-Mills theory* in $4d$ [5][6]. It is basically the ordinary Yang-Mills theory with no matter, but formulated in two complex dimensions and with the requirement that the field strength be holomorphic. This means that, in holomorphic notation,

$$F^{2,0} = F^{0,2} = 0 \tag{6.1}$$

and the only non-vanishing component of F is in the $F^{1,1}$ direction. Of course in general one cannot just set constraints such as (6.1) and expect to get a consistent field theory. However, it can be done in this case [5][6]. The idea is based on the link established between the $2d$ Yang-Mills and Donaldson theory in $2d$ [18]. It was shown there that ordinary $2d$ Yang-Mills theory can be viewed as the deformation of $2d$ topological Yang-Mills. Recall that topological Yang-Mills is a twisted version of $N = 2$ Yang-Mills. The four dimensional analog turns out to be the natural generalization of this construction: One starts from $N = 2$ Yang-Mills theory and twists it to obtain the Donaldson theory [19] and then perturb it using certain observables of Donaldson theory and in addition with some topologically trivial deformation. In this case one obtains the holomorphic Yang-Mills theory. Thus the theory makes sense as a quantum field theory.

Note that there are no local degrees of freedom in holomorphic Yang-Mills theory. To see this we have to go to a Minkowskian version of the theory. Consider signature $(1,1)$ in complex notation (which is the one which also appears for $N = 2$ strings). Let

$$A_i = \int \epsilon_i(p, \bar{p}) \exp(i(p \cdot \bar{x} + \bar{p} \cdot x))$$

$$\bar{A}_i = \int \bar{\epsilon}_i(p, \bar{p}) \exp(i(p \cdot \bar{x} + \bar{p} \cdot x))$$

where $i = 1, 2$ denote the holomorphic index. Then the linearized equations of motion imply that the support of ϵ is on $p \cdot p = 0$; moreover the lorentz gauge condition implies

$$p \cdot \bar{\epsilon} + \bar{p} \cdot \epsilon = 0$$

This cuts down the real degrees of freedom to 2. However we have the constraints (6.1) which imply

$$\epsilon_{(i p_j)} = 0$$

where we antisymmetrize in indices, and this further cuts down the number by two leaving us with no propagating degrees of freedom as desired⁸.

One other fact which suggests that large N version of gauge theory in 4d may lead to a string theory is the fact that for finite N we can turn on 't Hooft magnetic fluxes on the manifold which live on $H^2(M, \mathbf{Z}_N)$. As $N \rightarrow \infty$ the choice of the flux gets related to $H^2(M, U(1))$ which is precisely the choice of the antisymmetric field B that can be turned on for string theory that for a fixed N did not have a gauge theory analog.

Putting all this together we feel we have some evidence for the following conjecture:

The Large N limit of holomorphic Yang-Mills is equivalent to $N = 2$ strings.

How do we check this conjecture? The natural method should be first to solve the holomorphic Yang-Mills theory in 4d, just as Migdall solved the 2d theory. This should be an exactly solvable theory as there are no propagating modes, and steps in solving it have been taken [5]. Another related computation, given the relation of holomorphic Yang-Mills theory with the $N = 2$ $SU(N)$ Yang-Mills theory, is the large N computations of Douglas and Shenker [20].

Perhaps the simplest case to check would be holomorphic Yang-Mills on T^4 (or perhaps $K3$). Another test, also related to the computation we have done in this paper is to study the reduction of the holomorphic Yang-Mills to 2d on a small torus. Note that in string language a small torus and a big torus are equivalent by $R \rightarrow 1/R$ so the case we have been considering on $T^2 \times R^2$ can be viewed in this way. Thus we should get a gauge theory in 2d which for large N should reproduce the computations we have done in this paper for all N . Formally this theory should be a deformation of $N = 4$ topological theory in 2d, which at the topological level computes the Euler characteristic of Hitchin space [21]. Note that, being a deformation of a topological theory, it continues to have no propagating degrees

⁸ This same counting also work for one complex dimension, as there the constraints (6.1) are vacuous. In complex dimensions bigger than 2, for generic p , the constraints (6.1) will lead to 'negative' number of degrees of freedom. Thus complex dimensions 1 and 2 are critical for holomorphic Yang-Mills theory.

of freedom. We expect this deformation of 2 dimensional model be exactly solvable also, just as 2d Yang-Mills is exactly solvable. This would be very interesting to compute, as the results of this paper provide an all order prediction for its large N behaviour.

Another point we wish to comment on is whether we can sum up the perturbation series which we have computed for the example of $T^2 \times R^2$. Note that in this paper we computed the amplitudes for each g up to an overall g -dependent (but modulus independent) constant. It is tempting to speculate whether there is a natural choice of the overall constant which would make the summing up lead to a nice answer. Even though this may not be a strong test, we have found one particularly simple choice, which reproduces all g answers, which agrees with the normalizations we have obtained for $g = 1, 2$ (up to redefinition of string coupling constant). Let $\lambda = u_L^1 u_R^1$ and $\bar{\lambda} = u_L^2 u_R^2$ (we can absorb the definition of string coupling constant into this), then we have seen in this paper that

$$F^g(\lambda, \bar{\lambda}) \sim \sum_{(n,m) \neq (0,0)} |n + m\sigma|^{2g-4} \left(\frac{\lambda}{n + m\sigma} + \frac{\bar{\lambda}}{n + m\bar{\sigma}} \right)^{4g-4}$$

(note in particular that the limit $\tau \rightarrow \infty$ is proportional to $(\lambda + \bar{\lambda})^{4g-4}$, as expected, only the $m = 0$ contributed to the sum). The natural guess for summing up all g is thus a geometrical sum

$$\sum_g F^g(\lambda, \bar{\lambda}) = \sum_{(n,m) \neq (0,0)} \frac{1}{|n + m\sigma|^2 - (\lambda \sqrt{\frac{n+m\bar{\sigma}}{n+m\sigma}} + \bar{\lambda} \sqrt{\frac{n+m\sigma}{n+m\bar{\sigma}}})^4}$$

It would be interesting to find out whether this correctly captures the g -dependent constant.

In this paper we have talked about $N = 2$ strings which is equivalent to $N = 4$ topological strings, with critical dimension 4. It would be tempting to connect this with $N = 2$ topological strings which has critical dimension 6 (corresponding to topological sigma models on Calabi-Yau threefolds). One idea along this line, suggested to us by Yau, is to consider the twistor space. Recall that the twistor space includes a one parameter complex deformation of the complex structure of the manifold, without changing the metric. In particular if ω denotes the holomorphic 2-form and k the Kähler class, then we consider the new holomorphic 2-form $\Omega(t)$ to be

$$\Omega(t) = \omega + t k + t^2 \bar{\omega}$$

The total space of the manifold including the parameter t is a three dimensional complex manifold. It unfortunately does not have $c_1 = 0$. However, if we turn on an anti-symmetric 2-form on the three manifold defined by $B = \Omega(t)$ which is a 2-form, we can modify the condition for conformality from $c_1 = 0$ because we now have $H = dB \neq 0$. In fact we have checked that using the ideas of the construction of stringy cosmic strings [22] that the resulting theory would be a conformal theory. Thus it may be true that an $N = 2$ topological string on the twistor space, with the B -field turned on, is equivalent to $N = 4$ topological string on the 4 manifold. This we find an extremely interesting possibility, which deserves further study.

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Appendix A. Vanishing of Boundary Terms in the Harmonicity Equation

In section 3, we have proven that the $N = 2$ string amplitude F^g satisfies the harmonicity equation

$$\epsilon^{ab} u_R^c \frac{\partial}{\partial u_L^a} \frac{D}{Dt^{bc}} F^g(u_L, u_R) = 0$$

provided

$$\begin{aligned} & \int_{\mathcal{M}_g} \langle (\mu_{A'}, T_L) \prod_{A \neq A'} (\mu_A, \widehat{G}_L^-(u_L)) \prod_{A=1}^{3g-3} (\bar{\mu}_A, \widehat{G}_R^-(u_R)) \rangle \times \\ & \times \int_{\Sigma} J_L J_R \left[\int_{\Sigma} \widehat{G}_L^+(u_L) \widehat{G}_R^+(u_R) \right]^{g-1} \int_{\Sigma} [\widehat{Q}_R^+(u_R), \bar{\phi}] \rangle \end{aligned} \quad (\text{A.1})$$

and

$$\begin{aligned} & \int_{\mathcal{M}_g} \langle (\mu_{A'}, T_L) (\mu_{A''}, J_L^{--}) \prod_{A \neq A', A''} (\mu_A, \widehat{G}_L^-(u_L)) \prod_{A=1}^{3g-3} (\bar{\mu}_A, \widehat{G}_R^-(u_R)) \rangle \times \\ & \times \int_{\Sigma} J_L J_R \left[\int_{\Sigma} \widehat{G}_L^+(u_L) \widehat{G}_R^+(u_R) \right]^{g-1} \int_{\Sigma} \{ \widehat{Q}_L^+(u_L), [\widehat{Q}_R^+(u_R), \bar{\phi}] \} \rangle \end{aligned} \quad (\text{A.2})$$

vanish. Here we will show that this is indeed the case. Since the insertion of $(\mu_{A'}, T_L)$ generate a derivative $\partial/\partial m_{A'}$ on the moduli space \mathcal{M}_g in the direction of the Beltrami differential $\mu_{A'}$, we just need to check that

$$\begin{aligned}
W_{A'} &= \langle \prod_{A \neq A'} (\mu_A, \widehat{G}_L^-(u_L)) \prod_{A=1}^{3g-3} (\bar{\mu}_A, \widehat{G}_R^-(u_R)) \times \\
&\quad \times \int_{\Sigma} J_L J_R \left[\int_{\Sigma} \widehat{G}_L^+(u_L) \widehat{G}_R^+(u_R) \right]^{g-1} \int_{\Sigma} [\widehat{Q}_R^+(u_R), \bar{\phi}] \rangle \\
V_{A', A''} &= \langle (\mu_{A''}, J_L^-) \prod_{A \neq A', A''} (\mu_A, \widehat{G}_L^-(u_L)) \prod_{A=1}^{3g-3} (\bar{\mu}_A, \widehat{G}_R^-(u_R)) \int_{\Sigma} J_L J_R \times \\
&\quad \times \left[\int_{\Sigma} \widehat{G}_L^+(u_L) \widehat{G}_R^+(u_R) \right]^{g-1} \int_{\Sigma} \{ \widehat{Q}_L^+(u_L), [\widehat{Q}_R^+(u_R), \bar{\phi}] \} \rangle
\end{aligned}$$

vanish at the boundary of \mathcal{M}_g whose normal direction is $\partial_{m_{A'}}$.

As we approach the boundary, the Riemann surface will degenerate and acquire a node. By conformal invariance, we can transform the node into a cylinder whose length becomes infinite at the boundary. In this limit, $(\bar{\mu}_A, \widehat{G}_R^-)$ with $A = A'$ becomes a contour integral $\oint \widehat{G}_R^-$ around the homology cycle of the cylinder. Since states propagating along the cylinder are projected onto zero energy states as we approach the boundary, they will be annihilated by $\oint \widehat{G}_R^-$ unless there is another operator on the cylinder which does not (anti-) commute with \widehat{G}_R^- . The only operator in $V_{A', A''}$ and $W_{A'}$ which does not commute with \widehat{G}_R^- is J_R . However since the commutator of J_R and \widehat{G}_R^- is proportional to \widehat{G}_R^- itself, the zero energy states are still annihilated by $\oint \widehat{G}_R^-$ even if J_R is inserted on the cylinder. Thus we find that $V_{A', A''}$ and $W_{A'}$ vanish as we approach the boundary of the moduli space, and this is what we wanted to show.

Appendix B. Genus Two Amplitude on T^4

In this section, we will derive the following expression of F^g at $g = 2$ when the target space is T^4 .

$$\begin{aligned}
F^2(u_L, u_R) &= \sum_{P_L, P_R} \int_{\mathcal{M}_2} \left(\frac{\det g}{\det \text{Im} \Omega} \right)^2 \langle [\det(\widehat{P}_L + \widehat{r}_L) \det(\widehat{P}_R + \widehat{r}_R)]^2 \rangle \times \\
&\quad \times \exp[-S(P_L, P_R)] \frac{d^3 \Omega d^3 \bar{\Omega}}{[\det \text{Im} \Omega]^2}.
\end{aligned} \tag{B.1}$$

The notations will be explained in the following.

Let us start with the definition of F^2

$$F^2 = \int_{\mathcal{M}_2} d^3m d^3\bar{m} \langle (\mu_1, \widehat{G}_L^-)(\mu_2, \widehat{G}_L^-)(\bar{\mu}_1, \widehat{G}_R^-)(\bar{\mu}_2, \widehat{G}_R^-) \times (\mu_3, J_L^{--})(\bar{\mu}_3, J_R^{--}) [\int \widetilde{G}_L^+ \widetilde{G}_R^+]^2 \rangle.$$

This formula for F^2 contains *two* \widehat{G}_L^- , *one* J_L^{--} and *two* \widetilde{G}_L^+ given by

$$\begin{aligned} \widehat{G}_L^- &= g_{i\bar{j}} \psi_L^{\bar{j}} \partial \widehat{X}^i \\ J_L^{--} &= \epsilon_{i\bar{j}} \psi_L^{\bar{i}} \psi_L^{\bar{j}} \\ \widetilde{G}_L^+ &= \epsilon_{ij} \psi_L^i \partial \widehat{X}^j, \end{aligned} \tag{B.2}$$

where

$$\partial \widehat{X}^i = u_L^1 \partial X^i + u_L^2 \epsilon^{ij} g_{j\bar{k}} \partial X^{\bar{j}}.$$

Thus there are *two* ψ_L and *four* $\bar{\psi}_L$ in F^2 . When the target space is T^4 there are 2 zero modes for ψ_L and $2g$ zero modes for $\bar{\psi}_L$ on genus g since ψ_L 's are zero-forms and $\bar{\psi}_L$'s are one-forms. Therefore, at genus 2, the fermions in F^2 just absorb their zero modes and the computation of F^2 does not involve the Green function. This will simplify the computation at $g = 2$. For $g \geq 3$, we must deal with the Green function of the fermions.

For the bosonic field X^i , we use the decomposition

$$X^i(z, \bar{z}) = X_0^i(z, \bar{z}) + \phi^i(z, \bar{z})$$

where X_0^i is the classical part obeying $\partial_z \bar{\partial}_{\bar{z}} X_0^i = 0$ and ϕ^i is the quantum part which is single-valued on Σ . The classical part X_0^i is parametrized by a set of integers $n_a^i, m^{ia}, s_a^i, r^{ia}$ ($i, a = 1, 2$) as

$$\begin{aligned} X_0^i(z, \bar{z}) &= (m^{ia} + \rho_j^i r^{ja}) \text{Re} \int^z \omega_a + \\ &+ [(n_a^i + \rho_j^i s_a^j) - (m^{ib} + \rho_j^i r^{jb}) \text{Re} \Omega_{ba}] (\text{Im} \Omega^{-1})^{ac} \text{Im} \int^z \omega_c \end{aligned} \tag{B.3}$$

where ω_a is the holomorphic one-forms on Σ and Ω_{ab} is their period matrix. The matrix ρ_j^i characterize the complex structure of the target space torus T^4 as

$$T^4 = R^4 / (x^i \sim x^i + n^i + \rho_j^i m^j) \quad (n^i, m^i \in \mathbf{Z}).$$

The quantum part ϕ^i is the free boson whose propagator is given by

$$\begin{aligned}\langle \partial_z \phi^i \partial_w \phi^{\bar{j}} \rangle &= g^{i\bar{j}} [-\partial_z \partial_w \log E(z, w) + \omega_a(z) \omega_b(w) (\text{Im} \Omega^{-1})^{ab}] \\ \langle \partial_z \phi^i \partial_{\bar{w}} \phi^{\bar{j}} \rangle &= -g^{i\bar{j}} \omega_a(z) \bar{\omega}_b(\bar{w}) (\text{Im} \Omega^{-1})^{ab} \quad (z \neq w),\end{aligned}\tag{B.4}$$

where $E(z, w)$ is the prime form on Σ . In F^2 , ϕ^i and $\phi^{\bar{i}}$ appear in the combinations

$$\begin{aligned}\partial \widehat{\phi}^i &= u_L^1 \partial \phi^i + u_L^2 \epsilon^{ij} g_{j\bar{k}} \partial \phi^{\bar{k}} \\ \bar{\partial} \widehat{\phi}^{\bar{i}} &= u_R^1 \bar{\partial} \phi^{\bar{i}} + u_R^2 \epsilon^{\bar{i}\bar{j}} g_{\bar{j}k} \bar{\partial} \phi^k,\end{aligned}$$

and their Wick contraction rules are

$$\begin{aligned}\langle \partial_z \widehat{\phi}^i \partial_w \widehat{\phi}^{\bar{j}} \rangle &= \langle \partial_z \widehat{\phi}^{\bar{i}} \partial_{\bar{w}} \widehat{\phi}^{\bar{j}} \rangle = 0 \\ \langle \partial_z \widehat{\phi}^i \bar{\partial}_{\bar{w}} \widehat{\phi}^{\bar{i}} \rangle &= -g^{i\bar{j}} (u_L^1 u_R^1 + u_L^2 u_R^2) \omega_a(z) \bar{\omega}_b(\bar{w}) (\text{Im} \Omega^{-1})^{ab}.\end{aligned}$$

Therefore for the purpose of evaluating F^2 , we may write the bosonic field X^i as

$$\begin{aligned}\partial \widehat{X}^i &= (\widehat{P}_{La}^i + \widehat{r}_{La}^i) (\text{Im} \Omega^{-1})^{ab} \omega_b \\ \bar{\partial} \widehat{X}^{\bar{i}} &= (\widehat{P}_{Ra}^{\bar{i}} + \widehat{r}_{Ra}^{\bar{i}}) (\text{Im} \Omega^{-1})^{ab} \bar{\omega}_b,\end{aligned}\tag{B.5}$$

where

$$\begin{aligned}\widehat{P}_{La}^i &= u_L^1 P_{La}^i + u_L^2 \epsilon^{ij} g_{j\bar{k}} P_{La}^{\bar{k}} \\ \widehat{P}_{Ra}^{\bar{i}} &= u_R^1 P_{Ra}^{\bar{i}} + u_R^2 \epsilon^{\bar{i}\bar{j}} g_{\bar{j}k} P_{Ra}^k\end{aligned}\tag{B.6}$$

and

$$\begin{aligned}P_{La}^i &= (n_a^i + \rho_j^i s_a^j) - (m^{ib} + \rho_j^i r^{jb}) \bar{\Omega}_{ba} \\ P_{La}^{\bar{i}} &= (n_a^{\bar{i}} + \bar{\rho}_j^{\bar{i}} s_a^{\bar{j}}) - (m^{ib} + \bar{\rho}_j^{\bar{i}} r^{jb}) \bar{\Omega}_{ba} \\ P_{Ra}^i &= (n_a^i + \rho_j^i s_a^j) - (m^{ib} + \rho_j^i r^{jb}) \Omega_{ba} \\ P_{Ra}^{\bar{i}} &= (n_a^{\bar{i}} + \bar{\rho}_j^{\bar{i}} s_a^{\bar{j}}) - (m^{ib} + \bar{\rho}_j^{\bar{i}} r^{jb}) \Omega_{ba},\end{aligned}\tag{B.7}$$

and \widehat{r}_L and \widehat{r}_R are quantum variables obeying the Wick rules

$$\begin{aligned}\langle \widehat{r}_{La}^i \widehat{r}_{Lb}^{\bar{j}} \rangle &= \langle \widehat{r}_{Ra}^{\bar{i}} \widehat{r}_{Rb}^{\bar{j}} \rangle = 0 \\ \langle \widehat{r}_{La}^i \widehat{r}_{Rb}^{\bar{j}} \rangle &= -g^{i\bar{j}} (u_L^1 u_R^1 + u_L^2 u_R^2) (\text{Im} \Omega)_{ab}\end{aligned}\tag{B.8}$$

Now we are ready to evaluate F^2 . Since ψ_L^i and $\psi_R^{\bar{i}}$ are zero-forms, it is natural to normalize their zero modes as

$$\begin{aligned}\langle \psi_L^i(z_1) \psi_L^{\bar{j}}(z_2) \rangle &= \epsilon^{i\bar{j}} \\ \langle \psi_R^{\bar{i}}(\bar{z}_1) \psi_R^{\bar{j}}(\bar{z}_2) \rangle &= \epsilon^{\bar{i}\bar{j}}.\end{aligned}$$

Thus ψ_L^i and $\psi_R^{\bar{i}}$ may be regarded as constant on Σ , and we can perform the surface integral of

$$\begin{aligned}\widehat{\widetilde{G}}_L^+ &= \epsilon_{ij} \psi_L^i (\widehat{P}_{La}^j + \widehat{r}_{La}^j) (\text{Im}\Omega)^{ab} \omega_b \\ \widehat{\widetilde{G}}_R^+ &= \epsilon_{\bar{i}\bar{j}} \psi_R^{\bar{i}} (\widehat{P}_{Ra}^{\bar{j}} + \widehat{r}_{Ra}^{\bar{j}}) (\text{Im}\Omega^{-1})^{ab} \bar{\omega}_b,\end{aligned}$$

as

$$\int \widehat{\widetilde{G}}_L^+ \widehat{\widetilde{G}}_R^+ = \epsilon_{ij} \psi_L^i \epsilon_{\bar{i}\bar{j}} \psi_R^{\bar{i}} (\widehat{P}_{La}^j + \widehat{r}_{La}^j) (\widehat{P}_{Rb}^{\bar{j}} + \widehat{r}_{Rb}^{\bar{j}}) (\text{Im}\Omega^{-1})^{ab}.$$

The expectation value of these operators then becomes

$$\begin{aligned}\langle \int \widehat{\widetilde{G}}_L^+ \widehat{\widetilde{G}}_R^+ \int \widehat{\widetilde{G}}_L^+ \widehat{\widetilde{G}}_R^+ \rangle &= \\ &= \det(\widehat{P}_L + \widehat{r}_L) \det(\widehat{P}_R + \widehat{r}_R) (\det \text{Im}\Omega)^{-1}.\end{aligned}\tag{B.9}$$

For $\psi_L^{\bar{i}}$ and ψ_R^i zero modes, we can express them as a linear combination of the holomorphic and anti-holomorphic one-forms ω_a ($a = 1, 2$) as

$$\begin{aligned}\psi_L^{\bar{i}}(z) &= \theta_L^{\bar{i}a} \omega_a(z) \\ \psi_R^i(\bar{z}) &= \theta_R^{ia} \bar{\omega}_a(\bar{z})\end{aligned}\tag{B.10}$$

where $\theta_L^{\bar{i}a}$ and θ_R^{ia} are Grassmannian variables. We normalize them as

$$\langle \theta_L^{\bar{i}1} \theta_L^{\bar{j}1} \theta_L^{\bar{k}2} \theta_L^{\bar{l}2} \theta_R^{i1} \theta_R^{j1} \theta_R^{k2} \theta_R^{l2} \rangle = \epsilon^{\bar{i}\bar{j}} \epsilon^{\bar{k}\bar{l}} \epsilon^{ij} \epsilon^{kl}.\tag{B.11}$$

In evaluating F^2 , we may replace the fermions by their zero modes (B.10) as

$$\begin{aligned}\widehat{G}_L^- &= g_{\bar{i}\bar{j}} \theta_L^{\bar{j}a} (\widehat{P}_{Lb}^i + \widehat{r}_{Lb}^i) (\text{Im}\Omega^{-1})^{bc} \omega_a \omega_c \\ J_L^{--} &= \epsilon_{\bar{i}\bar{j}} \theta_L^{\bar{i}a} \theta_L^{\bar{j}b} \omega_a \omega_b.\end{aligned}$$

Here the following formula becomes useful.

$$(\mu_A, \omega_a \omega_b) = \int \mu_A \omega_a \omega_b = \frac{\partial \Omega_{ab}}{\partial m^A},$$

where $\partial/\partial m_A$ is a derivative on the moduli space \mathcal{M}_2 in the direction specified by the Beltrami-differential μ_A . Thanks to this formula, we can perform the surface integrals of \widehat{G}_L^- and J_L^{--} as

$$\begin{aligned}(\mu_A, \widehat{G}_L^-) &= g_{\bar{i}\bar{j}} \theta_L^{\bar{j}a} (\widehat{P}_{Lb}^i + \widehat{r}_{Lb}^i) (\text{Im}\Omega^{-1})^{bc} \frac{\partial \Omega_{ac}}{\partial m^A} \\ (\mu_A, J_L^{--}) &= \epsilon_{\bar{i}\bar{j}} \theta_L^{\bar{i}a} \theta_L^{\bar{j}b} \frac{\partial \Omega_{ab}}{\partial m^A}.\end{aligned}$$

By working out simple combinatorics, one finds

$$\begin{aligned} & \langle (\mu_1, \widehat{G}_L^-)(\mu_2, \widehat{G}_L^-)(\mu_3, J_L^{--}) \rangle = \\ & = \theta_L^{\bar{1}1} \theta_L^{\bar{1}2} \theta_L^{\bar{2}1} \theta_L^{\bar{2}2} \epsilon_{\bar{1}\bar{2}} \det g \det(\widehat{P}_L + \widehat{r}_L) (\det \text{Im}\Omega)^{-1} \frac{\partial(\Omega_{11}, \Omega_{12}, \Omega_{22})}{\partial(m_1, m_2, m_3)}. \end{aligned}$$

At genus 2, so called the Schottky problem is absent, and we can use the three components of the period matrix Ω_{ab} as coordinates on \mathcal{M}_2 . By taking into the normalization of θ_L and θ_R in (B.11), we obtain

$$\begin{aligned} & \langle (\mu_1, \widehat{G}_L^-)(\mu_2, \widehat{G}_L^-)(\bar{\mu}_1, \widehat{G}_R^-)(\bar{\mu}_2, \widehat{G}_R^-)(\mu_3, J_L^{--})(\bar{\mu}_3, J_R^{--}) \rangle d^3 m d^3 \bar{m} = \\ & = \det g \det(\widehat{P}_L + \widehat{r}_L) \det(\widehat{P}_R + \widehat{r}_R) \frac{d^3 \Omega d^3 \bar{\Omega}}{[\det \text{Im}\Omega]^3} \end{aligned} \quad (\text{B.12})$$

where we used $\det g = \epsilon_{12} \epsilon_{\bar{1}\bar{2}}$ to reduce the expression. This relation between $g_{i\bar{j}}$ and $\epsilon_{ij}, \epsilon_{\bar{i}\bar{j}}$ is required in order for the generators (B.2) to make the $N = 4$ superconformal algebra.

Finally, by combining (B.9) and (B.12), we derive

$$\begin{aligned} & \langle (\mu_1, \widehat{G}_L^-)(\mu_2, \widehat{G}_L^-)(\bar{\mu}_1, \widehat{G}_R^-)(\bar{\mu}_2, \widehat{G}_R^-) \times \\ & \times (\mu_3, J_L^{--})(\bar{\mu}_3, J_R^{--}) \left[\int \widehat{G}_L^+ \widehat{G}_R^+ \right]^2 \rangle d^3 m d^3 \bar{m} = \\ & = \left(\frac{\det g}{\det \text{Im}\Omega} \right)^2 \left[\det(\widehat{P}_L + \widehat{r}_L) \det(\widehat{P}_R + \widehat{r}_R) \right]^2 \frac{d^3 \Omega d^3 \bar{\Omega}}{[\det \text{Im}\Omega]^3}. \end{aligned}$$

One can easily check that this expression is covariant both on \mathcal{M}_2 and T^4 .

To complete the evaluation of F^2 , we need to contract \widehat{r}_L and \widehat{r}_R according to the rule (B.8), and multiply $\exp(-S)$ where S is the classical action for (B.3) given by

$$S = \left(t_{i\bar{j}} P_{La}^i P_{Rb}^{\bar{j}} + \bar{t}_{i\bar{j}} P_{La}^{\bar{j}} P_{Rb}^i \right) (\text{Im}\Omega)^{ab},$$

where

$$\begin{aligned} t_{i\bar{j}} &= g_{i\bar{j}} + i\theta^\alpha k_{i\bar{j}}^\alpha \\ \bar{t}_{i\bar{j}} &= g_{i\bar{j}} - i\theta^\alpha k_{i\bar{j}}^\alpha \end{aligned}$$

and k^α ($\alpha = 1, \dots, h^{1,1}$) are generators of $H^{1,1}(T^4, \mathbf{Z})$. The determinant factors of the bosons ϕ and the fermions ψ_L and ψ_R cancel out. By assembling the ingredients together, we obtain

$$\int_{\mathcal{M}_2} \left(\frac{\det g}{\det \text{Im}\Omega} \right)^2 \left[\det(\widehat{P}_L + \widehat{r}_L) \det(\widehat{P}_R + \widehat{r}_R) \right]^2 \frac{d^3 \Omega d^3 \bar{\Omega}}{[\det \text{Im}\Omega]^3} \exp(-S)$$

Finally we sum this over all n, m, r, s parametrizing P_L and P_R as (B.7). This way, we have derived the expression (B.1) for F^2 .

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